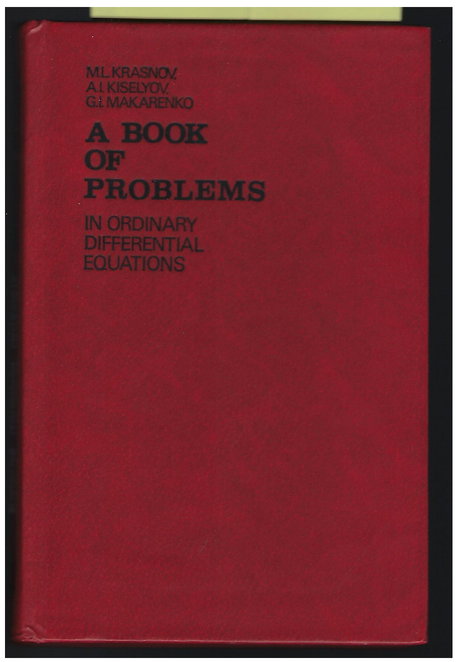
A Solution Manual For

A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983



Nasser M. Abbasi

May 15, 2024

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1.1 problem 2

Internal problem ID [14934]
Internal file name [`OUTPUT/14943_Monday_April_15_2024_12_04_02_AM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[[_Riccati, _special]]`

$$\boxed{y' - y^2 = x^2}$$

1.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^2 + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 + \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2\right) \sqrt{x}$$

The above shows that

$$u'(x) = x^{\frac{3}{2}} \left(\text{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 + \text{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{x\left(\text{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 + \text{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_2\right)}{\text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 + \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x\left(\text{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3 + \text{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right)\right)}{\text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_3 + \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x\left(\text{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3 + \text{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right)\right)}{\text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_3 + \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)} \tag{1}$$

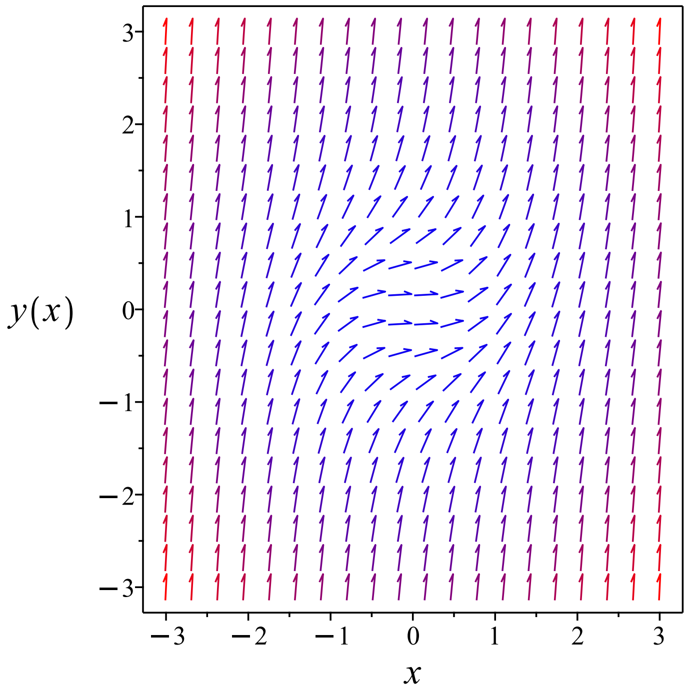


Figure 1: Slope field plot

Verification of solutions

$$y = -\frac{x\left(\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3 + \operatorname{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right)\right)}{\operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_3 + \operatorname{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x)=x^2+y(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{x\left(\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 + \operatorname{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right)\right)}{c_1 \operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 169

```
DSolve[y'[x]==x^2+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2\left(-2\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) + c_1\left(\operatorname{BesselJ}\left(\frac{3}{4}, \frac{x^2}{2}\right) - \operatorname{BesselJ}\left(-\frac{5}{4}, \frac{x^2}{2}\right)\right)\right) - c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{x^2}{2}\right)}{2x\left(\operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{x^2}{2}\right)\right)}$$

$$y(x) \to -\frac{x^2 \operatorname{BesselJ}\left(-\frac{5}{4}, \frac{x^2}{2}\right) - x^2 \operatorname{BesselJ}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{x^2}{2}\right)}{2x \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{x^2}{2}\right)}$$

1.2 problem 3

Internal problem ID [14935]
Internal file name [`OUTPUT/14944_Monday_April_15_2024_12_04_04_AM_33971096/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x}{y} = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = x\, dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int x\, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{x^2 + 2c_1}$$

$$y = -\sqrt{x^2 + 2c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{x^2 + 2c_1} \tag{2}$$

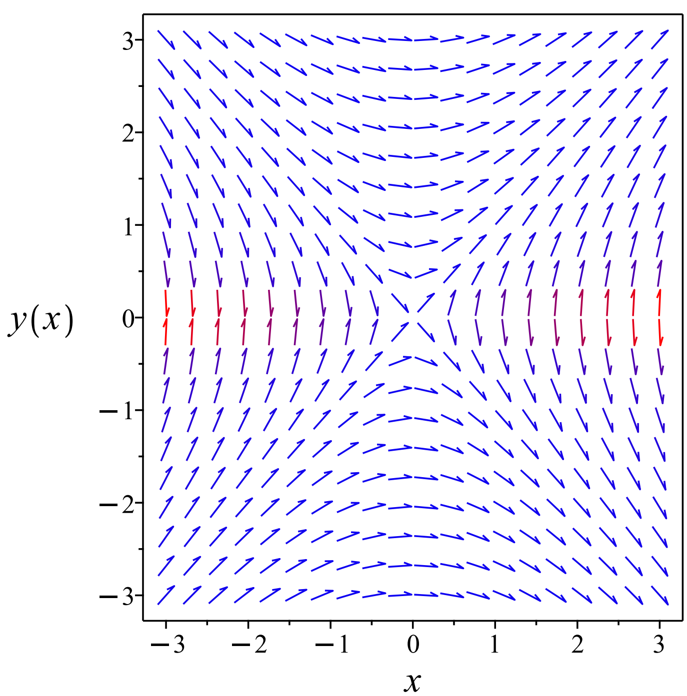


Figure 2: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1}$$

Verified OK.

1.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{1}{u\,(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 1}{xu} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln\,(u-1)}{2} + \frac{\ln\,(u+1)}{2} &= -\ln\,(x) + c_2 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln\,(u-1) + \ln\,(u+1)) &= -\ln\,(x) + 2c_2 \\ \ln\,(u-1) + \ln\,(u+1) &= (2)\,(-\ln\,(x) + 2c_2) \\ &= -2\ln\,(x) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned} u^2 - 1 &= \frac{2c_2}{x^2} \\ &= \frac{c_3}{x^2} \end{aligned}$$

The solution is

$$u(x)^2 - 1 = \frac{c_3}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x^2}$$
$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x^2}$$

Which simplifies to

$$-(-y + x)(y + x) = c_3$$

Summary

The solution(s) found are the following

$$-(-y + x)(y + x) = c_3 \tag{1}$$

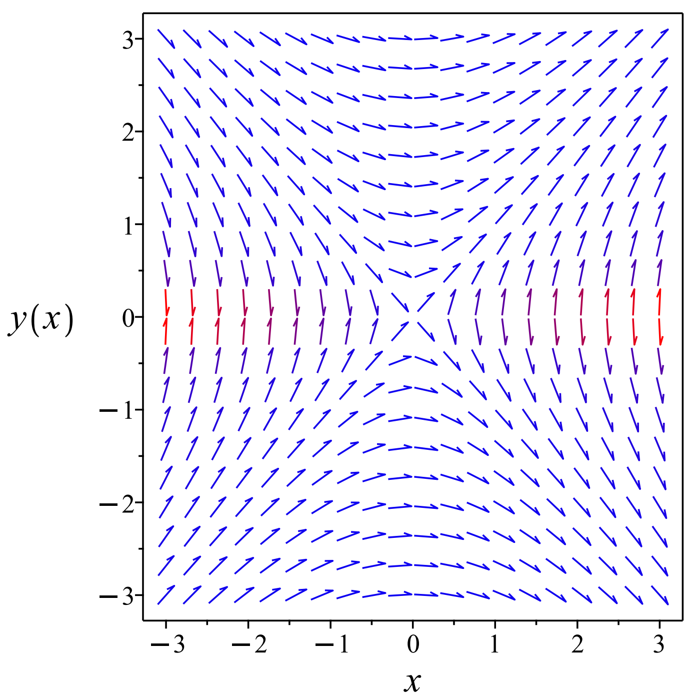


Figure 3: Slope field plot

Verification of solutions

$$-(-y+x)(y+x)=c_3$$

Verified OK.

1.2.3 Solving as differentialType ode

Writing the ode as

$$y'=\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\,dy=(x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(x)\,dx=d\left(\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy=d\left(\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y=\sqrt{x^2+2c_1}+c_1$$
$$y=-\sqrt{x^2+2c_1}+c_1$$

Summary
The solution(s) found are the following

$$y=\sqrt{x^2+2c_1}+c_1 \tag{1}$$
$$y=-\sqrt{x^2+2c_1}+c_1 \tag{2}$$

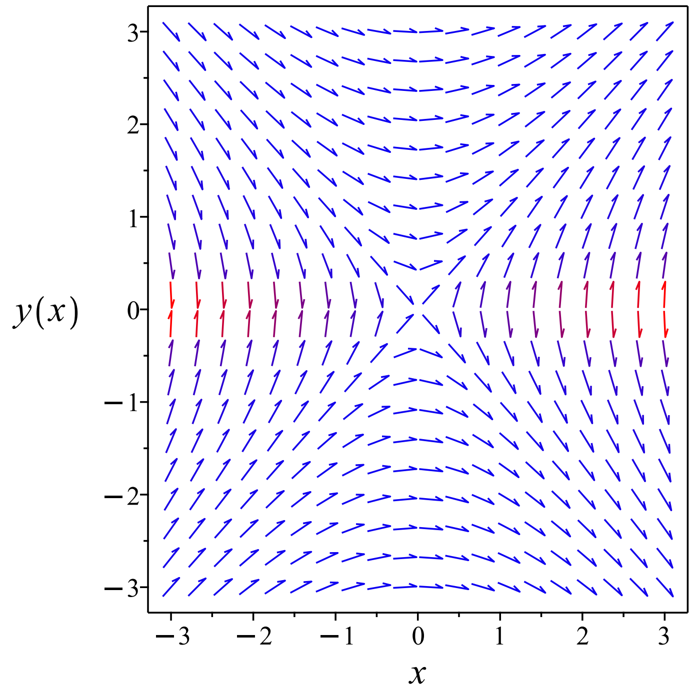


Figure 4: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

1.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= \frac{1}{x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

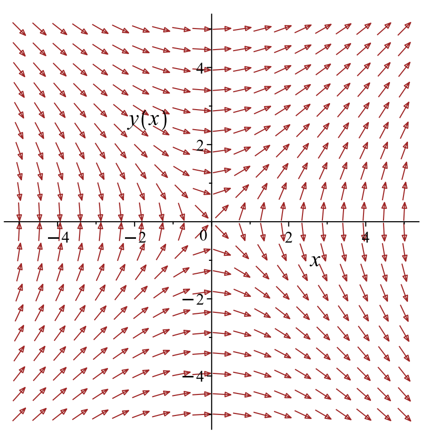
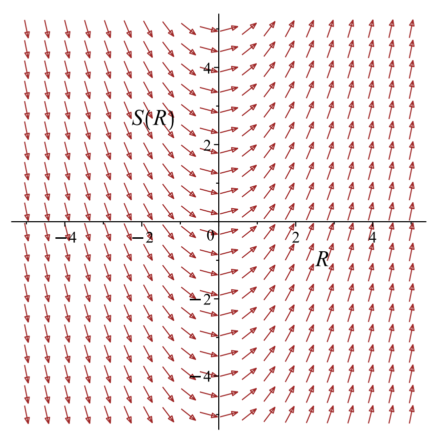
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

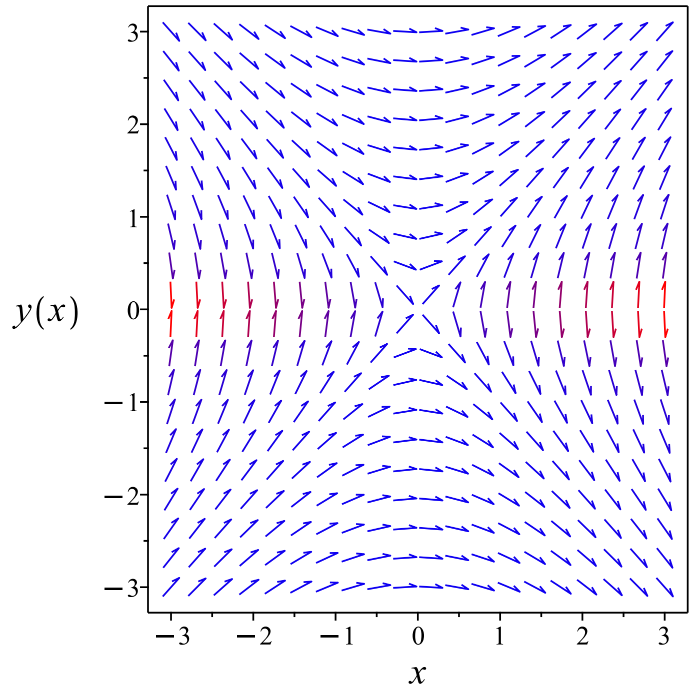


Figure 5: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + (y)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x \,\mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y) \,\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

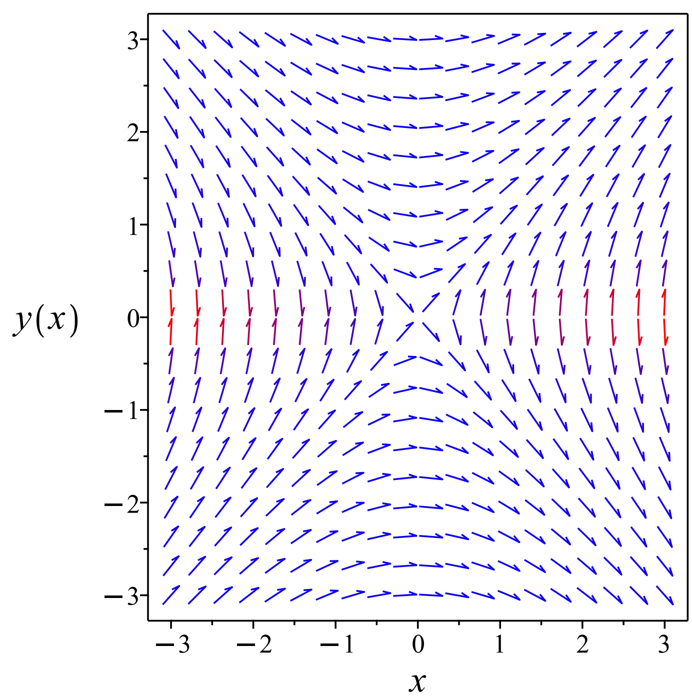


Figure 6: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$y' - \frac{x}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = x$

- Integrate both sides with respect to x

  $\int yy'dx = \int xdx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{x^2 + 2c_1}, y = -\sqrt{x^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=x/y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 35

```
DSolve[y'[x]==x/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x^2 + 2c_1}$$
$$y(x) \to \sqrt{x^2 + 2c_1}$$

1.3 problem 4

Internal problem ID [14936]
Internal file name [OUTPUT/14945_Monday_April_15_2024_12_04_06_AM_23389732/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y - 3y^{\frac{1}{3}} = 0}$$

1.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y + 3y^{\frac{1}{3}}} dy = \int dx$$

$$\frac{\ln(y^2 + 27)}{2} - \frac{\ln\left(y^{\frac{4}{3}} - 3y^{\frac{2}{3}} + 9\right)}{2} + \ln\left(y^{\frac{2}{3}} + 3\right) = x + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y^2+27)}{2} - \frac{\ln\left(y^{\frac{4}{3}} - 3y^{\frac{2}{3}} + 9\right)}{2} + \ln\left(y^{\frac{2}{3}} + 3\right)} = e^{x + c_1}$$

Which simplifies to

$$\frac{\sqrt{y^2 + 27}\left(y^{\frac{2}{3}} + 3\right)}{\sqrt{y^{\frac{4}{3}} - 3y^{\frac{2}{3}} + 9}} = c_2 e^x$$

Summary

The solution(s) found are the following

$$y = \left(\left(e^{2x} c_2^2\right)^{\frac{1}{3}} - 3\right)^{\frac{3}{2}} \tag{1}$$

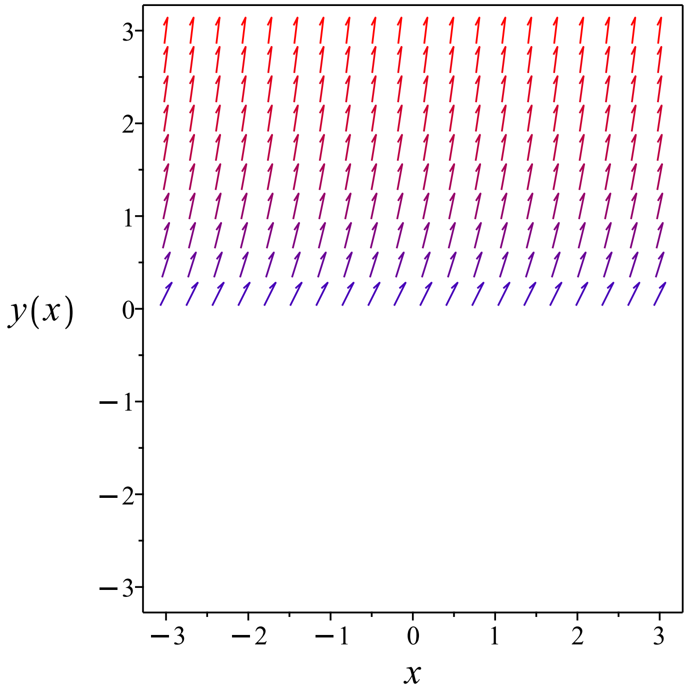


Figure 7: Slope field plot

Verification of solutions

$$y = \left(\left(e^{2x} c_2^2\right)^{\frac{1}{3}} - 3\right)^{\frac{3}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=y(x)+3*y(x)^(1/3),y(x), singsol=all)
```

$$3 + y(x)^{\frac{2}{3}} - e^{\frac{2x}{3}} c_1 = 0$$

✓ Solution by Mathematica

Time used: 2.285 (sec). Leaf size: 39

```
DSolve[y'[x]==y[x]+3*y[x]^(1/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-3 + e^{\frac{2(x+c_1)}{3}}\right)^{3/2}$$
$$y(x) \to 0$$
$$y(x) \to -3i\sqrt{3}$$

1.4 problem 5

Internal problem ID [14937]
Internal file name [`OUTPUT/14946_Monday_April_15_2024_12_04_07_AM_53634759/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sqrt{-y + x} = 0}$$

1.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \sqrt{-y + x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \sqrt{-y+x}\,(b_3 - a_2) - (-y+x)\,a_3 - \frac{xa_2 + ya_3 + a_1}{2\sqrt{-y+x}} + \frac{xb_2 + yb_3 + b_1}{2\sqrt{-y+x}} = 0 \quad (5E)$$

Putting the above in normal form gives

$$-\frac{2a_3\sqrt{-y+x}\,x - 2a_3\sqrt{-y+x}\,y - 2b_2\sqrt{-y+x} + 3xa_2 - xb_2 - 2b_3x - 2a_2y + ya_3 + yb_3 + a_1 - b_1}{2\sqrt{-y+x}} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-2a_3\sqrt{-y+x}\,x + 2a_3\sqrt{-y+x}\,y + 2b_2\sqrt{-y+x} \\ &- 3xa_2 + xb_2 + 2b_3x + 2a_2y - ya_3 - yb_3 - a_1 + b_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned} &-2(-y+x)\,a_2 + 2(-y+x)\,b_3 - 2a_3\sqrt{-y+x}\,x + 2a_3\sqrt{-y+x}\,y \\ &+ 2b_2\sqrt{-y+x} - xa_2 + xb_2 - ya_3 + yb_3 - a_1 + b_1 = 0 \end{aligned} \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-2a_3\sqrt{-y+x}\,x + 2a_3\sqrt{-y+x}\,y + 2b_2\sqrt{-y+x} \\ &- 3xa_2 + xb_2 + 2b_3x + 2a_2y - ya_3 - yb_3 - a_1 + b_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y+x}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y+x} = v_3\right\}$$

The above PDE (6E) now becomes

$$-2a_3v_3v_1 + 2a_3v_3v_2 - 3v_1a_2 + 2a_2v_2 - v_2a_3 + v_1b_2 + 2b_2v_3 + 2b_3v_1 - v_2b_3 - a_1 + b_1 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-2a_3v_3v_1 + (-3a_2 + b_2 + 2b_3)\, v_1 + 2a_3v_3v_2 + (2a_2 - a_3 - b_3)\, v_2 + 2b_2v_3 - a_1 + b_1 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_3 &= 0 \\ 2a_3 &= 0 \\ 2b_2 &= 0 \\ -a_1 + b_1 &= 0 \\ -3a_2 + b_2 + 2b_3 &= 0 \\ 2a_2 - a_3 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(\sqrt{-y + x}\right)(1) \\ &= 1 - \sqrt{-y + x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 - \sqrt{-y + x}} dy \end{aligned}$$

Which results in

$$S = \ln (x - y - 1) + 2\sqrt{-y + x} + \ln \left(\sqrt{-y + x} - 1\right) - \ln \left(1 + \sqrt{-y + x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sqrt{-y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{\sqrt{-y + x} - 1} \\ S_y &= \frac{1}{1 - \sqrt{-y + x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x-y-1) + 2\sqrt{-y+x} + \ln\left(\sqrt{-y+x}-1\right) - \ln\left(1+\sqrt{-y+x}\right) = -x + c_1$$

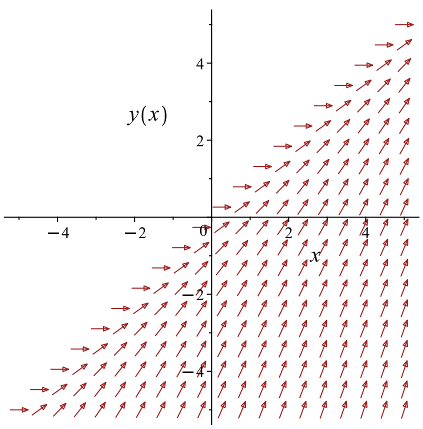
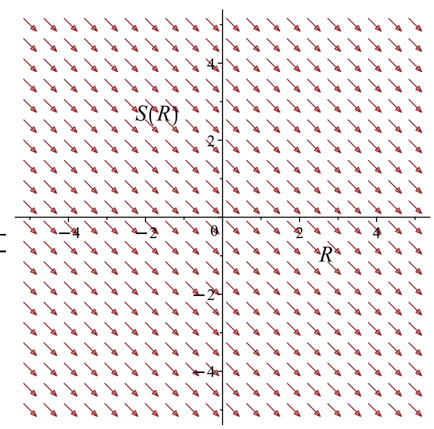
Which simplifies to

$$\ln(x-y-1) + 2\sqrt{-y+x} + \ln\left(\sqrt{-y+x}-1\right) - \ln\left(1+\sqrt{-y+x}\right) = -x + c_1$$

Which gives

$$y = -\mathrm{e}^{-2\,\mathrm{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}+\frac{c_1}{2}}\right)-2-x+c_1} - 2\,\mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}+\frac{c_1}{2}}\right)-1-\frac{x}{2}+\frac{c_1}{2}} + x - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{-y+x}$ 	$R = x$ $S = \ln(x-y-1) + 2\sqrt{-}$	$\frac{dS}{dR} = -1$  $(1+\sqrt{}$

Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{-2\,\mathrm{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}+\frac{c_1}{2}}\right)-2-x+c_1} - 2\,\mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}+\frac{c_1}{2}}\right)-1-\frac{x}{2}+\frac{c_1}{2}} + x - 1 \quad (1)$$

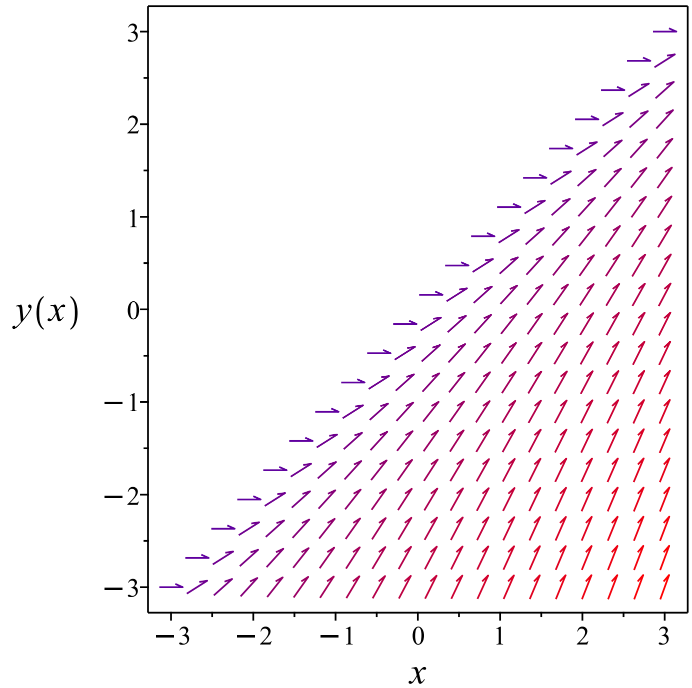


Figure 8: Slope field plot

Verification of solutions

$$y = -\mathrm{e}^{-2\,\mathrm{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}+\frac{c_1}{2}}\right)-2-x+c_1} - 2\,\mathrm{e}^{-\mathrm{LambertW}\left(\mathrm{e}^{-1-\frac{x}{2}+\frac{c_1}{2}}\right)-1-\frac{x}{2}+\frac{c_1}{2}} + x - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 50

```
dsolve(diff(y(x),x)=sqrt(x-y(x)),y(x), singsol=all)
```

$$x+\ln\left(-y(x)+x-1\right)+2\sqrt{-y\left(x\right)+x}+\ln\left(-1+\sqrt{-y\left(x\right)+x}\right)-\ln\left(1+\sqrt{-y\left(x\right)+x}\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 7.657 (sec). Leaf size: 53

```
DSolve[y'[x]==Sqrt[x-y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^2-2W\left(e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)+x-1$$
$$y(x)\to x-1$$

1.5 problem 6

Internal problem ID [14938]
Internal file name [`OUTPUT/14947_Monday_April_15_2024_12_04_09_AM_10457672/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y' - \sqrt{x^2 - y} = -x}$$

1.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \sqrt{x^2 - y} - x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \left(\sqrt{x^2-y}-x\right)(b_3-a_2) - \left(\sqrt{x^2-y}-x\right)^2 a_3 \\
&- \left(\frac{x}{\sqrt{x^2-y}}-1\right)(xa_2+ya_3+a_1) + \frac{xb_2+yb_3+b_1}{2\sqrt{x^2-y}} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2(x^2-y)^{\frac{3}{2}} a_3 + 2\sqrt{x^2-y}\,x^2a_3 - 4x^3a_3 - 4\sqrt{x^2-y}\,xa_2 + 2\sqrt{x^2-y}\,xb_3 - 2\sqrt{x^2-y}\,ya_3 + 4x^2a_2 - 2x^2}{2\sqrt{x^2-y}}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2\left(x^2-y\right)^{\frac{3}{2}} a_3 - 2\sqrt{x^2-y}\,x^2a_3 + 4x^3a_3 + 4\sqrt{x^2-y}\,xa_2 \\
&- 2\sqrt{x^2-y}\,xb_3 + 2\sqrt{x^2-y}\,ya_3 - 4x^2a_2 + 2x^2b_3 - 6xya_3 \\
&+ 2\sqrt{x^2-y}\,a_1 + 2b_2\sqrt{x^2-y} - 2xa_1 + xb_2 + 2ya_2 - yb_3 + b_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-2\left(x^2-y\right)^{\frac{3}{2}} a_3 + 4\left(x^2-y\right)xa_3 - 2\sqrt{x^2-y}\,x^2a_3 - 2\left(x^2-y\right)a_2 \\
&+ 2\left(x^2-y\right)b_3 + 4\sqrt{x^2-y}\,xa_2 - 2\sqrt{x^2-y}\,xb_3 + 2\sqrt{x^2-y}\,ya_3 - 2x^2a_2 \\
&- 2xya_3 + 2\sqrt{x^2-y}\,a_1 + 2b_2\sqrt{x^2-y} - 2xa_1 + xb_2 + yb_3 + b_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-4\sqrt{x^2-y}\,x^2a_3 + 4x^3a_3 + 4\sqrt{x^2-y}\,xa_2 - 2\sqrt{x^2-y}\,xb_3 + 4\sqrt{x^2-y}\,ya_3 - 4x^2a_2 \\
&+ 2x^2b_3 - 6xya_3 + 2\sqrt{x^2-y}\,a_1 + 2b_2\sqrt{x^2-y} - 2xa_1 + xb_2 + 2ya_2 - yb_3 + b_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x, y, \sqrt{x^2-y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2-y} = v_3\right\}$$

The above PDE (6E) now becomes

$$4v_1^3a_3 - 4v_3v_1^2a_3 - 4v_1^2a_2 + 4v_3v_1a_2 - 6v_1v_2a_3 + 4v_3v_2a_3 + 2v_1^2b_3 \\ - 2v_3v_1b_3 - 2v_1a_1 + 2v_3a_1 + 2v_2a_2 + v_1b_2 + 2b_2v_3 - v_2b_3 + b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$4v_1^3a_3 - 4v_3v_1^2a_3 + (-4a_2 + 2b_3)\, v_1^2 - 6v_1v_2a_3 + (4a_2 - 2b_3)\, v_1v_3 \\ + (-2a_1 + b_2)\, v_1 + 4v_3v_2a_3 + (2a_2 - b_3)\, v_2 + (2a_1 + 2b_2)\, v_3 + b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -6a_3 &= 0 \\ -4a_3 &= 0 \\ 4a_3 &= 0 \\ -2a_1 + b_2 &= 0 \\ 2a_1 + 2b_2 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ 2a_2 - b_3 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 2y - \left(\sqrt{x^2-y}-x\right)(x) \\ &= x^2 - x\sqrt{x^2-y}+2y \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy \\ &= \int \frac{1}{x^2 - x\sqrt{x^2-y}+2y}dy\end{aligned}$$

Which results in

$$S = \frac{3\ln(5x^2+4y)}{10} + \frac{\ln(y)}{5} - \frac{\ln\left(\sqrt{x^2-y}+x\right)}{5} - \frac{3\ln\left(2\sqrt{x^2-y}-3x\right)}{10} + \frac{\ln\left(\sqrt{x^2-y}-x\right)}{5} + \frac{3\ln\left(2\sqrt{\right.}}{}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sqrt{x^2-y} - x$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{2\sqrt{x^2-y}+3x} \\
S_y &= \frac{x\sqrt{x^2-y}+x^2+2y}{5x^2y+4y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

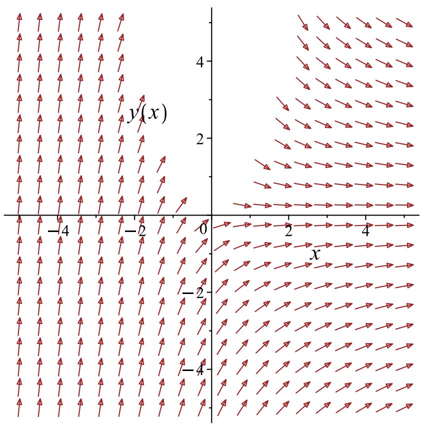
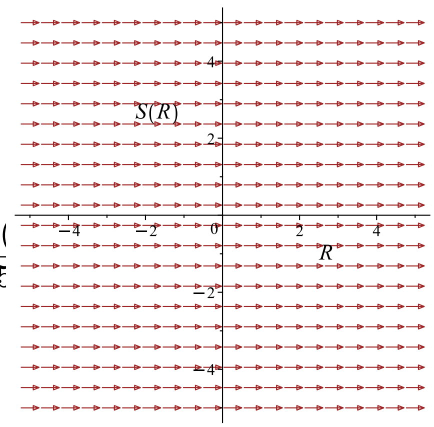
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(5x^2+4y)}{10}+\frac{\ln(y)}{5}-\frac{\ln\left(\sqrt{x^2-y}+x\right)}{5}-\frac{3\ln\left(2\sqrt{x^2-y}-3x\right)}{10}+\frac{\ln\left(\sqrt{x^2-y}-x\right)}{5}+\frac{3\ln\left(2\sqrt{x^2-\right.}}{10}$$

Which simplifies to

$$\frac{3\ln(5x^2+4y)}{10}+\frac{\ln(y)}{5}-\frac{\ln\left(\sqrt{x^2-y}+x\right)}{5}-\frac{3\ln\left(2\sqrt{x^2-y}-3x\right)}{10}+\frac{\ln\left(\sqrt{x^2-y}-x\right)}{5}+\frac{3\ln\left(2\sqrt{x^2-\right.}}{10}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{x^2 - y} - x$ 	$R = x$ $S = \frac{3\ln(5x^2+4y)}{10} + \frac{\ln(\ldots}{5} \ldots 2\sqrt{x^2 - \ldots}{10}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(5x^2+4y)}{10} + \frac{\ln(y)}{5} - \frac{\ln\left(\sqrt{x^2-y}+x\right)}{5} - \frac{3\ln\left(2\sqrt{x^2-y}-3x\right)}{10} + \frac{\ln\left(\sqrt{x^2-y}-x\right)}{5} + \frac{3\ln\left(2\sqrt{x^2-y}+3x\right)}{10} = c_1 \tag{1}$$

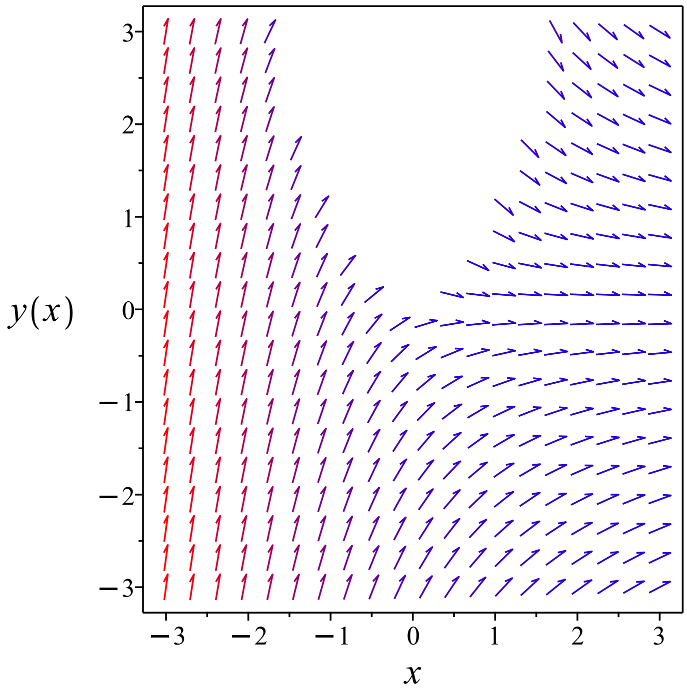


Figure 9: Slope field plot

Verification of solutions

$$\frac{3\ln\left(5x^{2}+4y\right)}{10}+\frac{\ln\left(y\right)}{5}-\frac{\ln\left(\sqrt{x^{2}-y}+x\right)}{5}-\frac{3\ln\left(2\sqrt{x^{2}-y}-3x\right)}{10}+\frac{\ln\left(\sqrt{x^{2}-y}-x\right)}{5}+\frac{3\ln\left(2\sqrt{x^{2}-y}+3x\right)}{10}=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 171

```
dsolve(diff(y(x),x)=sqrt(x^2-y(x))-x,y(x), singsol=all)
```

$$\frac{250\left(x^6c_1y(x)^2+\frac{12x^4c_1y(x)^3}{5}+\frac{48x^2c_1y(x)^4}{25}+\frac{64c_1y(x)^5}{125}-\frac{1}{125}\right)\left(x^2-y(x)\right)^{\frac{3}{2}}\left(x^2+4y(x)\right)-250\left(x^6c_1y(x)^2+\cdots\right.}{\left(5x^2+4y\left(x\right)\right)^3y\left(x\right)^2\left(-\sqrt{x^2-y\left(x\right)}+x\right)^2\left(3x+2\sqrt{\cdots}\right.}$$

$$=0$$

✓ Solution by Mathematica

Time used: 4.748 (sec). Leaf size: 416

```
DSolve[y'[x]==Sqrt[x^2-y[x]]-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \text{Root}\left[64\#1^5+240\#1^4x^2+300\#1^3x^4+\#1^2\left(125x^6-40e^{5c_1}x\right)-10\#1e^{5c_1}x^3-4e^{5c_1}x^5+e^{10c_1}\&,1\right]$$
$$y(x)\to \text{Root}\left[64\#1^5+240\#1^4x^2+300\#1^3x^4+\#1^2\left(125x^6-40e^{5c_1}x\right)-10\#1e^{5c_1}x^3-4e^{5c_1}x^5+e^{10c_1}\&,2\right]$$
$$y(x)\to \text{Root}\left[64\#1^5+240\#1^4x^2+300\#1^3x^4+\#1^2\left(125x^6-40e^{5c_1}x\right)-10\#1e^{5c_1}x^3-4e^{5c_1}x^5+e^{10c_1}\&,3\right]$$
$$y(x)\to \text{Root}\left[64\#1^5+240\#1^4x^2+300\#1^3x^4+\#1^2\left(125x^6-40e^{5c_1}x\right)-10\#1e^{5c_1}x^3-4e^{5c_1}x^5+e^{10c_1}\&,4\right]$$
$$y(x)\to \text{Root}\left[64\#1^5+240\#1^4x^2+300\#1^3x^4+\#1^2\left(125x^6-40e^{5c_1}x\right)-10\#1e^{5c_1}x^3-4e^{5c_1}x^5+e^{10c_1}\&,5\right]$$
$$y(x)\to 0$$

1.6 problem 7

Internal problem ID [14939]
Internal file name [`OUTPUT/14948_Monday_April_15_2024_12_04_12_AM_48849731/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{1-y^2} = 0}$$

1.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+1}} dy = x + c_1$$
$$\arcsin(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \sin(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \sin(x + c_1) \tag{1}$$

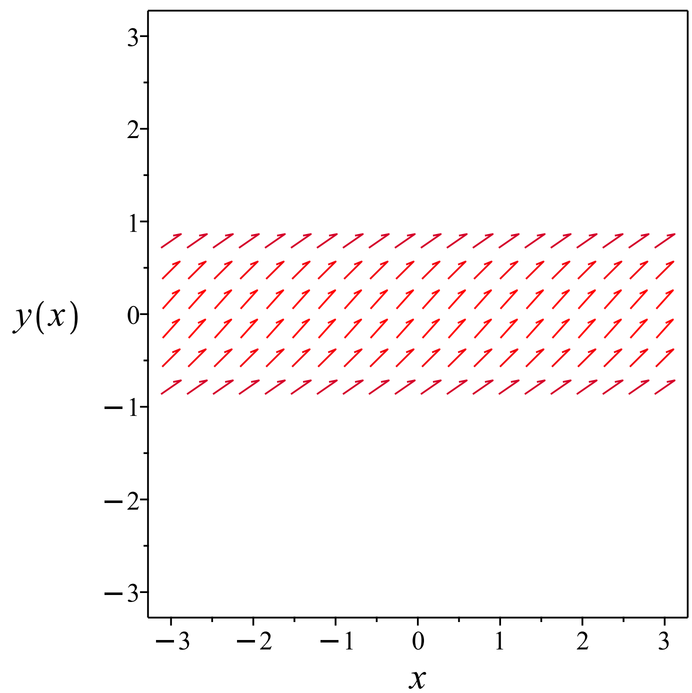


Figure 10: Slope field plot

Verification of solutions

$$y = \sin(x + c_1)$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$y' - \sqrt{1 - y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}} dx = \int 1 dx + c_1$

- Evaluate integral

$$\arcsin(y) = x + c_1$$

- Solve for y

$$y = \sin(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=sqrt(1-y(x)^2),y(x), singsol=all)
```

$$y(x) = \sin(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 28

```
DSolve[y'[x]==Sqrt[1-y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x + c_1)$$
$$y(x) \to -1$$
$$y(x) \to 1$$
$$y(x) \to \text{Interval}[\{-1, 1\}]$$

1.7 problem 8

Internal problem ID [14940]
Internal file name [`OUTPUT/14949_Monday_April_15_2024_12_04_12_AM_29529941/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y+1}{-y+x} = 0}$$

1.7.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{Y(X) + y_0 + 1}{Y(X) + y_0 - X - x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X)}{Y(X) - X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{Y}{Y - X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = Y$ and $N = -Y + X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X} X + u &= -\frac{u}{u - 1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-\frac{u(X)}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX} u(X) - \frac{-\frac{u(X)}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX} u(X)\right) X u(X) - \left(\frac{d}{dX} u(X)\right) X + u(X)^2 = 0$$

Or

$$X(u(X) - 1) \left(\frac{d}{dX} u(X)\right) + u(X)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X) g(u) \\ &= -\frac{u^2}{X (u - 1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{1}{u} + \ln(u) = -\ln(X) + c_2$$

The solution is

$$\frac{1}{u(X)} + \ln(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{X}{Y(X)} + \ln\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{X}{Y(X)} + \ln\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x - 1$$

Then the solution in y becomes

$$\frac{x+1}{y+1} + \ln\left(\frac{y+1}{x+1}\right) + \ln(x+1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{x+1}{y+1} + \ln\left(\frac{y+1}{x+1}\right) + \ln(x+1) - c_2 = 0 \tag{1}$$

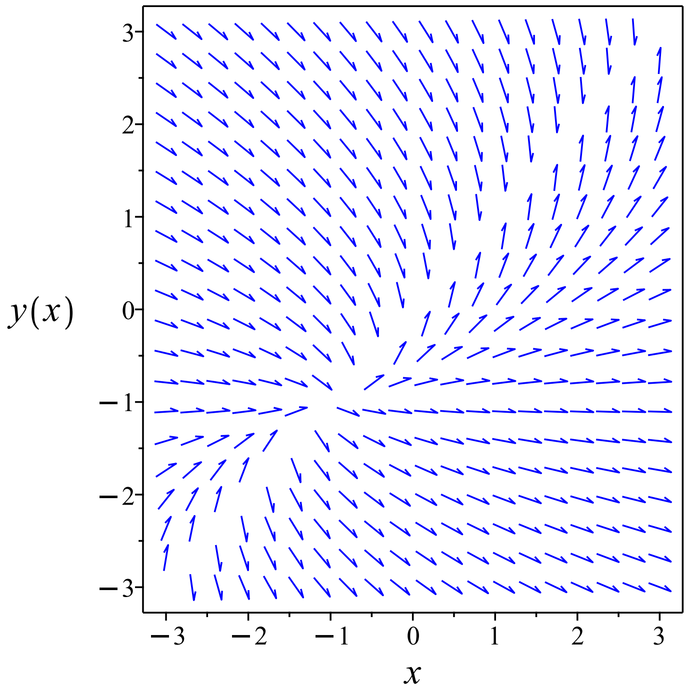


Figure 11: Slope field plot

Verification of solutions

$$\frac{x+1}{y+1}+\ln\left(\frac{y+1}{x+1}\right)+\ln(x+1)-c_2=0$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y+1}{y-x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y+1)(b_3-a_2)}{y-x} - \frac{(y+1)^2 a_3}{(y-x)^2} + \frac{(y+1)(xa_2+ya_3+a_1)}{(y-x)^2} - \left(-\frac{1}{y-x} + \frac{y+1}{(y-x)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2xyb_2 - y^2a_2 - y^2b_2 + y^2b_3 + xb_1 + xb_2 - xb_3 - ya_1 - ya_2 + ya_3 + 2yb_3 - a_1 + a_3 + b_1}{(-y+x)^2} = 0$$

Setting the numerator to zero gives

$$-2xyb_2 + y^2a_2 + y^2b_2 - y^2b_3 - xb_1 - xb_2 + xb_3 + ya_1 + ya_2 - ya_3 - 2yb_3 + a_1 - a_3 - b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a_2v_2^2 - 2b_2v_1v_2 + b_2v_2^2 - b_3v_2^2 + a_1v_2 + a_2v_2 - a_3v_2 - b_1v_1 - b_2v_1 + b_3v_1 - 2b_3v_2 + a_1 - a_3 - b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2b_2v_1v_2 + (-b_1 - b_2 + b_3)\, v_1 + (a_2 + b_2 - b_3)\, v_2^2 \\
&+ (a_1 + a_2 - a_3 - 2b_3)\, v_2 + a_1 - a_3 - b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2b_2 &= 0 \\
a_1 - a_3 - b_1 &= 0 \\
a_2 + b_2 - b_3 &= 0 \\
-b_1 - b_2 + b_3 &= 0 \\
a_1 + a_2 - a_3 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_3 + a_3 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + 1 \\
\eta &= y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y + 1 - \left(-\frac{y+1}{y-x}\right)(x + 1) \\
&= \frac{-y^2 - 2y - 1}{-y + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y^2-2y-1}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \ln (y+1) - \frac{-x-1}{y+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y+1}{y-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y+1} \\ S_y &= \frac{y-x}{(y+1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(y+1)\ln(y+1)+x+1}{y+1} = c_1$$

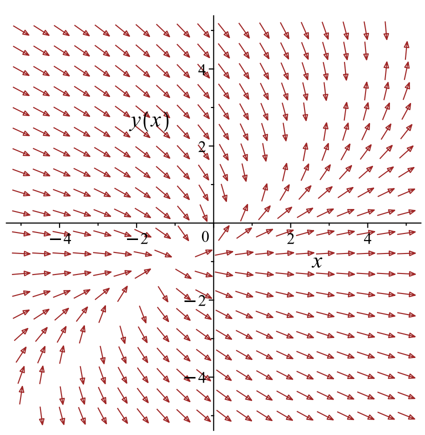
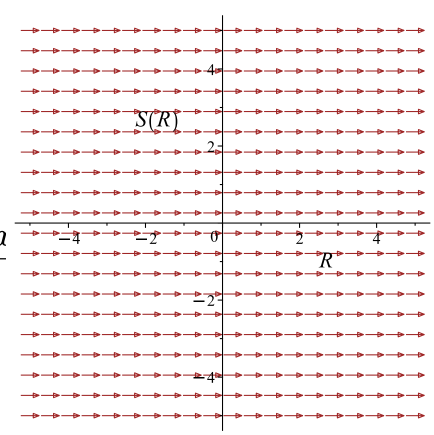
Which simplifies to

$$\frac{(y+1)\ln(y+1)+x+1}{y+1} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-(x+1)\mathrm{e}^{-c_1}\right)+c_1} - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+1}{y-x}$ 	$R = x$ $S = \frac{(y+1)\ln(y+1)+x}{y+1}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = e^{\operatorname{LambertW}\left(-(x+1)e^{-c_1}\right)+c_1} - 1 \tag{1}$$

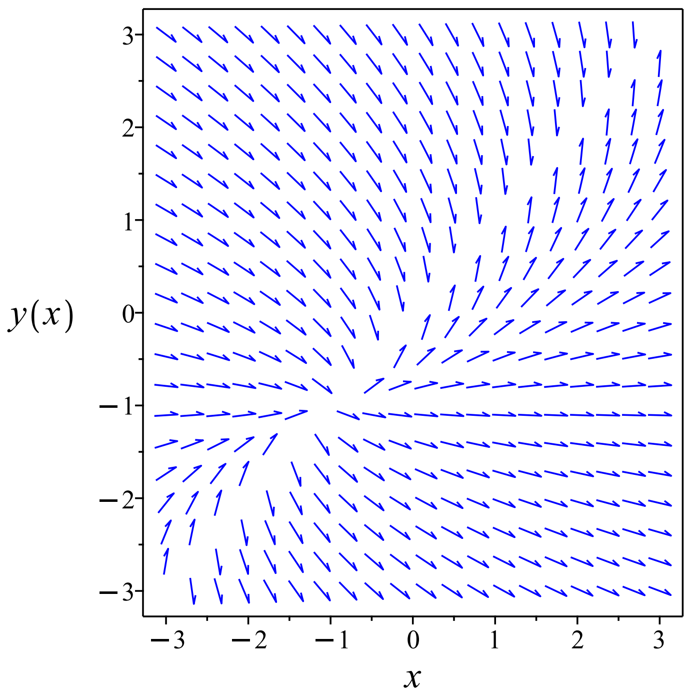


Figure 12: Slope field plot

Verification of solutions

$$y = e^{\operatorname{LambertW}\left(-(x+1)e^{-c_1}\right)+c_1} - 1$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y &= (-y-1)\,\mathrm{d}x\\ (y+1)\,\mathrm{d}x + (y-x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y+1\\ N(x,y) &= y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y+1)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y-x}((1)-(-1))\\ &= -\frac{2}{-y+x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{y+1}((-1)-(1))\\ &= -\frac{2}{y+1}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{2}{y+1} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y+1)}\\ &= \frac{1}{(y+1)^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(y+1)^2}(y+1)\\ &= \frac{1}{y+1}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(y+1)^2}(y-x) \\
&= \frac{y-x}{(y+1)^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{y+1}\right) + \left(\frac{y-x}{(y+1)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{1}{y+1}\,\mathrm{d}x \\
\phi &= \frac{x}{y+1} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{x}{(y+1)^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-x}{(y+1)^2}$. Therefore equation (4) becomes

$$
\frac{y-x}{(y+1)^2} = -\frac{x}{(y+1)^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{(y+1)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{y}{(y+1)^2}\right) \mathrm{d}y$$

$$f(y) = \ln(y+1) + \frac{1}{y+1} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{y+1} + \ln(y+1) + \frac{1}{y+1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{y+1} + \ln(y+1) + \frac{1}{y+1}$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-(x+1)\mathrm{e}^{-c_1}\right)+c_1} - 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-(x+1)\mathrm{e}^{-c_1}\right)+c_1} - 1 \tag{1}$$

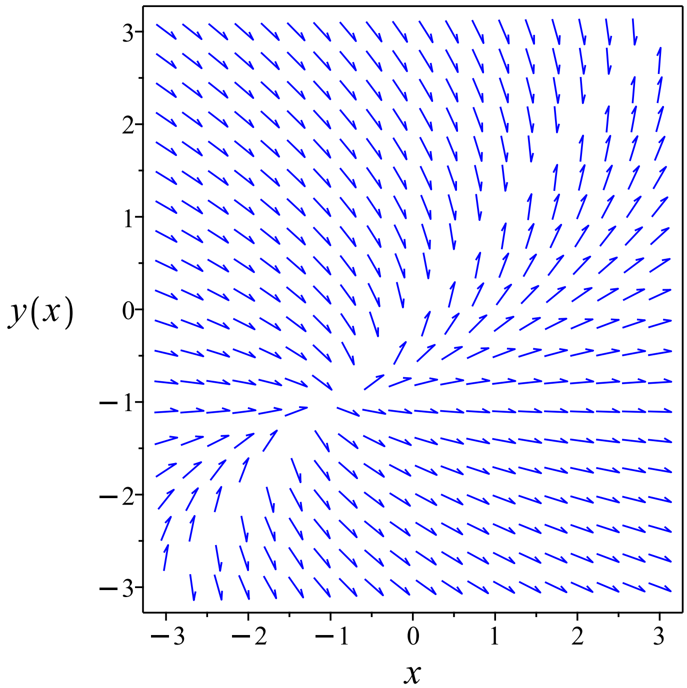


Figure 13: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-(x+1)e^{-c_1}\right)+c_1} - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(diff(y(x),x)=(y(x)+1)/(x-y(x)),y(x), singsol=all)
```

$$y(x) = \frac{-1 - x - \mathrm{LambertW}\left(-(1+x)\,\mathrm{e}^{-c_1}\right)}{\mathrm{LambertW}\left(-\,(1+x)\,\mathrm{e}^{-c_1}\right)}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 34

```
DSolve[y'[x]==(y[x]+1)/(x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[x = (y(x)+1)\left(-\frac{1}{y(x)+1} - \log(y(x)+1)\right) + c_1(y(x)+1), y(x)\right]$$

1.8 problem 9

Internal problem ID [14941]
Internal file name [`OUTPUT/14950_Monday_April_15_2024_12_04_14_AM_9773372/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{y' - \sin(y) = -\cos(x)}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=sin(y(x))-cos(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==Sin[y[x]]-Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

1.9 problem 10

Internal problem ID [14942]
Internal file name [`OUTPUT/14951_Monday_April_15_2024_12_04_15_AM_13746180/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + \cot(y) = 1}$$

1.9.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1-\cot(y)} dy = \int dx$$
$$\int^{y} \frac{1}{1-\cot(\_a)} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{1-\cot(\_a)} d\_a = x + c_1 \tag{1}$$

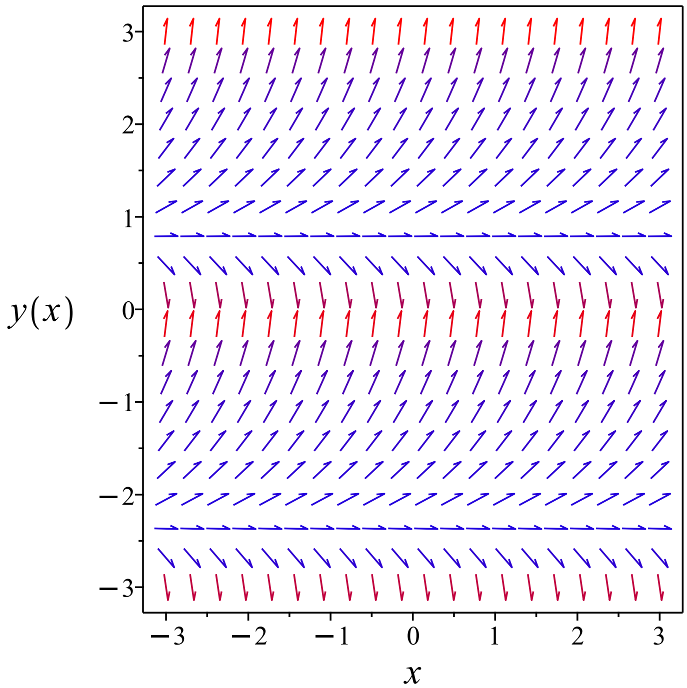


Figure 14: Slope field plot

Verification of solutions

$$\int^{y} \frac{1}{1-\cot(\_a)}d\_a = x + c_1$$

Verified OK.

1.9.2 Maple step by step solution

Let's solve

$y' + \cot(y) = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-\cot(y)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-\cot(y)}dx = \int 1dx + c_1$

- Evaluate integral

$$\frac{\ln(-1+\cot(y))}{2} - \frac{\ln\left(\cot(y)^2+1\right)}{4} - \frac{\pi}{4} + \frac{y}{2} = x + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)=1-cot(y(x)),y(x), singsol=all)
```

$$x + \frac{\ln\left(\csc\left(y(x)\right)^2\right)}{4} + \frac{\pi}{4} - \frac{\ln\left(-1+\cot\left(y(x)\right)\right)}{2} - \frac{y(x)}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.376 (sec). Leaf size: 69

```
DSolve[y'[x]==1-Cot[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\left(\frac{1}{4}+\frac{i}{4}\right)\log(-\tan(\#1)+i) - \frac{1}{2}\log(1-\tan(\#1)) + \left(\frac{1}{4}-\frac{i}{4}\right)\log(\tan(\#1)+i)\&\right][-x+c_1]$$

$$y(x) \to \frac{\pi}{4}$$

1.10 problem 11

Internal problem ID [14943]
Internal file name [`OUTPUT/14952_Monday_April_15_2024_12_04_16_AM_96670024/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - (3x - y)^{\frac{1}{3}} = -1}$$

1.10.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= (3x - y)^{\frac{1}{3}} - 1 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \left((3x-y)^{\frac{1}{3}}-1\right)(b_3-a_2) - \left((3x-y)^{\frac{1}{3}}-1\right)^2 a_3 \\ &- \frac{xa_2+ya_3+a_1}{(3x-y)^{\frac{2}{3}}} + \frac{xb_2+yb_3+b_1}{3\,(3x-y)^{\frac{2}{3}}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3(3x-y)^{\frac{4}{3}}\,a_3 - 3(3x-y)^{\frac{2}{3}}\,a_2 + 3a_3(3x-y)^{\frac{2}{3}} - 3b_2(3x-y)^{\frac{2}{3}} + 3(3x-y)^{\frac{2}{3}}\,b_3 + 12xa_2 - 18a_3x - xb_2 -}{3\,(3x-y)^{\frac{2}{3}}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-3(3x-y)^{\frac{4}{3}}\,a_3 + 3(3x-y)^{\frac{2}{3}}\,a_2 - 3a_3(3x-y)^{\frac{2}{3}} + 3b_2(3x-y)^{\frac{2}{3}} - 3(3x-y)^{\frac{2}{3}}\,b_3 \\ &- 12xa_2 + 18a_3x + xb_2 + 9b_3x + 3a_2y - 9ya_3 - 2yb_3 - 3a_1 + b_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-3(3x-y)^{\frac{4}{3}}\,a_3 - 3(3x-y)\,a_2 + 6(3x-y)\,a_3 + 3(3x-y)\,b_3 \\ &+ 3(3x-y)^{\frac{2}{3}}\,a_2 - 3a_3(3x-y)^{\frac{2}{3}} + 3b_2(3x-y)^{\frac{2}{3}} \\ &- 3(3x-y)^{\frac{2}{3}}\,b_3 - 3xa_2 + xb_2 - 3ya_3 + yb_3 - 3a_1 + b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &3(3x-y)^{\frac{2}{3}}\,a_2 - 3a_3(3x-y)^{\frac{2}{3}} + 3b_2(3x-y)^{\frac{2}{3}} - 3(3x-y)^{\frac{2}{3}}\,b_3 - 9(3x-y)^{\frac{1}{3}}\,a_3x \\ &+ 3(3x-y)^{\frac{1}{3}}\,a_3y - 12xa_2 + 18a_3x + xb_2 + 9b_3x + 3a_2y - 9ya_3 - 2yb_3 - 3a_1 + b_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, (3x-y)^{\frac{1}{3}}, (3x-y)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, (3x-y)^{\frac{1}{3}} = v_3, (3x-y)^{\frac{2}{3}} = v_4\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-9v_3a_3v_1 + 3v_3a_3v_2 - 12v_1a_2 + 3a_2v_2 + 3v_4a_2 + 18a_3v_1 - 9v_2a_3 \\
&- 3a_3v_4 + v_1b_2 + 3b_2v_4 + 9b_3v_1 - 2v_2b_3 - 3v_4b_3 - 3a_1 + b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-9v_3a_3v_1 + (-12a_2 + 18a_3 + b_2 + 9b_3)\, v_1 + 3v_3a_3v_2 \\
&+ (3a_2 - 9a_3 - 2b_3)\, v_2 + (3a_2 - 3a_3 + 3b_2 - 3b_3)\, v_4 - 3a_1 + b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-9a_3 &= 0 \\
3a_3 &= 0 \\
-3a_1 + b_1 &= 0 \\
3a_2 - 9a_3 - 2b_3 &= 0 \\
-12a_2 + 18a_3 + b_2 + 9b_3 &= 0 \\
3a_2 - 3a_3 + 3b_2 - 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 3a_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 3
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 3 - \left((3x-y)^{\frac{1}{3}} - 1\right)(1) \\
&= 4 - (3x-y)^{\frac{1}{3}} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{4 - (3x-y)^{\frac{1}{3}}} dy
\end{aligned}$$

Which results in

$$S = \frac{3(3x-y)^{\frac{2}{3}}}{2} + 32\ln\left(-4 + (3x-y)^{\frac{1}{3}}\right) - 16\ln\left((3x-y)^{\frac{2}{3}} + 4(3x-y)^{\frac{1}{3}} + 16\right) + 16\ln\left(-64 + 3x - y\right.$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (3x-y)^{\frac{1}{3}} - 1$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{-4 + (3x - y)^{\frac{1}{3}}} \\
S_y &= \frac{1}{4 - (3x - y)^{\frac{1}{3}}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

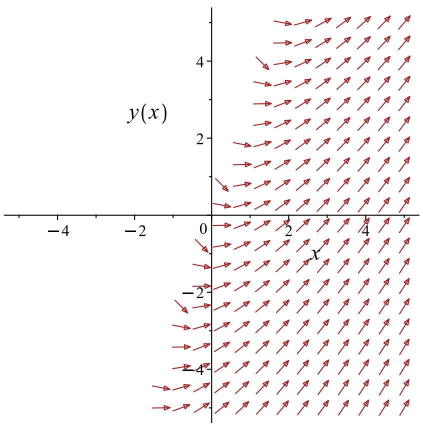
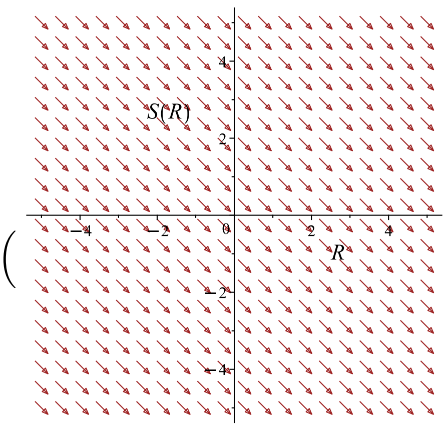
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3(3x - y)^{\frac{2}{3}}}{2} + 32\ln\left(-4 + (3x - y)^{\frac{1}{3}}\right) - 16\ln\left((3x - y)^{\frac{2}{3}} + 4(3x - y)^{\frac{1}{3}} + 16\right) + 16\ln(-64 + 3x - y) + 1$$

Which simplifies to

$$\frac{3(3x - y)^{\frac{2}{3}}}{2} + 32\ln\left(-4 + (3x - y)^{\frac{1}{3}}\right) - 16\ln\left((3x - y)^{\frac{2}{3}} + 4(3x - y)^{\frac{1}{3}} + 16\right) + 16\ln(-64 + 3x - y) + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (3x-y)^{\frac{1}{3}} - 1$ 	$R = x$ $S = \frac{3(3x-y)^{\frac{2}{3}}}{2} + 32\ln\Big($ ... $y)^{\frac{2}{3}} + 4$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$
\frac{3(3x-y)^{\frac{2}{3}}}{2} + 32\ln\left(-4+(3x-y)^{\frac{1}{3}}\right) - 16\ln\left((3x-y)^{\frac{2}{3}} + 4(3x-y)^{\frac{1}{3}} + 16\right) + 16\ln(-64+3x-y) + 12(3x-y)^{\frac{1}{3}} = -x + c_1 \tag{1}
$$

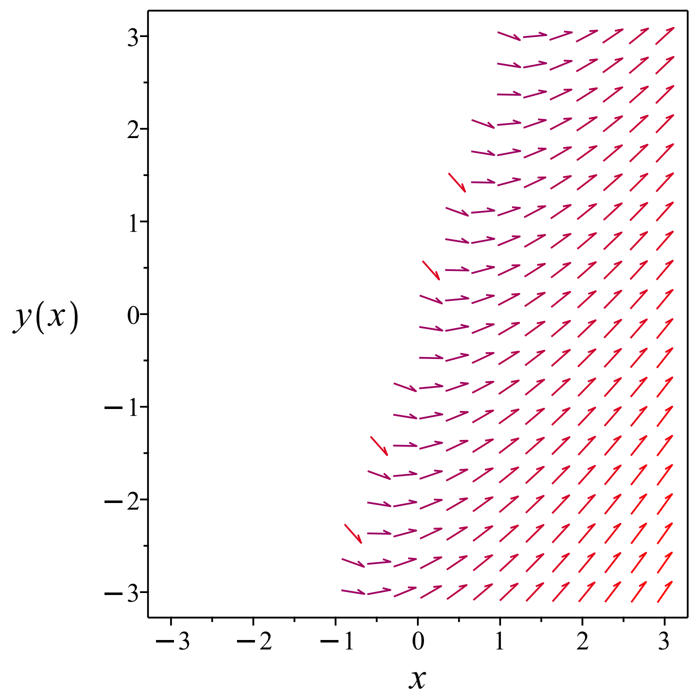


Figure 15: Slope field plot

Verification of solutions

$$\frac{3(3x-y)^{\frac{2}{3}}}{2}+32\ln\left(-4+(3x-y)^{\frac{1}{3}}\right)-16\ln\left((3x-y)^{\frac{2}{3}}+4(3x-y)^{\frac{1}{3}}+16\right)+16\ln\left(-64+3x-y\right)+12(3x-y)^{\frac{1}{3}}=-x+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 86

```
dsolve(diff(y(x),x)=(3*x-y(x))^(1/3)-1,y(x), singsol=all)
```

$$
x+\frac{3(3x-y(x))^{\frac{2}{3}}}{2}+32\ln\left(-4+(3x-y(x))^{\frac{1}{3}}\right)-16\ln\left((3x-y(x))^{\frac{2}{3}}+4(3x-y(x))^{\frac{1}{3}}+16\right)+16\ln\left(-64+3x-y(x)\right)+12(3x-y(x))^{\frac{1}{3}}-c_1=0
$$

✓ Solution by Mathematica

Time used: 0.203 (sec). Leaf size: 55

```
DSolve[y'[x]==(3*x-y[x])^(1/3)-1,y[x],x,IncludeSingularSolutions -> True]
```

$$
\text{Solve}\left[\frac{3}{2}(3x-y(x))^{2/3}+12\sqrt[3]{3x-y(x)}+48\log\left(\sqrt[3]{3x-y(x)}-4\right)+x=c_1,y(x)\right]
$$

1.11 problem 13

Internal problem ID [14944]
Internal file name [`OUTPUT/14953_Monday_April_15_2024_12_04_17_AM_18639985/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{y' - \sin(yx) = 0}$$

With initial conditions

$$[y(0) = 0]$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=sin(x*y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[x]==Sin[x*y[x]],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

1.12 problem 14

Internal problem ID [14945]
Internal file name [`OUTPUT/14954_Monday_April_15_2024_12_04_18_AM_83742478/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y'x + y = \cos(x)}$$

1.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{\cos(x)}{x}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{\cos(x)}{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\cos(x)}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= (x)\left(\frac{\cos(x)}{x}\right) \\
\mathrm{d}(xy) &= \cos(x)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int \cos(x)\,\mathrm{d}x \\
xy &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{\sin(x)}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
y = \frac{\sin(x) + c_1}{x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{\sin(x) + c_1}{x} \tag{1}
$$

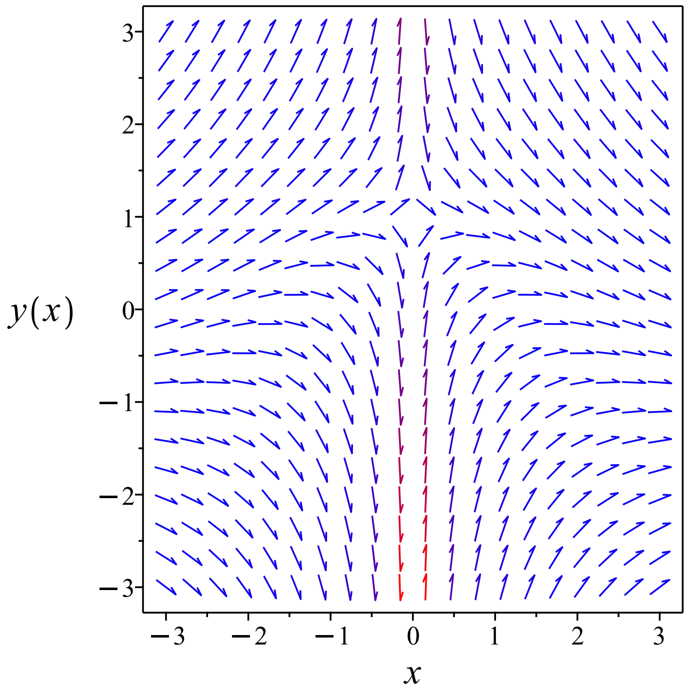


Figure 16: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + c_1}{x}$$

Verified OK.

1.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\cos(x) - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 6: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\cos(x) - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = \sin(x) + c_1$$

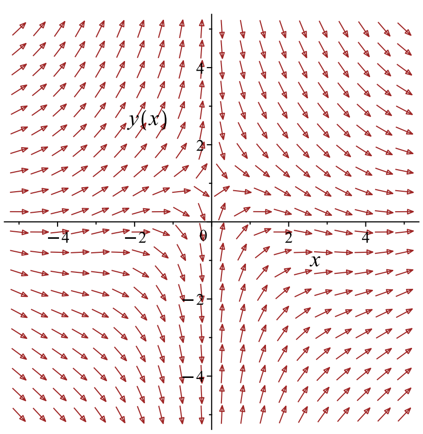
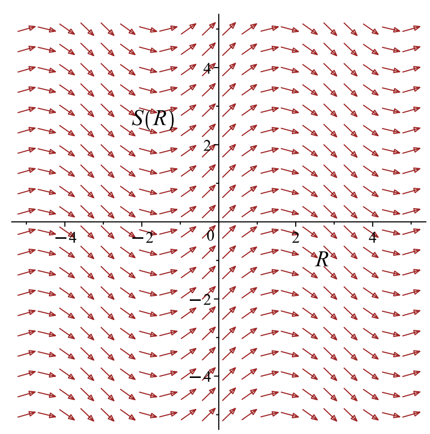
Which simplifies to

$$yx = \sin(x) + c_1$$

Which gives

$$y = \frac{\sin(x) + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)-y}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = \cos(R)$ 

Summary

The solution(s) found are the following

$$y = \frac{\sin(x) + c_1}{x} \tag{1}$$

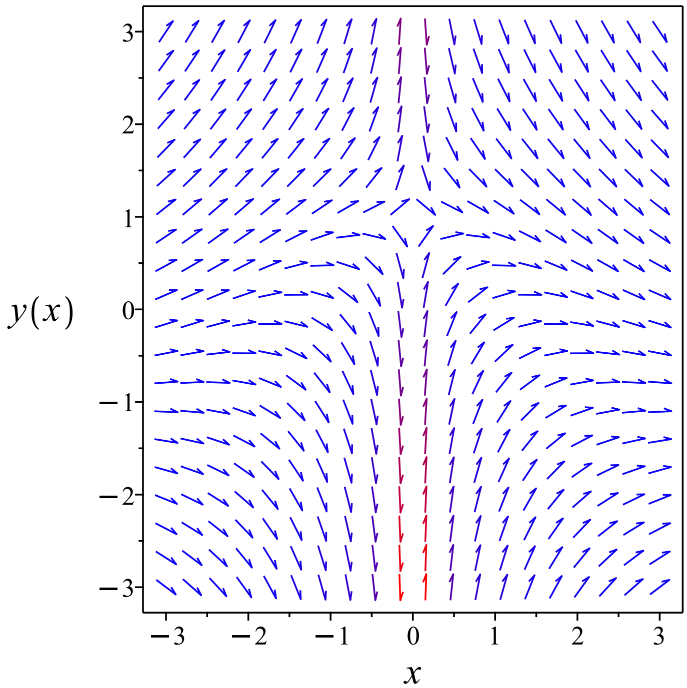


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + c_1}{x}$$

Verified OK.

1.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\, \mathrm{d}y &= (\cos(x) - y)\, \mathrm{d}x \\ (-\cos(x) + y)\, \mathrm{d}x + (x)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\cos(x) + y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x) + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\cos(x) + y \, \mathrm{d}x$$

$$\phi = xy - \sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \sin(x)$$

The solution becomes

$$y = \frac{\sin(x) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x) + c_1}{x} \tag{1}$$

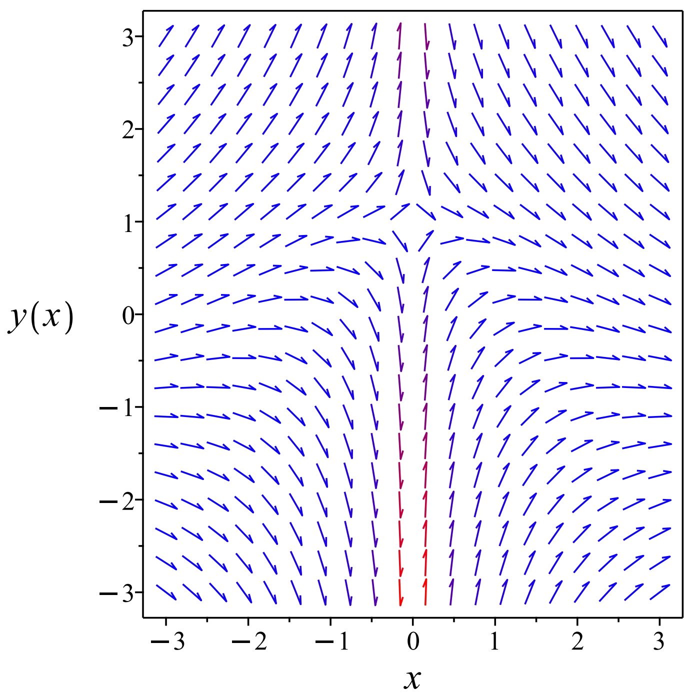


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + c_1}{x}$$

Verified OK.

1.12.4 Maple step by step solution

Let's solve

$y'x + y = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \frac{\cos(x)}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \frac{\cos(x)}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \frac{\mu(x)\cos(x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\cos(x)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\cos(x)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\cos(x)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \cos(x) dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 14

```
DSolve[x*y'[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x) + c_1}{x}$$

1.13 problem 15

Internal problem ID [14946]
Internal file name [`OUTPUT/14955_Monday_April_15_2024_12_04_19_AM_33886060/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = e^x}$$

1.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = e^x$$

Hence the ode is

$$y' + 2y = e^x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dx} \\ &= e^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^{x}\right) \\ \frac{d}{dx}\left(y\,e^{2x}\right) &= \left(e^{2x}\right)\left(e^{x}\right) \\ d\left(y\,e^{2x}\right) &= e^{3x}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,e^{2x} &= \int e^{3x}\,dx \\ y\,e^{2x} &= \frac{e^{3x}}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = \frac{e^{3x}e^{-2x}}{3} + c_1 e^{-2x}$$

which simplifies to

$$y = \frac{\left(e^{3x} + 3c_1\right)e^{-2x}}{3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\left(e^{3x} + 3c_1\right)e^{-2x}}{3} \tag{1}$$

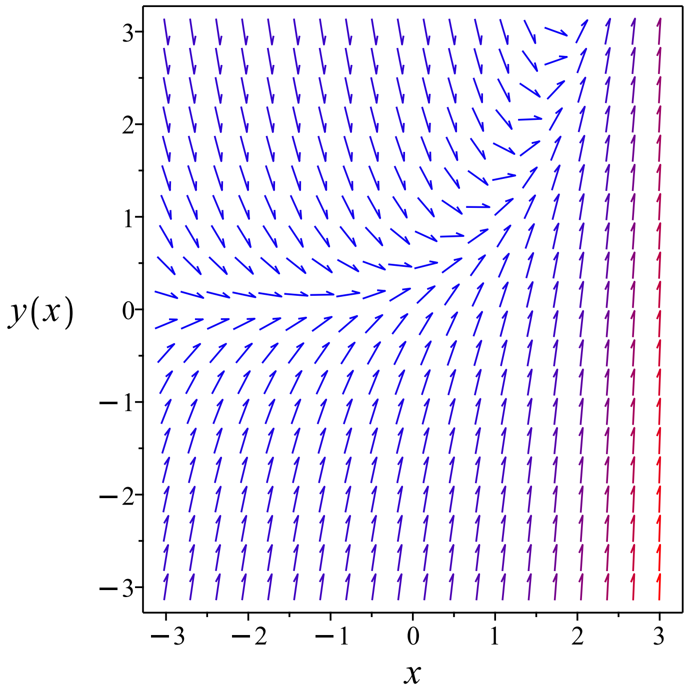


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{(e^{3x} + 3c_1) e^{-2x}}{3}$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2y + e^x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 9: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-2x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x}} dy \end{aligned}$$

Which results in

$$S = y\,e^{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + e^{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2y\,e^{2x} \\ S_y &= e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{3R}}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{2x} = \frac{e^{3x}}{3} + c_1$$

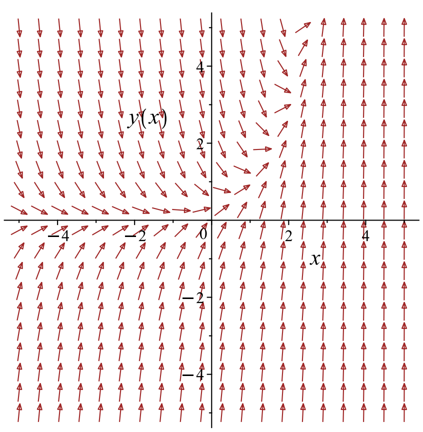
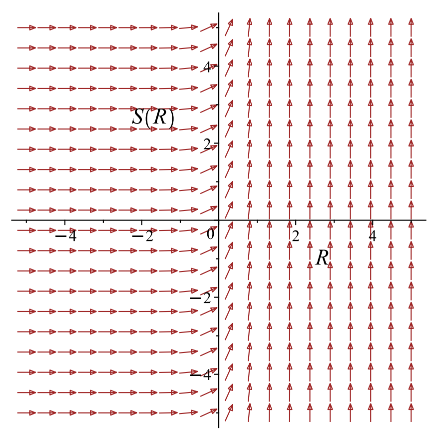
Which simplifies to

$$y\,e^{2x} = \frac{e^{3x}}{3} + c_1$$

Which gives

$$y = \frac{(e^{3x} + 3c_1)\,e^{-2x}}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + e^x$ 	$R = x$ $S = y\,e^{2x}$	$\frac{dS}{dR} = e^{3R}$ 

Summary
The solution(s) found are the following

$$y = \frac{(e^{3x} + 3c_1)\,e^{-2x}}{3} \tag{1}$$

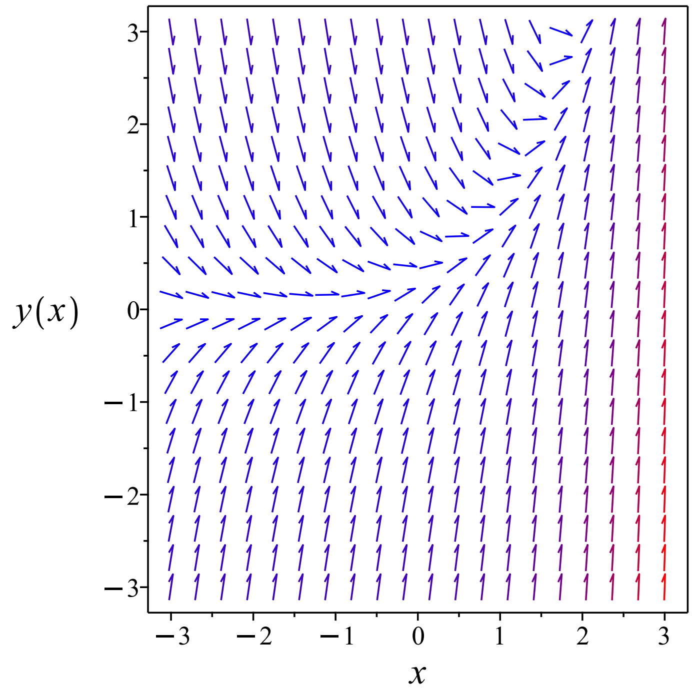


Figure 20: Slope field plot

Verification of solutions

$$y = \frac{(e^{3x} + 3c_1) e^{-2x}}{3}$$

Verified OK.

1.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned} dy &= (-2y + e^x) \, dx\\ (2y - e^x) \, dx + dy &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2y - e^x\\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - e^x)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((2) - (0))\\ &= 2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 2\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2x}\\ &= \mathrm{e}^{2x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{2x}(2y - \mathrm{e}^{x})\\ &= (2y - \mathrm{e}^{x})\,\mathrm{e}^{2x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{2x}(1)\\ &= \mathrm{e}^{2x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left((2y - \mathrm{e}^{x})\,\mathrm{e}^{2x}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int (2y - \mathrm{e}^{x})\,\mathrm{e}^{2x}\,\mathrm{d}x$$
$$\phi = -\frac{\mathrm{e}^{3x}}{3} + y\,\mathrm{e}^{2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{3x}}{3} + y\,\mathrm{e}^{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{3x}}{3} + y\,\mathrm{e}^{2x}$$

The solution becomes

$$y = \frac{(\mathrm{e}^{3x} + 3c_1)\,\mathrm{e}^{-2x}}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{3x} + 3c_1) e^{-2x}}{3} \tag{1}$$

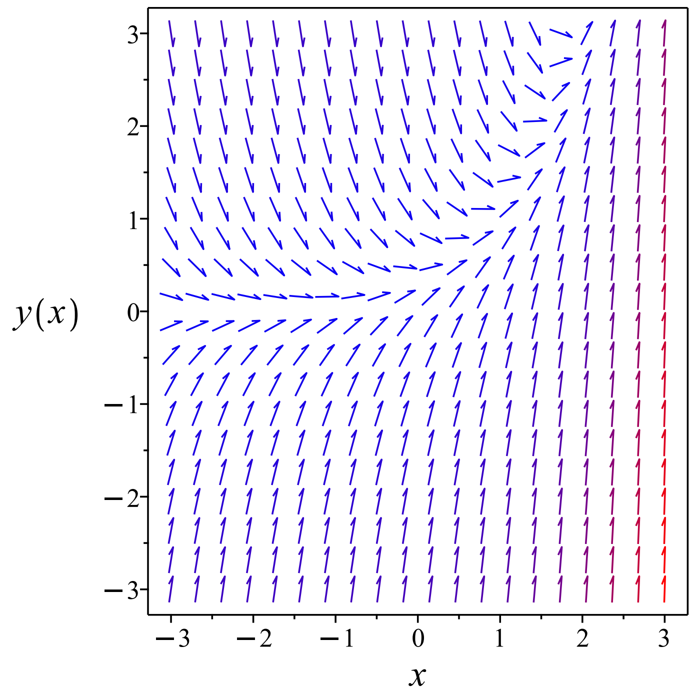


Figure 21: Slope field plot

Verification of solutions

$$y = \frac{(e^{3x} + 3c_1) e^{-2x}}{3}$$

Verified OK.

1.13.4 Maple step by step solution

Let's solve

$y' + 2y = e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -2y + e^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + 2y = e^{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y' + 2y\right) = \mu(x)\, e^{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\, y\right)$

$\mu(x)\left(y' + 2y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

$\mu'(x) = 2\mu(x)$

- Solve to find the integrating factor

$\mu(x) = e^{2x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}\left(\mu(x)\, y\right)\right) dx = \int \mu(x)\, e^{x} dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\, y = \int \mu(x)\, e^{x} dx + c_1$

- Solve for y

$y = \frac{\int \mu(x) e^{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{2x}$

$y = \frac{\int e^{x} e^{2x} dx + c_1}{e^{2x}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{e^{3x}}{3} + c_1}{e^{2x}}$

- Simplify

$y = \frac{\left(e^{3x} + 3c_1\right) e^{-2x}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+2*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(\mathrm{e}^{3x} + 3c_1)\,\mathrm{e}^{-2x}}{3}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 21

```
DSolve[y'[x]+2*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{3} + c_1 e^{-2x}$$

1.14 problem 16

Internal problem ID [14947]
Internal file name [`OUTPUT/14956_Monday_April_15_2024_12_04_19_AM_36373413/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 1. Basic concepts and definitions. Exercises page 18
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(-x^2+1)\, y' + yx = 2x}$$

1.14.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(y-2)}{x^2-1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2-1}$ and $g(y) = y-2$. Integrating both sides gives

$$\frac{1}{y-2}\, dy = \frac{x}{x^2-1}\, dx$$

$$\int \frac{1}{y-2}\, dy = \int \frac{x}{x^2-1}\, dx$$

$$\ln(y-2) = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1$$

Raising both side to exponential gives

$$y - 2 = e^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1}$$

Which simplifies to

$$y - 2 = c_2 e^{\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}}$$

Which simplifies to

$$y = c_2 \sqrt{x-1}\,\sqrt{x+1}\,e^{c_1} + 2$$

Summary

The solution(s) found are the following

$$y = c_2 \sqrt{x-1}\,\sqrt{x+1}\,e^{c_1} + 2 \tag{1}$$

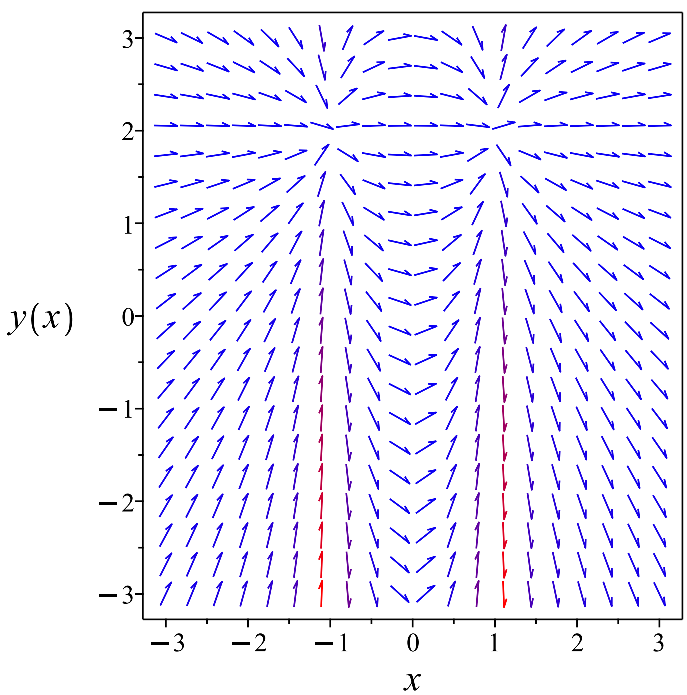


Figure 22: Slope field plot

Verification of solutions

$$y = c_2 \sqrt{x-1}\,\sqrt{x+1}\,e^{c_1} + 2$$

Verified OK.

1.14.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{x}{x^2-1}\\ q(x) &= -\frac{2x}{x^2-1}\end{aligned}$$

Hence the ode is

$$y' - \frac{xy}{x^2-1} = -\frac{2x}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{x}{x^2-1}dx}\\ &= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{2x}{x^2-1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(\frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\left(-\frac{2x}{x^2-1}\right)\\ \mathrm{d}\left(\frac{y}{\sqrt{x-1}\,\sqrt{x+1}}\right) &= \left(-\frac{2x}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} &= \int -\frac{2x}{(x^2-1)\sqrt{x-1}\,\sqrt{x+1}}\,\mathrm{d}x\\ \frac{y}{\sqrt{x-1}\,\sqrt{x+1}} &= \frac{2\sqrt{x-1}\,\sqrt{x+1}}{x^2-1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}$ results in

$$y = \frac{2(x-1)(x+1)}{x^2-1} + c_1\sqrt{x-1}\,\sqrt{x+1}$$

which simplifies to

$$y = 2 + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Summary

The solution(s) found are the following

$$y = 2 + c_1\sqrt{x-1}\,\sqrt{x+1} \tag{1}$$

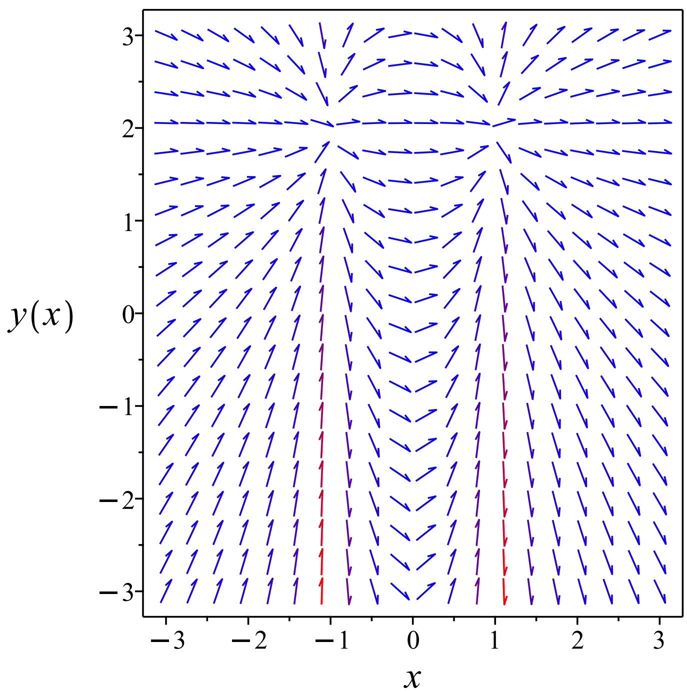


Figure 23: Slope field plot

Verification of solutions

$$y = 2 + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Verified OK.

1.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(y-2)}{x^2-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 12: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\ln\left(\frac{1}{\sqrt{x-1}}\right)+\ln\left(\frac{1}{\sqrt{x+1}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(y-2)}{x^2-1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{yx}{(x-1)^{\frac{3}{2}} (x+1)^{\frac{3}{2}}} \\
S_y &= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{2x}{(x-1)^{\frac{3}{2}} (x+1)^{\frac{3}{2}}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2R}{(R-1)^{\frac{3}{2}} (R+1)^{\frac{3}{2}}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2}{\sqrt{R-1}\,\sqrt{R+1}} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{2}{\sqrt{x-1}\,\sqrt{x+1}} + c_1
$$

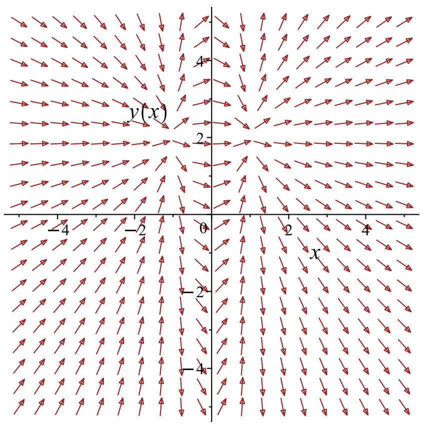
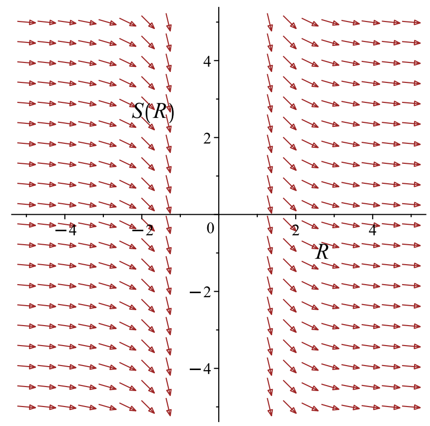
Which simplifies to

$$
\frac{y}{\sqrt{x-1}\,\sqrt{x+1}} = \frac{2}{\sqrt{x-1}\,\sqrt{x+1}} + c_1
$$

Which gives

$$
y = 2 + c_1\sqrt{x-1}\,\sqrt{x+1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(y-2)}{x^2-1}$ 	$R = x$ $S = \frac{y}{\sqrt{x-1}\,\sqrt{x+1}}$	$\frac{dS}{dR} = -\frac{2R}{(R-1)^{\frac{3}{2}}(R+1)^{\frac{3}{2}}}$ 

Summary

The solution(s) found are the following

$$y = 2 + c_1\sqrt{x-1}\,\sqrt{x+1} \tag{1}$$

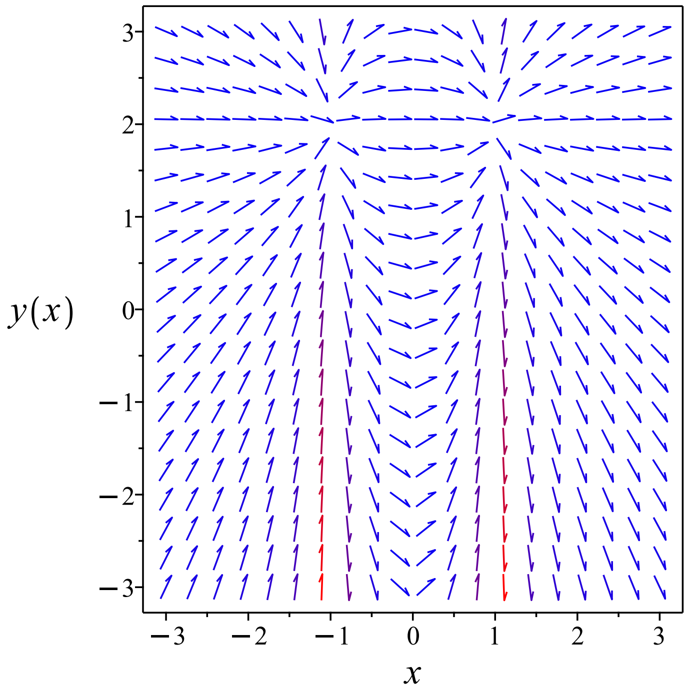


Figure 24: Slope field plot

Verification of solutions

$$y = 2 + c_1\sqrt{x-1}\,\sqrt{x+1}$$

Verified OK.

1.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y-2}\right) \mathrm{d}y &= \left(\frac{x}{x^2-1}\right) \mathrm{d}x \\
\left(-\frac{x}{x^2-1}\right) \mathrm{d}x + \left(\frac{1}{y-2}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{x^2-1} \\
N(x, y) &= \frac{1}{y-2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y-2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2-1}\,\mathrm{d}x \\
\phi &= -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y-2}$. Therefore equation (4) becomes

$$
\frac{1}{y-2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y-2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y-2}\right)\mathrm{d}y \\
f(y) &= \ln(y-2) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(y-2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(y-2)$$

Summary
The solution(s) found are the following

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(y-2) = c_1 \tag{1}$$

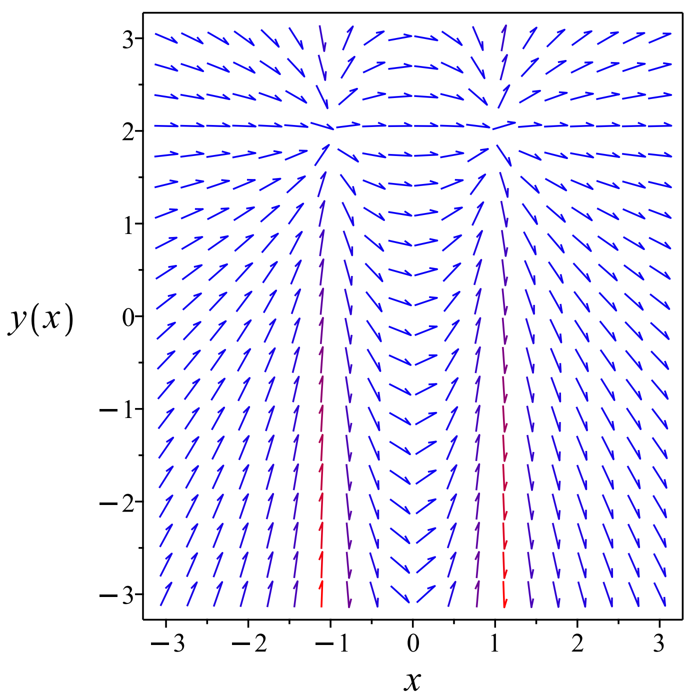


Figure 25: Slope field plot

Verification of solutions

$$-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + \ln(y-2) = c_1$$

Verified OK.

1.14.5 Maple step by step solution

Let's solve

$(-x^2+1)\,y' + yx = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-2} = \frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-2}dx = \int \frac{x}{(x-1)(x+1)}dx + c_1$

- Evaluate integral

  $\ln(y-2) = \frac{\ln((x-1)(x+1))}{2} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{\ln((x-1)(x+1))}{2}+c_1} + 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((1-x^2)*diff(y(x),x)+x*y(x)=2*x,y(x), singsol=all)
```

$$y(x) = \sqrt{-1+x}\,\sqrt{1+x}\,c_1 + 2$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 24

```
DSolve[(1-x^2)*y'[x]+x*y[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 + c_1\sqrt{x^2-1}$$
$$y(x) \to 2$$

2 Section 2. The method of isoclines. Exercises page 27

2.1 problem 21

Internal problem ID [14948]
Internal file name [`OUTPUT/14957_Monday_April_15_2024_12_04_20_AM_5802521/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = x + 1}$$

2.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x + 1 \, \mathrm{d}x \\ &= \frac{1}{2}x^2 + x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2}x^2 + x + c_1 \tag{1}$$

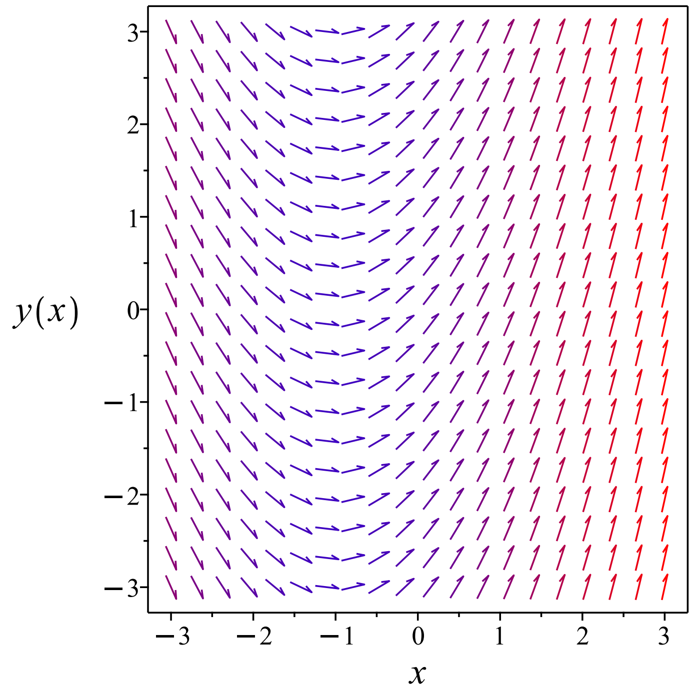


Figure 26: Slope field plot

Verification of solutions

$$y = \frac{1}{2}x^2 + x + c_1$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$y' = x + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (x + 1) \, dx + c_1$

- Evaluate integral

  $y = \frac{1}{2}x^2 + x + c_1$

- Solve for y

$$y = \frac{1}{2}x^2 + x + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=x+1,y(x), singsol=all)
```

$$y(x) = \frac{1}{2}x^2 + x + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
DSolve[y'[x]==x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + x + c_1$$

2.2 problem 22

Internal problem ID [14949]
Internal file name [`OUTPUT/14958_Monday_April_15_2024_12_04_21_AM_89108218/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x}$$

2.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = x$$

Hence the ode is

$$y' - y = x$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x)$$
$$\frac{d}{dx}(e^{-x} y) = (e^{-x})(x)$$
$$d(e^{-x} y) = (x e^{-x})\, dx$$

Integrating gives

$$e^{-x} y = \int x e^{-x}\, dx$$
$$e^{-x} y = -(x+1) e^{-x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}(x+1) e^{-x} + e^{x} c_1$$

which simplifies to

$$y = e^{x} c_1 - x - 1$$

Summary

The solution(s) found are the following

$$y = e^{x} c_1 - x - 1 \tag{1}$$

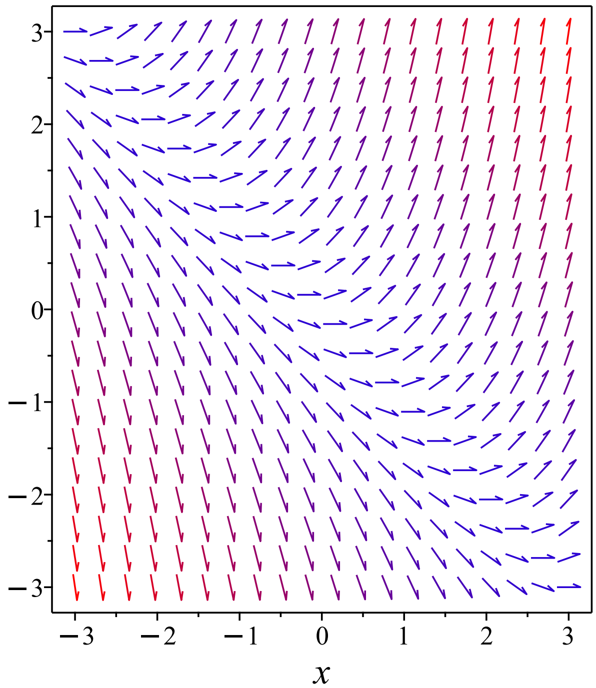


Figure 27: Slope field plot

Verification of solutions

$$y = e^x c_1 - x - 1$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = y + x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 16: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\,e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = -(x+1)\,\mathrm{e}^{-x} + c_1$$

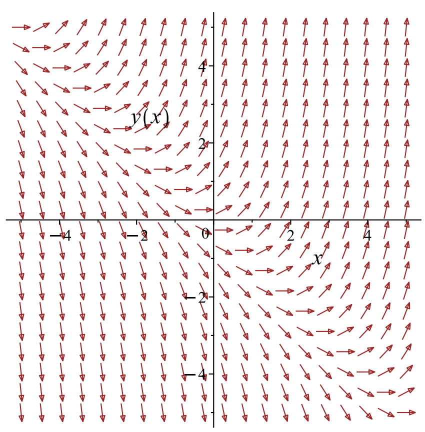
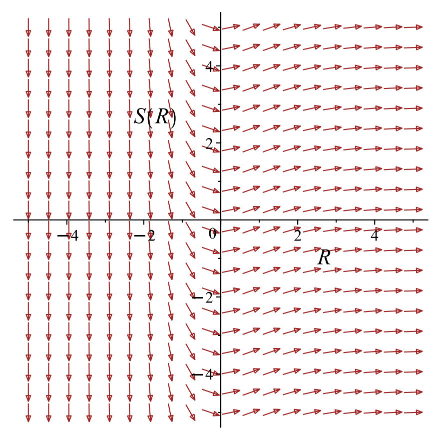
Which simplifies to

$$(x+y+1)\,\mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + x$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = R\,\mathrm{e}^{-R}$ 

Summary
The solution(s) found are the following

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

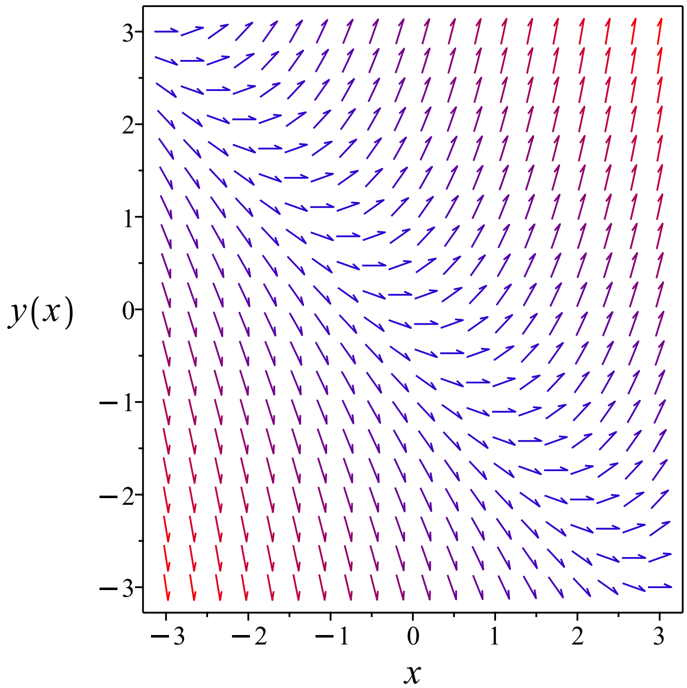


Figure 28: Slope field plot

Verification of solutions

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y + x)\, \mathrm{d}x\\ (-y - x)\, \mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y - x\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - x)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-y - x) \\
&= -\mathrm{e}^{-x}(y + x)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x}(y + x)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\mathrm{e}^{-x}(y+x) \mathrm{d}x$$
$$\phi = (x+y+1)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x+y+1)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x+y+1)\,\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

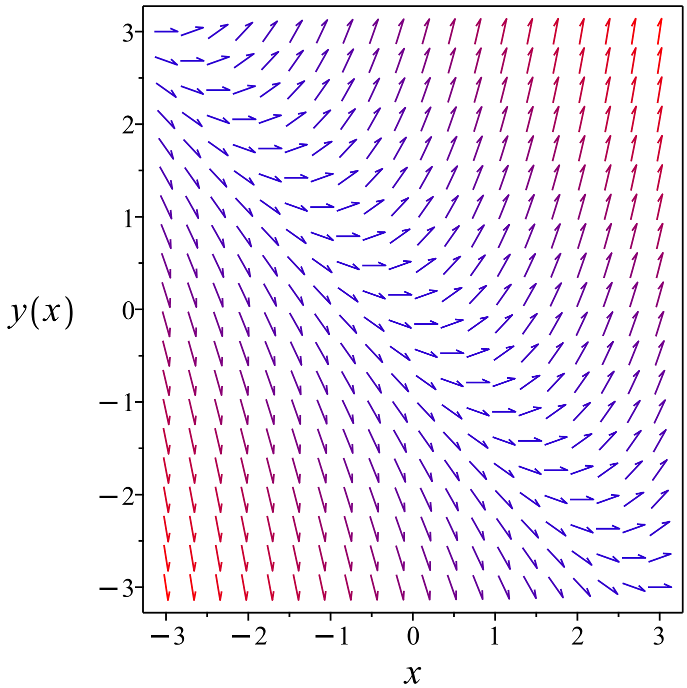


Figure 29: Slope field plot

Verification of solutions

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Verified OK.

2.2.4 Maple step by step solution

Let's solve

$y' - y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\, (y'-y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int x\, \mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(x+1)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{x} c_1 - x - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)=x+y(x),y(x), singsol=all)
```

$$y(x) = -x - 1 + c_1 \mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 16

```
DSolve[y'[x]==x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + c_1 e^x - 1$$

2.3 problem 23

Internal problem ID [14950]
Internal file name [`OUTPUT/14959_Monday_April_15_2024_12_04_21_AM_51000317/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = -x}$$

2.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = -x$$

Hence the ode is

$$y' - y = -x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(-x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x} y\right) &= \left(\mathrm{e}^{-x}\right)(-x) \\
\mathrm{d}\left(\mathrm{e}^{-x} y\right) &= \left(-x\,\mathrm{e}^{-x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-x} y &= \int -x\,\mathrm{e}^{-x}\,\mathrm{d}x \\
\mathrm{e}^{-x} y &= (x+1)\,\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$
y = \mathrm{e}^{x} (x+1)\,\mathrm{e}^{-x} + \mathrm{e}^{x} c_1
$$

which simplifies to

$$
y = \mathrm{e}^{x} c_1 + x + 1
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{x} c_1 + x + 1 \tag{1}
$$

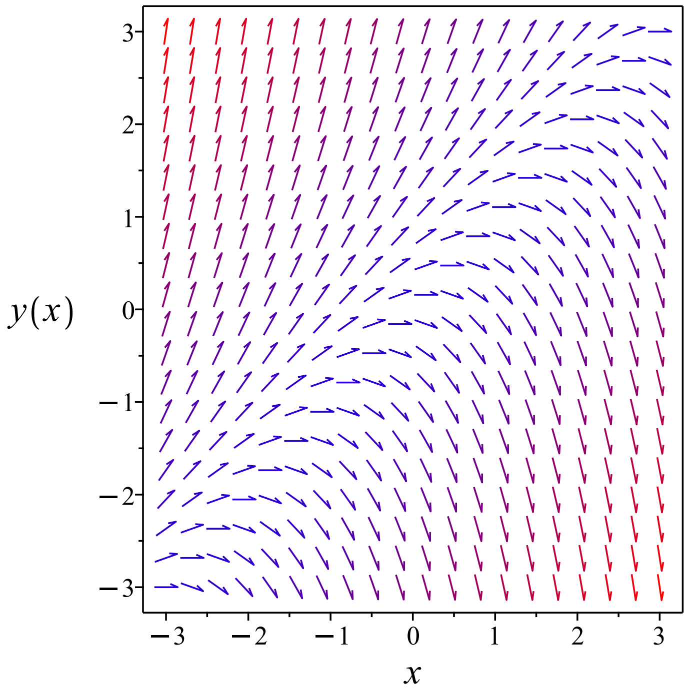


Figure 30: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{x} c_1 + x + 1$$

Verified OK.

2.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y - x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 19: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y - x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x\,e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R\,e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = (x+1)\,\mathrm{e}^{-x} + c_1$$

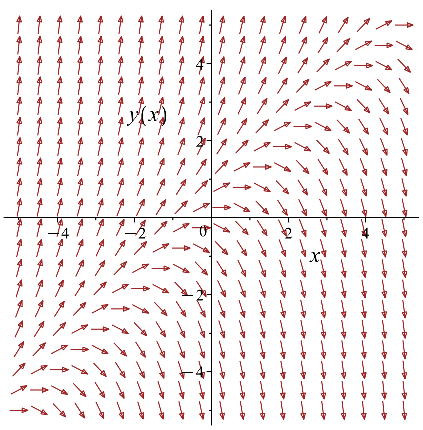
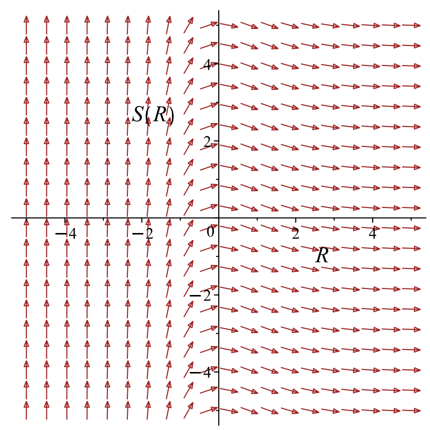
Which simplifies to

$$\mathrm{e}^{-x} y = (x+1)\,\mathrm{e}^{-x} + c_1$$

Which gives

$$y = \left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y - x$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = -R\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = \left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x} \tag{1}$$

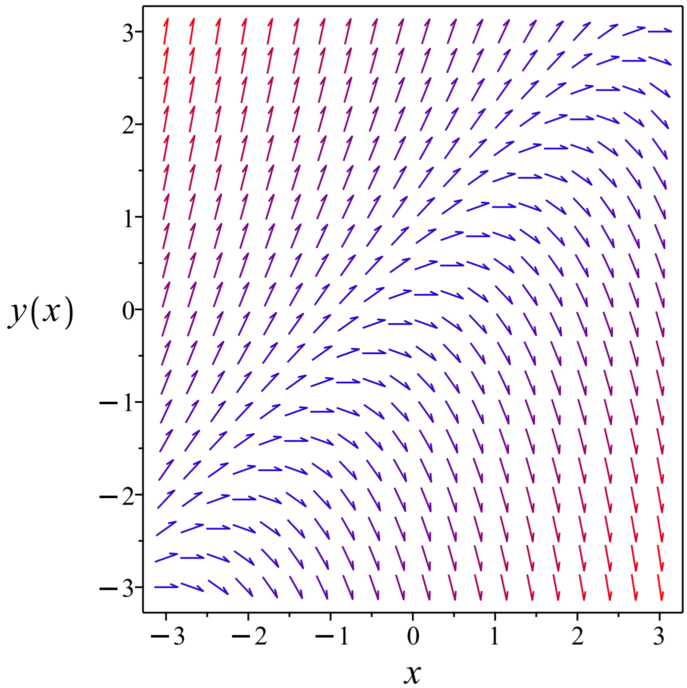


Figure 31: Slope field plot

Verification of solutions

$$y = \left(x\,e^{-x} + e^{-x} + c_1\right) e^{x}$$

Verified OK.

2.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (y - x) \,\mathrm{d}x$$
$$(-y + x) \,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y + x$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y + x) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -1 \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x} \\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-y + x) \\ &= (-y + x)\,\mathrm{e}^{-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((-y + x)\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int (-y+x) \mathrm{e}^{-x} \mathrm{d}x$$
$$\phi = -(-y+1+x) \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -(-y+1+x) \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -(-y+1+x) \mathrm{e}^{-x}$$

The solution becomes

$$y = \left(x \mathrm{e}^{-x} + \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \left(x \mathrm{e}^{-x} + \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x} \tag{1}$$

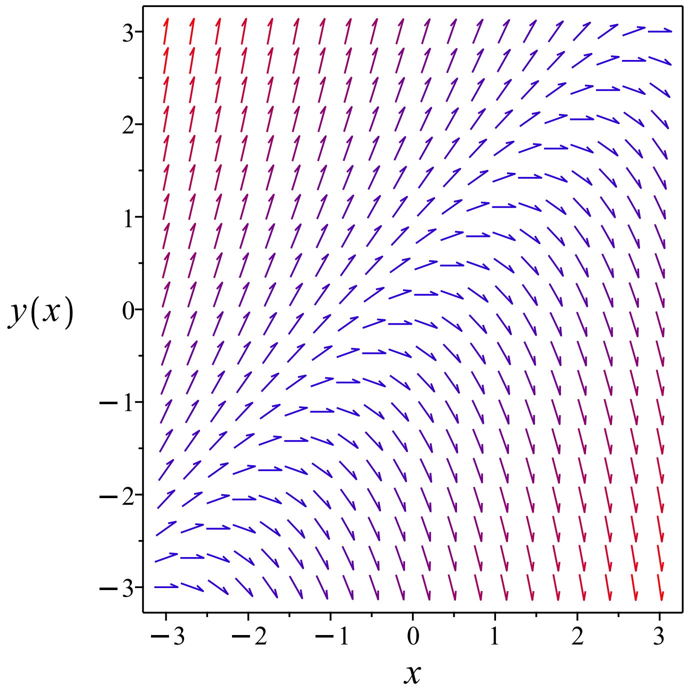


Figure 32: Slope field plot

Verification of solutions

$$y = \left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} + c_1\right)\mathrm{e}^{x}$$

Verified OK.

2.3.4 Maple step by step solution

Let's solve

$y' - y = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y - x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = -x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = -\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,x dx + c_1$
- Solve for y

  $y = \frac{\int -\mu(x)x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int -x\,\mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{(x+1)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{x} c_1 + x + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=y(x)-x,y(x), singsol=all)
```

$$y(x) = x + 1 + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 14

```
DSolve[y'[x]==y[x]-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^x + 1$$

2.4 problem 24

Internal problem ID [14951]
Internal file name [`OUTPUT/14960_Monday_April_15_2024_12_04_22_AM_31134022/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \frac{x}{2} + \frac{3}{2}}$$

2.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{x}{2} + \frac{3}{2} \end{aligned}$$

Hence the ode is

$$y' + y = \frac{x}{2} + \frac{3}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x}{2}+\frac{3}{2}\right) \\ \frac{d}{dx}(e^{x} y) &= (e^{x})\left(\frac{x}{2}+\frac{3}{2}\right) \\ d(e^{x} y) &= \left(\frac{(x+3)\,e^{x}}{2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x} y &= \int \frac{(x+3)\,e^{x}}{2}\, dx \\ e^{x} y &= \frac{(x+2)\,e^{x}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = \frac{e^{-x}(x+2)\,e^{x}}{2} + c_1 e^{-x}$$

which simplifies to

$$y = 1 + \frac{x}{2} + c_1 e^{-x}$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{x}{2} + c_1 e^{-x} \tag{1}$$

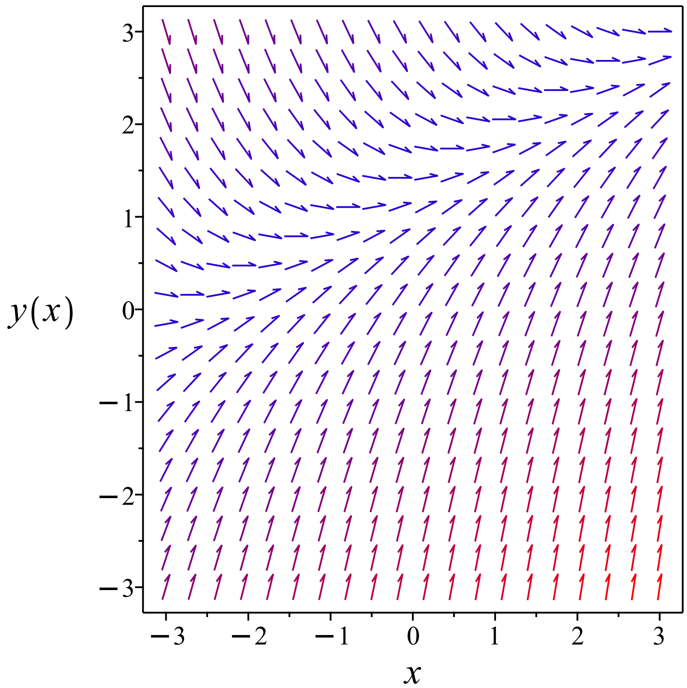


Figure 33: Slope field plot

Verification of solutions

$$y = 1 + \frac{x}{2} + c_1 e^{-x}$$

Verified OK.

2.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{2} - y + \frac{3}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 22: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x}{2} - y + \frac{3}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{(x + 3) e^x}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{(R + 3) e^R}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(R+2)\,e^{R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{x} = \frac{(x+2)\,e^{x}}{2} + c_1$$

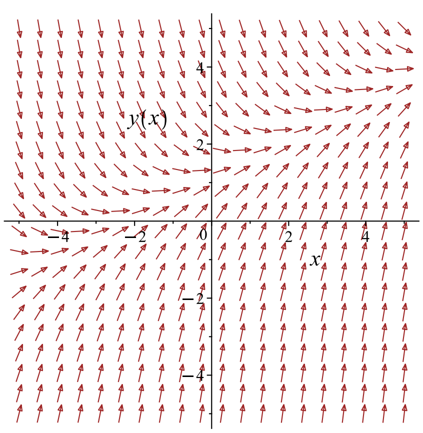
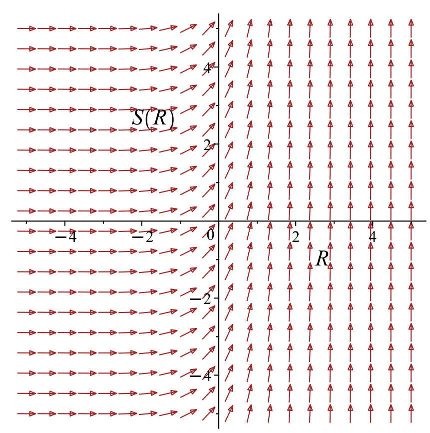
Which simplifies to

$$y\,e^{x} = \frac{(x+2)\,e^{x}}{2} + c_1$$

Which gives

$$y = \frac{(x\,e^{x} + 2\,e^{x} + 2c_1)\,e^{-x}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{2} - y + \frac{3}{2}$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = \frac{(R+3)e^{R}}{2}$ 

Summary

The solution(s) found are the following

$$y = \frac{(x\,e^{x} + 2\,e^{x} + 2c_1)\,e^{-x}}{2} \tag{1}$$

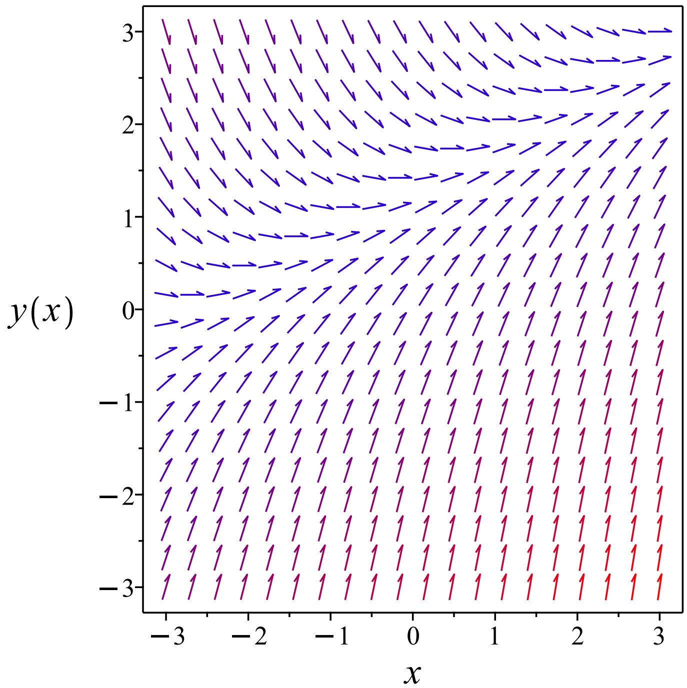


Figure 34: Slope field plot

Verification of solutions

$$y = \frac{(x\,e^{x} + 2\,e^{x} + 2c_1)\,e^{-x}}{2}$$

Verified OK.

2.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{x}{2} - y + \frac{3}{2}\right)\mathrm{d}x$$
$$\left(-\frac{x}{2} + y - \frac{3}{2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{2} + y - \frac{3}{2}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{2} + y - \frac{3}{2}\right)$$
$$= 1$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((1) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int 1 \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}\left(-\frac{x}{2} + y - \frac{3}{2}\right)\\ &= -\frac{(x - 2y + 3)\,\mathrm{e}^{x}}{2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{x}(1)\\ &= \mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{(x - 2y + 3)\,\mathrm{e}^{x}}{2}\right) + (\mathrm{e}^{x})\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial\phi}{\partial x} \, dx = \int -\frac{(x - 2y + 3) e^{x}}{2} \, dx$$
$$\phi = -\frac{(x - 2y + 2) e^{x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = e^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = e^{x}$. Therefore equation (4) becomes

$$e^{x} = e^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(x - 2y + 2) e^{x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(x - 2y + 2) e^{x}}{2}$$

The solution becomes

$$y = \frac{(x\,e^{x} + 2\,e^{x} + 2c_1)\,e^{-x}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(x\,e^{x} + 2\,e^{x} + 2c_1)\,e^{-x}}{2} \tag{1}$$

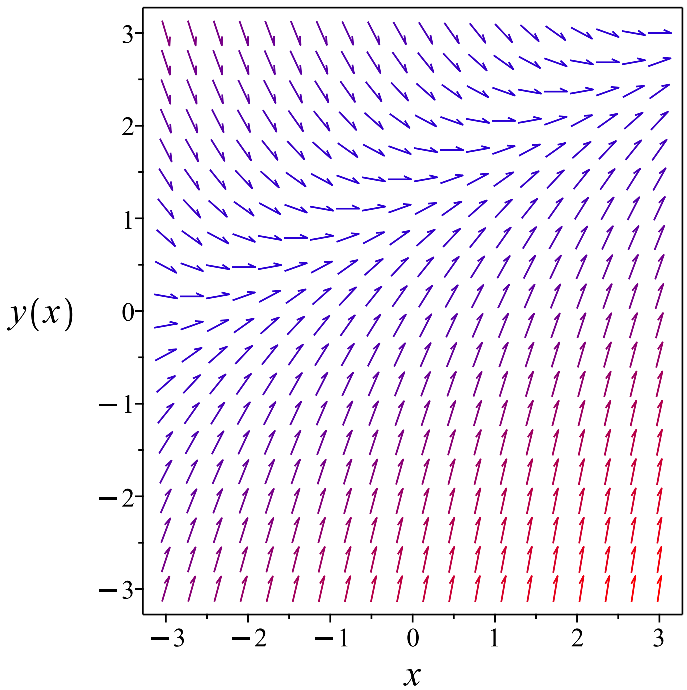


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{(x\,e^{x} + 2\,e^{x} + 2c_1)\,e^{-x}}{2}$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$y' + y = \frac{x}{2} + \frac{3}{2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{x}{2} - y + \frac{3}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = \frac{x}{2} + \frac{3}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\left(\frac{x}{2} + \frac{3}{2}\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\left(\frac{x}{2} + \frac{3}{2}\right) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\left(\frac{x}{2} + \frac{3}{2}\right) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\left(\frac{x}{2}+\frac{3}{2}\right)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int \left(\frac{x}{2}+\frac{3}{2}\right)\mathrm{e}^{x}dx + c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{(x+2)\,\mathrm{e}^{x}}{2} + c_1}{\mathrm{e}^{x}}$

- Simplify

$$y = 1 + \frac{x}{2} + c_1 e^{-x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=1/2*(x-2*y(x)+3),y(x), singsol=all)
```

$$y(x) = \frac{x}{2} + 1 + c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 20

```
DSolve[y'[x]==1/2*(x-2*y[x]+3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{2} + c_1 e^{-x} + 1$$

2.5 problem 25

Internal problem ID [14952]
Internal file name [`OUTPUT/14961_Monday_April_15_2024_12_04_23_AM_41410347/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - (y-1)^2 = 0}$$

2.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{(y-1)^2} dy = x + c_1$$
$$-\frac{1}{y-1} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{c_1 + x - 1}{x + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 + x - 1}{x + c_1} \tag{1}$$

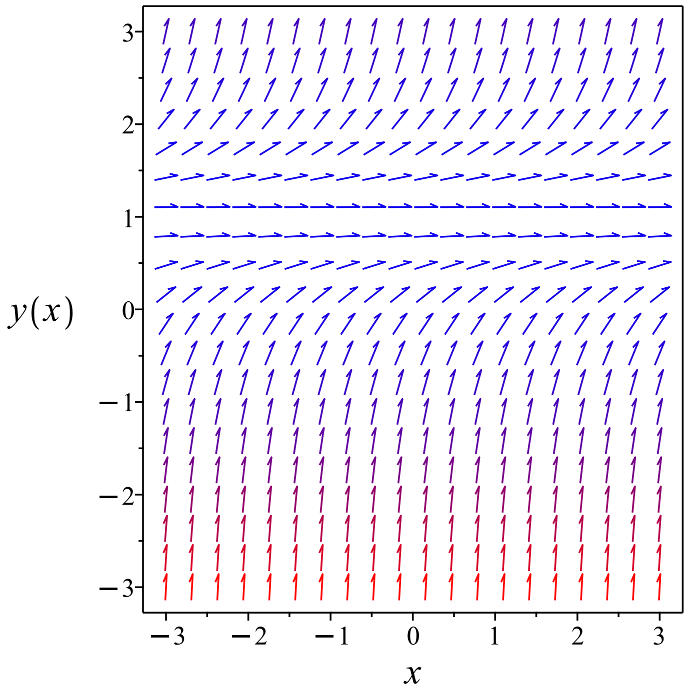


Figure 36: Slope field plot

Verification of solutions

$$y = \frac{c_1 + x - 1}{x + c_1}$$

Verified OK.

2.5.2 Maple step by step solution

Let's solve

$y' - (y-1)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(y-1)^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{(y-1)^2} dx = \int 1 dx + c_1$

- Evaluate integral

$$-\frac{1}{y-1} = x + c_1$$

- Solve for y

$$y = \frac{c_1 + x - 1}{x + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=(y(x)-1)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1 + x - 1}{c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.107 (sec). Leaf size: 22

```
DSolve[y'[x]==(y[x]-1)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x - 1 + c_1}{x + c_1}$$
$$y(x) \to 1$$

2.6 problem 26

Internal problem ID [14953]
Internal file name [`OUTPUT/14962_Monday_April_15_2024_12_04_23_AM_29212613/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - (y-1)\,x = 0}$$

2.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y-1)\,x \end{aligned}$$

Where $f(x) = x$ and $g(y) = y - 1$. Integrating both sides gives

$$\frac{1}{y-1}\,dy = x\,dx$$

$$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$$\ln(y-1) = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$y - 1 = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$y - 1 = c_2 e^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\frac{x^2}{2}+c_1} + 1 \tag{1}$$

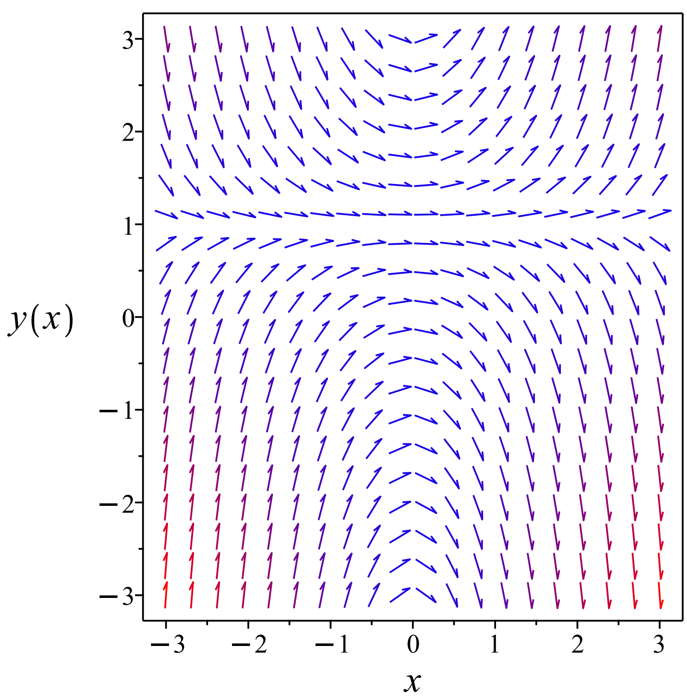


Figure 37: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{x^2}{2}+c_1} + 1$$

Verified OK.

2.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$y' - yx = -x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -x dx} \\ &= \mathrm{e}^{-\frac{x^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(-x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x^2}{2}} y\right) &= \left(\mathrm{e}^{-\frac{x^2}{2}}\right)(-x) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{x^2}{2}} y\right) &= \left(-x\,\mathrm{e}^{-\frac{x^2}{2}}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-\frac{x^2}{2}} y &= \int -x\,\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \\ \mathrm{e}^{-\frac{x^2}{2}} y &= \mathrm{e}^{-\frac{x^2}{2}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x^2}{2}}$ results in

$$y = \mathrm{e}^{\frac{x^2}{2}}\mathrm{e}^{-\frac{x^2}{2}} + c_1\mathrm{e}^{\frac{x^2}{2}}$$

which simplifies to

$$y = 1 + c_1\mathrm{e}^{\frac{x^2}{2}}$$

Summary
The solution(s) found are the following

$$y = 1 + c_1\mathrm{e}^{\frac{x^2}{2}} \tag{1}$$

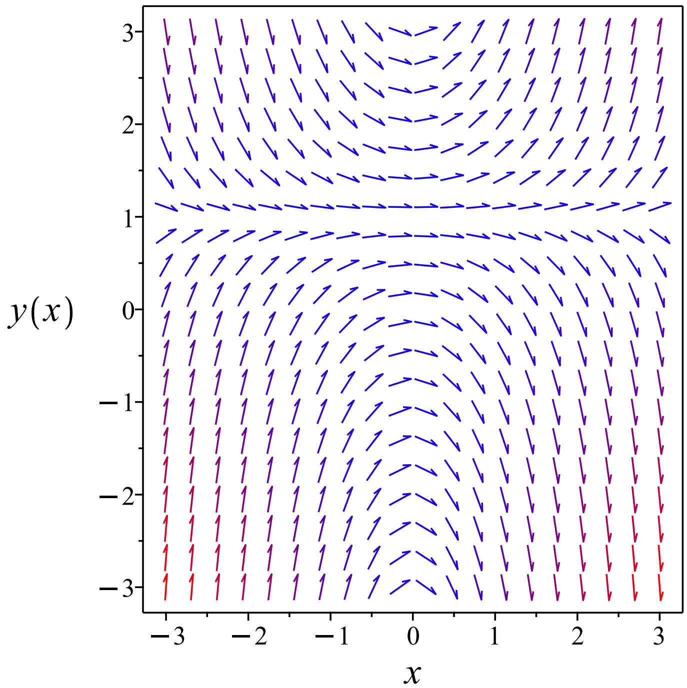


Figure 38: Slope field plot

Verification of solutions

$$y = 1 + c_1 \mathrm{e}^{\frac{x^2}{2}}$$

Verified OK.

2.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (y - 1)\,x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{\frac{x^2}{2}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (y - 1) x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x e^{-\frac{x^2}{2}} y \\ S_y &= e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x e^{-\frac{x^2}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R e^{-\frac{R^2}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{-\frac{R^2}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x^2}{2}} y = \mathrm{e}^{-\frac{x^2}{2}} + c_1$$

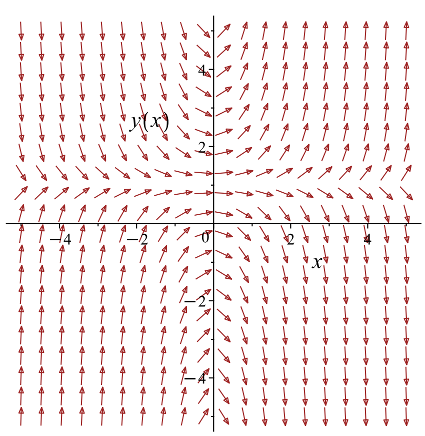
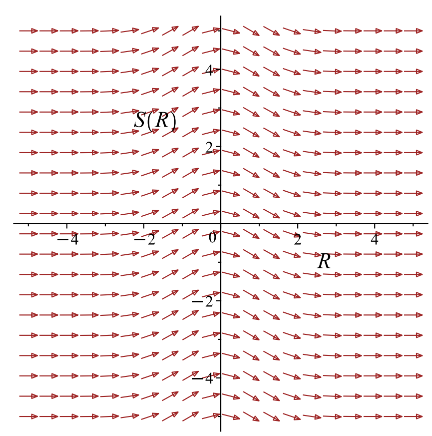
Which simplifies to

$$(y-1)\,\mathrm{e}^{-\frac{x^2}{2}} - c_1 = 0$$

Which gives

$$y = \left(\mathrm{e}^{-\frac{x^2}{2}} + c_1\right) \mathrm{e}^{\frac{x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (y-1)\,x$ 	$R = x$ $S = \mathrm{e}^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = -R\,\mathrm{e}^{-\frac{R^2}{2}}$ 

Summary

The solution(s) found are the following

$$y = \left(\mathrm{e}^{-\frac{x^2}{2}} + c_1\right) \mathrm{e}^{\frac{x^2}{2}} \tag{1}$$

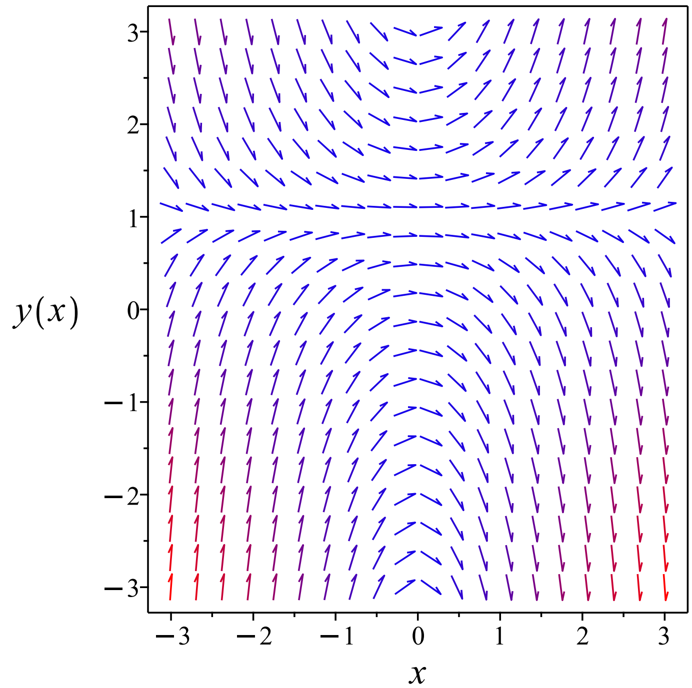


Figure 39: Slope field plot

Verification of solutions

$$y = \left(e^{-\frac{x^2}{2}} + c_1\right) e^{\frac{x^2}{2}}$$

Verified OK.

2.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y-1}\right) \mathrm{d}y &= (x)\, \mathrm{d}x \\
(-x)\, \mathrm{d}x + \left(\frac{1}{y-1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{y-1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y-1}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y-1}$. Therefore equation (4) becomes

$$
\frac{1}{y-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y-1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{y-1}\right) \mathrm{d}y\\
f(y) &= \ln(y-1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \ln(y-1)$$

The solution becomes

$$y = e^{\frac{x^2}{2}+c_1} + 1$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}+c_1} + 1 \tag{1}$$

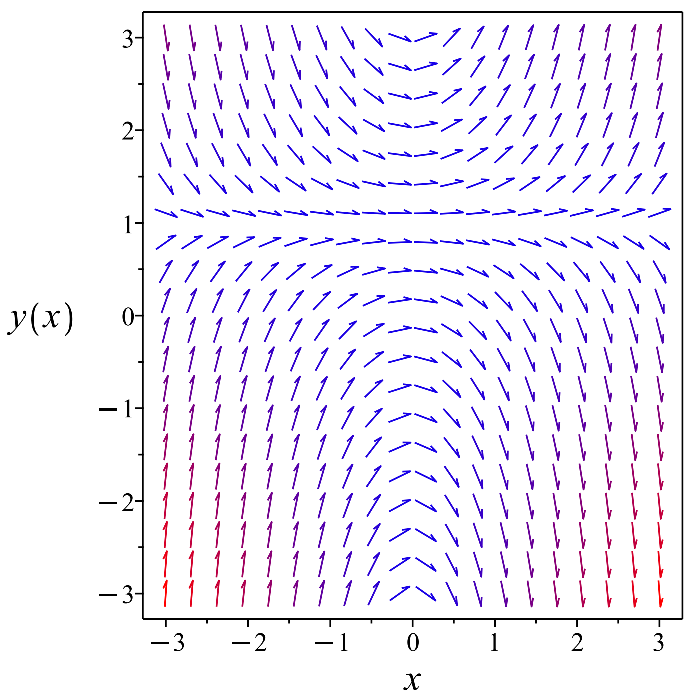


Figure 40: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}+c_1} + 1$$

Verified OK.

2.6.5 Maple step by step solution

Let's solve

$y' - (y-1)\,x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y-1) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{x^2}{2}+c_1} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=(y(x)-1)*x,y(x), singsol=all)
```

$$y(x) = 1 + c_1 e^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 24

```
DSolve[y'[x]==(y[x]-1)*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + c_1 e^{\frac{x^2}{2}}$$
$$y(x) \to 1$$

2.7 problem 27

Internal problem ID [14954]
Internal file name [`OUTPUT/14963_Monday_April_15_2024_12_04_23_AM_63877819/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' + y^2 = x^2}$$

2.7.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 - y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{x}\left(c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x^{\frac{3}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)} \tag{1}$$

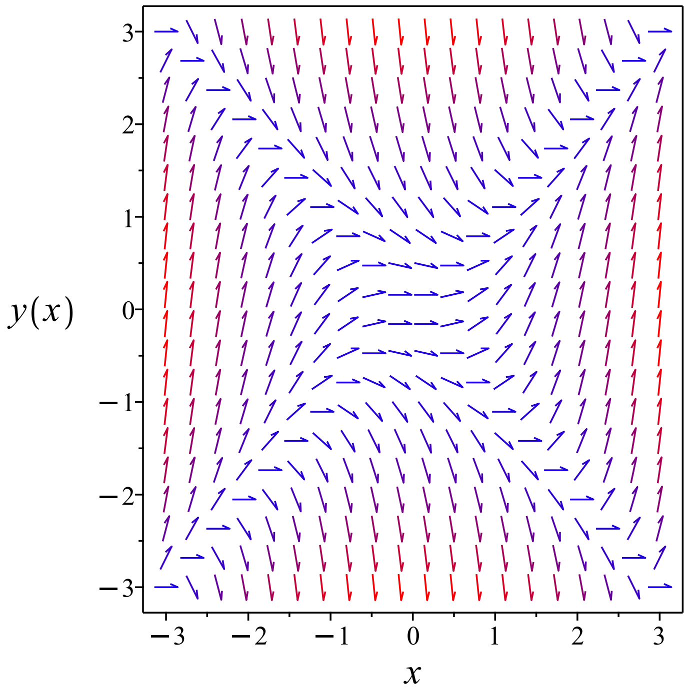


Figure 41: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(-\text{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \text{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \,\text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)=x^2-y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x\left(\operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 - \operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)\right)}{c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 197

```
DSolve[y'[x]==x^2-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-ix^2\left(2\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{ix^2}{2}\right) + c_1\left(\operatorname{BesselJ}\left(-\frac{5}{4}, \frac{ix^2}{2}\right) - \operatorname{BesselJ}\left(\frac{3}{4}, \frac{ix^2}{2}\right)\right)\right) - c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}{2x\left(\operatorname{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) + c_1 \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)\right)}$$

$$y(x) \to \frac{ix^2 \operatorname{BesselJ}\left(-\frac{5}{4}, \frac{ix^2}{2}\right) - ix^2 \operatorname{BesselJ}\left(\frac{3}{4}, \frac{ix^2}{2}\right) + \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}{2x \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}$$

2.8 problem 28

Internal problem ID [14955]
Internal file name [`OUTPUT/14964_Monday_April_15_2024_12_04_25_AM_1907743/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \cos(-y + x) = 0}$$

2.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \cos(-y + x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \cos(-y+x)(b_3 - a_2) - \cos(-y+x)^2 a_3 \\
& + \sin(-y+x)(x a_2 + y a_3 + a_1) - \sin(-y+x)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\begin{aligned}
& \sin(-y+x) x a_2 - \sin(-y+x) x b_2 + \sin(-y+x) y a_3 \\
& - \sin(-y+x) y b_3 - \cos(-y+x)^2 a_3 + \sin(-y+x) a_1 \\
& - \sin(-y+x) b_1 - \cos(-y+x) a_2 + \cos(-y+x) b_3 + b_2 = 0
\end{aligned}
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& \sin(-y+x) x a_2 - \sin(-y+x) x b_2 + \sin(-y+x) y a_3 \\
& - \sin(-y+x) y b_3 - \cos(-y+x)^2 a_3 + \sin(-y+x) a_1 \\
& - \sin(-y+x) b_1 - \cos(-y+x) a_2 + \cos(-y+x) b_3 + b_2 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& b_2 - \frac{a_3}{2} + \sin(-y+x) x a_2 - \sin(-y+x) x b_2 + \sin(-y+x) y a_3 \\
& - \sin(-y+x) y b_3 - \frac{a_3 \cos(-2y+2x)}{2} + \sin(-y+x) a_1 \\
& - \sin(-y+x) b_1 - \cos(-y+x) a_2 + \cos(-y+x) b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y, \cos(-2y+2x), \cos(-y+x), \sin(-y+x)\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2, \cos(-2y+2x) = v_3, \cos(-y+x) = v_4, \sin(-y+x) = v_5\}
$$

The above PDE (6E) now becomes

$$
b_2 - \frac{1}{2} a_3 + v_5 v_1 a_2 - v_5 v_1 b_2 + v_5 v_2 a_3 - v_5 v_2 b_3 - \frac{1}{2} a_3 v_3 + v_5 a_1 - v_5 b_1 - v_4 a_2 + v_4 b_3 = 0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (a_2 - b_2)\, v_1 v_5 + (a_3 - b_3)\, v_2 v_5 - \frac{a_3 v_3}{2} + (b_3 - a_2)\, v_4 + (a_1 - b_1)\, v_5 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -\frac{a_3}{2} &= 0 \\ a_1 - b_1 &= 0 \\ a_2 - b_2 &= 0 \\ a_3 - b_3 &= 0 \\ b_2 - \frac{a_3}{2} &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - (\cos(-y + x))\, (1) \\ &= 1 - \cos(y) \cos(x) - \sin(y) \sin(x) \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 - \cos(y)\cos(x) - \sin(y)\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{1}{\tan\left(-\frac{y}{2} + \frac{x}{2}\right)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \cos(-y + x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\csc\left(-\frac{y}{2} + \frac{x}{2}\right)^2}{2} \\ S_y &= \frac{\csc\left(-\frac{y}{2} + \frac{x}{2}\right)^2}{2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\csc\left(-\frac{y}{2}+\frac{x}{2}\right)^{2}\left(\cos\left(-y+x\right)-1\right)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\cot\left(-\frac{y}{2}+\frac{x}{2}\right) = -x + c_1$$

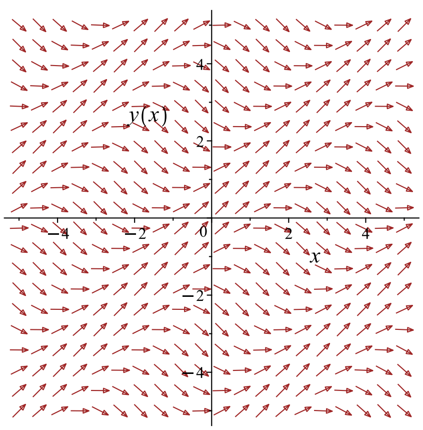
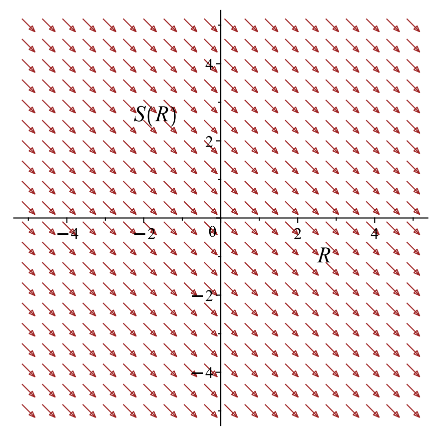
Which simplifies to

$$\cot\left(-\frac{y}{2}+\frac{x}{2}\right) = -x + c_1$$

Which gives

$$y = x - 2\operatorname{arccot}\left(-x + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(-y + x)$ 	$R = x$ $S = \cot\left(-\frac{y}{2} + \frac{x}{2}\right)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = x - 2 \operatorname{arccot}(-x + c_1) \tag{1}$$

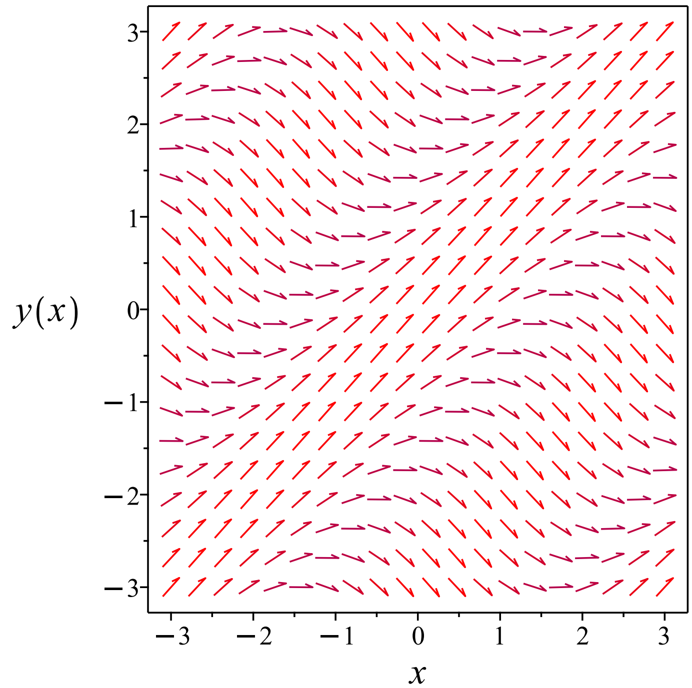


Figure 42: Slope field plot

Verification of solutions

$$y = x - 2 \operatorname{arccot}(-x + c_1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=cos(x-y(x)),y(x), singsol=all)
```

$$y(x) = x - 2 \operatorname{arccot}(c_1 - x)$$

✓ Solution by Mathematica

Time used: 0.45 (sec). Leaf size: 40

```
DSolve[y'[x]==Cos[x-y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + 2\cot^{-1}\left(x - \frac{c_1}{2}\right)$$
$$y(x) \to x + 2\cot^{-1}\left(x - \frac{c_1}{2}\right)$$
$$y(x) \to x$$

2.9 problem 29

Internal problem ID [14956]
Internal file name [`OUTPUT/14965_Monday_April_15_2024_12_04_26_AM_82658170/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = -x^2}$$

2.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -x^2 \end{aligned}$$

Hence the ode is

$$y' - y = -x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x} y\right) &= \left(\mathrm{e}^{-x}\right)\left(-x^2\right) \\
\mathrm{d}\left(\mathrm{e}^{-x} y\right) &= \left(-x^2 \mathrm{e}^{-x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-x} y &= \int -x^2 \mathrm{e}^{-x}\,\mathrm{d}x \\
\mathrm{e}^{-x} y &= \left(x^2 + 2x + 2\right)\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$
y = \mathrm{e}^{x}\left(x^2 + 2x + 2\right)\mathrm{e}^{-x} + \mathrm{e}^{x} c_1
$$

which simplifies to

$$
y = x^2 + 2x + 2 + \mathrm{e}^{x} c_1
$$

Summary

The solution(s) found are the following

$$
y = x^2 + 2x + 2 + \mathrm{e}^{x} c_1 \tag{1}
$$

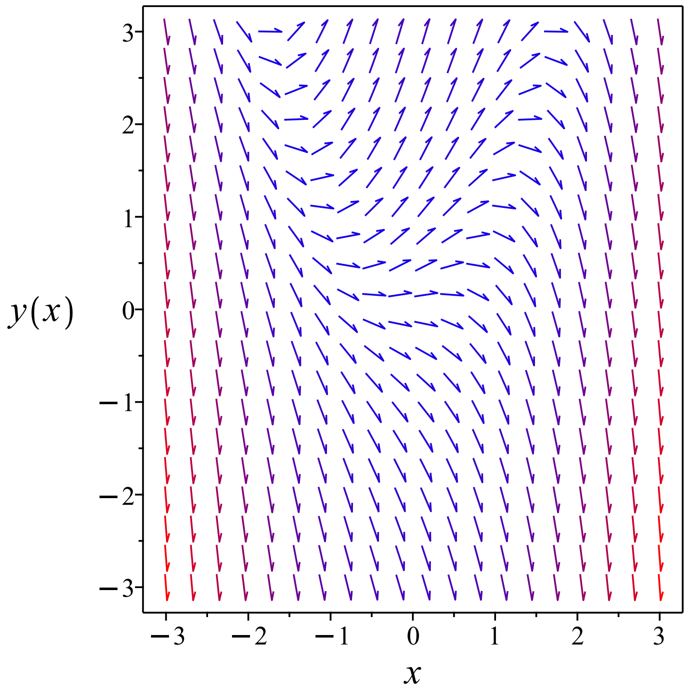


Figure 43: Slope field plot

Verification of solutions

$$y = x^2 + 2x + 2 + e^x c_1$$

Verified OK.

2.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -x^2 + y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 29: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^x} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -x^2 + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\mathrm{e}^{-x} y \\ S_y &= \mathrm{e}^{-x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^2 \mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^2 \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R^2 + 2R + 2)\,e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x}y = (x^2 + 2x + 2)\,e^{-x} + c_1$$

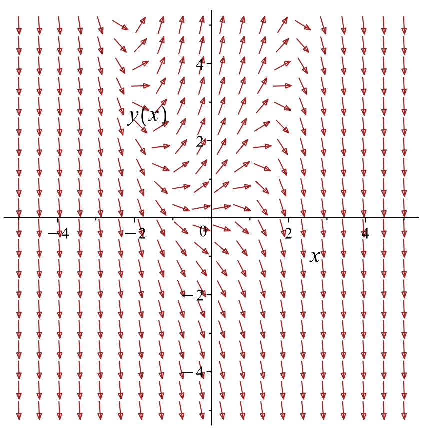
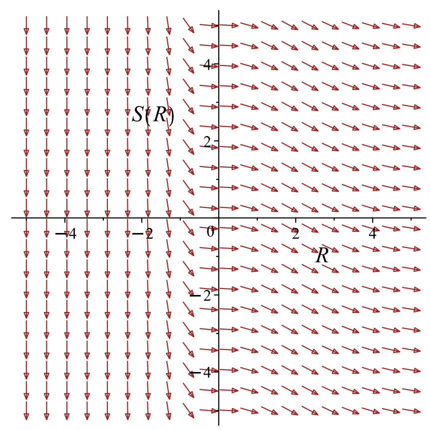
Which simplifies to

$$e^{-x}y = (x^2 + 2x + 2)\,e^{-x} + c_1$$

Which gives

$$y = (x^2 e^{-x} + 2x\,e^{-x} + 2\,e^{-x} + c_1)\,e^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x^2 + y$ 	$R = x$ $S = e^{-x}y$	$\frac{dS}{dR} = -R^2 e^{-R}$ 

Summary

The solution(s) found are the following

$$y = (x^2 e^{-x} + 2x\,e^{-x} + 2\,e^{-x} + c_1)\,e^{x} \tag{1}$$

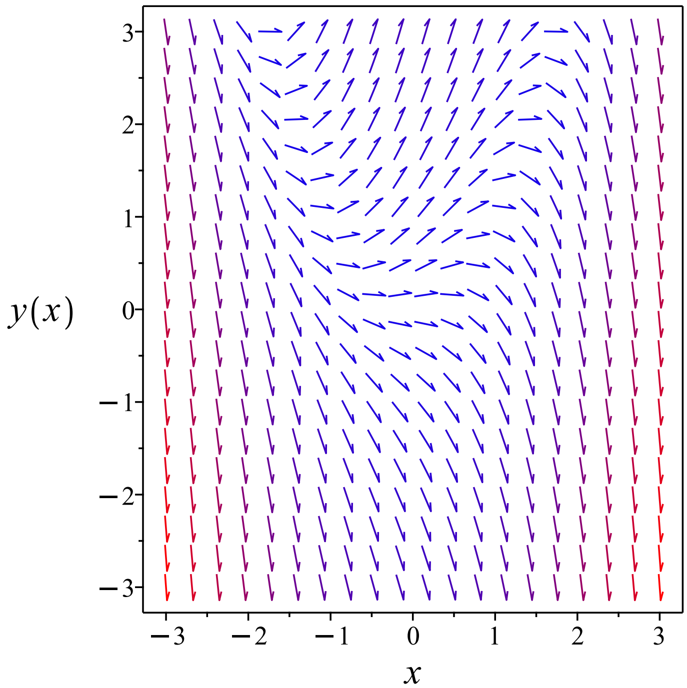


Figure 44: Slope field plot

Verification of solutions

$$y = \left(x^2 e^{-x} + 2x\, e^{-x} + 2\, e^{-x} + c_1\right) e^{x}$$

Verified OK.

2.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-x^2 + y\right) \mathrm{d}x$$
$$\left(x^2 - y\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x^2 - y$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(x^2 - y\right)$$
$$= -1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}\left(x^2 - y\right) \\ &= \left(x^2 - y\right) \mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(x^2 - y\right) \mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int (x^2 - y) \, \mathrm{e}^{-x} \, \mathrm{d}x$$

$$\phi = -\left(x^2 + 2x - y + 2\right) \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^2 + 2x - y + 2\right) \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^2 + 2x - y + 2\right) \mathrm{e}^{-x}$$

The solution becomes

$$y = \left(x^2 \mathrm{e}^{-x} + 2x \, \mathrm{e}^{-x} + 2 \, \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = (x^2 e^{-x} + 2x\, e^{-x} + 2\, e^{-x} + c_1)\, e^{x} \tag{1}$$

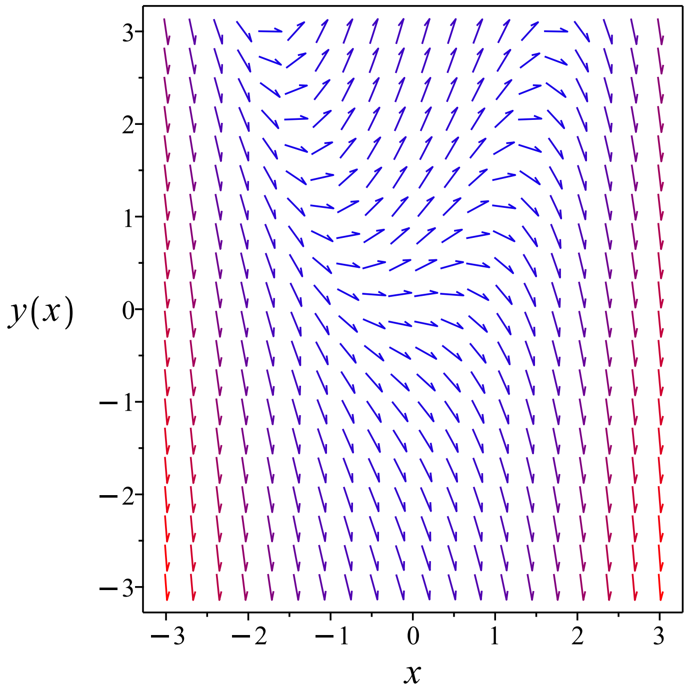


Figure 45: Slope field plot

Verification of solutions

$$y = (x^2 e^{-x} + 2x\, e^{-x} + 2\, e^{-x} + c_1)\, e^{x}$$

Verified OK.

2.9.4 Maple step by step solution

Let's solve

$y' - y = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -x^2 + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = -x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = -\mu(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,x^2 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,x^2 dx + c_1$
- Solve for y

  $y = \frac{\int -\mu(x)x^2 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int -x^2 \mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{(x^2+2x+2)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = x^2 + 2x + 2 + \mathrm{e}^{x} c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=y(x)-x^2,y(x), singsol=all)
```

$$y(x) = x^2 + 2x + 2 + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 19

```
DSolve[y'[x]==y[x]-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + 2x + c_1 e^x + 2$$

2.10 problem 30

Internal problem ID [14957]
Internal file name [`OUTPUT/14966_Monday_April_15_2024_12_04_27_AM_26971188/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x^2 + 2x}$$

2.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = x^2 + 2x$$

Hence the ode is

$$y' + y = x^2 + 2x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2+2x\right) \\
\frac{d}{dx}(e^{x} y) &= (e^{x})\left(x^2+2x\right) \\
d(e^{x} y) &= (x\,e^{x}(x+2))\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int x\,e^{x}(x+2)\,dx \\
e^{x} y &= x^2 e^{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = e^{-x} x^2 e^{x} + c_1 e^{-x}
$$

which simplifies to

$$
y = x^2 + c_1 e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = x^2 + c_1 e^{-x} \tag{1}
$$

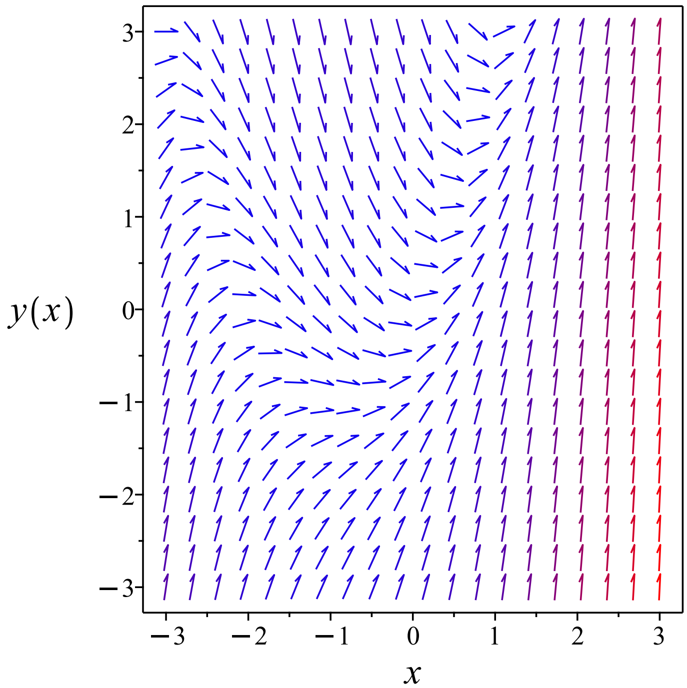


Figure 46: Slope field plot

Verification of solutions

$$y = x^2 + c_1 e^{-x}$$

Verified OK.

2.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^2 + 2x - y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 32: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy\end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 + 2x - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\, e^{x} (x + 2) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, e^{R} (R + 2)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\, e^x = x^2 e^x + c_1$$

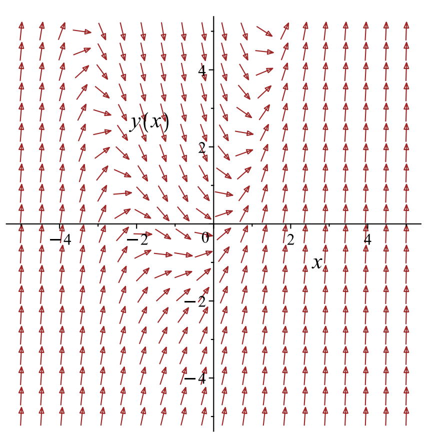
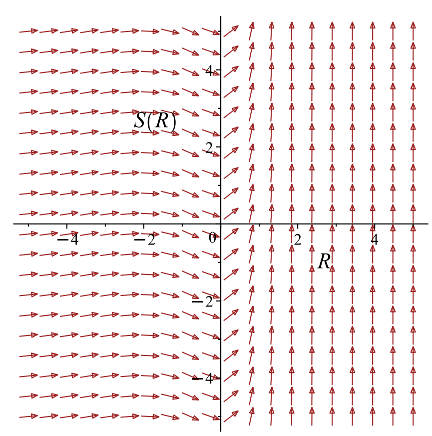
Which simplifies to

$$y\, e^x = x^2 e^x + c_1$$

Which gives

$$y = \left(x^2 e^x + c_1\right) e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + 2x - y$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = R e^R (R + 2)$ 

Summary
The solution(s) found are the following

$$y = \left(x^2 e^x + c_1\right) e^{-x} \tag{1}$$

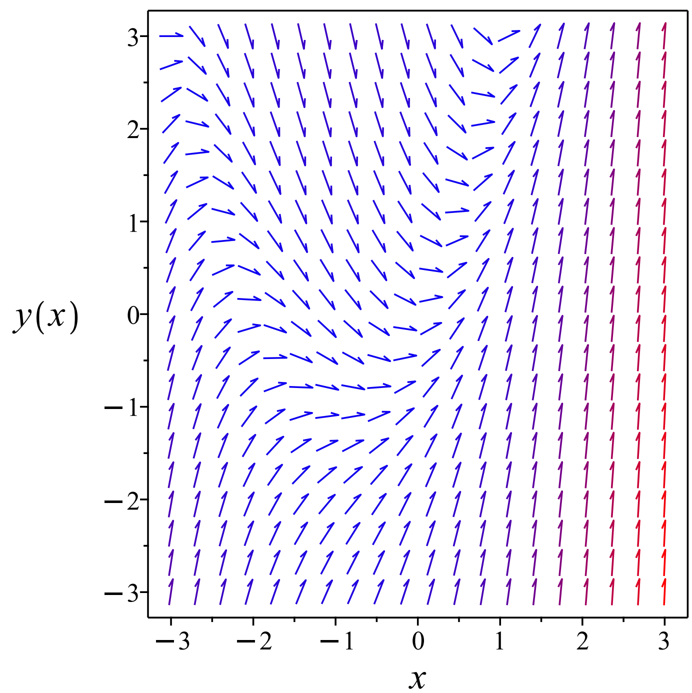


Figure 47: Slope field plot

Verification of solutions

$$y = (x^2 e^x + c_1) e^{-x}$$

Verified OK.

2.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(x^2 + 2x - y\right)\mathrm{d}x \\ \left(-x^2 - 2x + y\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 - 2x + y \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 2x + y\right) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(-x^2 - 2x + y) \\
&= -\mathrm{e}^{x}(x^2 + 2x - y)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{x}(x^2 + 2x - y)\right) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\mathrm{e}^{x}\left(x^{2}+2x-y\right) \mathrm{d}x$$
$$\phi = -\left(x^{2}-y\right)\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^{2}-y\right)\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^{2}-y\right)\mathrm{e}^{x}$$

The solution becomes

$$y = \left(x^{2}\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \left(x^{2}\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x} \tag{1}$$

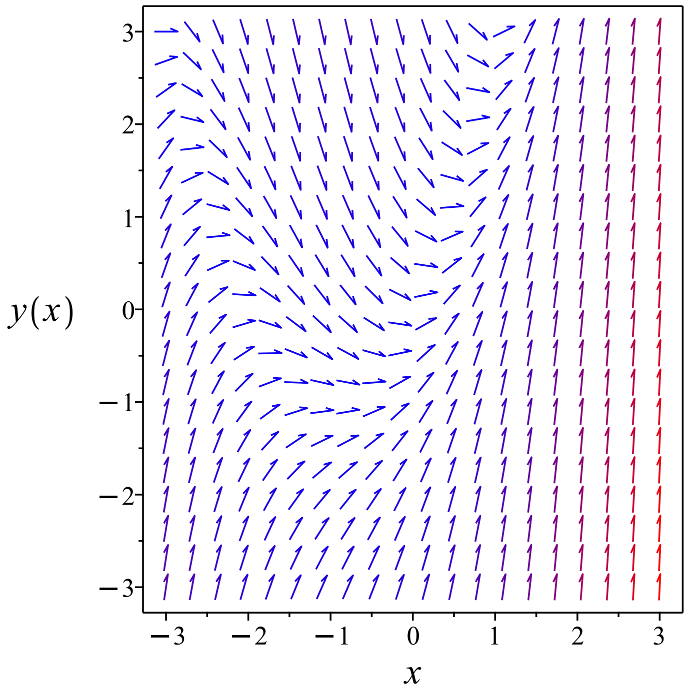


Figure 48: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x + c_1\right) e^{-x}$$

Verified OK.

2.10.4 Maple step by step solution

Let's solve

$y' + y = x^2 + 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = x^2 + 2x - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x^2 + 2x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\right) = \mu(x)\left(x^2 + 2x\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(x^2+2x)\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(x^2+2x)\,dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(x^2+2x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int (x^2+2x)\mathrm{e}^{x}dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{x^2\mathrm{e}^{x}+c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = x^2 + c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=x^2+2*x-y(x),y(x), singsol=all)
```

$$y(x) = x^2 + c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 17

```
DSolve[y'[x]==x^2+2*x-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 e^{-x}$$

2.11 problem 31

Internal problem ID [14958]
Internal file name [`OUTPUT/14967_Monday_April_15_2024_12_04_27_AM_221533/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y+1}{x-1} = 0}$$

2.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y+1}{x-1} \end{aligned}$$

Where $f(x) = \frac{1}{x-1}$ and $g(y) = y+1$. Integrating both sides gives

$$\frac{1}{y+1}\,dy = \frac{1}{x-1}\,dx$$
$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x-1}\,dx$$
$$\ln(y+1) = \ln(x-1) + c_1$$

Raising both side to exponential gives

$$y + 1 = e^{\ln(x-1)+c_1}$$

Which simplifies to

$$y + 1 = c_2(x-1)$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\ln(x-1)+c_1} - 1 \tag{1}$$

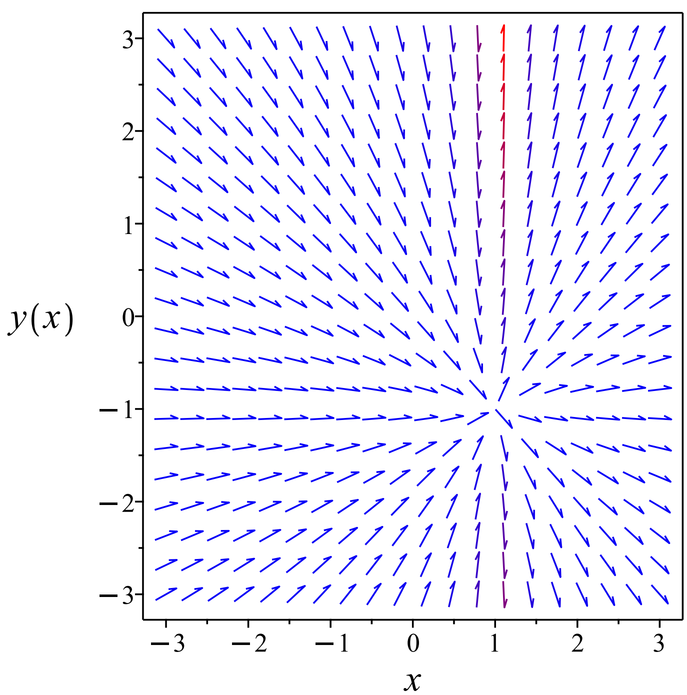


Figure 49: Slope field plot

Verification of solutions

$$y = c_2 e^{\ln(x-1)+c_1} - 1$$

Verified OK.

2.11.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x-1} \\ q(x) &= \frac{1}{x-1} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x-1} = \frac{1}{x-1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x-1}dx} \\ &= \frac{1}{x-1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{x-1}\right) \\ \frac{d}{dx}\left(\frac{y}{x-1}\right) &= \left(\frac{1}{x-1}\right)\left(\frac{1}{x-1}\right) \\ d\left(\frac{y}{x-1}\right) &= \frac{1}{(x-1)^2}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x-1} &= \int \frac{1}{(x-1)^2}\, dx \\ \frac{y}{x-1} &= -\frac{1}{x-1} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$y = c_1(x-1) - 1$$

Summary

The solution(s) found are the following

$$y = c_1(x-1) - 1 \tag{1}$$

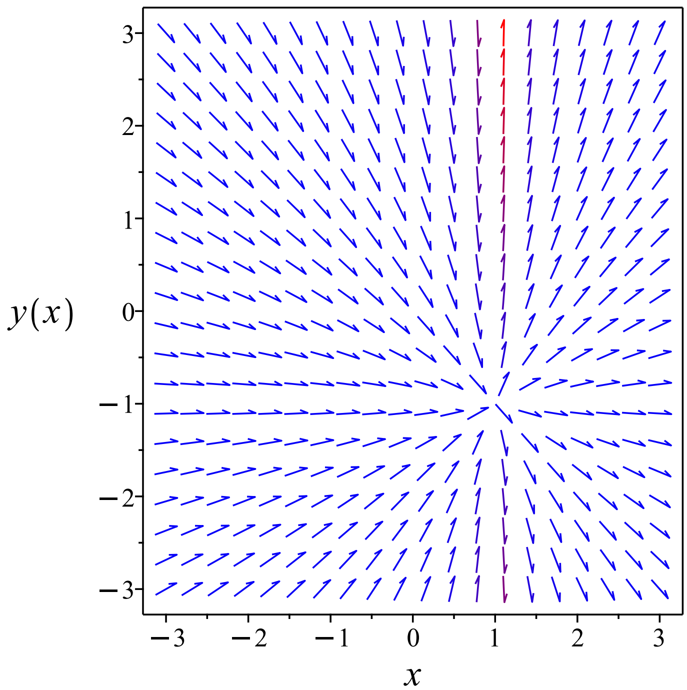


Figure 50: Slope field plot

Verification of solutions

$$y = c_1(x-1) - 1$$

Verified OK.

2.11.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x + 1}{x - 1} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u+1}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x(x-1)}$ and $g(u) = u + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u+1}\,du &= \frac{1}{x\,(x-1)}\,dx \\ \int \frac{1}{u+1}\,du &= \int \frac{1}{x\,(x-1)}\,dx \\ \ln(u+1) &= \ln(x-1) - \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$u + 1 = \mathrm{e}^{\ln(x-1) - \ln(x) + c_2}$$

Which simplifies to

$$u + 1 = c_3 \mathrm{e}^{\ln(x-1) - \ln(x)}$$

Which simplifies to

$$u(x) = c_3\left(\mathrm{e}^{c_2} - \frac{\mathrm{e}^{c_2}}{x}\right) - 1$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x\left(c_3\left(\mathrm{e}^{c_2} - \frac{\mathrm{e}^{c_2}}{x}\right) - 1\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x\left(c_3\left(\mathrm{e}^{c_2} - \frac{\mathrm{e}^{c_2}}{x}\right) - 1\right) \tag{1}$$

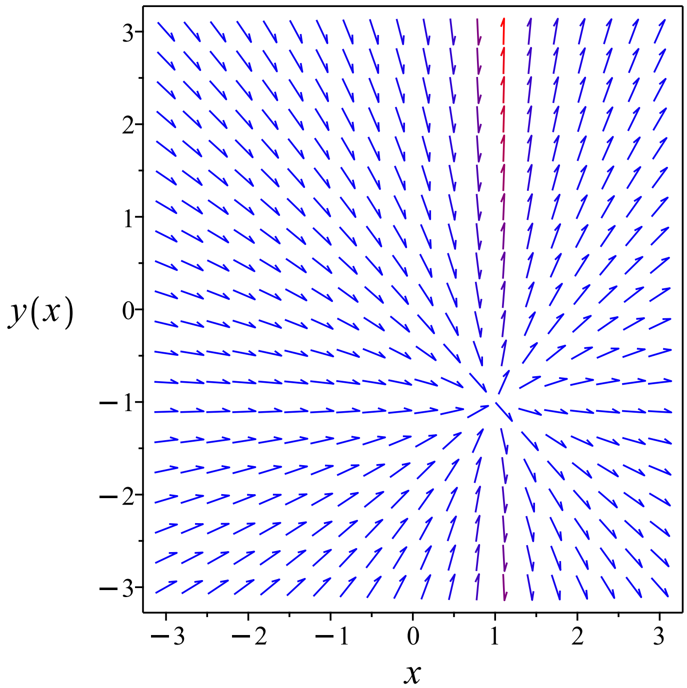


Figure 51: Slope field plot

Verification of solutions

$$y = x\left(c_3\left(e^{c_2} - \frac{e^{c_2}}{x}\right) - 1\right)$$

Verified OK.

2.11.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + 1}{X + x_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)}{X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{Y}{X}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= 0
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$
\begin{aligned}
u(X) &= \int 0 \, \mathrm{d}X \\
&= c_2
\end{aligned}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
Y(X) = X c_2
$$

Using the solution for $Y(X)$

$$
Y(X) = X c_2
$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes

$$y + 1 = c_2(x - 1)$$

Summary
The solution(s) found are the following

$$y + 1 = c_2(x - 1) \tag{1}$$

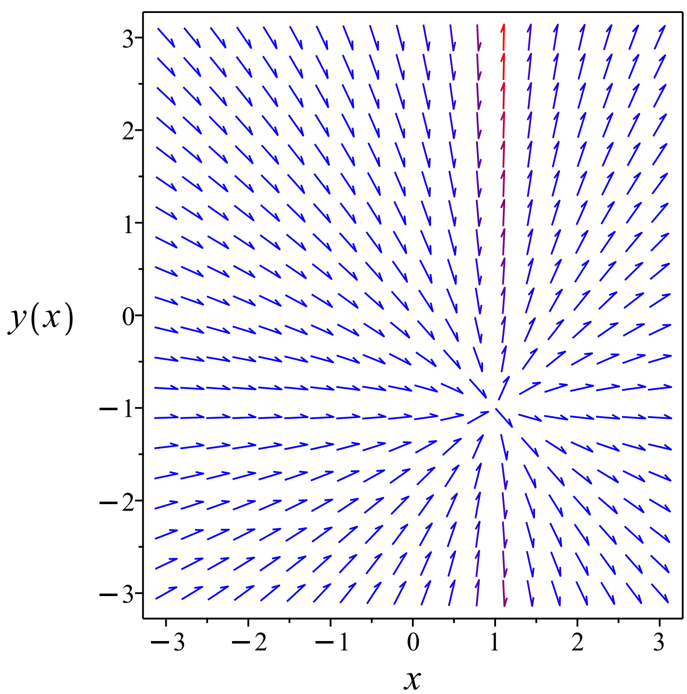


Figure 52: Slope field plot

Verification of solutions

$$y + 1 = c_2(x - 1)$$

Verified OK.

2.11.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y+1}{x-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 35: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= x-1\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{x-1} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+1}{x-1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{(x-1)^2} \\
S_y &= \frac{1}{x-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{(x-1)^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{(R-1)^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R-1} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x-1} = -\frac{1}{x-1} + c_1
$$

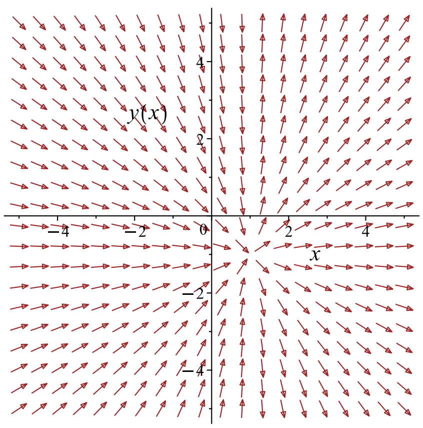
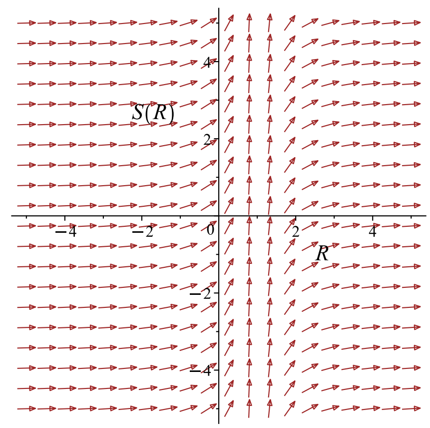
Which simplifies to

$$
\frac{-c_1 x + c_1 + y + 1}{x-1} = 0
$$

Which gives

$$
y = c_1 x - c_1 - 1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+1}{x-1}$ 	$R = x$ $S = \frac{y}{x-1}$	$\frac{dS}{dR} = \frac{1}{(R-1)^2}$ 

Summary
The solution(s) found are the following

$$y = c_1 x - c_1 - 1 \tag{1}$$

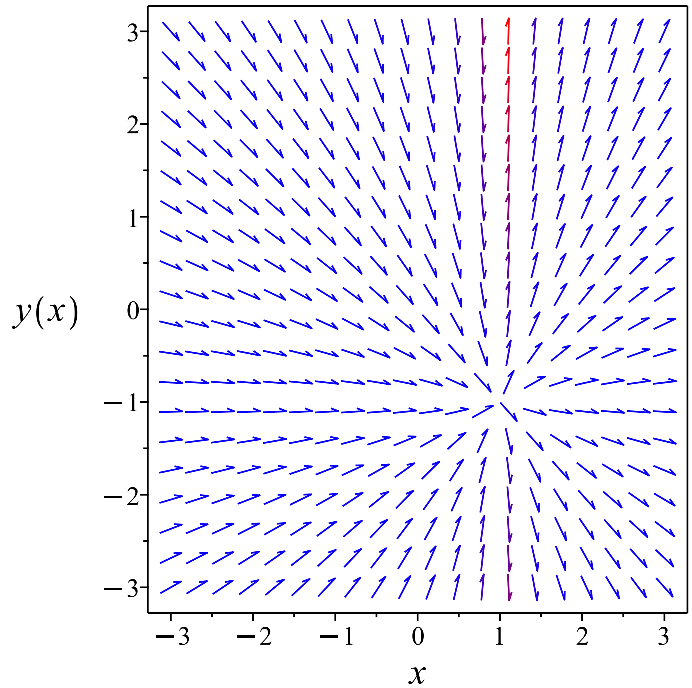


Figure 53: Slope field plot

Verification of solutions

$$y = c_1x - c_1 - 1$$

Verified OK.

2.11.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y+1}\right)\mathrm{d}y = \left(\frac{1}{x-1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x-1}\right)\mathrm{d}x + \left(\frac{1}{y+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x-1}$$
$$N(x,y) = \frac{1}{y+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x-1}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x-1}\,\mathrm{d}x \\
\phi &= -\ln(x-1) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$
\frac{1}{y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y+1}\right)\mathrm{d}y \\
f(y) &= \ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln (x-1) + \ln (y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln (x-1) + \ln (y+1)$$

The solution becomes

$$y = e^{c_1} x - e^{c_1} - 1$$

Summary

The solution(s) found are the following

$$y = e^{c_1} x - e^{c_1} - 1 \tag{1}$$

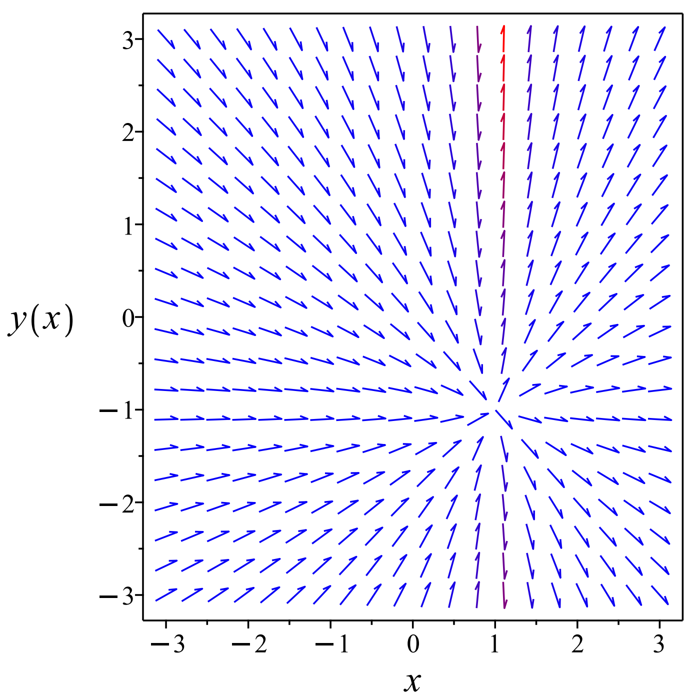


Figure 54: Slope field plot

Verification of solutions

$$y = e^{c_1} x - e^{c_1} - 1$$

Verified OK.

2.11.7 Maple step by step solution

Let's solve

$y' - \frac{y+1}{x-1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = \frac{1}{x-1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+1} dx = \int \frac{1}{x-1} dx + c_1$

- Evaluate integral

  $\ln (y+1) = \ln (x-1) + c_1$

- Solve for y

  $y = e^{c_1} x - e^{c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=(y(x)+1)/(x-1),y(x), singsol=all)
```

$$y(x) = -1 + (-1 + x) c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 18

```
DSolve[y'[x]==(y[x]+1)/(x-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + c_1(x-1)$$
$$y(x) \to -1$$

2.12 problem 32

Internal problem ID [14959]
Internal file name [`OUTPUT/14968_Monday_April_15_2024_12_04_28_AM_25636202/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y + x}{-y + x} = 0}$$

2.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x + x}{-u\,(x)\,x + x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(x) + c_2$$

The solution is

$$\frac{\ln\left(u(x)^2+1\right)}{2} - \arctan(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2} - \arctan\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

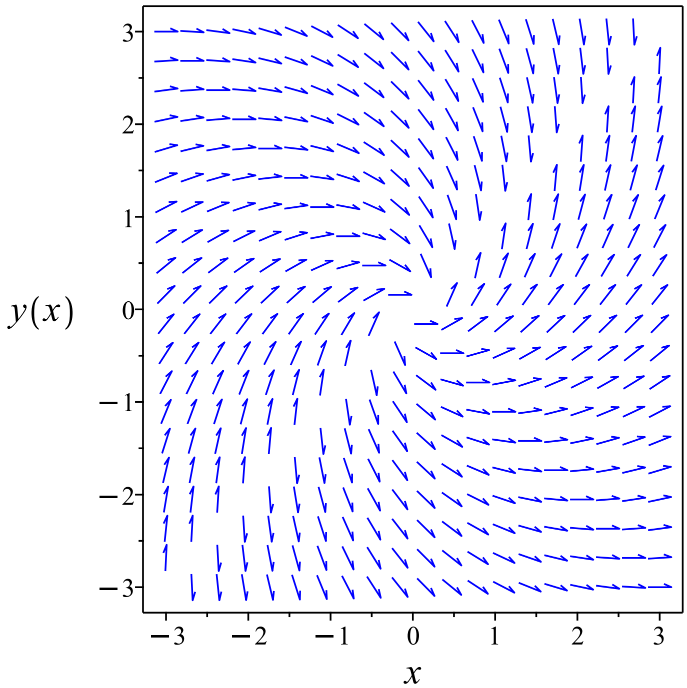


Figure 55: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

2.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y+x}{y-x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y+x)(b_3-a_2)}{y-x} - \frac{(y+x)^2 a_3}{(y-x)^2} - \left(-\frac{1}{y-x} - \frac{y+x}{(y-x)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{1}{y-x} + \frac{y+x}{(y-x)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(-y+x)^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1 v_2 \\
&- 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3)\, v_2^2 + 2a_1 v_2 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
-2b_1 &= 0 \\
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y + x}{y - x}\right)(x) \\
&= \frac{-x^2 - y^2}{-y + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2-y^2}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + x}{y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y + x}{x^2 + y^2} \\ S_y &= \frac{y - x}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

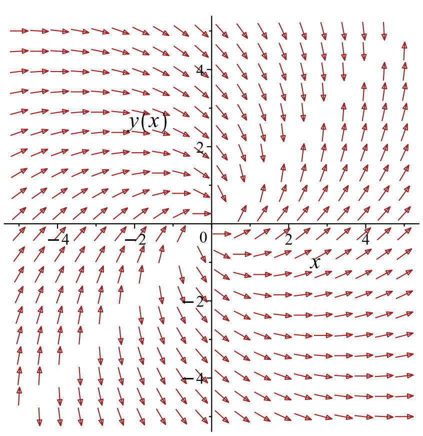
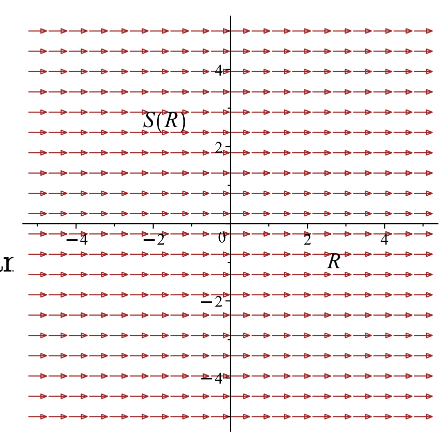
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+x}{y-x}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \text{arctar}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln (x^2+y^2)}{2}-\arctan\left(\frac{y}{x}\right)=c_1 \tag{1}$$

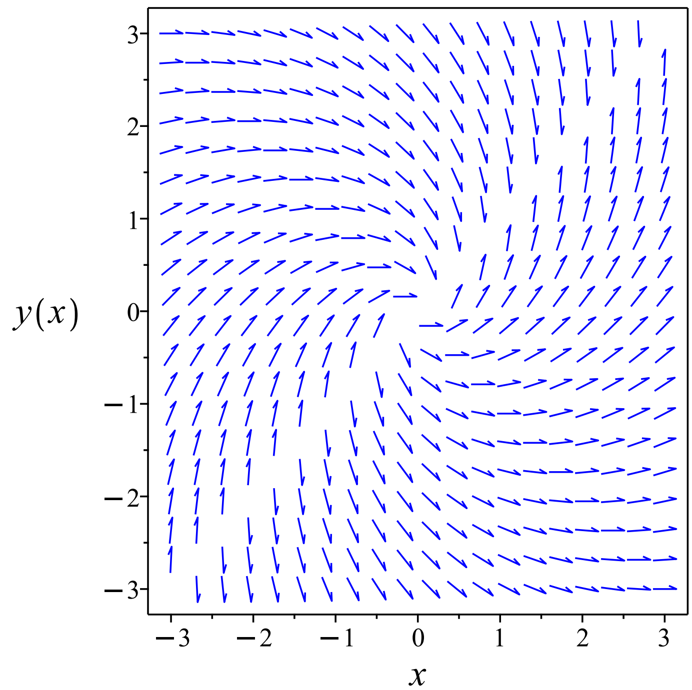


Figure 56: Slope field plot

Verification of solutions

$$\frac{\ln (x^2+y^2)}{2}-\arctan\left(\frac{y}{x}\right)=c_1$$

Verified OK.

2.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y&=(-y-x)\,\mathrm{d}x\\ (y+x)\,\mathrm{d}x+(y-x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y+x\\ N(x,y)&=y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y+x)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = y + x$ and $N = y - x$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{y+x}{x^2+y^2}\\ N &= \frac{y-x}{x^2+y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y-x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{y+x}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{y+x}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{y-x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y+x}{x^2+y^2}$$
$$N(x,y) = \frac{y-x}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y+x}{x^2+y^2}\right)$$
$$= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y-x}{x^2+y^2}\right)$$
$$= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{y+x}{x^2+y^2}\,\mathrm{d}x$$

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}$$

$$= \frac{y-x}{x^2+y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{y-x}{x^2+y^2} = \frac{y-x}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln\left(x^2+y^2\right)}{2} + \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(x^2+y^2\right)}{2}+\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

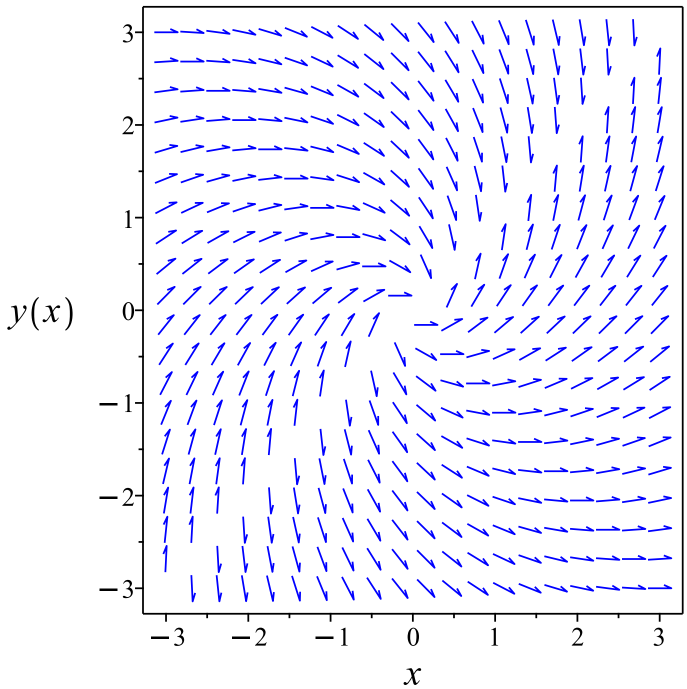


Figure 57: Slope field plot

Verification of solutions

$$\frac{\ln\left(x^2+y^2\right)}{2}+\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=(x+y(x))/(x-y(x)),y(x), singsol=all)
```

$$y(x) = \tan\left(\text{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 36

```
DSolve[y'[x]==(x+y[x])/(x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

2.13 problem 33

Internal problem ID [14960]
Internal file name [`OUTPUT/14969_Monday_April_15_2024_12_04_30_AM_89943328/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 1 - x}$$

2.13.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 1 - x \, \mathrm{d}x \\ &= x - \frac{1}{2}x^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x - \frac{1}{2}x^2 + c_1 \tag{1}$$

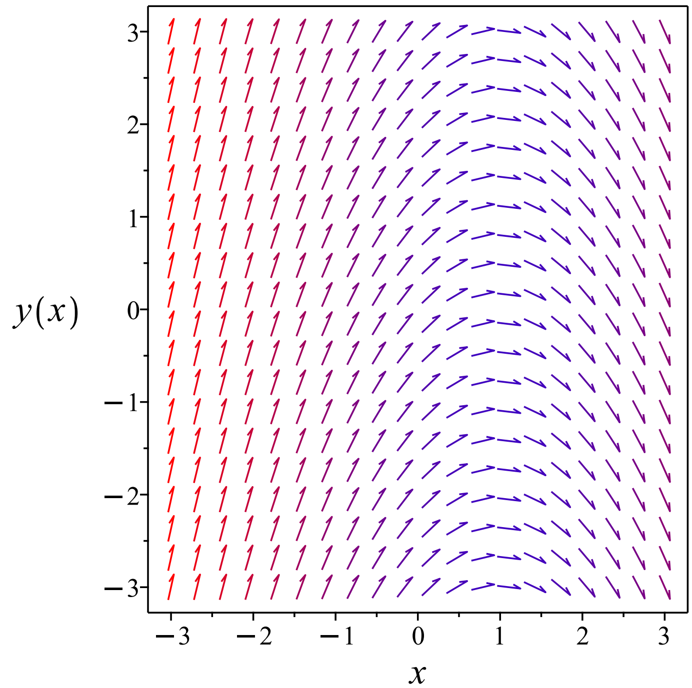


Figure 58: Slope field plot

Verification of solutions

$$y = x - \frac{1}{2}x^2 + c_1$$

Verified OK.

2.13.2 Maple step by step solution

Let's solve

$y' = 1 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (1 - x) dx + c_1$

- Evaluate integral

  $y = x - \frac{1}{2}x^2 + c_1$

- Solve for y

$$y = x - \frac{1}{2}x^2 + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=1-x,y(x), singsol=all)
```

$$y(x) = -\frac{1}{2}x^2 + x + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 16

```
DSolve[y'[x]==1-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{2} + x + c_1$$

2.14 problem 34

Internal problem ID [14961]
Internal file name [`OUTPUT/14970_Monday_April_15_2024_12_04_30_AM_70015247/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 2x}$$

2.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = 2x$$

Hence the ode is

$$y' + y = 2x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(2x) \\
\frac{d}{dx}(e^{x} y) &= (e^{x})(2x) \\
d(e^{x} y) &= (2x\, e^{x})\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int 2x\, e^{x}\, dx \\
e^{x} y &= 2\, e^{x}(x-1) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = 2\, e^{-x} e^{x}(x-1) + c_1 e^{-x}
$$

which simplifies to

$$
y = 2x - 2 + c_1 e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = 2x - 2 + c_1 e^{-x} \tag{1}
$$

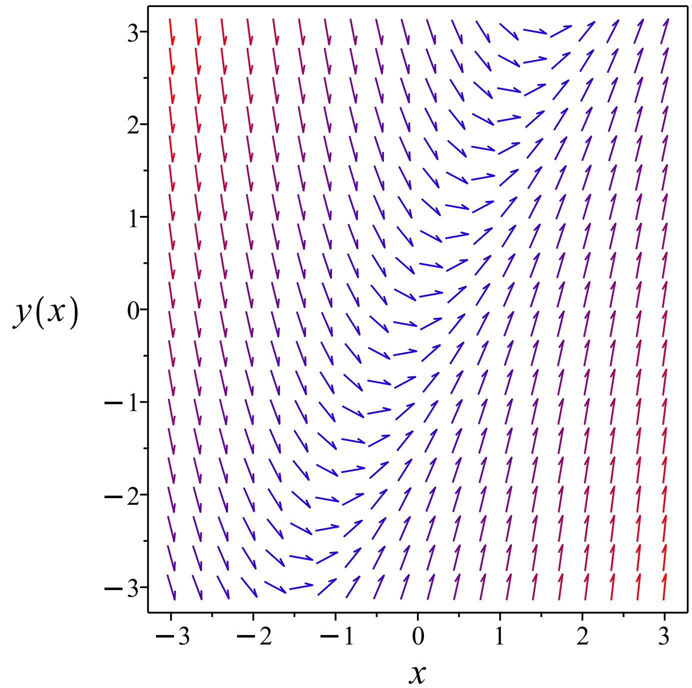


Figure 59: Slope field plot

Verification of solutions

$$y = 2x - 2 + c_1 e^{-x}$$

Verified OK.

2.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2x - y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2x - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x\,\mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R\,\mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2(R-1)\,e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{x} = 2\,e^{x}(x-1) + c_1$$

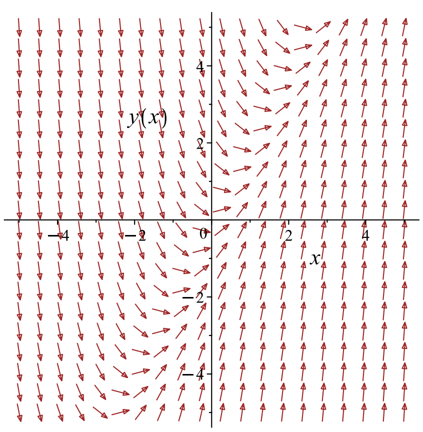
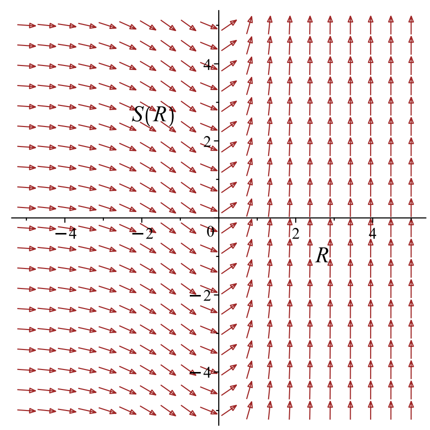
Which simplifies to

$$y\,e^{x} = 2\,e^{x}(x-1) + c_1$$

Which gives

$$y = (2x\,e^{x} - 2\,e^{x} + c_1)\,e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x - y$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = 2R\,e^{R}$ 

Summary

The solution(s) found are the following

$$y = (2x\,e^{x} - 2\,e^{x} + c_1)\,e^{-x} \tag{1}$$

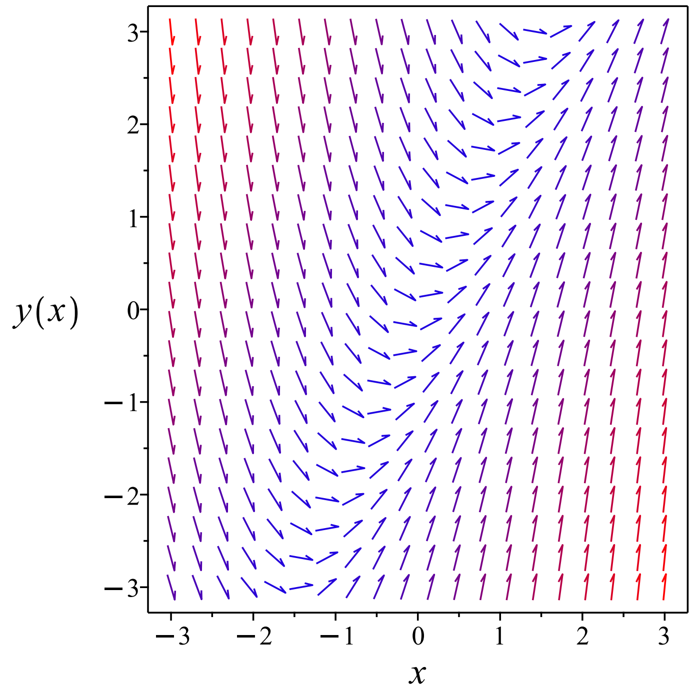


Figure 60: Slope field plot

Verification of solutions

$$y = (2x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

Verified OK.

2.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (2x - y) \, \mathrm{d}x \\ (-2x + y) \, \mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2x + y \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}(-2x + y) \\ &= (-2x + y)\,\mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((-2x + y)\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int (-2x + y) \mathrm{e}^{x} \mathrm{d}x$$
$$\phi = (-2x + y + 2) \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-2x + y + 2) \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-2x + y + 2) \mathrm{e}^{x}$$

The solution becomes

$$y = (2x \mathrm{e}^{x} - 2 \mathrm{e}^{x} + c_1) \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = (2x \mathrm{e}^{x} - 2 \mathrm{e}^{x} + c_1) \mathrm{e}^{-x} \tag{1}$$

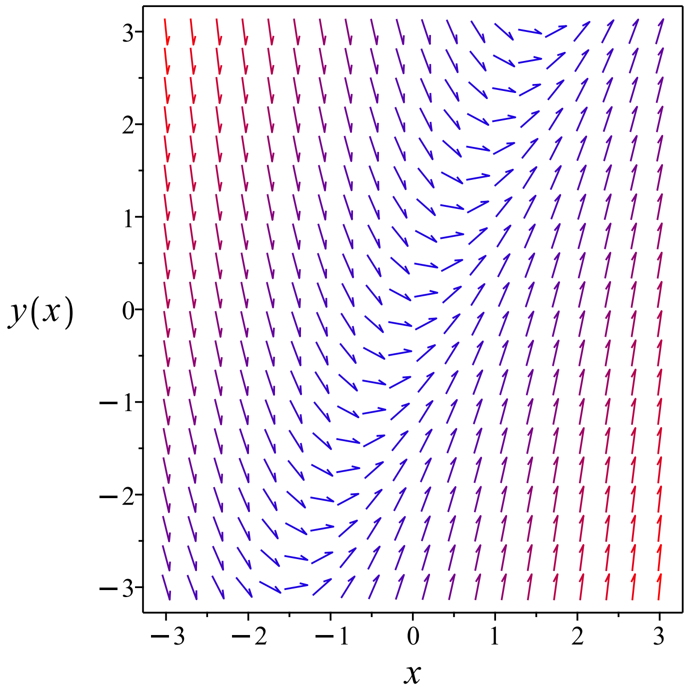


Figure 61: Slope field plot

Verification of solutions

$$y = (2x\,\mathrm{e}^{x} - 2\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

Verified OK.

2.14.4 Maple step by step solution

Let's solve

$y' + y = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2x - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 2x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = 2\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y'+y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 2\mu(x)\, x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 2\mu(x)\, x dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x)x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int 2x\, \mathrm{e}^{x} dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{2\, \mathrm{e}^{x}(x-1) + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = 2x - 2 + c_1 \mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=2*x-y(x),y(x), singsol=all)
```

$$y(x) = 2x - 2 + c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 18

```
DSolve[y'[x]==2*x-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x + c_1 e^{-x} - 2$$

2.15 problem 35

Internal problem ID [14962]
Internal file name [`OUTPUT/14971_Monday_April_15_2024_12_04_31_AM_83451682/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x^2}$$

2.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= x^2 \end{aligned}$$

Hence the ode is

$$y' - y = x^2$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2\right)$$
$$\frac{d}{dx}\left(e^{-x}y\right) = \left(e^{-x}\right)\left(x^2\right)$$
$$d\left(e^{-x}y\right) = \left(x^2 e^{-x}\right) dx$$

Integrating gives

$$e^{-x}y = \int x^2 e^{-x} dx$$
$$e^{-x}y = -\left(x^2 + 2x + 2\right) e^{-x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^x\left(x^2 + 2x + 2\right) e^{-x} + e^x c_1$$

which simplifies to

$$y = -x^2 - 2x - 2 + e^x c_1$$

Summary

The solution(s) found are the following

$$y = -x^2 - 2x - 2 + e^x c_1 \tag{1}$$

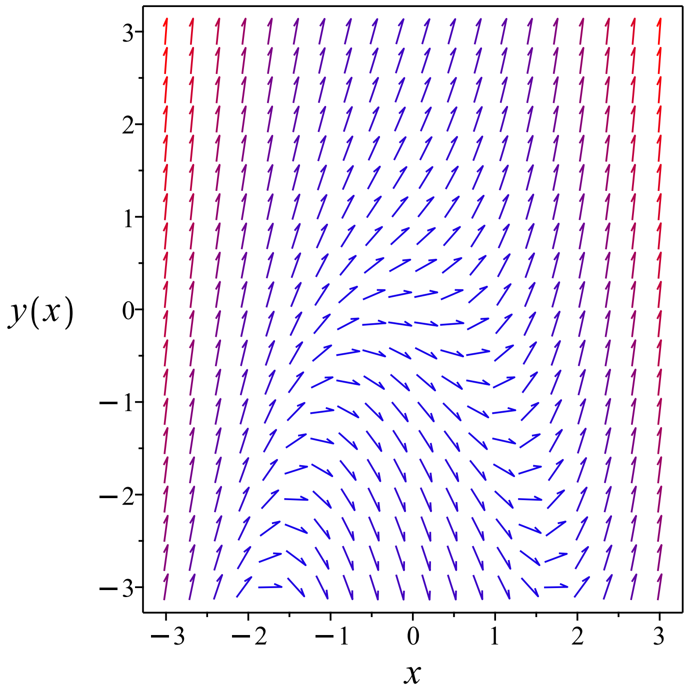


Figure 62: Slope field plot

Verification of solutions

$$y = -x^2 - 2x - 2 + e^x c_1$$

Verified OK.

2.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^2 + y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R^2 + 2R + 2)\,e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x}y = -(x^2 + 2x + 2)\,e^{-x} + c_1$$

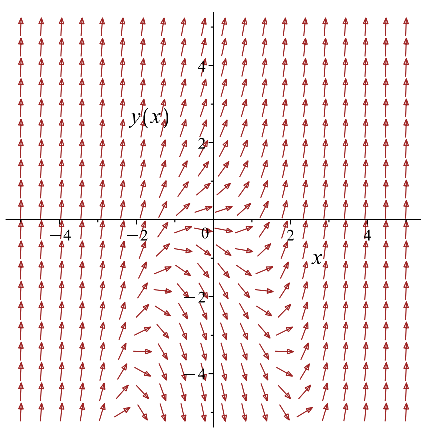
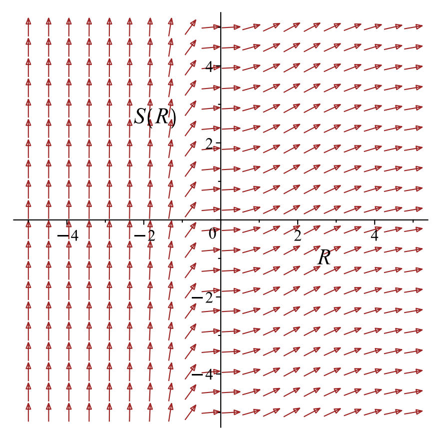
Which simplifies to

$$(x^2 + 2x + y + 2)\,e^{-x} - c_1 = 0$$

Which gives

$$y = -(x^2e^{-x} + 2x\,e^{-x} + 2\,e^{-x} - c_1)\,e^x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + y$ 	$R = x$ $S = e^{-x}y$	$\frac{dS}{dR} = R^2e^{-R}$ 

Summary

The solution(s) found are the following

$$y = -(x^2e^{-x} + 2x\,e^{-x} + 2\,e^{-x} - c_1)\,e^x \tag{1}$$

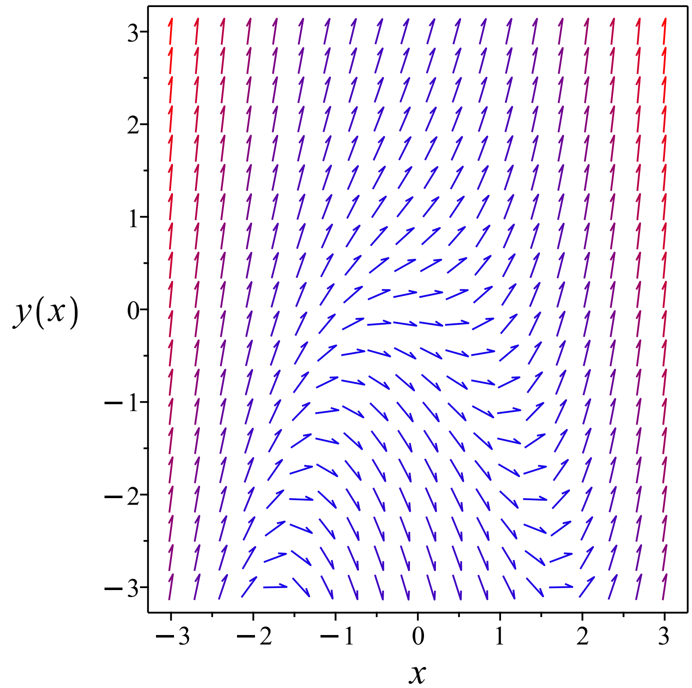


Figure 63: Slope field plot

Verification of solutions

$$y = -\left(x^2 e^{-x} + 2x\, e^{-x} + 2\, e^{-x} - c_1\right) e^{x}$$

Verified OK.

2.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(x^2 + y\right) \mathrm{d}x \\
\left(-x^2 - y\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 - y \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - y\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-x^2 - y) \\
&= -\mathrm{e}^{-x}(x^2 + y)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x}(x^2 + y)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-x} \left(x^2 + y\right) \mathrm{d}x$$

$$\phi = \left(x^2 + 2x + y + 2\right) \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^2 + 2x + y + 2\right) \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^2 + 2x + y + 2\right) \mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x^2 \mathrm{e}^{-x} + 2x \, \mathrm{e}^{-x} + 2 \, \mathrm{e}^{-x} - c_1\right) \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = -\left(x^2 e^{-x} + 2x e^{-x} + 2 e^{-x} - c_1\right) e^x \tag{1}$$

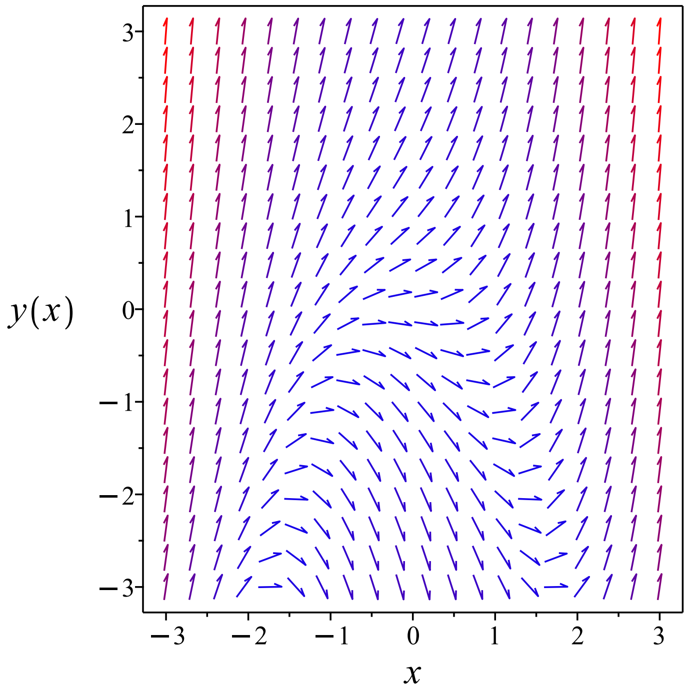


Figure 64: Slope field plot

Verification of solutions

$$y = -\left(x^2 e^{-x} + 2x e^{-x} + 2 e^{-x} - c_1\right) e^x$$

Verified OK.

2.15.4 Maple step by step solution

Let's solve

$y' - y = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = x^2 + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = \mu(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x^2 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x^2 dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)x^2 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int x^2 \mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(x^2+2x+2)\,\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = -x^2 - 2x - 2 + \mathrm{e}^{x} c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)=x^2+y(x),y(x), singsol=all)
```

$$y(x) = -x^2 - 2x - 2 + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 21

```
DSolve[y'[x]==x^2+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 - 2x + c_1 e^x - 2$$

2.16 problem 36

Internal problem ID [14963]
Internal file name [`OUTPUT/14972_Monday_April_15_2024_12_04_31_AM_22281195/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{y}{x} = 0}$$

2.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{1}{x}\,dx \\
\ln (y) &= -\ln (x) + c_1 \\
y &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} \tag{1}
$$

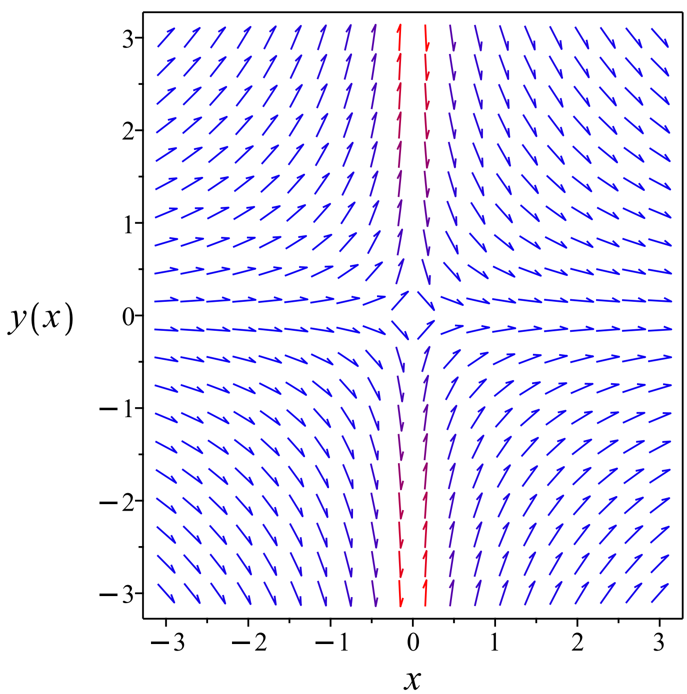


Figure 65: Slope field plot

Verification of solutions

$$
y = \frac{c_1}{x}
$$

Verified OK.

2.16.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{x}dx}\\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}(xy) &= 0\end{aligned}$$

Integrating gives

$$xy = c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

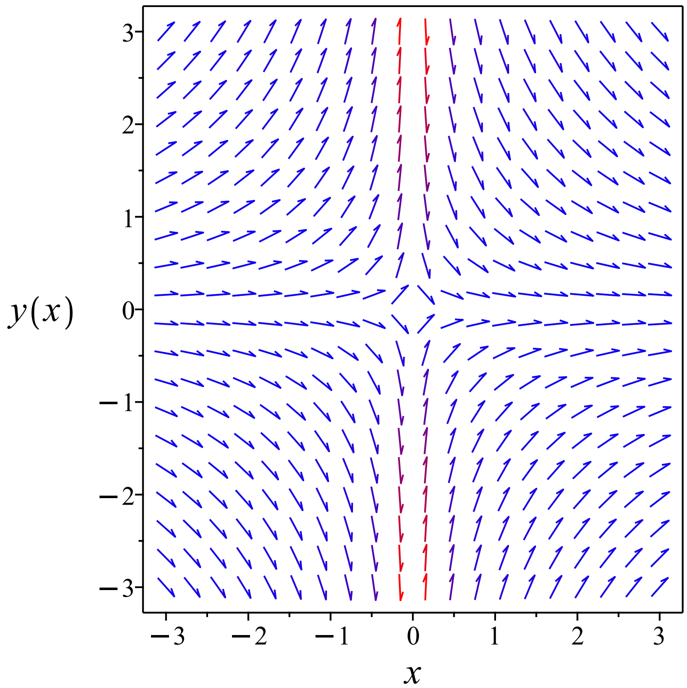


Figure 66: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

2.16.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + 2u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_2 \\
u &= e^{-2\ln(x)+c_2} \\
&= \frac{c_2}{x^2}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= \frac{c_2}{x}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_2}{x} \tag{1}$$

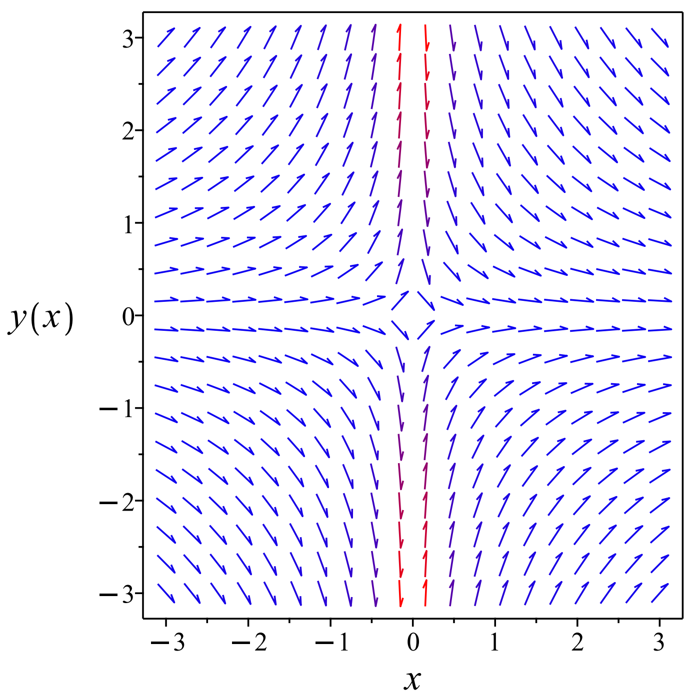


Figure 67: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x}$$

Verified OK.

2.16.4 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-xy)$$

Hence (2) becomes

$$0 = d(-xy)$$

Integrating both sides gives gives these solutions

$$y = \frac{c_1}{x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_1 \tag{1}$$

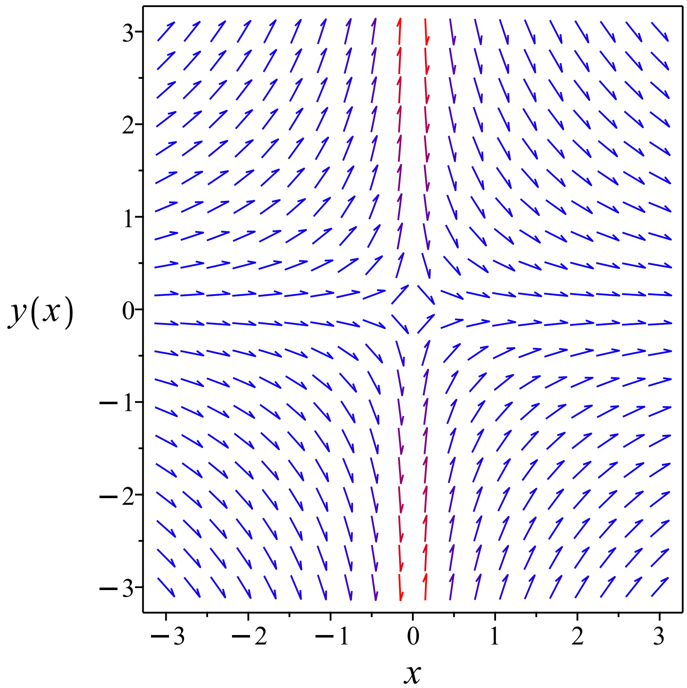


Figure 68: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x} + c_1$$

Verified OK.

2.16.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 45: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = c_1$$

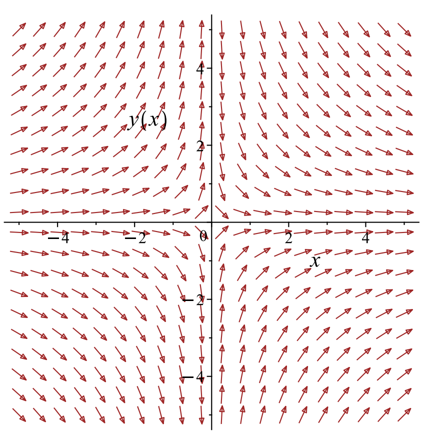
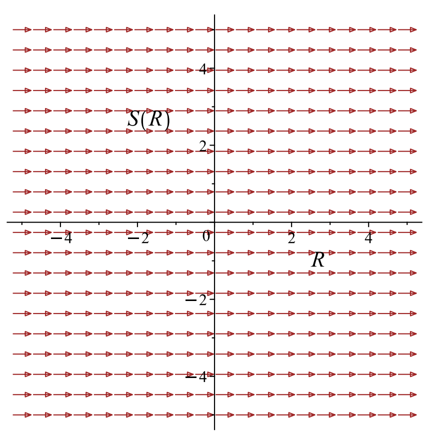
Which simplifies to

$$yx = c_1$$

Which gives

$$y = \frac{c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

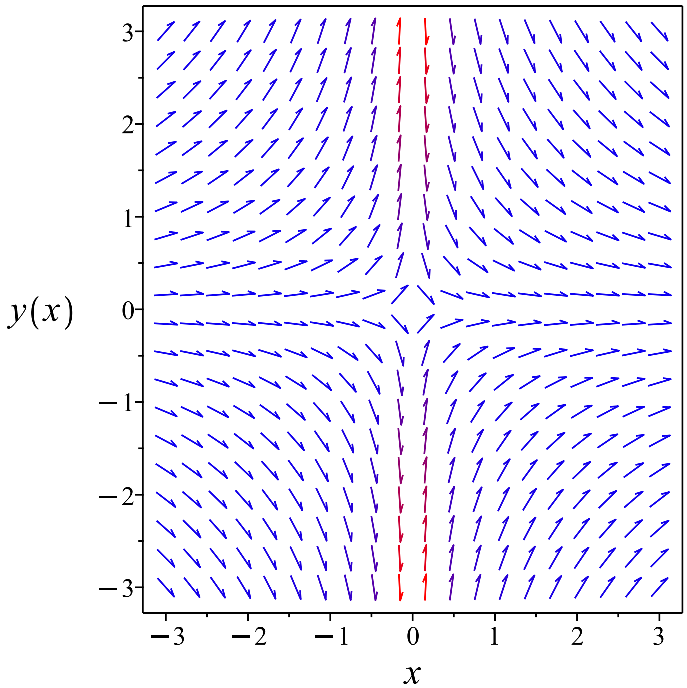


Figure 69: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

2.16.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\ \left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x} \\ N(x, y) &= -\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, dx &= \int M\, dx\\ \int \frac{\partial \phi}{\partial x}\, dx &= \int -\frac{1}{x}\, dx\\ \phi &= -\ln(x) + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\, dy &= \int \left(-\frac{1}{y}\right) dy\\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{e^{-c_1}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-c_1}}{x} \tag{1}$$

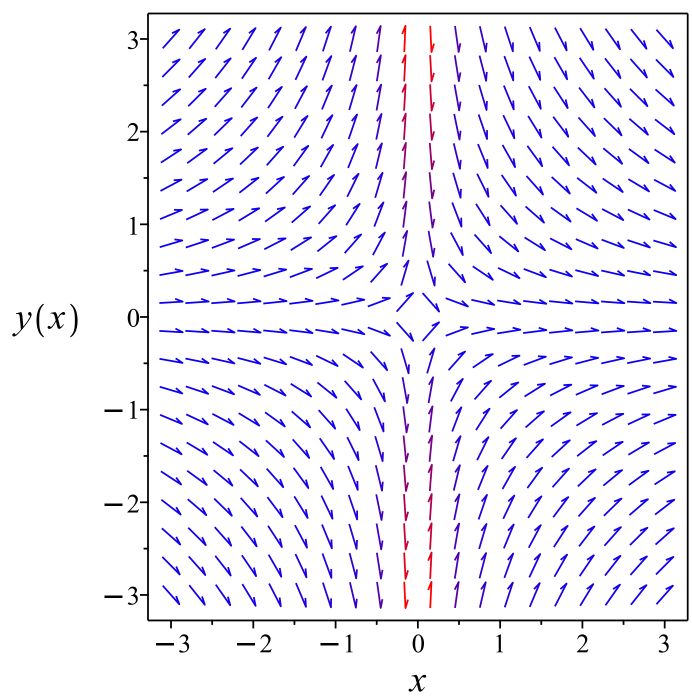


Figure 70: Slope field plot

Verification of solutions

$$y = \frac{e^{-c_1}}{x}$$

Verified OK.

2.16.7 Maple step by step solution

Let's solve

$y' + \frac{y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=-y(x)/x,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 16

```
DSolve[y'[x]==-y[x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to 0$$

2.17 problem 37

Internal problem ID [14964]
Internal file name [`OUTPUT/14973_Monday_April_15_2024_12_04_32_AM_88122069/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 1}$$

2.17.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 1 \, \mathrm{d}x \\ &= x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x + c_1 \tag{1}$$

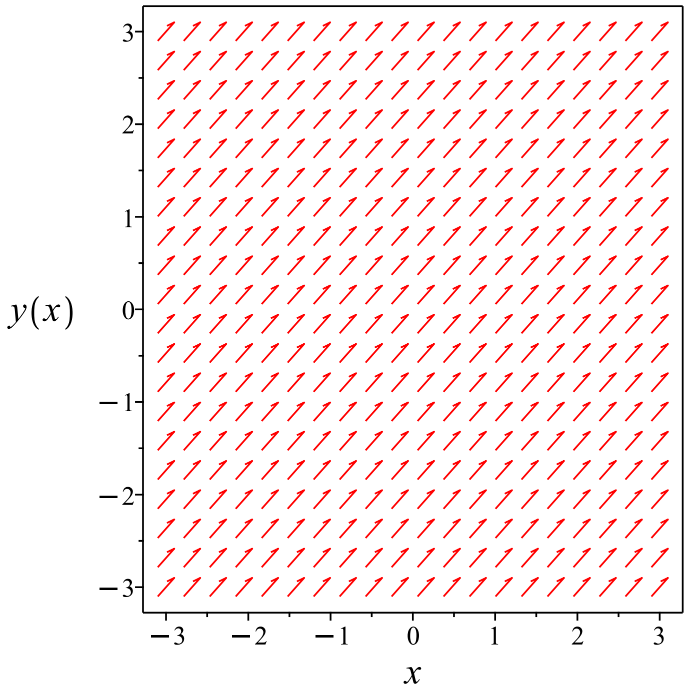


Figure 71: Slope field plot

Verification of solutions

$$y = x + c_1$$

Verified OK.

2.17.2 Maple step by step solution

Let's solve

$y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int 1 dx + c_1$

- Evaluate integral

  $y = x + c_1$

- Solve for y

  $y = x + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = c_1 + x$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 9

```
DSolve[y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1$$

2.18 problem 38

Internal problem ID [14965]
Internal file name [`OUTPUT/14974_Monday_April_15_2024_12_04_32_AM_35600974/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \frac{1}{x}}$$

2.18.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x}\, \mathrm{d}x \\ &= \ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x) + c_1 \tag{1}$$

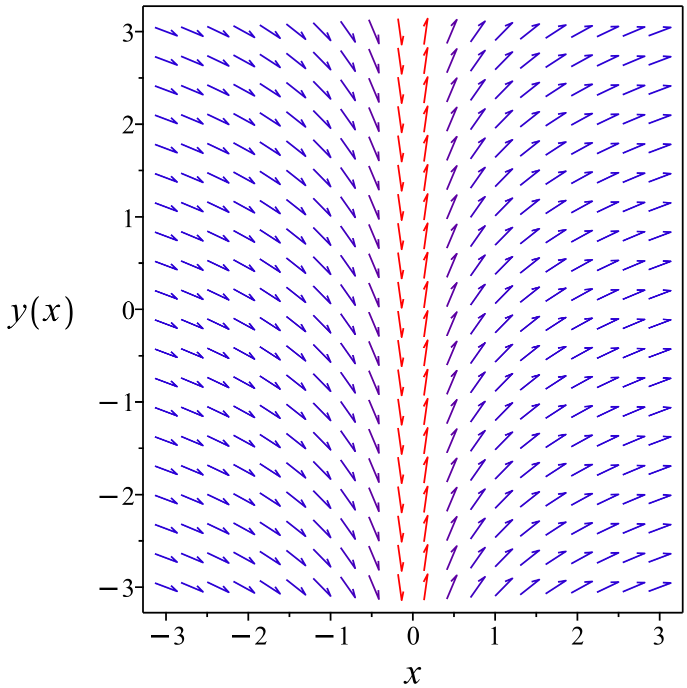


Figure 72: Slope field plot

Verification of solutions

$$y = \ln(x) + c_1$$

Verified OK.

2.18.2 Maple step by step solution

Let's solve

$y' = \frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $y = \ln(x) + c_1$

- Solve for y

$$y = \ln(x) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=1/x,y(x), singsol=all)
```

$$y(x) = \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 10

```
DSolve[y'[x]==1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x) + c_1$$

2.19 problem 39

Internal problem ID [14966]
Internal file name [`OUTPUT/14975_Monday_April_15_2024_12_04_33_AM_37740880/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

2.19.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{y} dy &= x + c_1 \\
\ln(y) &= x + c_1 \\
y &= e^{x+c_1} \\
y &= e^{x} c_1
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{x} c_1 \tag{1}$$

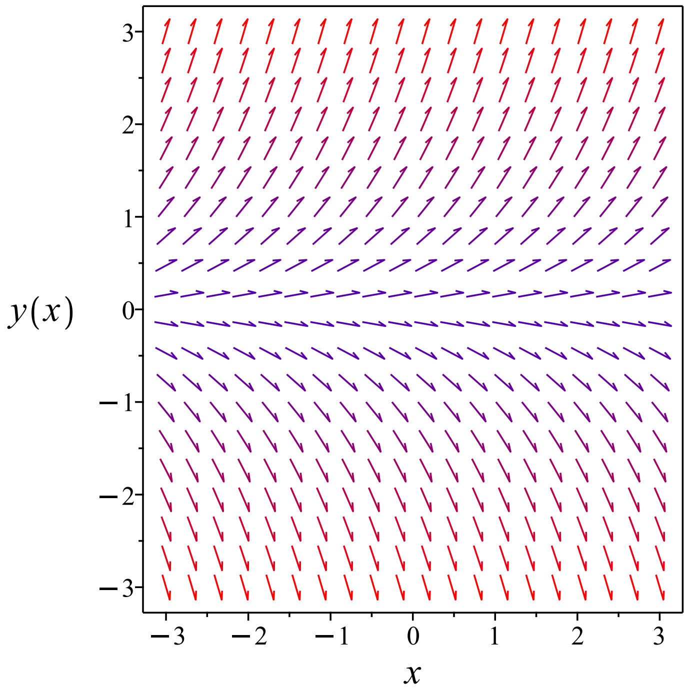


Figure 73: Slope field plot

Verification of solutions

$$y = e^{x} c_1$$

Verified OK.

2.19.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

$$\ln(y) = x + c_1$$

- Solve for y

$$y = e^{x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 16

```
DSolve[y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to 0$$

2.20 problem 40

Internal problem ID [14967]
Internal file name [`OUTPUT/14976_Monday_April_15_2024_12_04_33_AM_92472098/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 2. The method of isoclines. Exercises page 27
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y^2 = 0}$$

2.20.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2} dy = x + c_1$$
$$-\frac{1}{y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{x + c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{1}{x + c_1} \tag{1}$$

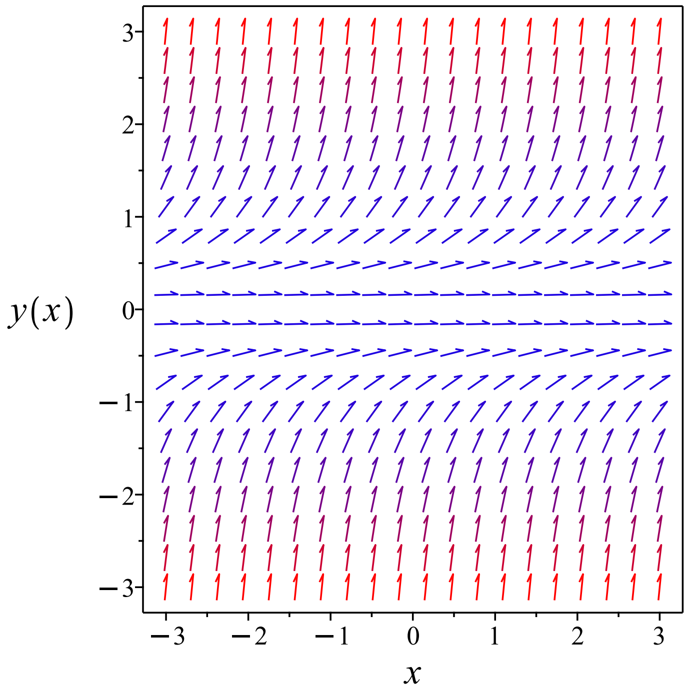


Figure 74: Slope field plot

Verification of solutions

$$y = -\frac{1}{x + c_1}$$

Verified OK.

2.20.2 Maple step by step solution

Let's solve

$y' - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int 1 dx + c_1$

- Evaluate integral

$$-\frac{1}{y} = x + c_1$$

- Solve for y

$$y = -\frac{1}{x + c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{c_1 - x}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 18

```
DSolve[y'[x]==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{x + c_1}$$
$$y(x) \to 0$$

3 Section 3. The method of successive approximation. Exercises page 31

3.1 problem 41

Internal problem ID [14968]
Internal file name [`OUTPUT/14977_Monday_April_15_2024_12_04_33_AM_68577942/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 3. The method of successive approximation. Exercises page 31
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + y^2 = x^2}$$

With initial conditions

$$[y(-1) = 0]$$

3.1.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x^2 - y^2 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^2 - y^2)\\ &= -2y\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.1.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= x^2 - y^2\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{-u} \qquad (1)\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)$$

But

$$\begin{aligned}f_2' &= 0\\ f_1 f_2 &= 0\\ f_2^2 f_0 &= x^2\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{x}\left(c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x^{\frac{3}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Initial conditions are used to solve for c_3. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\operatorname{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) - \operatorname{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right) c_3}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{1}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{1}{2}\right)}$$

$$c_3 = \frac{\operatorname{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right)}{\operatorname{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right)}$$

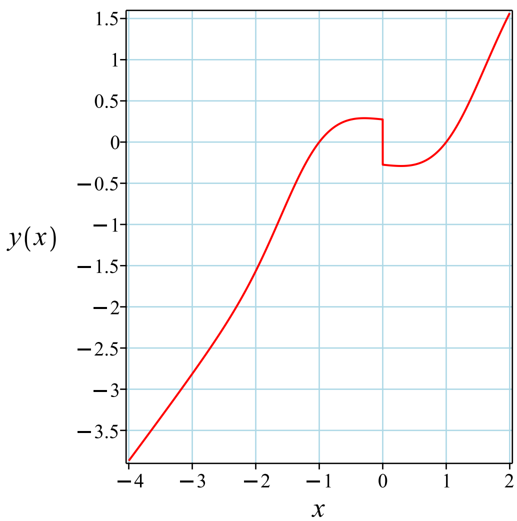
Substituting c_3 found above in the general solution gives

$$y = \frac{-x \operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) \operatorname{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right) + x \operatorname{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)}{\operatorname{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) \operatorname{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right)}$$

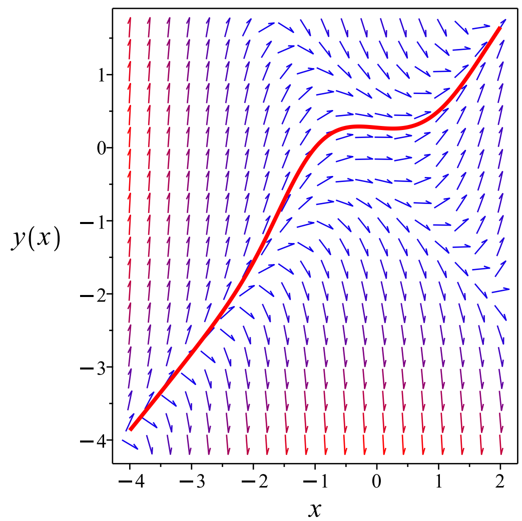
Summary

The solution(s) found are the following

$$y = \frac{-x\,\mathrm{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) \mathrm{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right) + x\,\mathrm{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) \mathrm{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)}{\mathrm{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) \mathrm{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \mathrm{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) \mathrm{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-x\,\mathrm{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) \mathrm{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right) + x\,\mathrm{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) \mathrm{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)}{\mathrm{BesselK}\left(\frac{3}{4}, \frac{1}{2}\right) \mathrm{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \mathrm{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) \mathrm{BesselI}\left(-\frac{3}{4}, \frac{1}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 55

```
dsolve([diff(y(x),x)=x^2-y(x)^2,y(-1) = 0],y(x), singsol=all)
```

$$y(x)=\frac{x\left(\operatorname{BesselI}\left(-\frac{3}{4},\frac{x^2}{2}\right)\operatorname{BesselK}\left(\frac{3}{4},\frac{1}{2}\right)-\operatorname{BesselK}\left(\frac{3}{4},\frac{x^2}{2}\right)\operatorname{BesselI}\left(-\frac{3}{4},\frac{1}{2}\right)\right)}{\operatorname{BesselK}\left(\frac{1}{4},\frac{x^2}{2}\right)\operatorname{BesselI}\left(-\frac{3}{4},\frac{1}{2}\right)+\operatorname{BesselK}\left(\frac{3}{4},\frac{1}{2}\right)\operatorname{BesselI}\left(\frac{1}{4},\frac{x^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 211

```
DSolve[{y'[x]==x^2-y[x]^2,{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{i\left(x^2\left(-\operatorname{BesselJ}\left(-\frac{5}{4},\frac{i}{2}\right)+i\operatorname{BesselJ}\left(-\frac{1}{4},\frac{i}{2}\right)+\operatorname{BesselJ}\left(\frac{3}{4},\frac{i}{2}\right)\right)\operatorname{BesselJ}\left(-\frac{3}{4},\frac{ix^2}{2}\right)+x^2\operatorname{BesselJ}\left(-\frac{3}{4},\frac{i}{2}\right)\operatorname{B}\right.}{x\left(2\operatorname{BesselJ}\left(-\frac{3}{4},\frac{i}{2}\right)\operatorname{BesselJ}\left(-\frac{1}{4},\frac{ix^2}{2}\right)+\left(-\operatorname{BesselJ}\left(-\frac{5}{4},\frac{i}{2}\right)+i\operatorname{Be}\right.\right.}$$

3.2 problem 42

Internal problem ID [14969]
Internal file name [`OUTPUT/14978_Monday_April_15_2024_12_04_35_AM_14496990/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 3. The method of successive approximation. Exercises page 31
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_Riccati, _special]]
```

$$\boxed{y' - y^2 = x}$$

With initial conditions

$$[y(0) = 0]$$

3.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 + x \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 + x) \\
&= 2y
\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.2.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= y^2 + x
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u} \qquad (1)
\end{aligned}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)$$

But

$$\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= x
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + xu(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x)$$

The above shows that

$$u'(x) = -c_1 \operatorname{AiryAi}(1, -x) - c_2 \operatorname{AiryBi}(1, -x)$$

Using the above in (1) gives the solution

$$y = -\frac{-c_1 \operatorname{AiryAi}(1, -x) - c_2 \operatorname{AiryBi}(1, -x)}{c_1 \operatorname{AiryAi}(-x) + c_2 \operatorname{AiryBi}(-x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \operatorname{AiryAi}(1, -x) + \operatorname{AiryBi}(1, -x)}{c_3 \operatorname{AiryAi}(-x) + \operatorname{AiryBi}(-x)}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{3\Gamma\left(\frac{2}{3}\right)^2 3^{\frac{2}{3}} - 3\Gamma\left(\frac{2}{3}\right)^2 c_3 3^{\frac{1}{6}}}{2\, 3^{\frac{5}{6}}\pi + 2\pi c_3 3^{\frac{1}{3}}}$$

$$c_3 = \sqrt{3}$$

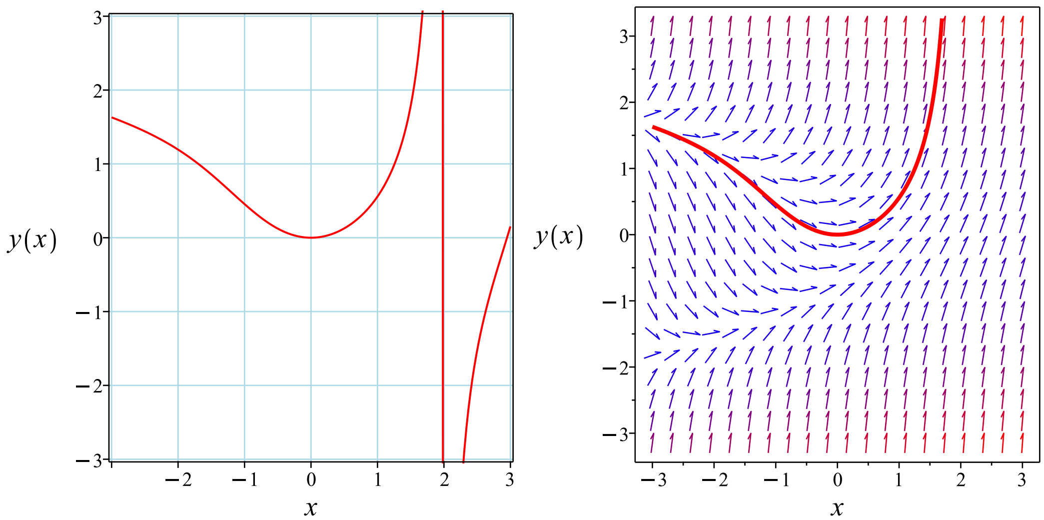
Substituting c_3 found above in the general solution gives

$$y = \frac{\operatorname{AiryAi}(1, -x)\sqrt{3} + \operatorname{AiryBi}(1, -x)}{\operatorname{AiryAi}(-x)\sqrt{3} + \operatorname{AiryBi}(-x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\operatorname{AiryAi}(1, -x)\sqrt{3} + \operatorname{AiryBi}(1, -x)}{\operatorname{AiryAi}(-x)\sqrt{3} + \operatorname{AiryBi}(-x)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\operatorname{AiryAi}(1, -x)\sqrt{3} + \operatorname{AiryBi}(1, -x)}{\operatorname{AiryAi}(-x)\sqrt{3} + \operatorname{AiryBi}(-x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 35

```
dsolve([diff(y(x),x)=x+y(x)^2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{3}\ \text{AiryAi}(1, -x) + \text{AiryBi}(1, -x)}{\sqrt{3}\ \text{AiryAi}(-x) + \text{AiryBi}(-x)}$$

✓ Solution by Mathematica

Time used: 1.269 (sec). Leaf size: 80

```
DSolve[{y'[x]==x+y[x]^2,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^{3/2}\,\text{BesselJ}\left(-\frac{4}{3}, \frac{2x^{3/2}}{3}\right) - x^{3/2}\,\text{BesselJ}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right) + \text{BesselJ}\left(-\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}{2x\,\text{BesselJ}\left(-\frac{1}{3}, \frac{2x^{3/2}}{3}\right)}$$

3.3 problem 43

Internal problem ID [14970]
Internal file name [`OUTPUT/14979_Monday_April_15_2024_12_04_36_AM_82294268/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 3. The method of successive approximation. Exercises page 31
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x}$$

With initial conditions

$$[y(0) = 1]$$

3.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = x$$

Hence the ode is

$$y' - y = x$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.3.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}\left(e^{-x}y\right) &= \left(e^{-x}\right)(x) \\ d\left(e^{-x}y\right) &= \left(x\,e^{-x}\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x}y &= \int x\,e^{-x}\,dx \\ e^{-x}y &= -(x+1)\,e^{-x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}(x+1)\,e^{-x} + e^{x}c_1$$

which simplifies to

$$y = e^{x}c_1 - x - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 + c_1$$

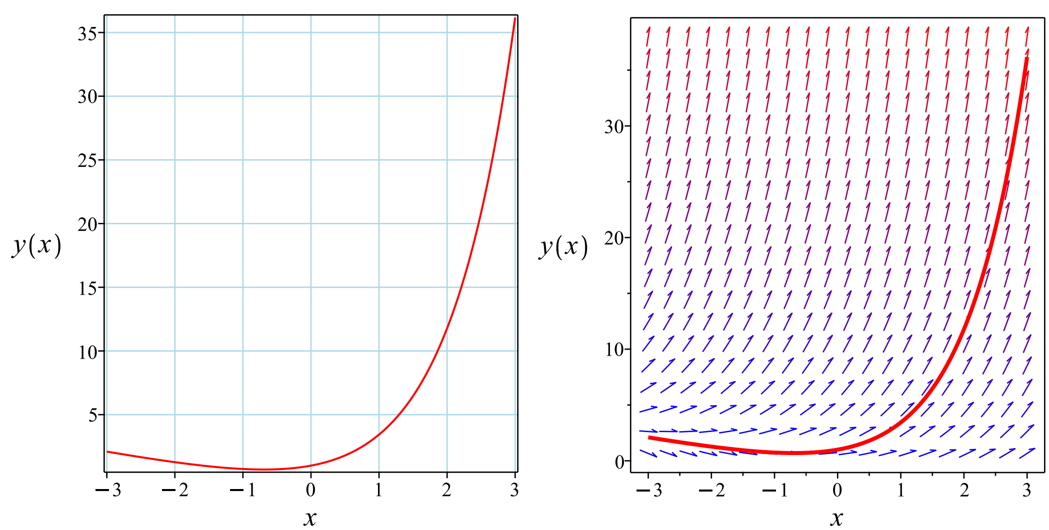
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = -1 + 2\,\mathrm{e}^{x} - x$$

Summary
The solution(s) found are the following

$$y = -1 + 2\,\mathrm{e}^{x} - x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -1 + 2\,\mathrm{e}^{x} - x$$

Verified OK.

3.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y + x \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 52: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0 \\ \eta(x,y) &= e^{x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy\end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\,\mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = -(x+1)\,\mathrm{e}^{-x} + c_1$$

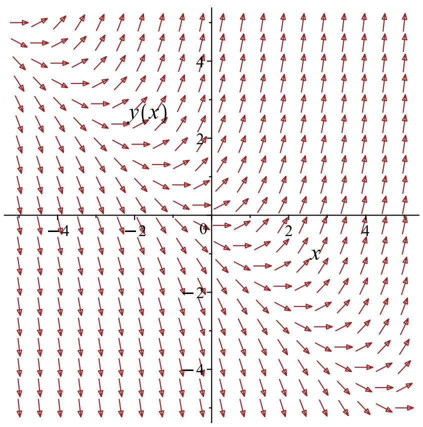
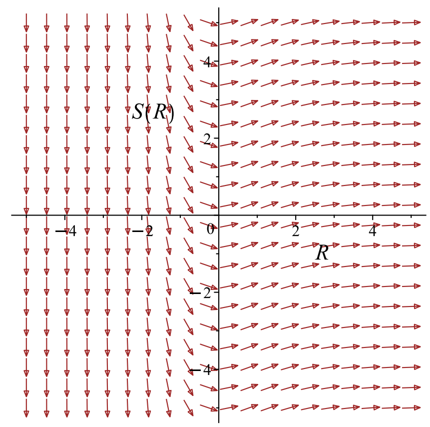
Which simplifies to

$$(x+y+1)\,\mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + x$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = R\,\mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 + c_1$$

$$c_1 = 2$$

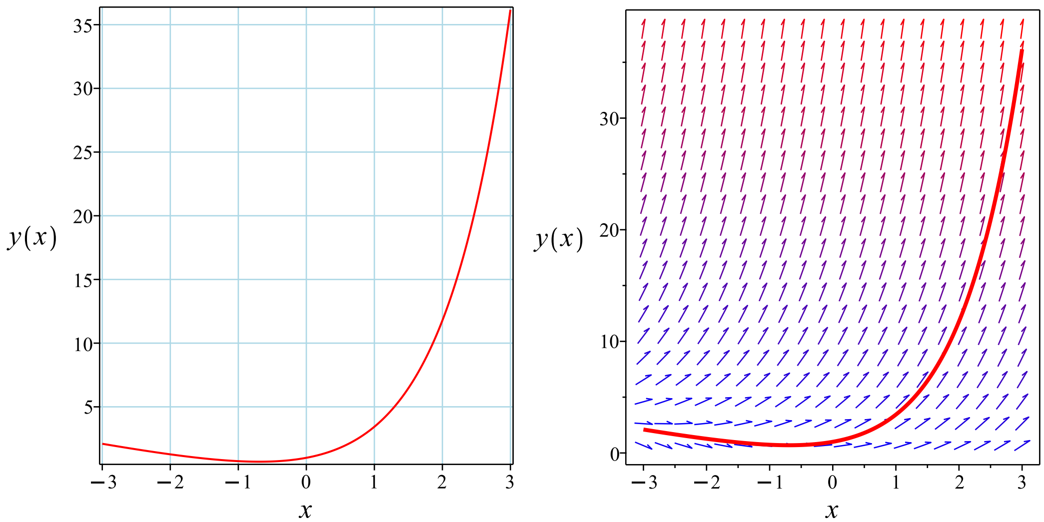
Substituting c_1 found above in the general solution gives

$$y = -1 + 2\,\mathrm{e}^{x} - x$$

Summary

The solution(s) found are the following

$$y = -1 + 2\,\mathrm{e}^{x} - x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -1 + 2\,e^{x} - x$$

Verified OK.

3.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(y+x)\,\mathrm{d}x\\(-y-x)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y-x\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-y-x)\\&=-1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=1((-1)-(0))\\&=-1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, dx} \\
&= e^{\int -1 \, dx}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-y - x) \\
&= -\mathrm{e}^{-x}(y + x)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x}(y + x)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\mathrm{e}^{-x}(y + x) \, \mathrm{d}x
\end{aligned}$$

$$\phi = (x + y + 1)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x + y + 1)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x + y + 1)\,\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 + c_1$$

$$c_1 = 2$$

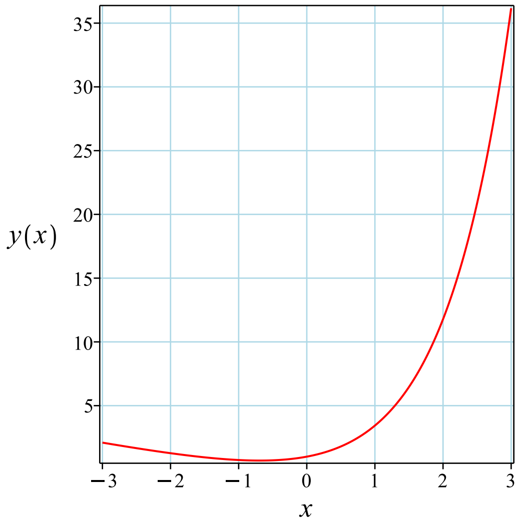
Substituting c_1 found above in the general solution gives

$$y = -1 + 2\,\mathrm{e}^{x} - x$$

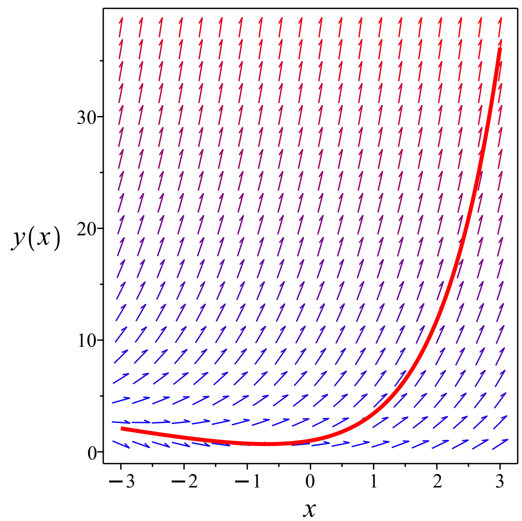
Summary

The solution(s) found are the following

$$y = -1 + 2\,e^{x} - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1 + 2\,e^{x} - x$$

Verified OK.

3.3.5 Maple step by step solution

Let's solve

$[y' - y = x, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-y\right)=\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-y\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{-x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,xdx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,xdx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{-x}$

  $y=\frac{\int x\,\mathrm{e}^{-x}dx+c_1}{\mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y=\frac{-(x+1)\mathrm{e}^{-x}+c_1}{\mathrm{e}^{-x}}$

- Simplify

  $y=\mathrm{e}^{x}c_1-x-1$

- Use initial condition $y(0)=1$

  $1=-1+c_1$

- Solve for c_1

  $c_1=2$

- Substitute $c_1=2$ into general solution and simplify

  $y=-1+2\,\mathrm{e}^{x}-x$

- Solution to the IVP

  $y=-1+2\,\mathrm{e}^{x}-x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=x+y(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = -x - 1 + 2\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 15

```
DSolve[{y'[x]==x+y[x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + 2e^{x} - 1$$

3.4 problem 44

Internal problem ID [14971]
Internal file name [`OUTPUT/14980_Monday_April_15_2024_12_04_37_AM_15956090/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 3. The method of successive approximation. Exercises page 31
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = -2x^2 - 3}$$

With initial conditions

$$[y(0) = 2]$$

3.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= -2x^2 - 3 \end{aligned}$$

Hence the ode is

$$y' - 2y = -2x^2 - 3$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2x^2 - 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.4.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-2)dx} \\ &= e^{-2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-2x^2 - 3\right) \\ \frac{d}{dx}\left(e^{-2x}y\right) &= \left(e^{-2x}\right)\left(-2x^2 - 3\right) \\ d\left(e^{-2x}y\right) &= \left(\left(-2x^2 - 3\right)e^{-2x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-2x}y &= \int \left(-2x^2 - 3\right)e^{-2x} dx \\ e^{-2x}y &= e^{-2x}\left(x^2 + x + 2\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = e^{2x}e^{-2x}\left(x^2 + x + 2\right) + c_1e^{2x}$$

which simplifies to

$$y = x^2 + x + 2 + c_1e^{2x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 2$$

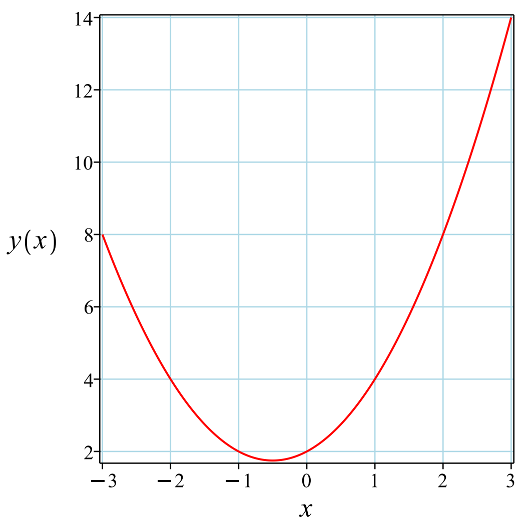
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

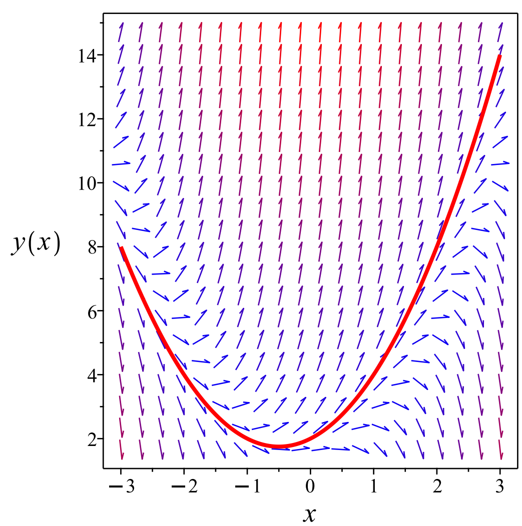
$$y = x^2 + x + 2$$

Summary
The solution(s) found are the following

$$y = x^2 + x + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 + x + 2$$

Verified OK.

3.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -2x^2 + 2y - 3 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{2x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{2x}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{-2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2x^2 + 2y - 3$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -2\,\mathrm{e}^{-2x} y\\ S_y &= \mathrm{e}^{-2x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(-2x^2 - 3\right) e^{-2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(-2R^2 - 3\right) e^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \left(R^2 + R + 2\right) e^{-2R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-2x} y = e^{-2x}\left(x^2 + x + 2\right) + c_1$$

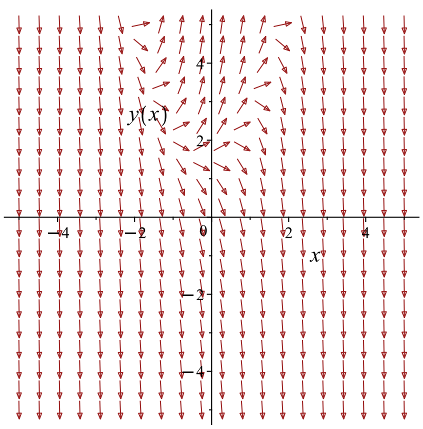
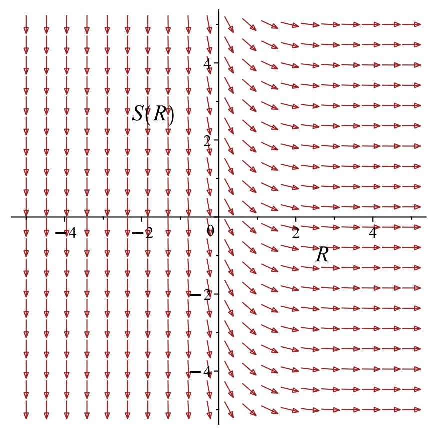
Which simplifies to

$$e^{-2x} y = e^{-2x}\left(x^2 + x + 2\right) + c_1$$

Which gives

$$y = \left(x^2 e^{-2x} + x\, e^{-2x} + 2\, e^{-2x} + c_1\right) e^{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2x^2 + 2y - 3$ 	$R = x$ $S = \mathrm{e}^{-2x} y$	$\frac{dS}{dR} = (-2R^2 - 3)\,\mathrm{e}^{-2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 2$$

$$c_1 = 0$$

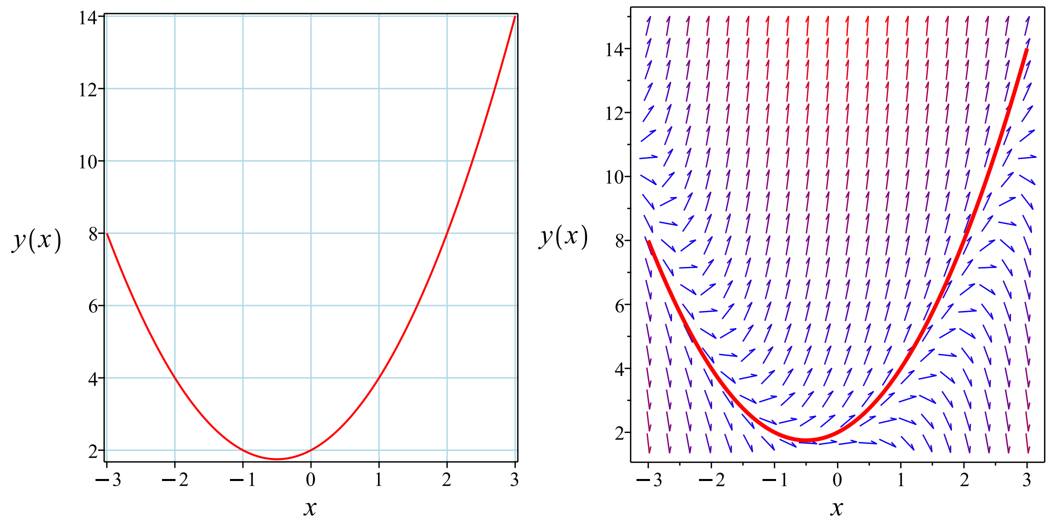
Substituting c_1 found above in the general solution gives

$$y = x^2 + x + 2$$

Summary

The solution(s) found are the following

$$y = x^2 + x + 2 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = x^2 + x + 2$$

Verified OK.

3.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(-2x^2+2y-3\right)\mathrm{d}x \\ \left(2x^2-2y+3\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= 2x^2-2y+3 \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x^2-2y+3\right) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-2)-(0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -2 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2x} \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-2x}\left(2x^2 - 2y + 3\right) \\
&= \left(2x^2 - 2y + 3\right) \mathrm{e}^{-2x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-2x}(1) \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(2x^2 - 2y + 3\right) \mathrm{e}^{-2x}\right) + \left(\mathrm{e}^{-2x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \left(2x^2 - 2y + 3\right) \mathrm{e}^{-2x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\left(x^2 + x - y + 2\right) \mathrm{e}^{-2x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2x} = \mathrm{e}^{-2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^2 + x - y + 2\right)\mathrm{e}^{-2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^2 + x - y + 2\right)\mathrm{e}^{-2x}$$

The solution becomes

$$y = \left(x^2\mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} + 2\,\mathrm{e}^{-2x} + c_1\right)\mathrm{e}^{2x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + 2$$

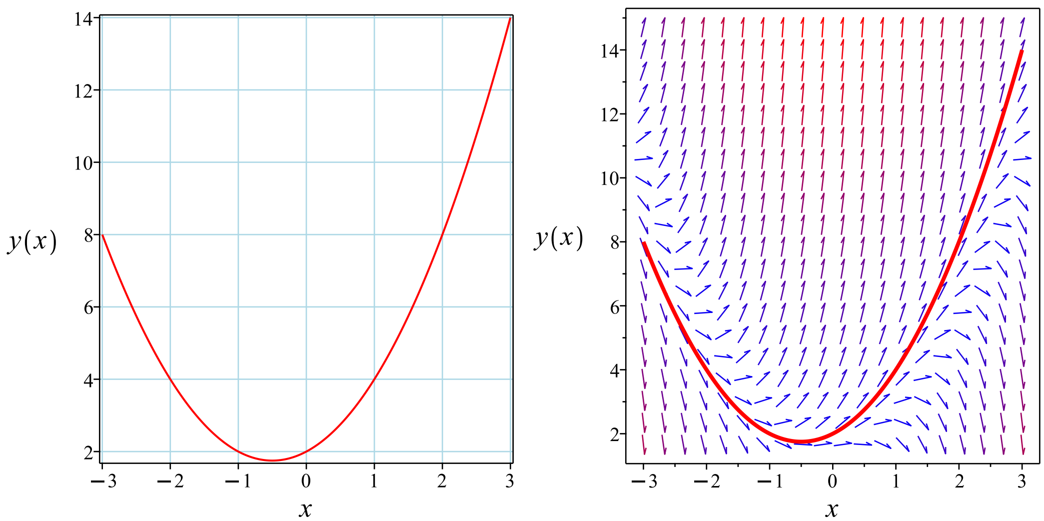
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = x^2 + x + 2$$

Summary
The solution(s) found are the following

$$y = x^2 + x + 2 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x^2 + x + 2$$

Verified OK.

3.4.5 Maple step by step solution

Let's solve

$[y' - 2y = -2x^2 - 3, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y - 2x^2 - 3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = -2x^2 - 3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\,(y'-2y)=\mu(x)\,(-2x^2-3)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\,(y'-2y)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x)=-2\mu(x)$

- Solve to find the integrating factor

$\mu(x)=\mathrm{e}^{-2x}$

- Integrate both sides with respect to x

$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,(-2x^2-3)\,dx+c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y=\int\mu(x)\,(-2x^2-3)\,dx+c_1$

- Solve for y

$y=\frac{\int\mu(x)(-2x^2-3)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{-2x}$

$y=\frac{\int(-2x^2-3)\mathrm{e}^{-2x}dx+c_1}{\mathrm{e}^{-2x}}$

- Evaluate the integrals on the rhs

$y=\frac{\mathrm{e}^{-2x}(x^2+x+2)+c_1}{\mathrm{e}^{-2x}}$

- Simplify

$y=x^2+x+2+c_1\mathrm{e}^{2x}$

- Use initial condition $y(0)=2$

$2=c_1+2$

- Solve for c_1

$c_1=0$

- Substitute $c_1=0$ into general solution and simplify

$y=x^2+x+2$

- Solution to the IVP

$y=x^2+x+2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=2*y(x)-2*x^2-3,y(0) = 2],y(x), singsol=all)
```

$$y(x) = x^2 + x + 2$$

✓ Solution by Mathematica

Time used: 0.091 (sec). Leaf size: 11

```
DSolve[{y'[x]==2*y[x]-2*x^2-3,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + x + 2$$

3.5 problem 45

Internal problem ID [14972]
Internal file name [`OUTPUT/14981_Monday_April_15_2024_12_04_38_AM_76303662/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 3. The method of successive approximation. Exercises page 31
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y'x + y = 2x}$$

With initial conditions

$$[y(1) = 2]$$

3.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= 2 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = 2$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.5.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2) \\ \frac{\mathrm{d}}{\mathrm{d}x}(xy) &= (x)(2) \\ \mathrm{d}(xy) &= (2x)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} xy &= \int 2x\,\mathrm{d}x \\ xy &= x^2 + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = x + \frac{c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 + c_1$$

$$c_1 = 1$$

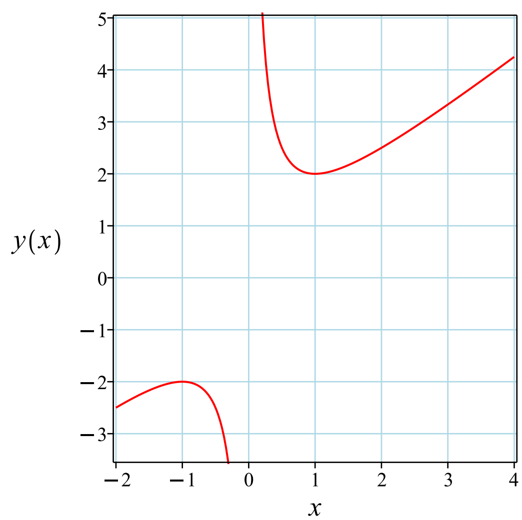
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + 1}{x}$$

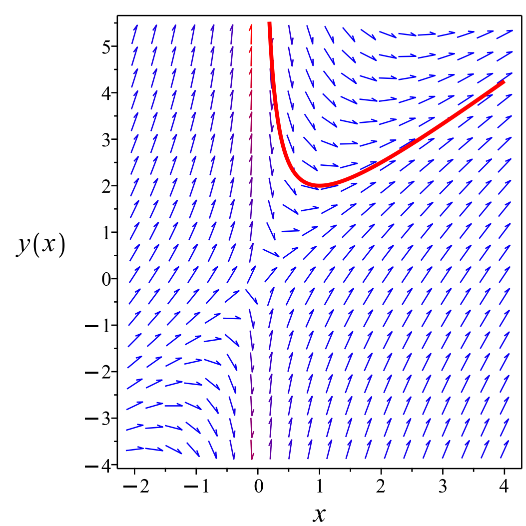
Summary

The solution(s) found are the following

$$y = \frac{x^2 + 1}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 + 1}{x}$$

Verified OK.

3.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x + u(x)\,x = 2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-2u+2}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -2u + 2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-2u+2}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{-2u+2}\,du &= \int \frac{1}{x}\,dx \\ -\frac{\ln(u-1)}{2} &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{u-1}} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{u-1}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \frac{(c_3^2\mathrm{e}^{2c_2}x^2+1)\,\mathrm{e}^{-2c_2}}{x\,c_3^2} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{\mathrm{e}^{2c_2}\mathrm{e}^{-2c_2}c_3^2 + \mathrm{e}^{-2c_2}}{c_3^2}$$

$$
c_2 = -\ln(c_3)
$$

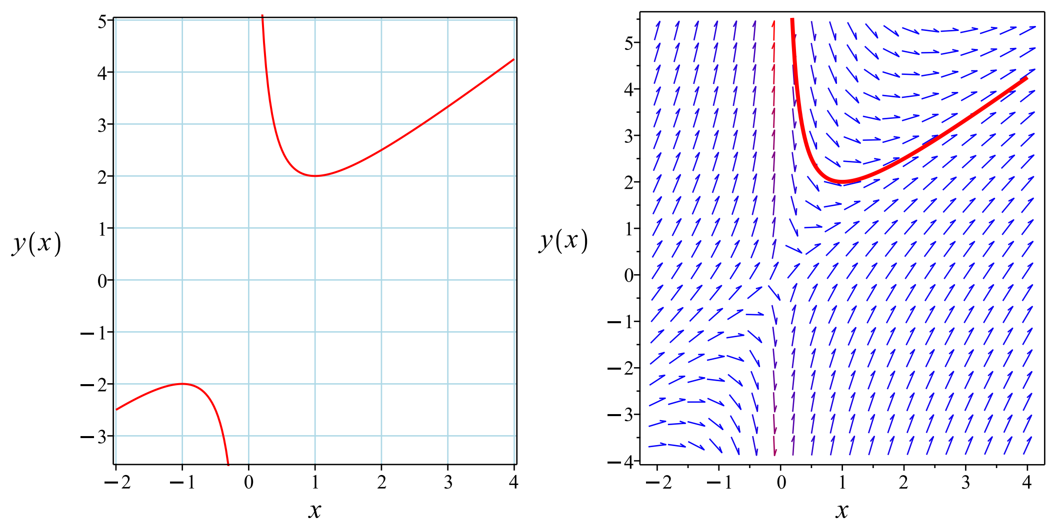
Substituting c_2 found above in the general solution gives

$$
y = \frac{x^2+1}{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^2+1}{x} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \frac{x^2+1}{x}
$$

Verified OK.

3.5.4 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x - y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + (2x - y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (2x - y)\, dx = d\left(x^2 - xy\right)$$

Hence (2) becomes

$$0 = d\left(x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + c_1}{x} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 2c_1 + 1$$

$$c_1 = \frac{1}{2}$$

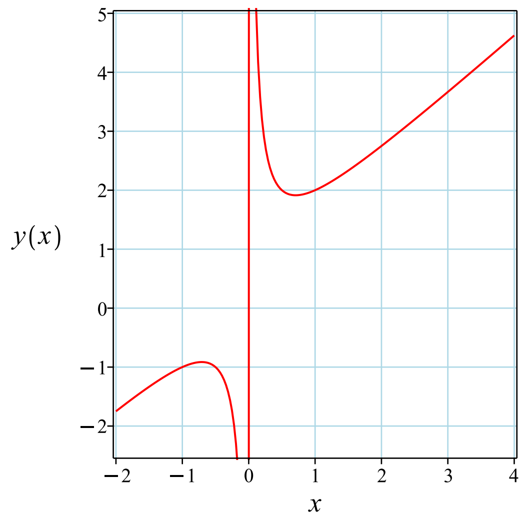
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^2 + x + 1}{2x}$$

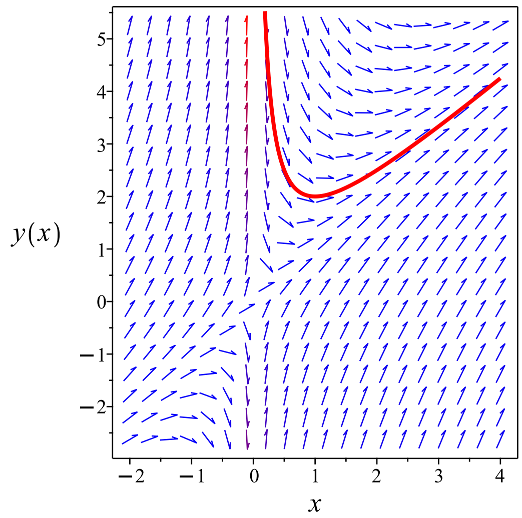
Summary

The solution(s) found are the following

$$y = \frac{2x^2 + x + 1}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2x^2 + x + 1}{2x}$$

Warning, solution could not be verified

3.5.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = x^2 + c_1$$

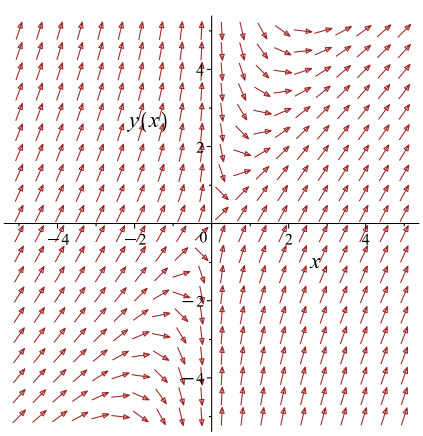
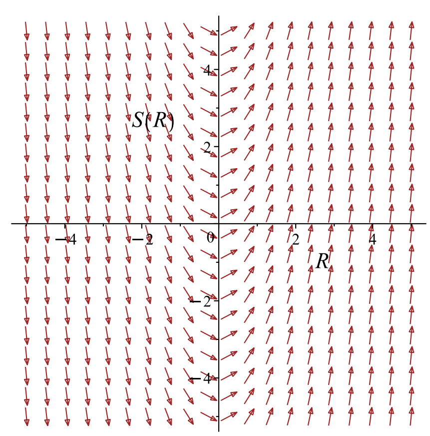
Which simplifies to

$$yx = x^2 + c_1$$

Which gives

$$y = \frac{x^2 + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-2x+y}{x}$

</td><td>$R = x$
$S = xy$</td><td>$\frac{dS}{dR} = 2R$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 + c_1$$

$$c_1 = 1$$

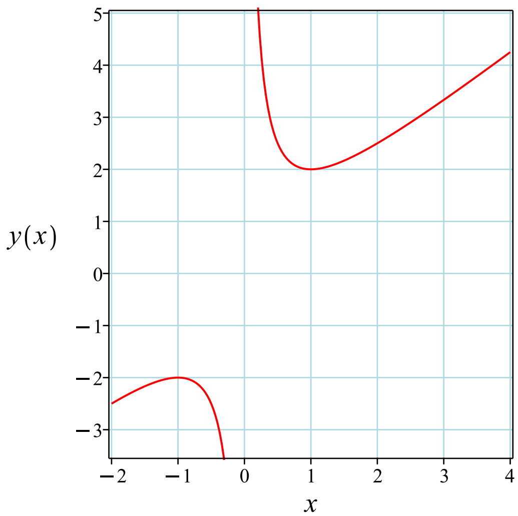
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + 1}{x}$$

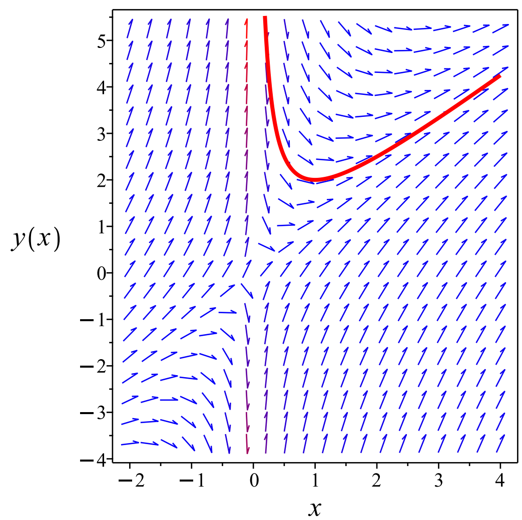
Summary

The solution(s) found are the following

$$y = \frac{x^2 + 1}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 + 1}{x}$$

Verified OK.

3.5.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(2x-y)\,\mathrm{d}x\\ (-2x+y)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-2x+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-2x+y)\\&=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\&=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x&=\int -2x+y\,\mathrm{d}x\end{aligned}$$

$$\phi=-x(-y+x)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x$. Therefore equation (4) becomes

$$x=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(-y + x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(-y + x)$$

The solution becomes

$$y = \frac{x^2 + c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 + c_1$$

$$c_1 = 1$$

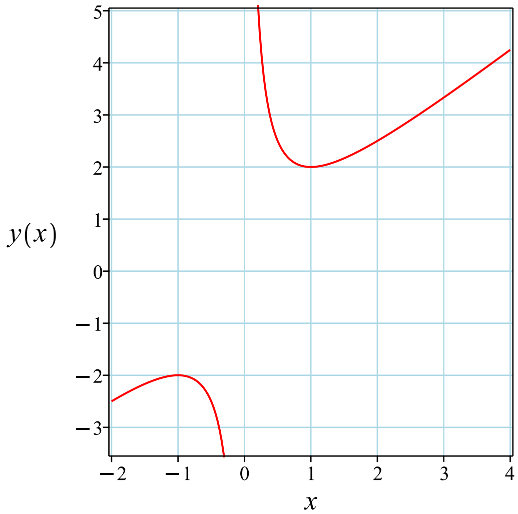
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2 + 1}{x}$$

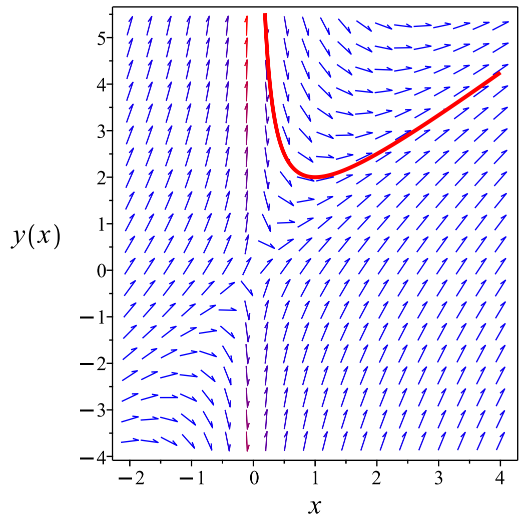
Summary

The solution(s) found are the following

$$y = \frac{x^2 + 1}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2 + 1}{x}$$

Verified OK.

3.5.7 Maple step by step solution

Let's solve

$[y'x + y = 2x, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2 - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = 2\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 2\mu(x)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 2\mu(x)\, dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x$

  $y = \frac{\int 2x dx + c_1}{x}$
- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{x}$
- Use initial condition $y(1) = 2$

  $2 = 1 + c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{x^2 + 1}{x}$
- Solution to the IVP

  $y = \frac{x^2 + 1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)=2*x-y(x),y(1) = 2],y(x), singsol=all)
```

$$y(x) = x + \frac{1}{x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 10

```
DSolve[{x*y'[x]==2*x-y[x],{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{1}{x}$$

4 Section 4. Equations with variables separable and equations reducible to them. Exercises page 38

4.1 problem 46

Internal problem ID [14973]
Internal file name [`OUTPUT/14982_Monday_April_15_2024_12_04_39_AM_68971325/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + \left(x^2+1\right) y' = -1}$$

4.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-y^2-1}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = -y^2-1$. Integrating both sides gives

$$\frac{1}{-y^2-1}\,dy = \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{-y^2-1}\,dy = \int \frac{1}{x^2+1}\,dx$$
$$-\arctan(y) = \arctan(x) + c_1$$

Which results in

$$y = -\tan(\arctan(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = -\tan(\arctan(x) + c_1) \tag{1}$$

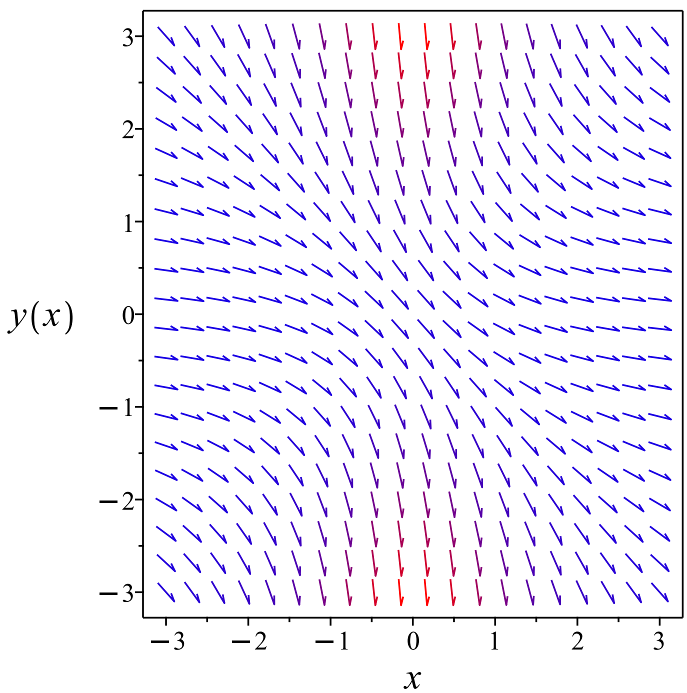


Figure 88: Slope field plot

Verification of solutions

$$y = -\tan(\arctan(x) + c_1)$$

Verified OK.

4.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2+1}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= x^2 + 1 \\ \eta(x, y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2 + 1} dx\end{aligned}$$

Which results in

$$S = \arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2 + 1}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(x) = -\arctan(y) + c_1$$

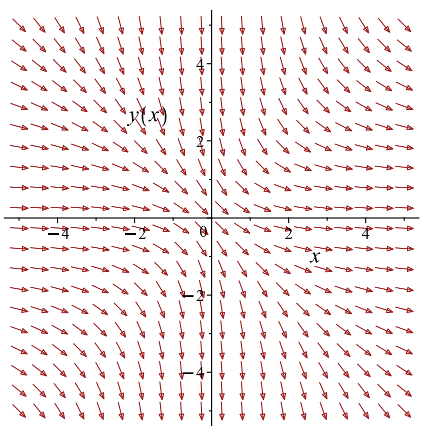
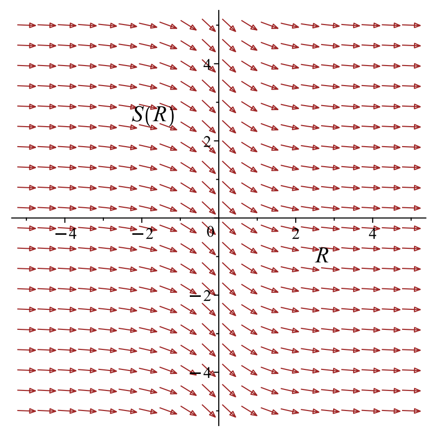
Which simplifies to

$$\arctan(x) = -\arctan(y) + c_1$$

Which gives

$$y = \tan(-\arctan(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+1}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = -\frac{1}{R^2+1}$ 

<u>Summary</u>

The solution(s) found are the following

$$y = \tan(-\arctan(x) + c_1) \tag{1}$$

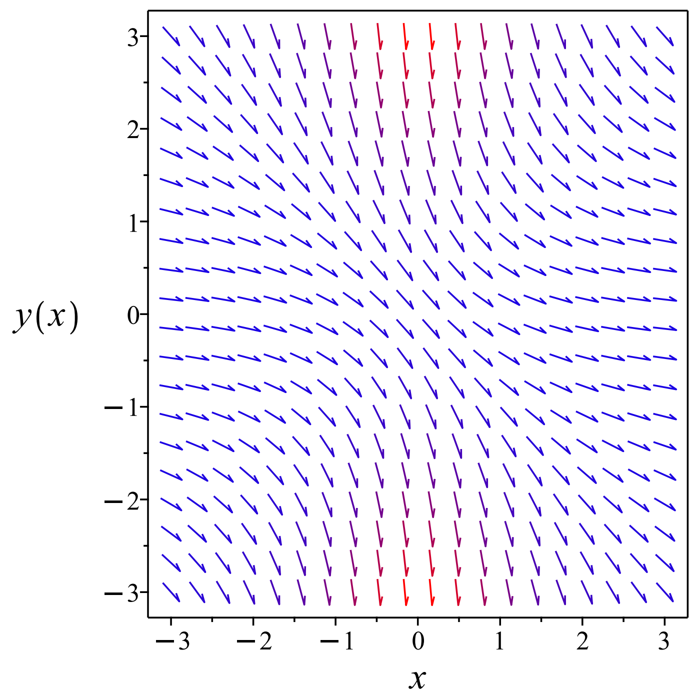


Figure 89: Slope field plot

Verification of solutions

$$y = \tan(-\arctan(x) + c_1)$$

Verified OK.

4.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\left(\frac{1}{-y^2 - 1}\right) \mathrm{d}y = \left(\frac{1}{x^2 + 1}\right) \mathrm{d}x
$$

$$
\left(-\frac{1}{x^2 + 1}\right) \mathrm{d}x + \left(\frac{1}{-y^2 - 1}\right) \mathrm{d}y = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x^2 + 1} \\
N(x, y) &= \frac{1}{-y^2 - 1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2 + 1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y^2-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x^2+1} \,\mathrm{d}x \\
\phi &= -\arctan(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y^2-1}$. Therefore equation (4) becomes

$$
\frac{1}{-y^2-1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= -\arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - \arctan(y)$$

Summary

The solution(s) found are the following

$$-\arctan(x) - \arctan(y) = c_1 \tag{1}$$

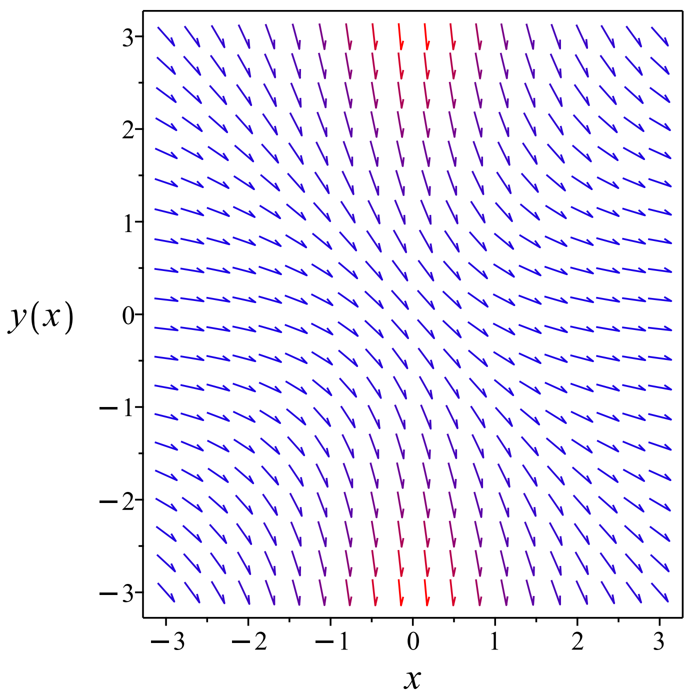


Figure 90: Slope field plot

Verification of solutions

$$-\arctan(x) - \arctan(y) = c_1$$

Verified OK.

4.1.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2+1}{x^2+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x^2+1} - \frac{1}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{x^2+1}$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{x^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x}{(x^2+1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{1}{(x^2+1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x^2+1} - \frac{2xu'(x)}{(x^2+1)^2} - \frac{u(x)}{(x^2+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1 x + c_2}{\sqrt{x^2+1}}$$

The above shows that

$$u'(x) = \frac{-c_2 x + c_1}{(x^2+1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$y = \frac{-c_2 x + c_1}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 - x}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 - x}{c_3 x + 1} \tag{1}$$

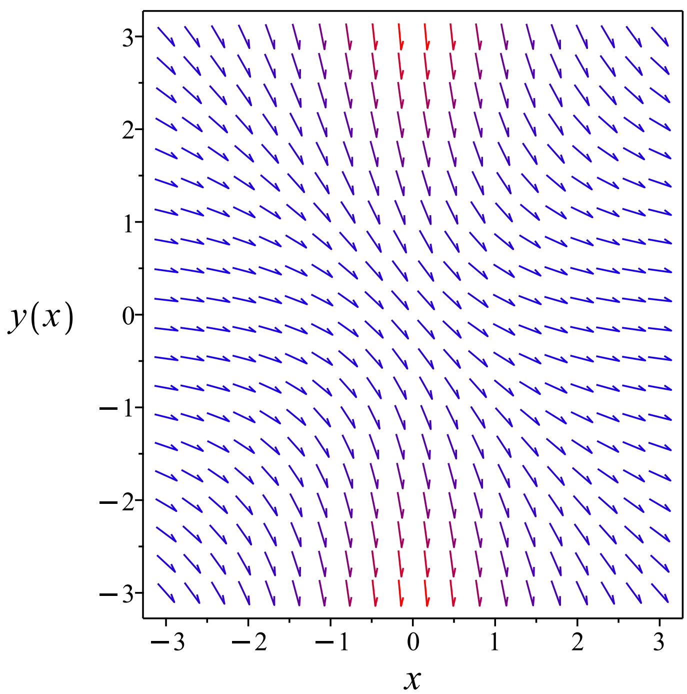


Figure 91: Slope field plot

Verification of solutions

$$y = \frac{c_3 - x}{c_3 x + 1}$$

Verified OK.

4.1.5 Maple step by step solution

Let's solve

$y^2 + (x^2+1)\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-1} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-1} dx = \int \frac{1}{x^2+1} dx + c_1$

- Evaluate integral

  $-\arctan(y) = \arctan(x) + c_1$

- Solve for y

  $y = -\tan(\arctan(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((1+y(x)^2)+(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\tan\left(\arctan\left(x\right) + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 29

```
DSolve[(1+y[x]^2)+(1+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\tan(\arctan(x) - c_1)$$
$$y(x) \to -i$$
$$y(x) \to i$$

4.2 problem 47

Internal problem ID [14974]
Internal file name [`OUTPUT/14983_Monday_April_15_2024_12_04_40_AM_61741925/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + xyy' = -1}$$

4.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2+1}{xy} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\,dy = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{y^2+1}{y}} dy = \int -\frac{1}{x} dx$$

$$\frac{\ln(y^2+1)}{2} = -\ln(x) + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{-\ln(x)+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = \frac{c_2}{x}$$

Which simplifies to

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{x}$$

The solution is

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{x}$$

<u>Summary</u>

The solution(s) found are the following

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{x} \tag{1}$$

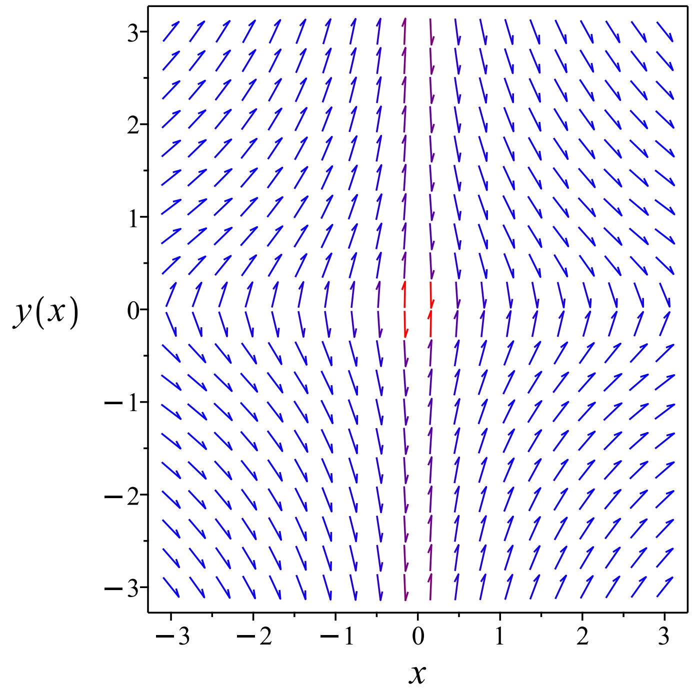


Figure 92: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{x}$$

Verified OK.

4.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2+1}{xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 64: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= -x \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x} dx \end{aligned}$$

Which results in

$$S = -\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2 + 1}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

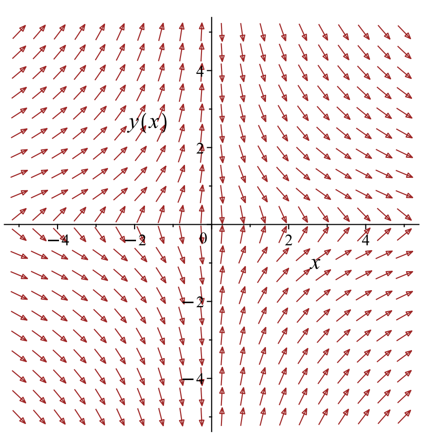
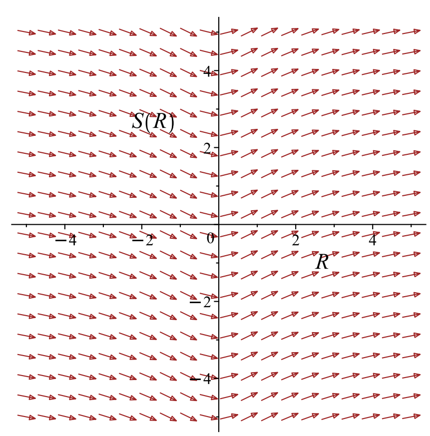
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$-\ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+1}{xy}$ 	$R = y$ $S = -\ln(x)$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary
The solution(s) found are the following

$$-\ln(x) = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

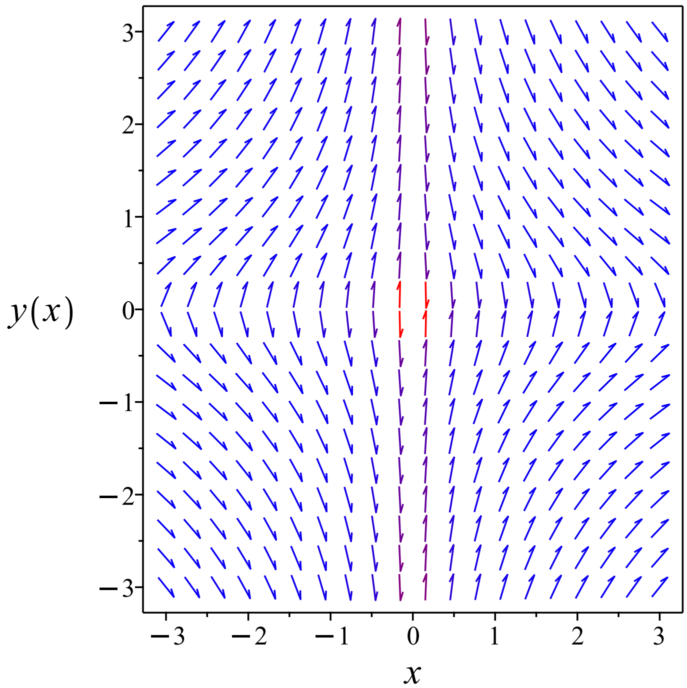


Figure 93: Slope field plot

<u>Verification of solutions</u>

$$-\ln(x) = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

4.2.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2+1}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{1}{x}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{1}{x} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} - \frac{1}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{x} - \frac{1}{x} \\ w' &= -\frac{2w}{x} - \frac{2}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{2}{x} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -\frac{2}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{2}{x}\right) \\ \frac{d}{dx}(x^2 w) &= (x^2)\left(-\frac{2}{x}\right) \\ d(x^2 w) &= (-2x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int -2x\, dx \\ x^2 w &= -x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -1 + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -1 + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{-x^2 + c_1}}{x} \\ y(x) &= -\frac{\sqrt{-x^2 + c_1}}{x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{-x^2 + c_1}}{x} \tag{1}$$

$$y = -\frac{\sqrt{-x^2 + c_1}}{x} \tag{2}$$

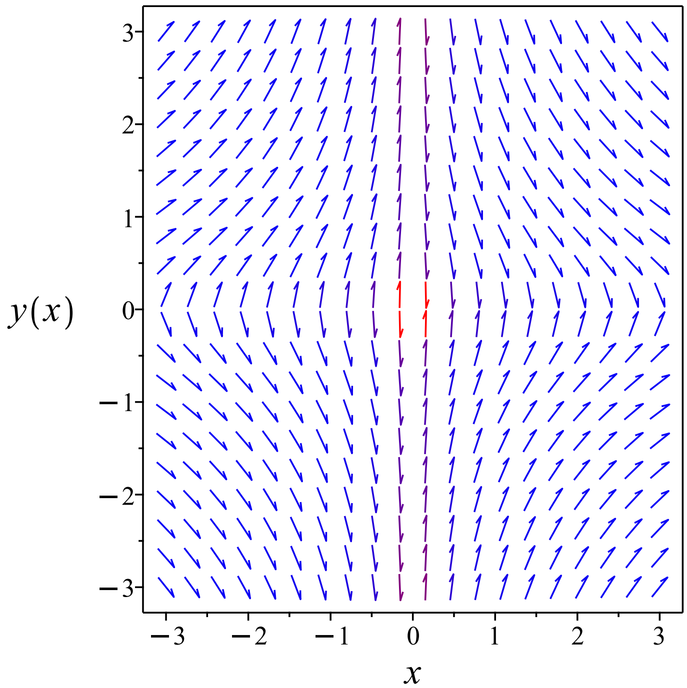


Figure 94: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-x^2 + c_1}}{x}$$

Verified OK.

$$y = -\frac{\sqrt{-x^2 + c_1}}{x}$$

Verified OK.

4.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x}&=M\\
\frac{\partial \phi}{\partial y}&=N
\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-\frac{y}{y^2+1}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\
\left(-\frac{1}{x}\right)\mathrm{d}x+\left(-\frac{y}{y^2+1}\right)\mathrm{d}y&=0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y)&=-\frac{1}{x}\\
N(x,y)&=-\frac{y}{y^2+1}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\
&=0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y^2+1}$. Therefore equation (4) becomes

$$
-\frac{y}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{y}{y^2+1}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(y^2+1)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\ln(x) - \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

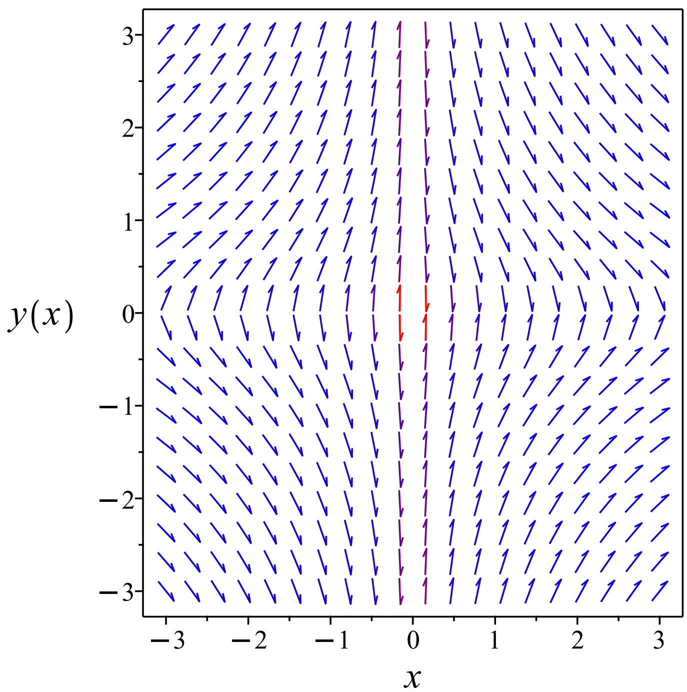


Figure 95: Slope field plot

Verification of solutions

$$-\ln(x) - \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

4.2.5 Maple step by step solution

Let's solve

$y^2 + xyy' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{-y^2-1} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{-y^2-1}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $-\frac{\ln(1+y^2)}{2} = \ln(x) + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{-(e^{c_1})^2 x^2+1}}{e^{c_1}x}, y = -\frac{\sqrt{-(e^{c_1})^2 x^2+1}}{e^{c_1}x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve((1+y(x)^2)+(x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-x^2 + c_1}}{x}$$
$$y(x) = -\frac{\sqrt{-x^2 + c_1}}{x}$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 96

```
DSolve[(1+y[x]^2)+(x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^2 + e^{2c_1}}}{x}$$
$$y(x) \to \frac{\sqrt{-x^2 + e^{2c_1}}}{x}$$
$$y(x) \to -i$$
$$y(x) \to i$$
$$y(x) \to \frac{x}{\sqrt{-x^2}}$$
$$y(x) \to \frac{\sqrt{-x^2}}{x}$$

4.3 problem 48

Internal problem ID [14975]
Internal file name [`OUTPUT/14984_Monday_April_15_2024_12_04_41_AM_98181315/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' \sin (x) - y \cos (x) = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

4.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\cot(x) \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - y\cot(x) = 0$$

The domain of $p(x) = -\cot(x)$ is

$$\{x < \pi\_Z796 \lor \pi\_Z796 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. Hence solution exists and is unique.

4.3.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y\cos(x)}{\sin(x)} \end{aligned}$$

Where $f(x) = \frac{\cos(x)}{\sin(x)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{\cos(x)}{\sin(x)}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{\cos(x)}{\sin(x)}\,dx \\ \ln(y) &= \ln(\sin(x)) + c_1 \\ y &= e^{\ln(\sin(x))+c_1} \\ &= \sin(x)\,c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

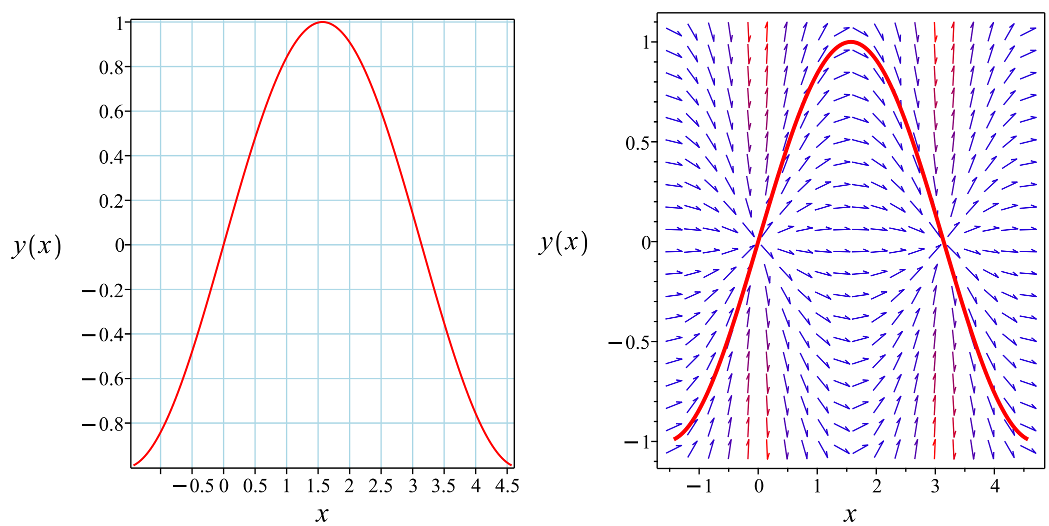
Substituting c_1 found above in the general solution gives

$$y = \sin(x)$$

Summary

The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

4.3.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)}\end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}(\csc(x)\, y) = 0$$

Integrating gives

$$\csc(x)\, y = c_1$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = \sin(x)\, c_1$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

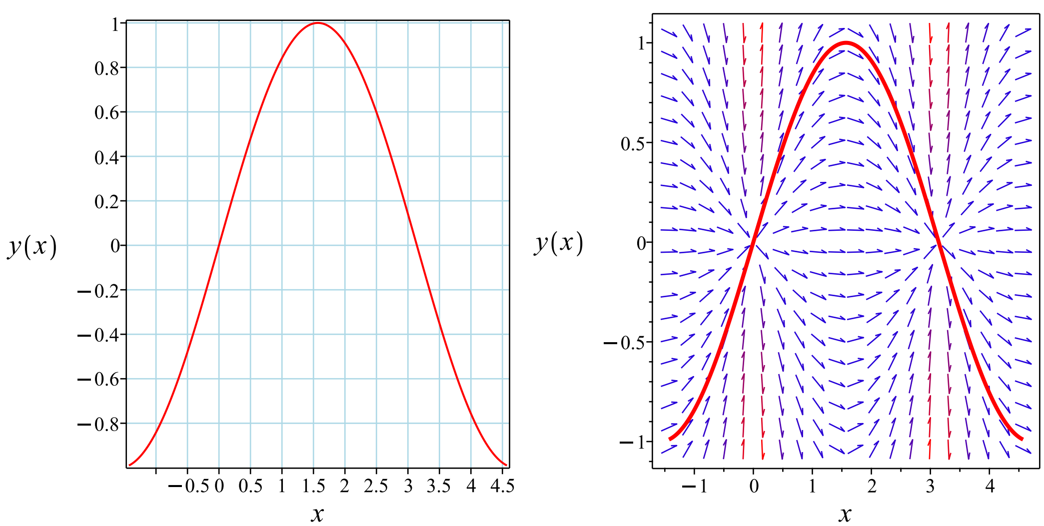
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \sin(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

4.3.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\sin(x) - u(x)\,x\cos(x) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(\cos(x)\,x - \sin(x))}{x\sin(x)}
\end{aligned}$$

Where $f(x) = \frac{\cos(x)x-\sin(x)}{x\sin(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{\cos(x)\,x - \sin(x)}{x\sin(x)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{\cos(x)\,x - \sin(x)}{x\sin(x)}\,dx \\
\ln(u) &= \ln(\sin(x)) - \ln(x) + c_2 \\
u &= e^{\ln(\sin(x))-\ln(x)+c_2} \\
&= c_2 e^{\ln(\sin(x))-\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2\sin(x)}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2\sin(x)
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2$$

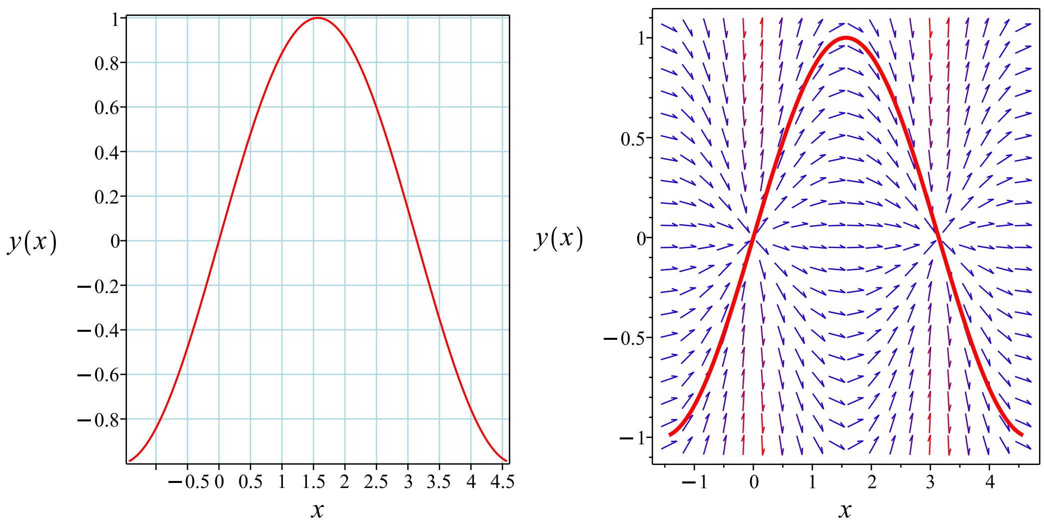
$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = \sin(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

4.3.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y\cos(x)}{\sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 67: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sin(x) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\cos(x)}{\sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x)\cot(x)\, y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = c_1$$

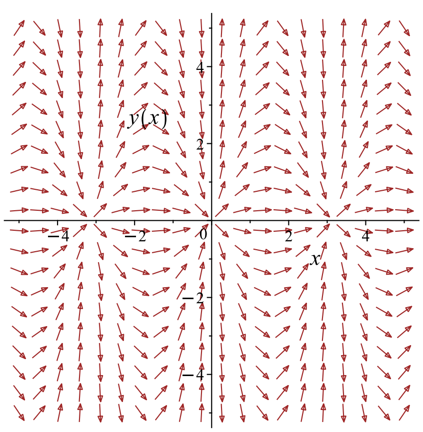
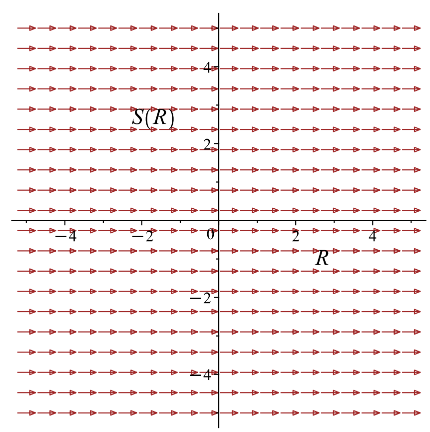
Which simplifies to

$$\csc(x)\, y = c_1$$

Which gives

$$y = \frac{c_1}{\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\cos(x)}{\sin(x)}$ 	$R = x$ $S = \csc(x)\, y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

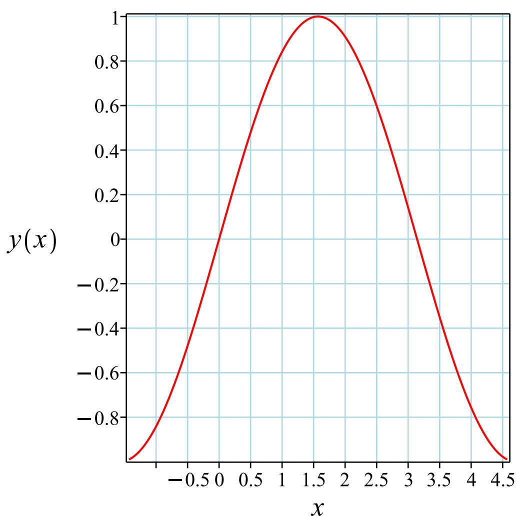
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

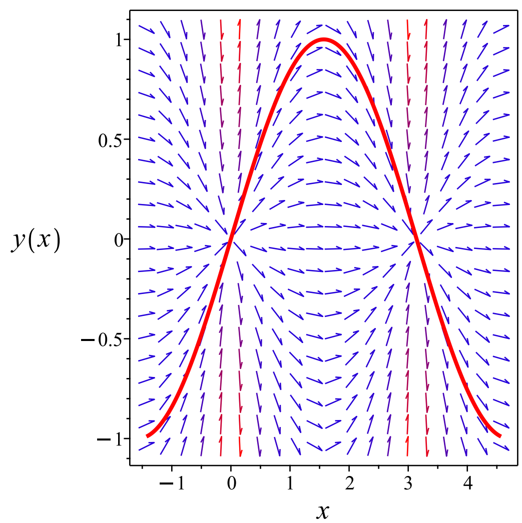
$$y = \sin(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

4.3.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^{2} \phi}{\partial x \partial y}=\frac{\partial^{2} \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^{2} \phi}{\partial x \partial y}=\frac{\partial^{2} \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d} y &= \left(\frac{\cos (x)}{\sin (x)}\right) \mathrm{d} x \\
\left(-\frac{\cos (x)}{\sin (x)}\right) \mathrm{d} x+\left(\frac{1}{y}\right) \mathrm{d} y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{\cos (x)}{\sin (x)} \\
N(x, y) &= \frac{1}{y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\cos (x)}{\sin (x)}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\cos(x)}{\sin(x)}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln(\sin(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y}\right)\mathrm{d}y\\ f(y) &= \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) + \ln(y)$$

The solution becomes

$$y = e^{c_1} \sin(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{c_1}$$

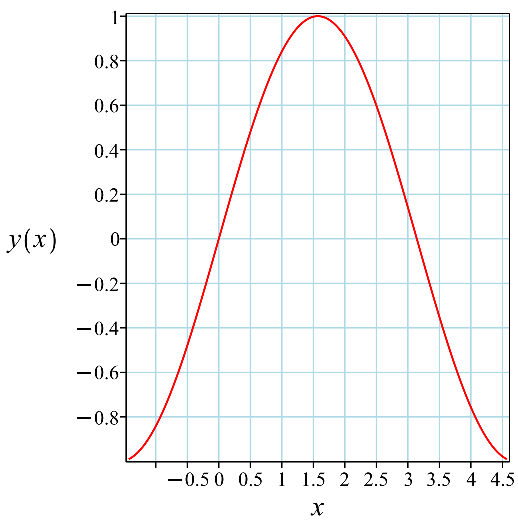
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

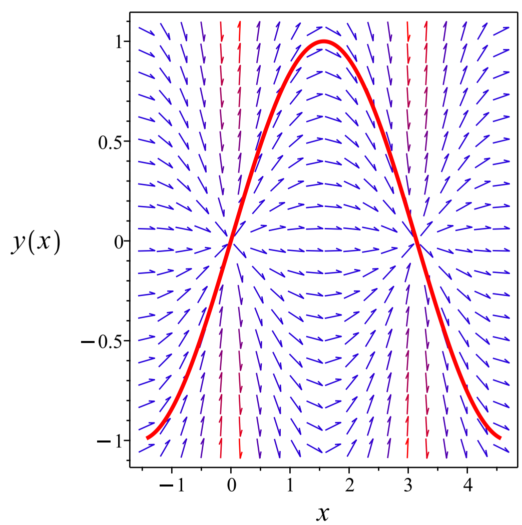
$$y = \sin(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)$$

Verified OK.

4.3.7 Maple step by step solution

Let's solve

$\left[y' \sin(x) - y \cos(x) = 0, y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{\cos(x)}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{\cos(x)}{\sin(x)} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(\sin(x)) + c_1$

- Solve for y

  $y = e^{c_1} \sin(x)$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 1$

  $1 = e^{c_1}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \sin(x)$

- Solution to the IVP

  $y = \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 6

```
dsolve([diff(y(x),x)*sin(x)-y(x)*cos(x)=0,y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \sin(x)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 7

```
DSolve[{y'[x]*Sin[x]-y[x]*Cos[x]==0,{y[Pi/2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)$$

4.4 problem 49

Internal problem ID [14976]
Internal file name [`OUTPUT/14985_Monday_April_15_2024_12_04_43_AM_23609599/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 - y'x = -1}$$

4.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int \frac{1}{x}\,dx$$
$$\arctan(y) = \ln(x) + c_1$$

Which results in

$$y = \tan(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \tan(\ln(x) + c_1) \tag{1}$$

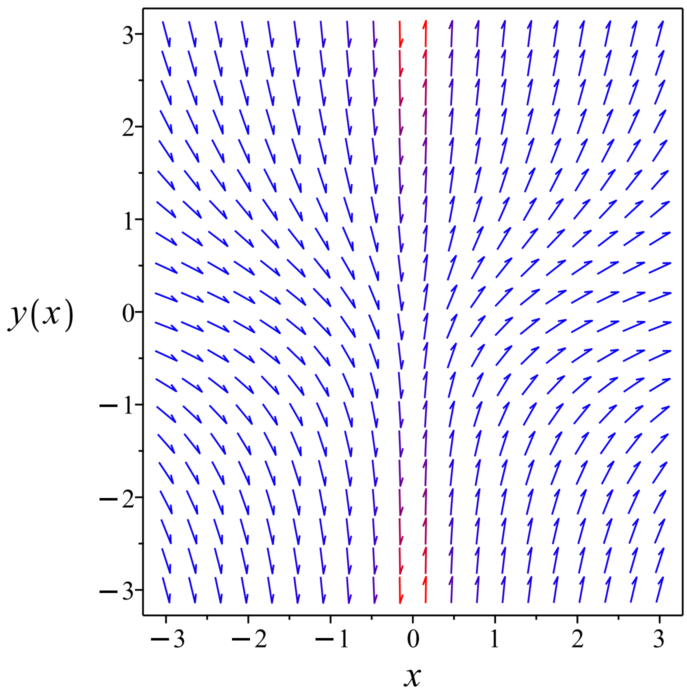


Figure 101: Slope field plot

Verification of solutions

$$y = \tan(\ln(x) + c_1)$$

Verified OK.

4.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y^2+1}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{y^2 + 1}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \arctan(y) + c_1$$

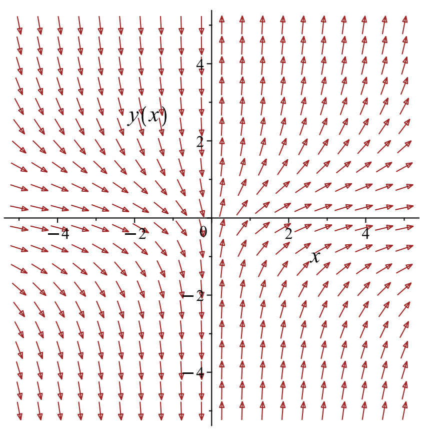
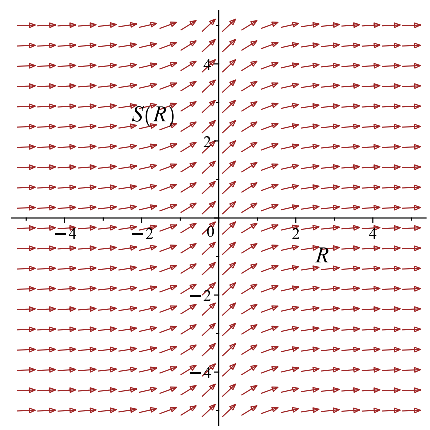
Which simplifies to

$$\ln(x) = \arctan(y) + c_1$$

Which gives

$$y = -\tan(-\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = -\tan(-\ln(x) + c_1) \tag{1}$$

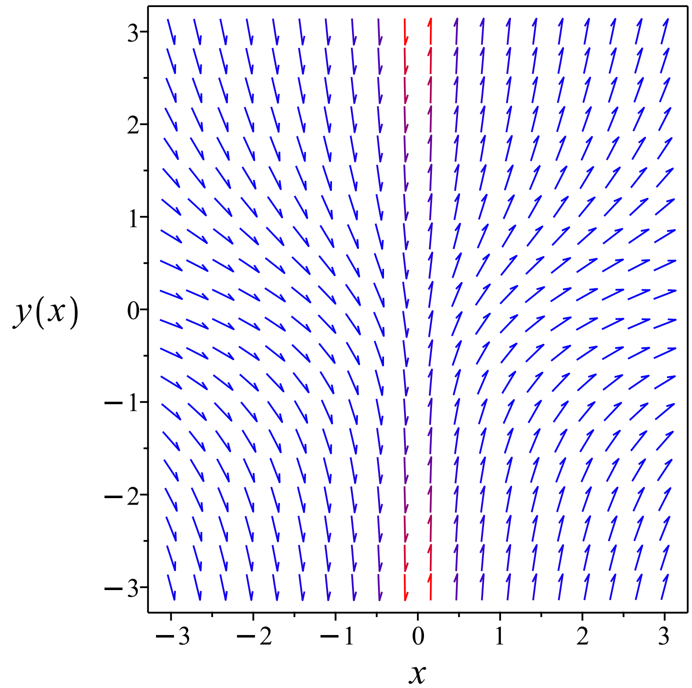


Figure 102: Slope field plot

Verification of solutions

$$y = -\tan(-\ln(x) + c_1)$$

Verified OK.

4.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2+1}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{y^2+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \arctan(y)$$

The solution becomes

$$y = \tan(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \tan(\ln(x) + c_1) \tag{1}$$

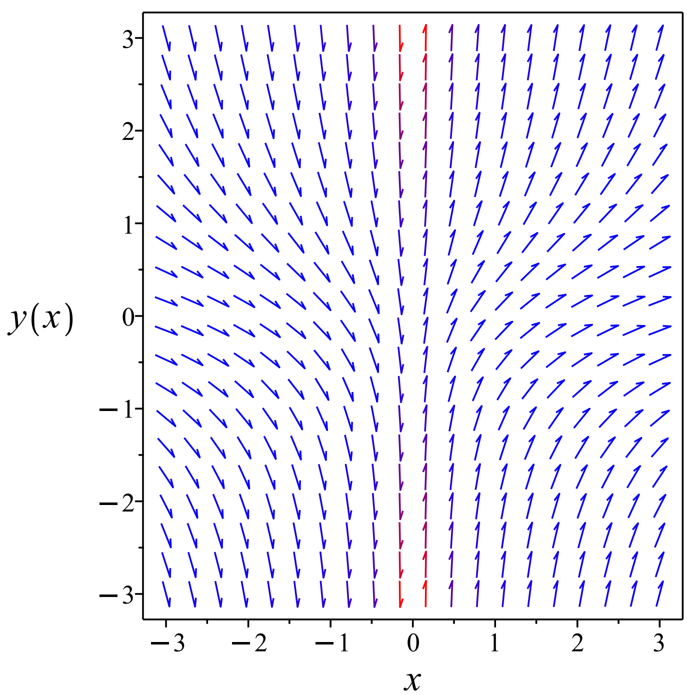


Figure 103: Slope field plot

Verification of solutions

$$y = \tan(\ln(x) + c_1)$$

Verified OK.

4.4.4 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{y^2+1}{x}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y^2}{x} + \frac{1}{x}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = \frac{1}{x}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{1}{x^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \frac{1}{x^3}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x} + \frac{u'(x)}{x^2} + \frac{u(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \sin(\ln(x))\, c_1 + c_2 \cos(\ln(x))
$$

The above shows that

$$u'(x) = \frac{\cos(\ln(x))\,c_1 - c_2 \sin(\ln(x))}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{\cos(\ln(x))\,c_1 - c_2 \sin(\ln(x))}{\sin(\ln(x))\,c_1 + c_2 \cos(\ln(x))}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\cos(\ln(x))\,c_3 + \sin(\ln(x))}{c_3 \sin(\ln(x)) + \cos(\ln(x))}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(\ln(x))\,c_3 + \sin(\ln(x))}{c_3 \sin(\ln(x)) + \cos(\ln(x))} \tag{1}$$

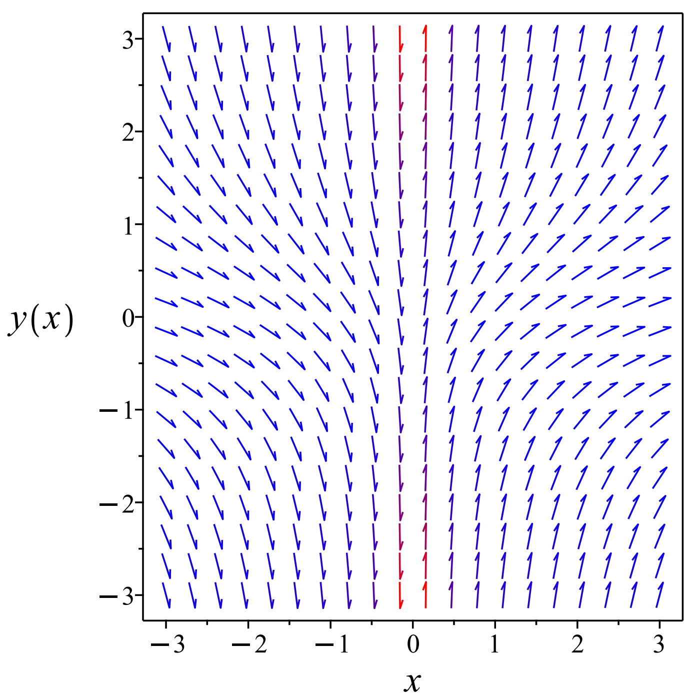


Figure 104: Slope field plot

Verification of solutions

$$y = \frac{-\cos(\ln(x)) c_3 + \sin(\ln(x))}{c_3 \sin(\ln(x)) + \cos(\ln(x))}$$

Verified OK.

4.4.5 Maple step by step solution

Let's solve

$y^2 - y'x = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y^2-1} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y^2-1} dx = \int -\frac{1}{x} dx + c_1$

- Evaluate integral

  $-\arctan(y) = -\ln(x) + c_1$

- Solve for y

  $y = -\tan(-\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve((1+y(x)^2)=x*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \tan(\ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.167 (sec). Leaf size: 25

```
DSolve[(1+y[x]^2)==x*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to \tan(\log(x) + c_1) \\ y(x) &\to -i \\ y(x) &\to i \end{aligned}$$

4.5 problem 50

Internal problem ID [14977]
Internal file name [`OUTPUT/14986_Monday_April_15_2024_12_04_43_AM_45110033/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 50.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\sqrt{1+y^2}+yy'\sqrt{x^2+1}=0}$$

4.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x\sqrt{y^2+1}}{y\sqrt{x^2+1}} \end{aligned}$$

Where $f(x)=-\frac{x}{\sqrt{x^2+1}}$ and $g(y)=\frac{\sqrt{y^2+1}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{y^2+1}}{y}}\,dy = -\frac{x}{\sqrt{x^2+1}}\,dx$$

$$\int \frac{1}{\frac{\sqrt{y^2+1}}{y}}\, dy = \int -\frac{x}{\sqrt{x^2+1}}\, dx$$

$$\sqrt{y^2+1} = -\sqrt{x^2+1} + c_1$$

The solution is

$$\sqrt{1+y^2} + \sqrt{x^2+1} - c_1 = 0$$

Summary

The solution(s) found are the following

$$\sqrt{1+y^2} + \sqrt{x^2+1} - c_1 = 0 \tag{1}$$

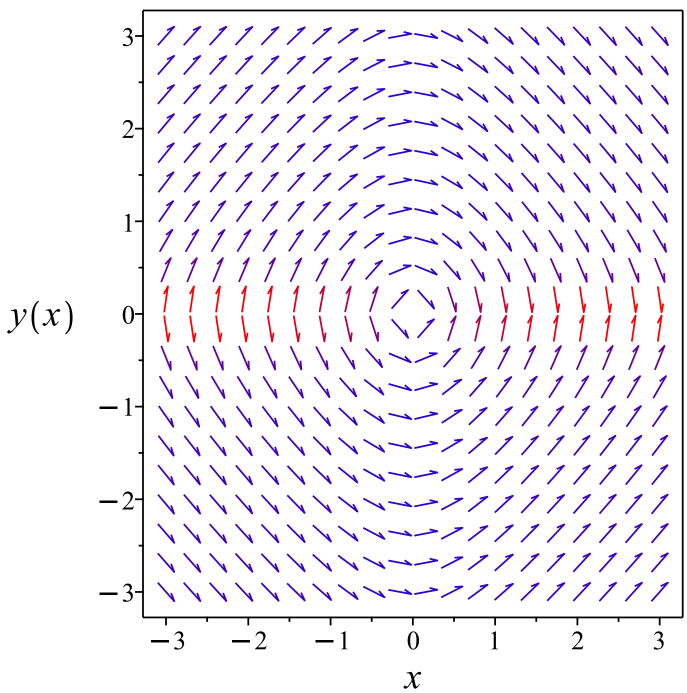


Figure 105: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} + \sqrt{x^2+1} - c_1 = 0$$

Verified OK.

4.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\sqrt{y^2+1}}{y\sqrt{x^2+1}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= -\frac{\sqrt{x^2+1}}{x} \\ \eta(x, y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{\sqrt{x^2+1}}{x}} dx\end{aligned}$$

Which results in

$$S = -\sqrt{x^2+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x\sqrt{y^2+1}}{y\sqrt{x^2+1}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{x}{\sqrt{x^2+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y}{\sqrt{y^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R}{\sqrt{R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sqrt{R^2+1} + c_1 \tag{4}
$$

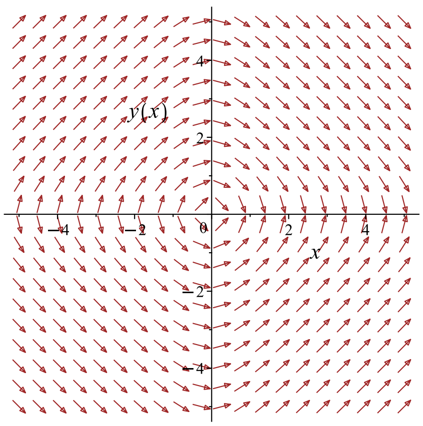
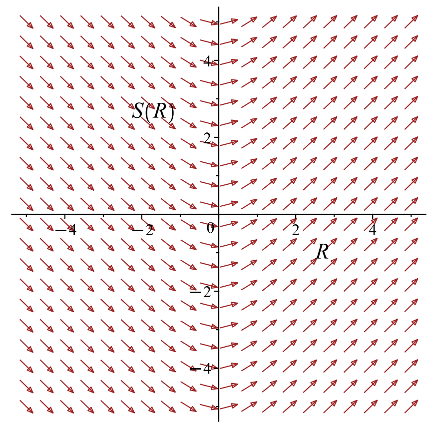
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\sqrt{x^2+1} = \sqrt{1+y^2} + c_1
$$

Which simplifies to

$$
-\sqrt{x^2+1} = \sqrt{1+y^2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\sqrt{y^2+1}}{y\sqrt{x^2+1}}$ 	$R = y$ $S = -\sqrt{x^2+1}$	$\frac{dS}{dR} = \frac{R}{\sqrt{R^2+1}}$ 

Summary

The solution(s) found are the following

$$-\sqrt{x^2+1} = \sqrt{1+y^2} + c_1 \tag{1}$$

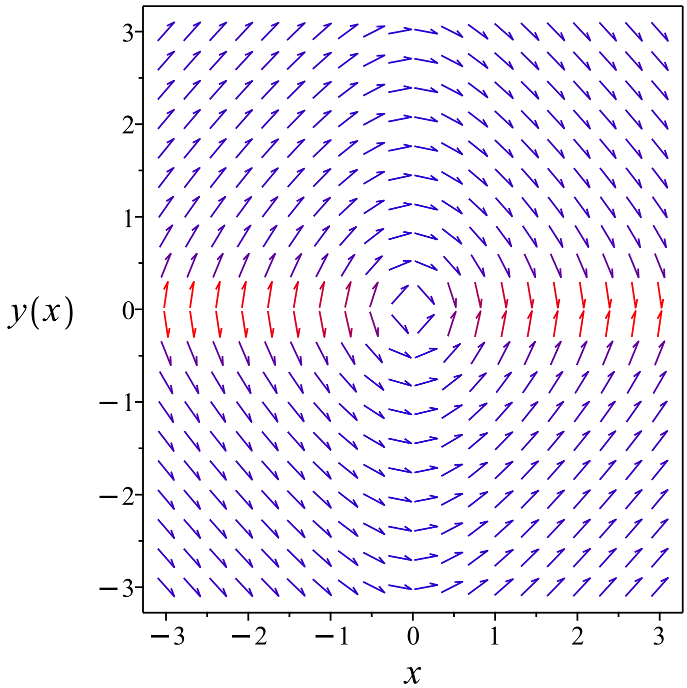


Figure 106: Slope field plot

Verification of solutions

$$-\sqrt{x^2+1} = \sqrt{1+y^2} + c_1$$

Verified OK.

4.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{y}{\sqrt{y^2+1}}\right) \mathrm{d}y &= \left(\frac{x}{\sqrt{x^2+1}}\right) \mathrm{d}x \\
\left(-\frac{x}{\sqrt{x^2+1}}\right) \mathrm{d}x + \left(-\frac{y}{\sqrt{y^2+1}}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{\sqrt{x^2+1}} \\
N(x, y) &= -\frac{y}{\sqrt{y^2+1}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{x^2+1}}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{\sqrt{y^2+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x\end{aligned}$$

$$\phi = -\sqrt{x^2+1} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{\sqrt{y^2+1}}$. Therefore equation (4) becomes

$$-\frac{y}{\sqrt{y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{\sqrt{y^2+1}}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{y}{\sqrt{y^2+1}}\right) \mathrm{d}y\\ f(y) &= -\sqrt{y^2+1} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sqrt{x^2+1} - \sqrt{y^2+1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sqrt{x^2+1} - \sqrt{y^2+1}$$

Summary

The solution(s) found are the following

$$-\sqrt{1+y^2} - \sqrt{x^2+1} = c_1 \tag{1}$$

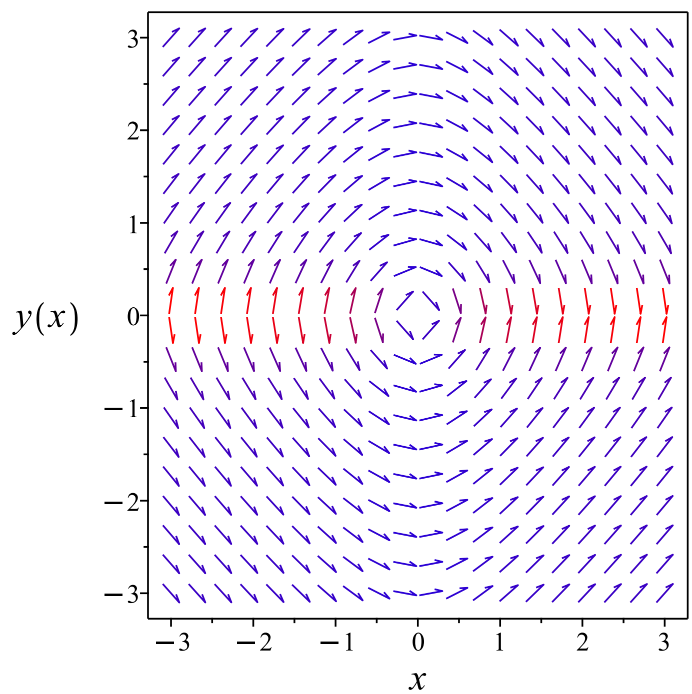


Figure 107: Slope field plot

Verification of solutions

$$-\sqrt{1+y^2} - \sqrt{x^2+1} = c_1$$

Verified OK.

4.5.4 Maple step by step solution

Let's solve

$x\sqrt{1+y^2}+yy'\sqrt{x^2+1}=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{yy'}{\sqrt{1+y^2}}=-\frac{x}{\sqrt{x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{yy'}{\sqrt{1+y^2}}dx=\int -\frac{x}{\sqrt{x^2+1}}dx+c_1$

- Evaluate integral

  $\sqrt{1+y^2}=-\sqrt{x^2+1}+c_1$

- Solve for y

  $\left\{y=\sqrt{x^2-2c_1\sqrt{x^2+1}+c_1^2}, y=-\sqrt{x^2-2c_1\sqrt{x^2+1}+c_1^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*sqrt(1+y(x)^2)+y(x)*diff(y(x),x)*sqrt(1+x^2)=0,y(x), singsol=all)
```

$$\sqrt{x^2+1}+\sqrt{1+y(x)^2}+c_1=0$$

✓ Solution by Mathematica

Time used: 0.29 (sec). Leaf size: 75

```
DSolve[x*Sqrt[1+y[x]^2]+y[x]*y'[x]*Sqrt[1+x^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x^2 + c_1 \left(-2\sqrt{x^2+1} + c_1\right)}$$
$$y(x) \to \sqrt{x^2 + c_1 \left(-2\sqrt{x^2+1} + c_1\right)}$$
$$y(x) \to -i$$
$$y(x) \to i$$

4.6 problem 51

Internal problem ID [14978]
Internal file name [`OUTPUT/14987_Monday_April_15_2024_12_04_44_AM_6549318/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\sqrt{1-y^2}+y\sqrt{-x^2+1}\,y'=0}$$

With initial conditions

$$[y(0)=1]$$

4.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}\right)\\ &= \frac{x}{\sqrt{-y^2+1}\,\sqrt{-x^2+1}} + \frac{x\sqrt{-y^2+1}}{y^2\sqrt{-x^2+1}}\end{aligned}$$

$\frac{\partial f}{\partial y}$ is not defined at $y = 1$ therefore existence and uniqueness theorem do not apply. Solution exist but not guaranteed to be unique.

4.6.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}\end{aligned}$$

Where $f(x) = -\frac{x}{\sqrt{-x^2+1}}$ and $g(y) = \frac{\sqrt{-y^2+1}}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy &= -\frac{x}{\sqrt{-x^2+1}}\,dx\\ \int\frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy &= \int -\frac{x}{\sqrt{-x^2+1}}\,dx\\ -\sqrt{-y^2+1} &= \sqrt{-x^2+1} + c_1\end{aligned}$$

The solution is

$$-\sqrt{1-y^2} - \sqrt{-x^2+1} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 - c_1 = 0$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$-\sqrt{-y^2+1} - \sqrt{-x^2+1} + 1 = 0$$

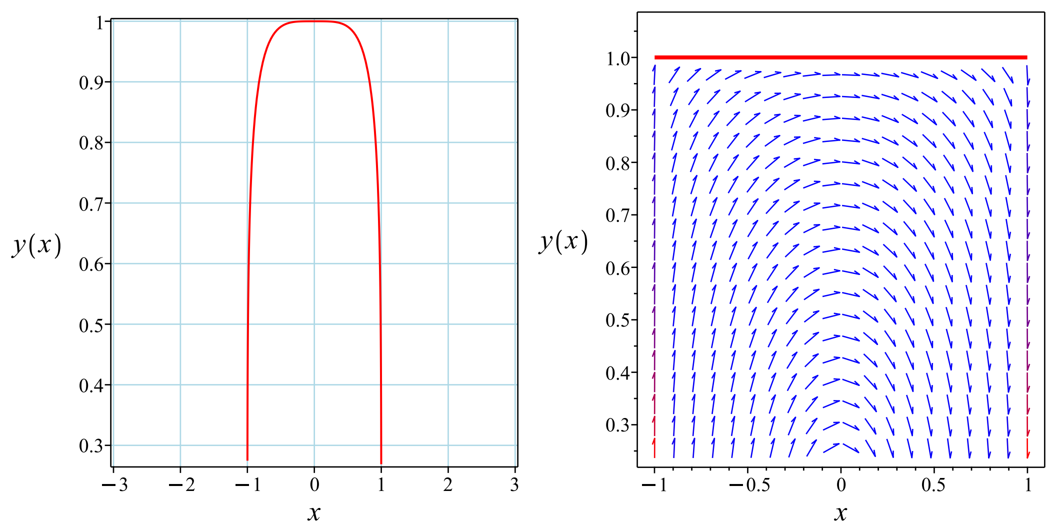
Solving for y from the above gives

$$y = \sqrt{x^2 - 1 + 2\sqrt{-x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 - 1 + 2\sqrt{-x^2+1}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{x^2 - 1 + 2\sqrt{-x^2+1}}$$

Verified OK.

4.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{\sqrt{-x^2+1}}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\sqrt{-x^2+1}}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{(x-1)(x+1)}{\sqrt{-x^2+1}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{x}{\sqrt{-x^2+1}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y}{\sqrt{-y^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R}{\sqrt{-R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(R-1)(R+1)}{\sqrt{-R^2+1}} + c_1 \tag{4}
$$

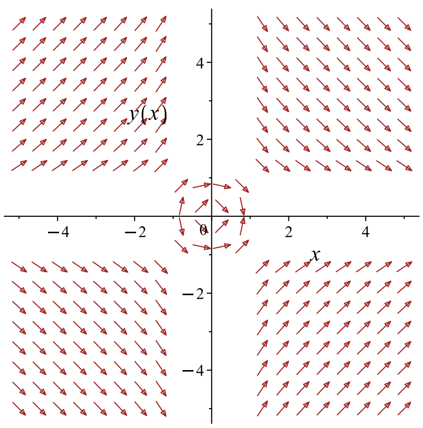
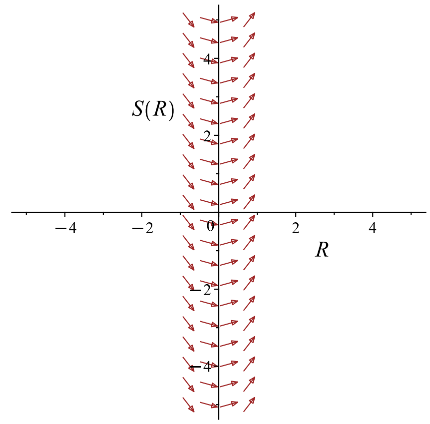
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sqrt{-x^2+1} = \frac{(y-1)(y+1)}{\sqrt{1-y^2}} + c_1
$$

Which simplifies to

$$
\sqrt{-x^2+1} = \frac{(y-1)(y+1)}{\sqrt{1-y^2}} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}$ 	$R = y$ $S = \sqrt{-x^2+1}$	$\frac{dS}{dR} = \frac{R}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$\sqrt{-x^2+1} = \frac{y^2 + \sqrt{-y^2+1} - 1}{\sqrt{-y^2+1}}$$

The above simplifies to

$$\sqrt{-y^2+1}\,\sqrt{-x^2+1} - y^2 - \sqrt{-y^2+1} + 1 = 0$$

Summary

The solution(s) found are the following

$$\left(\sqrt{-x^2+1} - 1\right)\sqrt{1-y^2} - y^2 + 1 = 0 \tag{1}$$

Verification of solutions

$$\left(\sqrt{-x^2+1} - 1\right)\sqrt{1-y^2} - y^2 + 1 = 0$$

Verified OK.

4.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y}{\sqrt{-y^2+1}}\right)\mathrm{d}y = \left(\frac{x}{\sqrt{-x^2+1}}\right)\mathrm{d}x$$
$$\left(-\frac{x}{\sqrt{-x^2+1}}\right)\mathrm{d}x + \left(-\frac{y}{\sqrt{-y^2+1}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{\sqrt{-x^2+1}} \\
N(x, y) &= -\frac{y}{\sqrt{-y^2+1}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{-x^2+1}}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{\sqrt{-y^2+1}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x}{\sqrt{-x^2+1}} \,\mathrm{d}x \\
\phi &= \sqrt{-x^2+1} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$-\frac{y}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{y}{\sqrt{-y^2+1}}\right) \mathrm{d}y$$

$$f(y) = -\frac{(y-1)(y+1)}{\sqrt{-y^2+1}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{-y^2+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{-y^2+1}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$\sqrt{-x^2+1} - \frac{(y-1)(y+1)}{\sqrt{-y^2+1}} = 1$$

The above simplifies to

$$\sqrt{-y^2+1}\,\sqrt{-x^2+1}-y^2-\sqrt{-y^2+1}+1=0$$

Summary

The solution(s) found are the following

$$\left(\sqrt{-x^2+1}-1\right)\sqrt{1-y^2}-y^2+1=0 \tag{1}$$

Verification of solutions

$$\left(\sqrt{-x^2+1}-1\right)\sqrt{1-y^2}-y^2+1=0$$

Verified OK.

4.6.5 Maple step by step solution

Let's solve

$\left[x\sqrt{1-y^2}+y\sqrt{-x^2+1}\,y'=0, y(0)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{1-y^2}}=-\frac{x}{\sqrt{-x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{1-y^2}}dx=\int -\frac{x}{\sqrt{-x^2+1}}dx+c_1$

- Evaluate integral

  $-\sqrt{1-y^2}=-\frac{(x-1)(x+1)}{\sqrt{-x^2+1}}+c_1$

- Solve for y

  $\left\{y=\sqrt{-2c_1\sqrt{-x^2+1}-c_1^2+x^2}, y=-\sqrt{-2c_1\sqrt{-x^2+1}-c_1^2+x^2}\right\}$

- Use initial condition $y(0)=1$

  $1=\sqrt{-c_1^2-2c_1}$

- Solve for c_1

  $c_1=-1$

- Substitute $c_1=-1$ into general solution and simplify

$$y = \sqrt{x^2 - 1 + 2\sqrt{-x^2 + 1}}$$

- Use initial condition $y(0) = 1$

$$1 = -\sqrt{-c_1^2 - 2c_1}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = \sqrt{x^2 - 1 + 2\sqrt{-x^2 + 1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([x*sqrt(1-y(x)^2)+y(x)*sqrt(1-x^2)*diff(y(x),x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 3.582 (sec). Leaf size: 32

```
DSolve[{x*Sqrt[1-y[x]^2]+y[x]*Sqrt[1-x^2]*y'[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolution
```

$$y(x) \to 1$$
$$y(x) \to \sqrt{x^2 + 2\sqrt{1 - x^2} - 1}$$

4.7 problem 52

Internal problem ID [14979]
Internal file name [`OUTPUT/14988_Monday_April_15_2024_12_04_46_AM_7008515/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{e^{-y}y' = 1}$$

4.7.1 Solving as quadrature ode

Integrating both sides gives

$$\int e^{-y}dy = x + c_1$$
$$-e^{-y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = -\ln(-x - c_1)$$

Summary
The solution(s) found are the following

$$y = -\ln(-x - c_1) \tag{1}$$

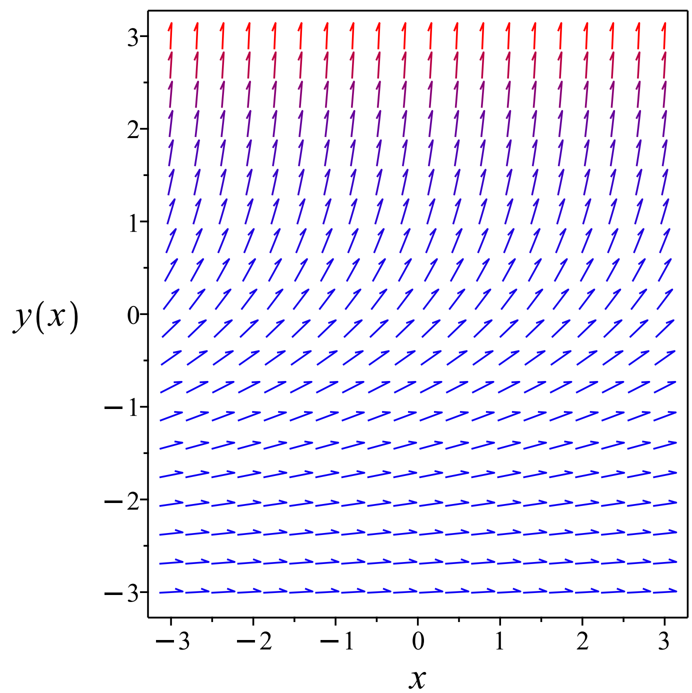


Figure 109: Slope field plot

Verification of solutions

$$y = -\ln(-x - c_1)$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$e^{-y}y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int e^{-y}y'dx = \int 1dx + c_1$

- Evaluate integral

  $-e^{-y} = x + c_1$

- Solve for y

  $y = -\ln(-x - c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(exp(-y(x))*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{c_1 + x}\right)$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 16

```
DSolve[Exp[-y[x]]*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(-x - c_1)$$

4.8 problem 53

Internal problem ID [14980]
Internal file name [`OUTPUT/14989_Monday_April_15_2024_12_04_46_AM_80955332/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 53.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\ln(y)\, y + y'x = 1}$$

With initial conditions

$$[y(1) = 1]$$

4.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{\ln(y)\, y - 1}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\ln(y)\,y - 1}{x}\right)\\ &= -\frac{1 + \ln(y)}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

4.8.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{-\ln(y)\,y + 1}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = -\ln(y)\,y + 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-\ln(y)\,y + 1}\,dy &= \frac{1}{x}\,dx\\ \int \frac{1}{-\ln(y)\,y + 1}\,dy &= \int \frac{1}{x}\,dx\\ \int^{y} \frac{1}{-\ln(\_a)\,\_a + 1}d\_a &= \ln(x) + c_1\end{aligned}$$

Which results in

$$\int^{y} \frac{1}{-\ln (\_a) \_a+1} d\_a = \ln (x) + c_1$$

The solution is

$$\int^{y} \frac{1}{-\ln (\_a) \_a+1} d\_a - \ln (x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\int^{1} -\frac{1}{\ln (\_a) \_a-1} d\_a - c_1 = 0$$

$$c_1 = -\left(\int^{1} \frac{1}{\ln (\_a) \_a-1} d\_a\right)$$

Substituting c_1 found above in the general solution gives

$$\int^{y} -\frac{1}{\ln (\_a) \_a-1} d\_a - \ln (x) + \int^{1} \frac{1}{\ln (\_a) \_a-1} d\_a = 0$$

Solving for y from the above gives

$$y = \operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\ln (\_a) \_a-1} d\_a + \ln (x) - \left(\int^{1} \frac{1}{\ln (\_a) \_a-1} d\_a\right)\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\ln (\_a) \_a-1} d\_a + \ln (x) - \left(\int^{1} \frac{1}{\ln (\_a) \_a-1} d\_a\right)\right) \tag{1}$$

Verification of solutions

$$y = \operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\ln (\_a) \_a-1} d\_a + \ln (x) - \left(\int^{1} \frac{1}{\ln (\_a) \_a-1} d\_a\right)\right)$$

Verified OK.

4.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\ln(y)\, y - 1}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 80: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x} dx \end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\ln(y)\, y - 1}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{\ln(y)\, y - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{\ln(R)\, R - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int -\frac{1}{\ln(R)\, R - 1} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int^{y} -\frac{1}{\ln(\_a)\_a - 1} d\_a + c_1$$

Which simplifies to

$$\ln(x) = \int^{y} -\frac{1}{\ln(\_a)\_a - 1} d\_a + c_1$$

This results in

$$\ln(x) = \int^{y} -\frac{1}{\ln(\_a)\_a - 1} d\_a + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \int^{1} -\frac{1}{\ln(\_a)\_a - 1} d\_a + c_1$$

$$c_1 = \int^{1} \frac{1}{\ln(\_a)\_a - 1} d\_a$$

Substituting c_1 found above in the general solution gives

$$\ln(x) = \int^{y} -\frac{1}{\ln(\_a)\_a - 1} d\_a + \int^{1} \frac{1}{\ln(\_a)\_a - 1} d\_a$$

Solving for y from the above gives

$$y = \text{RootOf}\left(\ln(x) + \int^{\_Z} \frac{1}{\ln(\_a)\_a - 1} d\_a - \left(\int^{1} \frac{1}{\ln(\_a)\_a - 1} d\_a\right)\right)$$

Summary

The solution(s) found are the following

$$y = \text{RootOf}\left(\ln(x) + \int^{\_Z} \frac{1}{\ln(\_a)\_a - 1} d\_a - \left(\int^{1} \frac{1}{\ln(\_a)\_a - 1} d\_a\right)\right) \tag{1}$$

Verification of solutions

$$y = \text{RootOf}\left(\ln(x) + \int^{\_Z} \frac{1}{\ln(\_a)\_a - 1} d\_a - \left(\int^{1} \frac{1}{\ln(\_a)\_a - 1} d\_a\right)\right)$$

Verified OK.

4.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-\ln(y)\,y+1}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$

$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{-\ln(y)\,y+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{1}{x} \\ N(x,y) &= \frac{1}{-\ln(y)\,y+1} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-\ln(y)\,y+1}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{x} \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-\ln(y)y+1}$. Therefore equation (4) becomes

$$\frac{1}{-\ln(y)\, y + 1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\ln(y)\, y - 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{1}{\ln(y)\, y - 1}\right) \mathrm{d}y$$
$$f(y) = \int_0^y -\frac{1}{\ln(\_a)\_a - 1} d\_a + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \int_0^y -\frac{1}{\ln(\_a)\_a - 1} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \int_0^y -\frac{1}{\ln(\_a)\_a - 1} d\_a$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\int_0^1 -\frac{1}{\ln(\_a)\_a - 1} d\_a = c_1$$

$$c_1 = -\left(\int_0^1 \frac{1}{\ln(\_a)\_a - 1} d\_a\right)$$

Substituting c_1 found above in the general solution gives

$$-\ln(x) + \int_0^y -\frac{1}{\ln(\_a)\_a - 1} d\_a = -\left(\int_0^1 \frac{1}{\ln(\_a)\_a - 1} d\_a\right)$$

Summary

The solution(s) found are the following

$$-\ln(x) - \left(\int_0^y \frac{1}{\ln(\_a)\_a - 1} d\_a\right) = -\left(\int_0^1 \frac{1}{\ln(\_a)\_a - 1} d\_a\right) \tag{1}$$

Verification of solutions

$$-\ln(x) - \left(\int_0^y \frac{1}{\ln(\_a)\_a - 1} d\_a\right) = -\left(\int_0^1 \frac{1}{\ln(\_a)\_a - 1} d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.922 (sec). Leaf size: 38

```
dsolve([y(x)*ln(y(x))+x*diff(y(x),x)=1,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \mathrm{RootOf}\left(\int_{1}^{\_Z}\frac{1}{\ln(\_a)\_a - 1}d\_a + \ln(x)\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y[x]*Log[y[x]]+x*y'[x]==1,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

{}

4.9 problem 54

Internal problem ID [14981]
Internal file name [`OUTPUT/14990_Monday_April_15_2024_12_04_49_AM_41359583/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 54.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - a^{y+x} = 0}$$

4.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= a^y a^x \end{aligned}$$

Where $f(x) = a^x$ and $g(y) = a^y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{a^y}\,dy &= a^x\,dx \\ \int \frac{1}{a^y}\,dy &= \int a^x\,dx \\ -\frac{a^{-y}}{\ln(a)} &= \frac{a^x}{\ln(a)} + c_1 \end{aligned}$$

The solution is

$$-\frac{a^{-y}}{\ln(a)}-\frac{a^{x}}{\ln(a)}-c_1=0$$

Summary

The solution(s) found are the following

$$-\frac{a^{-y}}{\ln(a)}-\frac{a^{x}}{\ln(a)}-c_1=0 \tag{1}$$

Verification of solutions

$$-\frac{a^{-y}}{\ln(a)}-\frac{a^{x}}{\ln(a)}-c_1=0$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=a^{y+x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= a^{-x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{a^{-x}} dx \end{aligned}$$

Which results in

$$S = \frac{a^x}{\ln(a)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = a^{y+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= a^x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = a^{-y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = a^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{a^{-R}}{\ln(a)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{a^x}{\ln(a)} = -\frac{a^{-y}}{\ln(a)} + c_1$$

Which simplifies to

$$\frac{a^x + a^{-y} - c_1 \ln(a)}{\ln(a)} = 0$$

Which gives

$$y = -\frac{\ln\left(-e^{\ln(a)x} + c_1 \ln(a)\right)}{\ln(a)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln\left(-e^{\ln(a)x} + c_1 \ln(a)\right)}{\ln(a)} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln\left(-e^{\ln(a)x} + c_1 \ln(a)\right)}{\ln(a)}$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(a^{-y})\,\mathrm{d}y&=(a^{x})\,\mathrm{d}x\\ (-a^{x})\,\mathrm{d}x+(a^{-y})\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-a^{x}\\ N(x,y)&=a^{-y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-a^{x})\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(a^{-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -a^{x} \mathrm{d}x$$

$$\phi = -\frac{a^{x}}{\ln(a)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = a^{-y}$. Therefore equation (4) becomes

$$a^{-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = a^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(a^{-y}\right) \mathrm{d}y$$

$$f(y) = -\frac{a^{-y}}{\ln(a)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{a^x}{\ln(a)} - \frac{a^{-y}}{\ln(a)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{a^x}{\ln(a)} - \frac{a^{-y}}{\ln(a)}$$

The solution becomes

$$y = -\frac{\ln\left(-c_1 \ln(a) - e^{\ln(a)x}\right)}{\ln(a)}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln\left(-c_1 \ln(a) - e^{\ln(a)x}\right)}{\ln(a)} \tag{1}$$

Verification of solutions

$$y = -\frac{\ln\left(-c_1 \ln(a) - e^{\ln(a)x}\right)}{\ln(a)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve(diff(y(x),x)=a^(x+y(x)),y(x), singsol=all)
```

$$y(x) = \frac{\ln\left(-\frac{1}{c_1 \ln(a)+a^x}\right)}{\ln(a)}$$

✓ Solution by Mathematica

Time used: 3.796 (sec). Leaf size: 24

```
DSolve[y'[x]==a^(x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\log(-a^x - c_1 \log(a))}{\log(a)}$$

4.10 problem 55

Internal problem ID [14982]
Internal file name [`OUTPUT/14991_Monday_April_15_2024_12_04_50_AM_88899256/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 55.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{y}\left(x^{2}+1\right) y'-2x(e^{y}+1)=0}$$

4.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{2x(e^{-y}+1)}{x^{2}+1} \end{aligned}$$

Where $f(x)=\frac{2x}{x^{2}+1}$ and $g(y)=e^{-y}+1$. Integrating both sides gives

$$\frac{1}{e^{-y}+1}\,dy = \frac{2x}{x^{2}+1}\,dx$$

$$\int \frac{1}{e^{-y}+1}\,dy = \int \frac{2x}{x^{2}+1}\,dx$$

$$\ln\left(\mathrm{e}^{-y}+1\right)-\ln\left(\mathrm{e}^{-y}\right)=\ln\left(x^2+1\right)+c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln\left(\mathrm{e}^{-y}+1\right)-\ln\left(\mathrm{e}^{-y}\right)}=\mathrm{e}^{\ln\left(x^2+1\right)+c_1}$$

Which simplifies to

$$1+\mathrm{e}^{y}=c_2\left(x^2+1\right)$$

Summary

The solution(s) found are the following

$$y=\ln\left(c_2x^2+c_2-1\right) \tag{1}$$

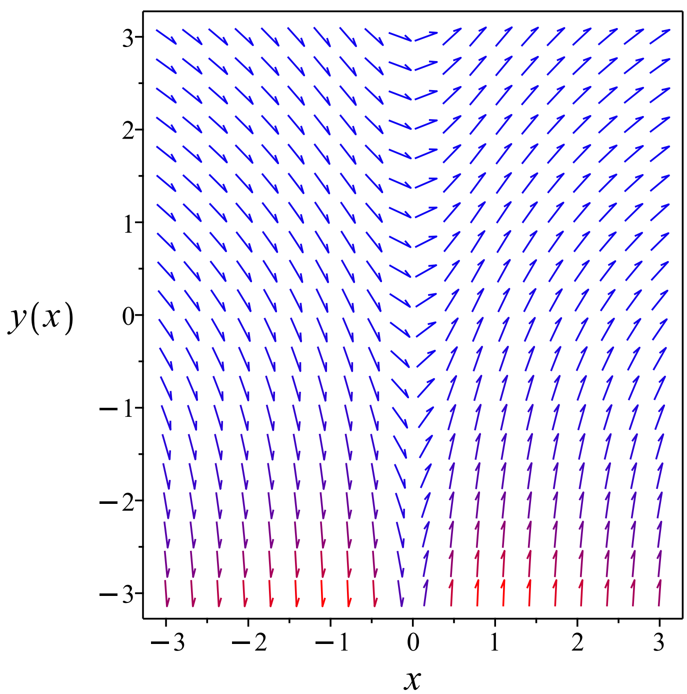


Figure 110: Slope field plot

Verification of solutions

$$y=\ln\left(c_2x^2+c_2-1\right)$$

Verified OK.

4.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2x(1+e^{y})\,e^{-y}}{x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2+1}{2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2+1}{2x}} dx
\end{aligned}
$$

Which results in

$$
S = \ln\left(x^2+1\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{2x(1+e^{y})\,e^{-y}}{x^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{2x}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{e^y}{1+e^y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{e^R}{1+e^R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln\left(1+e^R\right) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln\left(x^2+1\right) = \ln\left(e^y+1\right) + c_1
$$

Which simplifies to

$$
\ln\left(x^2+1\right) = \ln\left(e^y+1\right) + c_1
$$

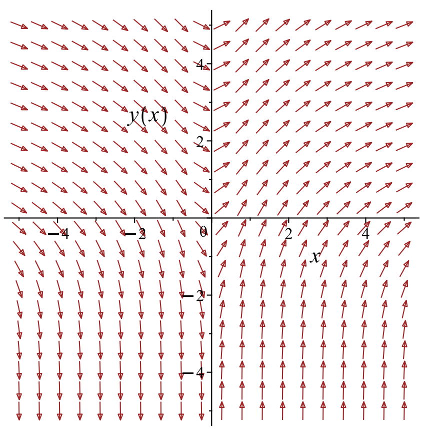
Which gives

$$
y = \ln\left(1+x^2-e^{c_1}\right) - c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2x(1+e^{y})e^{-y}}{x^2+1}$



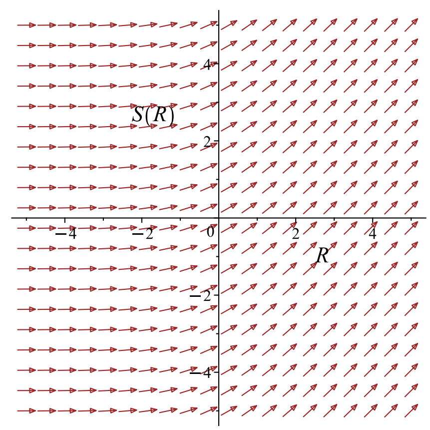
</td><td>

$R = y$

$S = \ln(x^2 + 1)$

</td><td>

$\frac{dS}{dR} = \frac{e^{R}}{1+e^{R}}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \ln(1 + x^2 - e^{c_1}) - c_1 \quad (1)$$

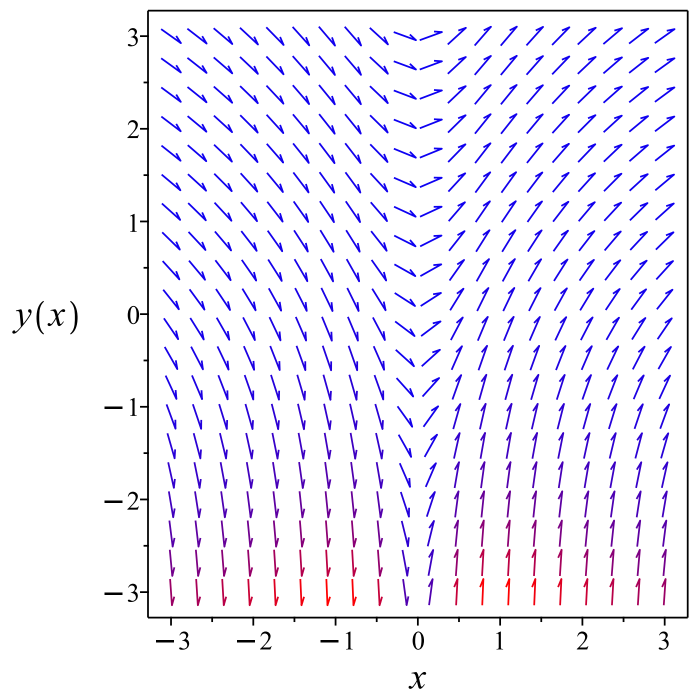


Figure 111: Slope field plot

Verification of solutions

$$y = \ln\left(1 + x^2 - e^{c_1}\right) - c_1$$

Verified OK.

4.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}\right) \mathrm{d}y &= \left(\frac{x}{x^{2}+1}\right) \mathrm{d}x \\
\left(-\frac{x}{x^{2}+1}\right) \mathrm{d}x + \left(\frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{x^{2}+1} \\
N(x, y) &= \frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^{2}+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^{2}+1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln\left(x^{2}+1\right)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}$. Therefore equation (4) becomes

$$
\frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{\mathrm{e}^{y}}{2+2\,\mathrm{e}^{y}}\right)\mathrm{d}y\\
f(y) &= \frac{\ln\left(1+\mathrm{e}^{y}\right)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{\ln(1+e^y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\ln(1+e^y)}{2}$$

The solution becomes

$$y = \ln\left(x^2 e^{2c_1} + e^{2c_1} - 1\right)$$

Summary

The solution(s) found are the following

$$y = \ln\left(x^2 e^{2c_1} + e^{2c_1} - 1\right) \tag{1}$$

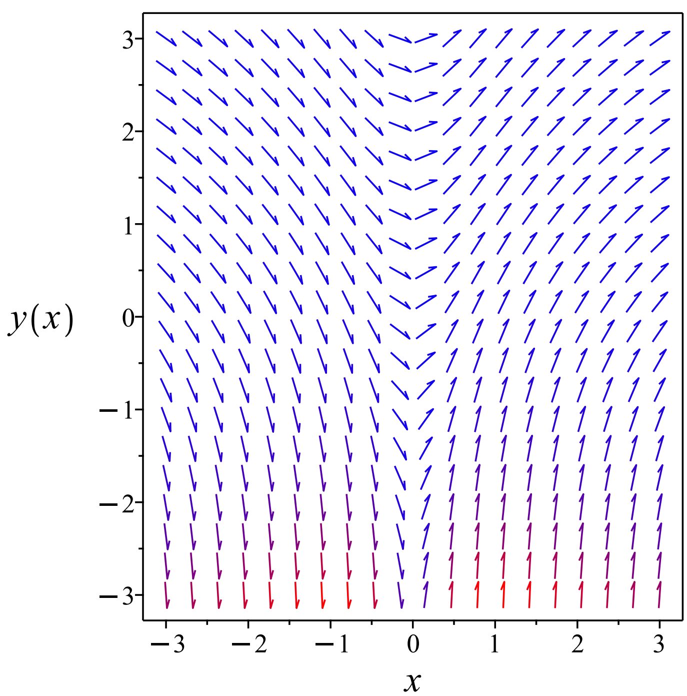


Figure 112: Slope field plot

Verification of solutions

$$y = \ln\left(x^2 e^{2c_1} + e^{2c_1} - 1\right)$$

Verified OK.

4.10.4 Maple step by step solution

Let's solve

$e^y (x^2+1)\, y' - 2x(e^y+1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' e^y}{e^y+1} = \frac{2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y' e^y}{e^y+1} dx = \int \frac{2x}{x^2+1} dx + c_1$

- Evaluate integral

  $\ln(e^y+1) = \ln(x^2+1) + c_1$

- Solve for y

  $y = \ln(x^2 e^{c_1} + e^{c_1} - 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve(exp(y(x))*(1+x^2)*diff(y(x),x)-2*x*(1+exp(y(x)))=0,y(x), singsol=all)
```

$$y(x) = \ln\left(c_1 x^2 + c_1 - 1\right)$$

✓ Solution by Mathematica

Time used: 0.638 (sec). Leaf size: 27

```
DSolve[Exp[y[x]]*(1+x^2)*y'[x]-2*x*(1+Exp[y[x]])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(-1 + e^{c_1}\left(x^2 + 1\right)\right)$$
$$y(x) \to i\pi$$

4.11 problem 56

Internal problem ID [14983]
Internal file name [`OUTPUT/14992_Monday_April_15_2024_12_04_51_AM_9686249/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 56.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2x\sqrt{1-y^2}-\left(x^2+1\right)y'=0}$$

4.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{2x\sqrt{-y^2+1}}{x^2+1} \end{aligned}$$

Where $f(x)=\frac{2x}{x^2+1}$ and $g(y)=\sqrt{-y^2+1}$. Integrating both sides gives

$$\frac{1}{\sqrt{-y^2+1}}\,dy=\frac{2x}{x^2+1}\,dx$$

$$\int \frac{1}{\sqrt{-y^2+1}}\, dy = \int \frac{2x}{x^2+1}\, dx$$

$$\arcsin(y) = \ln(x^2+1) + c_1$$

Which results in

$$y = \sin(\ln(x^2+1) + c_1)$$

Summary

The solution(s) found are the following

$$y = \sin(\ln(x^2+1) + c_1) \tag{1}$$

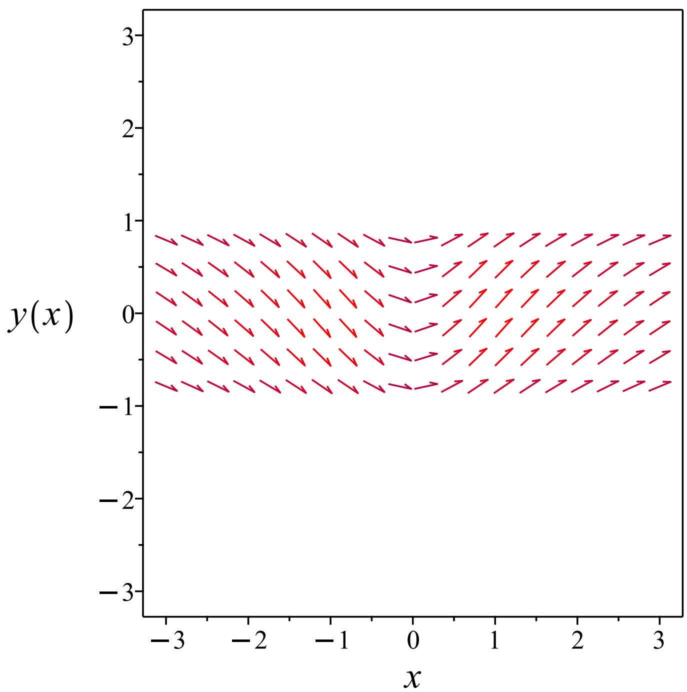


Figure 113: Slope field plot

Verification of solutions

$$y = \sin(\ln(x^2+1) + c_1)$$

Verified OK.

4.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2x\sqrt{-y^2+1}}{x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 87: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2+1}{2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2+1}{2x}} dx
\end{aligned}
$$

Which results in

$$
S = \ln\left(x^2+1\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{2x\sqrt{-y^2+1}}{x^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{2x}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-y^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \arcsin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln\left(x^2+1\right) = \arcsin(y) + c_1
$$

Which simplifies to

$$
\ln\left(x^2+1\right) = \arcsin(y) + c_1
$$

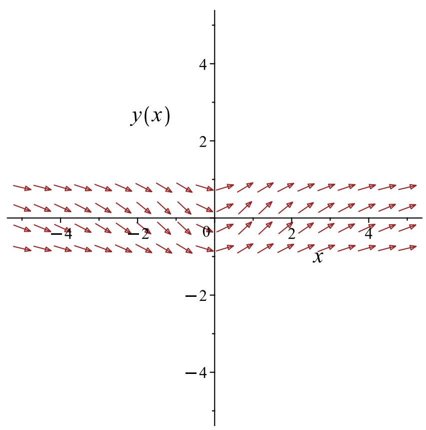
Which gives

$$
y = -\sin\left(-\ln\left(x^2+1\right) + c_1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{2x\sqrt{-y^2+1}}{x^2+1}$



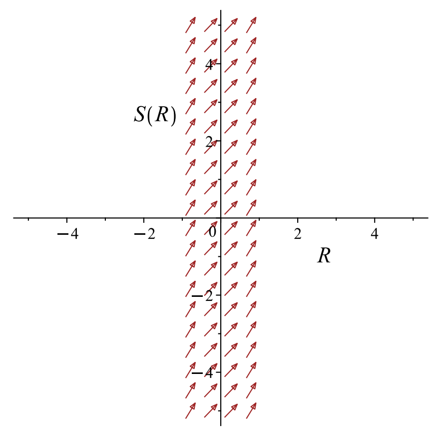
</td><td>

$R = y$

$S = \ln\left(x^2+1\right)$

</td><td>

$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\sin\left(-\ln\left(x^2+1\right)+c_1\right) \tag{1}$$

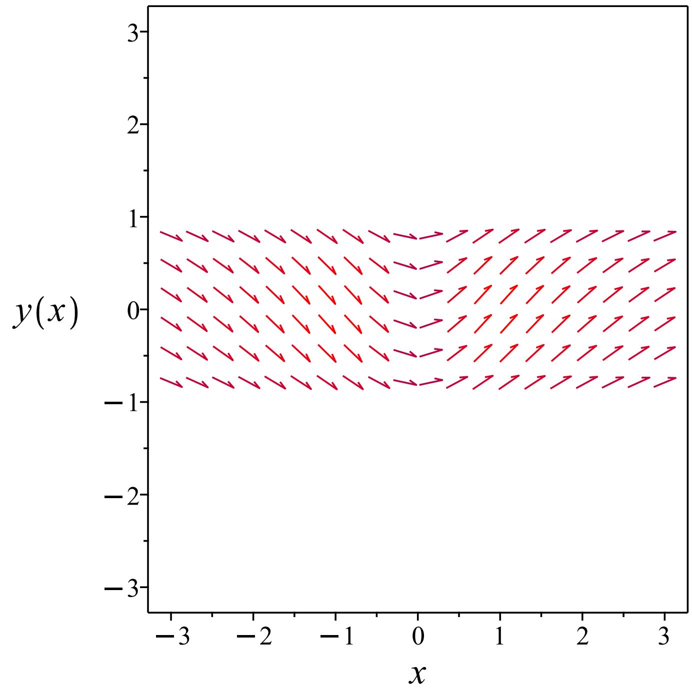


Figure 114: Slope field plot

Verification of solutions

$$y = -\sin\left(-\ln\left(x^2+1\right)+c_1\right)$$

Verified OK.

4.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2\sqrt{-y^2+1}}\right) \mathrm{d}y = \left(\frac{x}{x^2+1}\right) \mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right) \mathrm{d}x + \left(\frac{1}{2\sqrt{-y^2+1}}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{x^2+1}$$
$$N(x, y) = \frac{1}{2\sqrt{-y^2+1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{-y^2+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{x}{x^2+1}\, \mathrm{d}x\\ \phi &= -\frac{\ln(x^2+1)}{2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{2\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$
\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2\sqrt{-y^2+1}}\right)\mathrm{d}y
$$

$$
f(y) = \frac{\arcsin(y)}{2} + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{\ln(x^2+1)}{2} + \frac{\arcsin(y)}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\arcsin(y)}{2}
$$

The solution becomes

$$
y = \sin\left(2c_1 + \ln\left(x^2+1\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = \sin\left(2c_1 + \ln\left(x^2+1\right)\right) \tag{1}
$$

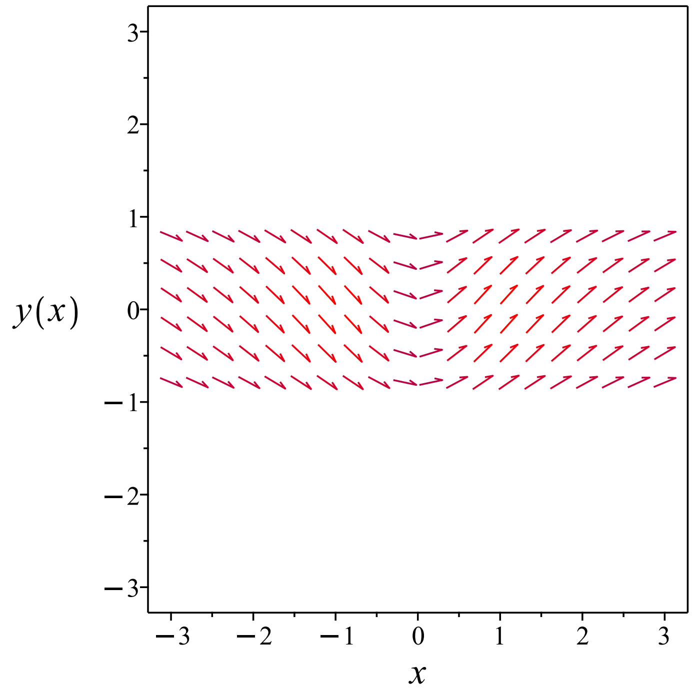


Figure 115: Slope field plot

Verification of solutions

$$y = \sin\left(2c_1 + \ln\left(x^2 + 1\right)\right)$$

Verified OK.

4.11.4 Maple step by step solution

Let's solve

$2x\sqrt{1 - y^2} - (x^2 + 1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = \frac{2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx = \int \frac{2x}{x^2+1}dx + c_1$

- Evaluate integral

$$\arcsin(y) = \ln(x^2+1) + c_1$$

- Solve for y

$$y = \sin(\ln(x^2+1) + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(2*x*sqrt(1-y(x)^2)=diff(y(x),x)*(1+x^2),y(x), singsol=all)
```

$$y(x) = \sin\left(\ln\left(x^2+1\right) + 2c_1\right)$$

✓ Solution by Mathematica

Time used: 0.271 (sec). Leaf size: 33

```
DSolve[2*x*Sqrt[1-y[x]^2]==y'[x]*(1+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to \cos\left(\log\left(x^2+1\right) + c_1\right) \\ y(x) &\to -1 \\ y(x) &\to 1 \\ y(x) &\to \text{Interval}[\{-1,1\}] \end{aligned}$$

4.12 problem 57

Internal problem ID [14984]
Internal file name [`OUTPUT/14993_Monday_April_15_2024_12_04_52_AM_36276539/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 57.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{x} \sin(y)^{3}+\left(e^{2 x}+1\right) \cos(y) y'=0}$$

4.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= -\frac{e^{x} \sin(y)^{2} \tan(y)}{e^{2 x}+1} \end{aligned}$$

Where $f(x)=-\frac{e^{x}}{e^{2x}+1}$ and $g(y)=\sin(y)^{2}\tan(y)$. Integrating both sides gives

$$\frac{1}{\sin(y)^{2} \tan(y)}\, dy = -\frac{e^{x}}{e^{2 x}+1}\, dx$$

$$\int \frac{1}{\sin(y)^2 \tan(y)}\, dy = \int -\frac{e^x}{e^{2x}+1}\, dx$$
$$-\frac{1}{2\tan(y)^2} = -\arctan(e^x) + c_1$$

Which results in

$$y = \arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{-c_1+\arctan(e^x)}}}{2}\right)$$
$$y = -\arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{-c_1+\arctan(e^x)}}}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{-c_1+\arctan(e^x)}}}{2}\right) \tag{1}$$
$$y = -\arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{-c_1+\arctan(e^x)}}}{2}\right) \tag{2}$$

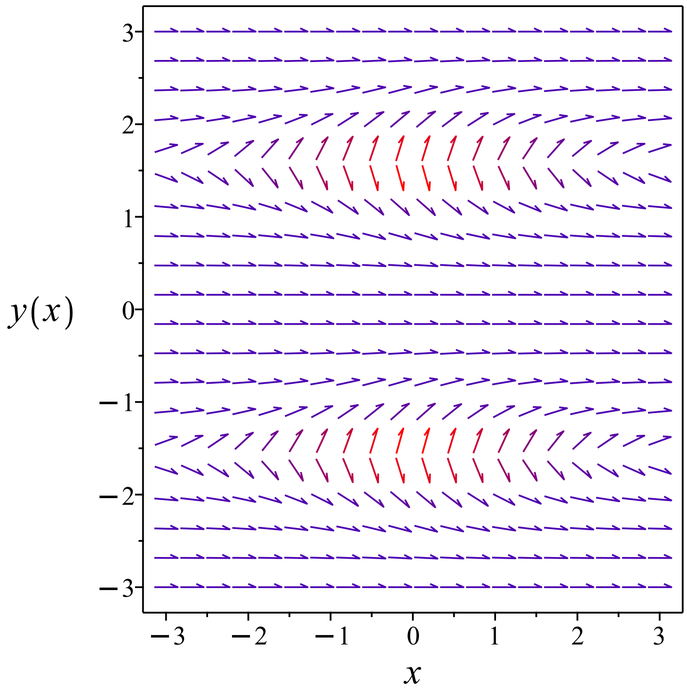


Figure 116: Slope field plot

Verification of solutions

$$y = \arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{-c_1+\arctan(e^x)}}}{2}\right)$$

Verified OK.

$$y = -\arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{-c_1+\arctan(e^x)}}}{2}\right)$$

Verified OK.

4.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^x \sin(y)^3}{(e^{2x}+1)\cos(y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 90: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= -\left(e^{2x} + 1\right) e^{-x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\left(e^{2x}+1\right) e^{-x}} dx \end{aligned}$$

Which results in

$$S = -\arctan\left(e^{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{e^{x} \sin(y)^{3}}{\left(e^{2x}+1\right) \cos(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{e^{x}}{e^{2x}+1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \csc(y)^{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)\csc(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cot(R)^2}{2} + c_1 \tag{4}$$

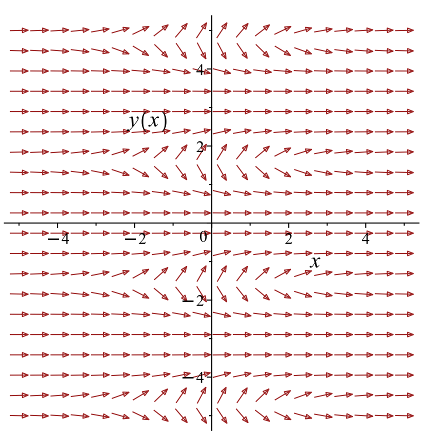
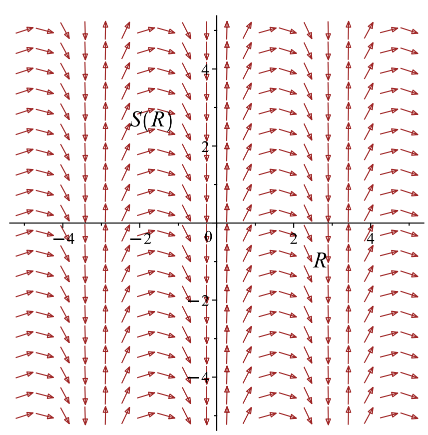
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\arctan(e^x) = -\frac{\cot(y)^2}{2} + c_1$$

Which simplifies to

$$-\arctan(e^x) = -\frac{\cot(y)^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^x \sin(y)^3}{(e^{2x}+1)\cos(y)}$ 	$R = y$ $S = -\arctan(e^x)$	$\frac{dS}{dR} = \cot(R)\csc(R)^2$ 

Summary

The solution(s) found are the following

$$-\arctan\left(\mathrm{e}^{x}\right) = -\frac{\cot\left(y\right)^{2}}{2}+c_{1} \tag{1}$$

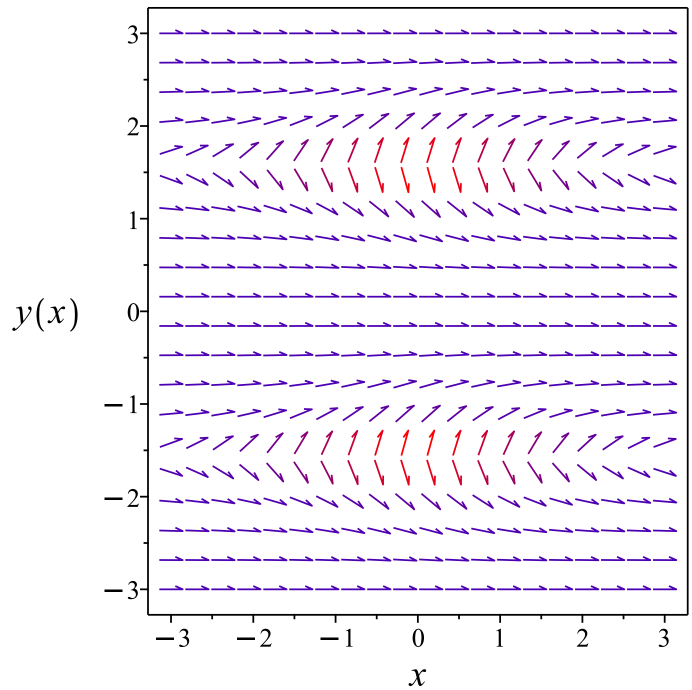


Figure 117: Slope field plot

Verification of solutions

$$-\arctan\left(\mathrm{e}^{x}\right) = -\frac{\cot\left(y\right)^{2}}{2}+c_{1}$$

Verified OK.

4.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{\cos(y)}{\sin(y)^3}\right)\mathrm{d}y&=\left(\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1}\right)\mathrm{d}x\\ \left(-\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1}\right)\mathrm{d}x+\left(-\frac{\cos(y)}{\sin(y)^3}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1}\\ N(x,y)&=-\frac{\cos(y)}{\sin(y)^3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\cos(y)}{\sin(y)^3}\right)\\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1}\, \mathrm{d}x
\end{aligned}$$

$$\phi = -\arctan(\mathrm{e}^x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\cos(y)}{\sin(y)^3}$. Therefore equation (4) becomes

$$-\frac{\cos(y)}{\sin(y)^3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\cos(y)}{\sin(y)^3}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(-\cot(y)\csc(y)^2\right) \mathrm{d}y\\
f(y) &= \frac{\cot(y)^2}{2} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan\left(e^{x}\right)+\frac{\cot\left(y\right)^{2}}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan\left(e^{x}\right)+\frac{\cot\left(y\right)^{2}}{2}$$

Summary

The solution(s) found are the following

$$-\arctan\left(e^{x}\right)+\frac{\cot\left(y\right)^{2}}{2} = c_1 \tag{1}$$

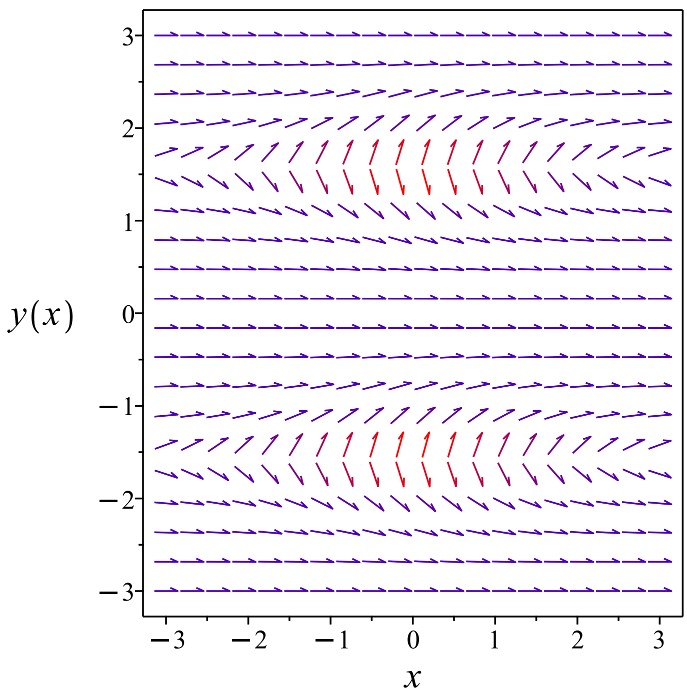


Figure 118: Slope field plot

Verification of solutions

$$-\arctan\left(e^{x}\right)+\frac{\cot\left(y\right)^{2}}{2} = c_1$$

Verified OK.

4.12.4 Maple step by step solution

Let's solve

$e^{x} \sin \left(y\right)^{3}+\left(e^{2 x}+1\right) \cos \left(y\right) y^{\prime}=0$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- Separate variables

  $\frac{y^{\prime} \cos\left(y\right)}{\sin\left(y\right)^{3}}=-\frac{e^{x}}{e^{2 x}+1}$

- Integrate both sides with respect to x

  $\int \frac{y^{\prime} \cos\left(y\right)}{\sin\left(y\right)^{3}}d x =\int -\frac{e^{x}}{e^{2 x}+1}d x +c_{1}$

- Evaluate integral

  $-\frac{1}{2 \sin\left(y\right)^{2}}=-\arctan \left(e^{x}\right)+c_{1}$

- Solve for y

  $\left\{y=-\arcsin \left(\frac{\sqrt{2}\, \sqrt{\frac{1}{-c_{1}+\arctan\left(e^{x}\right)}}}{2}\right), y=\arcsin \left(\frac{\sqrt{2}\, \sqrt{\frac{1}{-c_{1}+\arctan\left(e^{x}\right)}}}{2}\right)\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(exp(x)*sin(y(x))^3+(1+exp(2*x))*cos(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{c_1+\arctan(\mathrm{e}^x)}}}{2}\right)$$

$$y(x) = -\arctan\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{c_1+\arctan(\mathrm{e}^x)}}}{2}\right)$$

✓ Solution by Mathematica

Time used: 1.83 (sec). Leaf size: 56

```
DSolve[Exp[x]*Sin[y[x]]^3+(1+Exp[2*x])*Cos[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\csc^{-1}\left(\sqrt{2}\sqrt{\arctan\left(e^x\right)-4c_1}\right)$$

$$y(x) \to \csc^{-1}\left(\sqrt{2}\sqrt{\arctan\left(e^x\right)-4c_1}\right)$$

$$y(x) \to 0$$

4.13 problem 58

Internal problem ID [14985]
Internal file name [`OUTPUT/14994_Monday_April_15_2024_12_04_55_AM_73642098/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 58.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\sin(x)\, y^2 + \cos(x)^2 \ln(y)\, y' = 0}$$

4.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\sin(x)\, y^2}{\cos(x)^2 \ln(y)} \end{aligned}$$

Where $f(x) = -\frac{\sin(x)}{\cos(x)^2}$ and $g(y) = \frac{y^2}{\ln(y)}$. Integrating both sides gives

$$\frac{1}{\frac{y^2}{\ln(y)}}\, dy = -\frac{\sin(x)}{\cos(x)^2}\, dx$$

$$\int \frac{1}{\frac{y^2}{\ln(y)}}dy = \int -\frac{\sin(x)}{\cos(x)^2}dx$$

$$-\frac{\ln(y)}{y} - \frac{1}{y} = -\frac{1}{\cos(x)} + c_1$$

Which results in

$$y = e^{-\operatorname{LambertW}\left(\frac{(c_1\cos(x)-1)e^{-1}}{\cos(x)}\right)-1}$$

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(\frac{(c_1\cos(x)-1)e^{-1}}{\cos(x)}\right)-1} \tag{1}$$

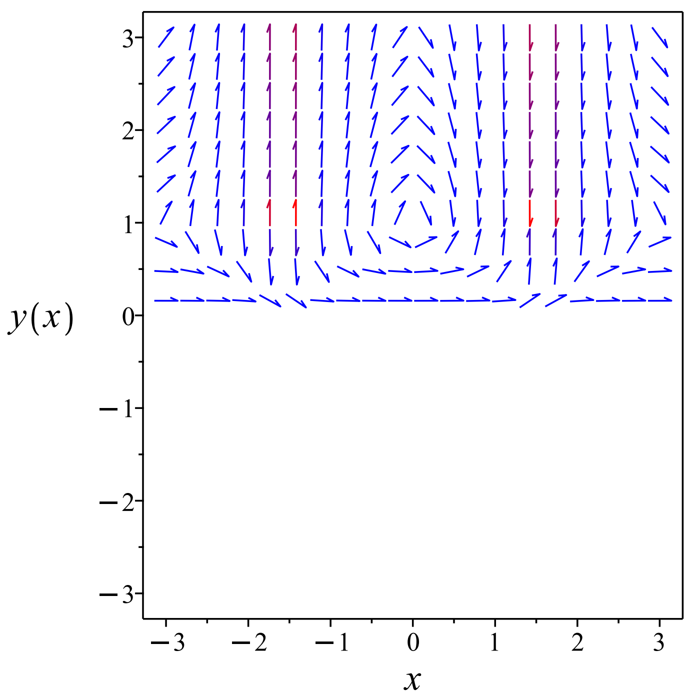


Figure 119: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(\frac{(c_1\cos(x)-1)e^{-1}}{\cos(x)}\right)-1}$$

Verified OK.

4.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\sin(x)\, y^2}{\cos(x)^2 \ln(y)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{\cos(x)^2}{\sin(x)} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\cos(x)^2}{\sin(x)}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{1}{\cos(x)}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{\sin(x)\, y^2}{\cos(x)^2 \ln(y)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0\\R_y &= 1\\S_x &= -\sec(x)\tan(x)\\S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(y)}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{R} - \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\sec(x) = -\frac{\ln(y)}{y} - \frac{1}{y} + c_1$$

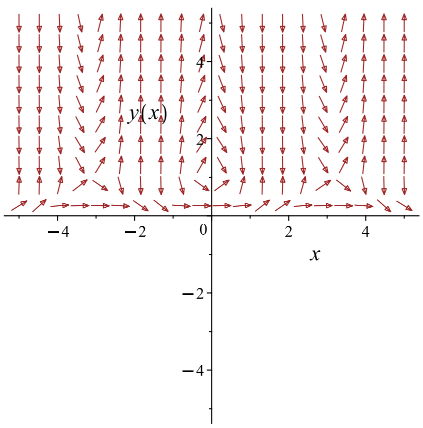
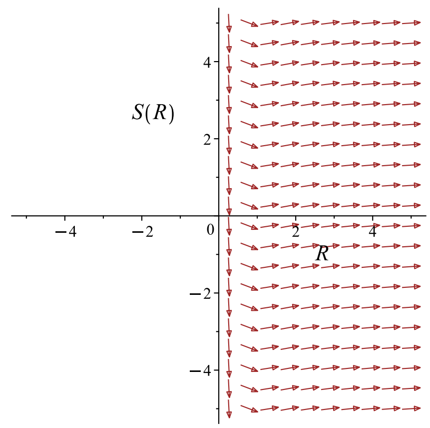
Which simplifies to

$$-\sec(x) = -\frac{\ln(y)}{y} - \frac{1}{y} + c_1$$

Which gives

$$y = e^{-\operatorname{LambertW}\left(-\frac{(c_1\cos(x)+1)e^{-1}}{\cos(x)}\right)-1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(x) y^2}{\cos(x)^2 \ln(y)}$ 	$R = y$ $S = -\sec(x)$	$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$ 

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{(c_1 \cos(x)+1)e^{-1}}{\cos(x)}\right)-1} \tag{1}$$

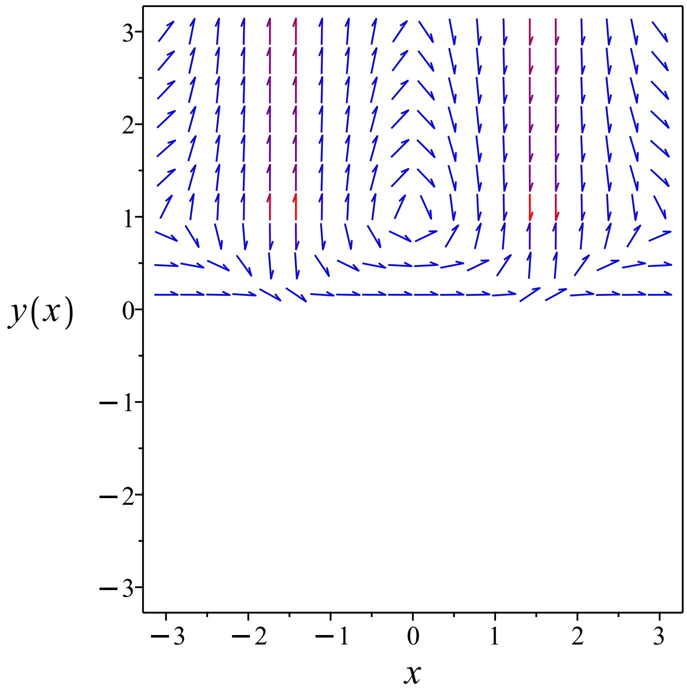


Figure 120: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-\operatorname{LambertW}\left(-\frac{(c_1 \cos(x)+1)e^{-1}}{\cos(x)}\right)-1}$$

Verified OK.

4.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{\ln(y)}{y^2}\right) \mathrm{d}y &= \left(\frac{\sin(x)}{\cos(x)^2}\right) \mathrm{d}x \\
\left(-\frac{\sin(x)}{\cos(x)^2}\right) \mathrm{d}x + \left(-\frac{\ln(y)}{y^2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{\sin(x)}{\cos(x)^2} \\
N(x, y) &= -\frac{\ln(y)}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(x)}{\cos(x)^2}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\ln(y)}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\sin(x)}{\cos(x)^2} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\sec(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\ln(y)}{y^2}$. Therefore equation (4) becomes

$$
-\frac{\ln(y)}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{\ln(y)}{y^2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{\ln(y)}{y^2}\right) \mathrm{d}y \\
f(y) &= \frac{\ln(y)}{y} + \frac{1}{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sec(x) + \frac{\ln(y)}{y} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sec(x) + \frac{\ln(y)}{y} + \frac{1}{y}$$

The solution becomes

$$y = e^{-\operatorname{LambertW}\left(-\frac{(c_1\cos(x)+1)e^{-1}}{\cos(x)}\right)-1}$$

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-\frac{(c_1\cos(x)+1)e^{-1}}{\cos(x)}\right)-1} \tag{1}$$

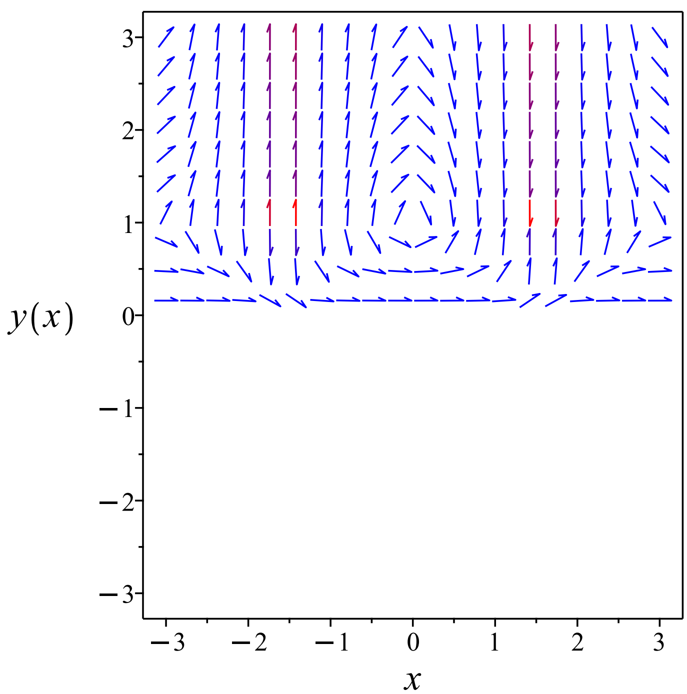


Figure 121: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-\frac{(c_1 \cos(x)+1)e^{-1}}{\cos(x)}\right)-1}$$

Verified OK.

4.13.4 Maple step by step solution

Let's solve

$\sin(x)\, y^2 + \cos(x)^2 \ln(y)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \ln(y)}{y^2} = -\frac{\sin(x)}{\cos(x)^2}$

- Integrate both sides with respect to x

  $\int \frac{y' \ln(y)}{y^2} dx = \int -\frac{\sin(x)}{\cos(x)^2} dx + c_1$

- Evaluate integral

  $-\frac{\ln(y)}{y} - \frac{1}{y} = -\frac{1}{\cos(x)} + c_1$

- Solve for y

  $y = e^{-LambertW\left(\frac{c_1 \cos(x)-1}{e \cos(x)}\right)-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 21

```
dsolve(y(x)^2*sin(x)+cos(x)^2*ln(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-(\sec(x) + c_1)\,\mathrm{e}^{-1}\right)}{\sec(x) + c_1}$$

✓ Solution by Mathematica

Time used: 60.174 (sec). Leaf size: 29

```
DSolve[y[x]^2*Sin[x]+Cos[x]^2*Log[y[x]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cos(x) W\left(\frac{-\sec(x)+c_1}{e}\right)}{-1 + c_1 \cos(x)}$$

4.14 problem 59

Internal problem ID [14986]
Internal file name [`OUTPUT/14995_Monday_April_15_2024_12_04_57_AM_27872046/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 59.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sin(-y + x) = 0}$$

4.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \sin(-y + x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \sin(-y+x)(b_3 - a_2) - \sin(-y+x)^2 a_3 \\ &- \cos(-y+x)(xa_2 + ya_3 + a_1) + \cos(-y+x)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}&-\cos(-y+x)xa_2 + \cos(-y+x)xb_2 - \cos(-y+x)ya_3 \\ &+ \cos(-y+x)yb_3 - \sin(-y+x)^2 a_3 - \cos(-y+x)a_1 \\ &+ \cos(-y+x)b_1 - \sin(-y+x)a_2 + \sin(-y+x)b_3 + b_2 = 0\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}&-\cos(-y+x)xa_2 + \cos(-y+x)xb_2 - \cos(-y+x)ya_3 \\ &+ \cos(-y+x)yb_3 - \sin(-y+x)^2 a_3 - \cos(-y+x)a_1 \\ &+ \cos(-y+x)b_1 - \sin(-y+x)a_2 + \sin(-y+x)b_3 + b_2 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&b_2 - \frac{a_3}{2} - \cos(-y+x)xa_2 + \cos(-y+x)xb_2 - \cos(-y+x)ya_3 \\ &+ \cos(-y+x)yb_3 + \frac{a_3\cos(-2y+2x)}{2} - \cos(-y+x)a_1 \\ &+ \cos(-y+x)b_1 - \sin(-y+x)a_2 + \sin(-y+x)b_3 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(-2y+2x), \cos(-y+x), \sin(-y+x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(-2y+2x) = v_3, \cos(-y+x) = v_4, \sin(-y+x) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2}a_3 - v_4v_1a_2 + v_4v_1b_2 - v_4v_2a_3 + v_4v_2b_3 + \frac{1}{2}a_3v_3 - v_4a_1 + v_4b_1 - v_5a_2 + v_5b_3 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (-a_2 + b_2)\,v_1v_4 + (-a_3 + b_3)\,v_2v_4 + \frac{a_3v_3}{2} + (-a_1 + b_1)\,v_4 + (b_3 - a_2)\,v_5 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
\frac{a_3}{2} &= 0 \\
-a_1 + b_1 &= 0 \\
-a_2 + b_2 &= 0 \\
-a_3 + b_3 &= 0 \\
b_2 - \frac{a_3}{2} &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - (\sin(-y+x))\,(1) \\
&= 1 + \sin(y)\cos(x) - \cos(y)\sin(x) \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{1 + \sin(y)\cos(x) - \cos(y)\sin(x)} dy
\end{aligned}$$

Which results in

$$S = -\frac{2}{-\tan\left(-\frac{y}{2} + \frac{x}{2}\right) + 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sin(-y+x)$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{\sin\left(-y+x\right)-1} \\
S_y &= -\frac{1}{\sin\left(-y+x\right)-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2}{\tan\left(-\frac{y}{2}+\frac{x}{2}\right)-1} = -x + c_1
$$

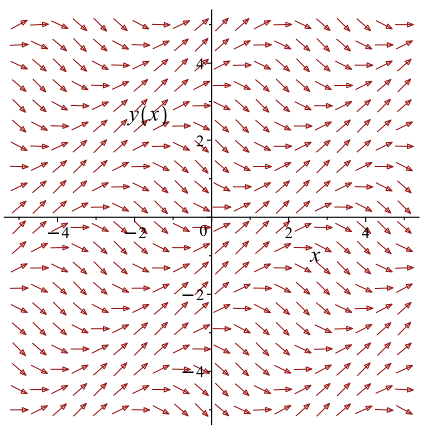
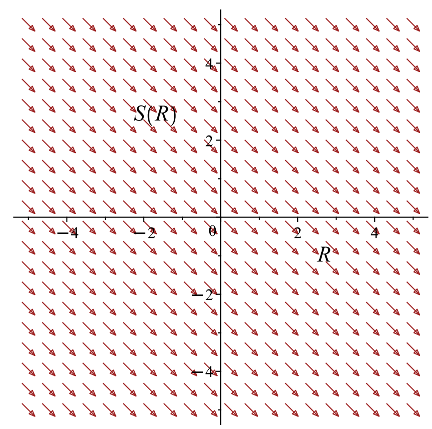
Which simplifies to

$$
\frac{2}{\tan\left(-\frac{y}{2}+\frac{x}{2}\right)-1} = -x + c_1
$$

Which gives

$$
y = x - 2\arctan\left(\frac{c_1 - x + 2}{-x + c_1}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(-y + x)$ 	$R = x$ $S = \frac{2}{\tan\left(-\frac{y}{2} + \frac{x}{2}\right) - 1}$	$\frac{dS}{dR} = -1$ 

Summary
The solution(s) found are the following

$$y = x - 2\arctan\left(\frac{c_1 - x + 2}{-x + c_1}\right) \tag{1}$$

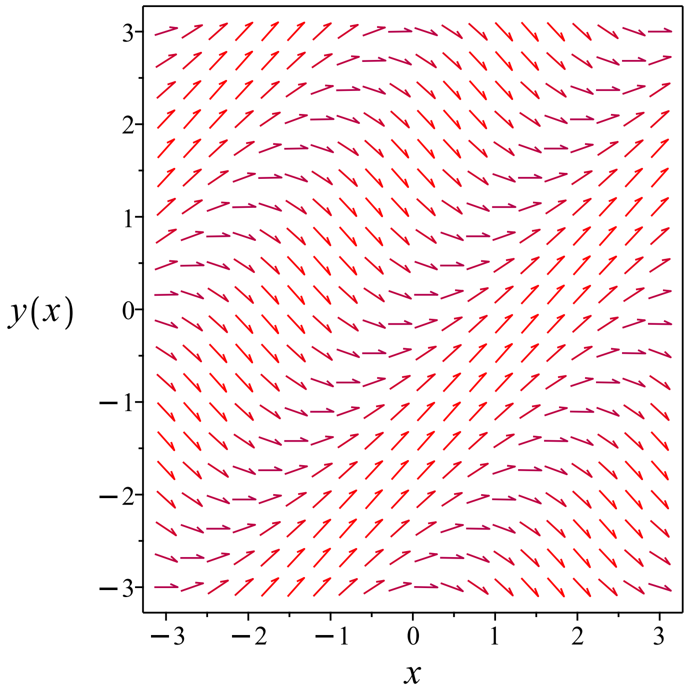


Figure 122: Slope field plot

Verification of solutions

$$y = x - 2\arctan\left(\frac{c_1 - x + 2}{-x + c_1}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=sin(x-y(x)),y(x), singsol=all)
```

$$y(x) = x - 2\arctan\left(\frac{2 - x + c_1}{c_1 - x}\right)$$

✓ Solution by Mathematica

Time used: 0.415 (sec). Leaf size: 64

```
DSolve[y'[x]==Sin[x-y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[y(x) - \sec(x - y(x))\left(2\sqrt{\cos^2(x - y(x))}\arcsin\left(\frac{\sqrt{1 - \sin(x - y(x))}}{\sqrt{2}}\right)\right.\right.$$
$$\left.\left. + \sin(x - y(x)) + 1\right) = c_1, y(x)\right]$$

4.15 problem 60

Internal problem ID [14987]
Internal file name [`OUTPUT/14996_Monday_April_15_2024_12_04_58_AM_31209479/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 60.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - yb = ax + c}$$

4.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -b$$
$$q(x) = ax + c$$

Hence the ode is

$$y' - yb = ax + c$$

The integrating factor μ is

$$\mu = e^{\int -b dx}$$
$$= e^{-bx}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(ax + c)$$
$$\frac{d}{dx}(e^{-bx} y) = (e^{-bx})(ax + c)$$
$$d(e^{-bx} y) = ((ax + c) e^{-bx}) dx$$

Integrating gives

$$e^{-bx} y = \int (ax + c) e^{-bx} dx$$
$$e^{-bx} y = -\frac{(abx + bc + a) e^{-bx}}{b^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-bx}$ results in

$$y = -\frac{e^{bx}(abx + bc + a) e^{-bx}}{b^2} + c_1 e^{bx}$$

which simplifies to

$$y = \frac{c_1 e^{bx} b^2 + (-ax - c) b - a}{b^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{bx} b^2 + (-ax - c) b - a}{b^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{bx} b^2 + (-ax - c) b - a}{b^2}$$

Verified OK.

4.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= ax + by + c \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 96: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{bx}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{bx}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{-bx} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = ax + by + c$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -b\,\mathrm{e}^{-bx} y\\ S_y &= \mathrm{e}^{-bx}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (ax + c)\,\mathrm{e}^{-bx} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (Ra + c)\,\mathrm{e}^{-bR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(Rab + bc + a)\,\mathrm{e}^{-bR}}{b^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-bx}y = -\frac{(abx + bc + a)\,\mathrm{e}^{-bx}}{b^2} + c_1$$

Which simplifies to

$$\mathrm{e}^{-bx}y = -\frac{(abx + bc + a)\,\mathrm{e}^{-bx}}{b^2} + c_1$$

Which gives

$$y = -\frac{\left(\mathrm{e}^{-bx}abx + \mathrm{e}^{-bx}bc - c_1 b^2 + a\,\mathrm{e}^{-bx}\right)\mathrm{e}^{bx}}{b^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(\mathrm{e}^{-bx}abx + \mathrm{e}^{-bx}bc - c_1 b^2 + a\,\mathrm{e}^{-bx}\right)\mathrm{e}^{bx}}{b^2} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(\mathrm{e}^{-bx}abx + \mathrm{e}^{-bx}bc - c_1 b^2 + a\,\mathrm{e}^{-bx}\right)\mathrm{e}^{bx}}{b^2}$$

Verified OK.

4.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(ax+by+c)\,\mathrm{d}x\\ (-ax-by-c)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-ax-by-c\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-ax - by - c) \\ &= -b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-b) - (0)) \\ &= -b\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -b \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-bx} \\ &= \mathrm{e}^{-bx}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-bx}(-ax - by - c) \\ &= -\mathrm{e}^{-bx}(ax + by + c)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-bx}(1) \\
&= \mathrm{e}^{-bx}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-bx}(ax+by+c)\right) + \left(\mathrm{e}^{-bx}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-bx}(ax+by+c)\,\mathrm{d}x \\
\phi &= \frac{(b^2y+(ax+c)\,b+a)\,\mathrm{e}^{-bx}}{b^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-bx} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-bx}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-bx} = \mathrm{e}^{-bx} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(b^2 y + (ax + c) b + a) e^{-bx}}{b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(b^2 y + (ax + c) b + a) e^{-bx}}{b^2}$$

The solution becomes

$$y = -\frac{(e^{-bx} abx + e^{-bx} bc - c_1 b^2 + a e^{-bx}) e^{bx}}{b^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{(e^{-bx} abx + e^{-bx} bc - c_1 b^2 + a e^{-bx}) e^{bx}}{b^2} \tag{1}$$

Verification of solutions

$$y = -\frac{(e^{-bx} abx + e^{-bx} bc - c_1 b^2 + a e^{-bx}) e^{bx}}{b^2}$$

Verified OK.

4.15.4 Maple step by step solution

Let's solve

$y' - yb = ax + c$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = ax + yb + c$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - yb = ax + c$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y' - yb) = \mu(x)\,(ax + c)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y' - yb) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\mu(x)\,b$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{-bx}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(ax + c)\,dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\,(ax + c)\,dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)(ax+c)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{-bx}$

$$y = \frac{\int (ax+c)\mathrm{e}^{-bx}dx + c_1}{\mathrm{e}^{-bx}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{(abx+bc+a)\mathrm{e}^{-bx}}{b^2} + c_1}{\mathrm{e}^{-bx}}$$

- Simplify

$$y = \frac{c_1\mathrm{e}^{bx}b^2 + (-ax-c)b - a}{b^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x)=a*x+b*y(x)+c,y(x), singsol=all)
```

$$y(x) = \frac{e^{bx} c_1 b^2 + (-ax - c) b - a}{b^2}$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 28

```
DSolve[y'[x]==a*x+b*y[x]+c,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{abx + a + bc}{b^2} + c_1 e^{bx}$$

4.16 problem 61

Internal problem ID [14988]
Internal file name [OUTPUT/14997_Monday_April_15_2024_12_04_59_AM_24706420/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 61.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{(y+x)^2 y' = a^2}$$

4.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{a^2}{x^2 + 2xy + y^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{a^2(b_3 - a_2)}{x^2 + 2xy + y^2} - \frac{a^4 a_3}{(x^2 + 2xy + y^2)^2} + \frac{a^2(2y + 2x)(x a_2 + y a_3 + a_1)}{(x^2 + 2xy + y^2)^2} + \frac{a^2(2y + 2x)(x b_2 + y b_3 + b_1)}{(x^2 + 2xy + y^2)^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^4 a_3 - a^2 x^2 a_2 - 2a^2 x^2 b_2 - a^2 x^2 b_3 - 2a^2 xy a_3 - 2a^2 xy b_2 - 4a^2 xy b_3 + a^2 y^2 a_2 - 2a^2 y^2 a_3 - 3a^2 y^2 b_3 - x^4 b}{(x^2 + 2xy + y^2)^2} = 0$$

Setting the numerator to zero gives

$$-a^4 a_3 + a^2 x^2 a_2 + 2a^2 x^2 b_2 + a^2 x^2 b_3 + 2a^2 xy a_3 + 2a^2 xy b_2 + 4a^2 xy b_3 - a^2 y^2 a_2 + 2a^2 y^2 a_3 + 3a^2 y^2 b_3 + x^4 b_2 + 4x^3 y b_2 + 6x^2 y^2 b_2 + 4x y^3 b_2 + y^4 b_2 + 2a^2 x a_1 + 2a^2 x b_1 + 2a^2 y a_1 + 2a^2 y b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a^4 a_3 + a^2 a_2 v_1^2 - a^2 a_2 v_2^2 + 2a^2 a_3 v_1 v_2 + 2a^2 a_3 v_2^2 + 2a^2 b_2 v_1^2 + 2a^2 b_2 v_1 v_2 + a^2 b_3 v_1^2 + 4a^2 b_3 v_1 v_2 + 3a^2 b_3 v_2^2 + b_2 v_1^4 + 4b_2 v_1^3 v_2 + 6b_2 v_1^2 v_2^2 + 4b_2 v_1 v_2^3 + b_2 v_2^4 + 2a^2 a_1 v_1 + 2a^2 a_1 v_2 + 2a^2 b_1 v_1 + 2a^2 b_1 v_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & b_2v_1^4 + 4b_2v_1^3v_2 + 6b_2v_1^2v_2^2 + \left(a^2a_2 + 2a^2b_2 + a^2b_3\right)v_1^2 + 4b_2v_1v_2^3 \\ & + \left(2a^2a_3 + 2a^2b_2 + 4a^2b_3\right)v_1v_2 + \left(2a^2a_1 + 2a^2b_1\right)v_1 + b_2v_2^4 \\ & + \left(-a^2a_2 + 2a^2a_3 + 3a^2b_3\right)v_2^2 + \left(2a^2a_1 + 2a^2b_1\right)v_2 - a^4a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ 4b_2 &= 0 \\ 6b_2 &= 0 \\ -a^4a_3 &= 0 \\ 2a^2a_1 + 2a^2b_1 &= 0 \\ -a^2a_2 + 2a^2a_3 + 3a^2b_3 &= 0 \\ a^2a_2 + 2a^2b_2 + a^2b_3 &= 0 \\ 2a^2a_3 + 2a^2b_2 + 4a^2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(\frac{a^2}{x^2+2xy+y^2}\right)(-1) \\
&= \frac{a^2+x^2+2xy+y^2}{x^2+2xy+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{a^2+x^2+2xy+y^2}{x^2+2xy+y^2}} dy
\end{aligned}$$

Which results in

$$S = y - a\arctan\left(\frac{2y+2x}{2a}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{a^2}{x^2+2xy+y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{a^2}{(y+x)^2 + a^2} \\
S_y &= 1 - \frac{a^2}{(y+x)^2 + a^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y - a \arctan\left(\frac{y+x}{a}\right) = c_1
$$

Which simplifies to

$$
y - a \arctan\left(\frac{y+x}{a}\right) = c_1
$$

Summary

The solution(s) found are the following

$$
y - a \arctan\left(\frac{y+x}{a}\right) = c_1 \tag{1}
$$

Verification of solutions

$$
y - a \arctan\left(\frac{y+x}{a}\right) = c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 24

```
dsolve((x+y(x))^2*diff(y(x),x)=a^2,y(x), singsol=all)
```

$$y(x) = a\,\mathrm{RootOf}\left(\tan\left(\_Z\right)a - \_Za + c_1 - x\right) - c_1$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 21

```
DSolve[(x+y[x])^2*y'[x]==a^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[y(x) - a\arctan\left(\frac{y(x)+x}{a}\right) = c_1, y(x)\right]$$

4.17 problem 62

Internal problem ID [14989]
Internal file name [`OUTPUT/14998_Monday_April_15_2024_12_05_02_AM_30619547/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 62.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x + y - a(yx + 1) = 0}$$

With initial conditions

$$\left[y\left(\frac{1}{a}\right) = -a\right]$$

4.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{ax - 1}{x}$$
$$q(x) = \frac{a}{x}$$

Hence the ode is

$$y' - \frac{(ax-1)\,y}{x} = \frac{a}{x}$$

The domain of $p(x) = -\frac{ax-1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = \frac{1}{a}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{ax-1}{x}dx} \\ &= e^{-ax+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = x\,e^{-ax}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{a}{x}\right) \\ \frac{d}{dx}\left(x\,e^{-ax}y\right) &= \left(x\,e^{-ax}\right)\left(\frac{a}{x}\right) \\ d\left(x\,e^{-ax}y\right) &= \left(a\,e^{-ax}\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x\,e^{-ax}y &= \int a\,e^{-ax}\,dx \\ x\,e^{-ax}y &= -e^{-ax} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x\,e^{-ax}$ results in

$$y = -\frac{e^{ax}e^{-ax}}{x} + \frac{c_1 e^{ax}}{x}$$

which simplifies to

$$y = \frac{c_1 e^{ax} - 1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{a}$ and $y = -a$ in the above solution gives an equation to solve for the constant of integration.

$$-a = a\,e c_1 - a$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{1}{x}$$

Verified OK.

4.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{axy + a - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 99: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{ax-\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{ax - \ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,e^{-ax} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{axy + a - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{-ax} y(-ax + 1) \\ S_y &= x\,e^{-ax} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = a\,e^{-ax} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = a\,e^{-aR}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-aR} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x\,e^{-ax}y = -e^{-ax} + c_1$$

Which simplifies to

$$x\,e^{-ax}y = -e^{-ax} + c_1$$

Which gives

$$y = -\frac{(e^{-ax} - c_1)\,e^{ax}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{a}$ and $y = -a$ in the above solution gives an equation to solve for the constant of integration.

$$-a = a\,e c_1 - a$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{1}{x}$$

Verified OK.

4.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(-y+a(xy+1))\,\mathrm{d}x\\ (y-a(xy+1))\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-a(xy+1)\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - a(xy+1)) \\ &= -ax + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-ax+1) - (1)) \\ &= -a\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -a \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-ax} \\ &= \mathrm{e}^{-ax}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-ax}(y - a(xy+1)) \\ &= -\mathrm{e}^{-ax}(axy + a - y)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-ax}(x) \\
&= x\,\mathrm{e}^{-ax}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-ax}(axy + a - y)\right) + \left(x\,\mathrm{e}^{-ax}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-ax}(axy + a - y)\,\mathrm{d}x \\
\phi &= (xy + 1)\,\mathrm{e}^{-ax} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{-ax} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{-ax}$. Therefore equation (4) becomes

$$
x\,\mathrm{e}^{-ax} = x\,\mathrm{e}^{-ax} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (xy + 1)\,\mathrm{e}^{-ax} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (xy + 1)\,\mathrm{e}^{-ax}$$

The solution becomes

$$y = -\frac{(\mathrm{e}^{-ax} - c_1)\,\mathrm{e}^{ax}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{a}$ and $y = -a$ in the above solution gives an equation to solve for the constant of integration.

$$-a = a\,\mathrm{e}c_1 - a$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{x} \tag{1}$$

Verification of solutions

$$y = -\frac{1}{x}$$

Verified OK.

4.17.5 Maple step by step solution

Let's solve

$\left[y'x+y-a(yx+1)=0, y\left(\frac{1}{a}\right)=-a\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{(ax-1)y}{x}+\frac{a}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{(ax-1)y}{x}=\frac{a}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-\frac{(ax-1)y}{x}\right)=\frac{\mu(x)a}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{(ax-1)y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)(ax-1)}{x}$

- Solve to find the integrating factor

  $\mu(x)=x\,\mathrm{e}^{-ax}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)a}{x}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)a}{x}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)a}{x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x\mathrm{e}^{-ax}$

  $y=\frac{\int a\,\mathrm{e}^{-ax}dx+c_1}{x\,\mathrm{e}^{-ax}}$

- Evaluate the integrals on the rhs

  $y=\frac{-\mathrm{e}^{-ax}+c_1}{x\,\mathrm{e}^{-ax}}$

- Simplify

$y = \frac{c_1 e^{ax} - 1}{x}$

- Use initial condition $y\left(\frac{1}{a}\right) = -a$

$-a = (e c_1 - 1)\, a$

- Solve for c_1

$c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$y = -\frac{1}{x}$

- Solution to the IVP

$y = -\frac{1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([y(x)+x*diff(y(x),x)=a*(1+x*y(x)),y(1/a) = -a],y(x), singsol=all)
```

$$y(x) = -\frac{1}{x}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 10

```
DSolve[{y[x]+x*y'[x]==a*(1+x*y[x]),{y[1/a]==-a}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{x}$$

4.18 problem 63

Internal problem ID [14990]
Internal file name [`OUTPUT/14999_Monday_April_15_2024_12_05_02_AM_85345288/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 63.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 + 2x\sqrt{ax - x^2}\, y' = -a^2}$$

With initial conditions

$$[y(a) = 0]$$

4.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{a^2 + y^2}{2x\sqrt{ax - x^2}} \end{aligned}$$

$f(x, y)$ is not defined at $x = a$ therefore existence and uniqueness theorem do not apply.

4.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{-\frac{a^2}{2}-\frac{y^2}{2}}{x\sqrt{ax-x^2}}
\end{aligned}$$

Where $f(x)=\frac{1}{x\sqrt{ax-x^2}}$ and $g(y)=-\frac{a^2}{2}-\frac{y^2}{2}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{-\frac{a^2}{2}-\frac{y^2}{2}}\,dy &= \frac{1}{x\sqrt{ax-x^2}}\,dx \\
\int \frac{1}{-\frac{a^2}{2}-\frac{y^2}{2}}\,dy &= \int \frac{1}{x\sqrt{ax-x^2}}\,dx \\
-\frac{2\arctan\left(\frac{y}{a}\right)}{a} &= -\frac{2\sqrt{x\,(a-x)}}{xa}+c_1
\end{aligned}$$

Which results in

$$y=\tan\left(\frac{-c_1ax+2\sqrt{x\,(a-x)}}{2x}\right)a$$

Initial conditions are used to solve for c_1. Substituting $x=a$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0=-\tan\left(\frac{c_1a}{2}\right)a$$

$$c_1=0$$

Substituting c_1 found above in the general solution gives

$$y=\tan\left(\frac{\sqrt{x\,(a-x)}}{x}\right)a$$

Summary

The solution(s) found are the following

$$y=\tan\left(\frac{\sqrt{x\,(a-x)}}{x}\right)a \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{\sqrt{x\,(a-x)}}{x}\right) a$$

Verified OK.

4.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{a^2+y^2}{2x\sqrt{a x-x^2}} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 102: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x\sqrt{ax - x^2} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x\sqrt{ax - x^2}} dx \end{aligned}$$

Which results in

$$S = -\frac{2(a - x)}{a\sqrt{ax - x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{a^2 + y^2}{2x\sqrt{ax - x^2}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{\sqrt{a - x}\, x^{\frac{3}{2}}} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\sqrt{x\,(a - x)}}{\sqrt{a - x}\,\sqrt{x}\,(a^2 + y^2)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R^2 + a^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\arctan\left(\frac{R}{a}\right)}{a} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2\sqrt{a-x}}{a\sqrt{x}} = -\frac{2\arctan\left(\frac{y}{a}\right)}{a} + c_1$$

Which simplifies to

$$-\frac{2\sqrt{a-x}}{a\sqrt{x}} = -\frac{2\arctan\left(\frac{y}{a}\right)}{a} + c_1$$

Which gives

$$y = \tan\left(\frac{c_1 a\sqrt{x} + 2\sqrt{a-x}}{2\sqrt{x}}\right) a$$

Initial conditions are used to solve for c_1. Substituting $x = a$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan\left(\frac{c_1 a}{2}\right) a$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) a$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) a \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) a$$

Verified OK.

4.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-\frac{a^2}{2}-\frac{y^2}{2}}\right)\mathrm{d}y=\left(\frac{1}{x\sqrt{ax-x^2}}\right)\mathrm{d}x$$

$$\left(-\frac{1}{x\sqrt{ax-x^2}}\right)\mathrm{d}x+\left(\frac{1}{-\frac{a^2}{2}-\frac{y^2}{2}}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x\sqrt{a x-x^{2}}}\\
N(x,y) &= \frac{1}{-\frac{a^{2}}{2}-\frac{y^{2}}{2}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x\sqrt{a x-x^{2}}}\right)\\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-\frac{a^{2}}{2}-\frac{y^{2}}{2}}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x\sqrt{a x-x^{2}}}\, \mathrm{d}x\\
\phi &= \frac{2a-2x}{a\sqrt{x\left(a-x\right)}}+f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-\frac{a^2}{2} - \frac{y^2}{2}}$. Therefore equation (4) becomes

$$\frac{1}{-\frac{a^2}{2} - \frac{y^2}{2}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{a^2 + y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{2}{a^2 + y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{2\arctan\left(\frac{y}{a}\right)}{a} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2a - 2x}{a\sqrt{x\,(a - x)}} - \frac{2\arctan\left(\frac{y}{a}\right)}{a} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2a - 2x}{a\sqrt{x\,(a - x)}} - \frac{2\arctan\left(\frac{y}{a}\right)}{a}$$

The solution becomes

$$y = -\tan\left(\frac{c_1 a\sqrt{x\,(a - x)} - 2a + 2x}{2\sqrt{x\,(a - x)}}\right) a$$

Initial conditions are used to solve for c_1. Substituting $x = a$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\tan\left(\frac{c_1 a}{2}\right) a$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \tan\left(\frac{a-x}{\sqrt{x(a-x)}}\right) a$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{a-x}{\sqrt{x(a-x)}}\right) a \tag{1}$$

Verification of solutions

$$y = \tan\left(\frac{a-x}{\sqrt{x(a-x)}}\right) a$$

Verified OK.

4.18.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{a^2+y^2}{2x\sqrt{ax-x^2}} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{a^2}{2x\sqrt{ax-x^2}} - \frac{y^2}{2x\sqrt{ax-x^2}}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{a^2}{2x\sqrt{ax-x^2}}$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{2x\sqrt{ax-x^2}}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{2x\sqrt{ax-x^2}}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{1}{2x^2\sqrt{ax-x^2}} + \frac{a-2x}{4x\,(ax-x^2)^{\frac{3}{2}}} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= -\frac{a^2}{8x^3\,(ax-x^2)^{\frac{3}{2}}}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{u''(x)}{2x\sqrt{ax-x^2}} - \left(\frac{1}{2x^2\sqrt{ax-x^2}} + \frac{a-2x}{4x\,(ax-x^2)^{\frac{3}{2}}}\right) u'(x) - \frac{a^2 u(x)}{8x^3\,(ax-x^2)^{\frac{3}{2}}} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \sin\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) + c_2 \cos\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)
$$

The above shows that

$$
u'(x) = \frac{a\left(c_2 \sin\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) - c_1 \cos\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)\right)}{2\sqrt{a-x}\, x^{\frac{3}{2}}}
$$

Using the above in (1) gives the solution

$$
y = \frac{a\left(c_2 \sin\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) - c_1 \cos\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)\right)\sqrt{ax-x^2}}{\sqrt{a-x}\,\sqrt{x}\,\left(c_1 \sin\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) + c_2 \cos\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)\right)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(\sin\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) - c_3 \cos\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)\right) a\sqrt{x\,(a-x)}}{\left(c_3 \sin\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) + \cos\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)\right) \sqrt{a-x}\,\sqrt{x}}$$

Initial conditions are used to solve for c_3. Substituting $x = a$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

Warning: Failed to find c_3 using initial conditions. Solution could be wrong or there is no solution that satisfies the given initial conditions.

Verification of solutions N/A

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.562 (sec). Leaf size: 22

```
dsolve([(a^2+y(x)^2)+2*x*sqrt(a*x-x^2)*diff(y(x),x)=0,y(a) = 0],y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{a-x}{\sqrt{x\,(a-x)}}\right) a$$

✓ Solution by Mathematica

Time used: 31.916 (sec). Leaf size: 23

```
DSolve[{(a^2+y[x]^2)+2*x*Sqrt[a*x-x^2]*y'[x]==0,{y[a]==0}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to a \tan \left(\frac{\sqrt{x(a-x)}}{x}\right)$$

4.19 problem 81

Internal problem ID [14991]
Internal file name [`OUTPUT/15000_Monday_April_15_2024_12_05_05_AM_16613591/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 81.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} = 0}$$

With initial conditions

$$[y(0) = 0]$$

4.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.19.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{1}{x}\, dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= e^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 0$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x$$

Verified OK.

4.19.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 0$$

This solution is valid for any c_1. Hence there are infinite number of solutions.
Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

Verification of solutions

$$y = c_1 x$$

Verified OK.

4.19.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 0$$

This solution is valid for any c_2. Hence there are infinite number of solutions.
Summary
The solution(s) found are the following

$$y = c_2 x \tag{1}$$

Verification of solutions

$$y = c_2 x$$

Verified OK.

4.19.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 104: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

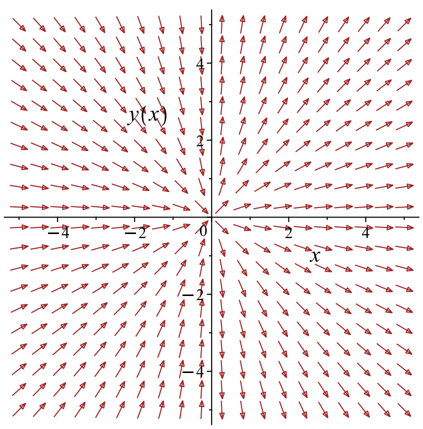
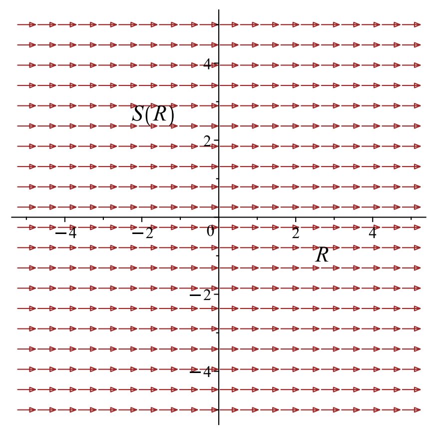
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 0$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

Verification of solutions

$$y = c_1 x$$

Verified OK.

4.19.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\N(x,y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\&= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\\\phi &= -\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{c_1} x$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 0$$

This solution is valid for any c_1. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{c_1} x \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{c_1} x$$

Verified OK.

4.19.7 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1}x$

- Use initial condition $y(0) = 0$

  $0 = 0$

- Solve for c_1

  $c_1 = c_1$

- Substitute $c_1 = c_1$ into general solution and simplify

  $y = \mathrm{e}^{c_1}x$

- Solution to the IVP

  $y = \mathrm{e}^{c_1}x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=y(x)/x,y(0) = 0],y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]/x,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

4.20 problem 85

Internal problem ID [14992]
Internal file name [`OUTPUT/15001_Monday_April_15_2024_12_05_06_AM_38929072/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 85.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\cos(y') = 0}$$

4.20.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\pi}{2}\,\mathrm{d}x \\ &= \frac{\pi x}{2} + c_1 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\pi x}{2} + c_1 \tag{1}$$

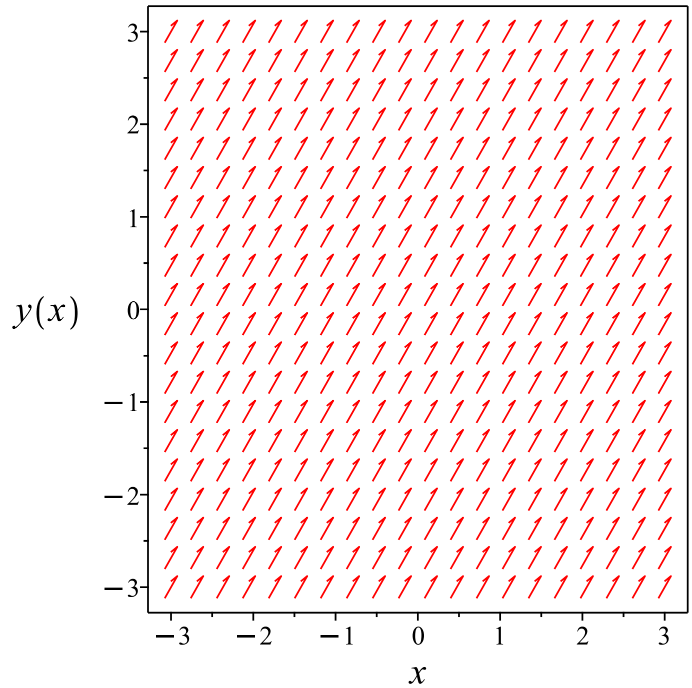


Figure 123: Slope field plot

Verification of solutions

$$y = \frac{\pi x}{2} + c_1$$

Verified OK.

4.20.2 Maple step by step solution

Let's solve

$\cos(y') = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \cos(y')\, dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \cos(y')\, dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(cos(diff(y(x),x))=0,y(x), singsol=all)
```

$$y(x) = \frac{\pi x}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 27

```
DSolve[Cos[y'[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\pi x}{2} + c_1$$
$$y(x) \rightarrow \frac{\pi x}{2} + c_1$$

4.21 problem 86

Internal problem ID [14993]
Internal file name [`OUTPUT/15002_Monday_April_15_2024_12_05_06_AM_91068066/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 86.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{e^{y'} = 1}$$

4.21.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, dx \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

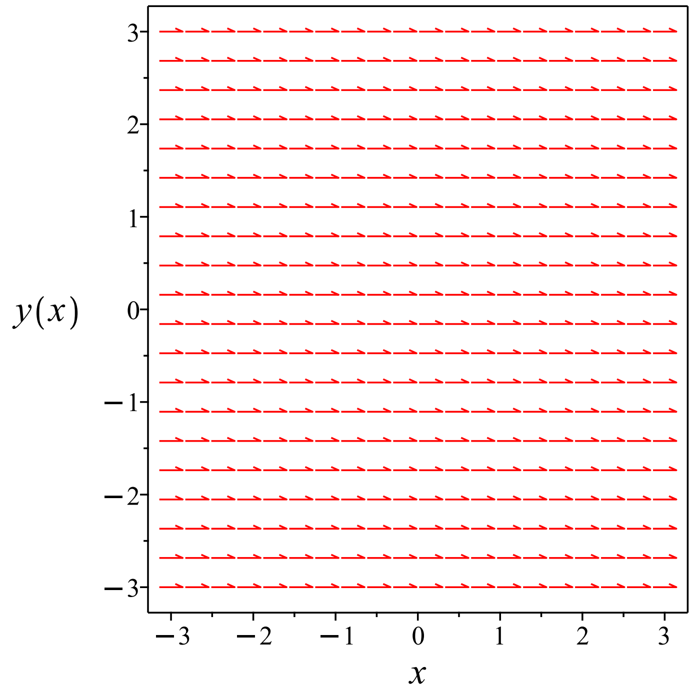


Figure 124: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

4.21.2 Maple step by step solution

Let's solve

$e^{y'} = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int e^{y'} dx = \int 1 dx + c_1$

- Cannot compute integral

  $\int e^{y'} dx = x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve(exp(diff(y(x),x))=1,y(x), singsol=all)
```

$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 7

```
DSolve[Exp[y'[x]]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

4.22 problem 87

Internal problem ID [14994]
Internal file name [`OUTPUT/15003_Monday_April_15_2024_12_05_06_AM_84758072/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 87.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\sin(y') = x}$$

4.22.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \arcsin(x)\, \mathrm{d}x \\ &= x \arcsin(x) + \sqrt{-x^2+1} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \arcsin(x) + \sqrt{-x^2+1} + c_1 \tag{1}$$

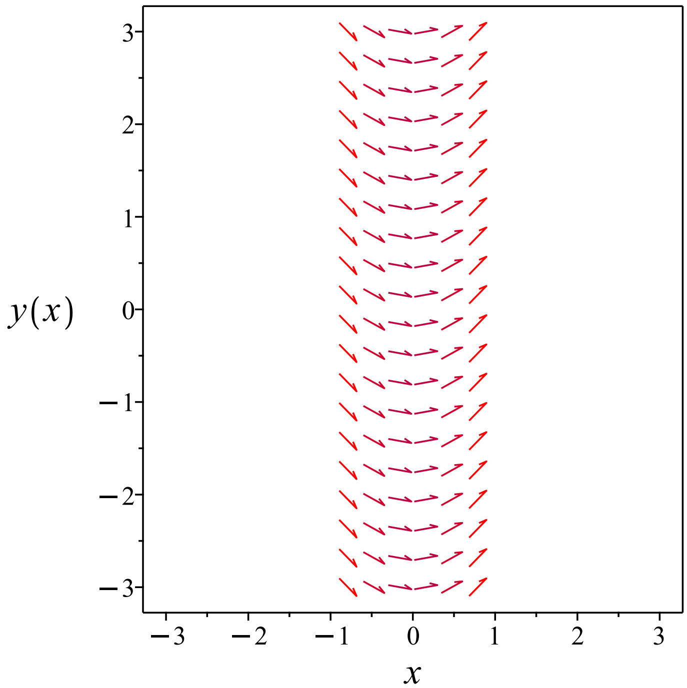


Figure 125: Slope field plot

Verification of solutions

$$y = x \arcsin(x) + \sqrt{-x^2 + 1} + c_1$$

Verified OK.

4.22.2 Maple step by step solution

Let's solve

$\sin(y') = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \sin(y')\, dx = \int x dx + c_1$

- Cannot compute integral

  $\int \sin(y')\, dx = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(sin(diff(y(x),x))=x,y(x), singsol=all)
```

$$y(x) = x \arcsin(x) + \sqrt{-x^2+1} + c_1$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 23

```
DSolve[Sin[y'[x]]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(x) + \sqrt{1-x^2} + c_1$$

4.23 problem 88

Internal problem ID [14995]
Internal file name [`OUTPUT/14995_Friday_April_19_2024_04_43_35_AM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 88.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\ln(y') = x}$$

4.23.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int e^x \, dx \\ &= e^x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^x + c_1 \tag{1}$$

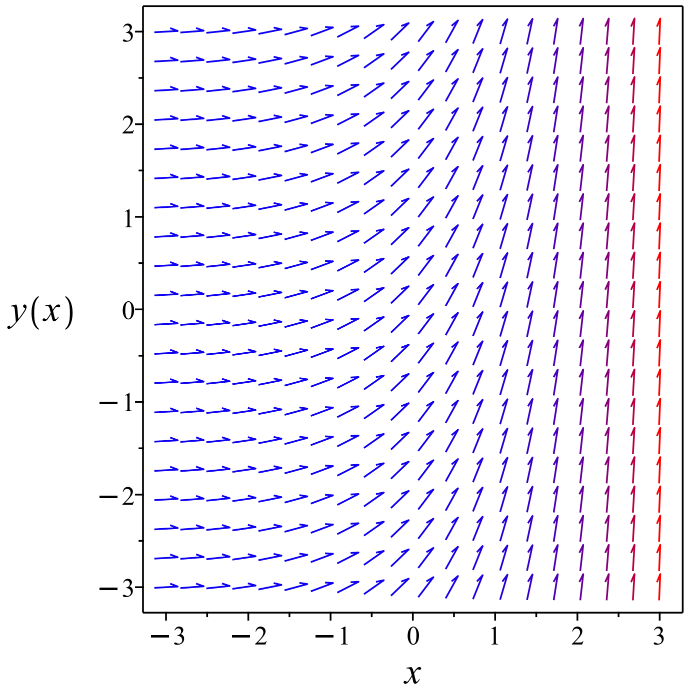


Figure 126: Slope field plot

Verification of solutions

$$y = e^x + c_1$$

Verified OK.

4.23.2 Maple step by step solution

Let's solve

$\ln(y') = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \ln(y')\, dx = \int x dx + c_1$

- Cannot compute integral

  $\int \ln(y')\, dx = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(ln(diff(y(x),x))=x,y(x), singsol=all)
```

$$y = \mathrm{e}^{x} + c_1$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 11

```
DSolve[Log[y'[x]]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x} + c_1$$

4.24 problem 89

Internal problem ID [14996]
Internal file name [`OUTPUT/14996_Friday_April_19_2024_04_43_37_AM_62813000/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 89.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\tan(y') = 0}$$

4.24.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

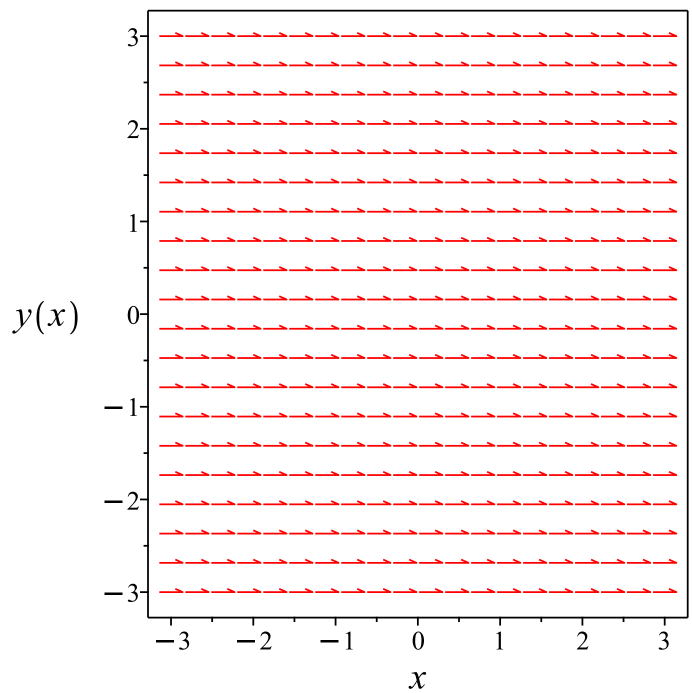


Figure 127: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

4.24.2 Maple step by step solution

Let's solve

$\tan(y') = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \tan(y')\, dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int \tan(y')\, dx = c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve(tan(diff(y(x),x))=0,y(x), singsol=all)
```

$$y = c_1$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 7

```
DSolve[Tan[y'[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1$$

4.25 problem 90

Internal problem ID [14997]
Internal file name [`OUTPUT/14997_Friday_April_19_2024_04_43_38_AM_7548041/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 90.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{e^{y'} = x}$$

4.25.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \ln(x)\, dx \\ &= x \ln(x) - x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x \ln(x) - x + c_1 \tag{1}$$

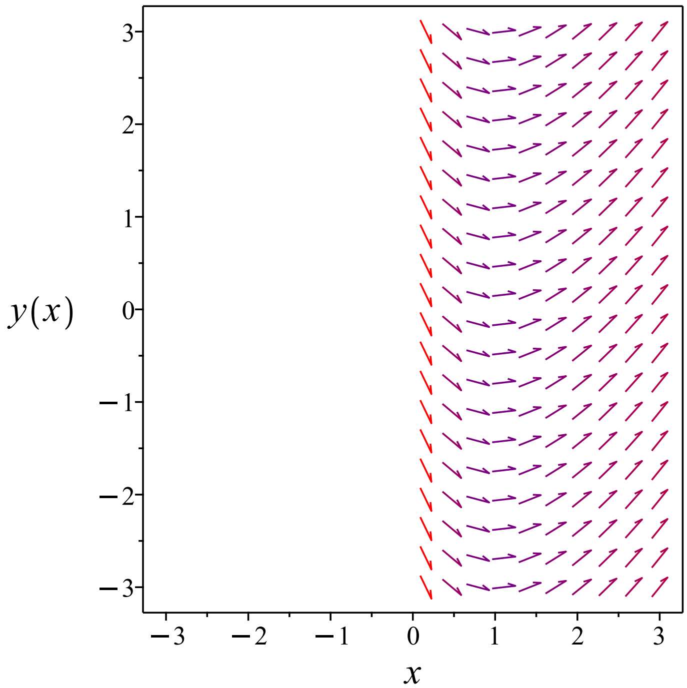


Figure 128: Slope field plot

Verification of solutions

$$y = x\ln(x) - x + c_1$$

Verified OK.

4.25.2 Maple step by step solution

Let's solve

$e^{y'} = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int e^{y'}dx = \int x dx + c_1$

- Cannot compute integral

  $\int e^{y'}dx = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(exp(diff(y(x),x))=x,y(x), singsol=all)
```

$$y = x\ln(x) - x + c_1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 15

```
DSolve[Exp[y'[x]]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + x\log(x) + c_1$$

4.26 problem 91

Internal problem ID [14998]
Internal file name [`OUTPUT/14998_Friday_April_19_2024_04_43_38_AM_25585316/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 91.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{\tan(y') = x}$$

4.26.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \arctan(x)\, \mathrm{d}x \\ &= -\frac{\ln(x^2+1)}{2} + x\arctan(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{\ln(x^2+1)}{2} + x\arctan(x) + c_1 \tag{1}$$

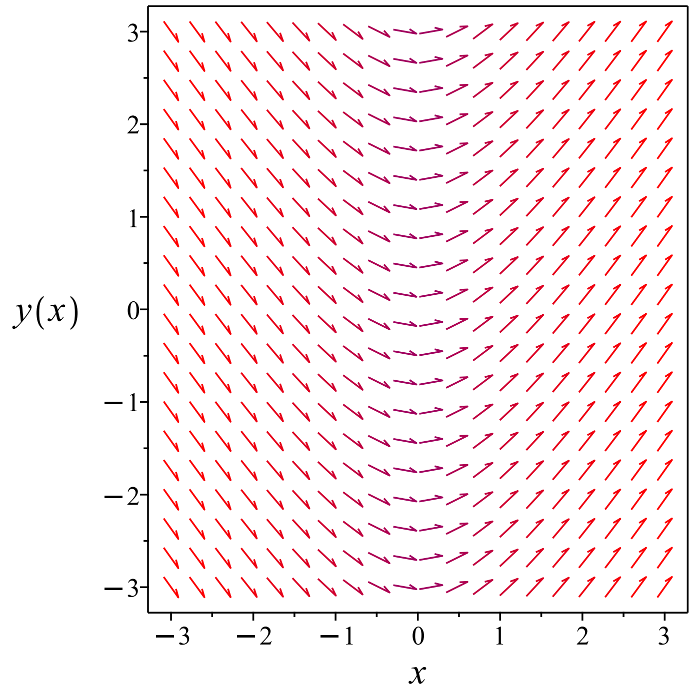


Figure 129: Slope field plot

Verification of solutions

$$y = -\frac{\ln\left(x^2+1\right)}{2} + x\arctan\left(x\right) + c_1$$

Verified OK.

4.26.2 Maple step by step solution

Let's solve

$\tan\left(y'\right) = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \tan\left(y'\right)dx = \int x dx + c_1$

- Cannot compute integral

  $\int \tan\left(y'\right)dx = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(tan(diff(y(x),x))=x,y(x), singsol=all)
```

$$y = x \arctan(x) - \frac{\ln(x^2+1)}{2} + c_1$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 163

```
DSolve[Tan[y'[x]]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x^2}\log(x^2+1)}{2x} - x\cos^{-1}\left(-\frac{1}{\sqrt{x^2+1}}\right) + c_1$$
$$y(x) \to -\frac{\sqrt{x^2}\log(x^2+1)}{2x} + x\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) + c_1$$
$$y(x) \to \frac{\sqrt{x^2}\log(x^2+1)}{2x} + x\cos^{-1}\left(-\frac{1}{\sqrt{x^2+1}}\right) + c_1$$
$$y(x) \to \frac{\sqrt{x^2}\log(x^2+1)}{2x} - x\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) + c_1$$

4.27 problem 92

Internal problem ID [14999]
Internal file name [`OUTPUT/14999_Friday_April_19_2024_04_43_38_AM_84918379/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 92.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

Unable to solve or complete the solution.

$$\boxed{x^2 y' \cos(y) = -1}$$

With initial conditions

$$\left[y(\infty) = \frac{16\pi}{3}\right]$$

4.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{1}{x^2 \cos (y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{16\pi}{3}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is inside this domain. The y domain of $f(x, y)$ when $x = \infty$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{16\pi}{3}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2 \cos (y)}\right) \\ &= -\frac{\sin (y)}{x^2 \cos (y)^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{16\pi}{3}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \infty$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{16\pi}{3}$ is inside this domain. Therefore solution exists and is unique.

4.27.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{\sec (y)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x^2}$ and $g(y) = \sec(y)$. Integrating both sides gives

$$\frac{1}{\sec(y)}\,dy = -\frac{1}{x^2}\,dx$$
$$\int \frac{1}{\sec(y)}\,dy = \int -\frac{1}{x^2}\,dx$$
$$\sin(y) = \frac{1}{x} + c_1$$

Which results in

$$y = \arcsin\left(\frac{c_1 x + 1}{x}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{16\pi}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{16\pi}{3} = \arcsin(c_1)$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{1}{x^2\cos(y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 114: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= -x^2 \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x^2} dx \end{aligned}$$

Which results in

$$S = \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{1}{x^2 \cos(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{x} = \sin(y) + c_1$$

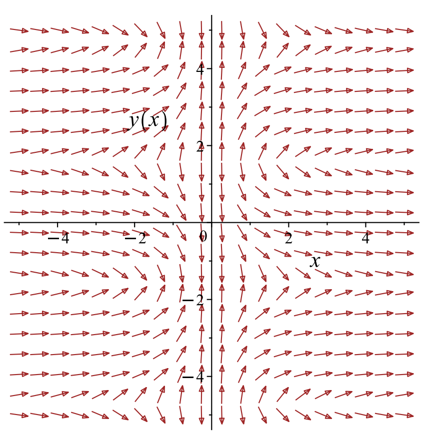
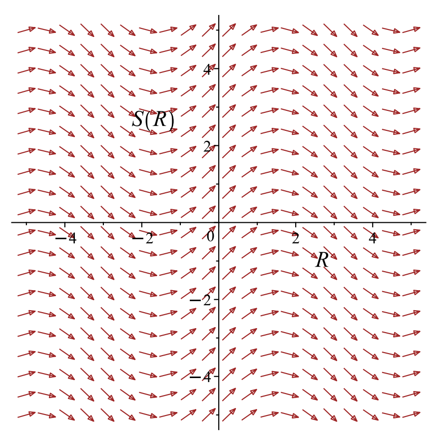
Which simplifies to

$$\frac{1}{x} = \sin(y) + c_1$$

Which gives

$$y = -\arcsin\left(\frac{c_1 x - 1}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{1}{x^2 \cos(y)}$ 	$R = y$ $S = \frac{1}{x}$	$\frac{dS}{dR} = \cos(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{16\pi}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{16\pi}{3} = -\arcsin(c_1)$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(-\cos(y))\,\mathrm{d}y &= \left(\frac{1}{x^2}\right)\mathrm{d}x \\
\left(-\frac{1}{x^2}\right)\mathrm{d}x + (-\cos(y))\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x^2} \\
N(x,y) &= -\cos(y)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\cos(y)) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x^2}\,\mathrm{d}x \\
\phi &= \frac{1}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\cos(y)$. Therefore equation (4) becomes

$$-\cos(y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\cos(y)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-\cos(y))\,\mathrm{d}y$$
$$f(y) = -\sin(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \sin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \sin(y)$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{16\pi}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{3}}{2} = c_1$$

$$c_1 = \frac{\sqrt{3}}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{x} - \sin(y) = \frac{\sqrt{3}}{2}$$

The above simplifies to

$$-2\sin(y)\,x - \sqrt{3}\,x + 2 = 0$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.27.5 Maple step by step solution

Let's solve

$\left[x^2 y' \cos(y) = -1, y(\infty) = \frac{16\pi}{3}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\cos(y)\,y' = -\frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \cos(y)\,y' dx = \int -\frac{1}{x^2} dx + c_1$

- Evaluate integral

  $\sin(y) = \frac{1}{x} + c_1$

- Solve for y

  $y = \arcsin\left(\frac{c_1 x + 1}{x}\right)$

- Use initial condition $y(\infty) = \frac{16\pi}{3}$

  $\frac{16\pi}{3} = 0$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 21

```
dsolve([x^2*diff(y(x),x)*cos(y(x))+1=0,y(infinity) = 16/3*Pi],y(x), singsol=all)
```

$$y = \arcsin\left(\frac{\sqrt{3}\, x - 2}{2x}\right) + 5\pi$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^2*y'[x]*Cos[y[x]]+1==0,{y[Infinity]==16/3*Pi}},y[x],x,IncludeSingularSolutions ->
```

{}

4.28 problem 93

Internal problem ID [15000]
Internal file name [`OUTPUT/15000_Friday_April_19_2024_04_43_40_AM_22920425/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 93.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

Unable to solve or complete the solution.

$$\boxed{x^2y' + \cos(2y) = 1}$$

With initial conditions

$$\left[y(\infty) = \frac{10\pi}{3}\right]$$

4.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y' &= f(x,y)\\ &= -\frac{-1+\cos(2y)}{x^2}\end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{10\pi}{3}$ is

$$\{x<0\lor 0<x\}$$

And the point $x_0=\infty$ is inside this domain. The y domain of $f(x,y)$ when $x=\infty$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=\frac{10\pi}{3}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-1+\cos(2y)}{x^2}\right)\\ &= \frac{2\sin(2y)}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=\frac{10\pi}{3}$ is

$$\{x<0\lor 0<x\}$$

And the point $x_0=\infty$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=\infty$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=\frac{10\pi}{3}$ is inside this domain. Therefore solution exists and is unique.

4.28.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{1-\cos(2y)}{x^2}\end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = 1 - \cos(2y)$. Integrating both sides gives

$$\frac{1}{1-\cos(2y)}\,dy = \frac{1}{x^2}\,dx$$
$$\int \frac{1}{1-\cos(2y)}\,dy = \int \frac{1}{x^2}\,dx$$
$$-\frac{1}{2\tan(y)} = -\frac{1}{x} + c_1$$

Which results in

$$y = -\arctan\left(\frac{x}{2c_1x-2}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{10\pi}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{10\pi}{3} = -\arctan\left(\frac{1}{2c_1}\right)$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-1+\cos(2y)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-1 + \cos(2y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{-1 + \cos(2y)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{-1 + \cos(2R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2\tan(R)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = -\frac{1}{2\tan(y)} + c_1$$

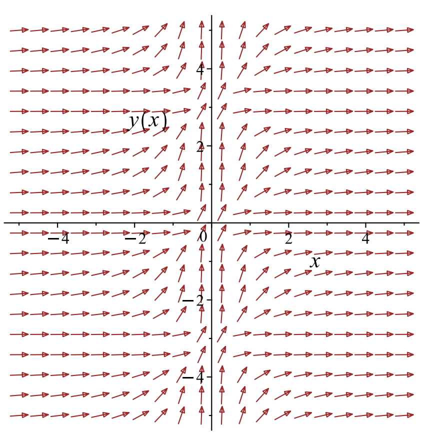
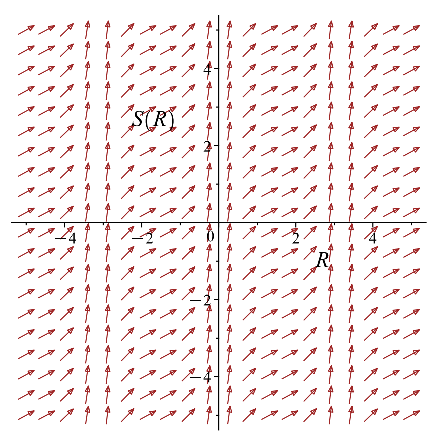
Which simplifies to

$$-\frac{1}{x} = -\frac{1}{2\tan(y)} + c_1$$

Which gives

$$y = \arctan\left(\frac{x}{2c_1x + 2}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-1+\cos(2y)}{x^2}$ 	$R = y$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\frac{1}{-1+\cos(2R)}$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{10\pi}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{10\pi}{3} = \arctan\left(\frac{1}{2c_1}\right)$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.28.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-\cos(2y)}\right)\mathrm{d}y &= \left(\frac{1}{x^2}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2}\right)\mathrm{d}x + \left(\frac{1}{1-\cos(2y)}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x^2}\\ N(x,y) &= \frac{1}{1-\cos(2y)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-\cos(2y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2} \, \mathrm{d}x \\
\phi &= \frac{1}{x} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-\cos(2y)}$. Therefore equation (4) becomes

$$\frac{1}{1 - \cos(2y)} = 0 + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{-1 + \cos(2y)}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{-1 + \cos(2y)} \right) \mathrm{d}y \\
f(y) &= -\frac{1}{2\tan(y)} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \frac{1}{2\tan(y)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \frac{1}{2\tan(y)}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{10\pi}{3}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\sqrt{3}}{6} = c_1$$

$$c_1 = -\frac{\sqrt{3}}{6}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{x} - \frac{1}{2\tan(y)} = -\frac{\sqrt{3}}{6}$$

The above simplifies to

$$\sqrt{3}\, x\tan(y) + 6\tan(y) - 3x = 0$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.28.5 Maple step by step solution

Let's solve

$\left[x^2 y' + \cos(2y) = 1, y(\infty) = \frac{10\pi}{3}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-\cos(2y)+1} = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-\cos(2y)+1} dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

  $$-\frac{1}{2\tan(y)} = -\frac{1}{x} + c_1$$

- Solve for y

  $$y = -\arctan\left(\frac{x}{2(c_1 x - 1)}\right)$$

- Use initial condition $y(\infty) = \frac{10\pi}{3}$

  $$\frac{10\pi}{3} = -\arctan\left(\frac{\infty}{c_1 \infty - 1}\right)$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 23

```
dsolve([x^2*diff(y(x),x)+cos(2*y(x))=1,y(infinity) = 10/3*Pi],y(x), singsol=all)
```

$$y = \frac{7\pi}{2} - \arctan\left(\frac{\sqrt{3}\,x + 6}{3x}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^2*y'[x]+Cos[2*y[x]]==1,{y[Infinity]==10/3*Pi}},y[x],x,IncludeSingularSolutions ->
```

{}

4.29 problem 94

Internal problem ID [15001]
Internal file name [`OUTPUT/15001_Friday_April_19_2024_04_43_43_AM_21036307/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 94.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

Unable to solve or complete the solution.

$$\boxed{y'x^3 - \sin(y) = 1}$$

With initial conditions

$$[y(\infty) = 5\pi]$$

4.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\sin(y)+1}{x^3} \end{aligned}$$

The x domain of $f(x,y)$ when $y=5\pi$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0=\infty$ is inside this domain. The y domain of $f(x,y)$ when $x=\infty$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=5\pi$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sin(y)+1}{x^3}\right) \\ &= \frac{\cos(y)}{x^3} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=5\pi$ is

$$\{x<0 \lor 0<x\}$$

And the point $x_0=\infty$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=\infty$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=5\pi$ is inside this domain. Therefore solution exists and is unique.

4.29.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sin(y)+1}{x^3} \end{aligned}$$

Where $f(x) = \frac{1}{x^3}$ and $g(y) = \sin(y) + 1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\sin(y) + 1}\, dy &= \frac{1}{x^3}\, dx \\
\int \frac{1}{\sin(y) + 1}\, dy &= \int \frac{1}{x^3}\, dx \\
-\frac{2}{\tan\left(\frac{y}{2}\right) + 1} &= -\frac{1}{2x^2} + c_1
\end{aligned}
$$

Which results in

$$
y = -2\arctan\left(\frac{2c_1x^2 + 4x^2 - 1}{2c_1x^2 - 1}\right)
$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 5\pi$ in the above solution gives an equation to solve for the constant of integration.

$$
5\pi = -2\arctan\left(\frac{c_1 + 2}{c_1}\right)
$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(y) + 1}{x^3} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= x^3 \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^3} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sin(y) + 1}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x^3} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sin(y) + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sin(R) + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2}{\tan\left(\frac{R}{2}\right)+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2x^2} = -\frac{2}{\tan\left(\frac{y}{2}\right)+1} + c_1$$

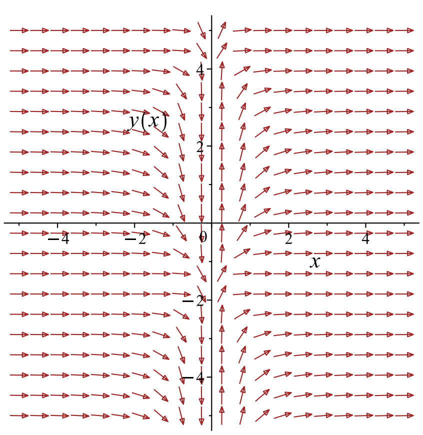
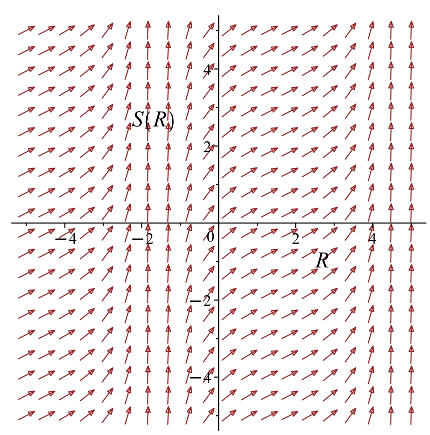
Which simplifies to

$$-\frac{1}{2x^2} = -\frac{2}{\tan\left(\frac{y}{2}\right)+1} + c_1$$

Which gives

$$y = -2\arctan\left(\frac{2c_1x^2 - 4x^2 + 1}{2c_1x^2 + 1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(y)+1}{x^3}$ 	$R = y$ $S = -\frac{1}{2x^2}$	$\frac{dS}{dR} = \frac{1}{\sin(R)+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 5\pi$ in the above solution gives an equation to solve for the constant of integration.

$$5\pi = -2\arctan\left(\frac{-2 + c_1}{c_1}\right)$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.29.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sin(y)+1}\right)\mathrm{d}y = \left(\frac{1}{x^3}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^3}\right)\mathrm{d}x + \left(\frac{1}{\sin(y)+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x^3}$$
$$N(x,y) = \frac{1}{\sin(y)+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^3}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sin(y)+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int M \mathop{\mathrm{d}x} \\
\int \frac{\partial \phi}{\partial x} \mathop{\mathrm{d}x} &= \int -\frac{1}{x^3} \mathop{\mathrm{d}x} \\
\phi &= \frac{1}{2x^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sin(y)+1}$. Therefore equation (4) becomes

$$
\frac{1}{\sin(y) + 1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\sin(y) + 1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathop{\mathrm{d}y} &= \int \left(\frac{1}{\sin(y) + 1}\right) \mathop{\mathrm{d}y} \\
f(y) &= -\frac{2}{\tan\left(\frac{y}{2}\right) + 1} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{1}{2x^2} - \frac{2}{\tan\left(\frac{y}{2}\right) + 1} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{2x^2} - \frac{2}{\tan\left(\frac{y}{2}\right) + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 5\pi$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{2x^2} - \frac{2}{\tan\left(\frac{y}{2}\right) + 1} = 0$$

The above simplifies to

$$-4x^2 + \tan\left(\frac{y}{2}\right) + 1 = 0$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.29.5 Maple step by step solution

Let's solve

$[y'x^3 - \sin(y) = 1, y(\infty) = 5\pi]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sin(y)+1} = \frac{1}{x^3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sin(y)+1}dx = \int \frac{1}{x^3}dx + c_1$

- Evaluate integral

$$-\frac{2}{\tan\left(\frac{y}{2}\right)+1} = -\frac{1}{2x^2} + c_1$$

- Solve for y

$$y = -2\arctan\left(\frac{2c_1x^2+4x^2-1}{2c_1x^2-1}\right)$$

- Use initial condition $y(\infty) = 5\pi$

$$5\pi = -2\arctan\left(\frac{c_1\infty+\infty}{c_1\infty-1}\right)$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✗ Solution by Maple

```
dsolve([x^3*diff(y(x),x)-sin(y(x))=1,y(infinity) = 5*Pi],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^3*y'[x]-Sin[y[x]]==1,{y[Infinity]==5*Pi}},y[x],x,IncludeSingularSolutions -> True]
```

{}

4.30 problem 95

Internal problem ID [15002]
Internal file name [`OUTPUT/15002_Friday_April_19_2024_04_43_44_AM_27100828/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 95.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\left(x^2+1\right) y'-\frac{\cos\left(2y\right)^2}{2}=0}$$

With initial conditions

$$\left[y(-\infty)=\frac{7\pi}{2}\right]$$

4.30.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\cos(2y)^2}{2x^2+2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{7\pi}{2}$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = -\infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.30.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{\cos(2y)^2}{2x^2+2} \end{aligned}$$

Where $f(x) = \frac{1}{2x^2+2}$ and $g(y) = \cos(2y)^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cos(2y)^2}\,dy &= \frac{1}{2x^2+2}\,dx \\ \int \frac{1}{\cos(2y)^2}\,dy &= \int \frac{1}{2x^2+2}\,dx \\ \frac{\tan(2y)}{2} &= \frac{\arctan(x)}{2} + c_1 \end{aligned}$$

Which results in

$$y = \frac{\arctan(\arctan(x) + 2c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = -\infty$ and $y = \frac{7\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{7\pi}{2} = \frac{\arctan\left(-\frac{\pi}{2} + 2c_1\right)}{2}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{\cos(2y)^2}{2x^2+2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 123: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 2x^2 + 2 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{2x^2 + 2} dx \end{aligned}$$

Which results in

$$S = \frac{\arctan(x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\cos(2y)^2}{2x^2 + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{2x^2 + 2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(2y)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(2R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\tan(2R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\arctan(x)}{2} = \frac{\tan(2y)}{2} + c_1$$

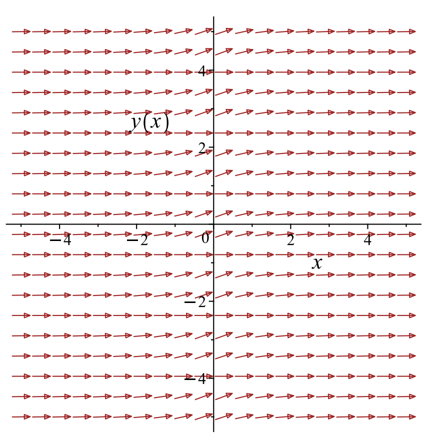
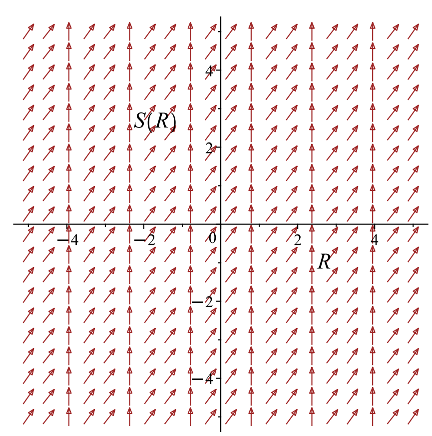
Which simplifies to

$$\frac{\arctan(x)}{2} = \frac{\tan(2y)}{2} + c_1$$

Which gives

$$y = -\frac{\arctan(-\arctan(x) + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(2y)^2}{2x^2+2}$ 	$R = y$ $S = \frac{\arctan(x)}{2}$	$\frac{dS}{dR} = \sec(2R)^2$ 

Initial conditions are used to solve for c_1. Substituting $x = -\infty$ and $y = \frac{7\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{7\pi}{2} = -\frac{\arctan\left(\frac{\pi}{2} + 2c_1\right)}{2}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2}{\cos(2y)^2}\right)\mathrm{d}y = \left(\frac{1}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^2+1}\right)\mathrm{d}x + \left(\frac{2}{\cos(2y)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x^2+1}$$
$$N(x,y) = \frac{2}{\cos(2y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2}{\cos(2y)^2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{x^2+1} \mathrm{d}x$$
$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{\cos(2y)^2}$. Therefore equation (4) becomes

$$\frac{2}{\cos(2y)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{\cos(2y)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(2\sec(2y)^2\right) \mathrm{d}y$$
$$f(y) = \tan(2y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) + \tan(2y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) + \tan(2y)$$

Initial conditions are used to solve for c_1. Substituting $x = -\infty$ and $y = \frac{7\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = c_1$$

$$c_1 = \frac{\pi}{2}$$

Substituting c_1 found above in the general solution gives

$$-\arctan(x) + \tan(2y) = \frac{\pi}{2}$$

Summary

The solution(s) found are the following

$$-\arctan(x) + \tan(2y) = \frac{\pi}{2} \tag{1}$$

Verification of solutions

$$-\arctan(x) + \tan(2y) = \frac{\pi}{2}$$

Verified OK.

4.30.5 Maple step by step solution

Let's solve

$\left[(x^2+1)\, y' - \frac{\cos(2y)^2}{2} = 0, y(-\infty) = \frac{7\pi}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(2y)^2} = \frac{1}{2(x^2+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(2y)^2} dx = \int \frac{1}{2(x^2+1)} dx + c_1$

- Evaluate integral

  $\frac{\tan(2y)}{2} = \frac{\arctan(x)}{2} + c_1$

- Solve for y

  $y = \frac{\arctan(\arctan(x)+2c_1)}{2}$

- Use initial condition $y(-\infty) = \frac{7\pi}{2}$

  $\frac{7\pi}{2} = \frac{\arctan\left(-\frac{\pi}{2}+2c_1\right)}{2}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.437 (sec). Leaf size: 17

```
dsolve([(1+x^2)*diff(y(x),x)-1/2*cos(2*y(x))^2=0,y(-infinity) = 7/2*Pi],y(x), singsol=all)
```

$$y = \frac{\arctan\left(\arctan\left(x\right) + \frac{\pi}{2}\right)}{2} + \frac{7\pi}{2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(1+x^2)*y'[x]-1/2*Cos[2*y[x]]^2==0,{y[-Infinity]==7/2*Pi}},y[x],x,IncludeSingularSol
```

{}

4.31 problem 96

Internal problem ID [15003]
Internal file name [`OUTPUT/15003_Friday_April_19_2024_04_43_46_AM_59032490/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 96.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{e^{y} - e^{4y} y' = 1}$$

4.31.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{e^{4y}}{e^{y} - 1} dy = \int dx$$
$$\int^{y} \frac{e^{4\_a}}{e^{\_a} - 1} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{e^{4\_a}}{e^{\_a} - 1} d\_a = x + c_1 \tag{1}$$

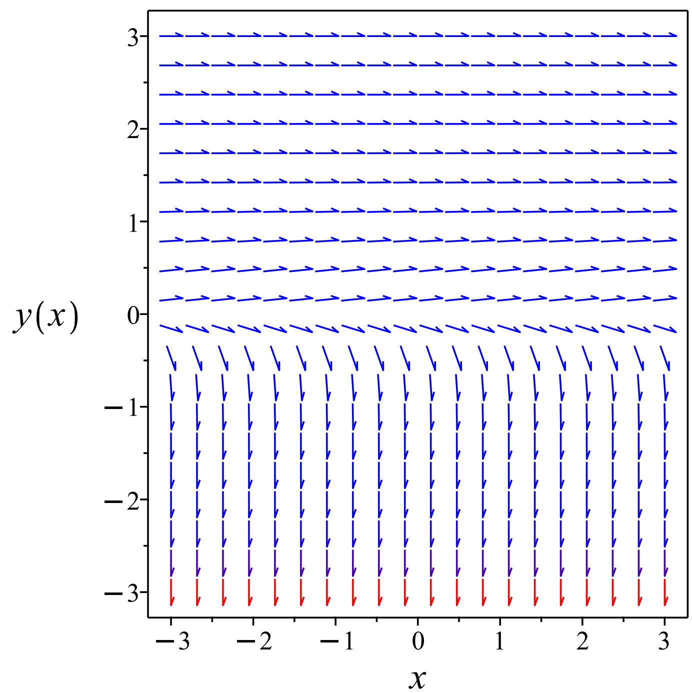


Figure 130: Slope field plot

Verification of solutions

$$\int^{y} \frac{e^{4\_a}}{e^{\_a} - 1} d\_a = x + c_1$$

Verified OK.

4.31.2 Maple step by step solution

Let's solve

$e^{y} - e^{4y} y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' e^{4y}}{-e^{y}+1} = -1$

- Integrate both sides with respect to x

  $\int \frac{y' e^{4y}}{-e^{y}+1} dx = \int (-1) dx + c_1$

- Evaluate integral

$$-\frac{(e^y)^3}{3}-\frac{(e^y)^2}{2}-e^y-\ln(e^y-1)=-x+c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(exp(y(x))=exp(4*y(x))*diff(y(x),x)+1,y(x), singsol=all)
```

$$x-\frac{e^{3y}}{3}-\frac{e^{2y}}{2}-e^y-\ln(e^y-1)+c_1=0$$

✓ Solution by Mathematica

Time used: 0.366 (sec). Leaf size: 48

```
DSolve[Exp[y[x]]==Exp[4*y[x]]*y'[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \text{InverseFunction}\left[\frac{1}{6}e^{\#1}\left(3e^{\#1}+2e^{2\#1}+6\right)+\log\left(e^{\#1}-1\right)\&\right][x+c_1]$$
$$y(x)\to 0$$

4.32 problem 97

Internal problem ID [15004]
Internal file name [`OUTPUT/15004_Friday_April_19_2024_04_43_46_AM_96550427/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 97.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x+1)\,y'-y=-1}$$

4.32.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y-1}{x+1} \end{aligned}$$

Where $f(x) = \frac{1}{x+1}$ and $g(y) = y - 1$. Integrating both sides gives

$$\frac{1}{y-1}\,dy = \frac{1}{x+1}\,dx$$
$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x+1}\,dx$$
$$\ln\left(y-1\right) = \ln\left(x+1\right) + c_1$$

Raising both side to exponential gives

$$y - 1 = \mathrm{e}^{\ln(x+1)+c_1}$$

Which simplifies to

$$y - 1 = (x+1)\,c_2$$

Summary

The solution(s) found are the following

$$y = c_2\mathrm{e}^{\ln(x+1)+c_1} + 1 \tag{1}$$

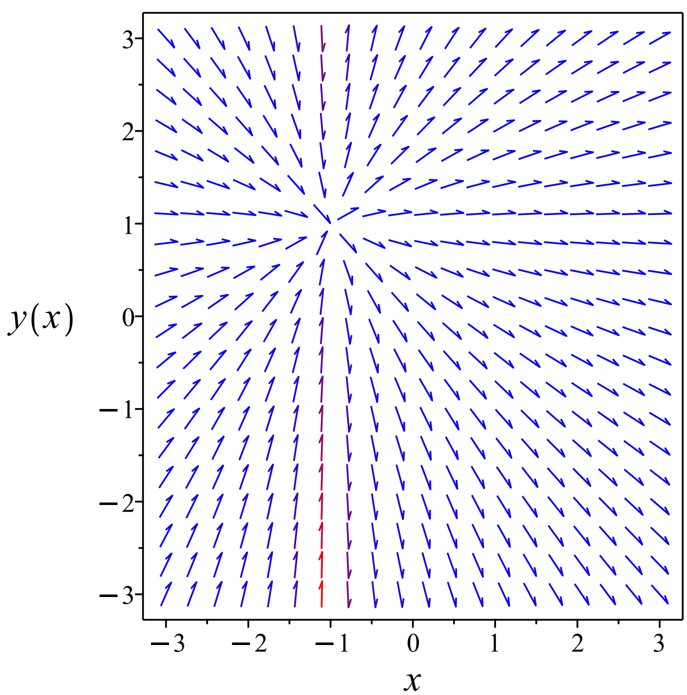


Figure 131: Slope field plot

Verification of solutions

$$y = c_2 e^{\ln(x+1)+c_1} + 1$$

Verified OK.

4.32.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x+1} \\ q(x) &= -\frac{1}{x+1} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x+1} = -\frac{1}{x+1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x+1}dx} \\ &= \frac{1}{x+1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{1}{x+1}\right) \\ \frac{d}{dx}\left(\frac{y}{x+1}\right) &= \left(\frac{1}{x+1}\right)\left(-\frac{1}{x+1}\right) \\ d\left(\frac{y}{x+1}\right) &= \left(-\frac{1}{(x+1)^2}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x+1} &= \int -\frac{1}{(x+1)^2} dx \\ \frac{y}{x+1} &= \frac{1}{x+1} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = 1 + c_1(x + 1)$$

which simplifies to

$$y = c_1x + c_1 + 1$$

Summary

The solution(s) found are the following

$$y = c_1x + c_1 + 1 \tag{1}$$

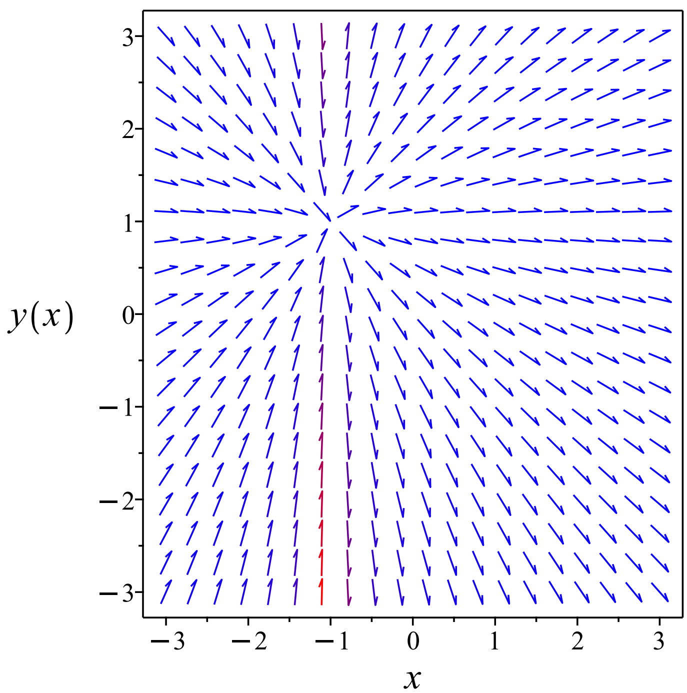


Figure 132: Slope field plot

Verification of solutions

$$y = c_1x + c_1 + 1$$

Verified OK.

4.32.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x+1)\left(u'(x)\,x+u(x)\right)-u(x)\,x=-1$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{-u-1}{x\,(x+1)}\end{aligned}$$

Where $f(x)=\frac{1}{x(x+1)}$ and $g(u)=-u-1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-u-1}\,du &= \frac{1}{x\,(x+1)}\,dx\\ \int \frac{1}{-u-1}\,du &= \int \frac{1}{x\,(x+1)}\,dx\\ -\ln\left(u+1\right) &= -\ln\left(x+1\right)+\ln\left(x\right)+c_2\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{u+1}=\mathrm{e}^{-\ln(x+1)+\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{u+1}=c_3\mathrm{e}^{-\ln(x+1)+\ln(x)}$$

Which simplifies to

$$u(x)=-\frac{\left(\frac{c_3\mathrm{e}^{c_2}x}{x+1}-1\right)(x+1)\,\mathrm{e}^{-c_2}}{c_3x}$$

Therefore the solution y is

$$\begin{aligned}y &= ux\\ &= -\frac{\left(\frac{c_3\mathrm{e}^{c_2}x}{x+1}-1\right)(x+1)\,\mathrm{e}^{-c_2}}{c_3}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(\frac{c_3 e^{c_2 x}}{x+1} - 1\right)(x+1)\,e^{-c_2}}{c_3} \tag{1}$$

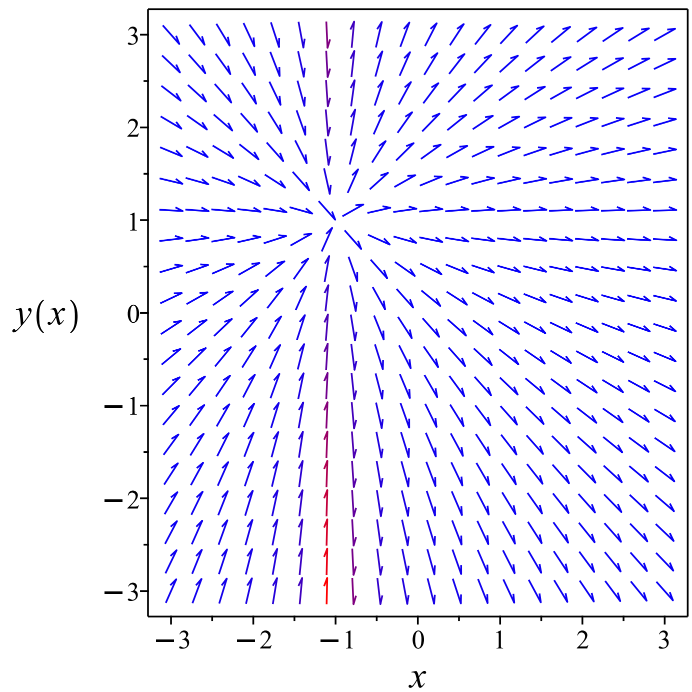


Figure 133: Slope field plot

Verification of solutions

$$y = -\frac{\left(\frac{c_3 e^{c_2 x}}{x+1} - 1\right)(x+1)\,e^{-c_2}}{c_3}$$

Verified OK.

4.32.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 - 1}{X + x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= 1
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X) = \frac{Y(X)}{X}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= \frac{Y}{X}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= 0
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$
\begin{aligned}
u(X) &= \int 0 \,\mathrm{d}X \\
&= c_2
\end{aligned}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = Xc_2$$

Using the solution for $Y(X)$

$$Y(X) = Xc_2$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x - 1$$

Then the solution in y becomes

$$y - 1 = (x + 1) c_2$$

Summary
The solution(s) found are the following

$$y - 1 = (x + 1) c_2 \tag{1}$$

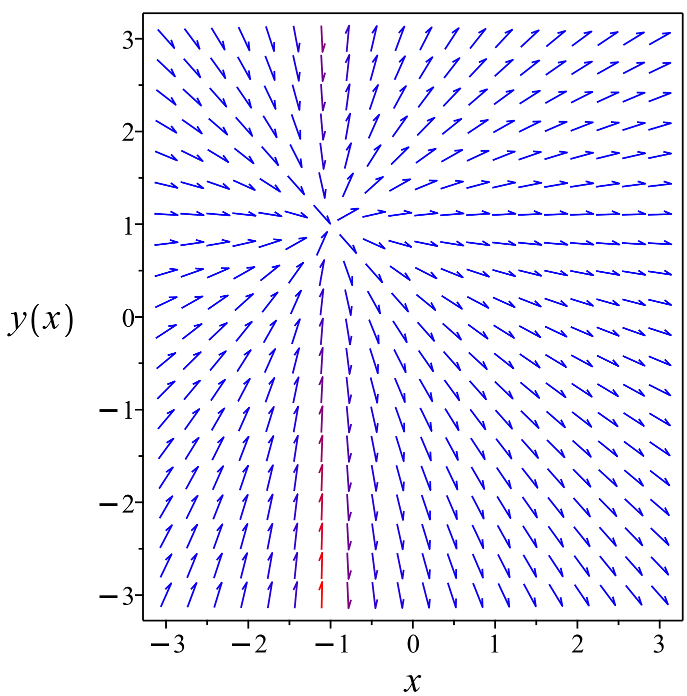


Figure 134: Slope field plot

Verification of solutions

$$
y - 1 = (x + 1) c_2
$$

Verified OK.

4.32.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = \frac{y - 1}{x + 1}
$$
$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 127: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x + 1 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y-1}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+1)^2} \\ S_y &= \frac{1}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{(x+1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{(R+1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{R+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x+1} = \frac{1}{x+1} + c_1$$

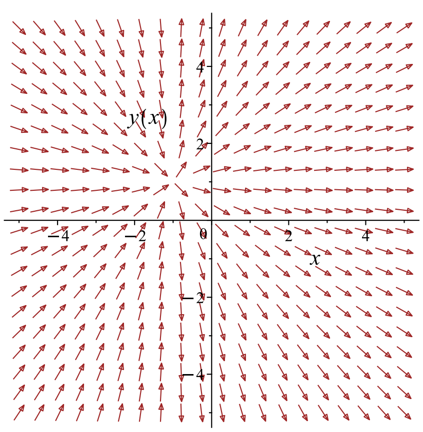
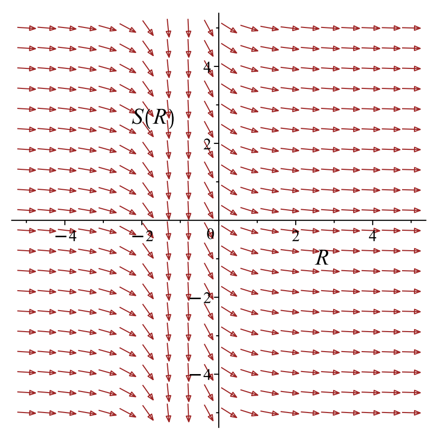
Which simplifies to

$$\frac{y}{x+1} = \frac{1}{x+1} + c_1$$

Which gives

$$y = c_1 x + c_1 + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y-1}{x+1}$ 	$R = x$ $S = \frac{y}{x+1}$	$\frac{dS}{dR} = -\frac{1}{(R+1)^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 + 1 \tag{1}$$

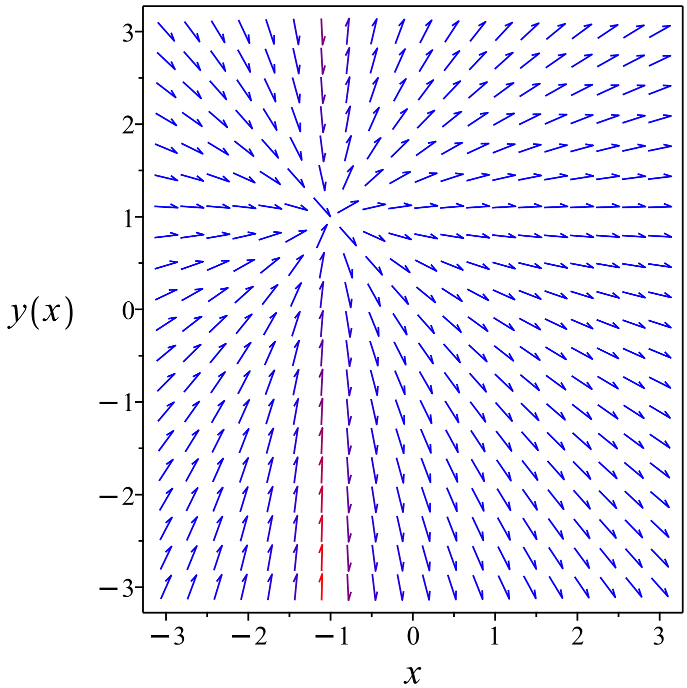


Figure 135: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x + c_1 + 1$$

Verified OK.

4.32.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y-1}\right)\mathrm{d}y = \left(\frac{1}{x+1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x+1}\right)\mathrm{d}x + \left(\frac{1}{y-1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x+1}$$
$$N(x,y) = \frac{1}{y-1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y-1}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x+1}\, \mathrm{d}x\\
\phi &= -\ln(x+1) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y-1}$. Therefore equation (4) becomes

$$
\frac{1}{y-1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y-1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{y-1}\right) \mathrm{d}y\\
f(y) &= \ln(y-1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) + \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) + \ln(y-1)$$

The solution becomes

$$y = e^{c_1}x + e^{c_1} + 1$$

Summary

The solution(s) found are the following

$$y = e^{c_1}x + e^{c_1} + 1 \tag{1}$$

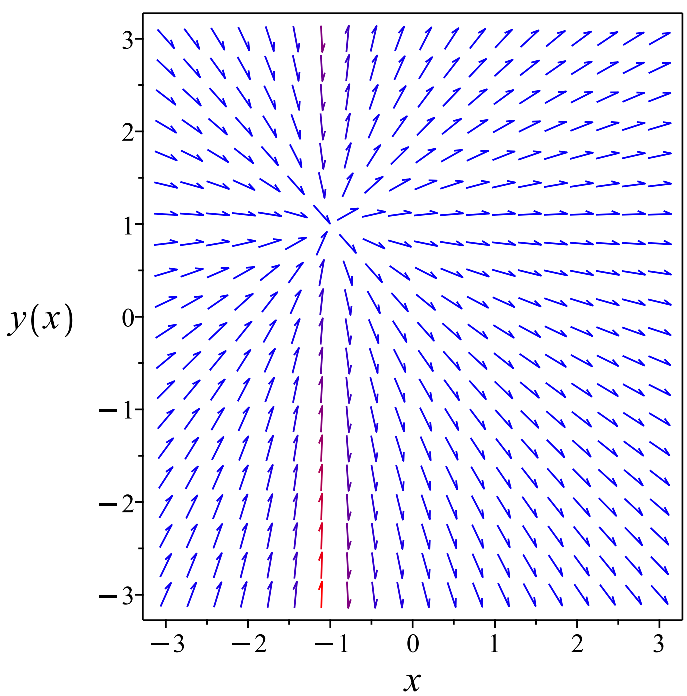


Figure 136: Slope field plot

Verification of solutions

$$y = e^{c_1}x + e^{c_1} + 1$$

Verified OK.

4.32.7 Maple step by step solution

Let's solve

$(x+1)\, y' - y = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = \frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int \frac{1}{x+1}dx + c_1$

- Evaluate integral

  $\ln\left(y-1\right) = \ln\left(x+1\right) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1} x + \mathrm{e}^{c_1} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((x+1)*diff(y(x),x)=y(x)-1,y(x), singsol=all)
```

$$y = c_1 x + c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 18

```
DSolve[(x+1)*y'[x]==y[x]-1,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to 1 + c_1(x+1) \\
y(x) &\to 1
\end{aligned}
$$

4.33 problem 98

Internal problem ID [15005]
Internal file name [`OUTPUT/15005_Friday_April_19_2024_04_43_47_AM_3534104/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 98.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2x(\pi + y) = 0}$$

4.33.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x(2\pi + 2y) \end{aligned}$$

Where $f(x) = x$ and $g(y) = 2\pi + 2y$. Integrating both sides gives

$$\frac{1}{2\pi + 2y}\, dy = x\, dx$$

$$\int \frac{1}{2\pi + 2y}\, dy = \int x\, dx$$

$$\frac{\ln(\pi + y)}{2} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{\pi + y} = \mathrm{e}^{\frac{x^2}{2} + c_1}$$

Which simplifies to

$$\sqrt{\pi + y} = c_2 \mathrm{e}^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = c_2^2 \mathrm{e}^{x^2 + 2c_1} - \pi \tag{1}$$

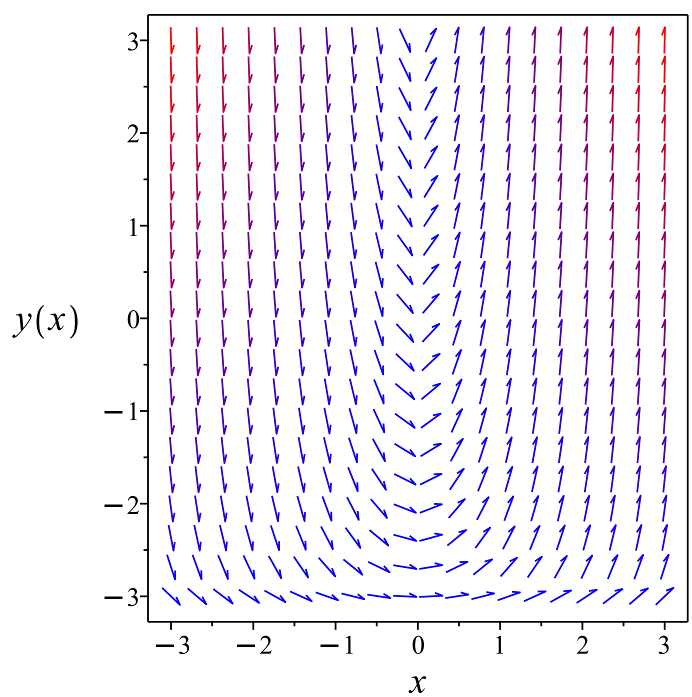


Figure 137: Slope field plot

Verification of solutions

$$y = c_2^2 \mathrm{e}^{x^2 + 2c_1} - \pi$$

Verified OK.

4.33.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -2x\\ q(x) &= 2\pi x\end{aligned}$$

Hence the ode is

$$y' - 2yx = 2\pi x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -2x dx}\\ &= e^{-x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(2\pi x)\\ \frac{d}{dx}\left(e^{-x^2}y\right) &= \left(e^{-x^2}\right)(2\pi x)\\ d\left(e^{-x^2}y\right) &= \left(2\pi x\, e^{-x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x^2}y &= \int 2\pi x\, e^{-x^2}\, dx\\ e^{-x^2}y &= -\pi\, e^{-x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = -e^{x^2}\pi\, e^{-x^2} + c_1 e^{x^2}$$

which simplifies to

$$y = -\pi + c_1 e^{x^2}$$

Summary

The solution(s) found are the following

$$y = -\pi + c_1 e^{x^2} \tag{1}$$

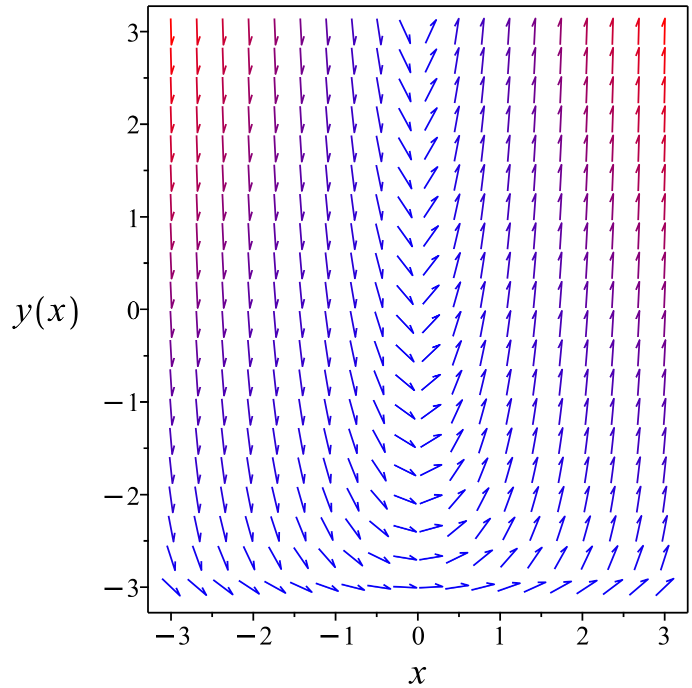


Figure 138: Slope field plot

Verification of solutions

$$y = -\pi + c_1 e^{x^2}$$

Verified OK.

4.33.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2x(\pi + y)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 130: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{x^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2x(\pi + y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2x\,\mathrm{e}^{-x^2} y \\ S_y &= \mathrm{e}^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\pi x\,\mathrm{e}^{-x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\pi R\,\mathrm{e}^{-R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\pi \,\mathrm{e}^{-R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x^2} y = -\pi \,\mathrm{e}^{-x^2} + c_1$$

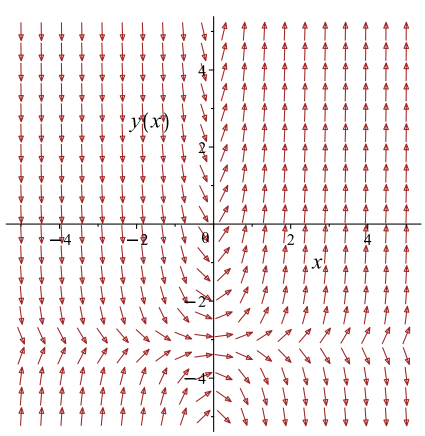
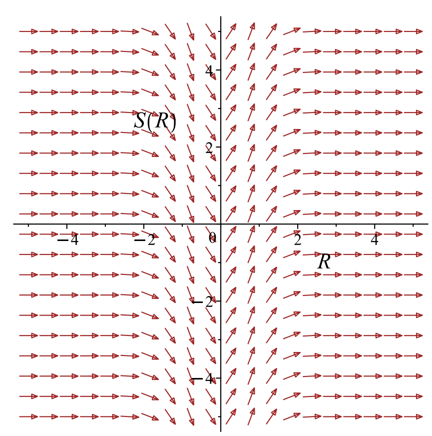
Which simplifies to

$$(\pi + y)\,\mathrm{e}^{-x^2} - c_1 = 0$$

Which gives

$$y = -\left(\pi \,\mathrm{e}^{-x^2} - c_1\right) \mathrm{e}^{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x(\pi + y)$ 	$R = x$ $S = \mathrm{e}^{-x^2} y$	$\frac{dS}{dR} = 2\pi R \,\mathrm{e}^{-R^2}$ 

Summary

The solution(s) found are the following

$$y = -\left(\pi \,\mathrm{e}^{-x^2} - c_1\right) \mathrm{e}^{x^2} \tag{1}$$

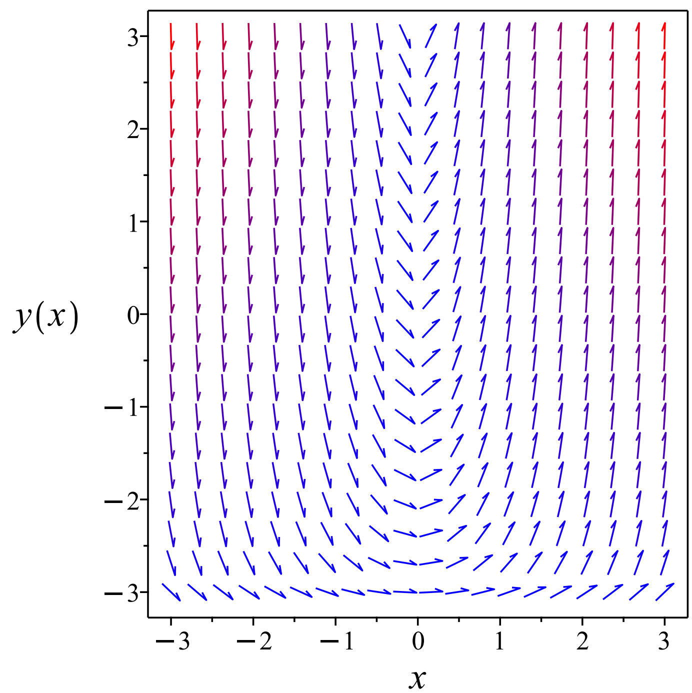


Figure 139: Slope field plot

Verification of solutions

$$y = -\left(\pi\, e^{-x^2} - c_1\right) e^{x^2}$$

Verified OK.

4.33.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2\pi + 2y}\right) \mathrm{d}y &= (x)\, \mathrm{d}x \\
(-x)\, \mathrm{d}x + \left(\frac{1}{2\pi + 2y}\right) \mathrm{d}y &= 0
\end{aligned} \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{2\pi + 2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2\pi + 2y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\\ \phi &= -\frac{x^2}{2} + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2\pi + 2y}$. Therefore equation (4) becomes

$$\frac{1}{2\pi + 2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2\pi + 2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2\pi + 2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(\pi + y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(\pi + y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(\pi + y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{x^2+2c_1} - \pi$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^2+2c_1} - \pi \tag{1}$$

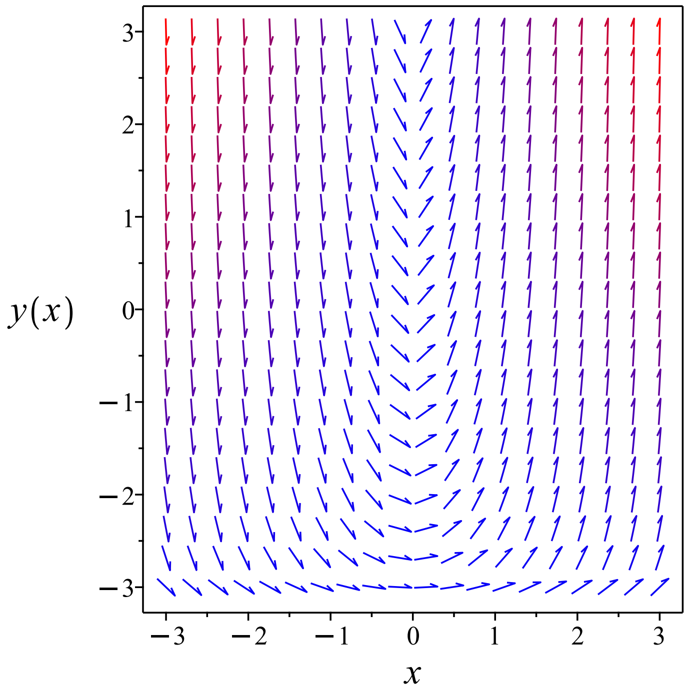


Figure 140: Slope field plot

<u>Verification of solutions</u>

$$y = e^{x^2+2c_1} - \pi$$

Verified OK.

4.33.5 Maple step by step solution

Let's solve

$y' - 2x(\pi + y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\pi+y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\pi+y} dx = \int 2x dx + c_1$

- Evaluate integral

$$\ln(\pi + y) = x^2 + c_1$$

- Solve for y

$$y = e^{x^2+c_1} - \pi$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=2*x*(Pi+y(x)),y(x), singsol=all)
```

$$y = -\pi + c_1 e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 24

```
DSolve[y'[x]==2*x*(Pi+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\pi + c_1 e^{x^2}$$
$$y(x) \to -\pi$$

4.34 problem 99

Internal problem ID [15006]
Internal file name [`OUTPUT/15006_Friday_April_19_2024_04_43_48_AM_20355238/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 4. Equations with variables separable and equations reducible to them. Exercises page 38
Problem number: 99.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

Unable to solve or complete the solution.

$$\boxed{x^2y' + \sin(2y) = 1}$$

With initial conditions

$$\left[y(\infty) = \frac{11\pi}{4}\right]$$

4.34.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y' &= f(x,y)\\ &= -\frac{\sin(2y)-1}{x^2}\end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{11\pi}{4}$ is

$$\{x<0\lor 0<x\}$$

And the point $x_0=\infty$ is inside this domain. The y domain of $f(x,y)$ when $x=\infty$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=\frac{11\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sin(2y)-1}{x^2}\right)\\ &= -\frac{2\cos(2y)}{x^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=\frac{11\pi}{4}$ is

$$\{-\infty<x<\infty\}$$

But the point $x_0=\infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

4.34.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{-\sin(2y)+1}{x^2}\end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = -\sin(2y) + 1$. Integrating both sides gives

$$\frac{1}{-\sin(2y) + 1} dy = \frac{1}{x^2} dx$$

$$\int \frac{1}{-\sin(2y) + 1} dy = \int \frac{1}{x^2} dx$$

$$-\frac{1}{\tan(y) - 1} = -\frac{1}{x} + c_1$$

Which results in

$$y = \arctan\left(\frac{c_1 x - x - 1}{c_1 x - 1}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{11\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{11\pi}{4} = \arctan\left(\frac{-1 + c_1}{c_1}\right)$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.34.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\sin(2y) - 1}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 133: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sin(2y) - 1}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{\sin(2y) - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{\sin(2R) - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{\tan(R) - 1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = -\frac{1}{\tan(y) - 1} + c_1$$

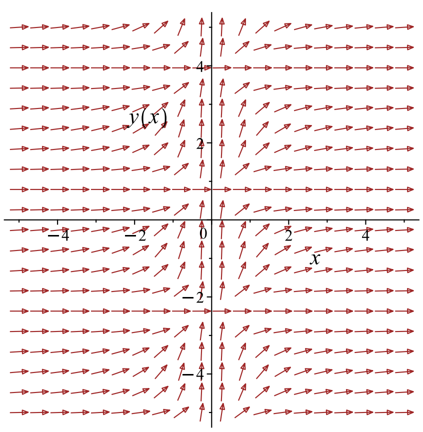
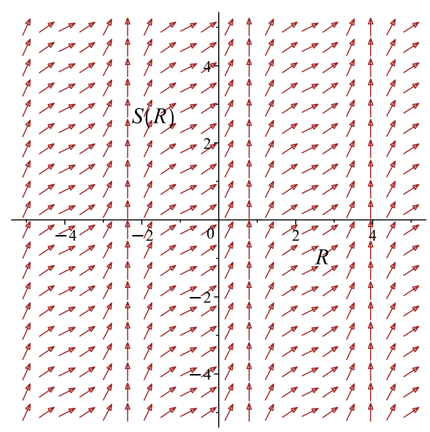
Which simplifies to

$$-\frac{1}{x} = -\frac{1}{\tan(y) - 1} + c_1$$

Which gives

$$y = \arctan\left(\frac{c_1 x + x + 1}{c_1 x + 1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sin(2y)-1}{x^2}$ 	$R = y$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\frac{1}{\sin(2R)-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{11\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{11\pi}{4} = \arctan\left(\frac{1+c_1}{c_1}\right)$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid.

Verification of solutions N/A

4.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-\sin(2y)+1}\right)\mathrm{d}y = \left(\frac{1}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^2}\right)\mathrm{d}x + \left(\frac{1}{-\sin(2y)+1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x^2}$$
$$N(x,y) = \frac{1}{-\sin(2y)+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-\sin(2y)+1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x^2} \, \mathrm{d}x$$
$$\phi = \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-\sin(2y)+1}$. Therefore equation (4) becomes

$$\frac{1}{-\sin(2y) + 1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{\sin(2y) - 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{\sin(2y) - 1}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{\tan(y) - 1} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \frac{1}{\tan(y) - 1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \frac{1}{\tan(y) - 1}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = \frac{11\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{x} - \frac{1}{\tan(y) - 1} = \frac{1}{2}$$

The above simplifies to

$$-\tan(y)\, x + 2\tan(y) - x - 2 = 0$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.34.5 Maple step by step solution

Let's solve

$\left[x^2 y' + \sin(2y) = 1, y(\infty) = \frac{11\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-\sin(2y)+1} = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-\sin(2y)+1} dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

$-\frac{1}{\tan(y)-1} = -\frac{1}{x} + c_1$

- Solve for y

  $y = \arctan\left(\frac{c_1 x - x - 1}{c_1 x - 1}\right)$

- Use initial condition $y(\infty) = \frac{11\pi}{4}$

  $\frac{11\pi}{4} = \arctan\left(\frac{c_1\infty + -\infty}{c_1\infty - 1}\right)$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 20

```
dsolve([x^2*diff(y(x),x)+sin(2*y(x))=1,y(infinity) = 11/4*Pi],y(x), singsol=all)
```

$$y = -\arctan\left(\frac{x+2}{x-2}\right) + 3\pi$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^2*y'[x]+Sin[2*y[x]]==1,{y[Infinity]==11/4*Pi}},y[x],x,IncludeSingularSolutions ->
```

{}

5 Section 5. Homogeneous equations. Exercises page 44

5.1 problem 100

Internal problem ID [15007]
Internal file name [`OUTPUT/15007_Friday_April_19_2024_04_43_57_AM_88411738/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 100.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x - y - x\cos\left(\frac{y}{x}\right)^2 = 0}$$

5.1.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \cos\left(\frac{y}{x}\right)^2 + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)
\end{aligned}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \cos\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\cos(u(x))^2}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{\cos(u)^2}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cos(u)^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\cos(u)^2}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\cos(u)^2}\, du &= \int \frac{1}{x}\, dx \\
\tan(u) &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
\tan(u(x)) - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
\tan\left(\frac{y}{x}\right) - \ln(x) - c_1 = 0
$$

Summary
The solution(s) found are the following

$$\tan\left(\frac{y}{x}\right)-\ln(x)-c_1=0 \tag{1}$$

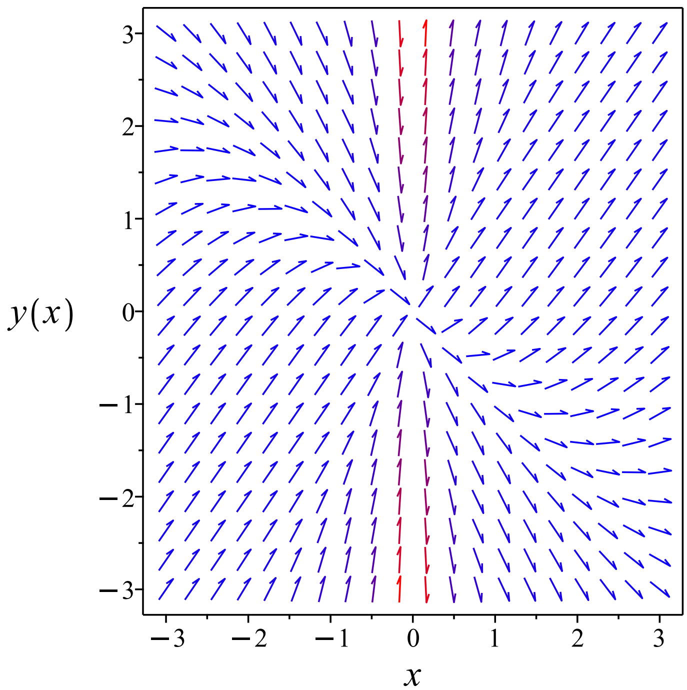


Figure 141: Slope field plot

Verification of solutions

$$\tan\left(\frac{y}{x}\right)-\ln(x)-c_1=0$$

Verified OK.

5.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x+u(x))\,x-u(x)\,x-x\cos(u(x))^2=0$$

In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=\frac{\cos(u)^2}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \cos(u)^2$. Integrating both sides gives

$$\frac{1}{\cos(u)^2}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{\cos(u)^2}\,du = \int \frac{1}{x}\,dx$$

$$\tan(u) = \ln(x) + c_2$$

The solution is

$$\tan(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\tan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

$$\tan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\tan\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

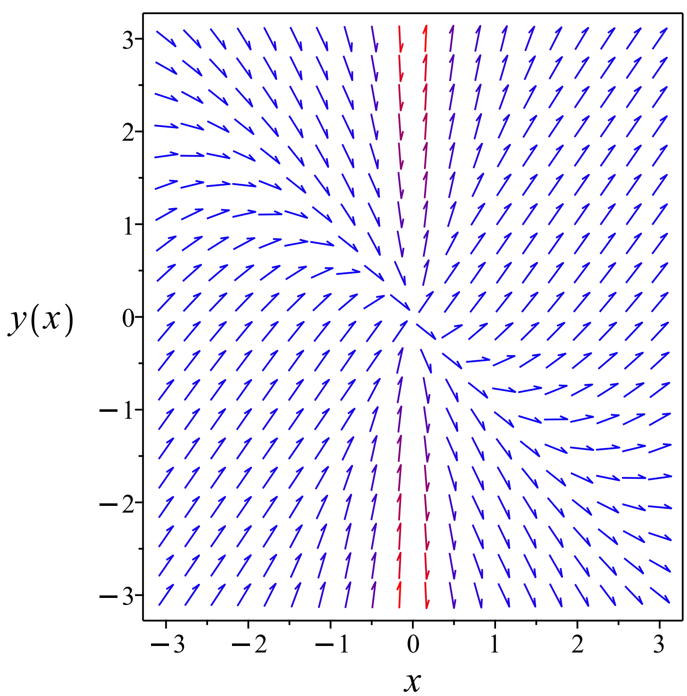


Figure 142: Slope field plot

Verification of solutions

$$\tan\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK.

5.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y+x\cos\left(\frac{y}{x}\right)^2}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 136: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2 \\
\eta(x,y) &= xy
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + x\cos\left(\frac{y}{x}\right)^2}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sec\left(\frac{y}{x}\right)^2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sec(R)^2 S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{-\tan(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{-\tan\left(\frac{y}{x}\right)}$$

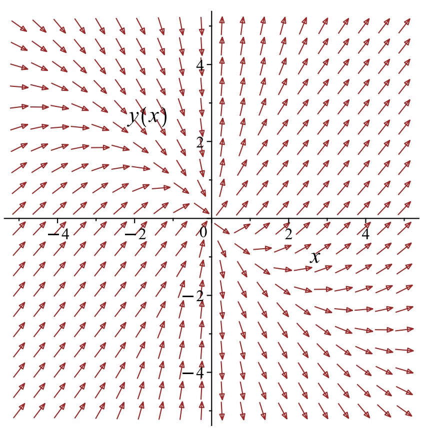
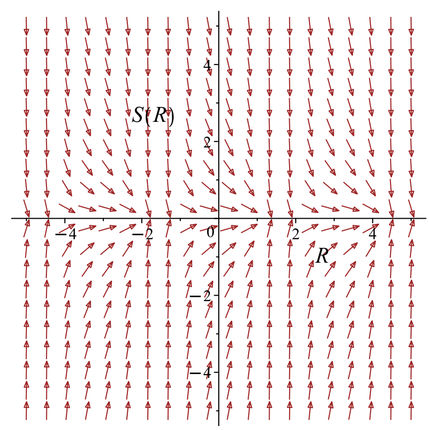
Which simplifies to

$$-\frac{1}{x} = c_1 e^{-\tan\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\arctan\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x\cos\left(\frac{y}{x}\right)^2}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -\sec(R)^2 S(R)$ 

Summary

The solution(s) found are the following

$$y = -\arctan\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

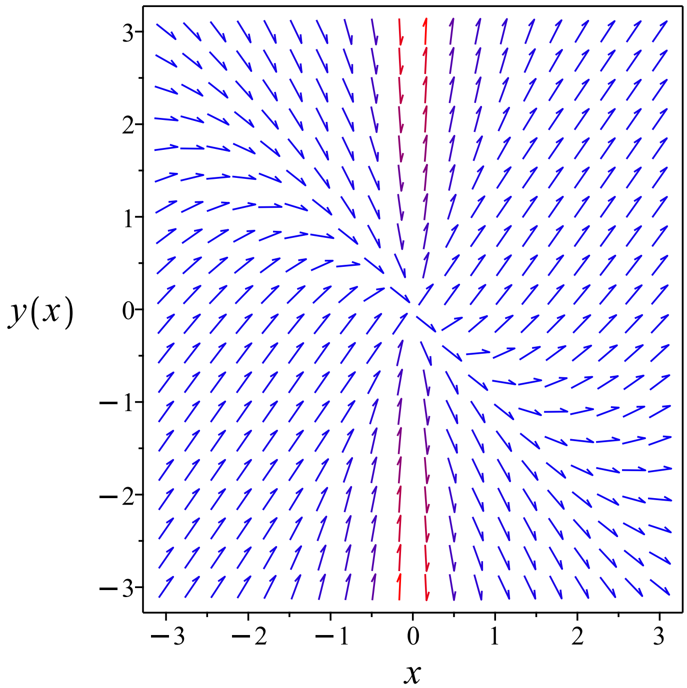


Figure 143: Slope field plot

Verification of solutions

$$y = -\arctan\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)=y(x)+x*cos(y(x)/x)^2,y(x), singsol=all)
```

$$y = \arctan\left(\ln\left(x\right) + c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.434 (sec). Leaf size: 35

```
DSolve[x*y'[x]==y[x]+x*Cos[y[x]/x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan^{-1}(\log(x) + 2c_1)$$
$$y(x) \to -\frac{\pi x}{2}$$
$$y(x) \to \frac{\pi x}{2}$$

5.2 problem 101

Internal problem ID [15008]
Internal file name [`OUTPUT/15008_Friday_April_19_2024_04_43_58_AM_39692444/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 101.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - y = -x}$$

5.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = -1$$

Hence the ode is

$$y' - \frac{y}{x} = -1$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-1)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)(-1)$$
$$d\left(\frac{y}{x}\right) = \left(-\frac{1}{x}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int -\frac{1}{x} dx$$
$$\frac{y}{x} = -\ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -x\ln(x) + c_1 x$$

which simplifies to

$$y = x(-\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(-\ln(x) + c_1) \tag{1}$$

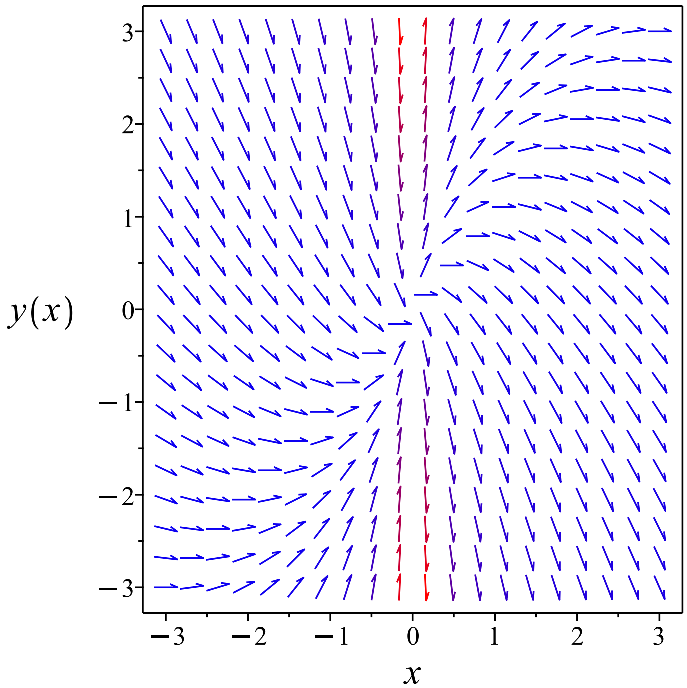


Figure 144: Slope field plot

Verification of solutions

$$y = x(-\ln(x) + c_1)$$

Verified OK.

5.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,x = -x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int -\frac{1}{x}\,\mathrm{d}x \\ &= -\ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(-\ln(x) + c_2) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x(-\ln(x) + c_2) \tag{1}$$

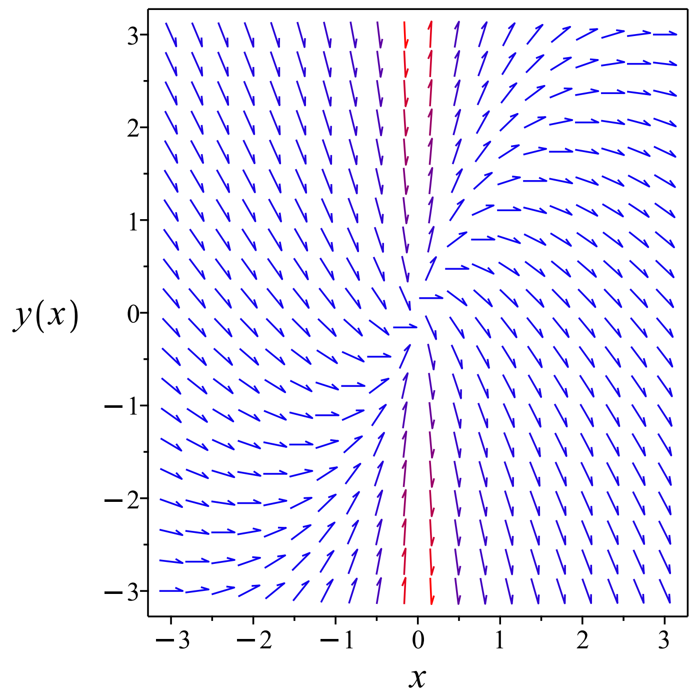


Figure 145: Slope field plot

Verification of solutions

$$y = x(-\ln(x) + c_2)$$

Verified OK.

5.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y - x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y - x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = -\ln(x) + c_1$$

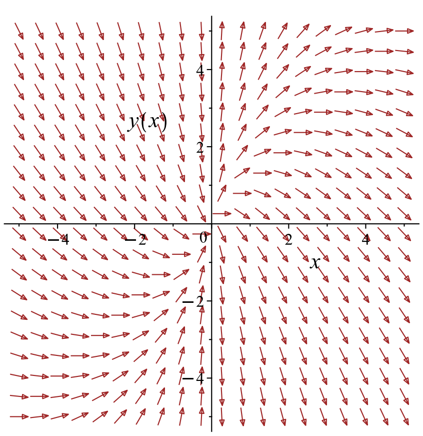
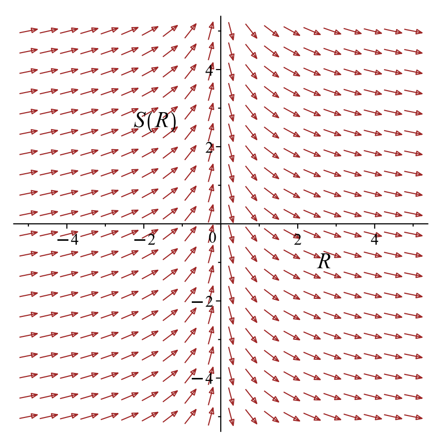
Which simplifies to

$$\frac{y}{x} = -\ln(x) + c_1$$

Which gives

$$y = -x(\ln(x) - c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y-x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -x(\ln(x) - c_1) \tag{1}$$

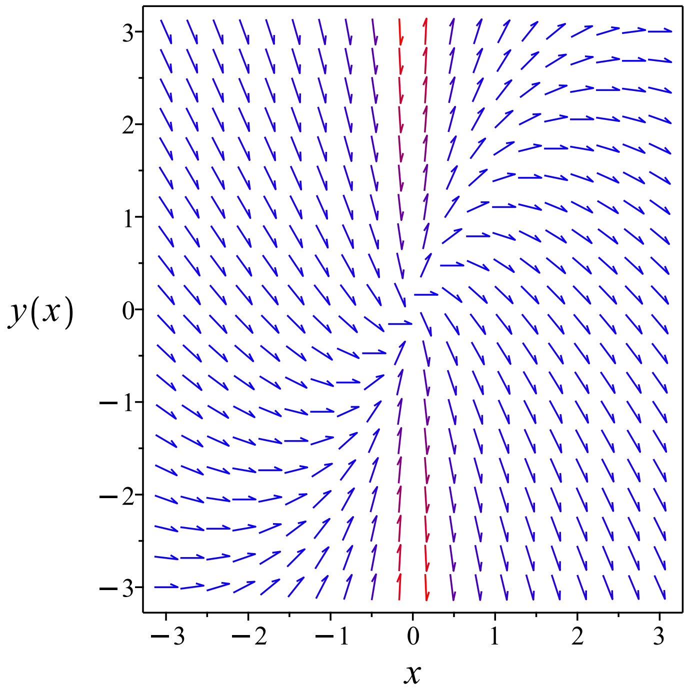


Figure 146: Slope field plot

Verification of solutions

$$y = -x(\ln(x) - c_1)$$

Verified OK.

5.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = (y - x)\,\mathrm{d}x$$
$$(-y + x)\,\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -y + x$$
$$N(x,y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y + x)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-1) - (1)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2}(-y + x) \\ &= \frac{-y + x}{x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2}(x) \\ &= \frac{1}{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-y + x}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-y+x}{x^2} \mathrm{d}x$$

$$\phi = \frac{y}{x} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} + \ln(x)$$

The solution becomes

$$y = -x(\ln(x) - c_1)$$

Summary

The solution(s) found are the following

$$y = -x(\ln(x) - c_1) \tag{1}$$

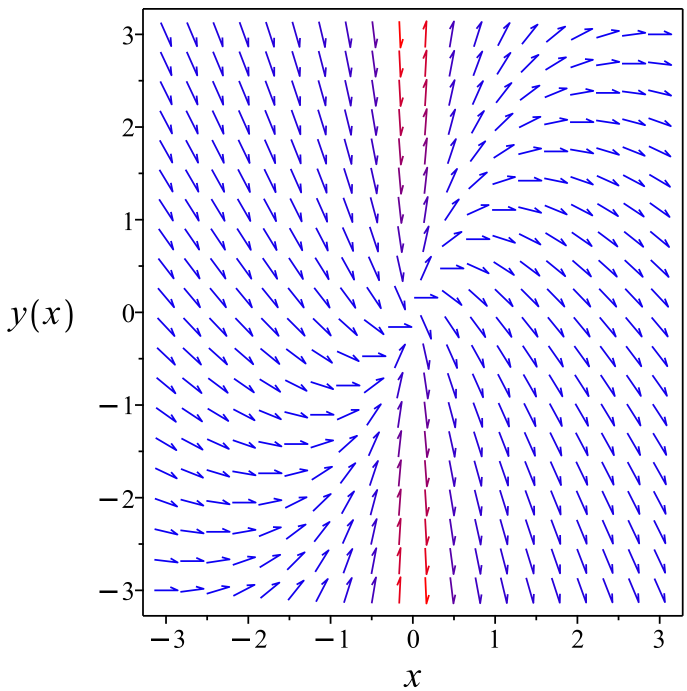


Figure 147: Slope field plot

Verification of solutions

$$y = -x(\ln(x) - c_1)$$

Verified OK.

5.2.5 Maple step by step solution

Let's solve

$y'x - y = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} - 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = -1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = -\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int -\frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(-\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-y(x))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = x(c_1 - \ln(x))$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 14

```
DSolve[(x-y[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(-\log(x) + c_1)$$

5.3 problem 102

Internal problem ID [15009]
Internal file name [`OUTPUT/15009_Friday_April_19_2024_04_43_59_AM_68741429/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 102.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x - y(\ln(y) - \ln(x)) = 0}$$

5.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(\ln(y) - \ln(x))}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \frac{y(\ln(y)-\ln(x))(b_3-a_2)}{x} - \frac{y^2(\ln(y)-\ln(x))^2 a_3}{x^2} \\ &- \left(-\frac{y}{x^2} - \frac{y(\ln(y)-\ln(x))}{x^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{\ln(y)-\ln(x)}{x} + \frac{1}{x}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\ln(x)^2 y^2 a_3 - 2\ln(x)\ln(y)y^2a_3 + \ln(y)^2 y^2 a_3 - \ln(x)x^2b_2 + \ln(x)y^2a_3 + \ln(y)x^2b_2 - \ln(y)y^2a_3 - \ln}{x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\ln(x)^2 y^2 a_3 + 2\ln(x)\ln(y)y^2a_3 - \ln(y)^2y^2a_3 + \ln(x)x^2b_2 \\ &- \ln(x)y^2a_3 - \ln(y)x^2b_2 + \ln(y)y^2a_3 + \ln(x)xb_1 - \ln(x)ya_1 \\ &- \ln(y)xb_1 + \ln(y)ya_1 + xya_2 - xyb_3 + y^2a_3 - xb_1 + ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(x), \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(x) = v_3, \ln(y) = v_4\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_3^2v_2^2a_3 + 2v_3v_4v_2^2a_3 - v_4^2v_2^2a_3 - v_3v_2^2a_3 + v_4v_2^2a_3 + v_3v_1^2b_2 - v_4v_1^2b_2 - v_3v_2a_1 \\ &+ v_4v_2a_1 + v_1v_2a_2 + v_2^2a_3 + v_3v_1b_1 - v_4v_1b_1 - v_1v_2b_3 + v_2a_1 - v_1b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$v_3 v_1^2 b_2 - v_4 v_1^2 b_2 + (-b_3 + a_2) v_1 v_2 + v_3 v_1 b_1 - v_4 v_1 b_1 - v_1 b_1 - v_3^2 v_2^2 a_3 + 2 v_3 v_4 v_2^2 a_3 - v_3 v_2^2 a_3 - v_4^2 v_2^2 a_3 + v_4 v_2^2 a_3 + v_2^2 a_3 - v_3 v_2 a_1 + v_4 v_2 a_1 + v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ -a_3 &= 0 \\ 2a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(\ln(y) - \ln(x))}{x}\right)(x) \\
&= y - \ln(y)\,y + y\ln(x) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y - \ln(y)\,y + y\ln(x)} dy
\end{aligned}
$$

Which results in

$$
S = -\ln(1 - \ln(y) + \ln(x))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y(\ln(y) - \ln(x))}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{x\left(1-\ln\left(y\right)+\ln\left(x\right)\right)} \\
S_y &= \frac{1}{y\left(1-\ln\left(y\right)+\ln\left(x\right)\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln\left(R\right)+c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln\left(1-\ln\left(y\right)+\ln\left(x\right)\right) = -\ln\left(x\right)+c_1
$$

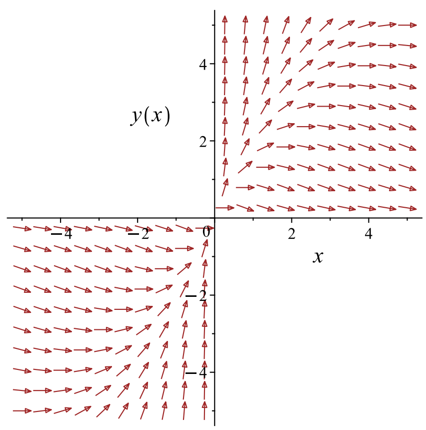
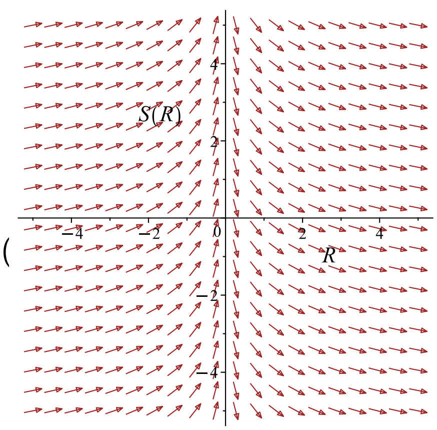
Which simplifies to

$$
-\ln\left(1-\ln\left(y\right)+\ln\left(x\right)\right) = -\ln\left(x\right)+c_1
$$

Which gives

$$
y = e^{\left(\ln(x)e^{c_1}+e^{c_1}-x\right)e^{-c_1}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(y)-\ln(x))}{x}$ 	$R = x$ $S = -\ln(1 - \ln(y) + \ln($	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{(\ln(x)\mathrm{e}^{c_1}+\mathrm{e}^{c_1}-x)\mathrm{e}^{-c_1}} \tag{1}$$

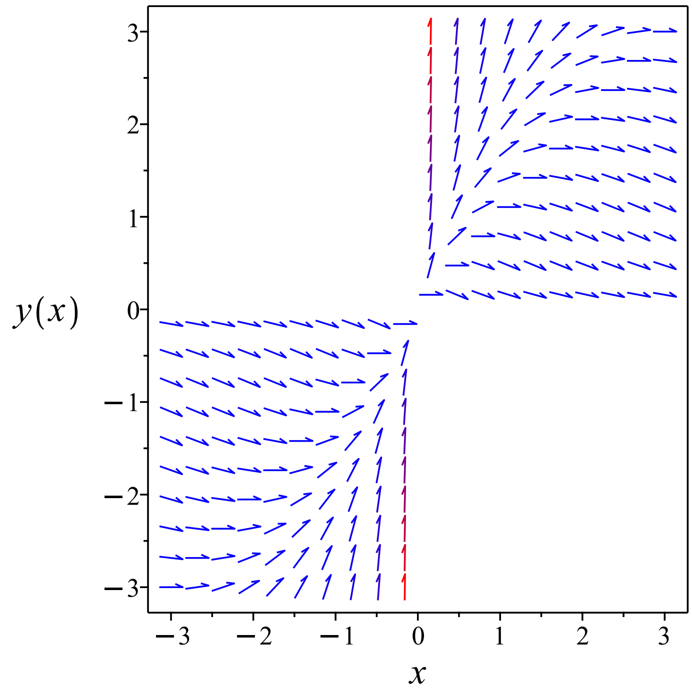


Figure 148: Slope field plot

Verification of solutions

$$y = e^{(\ln(x)e^{c_1}+e^{c_1}-x)e^{-c_1}}$$

Verified OK.

5.3.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\, \mathrm{d}y &= (y(\ln(y) - \ln(x)))\, \mathrm{d}x \\
(-y(\ln(y) - \ln(x)))\, \mathrm{d}x + (x)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y(\ln(y) - \ln(x)) \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(\ln(y) - \ln(x))) \\
&= -1 + \ln(x) - \ln(y)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = -y(\ln(y) - \ln(x))$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= -\frac{\ln(y) - \ln(x)}{x^2} \\
N &= \frac{1}{yx}
\end{aligned}
$$

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{1}{xy}\right) \mathrm{d}y = \left(\frac{\ln(y) - \ln(x)}{x^2}\right) \mathrm{d}x$$

$$\left(-\frac{\ln(y) - \ln(x)}{x^2}\right) \mathrm{d}x + \left(\frac{1}{xy}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\ln(y) - \ln(x)}{x^2}$$
$$N(x, y) = \frac{1}{xy}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\ln(y) - \ln(x)}{x^2}\right) = -\frac{1}{x^2 y}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{xy}\right) = -\frac{1}{x^2 y}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{\ln(y) - \ln(x)}{x^2} \mathrm{d}x$$
$$\phi = \frac{-1 + \ln(y) - \ln(x)}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{xy} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{xy}$. Therefore equation (4) becomes

$$\frac{1}{xy} = \frac{1}{xy} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-1 + \ln(y) - \ln(x)}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-1 + \ln(y) - \ln(x)}{x}$$

The solution becomes

$$y = e^{c_1 x + 1} x$$

Summary

The solution(s) found are the following

$$y = e^{c_1 x + 1} x \tag{1}$$

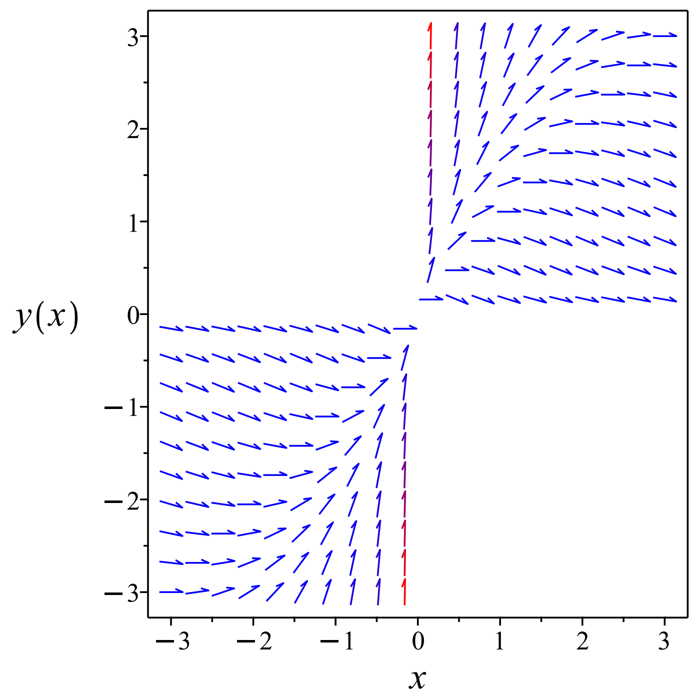


Figure 149: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{c_1 x+1} x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)=y(x)*( ln(y(x))-ln(x) ),y(x), singsol=all)
```

$$y = \mathrm{e}^{c_1 x+1} x$$

✓ Solution by Mathematica

Time used: 0.226 (sec). Leaf size: 24

```
DSolve[x*y'[x]==y[x]*( Log[y[x]]-Log[x] ),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x e^{1+e^{c_1 x}}$$
$$y(x) \to e x$$

5.4 problem 103

Internal problem ID [15010]
Internal file name [`OUTPUT/15010_Friday_April_19_2024_04_44_01_AM_77165017/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 103.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{x^2y' - y^2 + yx = x^2}$$

5.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) - u(x)^2\,x^2 + u(x)\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2 - 2u + 1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2 - 2u + 1$. Integrating both sides gives

$$\frac{1}{u^2 - 2u + 1} \, du = \frac{1}{x} \, dx$$
$$\int \frac{1}{u^2 - 2u + 1} \, du = \int \frac{1}{x} \, dx$$
$$-\frac{1}{u - 1} = \ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x) - 1} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{1}{\frac{y}{x} - 1} - \ln(x) - c_2 = 0$$
$$\frac{(\ln(x) + c_2) y - x(c_2 + \ln(x) - 1)}{-y + x} = 0$$

Summary

The solution(s) found are the following

$$\frac{(\ln(x) + c_2) y - x(c_2 + \ln(x) - 1)}{-y + x} = 0 \tag{1}$$

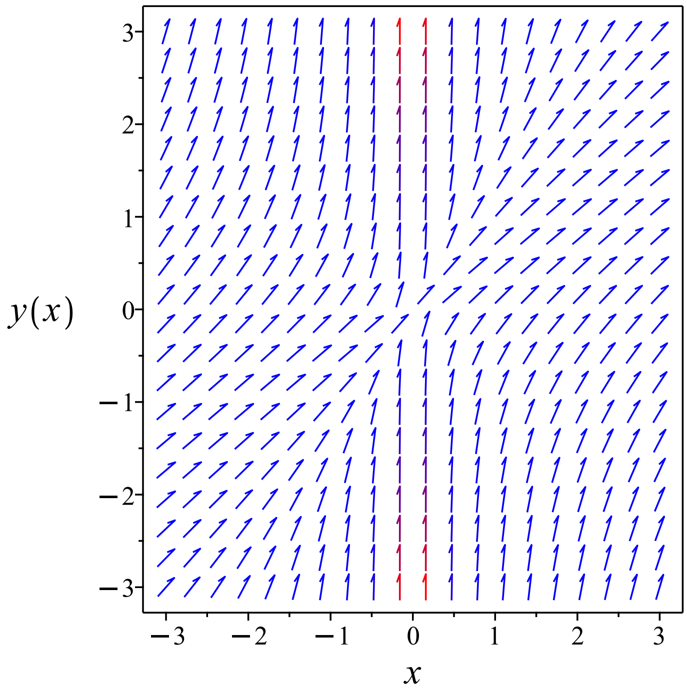


Figure 150: Slope field plot

Verification of solutions

$$\frac{(\ln(x)+c_2)\,y-x(c_2+\ln(x)-1)}{-y+x}=0$$

Verified OK.

5.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^2-xy+y^2}{x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{(x^2-xy+y^2)\,(b_3-a_2)}{x^2}-\frac{(x^2-xy+y^2)^2\,a_3}{x^4}-\left(\frac{2x-y}{x^2}-\frac{2(x^2-xy+y^2)}{x^3}\right)(xa_2+ya_3+a_1)-\frac{(-x+2y)\,(xb_2+yb_3+b_1)}{x^2}=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2+x^4a_3-2b_2x^4-x^4b_3-2x^3ya_3+2x^3yb_2-x^2y^2a_2+4x^2y^2a_3+x^2y^2b_3-4x\,y^3a_3+y^4a_3-x^3b_1+}{x^4}$$
$$=0$$

Setting the numerator to zero gives

$$-x^4a_2-x^4a_3+2b_2x^4+x^4b_3+2x^3ya_3-2x^3yb_2+x^2y^2a_2-4x^2y^2a_3-x^2y^2b_3+4x\,y^3a_3-y^4a_3+x^3b_1-x^2ya_1-2x^2yb_1+2x\,y^2a_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^4+a_2v_1^2v_2^2-a_3v_1^4+2a_3v_1^3v_2-4a_3v_1^2v_2^2+4a_3v_1v_2^3-a_3v_2^4+2b_2v_1^4-2b_2v_1^3v_2+b_3v_1^4-b_3v_1^2v_2^2-a_1v_1^2v_2+2a_1v_1v_2^2+b_1v_1^3-2b_1v_1^2v_2=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + 2b_2 + b_3) v_1^4 + (2a_3 - 2b_2) v_1^3 v_2 + b_1 v_1^3 + (a_2 - 4a_3 - b_3) v_1^2 v_2^2 + (-a_1 - 2b_1) v_1^2 v_2 + 4a_3 v_1 v_2^3 + 2a_1 v_1 v_2^2 - a_3 v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
2a_1 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-a_1 - 2b_1 &= 0 \\
2a_3 - 2b_2 &= 0 \\
a_2 - 4a_3 - b_3 &= 0 \\
-a_2 - a_3 + 2b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{x^2 - xy + y^2}{x^2}\right)(x) \\ &= \frac{-x^2 + 2xy - y^2}{x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2+2xy-y^2}{x}} dy\end{aligned}$$

Which results in

$$S = \frac{x}{y - x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 - xy + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(-y+x)^2} \\
S_y &= -\frac{x}{(-y+x)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x}{y-x} = -\ln(x) + c_1
$$

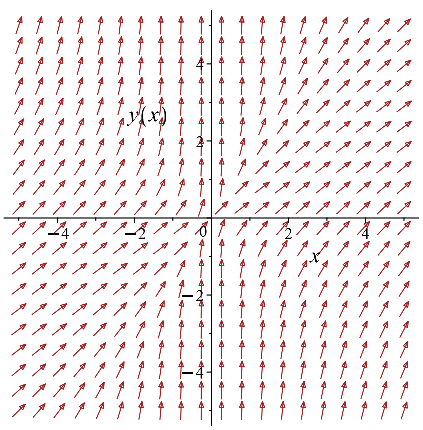
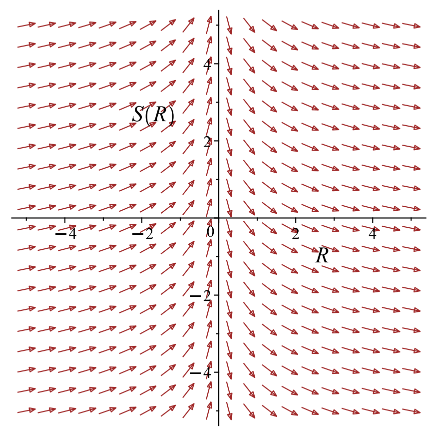
Which simplifies to

$$
\frac{x}{y-x} = -\ln(x) + c_1
$$

Which gives

$$
y = \frac{x(\ln(x) - c_1 - 1)}{\ln(x) - c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2-xy+y^2}{x^2}$ 	$R = x$ $S = \frac{x}{y-x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x(\ln(x) - c_1 - 1)}{\ln(x) - c_1} \tag{1}$$

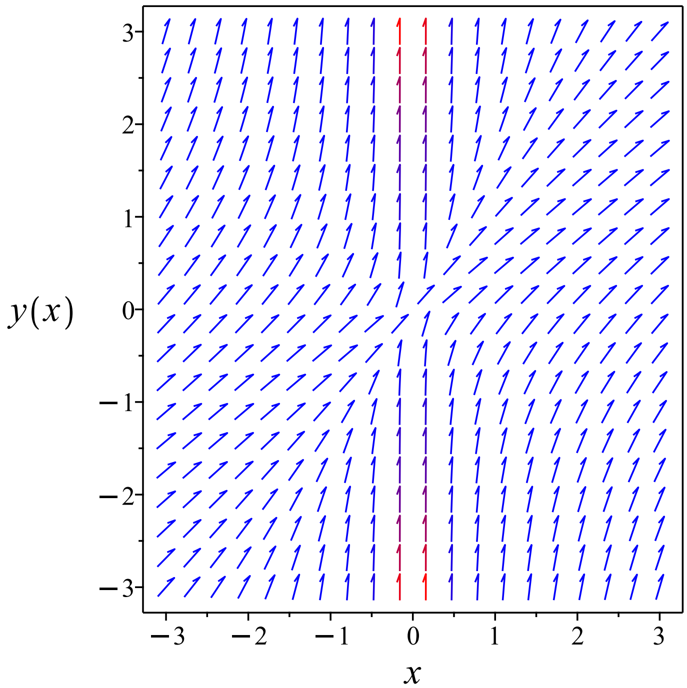


Figure 151: Slope field plot

Verification of solutions

$$y = \frac{x(\ln(x) - c_1 - 1)}{\ln(x) - c_1}$$

Verified OK.

5.4.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 - xy + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 1 - \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= -\frac{1}{x^3} \\
f_2^2 f_0 &= \frac{1}{x^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} + \frac{3u'(x)}{x^3} + \frac{u(x)}{x^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_1 + \ln(x) c_2}{x}
$$

The above shows that

$$
u'(x) = \frac{c_2 - c_1 - \ln(x) c_2}{x^2}
$$

Using the above in (1) gives the solution

$$
y = -\frac{x(c_2 - c_1 - \ln(x) c_2)}{c_1 + \ln(x) c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{x(-1 + c_3 + \ln(x))}{c_3 + \ln(x)}
$$

Summary
The solution(s) found are the following

$$y = \frac{x(-1 + c_3 + \ln(x))}{c_3 + \ln(x)} \tag{1}$$

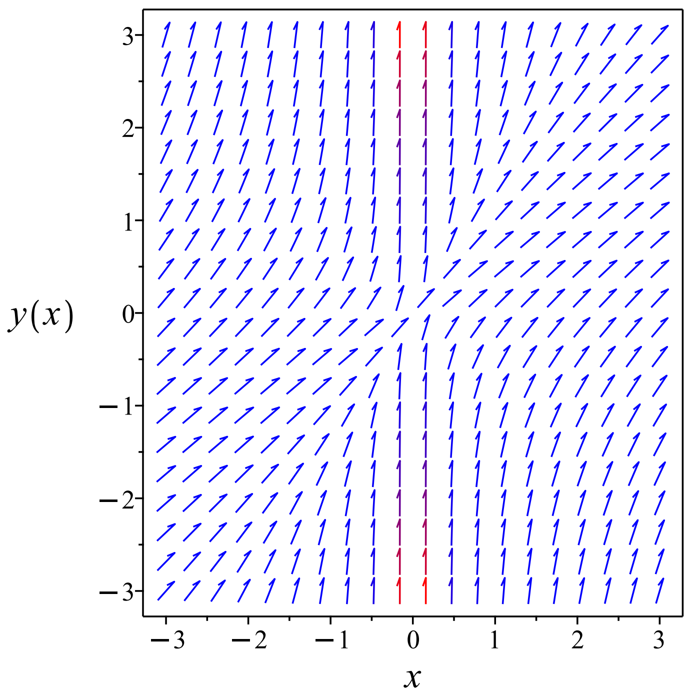


Figure 152: Slope field plot

Verification of solutions

$$y = \frac{x(-1 + c_3 + \ln(x))}{c_3 + \ln(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)=y(x)^2-x*y(x)+x^2,y(x), singsol=all)
```

$$y = \frac{x(\ln(x) + c_1 - 1)}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 25

```
DSolve[x^2*y'[x]==y[x]^2-x*y[x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x(\log(x) - 1 + c_1)}{\log(x) + c_1}$$
$$y(x) \to x$$

5.5 problem 104

Internal problem ID [15011]
Internal file name [`OUTPUT/15011_Friday_April_19_2024_04_44_02_AM_63147948/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 104.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'x - y - \sqrt{y^2 - x^2} = 0}$$

5.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{-x^2 + y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{\left(y+\sqrt{-x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{-x^2+y^2}\right)^2 a_3}{x^2} \\ & - \left(-\frac{1}{\sqrt{-x^2+y^2}} - \frac{y+\sqrt{-x^2+y^2}}{x^2}\right)(x a_2 + y a_3 + a_1) \\ & - \frac{\left(1+\frac{y}{\sqrt{-x^2+y^2}}\right)(x b_2 + y b_3 + b_1)}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(-x^2+y^2)^{\frac{3}{2}} a_3 - x^3 a_2 + x^3 b_3 - 2x^2 y a_3 + x^2 y b_2 + y^3 a_3 + \sqrt{-x^2+y^2}\, x b_1 - \sqrt{-x^2+y^2}\, y a_1 + x y b_1 - y^2}{\sqrt{-x^2+y^2}\, x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -\left(-x^2+y^2\right)^{\frac{3}{2}} a_3 + x^3 a_2 - x^3 b_3 + 2x^2 y a_3 - x^2 y b_2 - y^3 a_3 \\ & - \sqrt{-x^2+y^2}\, x b_1 + \sqrt{-x^2+y^2}\, y a_1 - x y b_1 + y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & -\left(-x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(-x^2+y^2\right) x b_3 - \left(-x^2+y^2\right) y a_3 + x^3 a_2 + x^2 y a_3 - x^2 y b_2 \\ & - x\, y^2 b_3 + \left(-x^2+y^2\right) a_1 - \sqrt{-x^2+y^2}\, x b_1 + \sqrt{-x^2+y^2}\, y a_1 + x^2 a_1 - x y b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} & x^3 a_2 - x^3 b_3 + x^2 \sqrt{-x^2+y^2}\, a_3 + 2x^2 y a_3 - x^2 y b_2 - \sqrt{-x^2+y^2}\, y^2 a_3 \\ & - y^3 a_3 - \sqrt{-x^2+y^2}\, x b_1 - x y b_1 + \sqrt{-x^2+y^2}\, y a_1 + y^2 a_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} v_1^3 a_2 + 2v_1^2 v_2 a_3 + v_1^2 v_3 a_3 - v_2^3 a_3 - v_3 v_2^2 a_3 - v_1^2 v_2 b_2 \\ - v_1^3 b_3 + v_2^2 a_1 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} (-b_3 + a_2) v_1^3 + (2a_3 - b_2) v_1^2 v_2 + v_1^2 v_3 a_3 - v_1 v_2 b_1 \\ - v_3 v_1 b_1 - v_2^3 a_3 - v_3 v_2^2 a_3 + v_2^2 a_1 + v_3 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ 2a_3 - b_2 &= 0 \\ -b_3 + a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y + \sqrt{-x^2 + y^2}}{x}\right)(x) \\
&= -\sqrt{-x^2 + y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{-x^2 + y^2}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln\left(y + \sqrt{-x^2 + y^2}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+\sqrt{-x^2+y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{\sqrt{-x^2+y^2}\,\left(y+\sqrt{-x^2+y^2}\right)} \\
S_y &= -\frac{1}{\sqrt{-x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x^2-2y^2-2\sqrt{-x^2+y^2}\,y}{x\sqrt{-x^2+y^2}\,\left(y+\sqrt{-x^2+y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y+\sqrt{y^2-x^2}\right) = -2\ln(x) + c_1$$

Which simplifies to

$$-\ln\left(y+\sqrt{y^2-x^2}\right) = -2\ln(x) + c_1$$

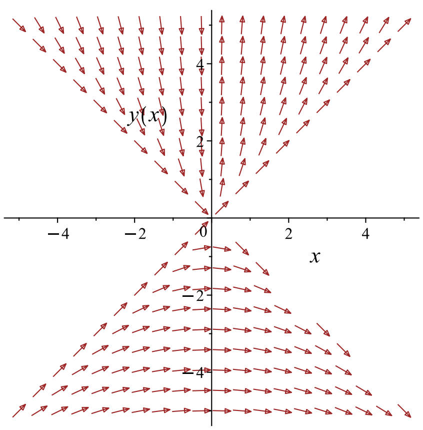
Which gives

$$y = \frac{\left(e^{2c_1}+x^2\right)e^{-c_1}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y+\sqrt{-x^2+y^2}}{x}$



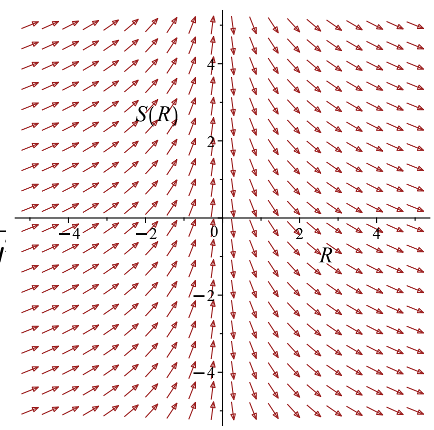
</td><td>

$R = x$

$S = -\ln\left(y + \sqrt{-x^2 + y}\right.$

</td><td>

$\frac{dS}{dR} = -\frac{2}{R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{\left(e^{2c_1} + x^2\right) e^{-c_1}}{2} \tag{1}$$

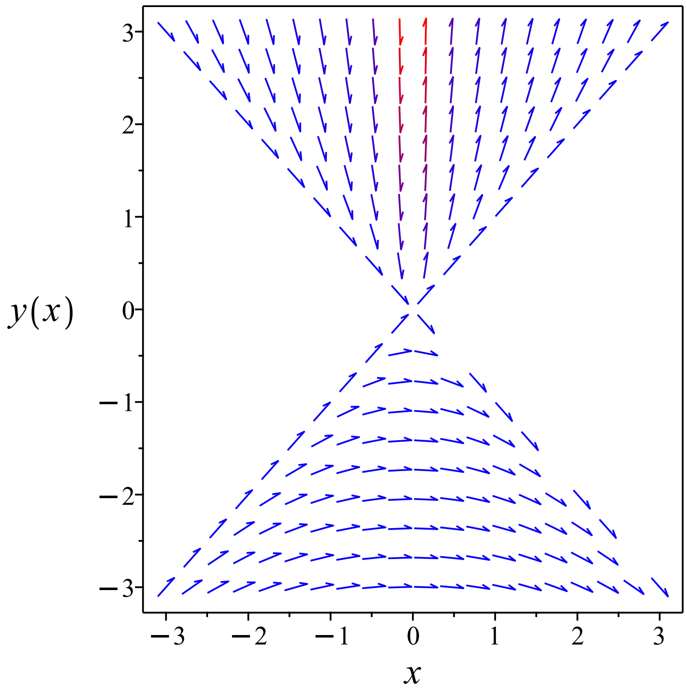


Figure 153: Slope field plot

Verification of solutions

$$y = \frac{\left(e^{2c_1} + x^2\right) e^{-c_1}}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(x*diff(y(x),x)=y(x)+sqrt(y(x)^2-x^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + y + \sqrt{y^2 - x^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.409 (sec). Leaf size: 14

```
DSolve[x*y'[x]==y[x]+Sqrt[y[x]^2-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x \cosh(\log(x) + c_1)$$

5.6 problem 105

Internal problem ID [15012]
Internal file name [`OUTPUT/15012_Friday_April_19_2024_04_44_04_AM_54216949/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 105.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{2x^2y' - y^2 = x^2}$$

5.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2(u'(x)\,x + u(x)) - u(x)^2\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\frac{1}{2}u^2 - u + \frac{1}{2}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{2}u^2 - u + \frac{1}{2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{2}u^2 - u + \frac{1}{2}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\frac{1}{2}u^2 - u + \frac{1}{2}}\,du &= \int \frac{1}{x}\,dx \\
-\frac{2}{u-1} &= \ln(x) + c_2
\end{aligned}
$$

The solution is

$$
-\frac{2}{u(x)-1} - \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{2}{\frac{y}{x}-1} - \ln(x) - c_2 &= 0 \\
\frac{(\ln(x) + c_2)\,y - x(c_2 + \ln(x) - 2)}{-y + x} &= 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\frac{(\ln(x) + c_2)\,y - x(c_2 + \ln(x) - 2)}{-y + x} = 0 \tag{1}
$$

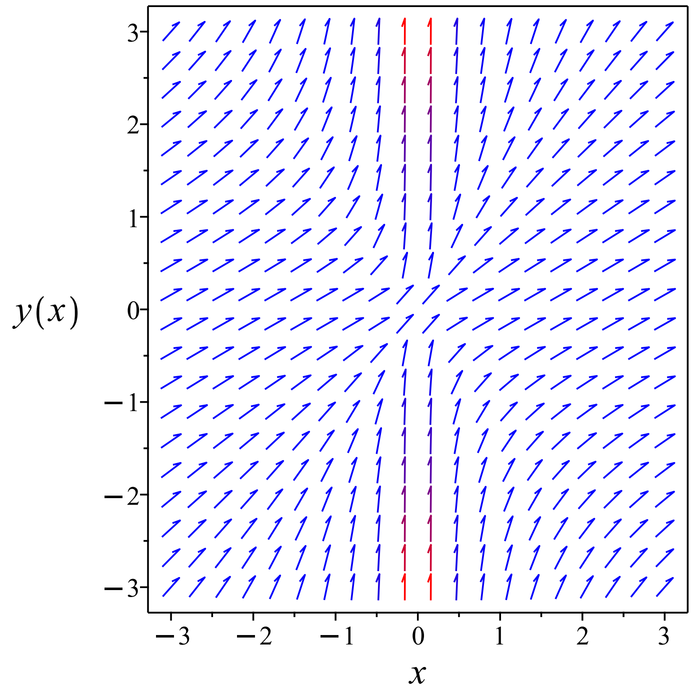


Figure 154: Slope field plot

Verification of solutions

$$
\frac{(\ln (x)+c_2)\, y-x(c_2+\ln (x)-2)}{-y+x}=0
$$

Verified OK.

5.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^2+y^2}{2x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(x^2+y^2)(b_3-a_2)}{2x^2} - \frac{(x^2+y^2)^2 a_3}{4x^4} - \left(\frac{1}{x} - \frac{x^2+y^2}{x^3}\right)(xa_2 + ya_3 + a_1) - \frac{y(xb_2 + yb_3 + b_1)}{x^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^4a_2 + x^4a_3 - 4b_2x^4 - 2x^4b_3 + 4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2a_3 + 2x^2y^2b_3 - 4x\,y^3a_3 + y^4a_3 + 4x^2yb_1 - 4x}{4x^4} = 0$$

Setting the numerator to zero gives

$$-2x^4a_2 - x^4a_3 + 4b_2x^4 + 2x^4b_3 - 4x^3yb_2 + 2x^2y^2a_2 - 2x^2y^2a_3 - 2x^2y^2b_3 + 4x\,y^3a_3 - y^4a_3 - 4x^2yb_1 + 4x\,y^2a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1^4 + 2a_2v_1^2v_2^2 - a_3v_1^4 - 2a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 + 4b_2v_1^4 - 4b_2v_1^3v_2 + 2b_3v_1^4 - 2b_3v_1^2v_2^2 + 4a_1v_1v_2^2 - 4b_1v_1^2v_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-2a_2 - a_3 + 4b_2 + 2b_3)\, v_1^4 - 4b_2 v_1^3 v_2 + (2a_2 - 2a_3 - 2b_3)\, v_1^2 v_2^2 \\
&- 4b_1 v_1^2 v_2 + 4a_3 v_1 v_2^3 + 4a_1 v_1 v_2^2 - a_3 v_2^4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
4a_1 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-4b_1 &= 0 \\
-4b_2 &= 0 \\
2a_2 - 2a_3 - 2b_3 &= 0 \\
-2a_2 - a_3 + 4b_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{x^2 + y^2}{2x^2}\right)(x) \\
&= \frac{-x^2 + 2xy - y^2}{2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2+2xy-y^2}{2x}} dy \end{aligned}$$

Which results in

$$S = \frac{2x}{y-x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2+y^2}{2x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2y}{(-y+x)^2} \\ S_y &= -\frac{2x}{(-y+x)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{2x}{-y + x} = -\ln(x) + c_1$$

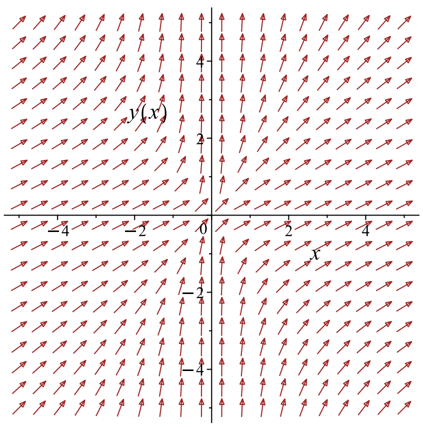
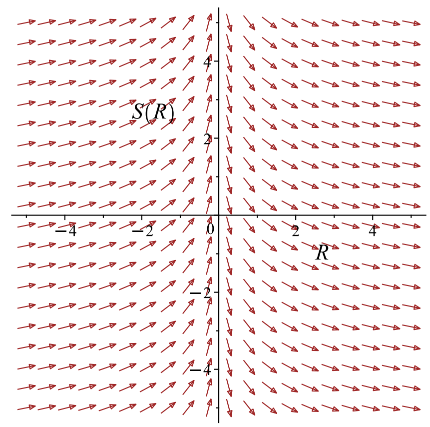
Which simplifies to

$$-\frac{2x}{-y + x} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x(\ln(x) - c_1 - 2)}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2x^2}$ 	$R = x$ $S = -\frac{2x}{-y+x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x(\ln(x) - c_1 - 2)}{\ln(x) - c_1} \quad (1)$$

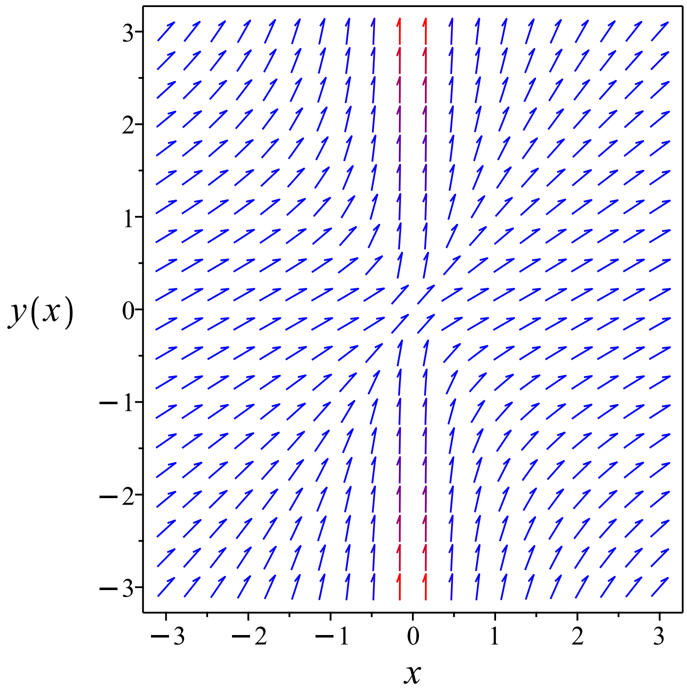


Figure 155: Slope field plot

Verification of solutions

$$y = \frac{x(\ln(x) - c_1 - 2)}{\ln(x) - c_1}$$

Verified OK.

5.6.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{1}{2} + \frac{y^2}{2x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{2}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{2x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{2x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{1}{x^3} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \frac{1}{8x^4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{2x^2} + \frac{u'(x)}{x^3} + \frac{u(x)}{8x^4} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_1 + \ln(x) c_2}{\sqrt{x}}
$$

The above shows that

$$
u'(x) = -\frac{\ln(x) c_2 + c_1 - 2c_2}{2x^{\frac{3}{2}}}
$$

Using the above in (1) gives the solution

$$
y = \frac{(\ln(x) c_2 + c_1 - 2c_2) x}{c_1 + \ln(x) c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{(\ln(x) + c_3 - 2) x}{c_3 + \ln(x)}
$$

Summary
The solution(s) found are the following

$$y = \frac{(\ln (x) + c_3 - 2) x}{c_3 + \ln (x)} \tag{1}$$

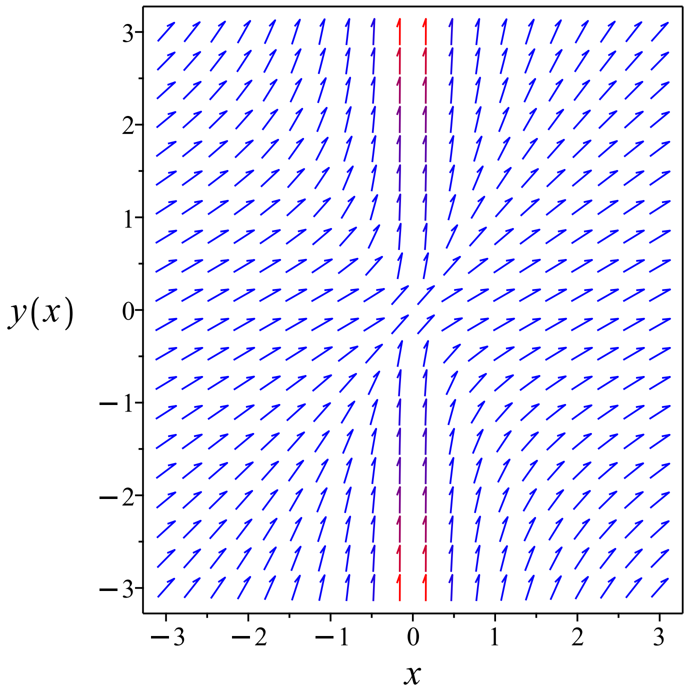


Figure 156: Slope field plot

Verification of solutions

$$y = \frac{(\ln (x) + c_3 - 2) x}{c_3 + \ln (x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(2*x^2*diff(y(x),x)=x^2+y(x)^2,y(x), singsol=all)
```

$$y = \frac{x(\ln (x) + c_1 - 2)}{\ln (x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 29

```
DSolve[2*x^2*y'[x]==x^2+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x(\log(x) - 2 + 2c_1)}{\log(x) + 2c_1}$$

$$y(x) \rightarrow x$$

5.7 problem 106

Internal problem ID [15013]
Internal file name [`OUTPUT/15013_Friday_April_19_2024_04_44_05_AM_48680824/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 106.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-3y + (2y - 3x)\, y' = -4x}$$

5.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-3u(x)\, x + (2u(x)\, x - 3x)\, (u'(x)\, x + u(x)) = -4x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(u^2 - 3u + 2)}{x\,(2u - 3)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2-3u+2}{2u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-3u+2}{2u-3}}\, du = -\frac{2}{x}\, dx$$
$$\int \frac{1}{\frac{u^2-3u+2}{2u-3}}\, du = \int -\frac{2}{x}\, dx$$
$$\ln\left(u^2-3u+2\right) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$u^2-3u+2 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$u^2-3u+2 = \frac{c_3}{x^2}$$

Which simplifies to

$$u(x)^2-3u(x)+2 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

The solution is

$$u(x)^2-3u(x)+2 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2}-\frac{3y}{x}+2 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$
$$\frac{y^2}{x^2}-\frac{3y}{x}+2 = \frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$(2x-y)(-y+x) = c_3\mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$(2x-y)(-y+x) = c_3\mathrm{e}^{c_2} \tag{1}$$

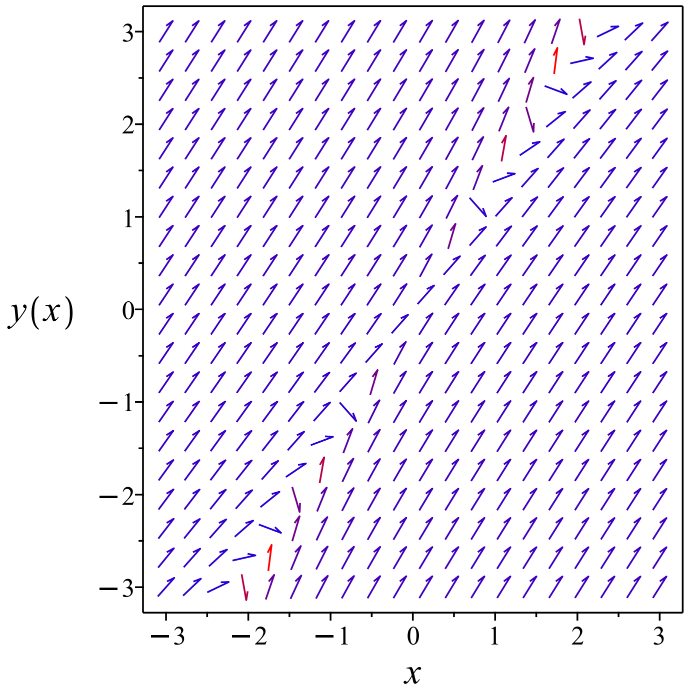


Figure 157: Slope field plot

<u>Verification of solutions</u>

$$(2x - y)(-y + x) = c_3 e^{c_2}$$

Verified OK.

5.7.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-4x + 3y}{2y - 3x} \tag{1}$$

Which becomes

$$(-2y)\, dy = (-3x)\, dy + (4x - 3y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-3x)\, dy + (4x - 3y)\, dx = d\left(2x^2 - 3xy\right)$$

Hence (2) becomes

$$(-2y)\, dy = d\left(2x^2 - 3xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{3x}{2} + \frac{\sqrt{x^2 - 4c_1}}{2} + c_1$$
$$y = \frac{3x}{2} - \frac{\sqrt{x^2 - 4c_1}}{2} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{3x}{2} + \frac{\sqrt{x^2 - 4c_1}}{2} + c_1 \tag{1}$$
$$y = \frac{3x}{2} - \frac{\sqrt{x^2 - 4c_1}}{2} + c_1 \tag{2}$$

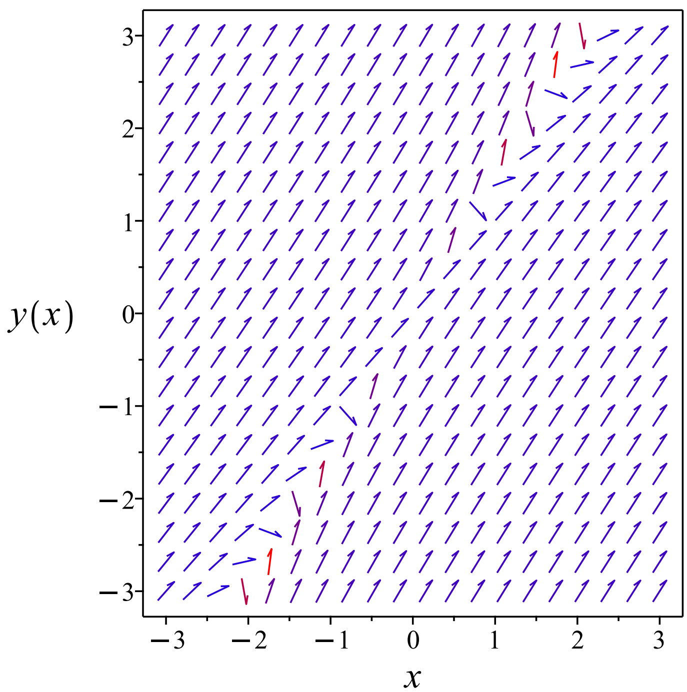


Figure 158: Slope field plot

Verification of solutions

$$y = \frac{3x}{2} + \frac{\sqrt{x^2 - 4c_1}}{2} + c_1$$

Verified OK.

$$y = \frac{3x}{2} - \frac{\sqrt{x^2 - 4c_1}}{2} + c_1$$

Verified OK.

5.7.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-4x + 3y}{2y - 3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-4x + 3y)(b_3 - a_2)}{2y - 3x} - \frac{(-4x + 3y)^2 a_3}{(2y - 3x)^2} - \left(-\frac{4}{2y - 3x} + \frac{-12x + 9y}{(2y - 3x)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{3}{2y - 3x} - \frac{2(-4x + 3y)}{(2y - 3x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{12x^2a_2+16x^2a_3-10x^2b_2-12x^2b_3-16xya_2-24xya_3+12xyb_2+16xyb_3+6y^2a_2+10y^2a_3-4y^2b_2-}{(3x-2y)^2}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}&-12x^2a_2-16x^2a_3+10x^2b_2+12x^2b_3+16xya_2+24xya_3-12xyb_2\\&-16xyb_3-6y^2a_2-10y^2a_3+4y^2b_2+6y^2b_3+xb_1-ya_1=0\end{aligned}\tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1,y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-12a_2v_1^2+16a_2v_1v_2-6a_2v_2^2-16a_3v_1^2+24a_3v_1v_2-10a_3v_2^2+10b_2v_1^2\\&-12b_2v_1v_2+4b_2v_2^2+12b_3v_1^2-16b_3v_1v_2+6b_3v_2^2-a_1v_2+b_1v_1=0\end{aligned}\tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-12a_2-16a_3+10b_2+12b_3)\,v_1^2+(16a_2+24a_3-12b_2-16b_3)\,v_1v_2\\&+b_1v_1+(-6a_2-10a_3+4b_2+6b_3)\,v_2^2-a_1v_2=0\end{aligned}\tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}b_1&=0\\-a_1&=0\\-12a_2-16a_3+10b_2+12b_3&=0\\-6a_2-10a_3+4b_2+6b_3&=0\\16a_2+24a_3-12b_2-16b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -3a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= -2a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{-4x + 3y}{2y - 3x}\right)(x) \\
&= \frac{-4x^2 + 6xy - 2y^2}{3x - 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-4x^2 + 6xy - 2y^2}{3x - 2y}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(2x^2 - 3xy + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-4x + 3y}{2y - 3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x - y} + \frac{1}{-2y + 2x} \\ S_y &= \frac{2y - 3x}{2(2x - y)(-y + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

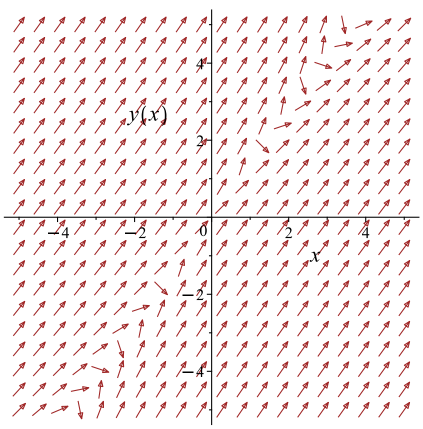
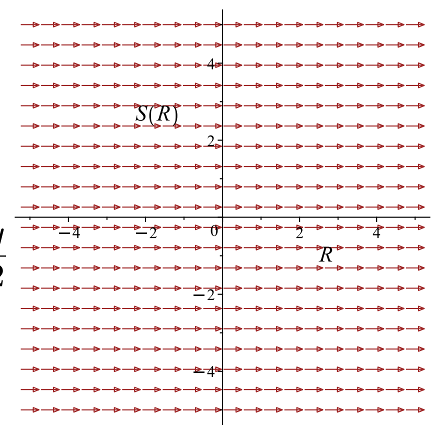
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2x - y)}{2} + \frac{\ln(-y + x)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(2x-y)}{2}+\frac{\ln(-y+x)}{2}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{-4x+3y}{2y-3x}$ 	$R = x$ $S = \frac{\ln(2x-y)}{2} + \frac{\ln(-y}{2}$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2x-y)}{2}+\frac{\ln(-y+x)}{2}=c_1 \tag{1}$$

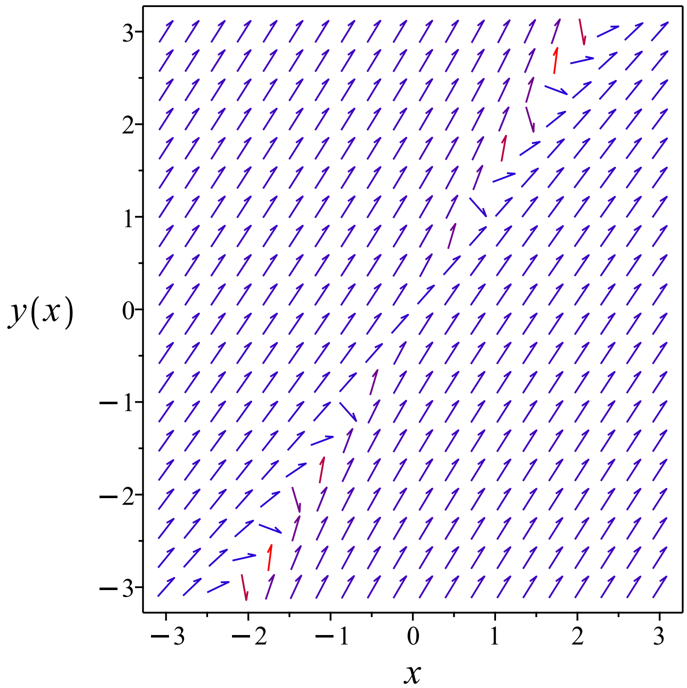


Figure 159: Slope field plot

Verification of solutions

$$\frac{\ln(2x-y)}{2}+\frac{\ln(-y+x)}{2}=c_1$$

Verified OK.

5.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(2y - 3x)\, \mathrm{d}y = (-4x + 3y)\, \mathrm{d}x$$
$$(4x - 3y)\, \mathrm{d}x + (2y - 3x)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 4x - 3y$$
$$N(x, y) = 2y - 3x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(4x - 3y)$$
$$= -3$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y - 3x)$$
$$= -3$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 4x - 3y \,\mathrm{d}x$$
$$\phi = x(2x - 3y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y - 3x$. Therefore equation (4) becomes

$$2y - 3x = -3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (2y) \,\mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(2x - 3y) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(2x - 3y) + y^2$$

Summary
The solution(s) found are the following

$$x(2x - 3y) + y^2 = c_1 \tag{1}$$

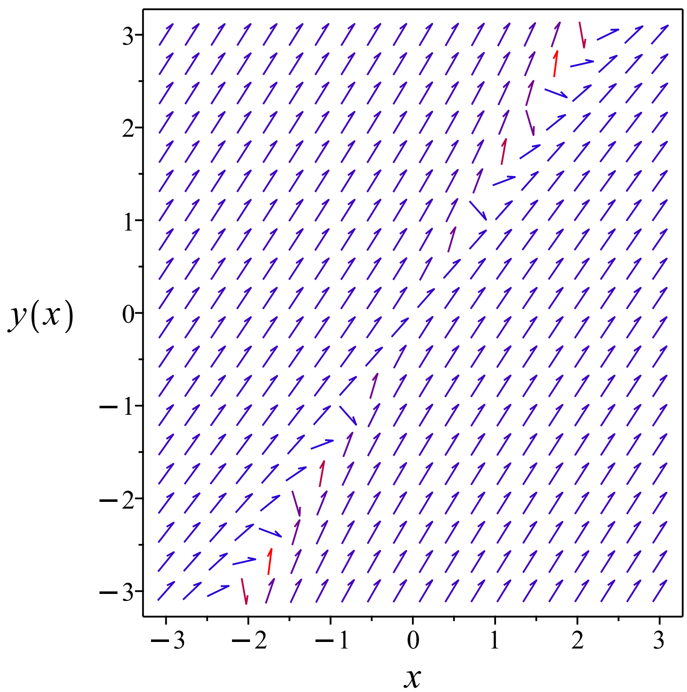


Figure 160: Slope field plot

Verification of solutions

$$x(2x - 3y) + y^2 = c_1$$

Verified OK.

5.7.5 Maple step by step solution

Let's solve

$-3y + (2y - 3x)\, y' = -4x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

  ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

  $-3 = -3$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (4x - 3y)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = 2x^2 - 3xy + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2y - 3x = -3x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = 2x^2 - 3xy + y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$2x^2 - 3xy + y^2 = c_1$$

- Solve for y

$$\left\{y = \frac{3x}{2} - \frac{\sqrt{x^2+4c_1}}{2}, y = \frac{3x}{2} + \frac{\sqrt{x^2+4c_1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve((4*x-3*y(x))+(2*y(x)-3*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{3c_1x - \sqrt{c_1^2x^2 + 4}}{2c_1}$$
$$y = \frac{3c_1x + \sqrt{c_1^2x^2 + 4}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.388 (sec). Leaf size: 95

```
DSolve[(4*x-3*y[x])+(2*y[x]-3*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(3x - \sqrt{x^2 + 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(3x + \sqrt{x^2 + 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(3x - \sqrt{x^2}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{x^2} + 3x\right)$$

5.8 problem 107

Internal problem ID [15014]
Internal file name [`OUTPUT/15014_Friday_April_19_2024_04_44_07_AM_27380750/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 107.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (y + x)\, y' = x}$$

5.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (u(x)\, x + x)\, (u'(x)\, x + u(x)) = x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 + 2u - 1}{x\, (u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+2u-1}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2u-1}{u+1}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2+2u-1}{u+1}}\,du = \int -\frac{1}{x}\,dx$$
$$\frac{\ln\left(u^2+2u-1\right)}{2} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+2u-1} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+2u-1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2+2u\left(x\right)-1} = \frac{c_3\,\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)^2+2u\left(x\right)-1} = \frac{c_3\,\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+\frac{2y}{x}-1} = \frac{c_3\,\mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{y^2+2yx-x^2}{x^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y^2+2yx-x^2}{x^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{x} \tag{1}$$

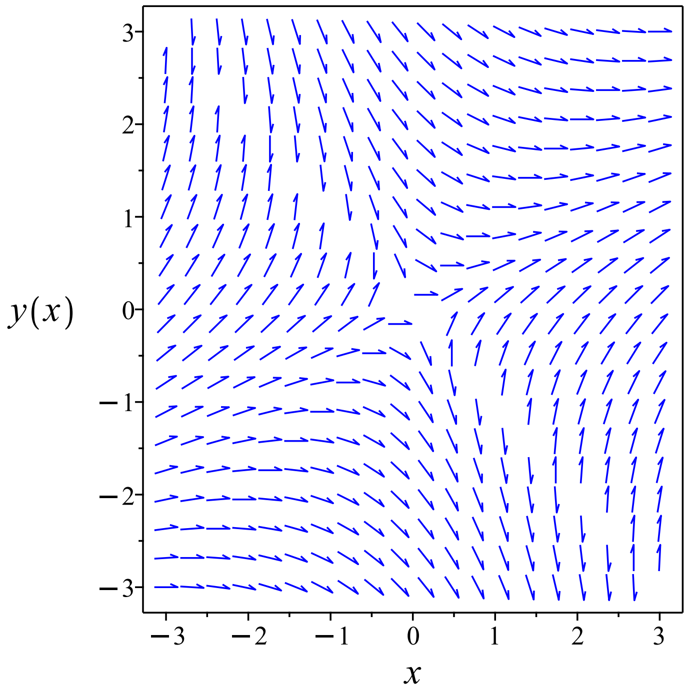


Figure 161: Slope field plot

<u>Verification of solutions</u>

$$\sqrt{\frac{y^2+2yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Verified OK.

5.8.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-y+x}{y+x} \tag{1}$$

Which becomes

$$(y)\,dy = (-x)\,dy + (-y+x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y+x)\,dx = d\left(\frac{1}{2}x^2 - xy\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(\frac{1}{2}x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = -x + \sqrt{2x^2 + 2c_1} + c_1$$
$$y = -x - \sqrt{2x^2 + 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = -x + \sqrt{2x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -x - \sqrt{2x^2 + 2c_1} + c_1 \tag{2}$$

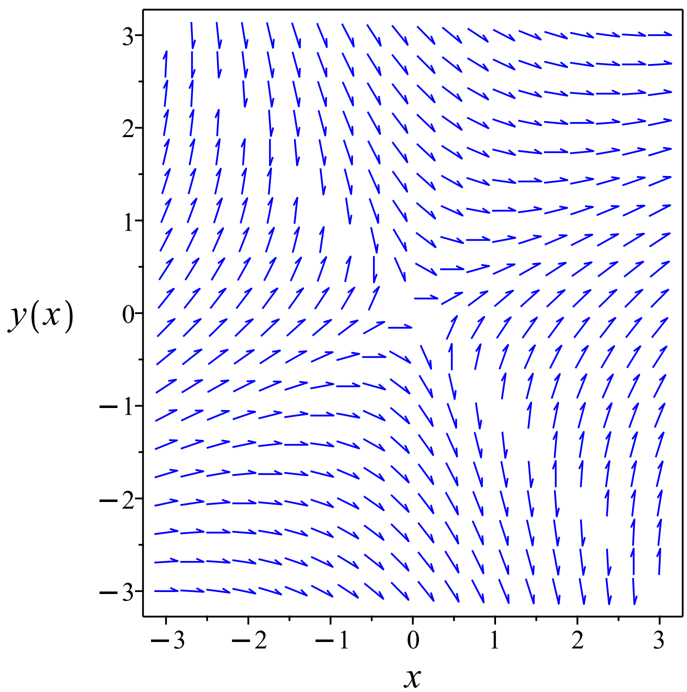


Figure 162: Slope field plot

Verification of solutions

$$y = -x + \sqrt{2x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -x - \sqrt{2x^2 + 2c_1} + c_1$$

Verified OK.

5.8.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y-x}{y+x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(y-x)(b_3-a_2)}{y+x} - \frac{(y-x)^2 a_3}{(y+x)^2} - \left(\frac{1}{y+x} + \frac{y-x}{(y+x)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{1}{y+x} + \frac{y-x}{(y+x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^2 a_2 + x^2 a_3 - 3x^2 b_2 - x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 - 2xy b_3 - y^2 a_2 + 3y^2 a_3 - y^2 b_2 + y^2 b_3 - 2x b_1 + 2}{(y+x)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^2 a_2 - x^2 a_3 + 3x^2 b_2 + x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 \\
&+ 2xy b_3 + y^2 a_2 - 3y^2 a_3 + y^2 b_2 - y^2 b_3 + 2x b_1 - 2y a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 - 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 + 2a_3 v_1 v_2 - 3a_3 v_2^2 + 3b_2 v_1^2 + 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_1^2 + 2b_3 v_1 v_2 - b_3 v_2^2 - 2a_1 v_2 + 2b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + 3b_2 + b_3) v_1^2 + (-2a_2 + 2a_3 + 2b_2 + 2b_3) v_1 v_2 + 2b_1 v_1 + (a_2 - 3a_3 + b_2 - b_3) v_2^2 - 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0 \\
2b_1 &= 0 \\
-2a_2 + 2a_3 + 2b_2 + 2b_3 &= 0 \\
-a_2 - a_3 + 3b_2 + b_3 &= 0 \\
a_2 - 3a_3 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_2 + b_3 \\
a_3 &= b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y-x}{y+x}\right)(x) \\
&= \frac{-x^2 + 2xy + y^2}{y+x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2+2xy+y^2}{y+x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(-x^2 + 2xy + y^2\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - x}{y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-y + x}{x^2 - 2xy - y^2} \\
S_y &= \frac{-y - x}{x^2 - 2xy - y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + 2yx - x^2)}{2} = c_1$$

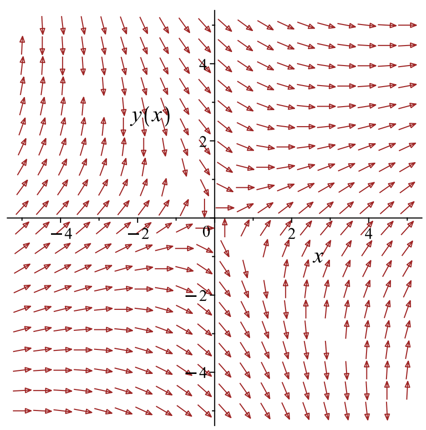
Which simplifies to

$$\frac{\ln(y^2 + 2yx - x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{y-x}{y+x}$



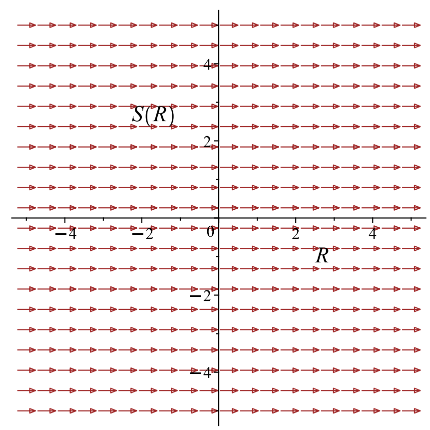
</td><td>

$R = x$

$S = \frac{\ln(-x^2 + 2xy + y^2)}{2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 + 2yx - x^2)}{2} = c_1 \tag{1}$$

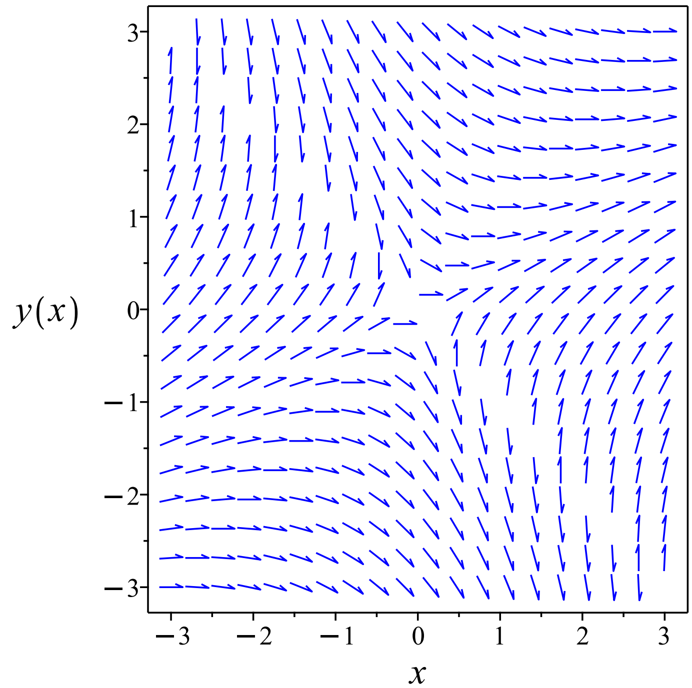


Figure 163: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^2+2yx-x^2\right)}{2}=c_1$$

Verified OK.

5.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y + x) \,\mathrm{d}y = (-y + x) \,\mathrm{d}x$$
$$(y - x) \,\mathrm{d}x + (y + x) \,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - x$$
$$N(x, y) = y + x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y - x)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y + x)$$
$$= 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y - x \, \mathrm{d}x$$
$$\phi = -\frac{x(x - 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + x$. Therefore equation (4) becomes

$$y + x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(x-2y)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(x-2y)}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x(x-2y)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

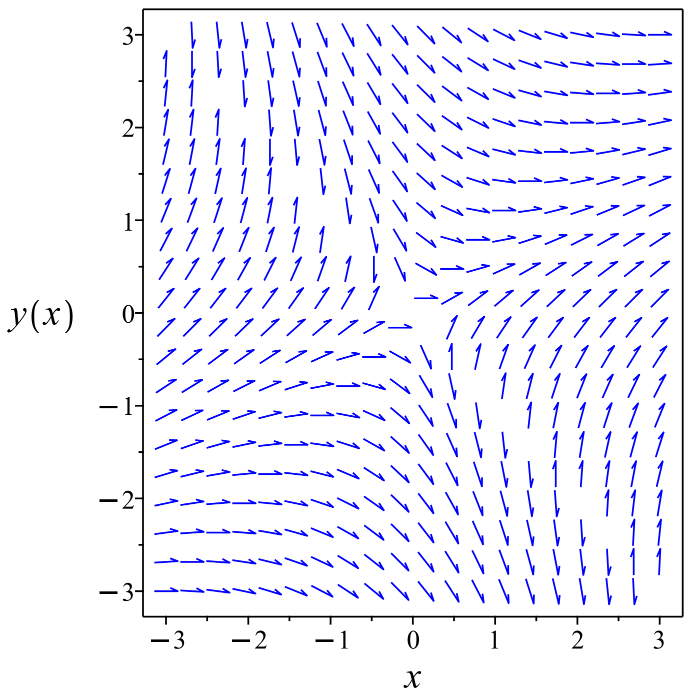


Figure 164: Slope field plot

Verification of solutions

$$-\frac{x(x-2y)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

5.8.5 Maple step by step solution

Let's solve

$y+(y+x)\,y'=x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- ◦ Evaluate derivatives

  $1=1$

- ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1,M(x,y)=F'(x,y)\,,N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(y-x)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=xy-\frac{x^2}{2}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y+x=x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=y$

- Solve for $f_1(y)$

  $f_1(y)=\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = xy - \frac{1}{2}x^2 + \frac{1}{2}y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $xy - \frac{1}{2}x^2 + \frac{1}{2}y^2 = c_1$

- Solve for y

  $\left\{y = -x - \sqrt{2x^2 + 2c_1}, y = -x + \sqrt{2x^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 51

```
dsolve((y(x)-x)+(y(x)+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{-c_1x - \sqrt{2c_1^2x^2 + 1}}{c_1}$$
$$y = \frac{-c_1x + \sqrt{2c_1^2x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.427 (sec). Leaf size: 94

```
DSolve[(y[x]-x)+(y[x]+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -x - \sqrt{2x^2 + e^{2c_1}} \\
y(x) &\to -x + \sqrt{2x^2 + e^{2c_1}} \\
y(x) &\to -\sqrt{2}\sqrt{x^2} - x \\
y(x) &\to \sqrt{2}\sqrt{x^2} - x
\end{aligned}
$$

5.9 problem 108

Internal problem ID [15015]
Internal file name [`OUTPUT/15015_Friday_April_19_2024_04_44_09_AM_32587077/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 108.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y + (1 - x)\, y' = -x + 2}$$

5.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x-1}$$
$$q(x) = \frac{x-2}{x-1}$$

Hence the ode is

$$y' - \frac{y}{x-1} = \frac{x-2}{x-1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x-1}dx}$$
$$= \frac{1}{x-1}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x-2}{x-1}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x-1}\right) = \left(\frac{1}{x-1}\right)\left(\frac{x-2}{x-1}\right)$$
$$d\left(\frac{y}{x-1}\right) = \left(\frac{x-2}{(x-1)^2}\right) dx$$

Integrating gives

$$\frac{y}{x-1} = \int \frac{x-2}{(x-1)^2} dx$$
$$\frac{y}{x-1} = \frac{1}{x-1} + \ln(x-1) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$y = (x-1)\left(\frac{1}{x-1} + \ln(x-1)\right) + c_1(x-1)$$

which simplifies to

$$y = (x-1)\ln(x-1) + 1 + c_1(x-1)$$

Summary

The solution(s) found are the following

$$y = (x-1)\ln(x-1) + 1 + c_1(x-1) \tag{1}$$

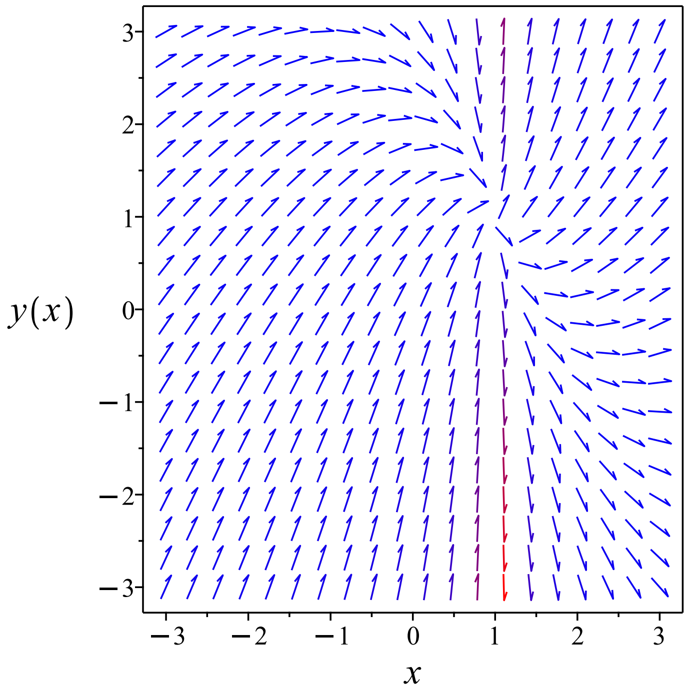


Figure 165: Slope field plot

Verification of solutions

$$y = (x-1)\ln(x-1) + 1 + c_1(x-1)$$

Verified OK.

5.9.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{X + x_0 + Y(X) + y_0 - 2}{X + x_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X + Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{X + Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X + Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 1 + u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{1}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{1}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$\begin{aligned} u(X) &= \int \frac{1}{X}\,\mathrm{d}X \\ &= \ln(X) + c_2 \end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X(\ln(X) + c_2)$$

Using the solution for $Y(X)$

$$Y(X) = X(\ln(X) + c_2)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 1$$
$$X = x + 1$$

Then the solution in y becomes

$$y - 1 = (x - 1)(\ln(x - 1) + c_2)$$

Summary

The solution(s) found are the following

$$y - 1 = (x - 1)(\ln(x - 1) + c_2) \tag{1}$$

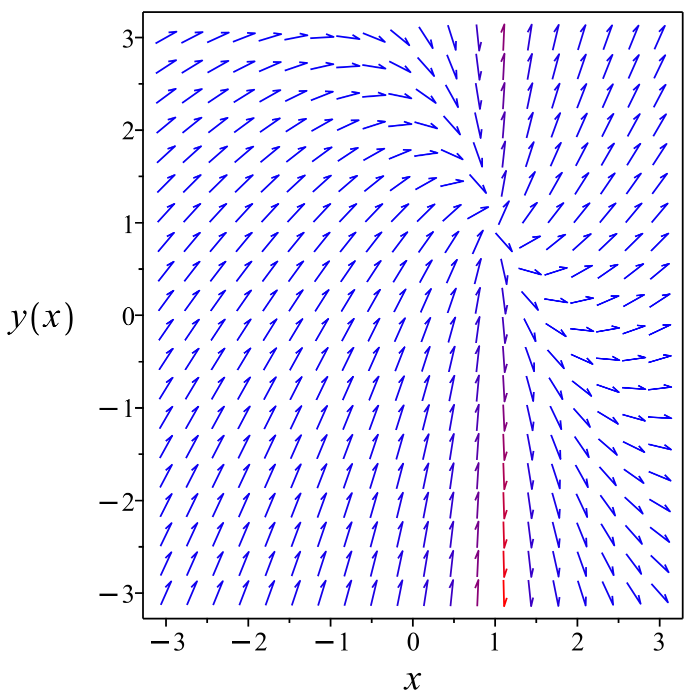


Figure 166: Slope field plot

Verification of solutions

$$y - 1 = (x - 1)(\ln(x - 1) + c_2)$$

Verified OK.

5.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x+y-2}{x-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 143: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x-1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x-1} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y}{x-1}
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}
$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x+y-2}{x-1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{(x-1)^2} \\
S_y &= \frac{1}{x-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{x-2}{(x-1)^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R-2}{(R-1)^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{1}{R-1} + \ln(R-1) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x-1} = \frac{1}{x-1} + \ln(x-1) + c_1
$$

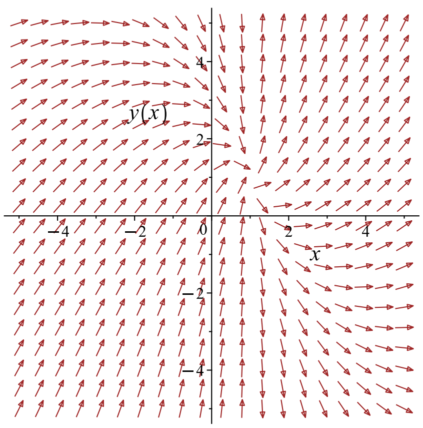
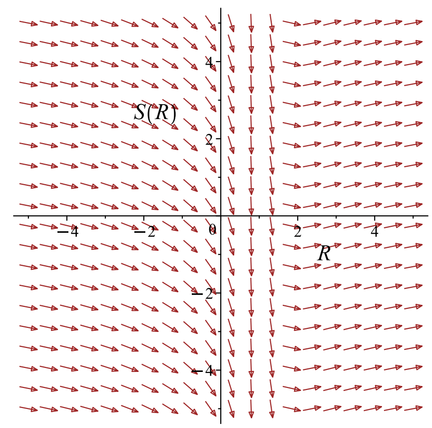
Which simplifies to

$$
\frac{y}{x-1} = \frac{1}{x-1} + \ln(x-1) + c_1
$$

Which gives

$$
y = \ln(x-1)\, x + c_1 x - \ln(x-1) - c_1 + 1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y-2}{x-1}$ 	$R = x$ $S = \frac{y}{x-1}$	$\frac{dS}{dR} = \frac{R-2}{(R-1)^2}$ 

Summary

The solution(s) found are the following

$$y = \ln(x-1)\,x + c_1 x - \ln(x-1) - c_1 + 1 \tag{1}$$

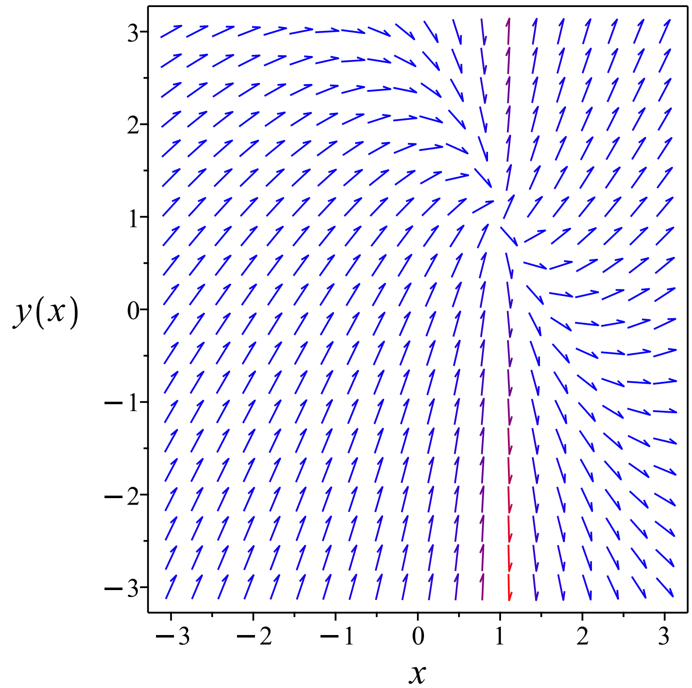


Figure 167: Slope field plot

Verification of solutions

$$y = \ln(x-1)\, x + c_1 x - \ln(x-1) - c_1 + 1$$

Verified OK.

5.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(1 - x)\,\mathrm{d}y &= (-x - y + 2)\,\mathrm{d}x \\
(x + y - 2)\,\mathrm{d}x + (1 - x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x + y - 2 \\
N(x, y) &= 1 - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y - 2) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1 - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{1-x}((1) - (-1)) \\
&= -\frac{2}{x-1}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x-1} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(x-1)} \\
&= \frac{1}{(x-1)^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(x-1)^2}(x+y-2) \\
&= \frac{x+y-2}{(x-1)^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(x-1)^2}(1-x) \\
&= -\frac{1}{x-1}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x+y-2}{(x-1)^2}\right) + \left(-\frac{1}{x-1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x+y-2}{(x-1)^2} \, \mathrm{d}x$$

$$\phi = \frac{1-y}{x-1} + \ln(x-1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x-1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x-1}$. Therefore equation (4) becomes

$$-\frac{1}{x-1} = -\frac{1}{x-1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1-y}{x-1} + \ln(x-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1-y}{x-1} + \ln(x-1)$$

The solution becomes

$$y = \ln(x-1)\,x - c_1 x - \ln(x-1) + c_1 + 1$$

Summary

The solution(s) found are the following

$$y = \ln(x-1)\,x - c_1 x - \ln(x-1) + c_1 + 1 \tag{1}$$

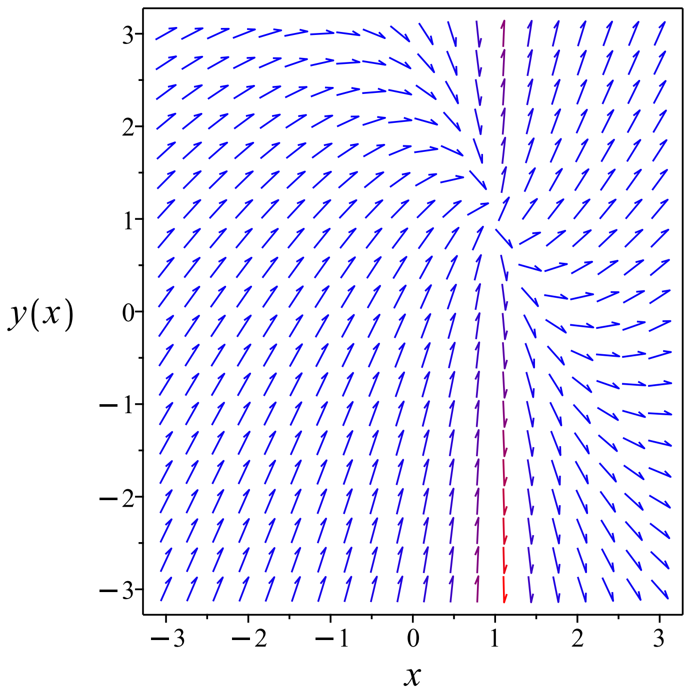


Figure 168: Slope field plot

Verification of solutions

$$y = \ln(x-1)\,x - c_1 x - \ln(x-1) + c_1 + 1$$

Verified OK.

5.9.5 Maple step by step solution

Let's solve

$y + (1 - x)\, y' = -x + 2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x-1} + \frac{x-2}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x-1} = \frac{x-2}{x-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x-1}\right) = \frac{\mu(x)(x-2)}{x-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x-1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(x-2)}{x-1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(x-2)}{x-1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)(x-2)}{x-1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x-1}$

  $y = (x-1)\left(\int \frac{x-2}{(x-1)^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = (x-1)\left(\frac{1}{x-1} + \ln(x-1) + c_1\right)$

- Simplify

$$y = (x-1)\ln(x-1) + 1 + c_1(x-1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x+y(x)-2+(1-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = (-1+x)\ln(-1+x) + 1 + c_1(-1+x)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 21

```
DSolve[x+y[x]-2+(1-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x-1)\left(\frac{1}{x-1} + \log(x-1) + c_1\right)$$

5.10 problem 109

Internal problem ID [15016]
Internal file name [`OUTPUT/15016_Friday_April_19_2024_04_44_10_AM_61812171/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 109.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y-(3x-7y-3)\,y'=7x-7}$$

5.10.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{3Y(X)+3y_0-7X-7x_0+7}{-3X-3x_0+7Y(X)+7y_0+3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=1$$
$$y_0=0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=-\frac{3Y(X)-7X}{-3X+7Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{3Y-7X}{-3X+7Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 3Y - 7X$ and $N = 3X - 7Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-3u+7}{7u-3} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} = 0$$

Or

$$7\left(\frac{d}{dX}u(X)\right)Xu(X) - 3\left(\frac{d}{dX}u(X)\right)X + 7u(X)^2 - 7 = 0$$

Or

$$-7 + X(7u(X)-3)\left(\frac{d}{dX}u(X)\right) + 7u(X)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{7(u^2-1)}{X\,(7u-3)} \end{aligned}$$

Where $f(X) = -\frac{7}{X}$ and $g(u) = \frac{u^2-1}{7u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{7u-3}}\,du = -\frac{7}{X}\,dX$$
$$\int \frac{1}{\frac{u^2-1}{7u-3}}\,du = \int -\frac{7}{X}\,dX$$
$$2\ln(u-1) + 5\ln(u+1) = -7\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-1)+5\ln(u+1)} = e^{-7\ln(X)+c_2}$$

Which simplifies to

$$(u-1)^2 (u+1)^5 = \frac{c_3}{X^7}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \mathrm{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

Using the solution for $Y(X)$

$$Y(X) = \mathrm{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$y = \mathrm{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + \left(x^2 - 2x + 1\right)\_Z^5 + \left(-5x^3 + 15x^2 - 15x + 5\right)\_Z^4 + \left(-5x^4 + 20x\right.$$

Summary
The solution(s) found are the following

$$y = \operatorname{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 + (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 + (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7-7x^6+21x^5-35x^4 + 35x^3-21x^2-c_3+7x-1\right) \tag{1}$$

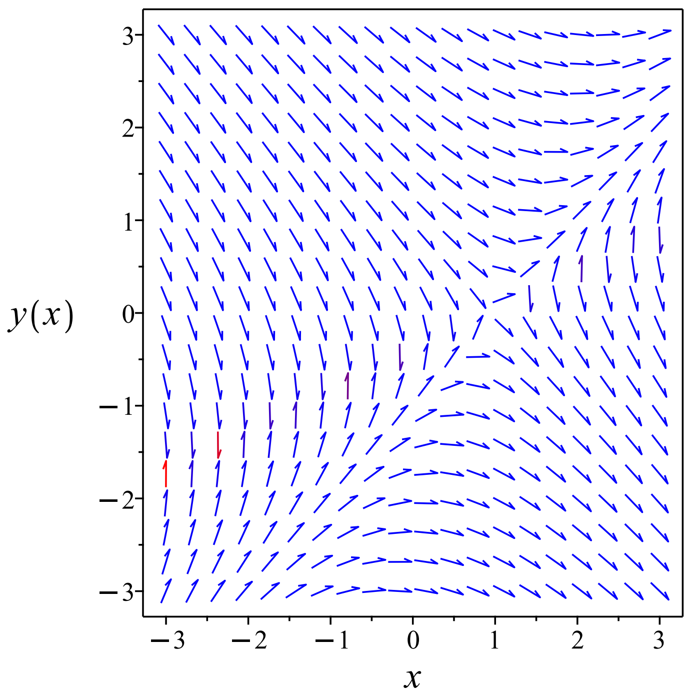


Figure 169: Slope field plot

Verification of solutions

$$y = \operatorname{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 + (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 + (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7-7x^6+21x^5-35x^4 + 35x^3-21x^2-c_3+7x-1\right)$$

Verified OK.

5.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3y - 7x + 7}{-3x + 7y + 3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{(3y - 7x + 7)(b_3 - a_2)}{-3x + 7y + 3} - \frac{(3y - 7x + 7)^2 a_3}{(-3x + 7y + 3)^2} \\ &- \left(\frac{7}{-3x + 7y + 3} - \frac{3(3y - 7x + 7)}{(-3x + 7y + 3)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{3}{-3x + 7y + 3} + \frac{21y - 49x + 49}{(-3x + 7y + 3)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 + 21y^2a_2 - 49y^2a_3 + 49y^2b_2 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 \\ &+ 21y^2a_2 - 49y^2a_3 + 49y^2b_2 - 21y^2b_3 - 42xa_2 + 98xa_3 + 40xb_1 - 58xb_2 + 42xb_3 \\ &- 40ya_1 + 58ya_2 - 42ya_3 + 42yb_2 - 98yb_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&21a_2v_1^2 - 98a_2v_1v_2 + 21a_2v_2^2 - 49a_3v_1^2 + 42a_3v_1v_2 - 49a_3v_2^2 + 49b_2v_1^2 \\
&- 42b_2v_1v_2 + 49b_2v_2^2 - 21b_3v_1^2 + 98b_3v_1v_2 - 21b_3v_2^2 - 40a_1v_2 \\
&- 42a_2v_1 + 58a_2v_2 + 98a_3v_1 - 42a_3v_2 + 40b_1v_1 - 58b_2v_1 + 42b_2v_2 \\
&+ 42b_3v_1 - 98b_3v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_1^2 + (-98a_2 + 42a_3 - 42b_2 + 98b_3)\, v_1v_2 \\
&+ (-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3)\, v_1 + (21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_2^2 \\
&+ (-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3)\, v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-98a_2 + 42a_3 - 42b_2 + 98b_3 &= 0 \\
21a_2 - 49a_3 + 49b_2 - 21b_3 &= 0 \\
-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3 &= 0 \\
-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3 &= 0 \\
21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_3 \\
a_2 &= b_3 \\
a_3 &= b_2 \\
b_1 &= -b_2 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = y$$
$$\eta = x - 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - 1 - \left(-\frac{3y - 7x + 7}{-3x + 7y + 3}\right)(y) \\
&= \frac{3x^2 - 3y^2 - 6x + 3}{3x - 7y - 3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2 - 3y^2 - 6x + 3}{3x - 7y - 3}} dy
\end{aligned}$$

Which results in

$$S = \frac{2 \ln (1 - x + y)}{3} + \frac{5 \ln (x - 1 + y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y - 7x + 7}{-3x + 7y + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{3x - 3 - 3y} + \frac{5}{3x - 3 + 3y} \\ S_y &= -\frac{2}{3x - 3 - 3y} + \frac{5}{3x - 3 + 3y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

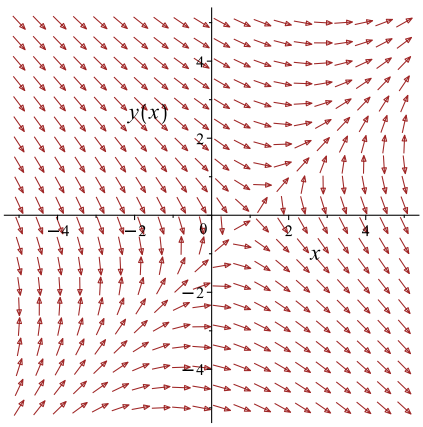
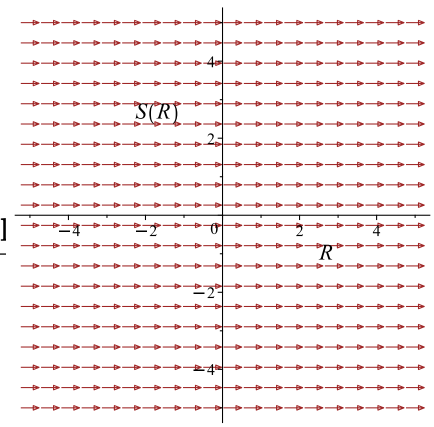
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(1 - x + y)}{3} + \frac{5\ln(y - 1 + x)}{3} = c_1$$

Which simplifies to

$$\frac{2\ln(1 - x + y)}{3} + \frac{5\ln(y - 1 + x)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y-7x+7}{-3x+7y+3}$ 	$R = x$ $S = \frac{2\ln(1-x+y)}{3} + \frac{5\mathrm{l}}{}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(1-x+y)}{3} + \frac{5\ln(y-1+x)}{3} = c_1 \tag{1}$$

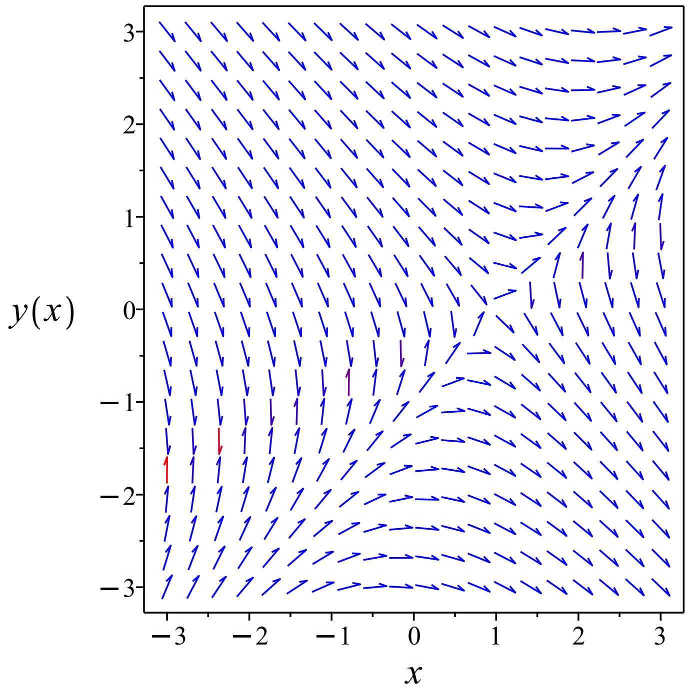


Figure 170: Slope field plot

Verification of solutions

$$\frac{2\ln(1-x+y)}{3}+\frac{5\ln(y-1+x)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 1814

```
dsolve((3*y(x)-7*x+7)-(3*x-7*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 60.866 (sec). Leaf size: 7785

```
DSolve[(3*y[x]-7*x+7)-(3*x-7*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

5.11 problem 110

Internal problem ID [15017]
Internal file name [`OUTPUT/15017_Friday_April_19_2024_04_44_12_AM_70258455/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 110.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x - y + 4)\, y' = -x + 2}$$

5.11.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-x - y + 2}{x - y + 4} \tag{1}$$

Which becomes

$$(-y + 4)\, dy = (-x)\, dy + (-x - y + 2)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-x - y + 2)\, dx = d\left(-\frac{1}{2}x^2 - xy + 2x\right)$$

Hence (2) becomes

$$(-y+4)\,dy = d\left(-\frac{1}{2}x^2 - xy + 2x\right)$$

Integrating both sides gives gives these solutions

$$y = x + 4 + \sqrt{2x^2 - 2c_1 + 4x + 16} + c_1$$
$$y = x + 4 - \sqrt{2x^2 - 2c_1 + 4x + 16} + c_1$$

Summary

The solution(s) found are the following

$$y = x + 4 + \sqrt{2x^2 - 2c_1 + 4x + 16} + c_1 \tag{1}$$
$$y = x + 4 - \sqrt{2x^2 - 2c_1 + 4x + 16} + c_1 \tag{2}$$

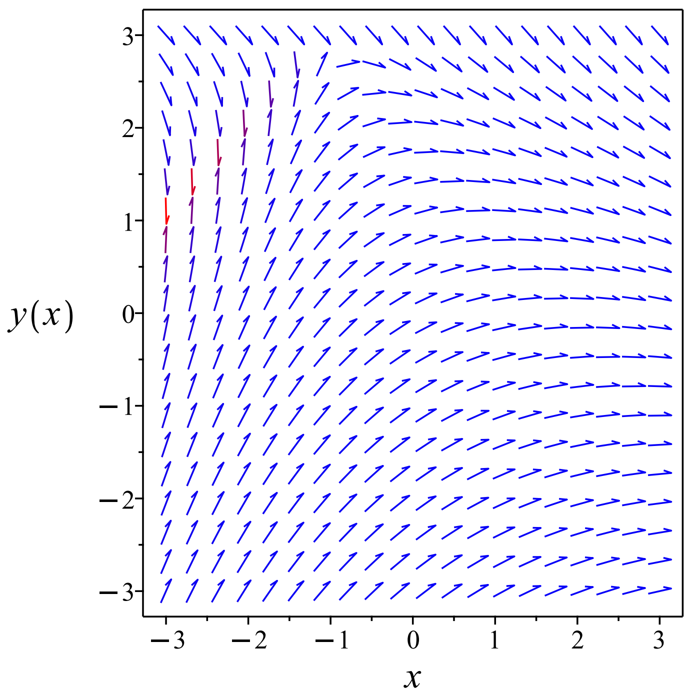


Figure 171: Slope field plot

Verification of solutions

$$y = x + 4 + \sqrt{2x^2 - 2c_1 + 4x + 16} + c_1$$

Verified OK.

$$y = x + 4 - \sqrt{2x^2 - 2c_1 + 4x + 16} + c_1$$

Verified OK.

5.11.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{X + x_0 + Y(X) + y_0 - 2}{-X - x_0 + Y(X) + y_0 - 4}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= -1 \\ y_0 &= 3 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X + Y(X)}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{X + Y}{-X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -X - Y$ and $N = X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{u + 1}{u - 1} \\ \frac{du}{dX} &= \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 - 2u(X) - 1 = 0$$

Or

$$X(u(X) - 1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 2u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 - 2u - 1}{X\,(u - 1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-2u-1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-2u-1}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-2u-1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln\left(u^2 - 2u - 1\right)}{2} = -\ln\left(X\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2 - 2u - 1} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2 - 2u - 1} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u\left(X\right)^2 - 2u\left(X\right) - 1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u\left(X\right)^2 - 2u\left(X\right) - 1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^{2}}{X^{2}}-\frac{2Y(X)}{X}-1} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^{2}-2Y(X)X-X^{2}}{X^{2}}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= 3 + y \\ X &= x - 1 \end{aligned}$$

Then the solution in y becomes

$$\sqrt{\frac{(y-3)^{2}-2(y-3)(x+1)-(x+1)^{2}}{(x+1)^{2}}} = \frac{c_3 e^{c_2}}{x+1}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{(y-3)^{2}-2(y-3)(x+1)-(x+1)^{2}}{(x+1)^{2}}} = \frac{c_3 e^{c_2}}{x+1} \tag{1}$$

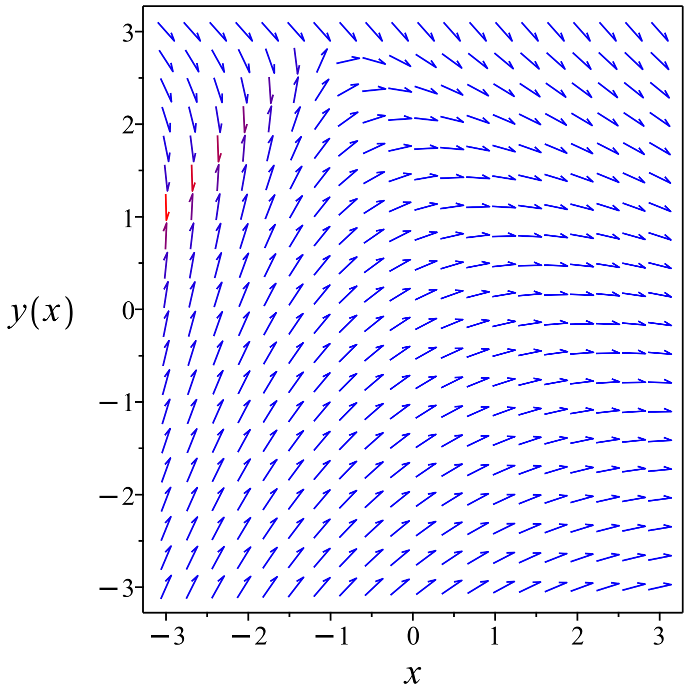


Figure 172: Slope field plot

Verification of solutions

$$\sqrt{\frac{(y-3)^2-2(y-3)(x+1)-(x+1)^2}{(x+1)^2}}=\frac{c_3 e^{c_2}}{x+1}$$

Verified OK.

5.11.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x+y-2}{-x+y-4}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(x+y-2)(b_3-a_2)}{-x+y-4} - \frac{(x+y-2)^2 a_3}{(-x+y-4)^2} \\ & - \left(\frac{1}{-x+y-4} + \frac{x+y-2}{(-x+y-4)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(\frac{1}{-x+y-4} - \frac{x+y-2}{(-x+y-4)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 - 2xyb_2 + 2xyb_3 - y^2a_2 - 3y^2a_3 + y^2b_2 + y^2b_3 + 8xa_2 + 4x}{(x-y+} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 - 2xyb_2 + 2xyb_3 - y^2a_2 \\ & - 3y^2a_3 + y^2b_2 + y^2b_3 + 8xa_2 + 4xa_3 + 2xb_1 + 10xb_2 - 2xb_3 - 2ya_1 \\ & + 6ya_2 + 10ya_3 - 8yb_2 - 4yb_3 + 6a_1 - 8a_2 - 4a_3 + 2b_1 + 16b_2 + 8b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_2v_1^2 - 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 \\ & - b_3v_1^2 + 2b_3v_1v_2 + b_3v_2^2 - 2a_1v_2 + 8a_2v_1 + 6a_2v_2 + 4a_3v_1 + 10a_3v_2 + 2b_1v_1 \\ & + 10b_2v_1 - 8b_2v_2 - 2b_3v_1 - 4b_3v_2 + 6a_1 - 8a_2 - 4a_3 + 2b_1 + 16b_2 + 8b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(a_2 - a_3 + 3b_2 - b_3) v_1^2 + (-2a_2 - 2a_3 - 2b_2 + 2b_3) v_1 v_2 \\
&+ (8a_2 + 4a_3 + 2b_1 + 10b_2 - 2b_3) v_1 + (-a_2 - 3a_3 + b_2 + b_3) v_2^2 \\
&+ (-2a_1 + 6a_2 + 10a_3 - 8b_2 - 4b_3) v_2 + 6a_1 - 8a_2 - 4a_3 + 2b_1 + 16b_2 + 8b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_2 - 2a_3 - 2b_2 + 2b_3 &= 0 \\
-a_2 - 3a_3 + b_2 + b_3 &= 0 \\
a_2 - a_3 + 3b_2 - b_3 &= 0 \\
-2a_1 + 6a_2 + 10a_3 - 8b_2 - 4b_3 &= 0 \\
8a_2 + 4a_3 + 2b_1 + 10b_2 - 2b_3 &= 0 \\
6a_1 - 8a_2 - 4a_3 + 2b_1 + 16b_2 + 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -5b_2 + b_3 \\
a_2 &= -2b_2 + b_3 \\
a_3 &= b_2 \\
b_1 &= b_2 - 3b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x + 1 \\
\eta &= y - 3
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - 3 - \left(\frac{x + y - 2}{-x + y - 4}\right)(x + 1) \\
&= \frac{x^2 + 2xy - y^2 - 4x + 8y - 14}{x - y + 4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+2xy-y^2-4x+8y-14}{x-y+4}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(-x^2 - 2xy + y^2 + 4x - 8y + 14\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + y - 2}{-x + y - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y - 2}{x^2 + (2y - 4)x - y^2 + 8y - 14} \\ S_y &= \frac{x - y + 4}{x^2 + (2y - 4)x - y^2 + 8y - 14} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

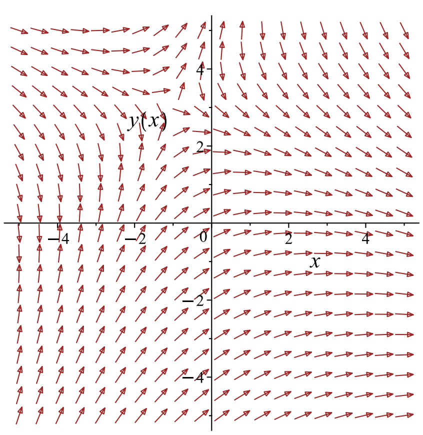
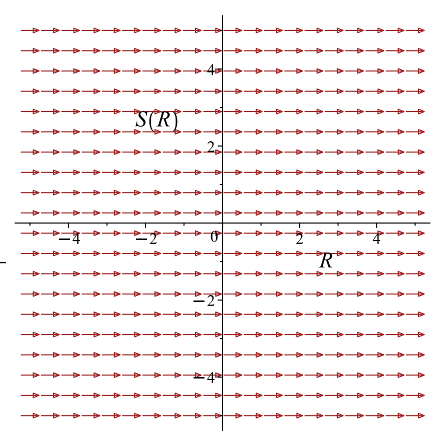
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(-x^2 + (-2y+4)\,x + y^2 - 8y + 14\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(-x^2 + (-2y+4)\,x + y^2 - 8y + 14\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y-2}{-x+y-4}$ 	$R = x$ $S = \frac{\ln\left(-x^2 + (-2y+4)\right)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(-x^2+(-2y+4)\,x+y^2-8y+14\right)}{2}=c_1 \tag{1}$$

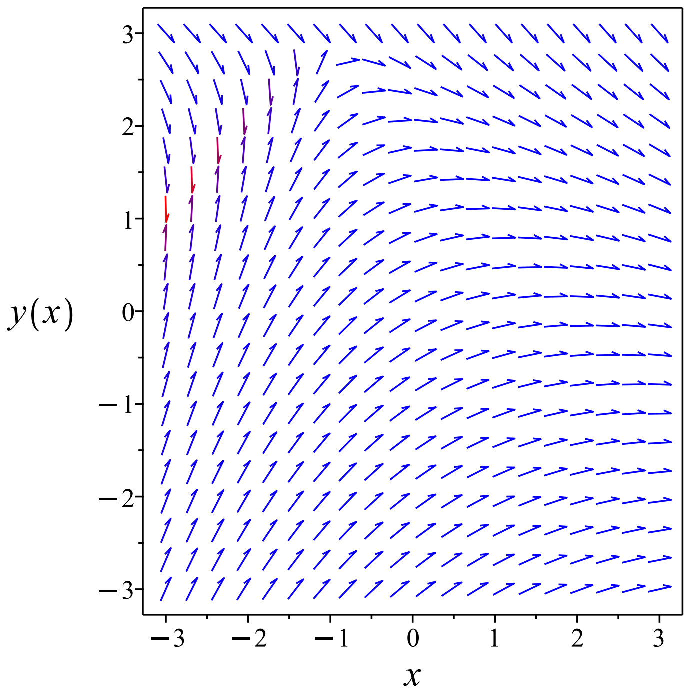


Figure 173: Slope field plot

Verification of solutions

$$\frac{\ln\left(-x^2+(-2y+4)\,x+y^2-8y+14\right)}{2}=c_1$$

Verified OK.

5.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x-y+4) \mathrm{d} y &= (-x-y+2) \mathrm{d} x \\
(x+y-2) \mathrm{d} x+(x-y+4) \mathrm{d} y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x+y-2 \\
N(x, y) &= x-y+4
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x+y-2) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-y+4)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x+y-2\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(x+2y-4)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - y + 4$. Therefore equation (4) becomes

$$x - y + 4 = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y + 4$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int (-y+4)\,\mathrm{d}y\\ f(y) &= -\frac{1}{2}y^2 + 4y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x+2y-4)}{2} - \frac{y^2}{2} + 4y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y-4)}{2} - \frac{y^2}{2} + 4y$$

Summary

The solution(s) found are the following

$$\frac{x(x+2y-4)}{2} - \frac{y^2}{2} + 4y = c_1 \tag{1}$$

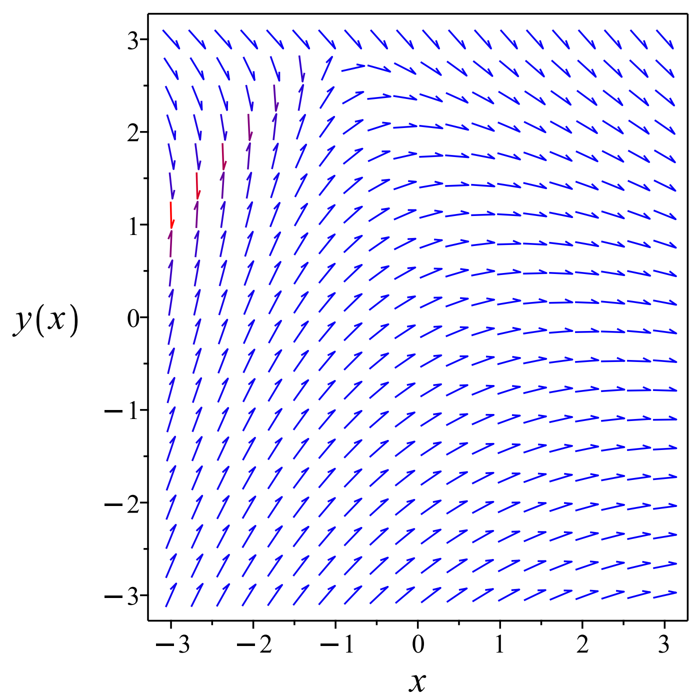


Figure 174: Slope field plot

Verification of solutions

$$\frac{x(x+2y-4)}{2} - \frac{y^2}{2} + 4y = c_1$$

Verified OK.

5.11.5 Maple step by step solution

Let's solve

$y+(x-y+4)\,y'=-x+2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $1=1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(x+y-2)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2}{2}+xy-2x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x-y+4=x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-y+4$

- Solve for $f_1(y)$

  $f_1(y)=-\frac{1}{2}y^2+4y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x, y) = \frac{1}{2}x^2 + xy - 2x - \frac{1}{2}y^2 + 4y$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{1}{2}x^2 + xy - 2x - \frac{1}{2}y^2 + 4y = c_1$

- Solve for y

  $\left\{y = x + 4 - \sqrt{2x^2 - 2c_1 + 4x + 16}, y = x + 4 + \sqrt{2x^2 - 2c_1 + 4x + 16}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 30

```
dsolve((x+y(x)-2)+(x-y(x)+4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{-\sqrt{2\left(x+1\right)^2 c_1^2 + 1} + \left(x+4\right) c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 59

```
DSolve[(x+y[x]-2)+(x-y[x]+4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{-2x^2 - 4x - 16 - c_1} + x + 4$$
$$y(x) \to i\sqrt{-2x^2 - 4x - 16 - c_1} + x + 4$$

5.12 problem 111

Internal problem ID [15018]
Internal file name [`OUTPUT/15018_Friday_April_19_2024_04_44_14_AM_38866116/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 111.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (x - y - 2)\, y' = -x}$$

5.12.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-y - x}{x - y - 2} \tag{1}$$

Which becomes

$$(-y - 2)\, dy = (-x)\, dy + (-y - x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-y - x)\, dx = d\left(-\frac{1}{2}x^2 - xy\right)$$

Hence (2) becomes

$$(-y-2)\,dy = d\left(-\frac{1}{2}x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = x - 2 + \sqrt{2x^2 - 2c_1 - 4x + 4} + c_1$$
$$y = x - 2 - \sqrt{2x^2 - 2c_1 - 4x + 4} + c_1$$

Summary

The solution(s) found are the following

$$y = x - 2 + \sqrt{2x^2 - 2c_1 - 4x + 4} + c_1 \tag{1}$$
$$y = x - 2 - \sqrt{2x^2 - 2c_1 - 4x + 4} + c_1 \tag{2}$$

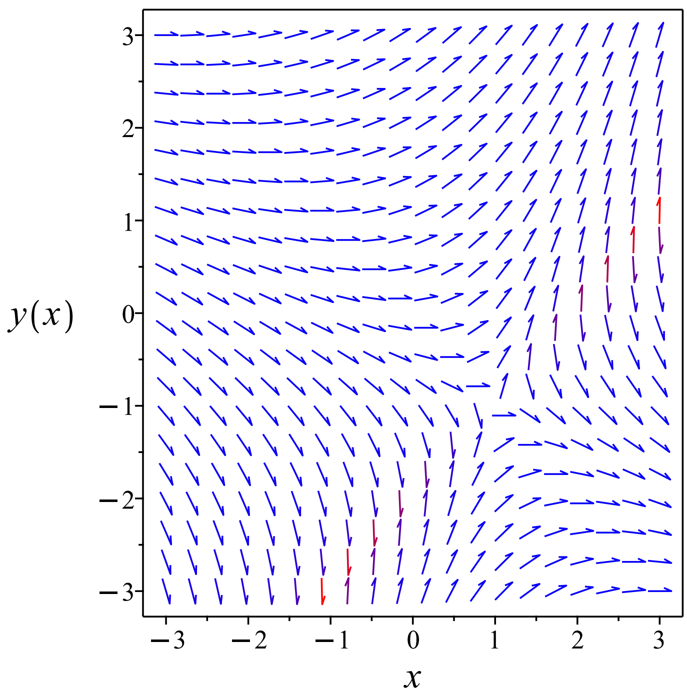


Figure 175: Slope field plot

Verification of solutions

$$y = x - 2 + \sqrt{2x^2 - 2c_1 - 4x + 4} + c_1$$

Verified OK.

$$y = x - 2 - \sqrt{2x^2 - 2c_1 - 4x + 4} + c_1$$

Verified OK.

5.12.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + X + x_0}{-X - x_0 + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 1 \\ y_0 &= -1 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X) + X}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{Y + X}{-X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -Y - X$ and $N = X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u+1}{u-1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 - 2u(X) - 1 = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 2u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2-2u-1}{X(u-1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-2u-1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-2u-1}{u-1}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{u^2-2u-1}{u-1}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln(u^2-2u-1)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2-2u-1} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2-2u-1} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2-2u(X)-1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2-2u(X)-1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2} - \frac{2Y(X)}{X} - 1} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2 - 2Y(X)X - X^2}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes

$$\sqrt{\frac{(y+1)^2 - 2(y+1)(x-1) - (x-1)^2}{(x-1)^2}} = \frac{c_3 e^{c_2}}{x-1}$$

<u>Summary</u>
The solution(s) found are the following

$$\sqrt{\frac{(y+1)^2 - 2(y+1)(x-1) - (x-1)^2}{(x-1)^2}} = \frac{c_3 e^{c_2}}{x-1} \tag{1}$$

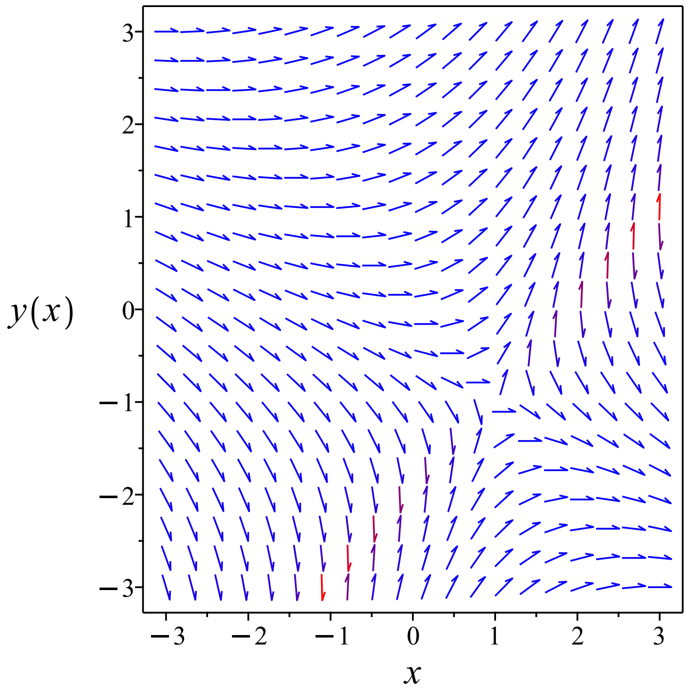


Figure 176: Slope field plot

Verification of solutions

$$\sqrt{\frac{(y+1)^2 - 2(y+1)(x-1) - (x-1)^2}{(x-1)^2}} = \frac{c_3 e^{c_2}}{x-1}$$

Verified OK.

5.12.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y+x}{-x+y+2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \frac{(y+x)(b_3-a_2)}{-x+y+2} - \frac{(y+x)^2 a_3}{(-x+y+2)^2} \\ &- \left(\frac{1}{-x+y+2} + \frac{y+x}{(-x+y+2)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{1}{-x+y+2} - \frac{y+x}{(-x+y+2)^2}\right)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 - 2xyb_2 + 2xyb_3 - y^2a_2 - 3y^2a_3 + y^2b_2 + y^2b_3 - 4xa_2 + 2x}{(x-y-2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}&x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 - 2xyb_2 + 2xyb_3 \\ &- y^2a_2 - 3y^2a_3 + y^2b_2 + y^2b_3 - 4xa_2 + 2xb_1 - 6xb_2 + 2xb_3 \\ &- 2ya_1 - 2ya_2 - 2ya_3 + 4yb_2 - 2a_1 - 2b_1 + 4b_2 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&a_2v_1^2 - 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 - 2b_2v_1v_2 \\ &+ b_2v_2^2 - b_3v_1^2 + 2b_3v_1v_2 + b_3v_2^2 - 2a_1v_2 - 4a_2v_1 - 2a_2v_2 \\ &- 2a_3v_2 + 2b_1v_1 - 6b_2v_1 + 4b_2v_2 + 2b_3v_1 - 2a_1 - 2b_1 + 4b_2 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(a_2-a_3+3b_2-b_3) v_1^2+(-2a_2-2a_3-2b_2+2b_3) v_1 v_2+(-4a_2+2b_1-6b_2+2b_3) v_1 \\ +(-a_2-3a_3+b_2+b_3) v_2^2+(-2a_1-2a_2-2a_3+4b_2) v_2-2a_1-2b_1+4b_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1-2b_1+4b_2&=0\\
-2a_1-2a_2-2a_3+4b_2&=0\\
-4a_2+2b_1-6b_2+2b_3&=0\\
-2a_2-2a_3-2b_2+2b_3&=0\\
-a_2-3a_3+b_2+b_3&=0\\
a_2-a_3+3b_2-b_3&=0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1&=3b_2-b_3\\
a_2&=-2b_2+b_3\\
a_3&=b_2\\
b_1&=-b_2+b_3\\
b_2&=b_2\\
b_3&=b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi&=x-1\\
\eta&=y+1
\end{aligned}$$

Shifting is now applied to make $\xi=0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta&=\eta-\omega(x,y)\,\xi\\
&=y+1-\left(\frac{y+x}{-x+y+2}\right)(x-1)\\
&=\frac{x^2+2xy-y^2-4y-2}{x-y-2}\\
\xi&=0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+2xy-y^2-4y-2}{x-y-2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - 2xy + y^2 + 4y + 2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + x}{-x + y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y + x}{-y^2 + (2x - 4) y + x^2 - 2} \\ S_y &= \frac{x - y - 2}{-y^2 + (2x - 4) y + x^2 - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

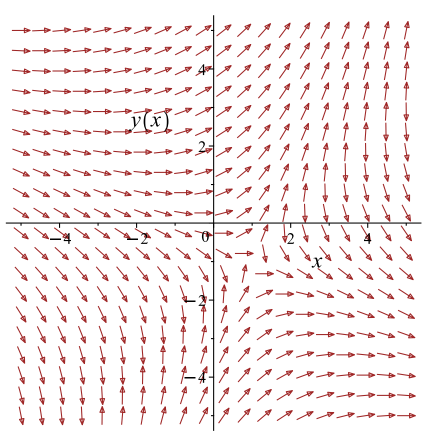
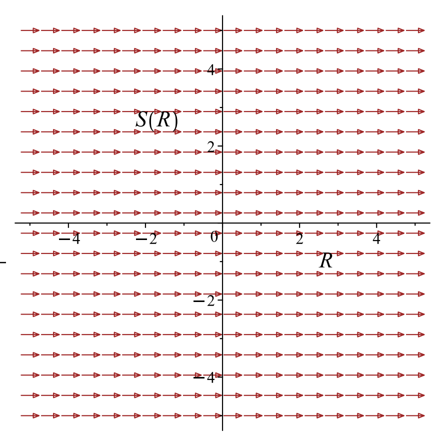
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y^2 + (-2x+4)\,y - x^2 + 2\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(y^2 + (-2x+4)\,y - x^2 + 2\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x}{-x+y+2}$ 	$R = x$ $S = \frac{\ln\left(y^2 + (-2x+4)\,y\right.}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln \left(y^{2}+\left(-2 x+4\right) y-x^{2}+2\right)}{2}=c_{1} \tag{1}$$

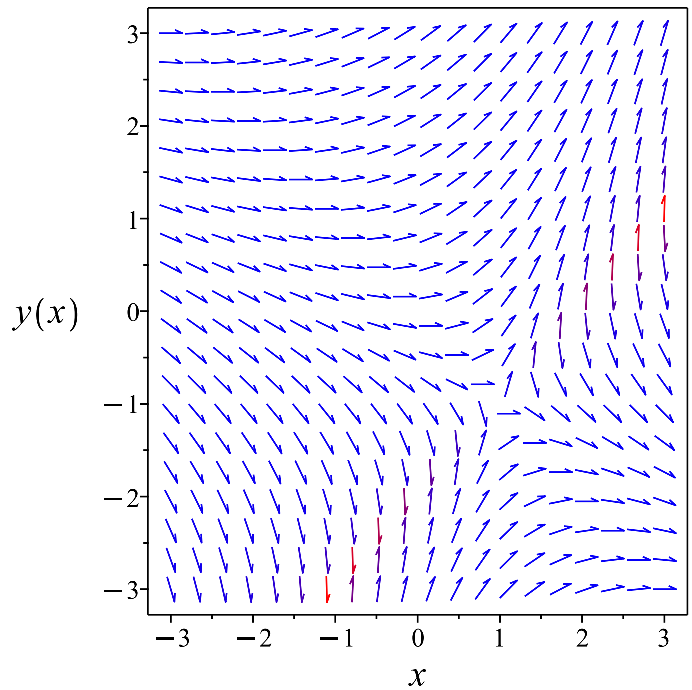


Figure 177: Slope field plot

Verification of solutions

$$\frac{\ln \left(y^{2}+\left(-2 x+4\right) y-x^{2}+2\right)}{2}=c_{1}$$

Verified OK.

5.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x-y-2)\,\mathrm{d}y&=(-y-x)\,\mathrm{d}x\\ (y+x)\,\mathrm{d}x+(x-y-2)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y+x\\ N(x,y)&=x-y-2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y+x)\\ &=1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - y - 2) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int y + x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(x + 2y)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - y - 2$. Therefore equation (4) becomes

$$
x - y - 2 = x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -y - 2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (-y - 2) \, \mathrm{d}y \\
f(y) &= -\frac{1}{2}y^2 - 2y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x+2y)}{2} - \frac{y^2}{2} - 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y)}{2} - \frac{y^2}{2} - 2y$$

Summary

The solution(s) found are the following

$$\frac{x(x+2y)}{2} - \frac{y^2}{2} - 2y = c_1 \tag{1}$$

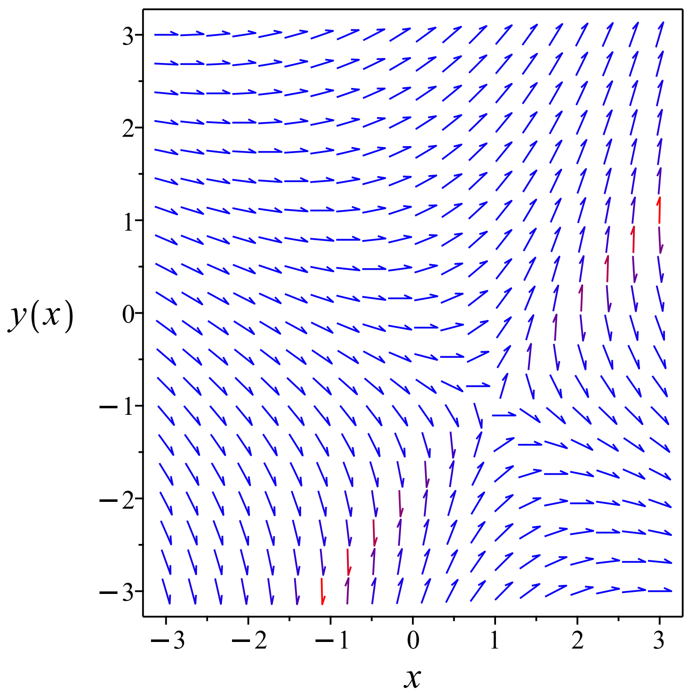


Figure 178: Slope field plot

Verification of solutions

$$\frac{x(x+2y)}{2} - \frac{y^2}{2} - 2y = c_1$$

Verified OK.

5.12.5 Maple step by step solution

Let's solve

$y + (x - y - 2)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $1 = 1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y + x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = xy + \frac{x^2}{2} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $x - y - 2 = x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -y - 2$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{1}{2}y^2 - 2y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x, y) = xy + \frac{1}{2}x^2 - \frac{1}{2}y^2 - 2y$

- Substitute $F(x, y)$ into the solution of the ODE

  $xy + \frac{1}{2}x^2 - \frac{1}{2}y^2 - 2y = c_1$

- Solve for y

  $\left\{y = x - 2 - \sqrt{2x^2 - 2c_1 - 4x + 4}, y = x - 2 + \sqrt{2x^2 - 2c_1 - 4x + 4}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 30

```
dsolve((x+y(x))+(x-y(x)-2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{-\sqrt{2\left(-1+x\right)^{2} c_1^{2}+1}+\left(x-2\right) c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 59

```
DSolve[(x+y[x])+(x-y[x]-2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{-2x^2 + 4x - 4 - c_1} + x - 2$$
$$y(x) \to i\sqrt{-2x^2 + 4x - 4 - c_1} + x - 2$$

5.13 problem 112

Internal problem ID [15019]
Internal file name [`OUTPUT/15019_Friday_April_19_2024_04_44_17_AM_74029769/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 112.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{3y + (3x + 2y - 5)\, y' = -2x + 5}$$

5.13.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2x - 3y + 5}{3x + 2y - 5} \tag{1}$$

Which becomes

$$(-5 + 2y)\, dy = (-3x)\, dy + (-2x - 3y + 5)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-3x)\, dy + (-2x - 3y + 5)\, dx = d\left(-x^2 - 3xy + 5x\right)$$

Hence (2) becomes

$$(-5+2y)\,dy = d\left(-x^2-3xy+5x\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{3x}{2}+\frac{5}{2}+\frac{\sqrt{5x^2+4c_1-10x+25}}{2}+c_1$$
$$y = -\frac{3x}{2}+\frac{5}{2}-\frac{\sqrt{5x^2+4c_1-10x+25}}{2}+c_1$$

Summary
The solution(s) found are the following

$$y = -\frac{3x}{2}+\frac{5}{2}+\frac{\sqrt{5x^2+4c_1-10x+25}}{2}+c_1 \tag{1}$$
$$y = -\frac{3x}{2}+\frac{5}{2}-\frac{\sqrt{5x^2+4c_1-10x+25}}{2}+c_1 \tag{2}$$

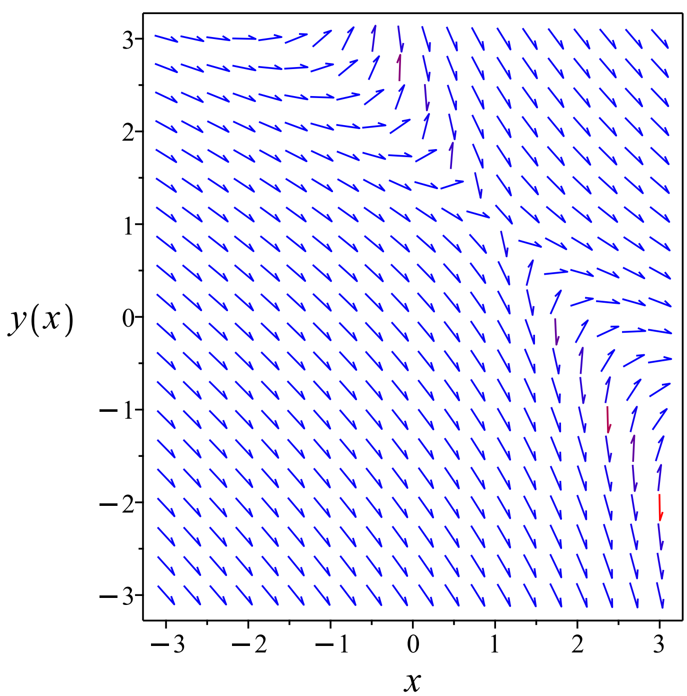


Figure 179: Slope field plot

Verification of solutions

$$y = -\frac{3x}{2} + \frac{5}{2} + \frac{\sqrt{5x^2 + 4c_1 - 10x + 25}}{2} + c_1$$

Verified OK.

$$y = -\frac{3x}{2} + \frac{5}{2} - \frac{\sqrt{5x^2 + 4c_1 - 10x + 25}}{2} + c_1$$

Verified OK.

5.13.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2X + 2x_0 + 3Y(X) + 3y_0 - 5}{3X + 3x_0 + 2Y(X) + 2y_0 - 5}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 1 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2X + 3Y(X)}{3X + 2Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{2X + 3Y}{3X + 2Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -2X - 3Y$ and $N = 3X + 2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = \frac{-3u-2}{2u+3}$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{\frac{-3u(X)-2}{2u(X)+3} - u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)-2}{2u(X)+3} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)Xu(X) + 3\left(\frac{d}{dX}u(X)\right)X + 2u(X)^2 + 6u(X) + 2 = 0$$

Or

$$2 + X(2u(X)+3)\left(\frac{d}{dX}u(X)\right) + 2u(X)^2 + 6u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{2(u^2+3u+1)}{X\,(2u+3)} \end{aligned}$$

Where $f(X) = -\frac{2}{X}$ and $g(u) = \frac{u^2+3u+1}{2u+3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u+1}{2u+3}}\,du = -\frac{2}{X}\,dX$$
$$\int \frac{1}{\frac{u^2+3u+1}{2u+3}}\,du = \int -\frac{2}{X}\,dX$$
$$\ln\left(u^2+3u+1\right) = -2\ln(X) + c_2$$

Raising both side to exponential gives

$$u^2 + 3u + 1 = \mathrm{e}^{-2\ln(X)+c_2}$$

Which simplifies to

$$u^2 + 3u + 1 = \frac{c_3}{X^2}$$

Which simplifies to

$$u(X)^2 + 3u(X) + 1 = \frac{c_3 e^{c_2}}{X^2}$$

The solution is

$$u(X)^2 + 3u(X) + 1 = \frac{c_3 e^{c_2}}{X^2}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{Y(X)^2}{X^2} + \frac{3Y(X)}{X} + 1 = \frac{c_3 e^{c_2}}{X^2}$$

Which simplifies to

$$Y(X)^2 + 3Y(X)\,X + X^2 = c_3 e^{c_2}$$

Using the solution for $Y(X)$

$$Y(X)^2 + 3Y(X)\,X + X^2 = c_3 e^{c_2}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y + 1 \\ X &= x + 1 \end{aligned}$$

Then the solution in y becomes

$$(y-1)^2 + 3(y-1)(x-1) + (x-1)^2 = c_3 e^{c_2}$$

Summary

The solution(s) found are the following

$$(y-1)^2 + 3(y-1)(x-1) + (x-1)^2 = c_3 e^{c_2} \tag{1}$$

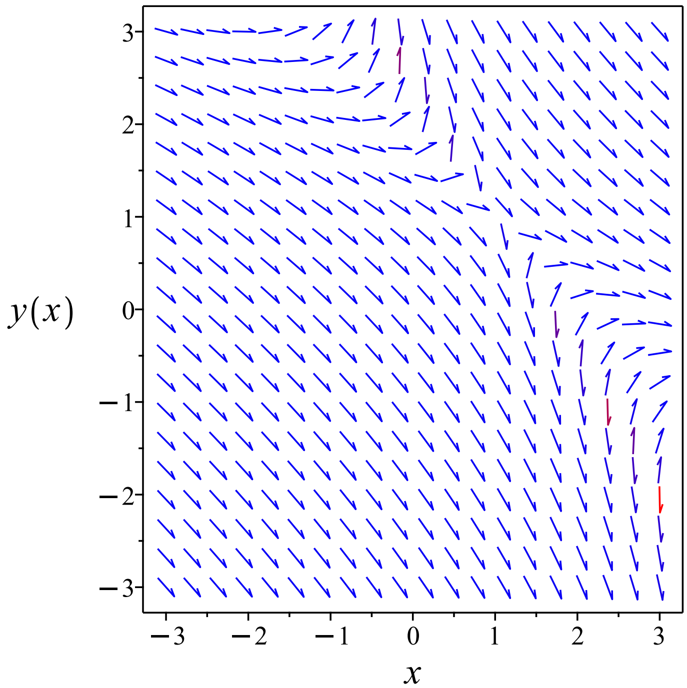


Figure 180: Slope field plot

Verification of solutions

$$(y-1)^2+3(y-1)(x-1)+(x-1)^2=c_3\mathrm{e}^{c_2}$$

Verified OK.

5.13.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{2x+3y-5}{3x+2y-5}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(2x+3y-5)(b_3-a_2)}{3x+2y-5} - \frac{(2x+3y-5)^2 a_3}{(3x+2y-5)^2} - \left(-\frac{2}{3x+2y-5} + \frac{6x+9y-15}{(3x+2y-5)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3}{3x+2y-5} + \frac{4x+6y-10}{(3x+2y-5)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2a_2 - 4x^2a_3 + 14x^2b_2 - 6x^2b_3 + 8xya_2 - 12xya_3 + 12xyb_2 - 8xyb_3 + 6y^2a_2 - 14y^2a_3 + 4y^2b_2 - 6y^2b_3 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$6x^2a_2 - 4x^2a_3 + 14x^2b_2 - 6x^2b_3 + 8xya_2 - 12xya_3 + 12xyb_2 - 8xyb_3 + 6y^2a_2 - 14y^2a_3 + 4y^2b_2 - 6y^2b_3 - 20xa_2 + 20xa_3 + 5xb_1 - 35xb_2 + 25xb_3 - 5ya_1 - 25ya_2 + 35ya_3 - 20yb_2 + 20yb_3 + 5a_1 + 25a_2 - 25a_3 - 5b_1 + 25b_2 - 25b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$6a_2v_1^2 + 8a_2v_1v_2 + 6a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 14a_3v_2^2 + 14b_2v_1^2 + 12b_2v_1v_2 + 4b_2v_2^2 - 6b_3v_1^2 - 8b_3v_1v_2 - 6b_3v_2^2 - 5a_1v_2 - 20a_2v_1 - 25a_2v_2 + 20a_3v_1 + 35a_3v_2 + 5b_1v_1 - 35b_2v_1 - 20b_2v_2 + 25b_3v_1 + 20b_3v_2 + 5a_1 + 25a_2 - 25a_3 - 5b_1 + 25b_2 - 25b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(6a_2 - 4a_3 + 14b_2 - 6b_3) v_1^2 + (8a_2 - 12a_3 + 12b_2 - 8b_3) v_1 v_2 \\
&+ (-20a_2 + 20a_3 + 5b_1 - 35b_2 + 25b_3) v_1 + (6a_2 - 14a_3 + 4b_2 - 6b_3) v_2^2 \\
&+ (-5a_1 - 25a_2 + 35a_3 - 20b_2 + 20b_3) v_2 + 5a_1 \\
&+ 25a_2 - 25a_3 - 5b_1 + 25b_2 - 25b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_2 - 14a_3 + 4b_2 - 6b_3 &= 0 \\
6a_2 - 4a_3 + 14b_2 - 6b_3 &= 0 \\
8a_2 - 12a_3 + 12b_2 - 8b_3 &= 0 \\
-5a_1 - 25a_2 + 35a_3 - 20b_2 + 20b_3 &= 0 \\
-20a_2 + 20a_3 + 5b_1 - 35b_2 + 25b_3 &= 0 \\
5a_1 + 25a_2 - 25a_3 - 5b_1 + 25b_2 - 25b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 4b_2 - b_3 \\
a_2 &= -3b_2 + b_3 \\
a_3 &= -b_2 \\
b_1 &= -b_2 - b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x - 1 \\
\eta &= y - 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - 1 - \left(-\frac{2x+3y-5}{3x+2y-5}\right)(x-1) \\
&= \frac{2x^2 + 6xy + 2y^2 - 10x - 10y + 10}{3x+2y-5} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^2+6xy+2y^2-10x-10y+10}{3x+2y-5}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + 3xy + y^2 - 5x - 5y + 5)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x+3y-5}{3x+2y-5}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x+3y-5}{2x^2+(6y-10)\,x+2y^2-10y+10} \\
S_y &= \frac{3x+2y-5}{2x^2+(6y-10)\,x+2y^2-10y+10}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

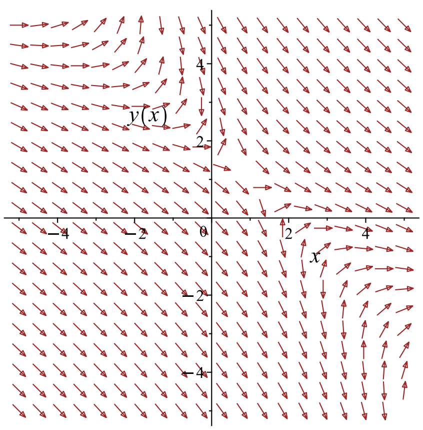
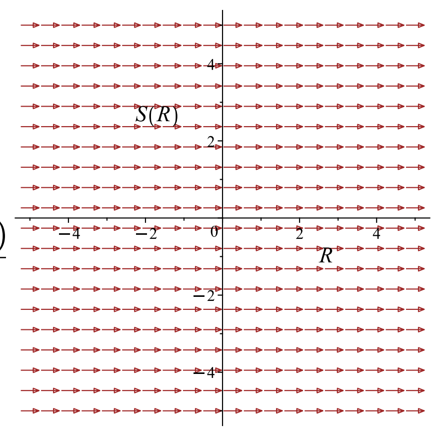
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(5+x^2+(3y-5)\,x+y^2-5y\right)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln\left(5+x^2+(3y-5)\,x+y^2-5y\right)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y-5}{3x+2y-5}$ 	$R = x$ $S = \frac{\ln(5 + x^2 + (3y - 5)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(5 + x^2 + (3y - 5)x + y^2 - 5y)}{2} = c_1 \tag{1}$$

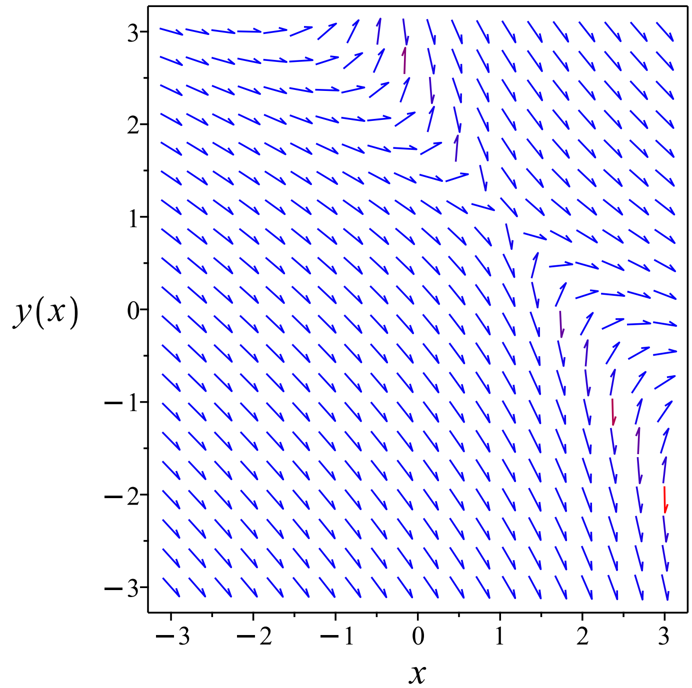


Figure 181: Slope field plot

Verification of solutions

$$\frac{\ln\left(5+x^2+(3y-5)\,x+y^2-5y\right)}{2}=c_1$$

Verified OK.

5.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x + 2y - 5)\,\mathrm{d}y &= (-2x - 3y + 5)\,\mathrm{d}x\\ (2x + 3y - 5)\,\mathrm{d}x + (3x + 2y - 5)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x + 3y - 5\\ N(x,y) &= 3x + 2y - 5\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x + 3y - 5)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x + 2y - 5)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 2x + 3y - 5 \,\mathrm{d}x$$
$$\phi = x(x + 3y - 5) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x + 2y - 5$. Therefore equation (4) becomes

$$3x + 2y - 5 = 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -5 + 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-5 + 2y) \,\mathrm{d}y$$
$$f(y) = y^2 - 5y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x + 3y - 5) + y^2 - 5y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x + 3y - 5) + y^2 - 5y$$

Summary

The solution(s) found are the following

$$x(x + 3y - 5) + y^2 - 5y = c_1 \tag{1}$$

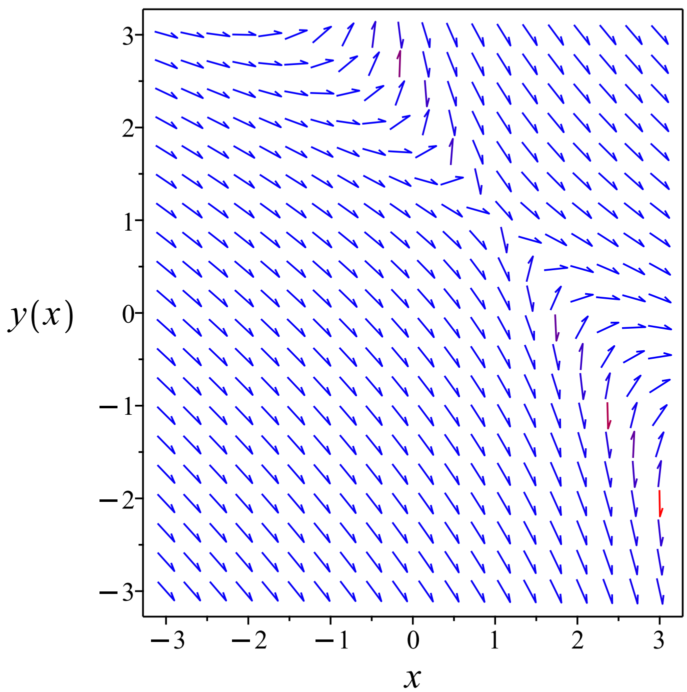


Figure 182: Slope field plot

Verification of solutions

$$x(x + 3y - 5) + y^2 - 5y = c_1$$

Verified OK.

5.13.5 Maple step by step solution

Let's solve

$3y + (3x + 2y - 5)\, y' = -2x + 5$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $3 = 3$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y)\,, N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (2x + 3y - 5)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x^2 + 3xy - 5x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3x + 2y - 5 = 3x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -5 + 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2 - 5y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = x^2 + 3xy + y^2 - 5x - 5y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$x^2 + 3xy + y^2 - 5x - 5y = c_1$$

- Solve for y

$$\left\{y = -\frac{3x}{2} + \frac{5}{2} - \frac{\sqrt{5x^2+4c_1-10x+25}}{2}, y = -\frac{3x}{2} + \frac{5}{2} + \frac{\sqrt{5x^2+4c_1-10x+25}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 33

```
dsolve((2*x+3*y(x)-5)+(3*x+2*y(x)-5)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{-\sqrt{5\left(-1+x\right)^2 c_1^2 + 4} + \left(-3x+5\right)c_1}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 65

```
DSolve[(2*x+3*y[x]-5)+(3*x+2*y[x]-5)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\sqrt{5x^2-10x+25+4c_1}-3x+5\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{5x^2-10x+25+4c_1}-3x+5\right)$$

5.14 problem 113

Internal problem ID [15020]
Internal file name [`OUTPUT/15020_Friday_April_19_2024_04_44_19_AM_34107603/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 113.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{4y + (4x + 2y + 1)\, y' = -8x - 1}$$

5.14.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-8x - 4y - 1}{4x + 2y + 1} \tag{1}$$

Which becomes

$$(2y + 1)\, dy = (-4x)\, dy + (-8x - 4y - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-4x)\, dy + (-8x - 4y - 1)\, dx = d\big(-4x^2 - 4xy - x\big)$$

Hence (2) becomes

$$(2y+1)\,dy = d\left(-4x^2 - 4xy - x\right)$$

Integrating both sides gives gives these solutions

$$y = -2x - \frac{1}{2} + \frac{\sqrt{4c_1 + 4x + 1}}{2} + c_1$$
$$y = -2x - \frac{1}{2} - \frac{\sqrt{4c_1 + 4x + 1}}{2} + c_1$$

Summary
The solution(s) found are the following

$$y = -2x - \frac{1}{2} + \frac{\sqrt{4c_1 + 4x + 1}}{2} + c_1 \tag{1}$$
$$y = -2x - \frac{1}{2} - \frac{\sqrt{4c_1 + 4x + 1}}{2} + c_1 \tag{2}$$

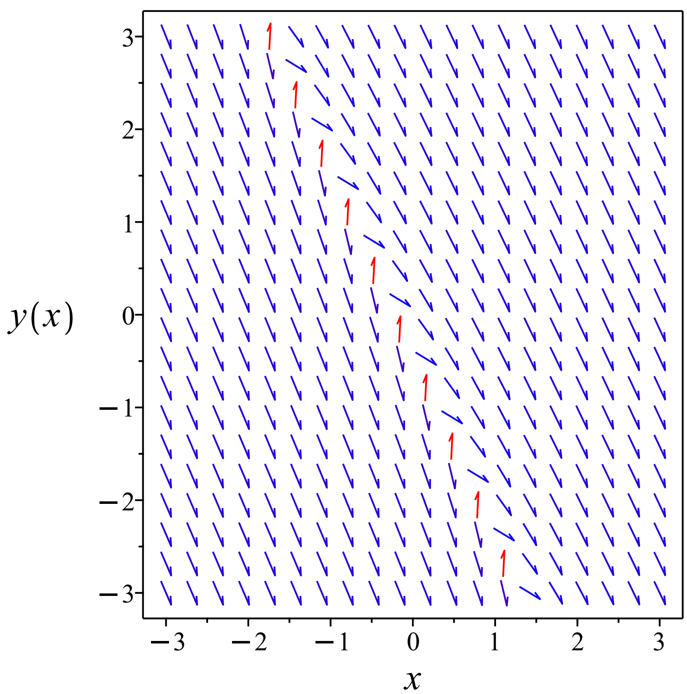


Figure 183: Slope field plot

Verification of solutions

$$y = -2x - \frac{1}{2} + \frac{\sqrt{4c_1 + 4x + 1}}{2} + c_1$$

Verified OK.

$$y = -2x - \frac{1}{2} - \frac{\sqrt{4c_1 + 4x + 1}}{2} + c_1$$

Verified OK.

5.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{8x + 4y + 1}{4x + 2y + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(8x + 4y + 1)(b_3 - a_2)}{4x + 2y + 1} - \frac{(8x + 4y + 1)^2 a_3}{(4x + 2y + 1)^2} - \left(-\frac{8}{4x + 2y + 1} + \frac{32x + 16y + 4}{(4x + 2y + 1)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{4}{4x + 2y + 1} + \frac{16x + 8y + 2}{(4x + 2y + 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{32x^2a_2 - 64x^2a_3 + 16x^2b_2 - 32x^2b_3 + 32xya_2 - 64xya_3 + 16xyb_2 - 32xyb_3 + 8y^2a_2 - 16y^2a_3 + 4y^2b_2 - 8}{(4x + 2y}$$
$= 0$

Setting the numerator to zero gives

$$32x^2a_2 - 64x^2a_3 + 16x^2b_2 - 32x^2b_3 + 32xya_2 - 64xya_3 + 16xyb_2 - 32xyb_3 + 8y^2a_2 - 16y^2a_3 + 4y^2b_2 - 8y^2b_3 + 16xa_2 - 16xa_3 + 10xb_2 - 12xb_3 + 6ya_2 - 4ya_3 + 4yb_2 - 4yb_3 + 4a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$32a_2v_1^2 + 32a_2v_1v_2 + 8a_2v_2^2 - 64a_3v_1^2 - 64a_3v_1v_2 - 16a_3v_2^2 + 16b_2v_1^2 + 16b_2v_1v_2 + 4b_2v_2^2 - 32b_3v_1^2 - 32b_3v_1v_2 - 8b_3v_2^2 + 16a_2v_1 + 6a_2v_2 - 16a_3v_1 - 4a_3v_2 + 10b_2v_1 + 4b_2v_2 - 12b_3v_1 - 4b_3v_2 + 4a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(32a_2 - 64a_3 + 16b_2 - 32b_3)\, v_1^2 + (32a_2 - 64a_3 + 16b_2 - 32b_3)\, v_1v_2 + (16a_2 - 16a_3 + 10b_2 - 12b_3)\, v_1 + (8a_2 - 16a_3 + 4b_2 - 8b_3)\, v_2^2 + (6a_2 - 4a_3 + 4b_2 - 4b_3)\, v_2 + 4a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}6a_2 - 4a_3 + 4b_2 - 4b_3 &= 0\\ 8a_2 - 16a_3 + 4b_2 - 8b_3 &= 0\\ 16a_2 - 16a_3 + 10b_2 - 12b_3 &= 0\\ 32a_2 - 64a_3 + 16b_2 - 32b_3 &= 0\\ 4a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1\\ a_2 &= 8a_1 + 4b_1\\ a_3 &= a_3\\ b_1 &= b_1\\ b_2 &= -8a_1 - 4b_1 - 2a_3\\ b_3 &= 4a_1 - 3a_3 + 2b_1\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= y\\ \eta &= -2x - 3y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= -2x - 3y - \left(-\frac{8x + 4y + 1}{4x + 2y + 1}\right)(y)\\ &= \frac{-8x^2 - 8xy - 2y^2 - 2x - 2y}{4x + 2y + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{-8x^2-8xy-2y^2-2x-2y}{4x+2y+1}} dy\end{aligned}$$

Which results in

$$S = -\frac{\ln\left(4x^2+4xy+y^2+x+y\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{8x+4y+1}{4x+2y+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= \frac{-8x-4y-1}{8x^2+(8y+2)\,x+2y^2+2y}\\ S_y &= \frac{-4x-2y-1}{8x^2+(8y+2)\,x+2y^2+2y}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

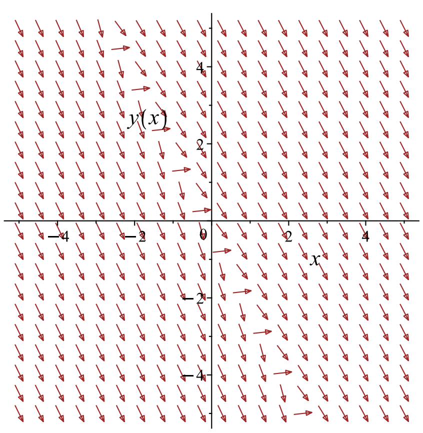
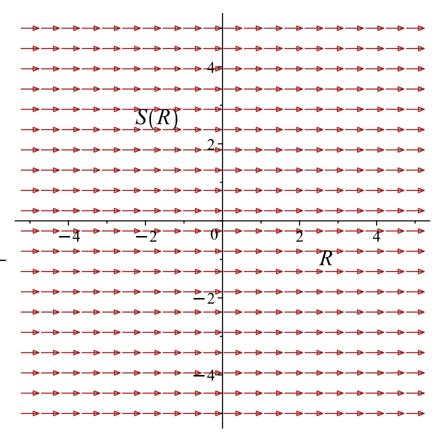
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(y^2+(4x+1)\,y+4x^2+x\right)}{2} = c_1$$

Which simplifies to

$$-\frac{\ln\left(y^2+(4x+1)\,y+4x^2+x\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{8x+4y+1}{4x+2y+1}$ 	$R = x$ $S = -\frac{\ln\left(y^2+(4x+1)\,y\right.}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$-\frac{\ln\left(y^2+(4x+1)\,y+4x^2+x\right)}{2} = c_1 \tag{1}$$

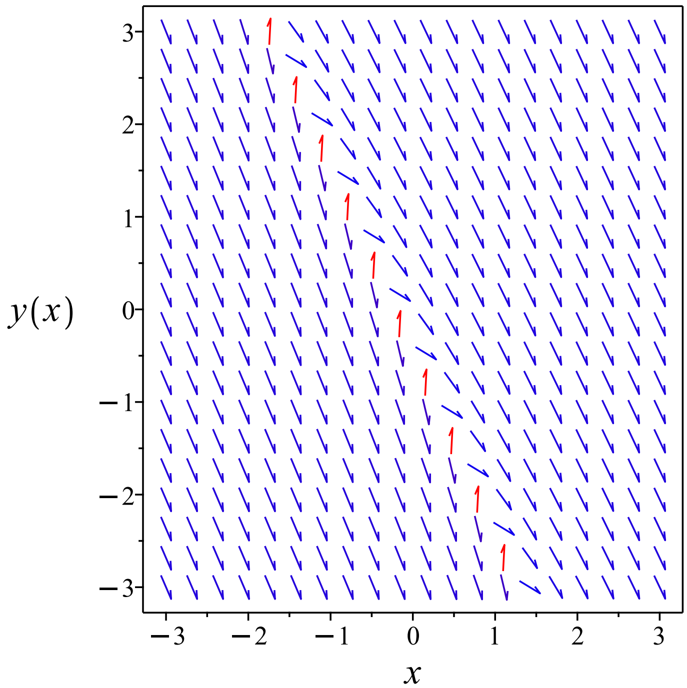


Figure 184: Slope field plot

Verification of solutions

$$
-\frac{\ln \left(y^{2}+\left(4 x+1\right) y+4 x^{2}+x\right)}{2}=c_{1}
$$

Verified OK.

5.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(4x + 2y + 1)\,\mathrm{d}y &= (-8x - 4y - 1)\,\mathrm{d}x \\
(8x + 4y + 1)\,\mathrm{d}x + (4x + 2y + 1)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 8x + 4y + 1 \\
N(x, y) &= 4x + 2y + 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(8x + 4y + 1) \\
&= 4
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4x + 2y + 1) \\
&= 4
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int 8x + 4y + 1 \, dx$$

$$\phi = x(4x + 4y + 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x + 2y + 1$. Therefore equation (4) becomes

$$4x + 2y + 1 = 4x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y + 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (2y + 1) \, dy$$

$$f(y) = y^2 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(4x + 4y + 1) + y^2 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(4x + 4y + 1) + y^2 + y$$

Summary
The solution(s) found are the following

$$x(4x + 4y + 1) + y^2 + y = c_1 \tag{1}$$

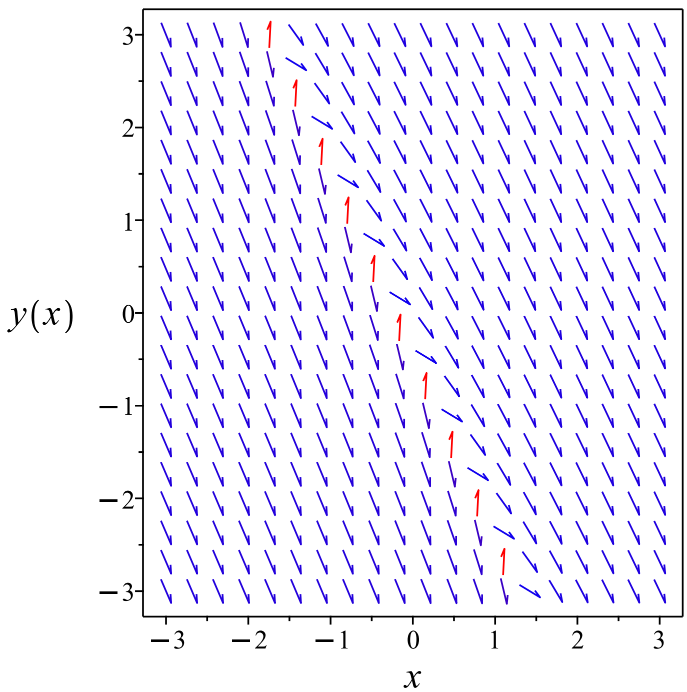


Figure 185: Slope field plot

Verification of solutions

$$x(4x + 4y + 1) + y^2 + y = c_1$$

Verified OK.

5.14.4 Maple step by step solution

Let's solve

$4y + (4x + 2y + 1)\, y' = -8x - 1$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - ∘ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - ∘ Evaluate derivatives

    $4 = 4$

  - ∘ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (8x + 4y + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = 4x^2 + 4xy + x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $4x + 2y + 1 = 4x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y + 1$

- Solve for $f_1(y)$

  $f_1(y) = y^2 + y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = 4x^2 + 4xy + y^2 + x + y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$4x^2 + 4xy + y^2 + x + y = c_1$$

- Solve for y

$$\left\{y = -2x - \frac{1}{2} - \frac{\sqrt{4c_1 + 4x + 1}}{2}, y = -2x - \frac{1}{2} + \frac{\sqrt{4c_1 + 4x + 1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve((8*x+4*y(x)+1)+(4*x+2*y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = -2x - \frac{1}{2} - \frac{\sqrt{-4c_1 + 4x + 1}}{2}$$
$$y = -2x - \frac{1}{2} + \frac{\sqrt{-4c_1 + 4x + 1}}{2}$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 55

```
DSolve[(8*x+4*y[x]+1)+(4*x+2*y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-4x - \sqrt{4x+1+4c_1} - 1\right)$$
$$y(x) \to \frac{1}{2}\left(-4x + \sqrt{4x+1+4c_1} - 1\right)$$

5.15 problem 114

Internal problem ID [15021]
Internal file name [`OUTPUT/15021_Friday_April_19_2024_04_44_20_AM_58728981/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 114.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y+(3x-6y+2)\,y'=1-x}$$

5.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-x+2y+1}{-3x+6y-2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(-x+2y+1)(b_3-a_2)}{-3x+6y-2} - \frac{(-x+2y+1)^2 a_3}{(-3x+6y-2)^2} \\
& - \left(\frac{1}{-3x+6y-2} - \frac{3(-x+2y+1)}{(-3x+6y-2)^2}\right)(xa_2+ya_3+a_1) \\
& - \left(-\frac{2}{-3x+6y-2} + \frac{-6x+12y+6}{(-3x+6y-2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 - 12xya_2 + 4xya_3 - 36xyb_2 + 12xyb_3 + 12y^2a_2 - 4y^2a_3 + 36y^2b_2 - 12y^2b_3}{(3x - 6y + 2}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
& 3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 - 12xya_2 + 4xya_3 - 36xyb_2 + 12xyb_3 \\
& + 12y^2a_2 - 4y^2a_3 + 36y^2b_2 - 12y^2b_3 + 4xa_2 + 2xa_3 + 2xb_2 + xb_3 \\
& + 2ya_2 + ya_3 - 24yb_2 - 12yb_3 + 5a_1 - 2a_2 - a_3 - 10b_1 + 4b_2 + 2b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& 3a_2v_1^2 - 12a_2v_1v_2 + 12a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 4a_3v_2^2 + 9b_2v_1^2 - 36b_2v_1v_2 \\
& + 36b_2v_2^2 - 3b_3v_1^2 + 12b_3v_1v_2 - 12b_3v_2^2 + 4a_2v_1 + 2a_2v_2 + 2a_3v_1 + a_3v_2 \\
& + 2b_2v_1 - 24b_2v_2 + b_3v_1 - 12b_3v_2 + 5a_1 - 2a_2 - a_3 - 10b_1 + 4b_2 + 2b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(3a_2 - a_3 + 9b_2 - 3b_3)\, v_1^2 + (-12a_2 + 4a_3 - 36b_2 + 12b_3)\, v_1 v_2 \\ + (4a_2 + 2a_3 + 2b_2 + b_3)\, v_1 + (12a_2 - 4a_3 + 36b_2 - 12b_3)\, v_2^2 \\ + (2a_2 + a_3 - 24b_2 - 12b_3)\, v_2 + 5a_1 - 2a_2 - a_3 - 10b_1 + 4b_2 + 2b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-12a_2 + 4a_3 - 36b_2 + 12b_3 &= 0 \\ 2a_2 + a_3 - 24b_2 - 12b_3 &= 0 \\ 3a_2 - a_3 + 9b_2 - 3b_3 &= 0 \\ 4a_2 + 2a_3 + 2b_2 + b_3 &= 0 \\ 12a_2 - 4a_3 + 36b_2 - 12b_3 &= 0 \\ 5a_1 - 2a_2 - a_3 - 10b_1 + 4b_2 + 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 2b_1 \\ a_2 &= -3b_2 \\ a_3 &= 6b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= -2b_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 2 \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(-\frac{-x + 2y + 1}{-3x + 6y - 2}\right)(2) \\ &= \frac{5x - 10y}{3x - 6y + 2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{5x-10y}{3x-6y+2}} dy \end{aligned}$$

Which results in

$$S = \frac{3y}{5} - \frac{\ln(2y - x)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x + 2y + 1}{-3x + 6y - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{5x - 10y} \\ S_y &= \frac{3}{5} + \frac{2}{5x - 10y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3y}{5} - \frac{\ln(-x+2y)}{5} = -\frac{x}{5} + c_1$$

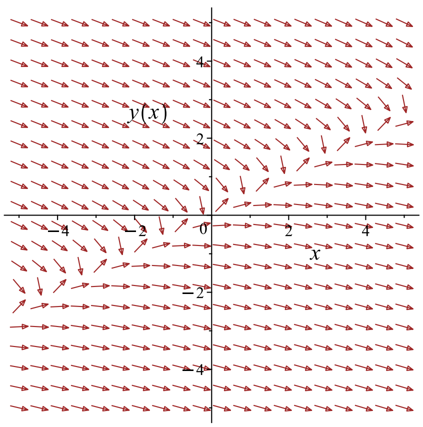
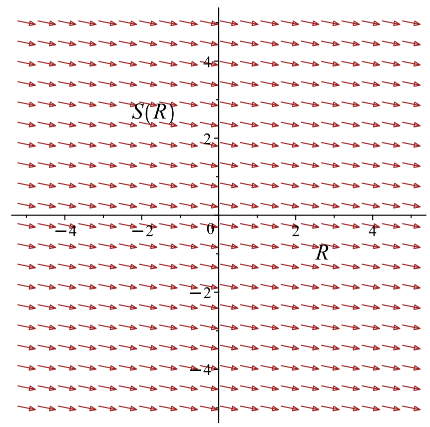
Which simplifies to

$$\frac{3y}{5} - \frac{\ln(-x+2y)}{5} = -\frac{x}{5} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{-\operatorname{LambertW}\left(-\frac{3\,\mathrm{e}^{\frac{5x}{2}-5c_1}}{2}\right)+\frac{5x}{2}-5c_1}}{2} + \frac{x}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x+2y+1}{-3x+6y-2}$ 	$R = x$ $S = \frac{3y}{5} - \frac{\ln(2y-x)}{5}$	$\frac{dS}{dR} = -\frac{1}{5}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-\operatorname{LambertW}\left(-\frac{3\,e^{\frac{5x}{2}-5c_1}}{2}\right)+\frac{5x}{2}-5c_1}}{2} + \frac{x}{2} \tag{1}$$

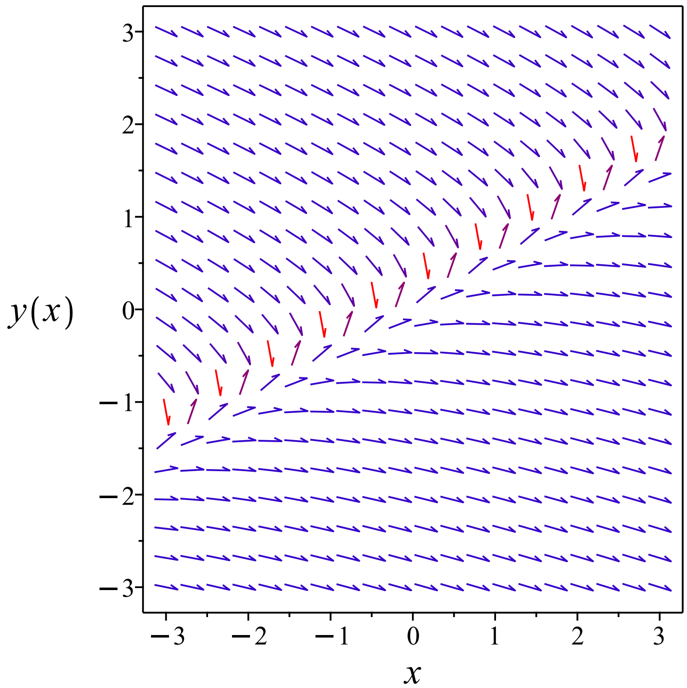


Figure 186: Slope field plot

Verification of solutions

$$y = \frac{e^{-\operatorname{LambertW}\left(-\frac{3\,e^{\frac{5x}{2}-5c_1}}{2}\right)+\frac{5x}{2}-5c_1}}{2} + \frac{x}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((x-2*y(x)-1)+(3*x-6*y(x)+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = -\frac{\text{LambertW}\left(-3\,\mathrm{e}^{\frac{5x}{2}-\frac{5c_1}{2}}\right)}{3} + \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 4.144 (sec). Leaf size: 38

```
DSolve[(x-2*y[x]-1)+(3*x-6*y[x]+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(3x - 2W\left(-e^{\frac{5x}{2}-1+c_1}\right)\right)$$
$$y(x) \to \frac{x}{2}$$

5.16 problem 115

Internal problem ID [15022]
Internal file name [`OUTPUT/15022_Friday_April_19_2024_04_44_22_AM_61324497/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 115.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y + (y - 1 + x)\, y' = -x}$$

5.16.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-y - x}{y - 1 + x} \tag{1}$$

Which becomes

$$(y - 1)\, dy = (-x)\, dy + (-y - x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (-y - x)\, dx = d\left(-\frac{1}{2}x^2 - xy\right)$$

Hence (2) becomes

$$(y-1)\,dy = d\left(-\frac{1}{2}x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = -x + 1 + \sqrt{2c_1 - 2x + 1} + c_1$$
$$y = -x + 1 - \sqrt{2c_1 - 2x + 1} + c_1$$

Summary

The solution(s) found are the following

$$y = -x + 1 + \sqrt{2c_1 - 2x + 1} + c_1 \tag{1}$$
$$y = -x + 1 - \sqrt{2c_1 - 2x + 1} + c_1 \tag{2}$$

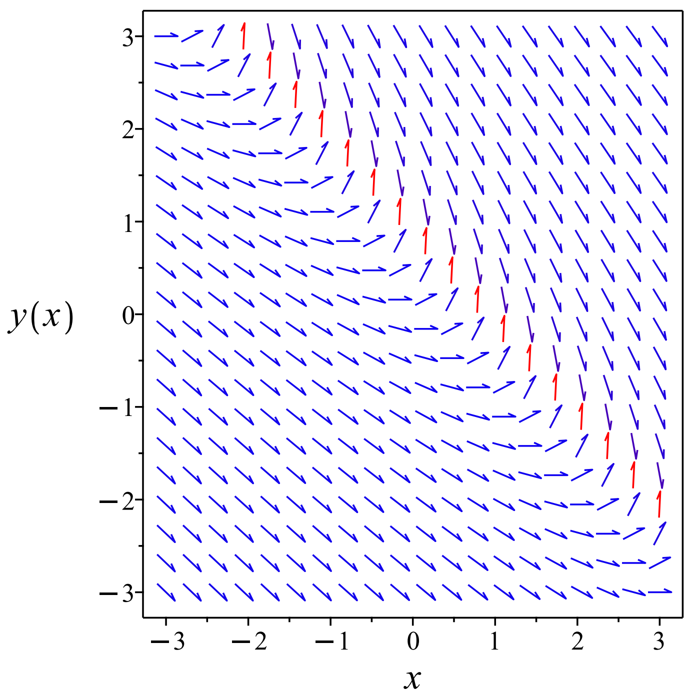


Figure 187: Slope field plot

Verification of solutions

$$y = -x + 1 + \sqrt{2c_1 - 2x + 1} + c_1$$

Verified OK.

$$y = -x + 1 - \sqrt{2c_1 - 2x + 1} + c_1$$

Verified OK.

5.16.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y+x}{x-1+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(y+x)(b_3 - a_2)}{x-1+y} - \frac{(y+x)^2 a_3}{(x-1+y)^2} \\
& - \left(-\frac{1}{x-1+y} + \frac{y+x}{(x-1+y)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(-\frac{1}{x-1+y} + \frac{y+x}{(x-1+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^2 a_2 - x^2 a_3 + x^2 b_2 - x^2 b_3 + 2xya_2 - 2xya_3 + 2xyb_2 - 2xyb_3 + y^2 a_2 - y^2 a_3 + y^2 b_2 - y^2 b_3 - 2xa_2 - 3xb_2}{(x-1+y)^2}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
& x^2 a_2 - x^2 a_3 + x^2 b_2 - x^2 b_3 + 2xya_2 - 2xya_3 + 2xyb_2 - 2xyb_3 + y^2 a_2 - y^2 a_3 + y^2 b_2 \\
& - y^2 b_3 - 2xa_2 - 3xb_2 + xb_3 - ya_2 - ya_3 - 2yb_2 - a_1 - b_1 + b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a_2 v_1^2 + 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 - 2a_3 v_1 v_2 - a_3 v_2^2 + b_2 v_1^2 + 2b_2 v_1 v_2 + b_2 v_2^2 - b_3 v_1^2 - 2b_3 v_1 v_2 - b_3 v_2^2 - 2a_2 v_1 - a_2 v_2 - a_3 v_2 - 3b_2 v_1 - 2b_2 v_2 + b_3 v_1 - a_1 - b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(a_2 - a_3 + b_2 - b_3) v_1^2 + (2a_2 - 2a_3 + 2b_2 - 2b_3) v_1 v_2 + (-2a_2 - 3b_2 + b_3) v_1 + (a_2 - a_3 + b_2 - b_3) v_2^2 + (-a_2 - a_3 - 2b_2) v_2 - a_1 - b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_1 - b_1 + b_2 &= 0 \\
-2a_2 - 3b_2 + b_3 &= 0 \\
-a_2 - a_3 - 2b_2 &= 0 \\
a_2 - a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 + 2b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_1 + b_2 \\
a_2 &= -a_3 - 2b_2 \\
a_3 &= a_3 \\
b_1 &= b_1 \\
b_2 &= b_2 \\
b_3 &= -2a_3 - b_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(-\frac{y+x}{x-1+y}\right)(-1) \\
&= -\frac{1}{x-1+y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{1}{x-1+y}} dy
\end{aligned}
$$

Which results in

$$
S = -xy + y - \frac{1}{2}y^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y+x}{x-1+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -x+1-y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

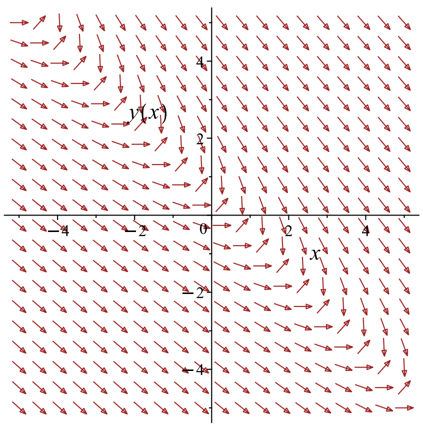
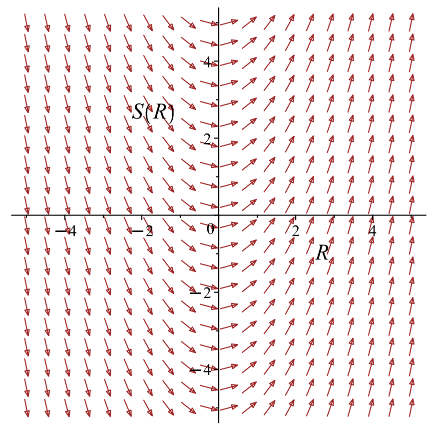
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y(2x+y-2)}{2} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{y(2x+y-2)}{2} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+x}{x-1+y}$ 	$R = x$ $S = -\frac{y(2x+y-2)}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-\frac{y(2x+y-2)}{2} = \frac{x^2}{2} + c_1 \tag{1}$$

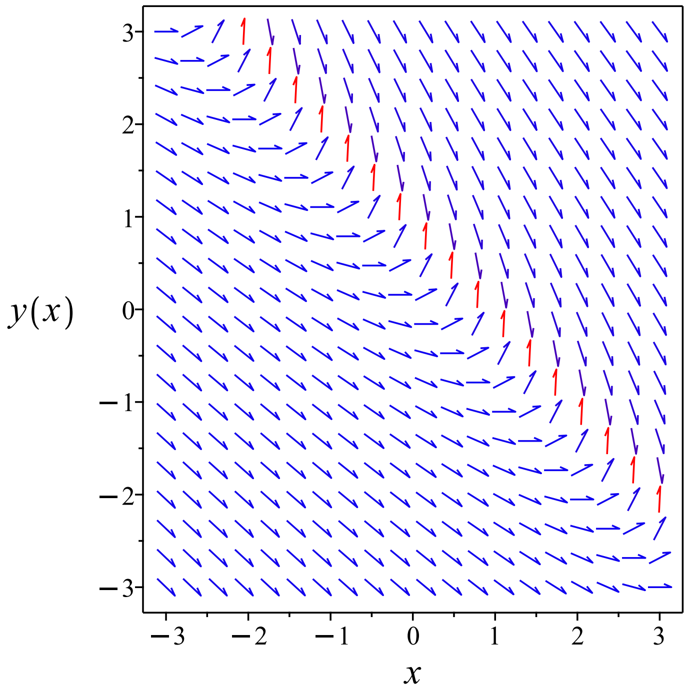


Figure 188: Slope field plot

Verification of solutions

$$-\frac{y(2x+y-2)}{2}=\frac{x^2}{2}+c_1$$

Verified OK.

5.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x - 1 + y)\, \mathrm{d}y = (-y - x)\, \mathrm{d}x$$
$$(y + x)\, \mathrm{d}x + (x - 1 + y)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y + x$$
$$N(x, y) = x - 1 + y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + x) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - 1 + y) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y + x \, \mathrm{d}x$$

$$\phi = \frac{x(x + 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - 1 + y$. Therefore equation (4) becomes

$$x - 1 + y = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y - 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y - 1) \, \mathrm{d}y$$

$$f(y) = \frac{1}{2}y^2 - y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x + 2y)}{2} + \frac{y^2}{2} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y)}{2} + \frac{y^2}{2} - y$$

Summary
The solution(s) found are the following

$$\frac{x(x+2y)}{2} + \frac{y^2}{2} - y = c_1 \tag{1}$$

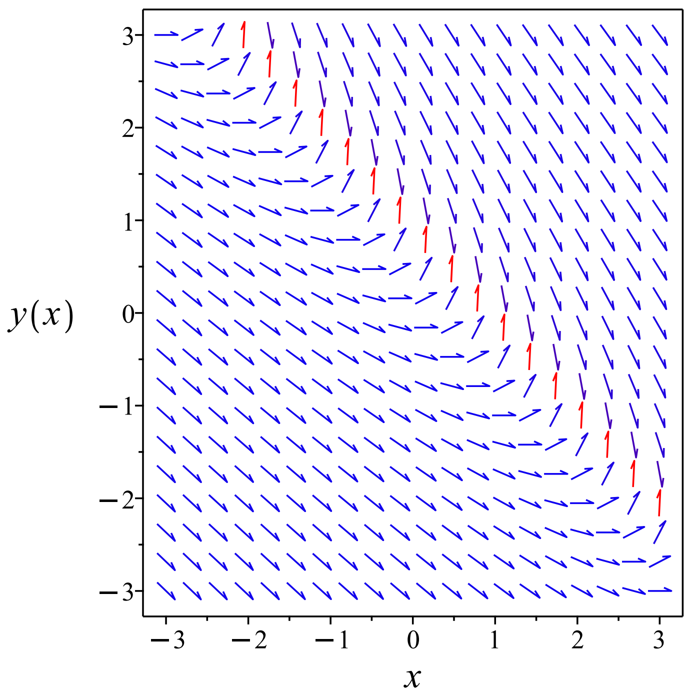


Figure 189: Slope field plot

Verification of solutions

$$\frac{x(x+2y)}{2} + \frac{y^2}{2} - y = c_1$$

Verified OK.

5.16.4 Maple step by step solution

Let's solve

$y + (y - 1 + x)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $1 = 1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\, , N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (y + x)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = xy + \frac{x^2}{2} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x - 1 + y = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y - 1$

- Solve for $f_1(y)$

  $f_1(y) = \frac{1}{2} y^2 - y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = xy + \frac{1}{2}x^2 + \frac{1}{2}y^2 - y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$xy + \frac{1}{2}x^2 + \frac{1}{2}y^2 - y = c_1$$

- Solve for y

$$\left\{y = -x + 1 - \sqrt{2c_1 - 2x + 1}, y = -x + 1 + \sqrt{2c_1 - 2x + 1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve((x+y(x))+(x+y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = 1 - x - \sqrt{2c_1 - 2x + 1}$$
$$y = 1 - x + \sqrt{2c_1 - 2x + 1}$$

✓ Solution by Mathematica

Time used: 0.123 (sec). Leaf size: 43

```
DSolve[(x+y[x])+(x+y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \sqrt{-2x + 1 + c_1} + 1$$
$$y(x) \to -x + \sqrt{-2x + 1 + c_1} + 1$$

5.17 problem 116

Internal problem ID [15023]
Internal file name [`OUTPUT/15023_Sunday_April_21_2024_01_20_48_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 116.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{2xy'(-y^2+x)+y^3=0}$$

5.17.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^3}{2x(y^2-x)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y^3(b_3 - a_2)}{2x\,(y^2 - x)} - \frac{y^6 a_3}{4x^2\,(y^2 - x)^2} \\ & - \left(-\frac{y^3}{2x^2\,(y^2 - x)} + \frac{y^3}{2x\,(y^2 - x)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(\frac{3y^2}{2x\,(y^2 - x)} - \frac{y^4}{x\,(y^2 - x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2y^4b_2 + y^6a_3 - 2x^3y^2b_2 - 2x^2y^3a_2 + 4x^2y^3b_3 - 4x\,y^4a_3 - 2x\,y^4b_1 + 2y^5a_1 + 4x^4b_2 + 6x^2y^2b_1 - 4x\,y^3a_1}{4x^2\,(-y^2 + x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^2y^4b_2 + y^6a_3 - 2x^3y^2b_2 - 2x^2y^3a_2 + 4x^2y^3b_3 - 4x\,y^4a_3 \\ & - 2x\,y^4b_1 + 2y^5a_1 + 4x^4b_2 + 6x^2y^2b_1 - 4x\,y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_3v_2^6 + 2b_2v_1^2v_2^4 + 2a_1v_2^5 - 2a_2v_1^2v_2^3 - 4a_3v_1v_2^4 - 2b_1v_1v_2^4 \\ & - 2b_2v_1^3v_2^2 + 4b_3v_1^2v_2^3 - 4a_1v_1v_2^3 + 6b_1v_1^2v_2^2 + 4b_2v_1^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^4 - 2b_2v_1^3v_2^2 + 2b_2v_1^2v_2^4 + (-2a_2 + 4b_3)\, v_1^2v_2^3 + 6b_1v_1^2v_2^2 \\ + (-4a_3 - 2b_1)\, v_1v_2^4 - 4a_1v_1v_2^3 + a_3v_2^6 + 2a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_3 &= 0 \\ -4a_1 &= 0 \\ 2a_1 &= 0 \\ 6b_1 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ 4b_2 &= 0 \\ -2a_2 + 4b_3 &= 0 \\ -4a_3 - 2b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^3}{2x\,(y^2 - x)}\right)(2x) \\
&= \frac{xy}{-y^2 + x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{xy}{-y^2+x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{y^2}{2x} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^3}{2x\,(y^2 - x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y^2}{2x^2} \\
S_y &= \frac{-y^2 + x}{xy}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2 \ln (y) x - y^2}{2x} = c_1
$$

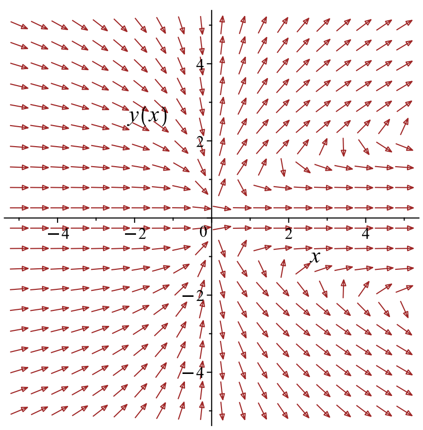
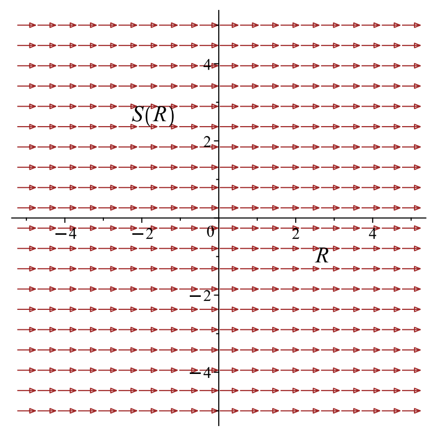
Which simplifies to

$$
\frac{2 \ln (y) x - y^2}{2x} = c_1
$$

Which gives

$$
y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{2c_1}}{x}\right)}{2} + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^3}{2x(y^2-x)}$  	$R = x$ $S = \frac{2\ln(y)\,x - y^2}{2x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\operatorname{LambertW}\left(-\frac{\mathrm{e}^{2c_1}}{x}\right)}{2}+c_1} \tag{1}$$

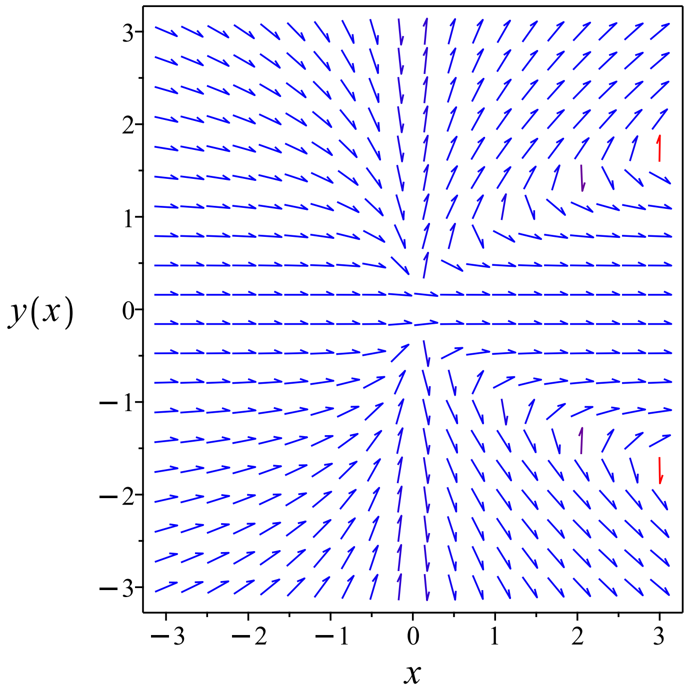


Figure 190: Slope field plot

Verification of solutions

$$y = e^{-\frac{\text{LambertW}\left(-\frac{e^{2c_1}}{x}\right)}{2}+c_1}$$

Verified OK.

5.17.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2x\left(-y^2+x\right)\right)\mathrm{d}y &= \left(-y^3\right)\mathrm{d}x \\
\left(y^3\right)\mathrm{d}x + \left(2x\left(-y^2+x\right)\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= y^3 \\
N(x,y) &= 2x\left(-y^2+x\right)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^3\right) \\
&= 3y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x\left(-y^2+x\right)\right) \\
&= -2y^2+4x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = y^3$ and $N = 2x(-y^2 + x)$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y^2}{x^2}$$
$$N = \frac{-2y^2 + 2x}{yx}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-2y^2+2x}{xy}\right)\mathrm{d}y = \left(-\frac{y^2}{x^2}\right)\mathrm{d}x$$

$$\left(\frac{y^2}{x^2}\right)\mathrm{d}x + \left(\frac{-2y^2+2x}{xy}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y^2}{x^2}$$
$$N(x,y) = \frac{-2y^2+2x}{xy}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2}{x^2}\right)\\ &= \frac{2y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-2y^2+2x}{xy}\right)\\ &= \frac{2y}{x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{y^2}{x^2}\,\mathrm{d}x$$

$$\phi = -\frac{y^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2y^2+2x}{xy}$. Therefore equation (4) becomes

$$\frac{-2y^2 + 2x}{xy} = -\frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$

$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{x} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{x} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)}{2} + \frac{c_1}{2}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)}{2} + \frac{c_1}{2}} \tag{1}$$

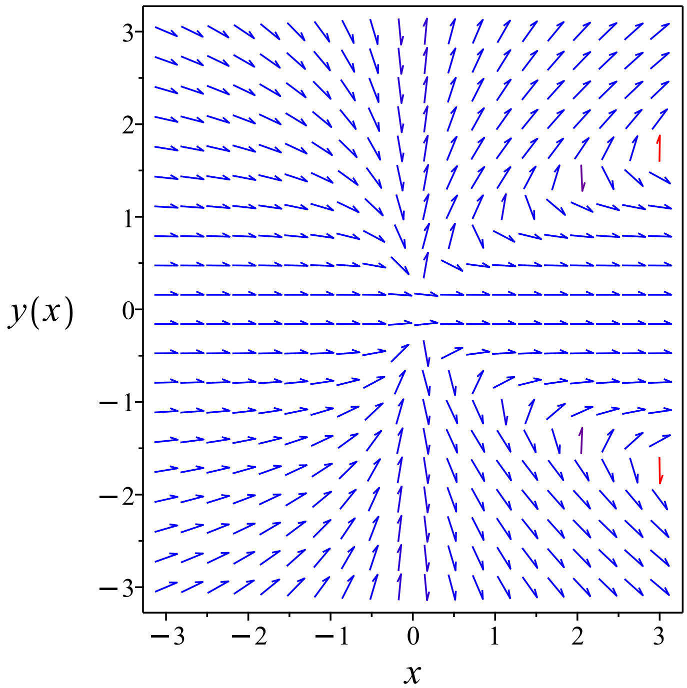


Figure 191: Slope field plot

Verification of solutions

$$y = e^{-\frac{\mathrm{LambertW}\left(-\frac{e^{c_1}}{x}\right)}{2}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(2*x*diff(y(x),x)*(x-y(x)^2)+y(x)^3=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{c_1}{2}}}{\sqrt{-\frac{e^{c_1}}{x \operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)}}}$$

✓ Solution by Mathematica

Time used: 6.48 (sec). Leaf size: 60

```
DSolve[2*x*y'[x]*(x-y[x]^2)+y[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{x}\sqrt{W\left(-\frac{e^{c_1}}{x}\right)}$$
$$y(x) \to i\sqrt{x}\sqrt{W\left(-\frac{e^{c_1}}{x}\right)}$$
$$y(x) \to 0$$

5.18 problem 117

Internal problem ID [15024]
Internal file name [`OUTPUT/15024_Sunday_April_21_2024_01_20_52_PM_33949059/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 117.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{4y^6 - 6y^5xy' = -x^3}$$

5.18.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{4y^6 + x^3}{6x\,y^5}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 151: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^4}{y^5}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^4}{y^5}} dy \end{aligned}$$

Which results in

$$S = \frac{y^6}{6x^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{4y^6 + x^3}{6x\, y^5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y^6}{3x^5} \\ S_y &= \frac{y^5}{x^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{6x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{6R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{6R} + c_1 \tag{4}$$

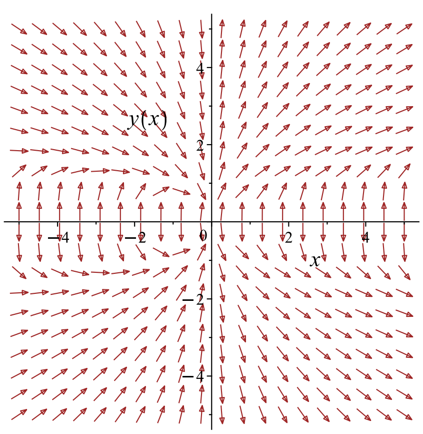
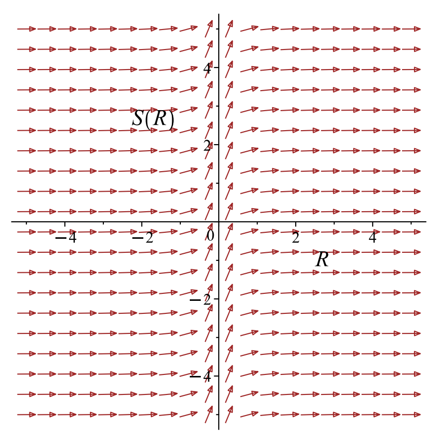
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^6}{6x^4} = -\frac{1}{6x} + c_1$$

Which simplifies to

$$\frac{y^6}{6x^4} = -\frac{1}{6x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{4y^6+x^3}{6x\,y^5}$ 	$R = x$ $S = \frac{y^6}{6x^4}$	$\frac{dS}{dR} = \frac{1}{6R^2}$ 

Summary

The solution(s) found are the following

$$\frac{y^6}{6x^4} = -\frac{1}{6x} + c_1 \tag{1}$$

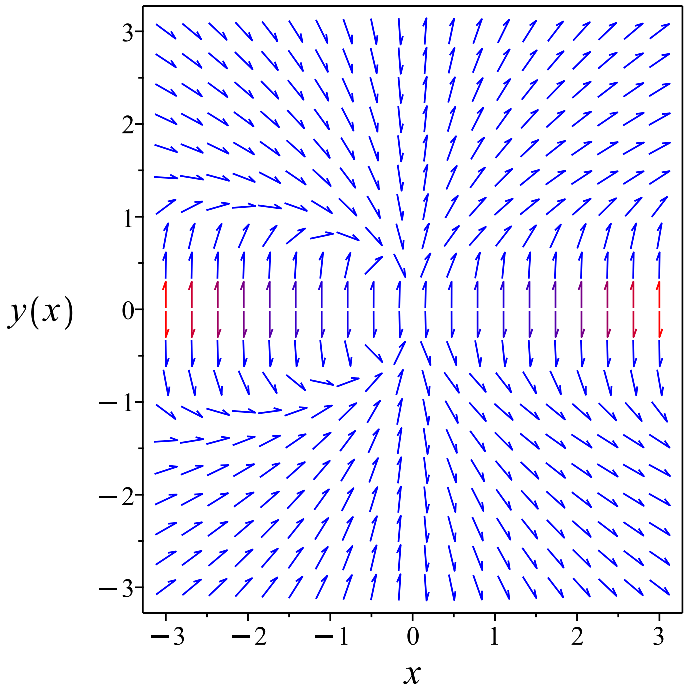


Figure 192: Slope field plot

Verification of solutions

$$\frac{y^6}{6x^4} = -\frac{1}{6x} + c_1$$

Verified OK.

5.18.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{4y^6 + x^3}{6x\,y^5} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{3x}y + \frac{x^2}{6}\frac{1}{y^5} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{2}{3x} \\
f_1(x) &= \frac{x^2}{6} \\
n &= -5
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^5}$ gives

$$
y'y^5 = \frac{2y^6}{3x} + \frac{x^2}{6} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^6 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 6y^5y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{6} &= \frac{2w(x)}{3x} + \frac{x^2}{6} \\
w' &= \frac{4w}{x} + x^2 \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{4w(x)}{x} = x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{4}{x}dx} \\ &= \frac{1}{x^4}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(\frac{w}{x^4}\right) &= \left(\frac{1}{x^4}\right)\left(x^2\right) \\ d\left(\frac{w}{x^4}\right) &= \frac{1}{x^2}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^4} &= \int \frac{1}{x^2}\, dx \\ \frac{w}{x^4} &= -\frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$w(x) = c_1 x^4 - x^3$$

Replacing w in the above by y^6 using equation (5) gives the final solution.

$$y^6 = c_1 x^4 - x^3$$

Solving for y gives

$$\begin{aligned}y(x) &= \left(x^3(c_1 x - 1)\right)^{\frac{1}{6}} \\ y(x) &= \frac{\left(1 + i\sqrt{3}\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \\ y(x) &= \frac{\left(i\sqrt{3} - 1\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \\ y(x) &= -\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}} \\ y(x) &= -\frac{\left(1 + i\sqrt{3}\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \\ y(x) &= -\frac{\left(i\sqrt{3} - 1\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(x^3(c_1 x - 1)\right)^{\frac{1}{6}} \tag{1}$$

$$y = \frac{\left(1 + i\sqrt{3}\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \tag{2}$$

$$y = \frac{\left(i\sqrt{3} - 1\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \tag{3}$$

$$y = -\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}} \tag{4}$$

$$y = -\frac{\left(1 + i\sqrt{3}\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \tag{5}$$

$$y = -\frac{\left(i\sqrt{3} - 1\right)\left(x^3(c_1 x - 1)\right)^{\frac{1}{6}}}{2} \tag{6}$$

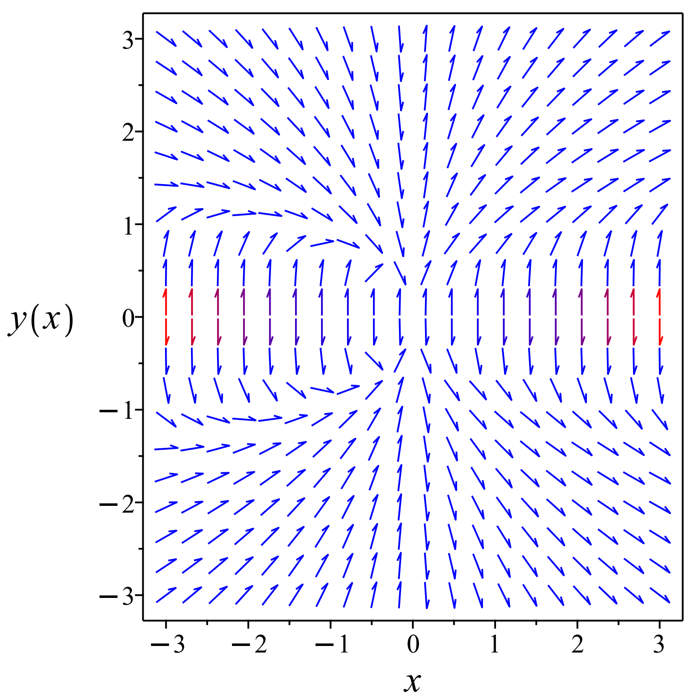


Figure 193: Slope field plot

Verification of solutions

$$y = \left(x^3(c_1x-1)\right)^{\frac{1}{6}}$$

Verified OK.

$$y = \frac{\left(1+i\sqrt{3}\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$

Verified OK.

$$y = \frac{\left(i\sqrt{3}-1\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$

Verified OK.

$$y = -\left(x^3(c_1x-1)\right)^{\frac{1}{6}}$$

Verified OK.

$$y = -\frac{\left(1+i\sqrt{3}\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$

Verified OK.

$$y = -\frac{\left(i\sqrt{3}-1\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$

Verified OK.

5.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-6x\,y^5)\,\mathrm{d}y &= (-4y^6 - x^3)\,\mathrm{d}x \\
(4y^6 + x^3)\,\mathrm{d}x + (-6x\,y^5)\,\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 4y^6 + x^3$$
$$N(x, y) = -6x\,y^5$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4y^6 + x^3) \\
&= 24y^5
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-6x\,y^5) \\
&= -6y^5
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{6y^5 x}\left(\left(24y^5\right) - \left(-6y^5\right)\right) \\ &= -\frac{5}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{5}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-5\ln(x)} \\ &= \frac{1}{x^5} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^5}\left(4y^6 + x^3\right) \\ &= \frac{4y^6 + x^3}{x^5} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^5}\left(-6x\,y^5\right) \\ &= -\frac{6y^5}{x^4} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{4y^6 + x^3}{x^5}\right) + \left(-\frac{6y^5}{x^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{4y^6 + x^3}{x^5} \mathrm{d}x$$
$$\phi = \frac{-y^6 - x^3}{x^4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{6y^5}{x^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{6y^5}{x^4}$. Therefore equation (4) becomes

$$-\frac{6y^5}{x^4} = -\frac{6y^5}{x^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-y^6 - x^3}{x^4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-y^6 - x^3}{x^4}$$

Summary

The solution(s) found are the following

$$\frac{-y^6 - x^3}{x^4} = c_1 \tag{1}$$

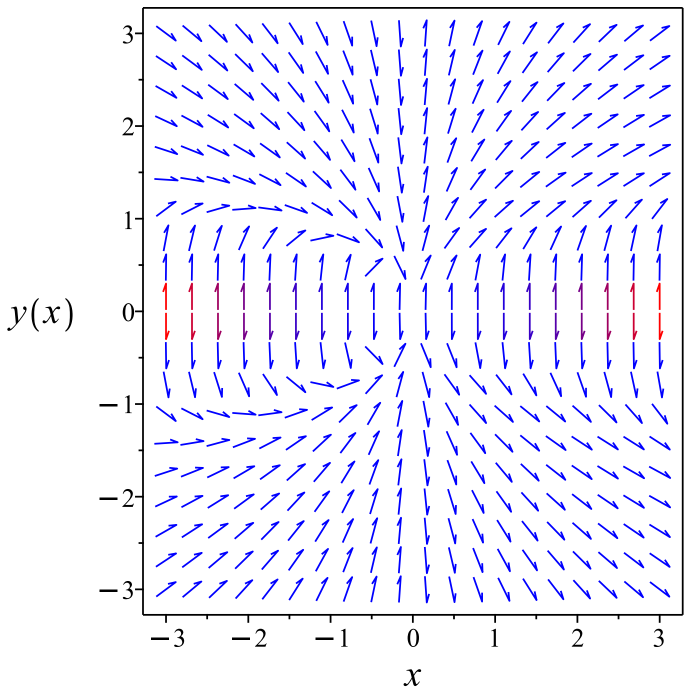


Figure 194: Slope field plot

Verification of solutions

$$\frac{-y^6 - x^3}{x^4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 127

```
dsolve(4*y(x)^6+x^3=6*x*y(x)^5*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \left(x^3(c_1x-1)\right)^{\frac{1}{6}}$$
$$y(x) = -\left(x^3(c_1x-1)\right)^{\frac{1}{6}}$$
$$y(x) = -\frac{\left(1+i\sqrt{3}\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$
$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$
$$y(x) = -\frac{\left(i\sqrt{3}-1\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$
$$y(x) = \frac{\left(1+i\sqrt{3}\right)\left(x^3(c_1x-1)\right)^{\frac{1}{6}}}{2}$$

✓ Solution by Mathematica

Time used: 0.276 (sec). Leaf size: 144

```
DSolve[4*y[x]^6+x^3==6*x*y[x]^5*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt[6]{-1+c_1x}$$
$$y(x) \to \sqrt{x}\sqrt[6]{-1+c_1x}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt{x}\sqrt[6]{-1+c_1x}$$
$$y(x) \to \sqrt[3]{-1}\sqrt{x}\sqrt[6]{-1+c_1x}$$
$$y(x) \to -(-1)^{2/3}\sqrt{x}\sqrt[6]{-1+c_1x}$$
$$y(x) \to (-1)^{2/3}\sqrt{x}\sqrt[6]{-1+c_1x}$$

5.19 problem 118

Internal problem ID [15025]
Internal file name [`OUTPUT/15025_Sunday_April_21_2024_01_20_53_PM_46894237/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 118.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y\left(1+\sqrt{y^4x^2+1}\right)+2y'x=0}$$

5.19.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y\left(1+\sqrt{x^2y^4+1}\right)}{2x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y\left(1+\sqrt{x^2y^4+1}\right)(b_3-a_2)}{2x} - \frac{y^2\left(1+\sqrt{x^2y^4+1}\right)^2 a_3}{4x^2} \\ &- \left(-\frac{y^5}{2\sqrt{x^2y^4+1}} + \frac{y\left(1+\sqrt{x^2y^4+1}\right)}{2x^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{1+\sqrt{x^2y^4+1}}{2x} - \frac{x\,y^4}{\sqrt{x^2y^4+1}}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-6x^4y^4b_2 - 2x^3y^5a_2 - 4x^3y^5b_3 + 2x^2y^6a_3 - 6x^3y^4b_1 + (x^2y^4+1)^{\frac{3}{2}}\,y^2a_3 - 6b_2\sqrt{x^2y^4+1}\,x^2 + 3y^2a_3\sqrt{x^2}}{4\sqrt{x^2y^4+1}\,x^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &6x^4y^4b_2 + 2x^3y^5a_2 + 4x^3y^5b_3 - 2x^2y^6a_3 + 6x^3y^4b_1 - \left(x^2y^4+1\right)^{\frac{3}{2}} y^2a_3 \\ &+ 6b_2\sqrt{x^2y^4+1}\,x^2 - 3y^2a_3\sqrt{x^2y^4+1} + 2\sqrt{x^2y^4+1}\,xb_1 \\ &- 2\sqrt{x^2y^4+1}\,ya_1 + 2x^2b_2 - 4y^2a_3 + 2xb_1 - 2ya_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &4x^4y^4b_2 + 2x^3y^5a_2 + 4x^3y^5b_3 + 2x^2y^6a_3 + 4x^3y^4b_1 + 2x^2y^5a_1 \\ &- \left(x^2y^4+1\right)^{\frac{3}{2}} y^2a_3 + 2\left(x^2y^4+1\right)x^2b_2 - 4\left(x^2y^4+1\right)y^2a_3 \\ &+ 2\left(x^2y^4+1\right)xb_1 - 2\left(x^2y^4+1\right)ya_1 + 6b_2\sqrt{x^2y^4+1}\,x^2 \\ &- 3y^2a_3\sqrt{x^2y^4+1} + 2\sqrt{x^2y^4+1}\,xb_1 - 2\sqrt{x^2y^4+1}\,ya_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-x^2\sqrt{x^2y^4+1}\,y^6a_3 + 6x^4y^4b_2 + 2x^3y^5a_2 + 4x^3y^5b_3 - 2x^2y^6a_3 \\ &+ 6x^3y^4b_1 + 6b_2\sqrt{x^2y^4+1}\,x^2 - 4y^2a_3\sqrt{x^2y^4+1} + 2x^2b_2 \\ &+ 2\sqrt{x^2y^4+1}\,xb_1 - 2\sqrt{x^2y^4+1}\,ya_1 - 4y^2a_3 + 2xb_1 - 2ya_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2y^4+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2y^4+1} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}-v_1^2v_3v_2^6a_3 + 2v_1^3v_2^5a_2 - 2v_1^2v_2^6a_3 + 6v_1^4v_2^4b_2 + 4v_1^3v_2^5b_3 + 6v_1^3v_2^4b_1 - 4v_2^2a_3v_3 \\ + 6b_2v_3v_1^2 - 2v_3v_2a_1 - 4v_2^2a_3 + 2v_3v_1b_1 + 2v_1^2b_2 - 2v_2a_1 + 2v_1b_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}6v_1^4v_2^4b_2 + (2a_2 + 4b_3)\,v_1^3v_2^5 + 6v_1^3v_2^4b_1 - v_1^2v_3v_2^6a_3 - 2v_1^2v_2^6a_3 + 6b_2v_3v_1^2 \\ + 2v_1^2b_2 + 2v_3v_1b_1 + 2v_1b_1 - 4v_2^2a_3v_3 - 4v_2^2a_3 - 2v_3v_2a_1 - 2v_2a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-2a_1 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ 2b_1 &= 0 \\ 6b_1 &= 0 \\ 2b_2 &= 0 \\ 6b_2 &= 0 \\ 2a_2 + 4b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -2b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= y - \left(-\frac{y\left(1+\sqrt{x^2y^4+1}\right)}{2x}\right)(-2x)\\
&= -y\sqrt{x^2y^4+1}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{-y\sqrt{x^2y^4+1}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2 y^4+1}}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y\left(1+\sqrt{x^2 y^4+1}\right)}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{2x\sqrt{x^2 y^4+1}} \\
S_y &= -\frac{1}{y\sqrt{x^2 y^4+1}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

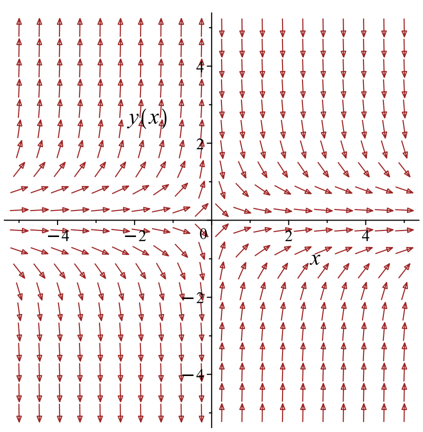
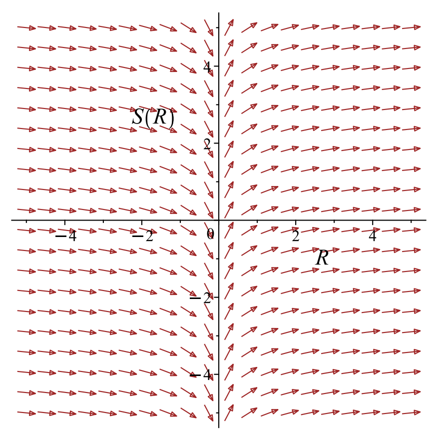
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{y^4x^2+1}}\right)}{2} = \frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{y^4x^2+1}}\right)}{2} = \frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\left(1+\sqrt{x^2y^4+1}\right)}{2x}$ 	$R = x$ $S = \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{x^2y^4+1}}\right)}{2}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{y^4x^2+1}}\right)}{2} = \frac{\ln(x)}{2} + c_1 \tag{1}$$

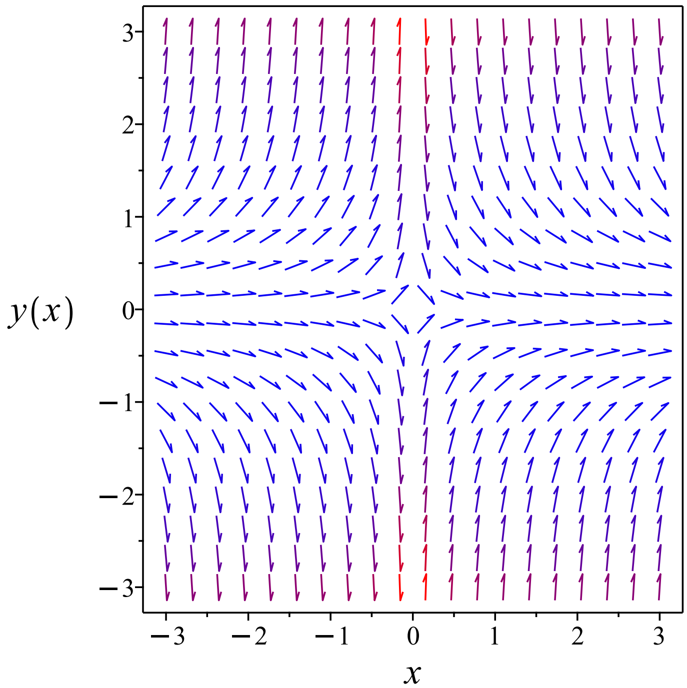


Figure 195: Slope field plot

Verification of solutions

$$\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{y^4x^2+1}}\right)}{2} = \frac{\ln(x)}{2} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(y(x)*(1+sqrt(x^2*y(x)^4+1))+2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{RootOf}\left(-\ln(x) + c_1 - 2\left(\int^{\_Z} \frac{1}{\_a\sqrt{\_a^4+1}} d\_a\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.624 (sec). Leaf size: 80

```
DSolve[y[x]*(1+Sqrt[x^2*y[x]^4+1])+2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i\sqrt{2}e^{\frac{c_1}{2}}}{\sqrt{-x^2 + e^{2c_1}}}$$
$$y(x) \to \frac{i\sqrt{2}e^{\frac{c_1}{2}}}{\sqrt{-x^2 + e^{2c_1}}}$$
$$y(x) \to 0$$

5.20 problem 119

Internal problem ID [15026]
Internal file name [`OUTPUT/15026_Sunday_April_21_2024_01_20_55_PM_73862004/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 5. Homogeneous equations. Exercises page 44
Problem number: 119.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y^3+3(y^3-x)\,y^2y'=-x}$$

5.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y^3+x}{3\,(y^3-x)\,y^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(y^3+x)(b_3-a_2)}{3(y^3-x)y^2} - \frac{(y^3+x)^2 a_3}{9(y^3-x)^2 y^4} \\
& - \left(-\frac{1}{3(y^3-x)y^2} - \frac{y^3+x}{3(y^3-x)^2 y^2}\right)(xa_2+ya_3+a_1) \\
& - \left(-\frac{1}{y^3-x} + \frac{y^3+x}{(y^3-x)^2} + \frac{\frac{2y^3}{3}+\frac{2x}{3}}{y^3(y^3-x)}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$\frac{9y^{10}b_2 - 24x\,y^7b_2 + 3y^8a_2 - 9y^8b_3 - 6y^7b_1 - 9x^2y^4b_2 + 6x\,y^5a_2 - 18x\,y^5b_3 + 5y^6a_3 - 18x\,y^4b_1 + 6y^5a_1 + \cdots}{9(-y^3+x)^2 y^4} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 9y^{10}b_2 - 24x\,y^7b_2 + 3y^8a_2 - 9y^8b_3 - 6y^7b_1 - 9x^2y^4b_2 + 6x\,y^5a_2 - 18x\,y^5b_3 + 5y^6a_3 \\
& - 18x\,y^4b_1 + 6y^5a_1 + 6x^3yb_2 - 3x^2y^2a_2 + 9x^2y^2b_3 - 2x\,y^3a_3 + 6x^2yb_1 - x^2a_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& 9b_2v_2^{10} + 3a_2v_2^8 - 24b_2v_1v_2^7 - 9b_3v_2^8 - 6b_1v_2^7 + 6a_2v_1v_2^5 + 5a_3v_2^6 - 9b_2v_1^2v_2^4 \\
& - 18b_3v_1v_2^5 + 6a_1v_2^5 - 18b_1v_1v_2^4 - 3a_2v_1^2v_2^2 - 2a_3v_1v_2^3 + 6b_2v_1^3v_2 + 9b_3v_1^2v_2^2 \\
& + 6b_1v_1^2v_2 - a_3v_1^2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &6b_2v_1^3v_2 - 9b_2v_1^2v_2^4 + (-3a_2 + 9b_3)\, v_1^2v_2^2 + 6b_1v_1^2v_2 - a_3v_1^2 \\ &- 24b_2v_1v_2^7 + (6a_2 - 18b_3)\, v_1v_2^5 - 18b_1v_1v_2^4 - 2a_3v_1v_2^3 \\ &+ 9b_2v_2^{10} + (3a_2 - 9b_3)\, v_2^8 - 6b_1v_2^7 + 5a_3v_2^6 + 6a_1v_2^5 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 6a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ 5a_3 &= 0 \\ -18b_1 &= 0 \\ -6b_1 &= 0 \\ 6b_1 &= 0 \\ -24b_2 &= 0 \\ -9b_2 &= 0 \\ 6b_2 &= 0 \\ 9b_2 &= 0 \\ -3a_2 + 9b_3 &= 0 \\ 3a_2 - 9b_3 &= 0 \\ 6a_2 - 18b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 3b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 3x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y^3 + x}{3\,(y^3 - x)\,y^2}\right)(3x) \\
&= \frac{-y^6 - x^2}{-y^5 + y^2 x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^6 - x^2}{-y^5 + y^2 x}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y^6 + x^2)}{6} - \frac{\arctan\left(\frac{y^3}{x}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^3+x}{3\left(y^3-x\right)y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3+x}{3y^6+3x^2} \\ S_y &= -\frac{\left(-y^3+x\right)y^2}{y^6+x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

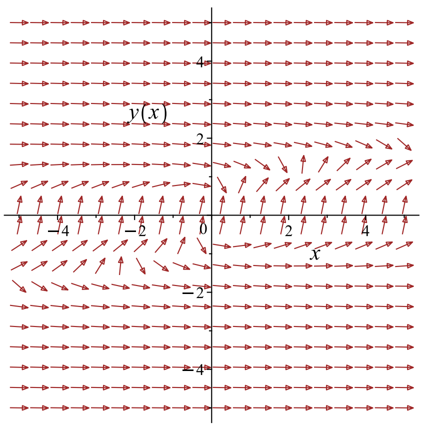
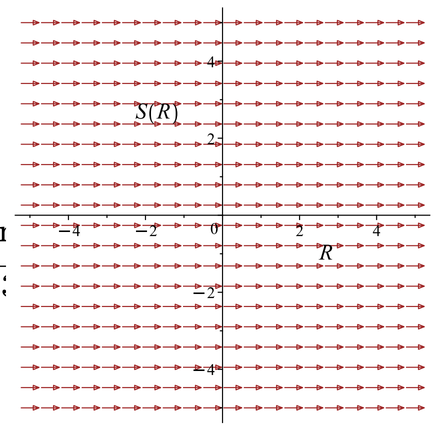
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y^6+x^2\right)}{6} - \frac{\arctan\left(\frac{y^3}{x}\right)}{3} = c_1$$

Which simplifies to

$$\frac{\ln\left(y^6+x^2\right)}{6} - \frac{\arctan\left(\frac{y^3}{x}\right)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^3+x}{3(y^3-x)y^2}$ 	$R = x$ $S = \frac{\ln(y^6+x^2)}{6} - \frac{\text{arctan}}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(y^6+x^2\right)}{6} - \frac{\arctan\left(\frac{y^3}{x}\right)}{3} = c_1 \tag{1}$$

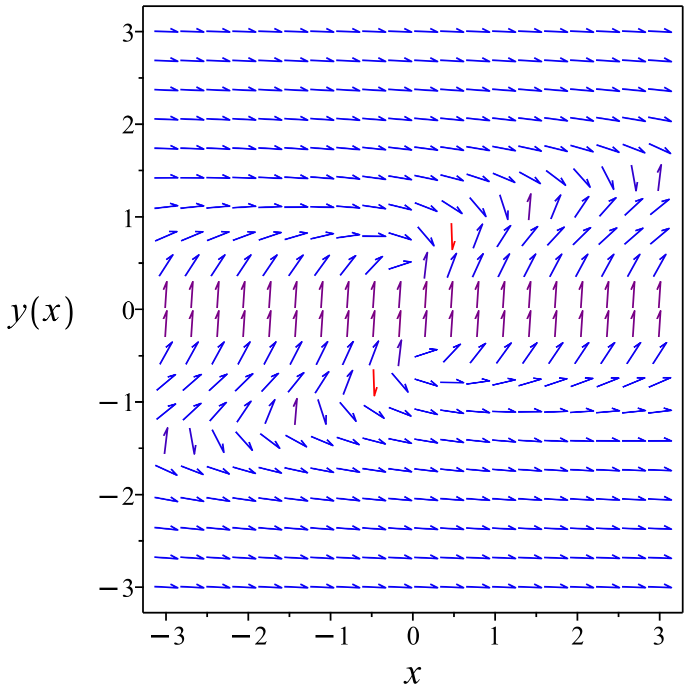


Figure 196: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^6+x^2\right)}{6}-\frac{\arctan\left(\frac{y^3}{x}\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 35

```
dsolve((x+y(x)^3)+3*(y(x)^3-x)*y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$\ln(x) - c_1 + \frac{\ln\left(\frac{y(x)^6+x^2}{x^2}\right)}{2} - \arctan\left(\frac{y(x)^3}{x}\right) = 0$$

✓ Solution by Mathematica

Time used: 0.154 (sec). Leaf size: 27

```
DSolve[(x+y[x]^3)+3*(y[x]^3-x)*y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{x}{y(x)^3}\right) + \frac{1}{2}\log\left(x^2 + y(x)^6\right) = c_1, y(x)\right]$$

6 Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54

6.1 problem 125

Internal problem ID [15027]
Internal file name [`OUTPUT/15027_Sunday_April_21_2024_01_20_56_PM_92513105/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 125.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = e^{-x}}$$

6.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = e^{-x}$$

Hence the ode is

$$y' + 2y = e^{-x}$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{-x}\right)$$
$$\frac{d}{dx}\left(y\, e^{2x}\right) = \left(e^{2x}\right)\left(e^{-x}\right)$$
$$d\left(y\, e^{2x}\right) = e^{x}\, dx$$

Integrating gives

$$y\, e^{2x} = \int e^{x}\, dx$$
$$y\, e^{2x} = e^{x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x} e^{x} + c_1 e^{-2x}$$

which simplifies to

$$y = e^{-x} + c_1 e^{-2x}$$

Summary
The solution(s) found are the following

$$y = e^{-x} + c_1 e^{-2x} \tag{1}$$

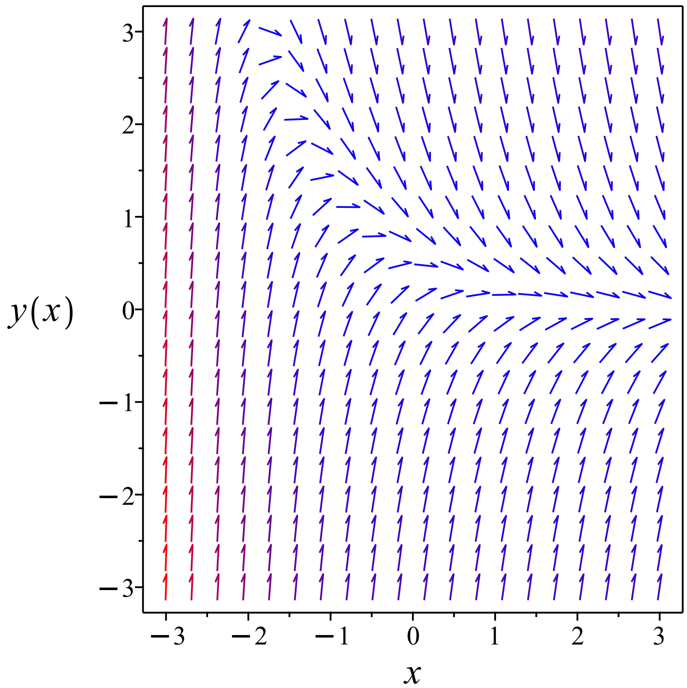


Figure 197: Slope field plot

Verification of solutions

$$y = e^{-x} + c_1 e^{-2x}$$

Verified OK.

6.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2y + e^{-x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 153: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-2x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x}} dy \end{aligned}$$

Which results in

$$S = y\, e^{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + e^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2y\, e^{2x} \\ S_y &= e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\, e^{2x} = e^{x} + c_1$$

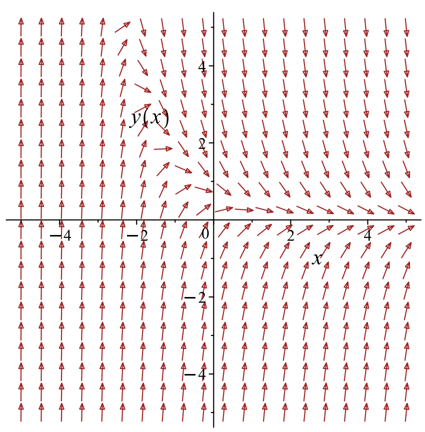
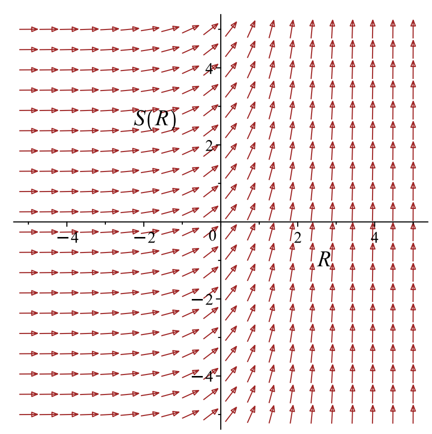
Which simplifies to

$$y\, e^{2x} = e^{x} + c_1$$

Which gives

$$y = (e^{x} + c_1)\, e^{-2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + e^{-x}$ 	$R = x$ $S = y\, e^{2x}$	$\frac{dS}{dR} = e^{R}$ 

<u>Summary</u>
The solution(s) found are the following

$$y = (e^{x} + c_1)\, e^{-2x} \tag{1}$$

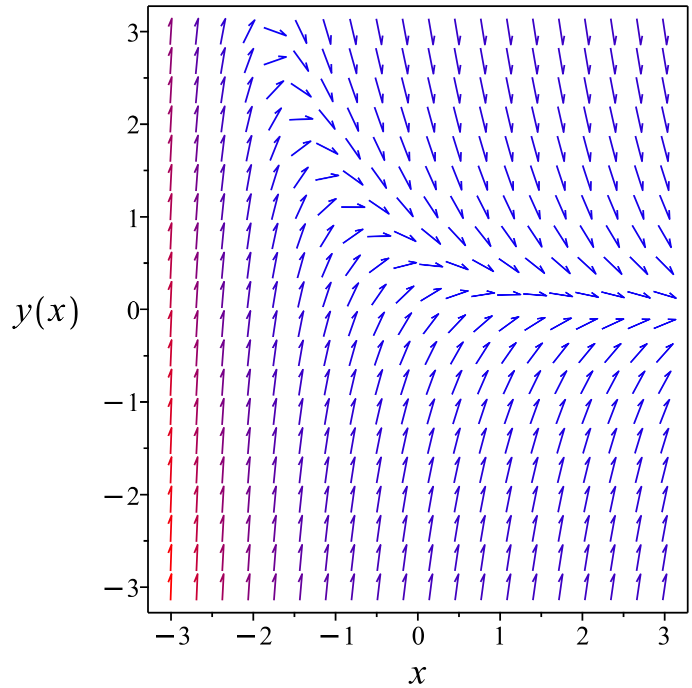


Figure 198: Slope field plot

Verification of solutions

$$y = (e^x + c_1) e^{-2x}$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-2y + \mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(2y - \mathrm{e}^{-x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2y - \mathrm{e}^{-x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - \mathrm{e}^{-x}\right)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2) - (0)) \\ &= 2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2x} \\ &= \mathrm{e}^{2x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{2x}\left(2y - \mathrm{e}^{-x}\right) \\ &= 2y\,\mathrm{e}^{2x} - \mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{2x}(1) \\ &= \mathrm{e}^{2x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(2y\,\mathrm{e}^{2x} - \mathrm{e}^{x}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2y \,\mathrm{e}^{2x} - \mathrm{e}^{x} \mathrm{d}x$$
$$\phi = y \,\mathrm{e}^{2x} - \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y \,\mathrm{e}^{2x} - \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y \,\mathrm{e}^{2x} - \mathrm{e}^{x}$$

The solution becomes

$$y = (\mathrm{e}^{x} + c_1) \,\mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = (\mathrm{e}^{x} + c_1) \,\mathrm{e}^{-2x} \tag{1}$$

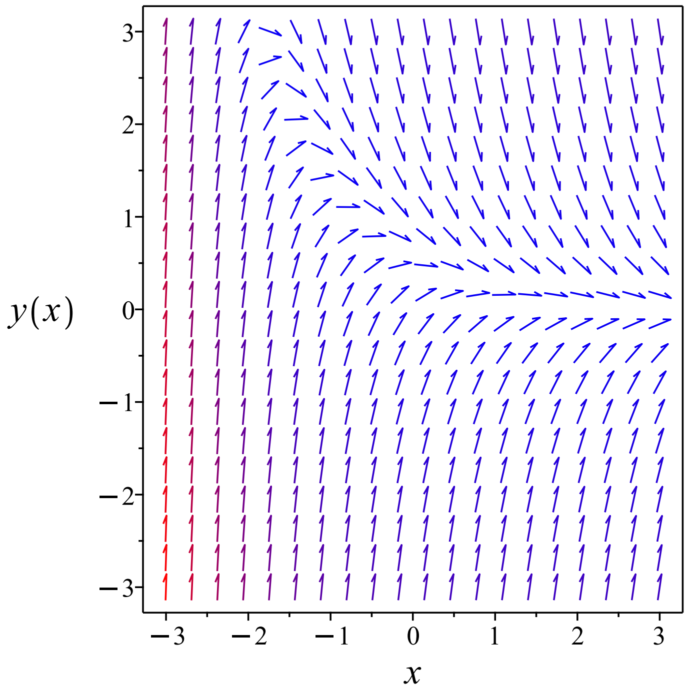


Figure 199: Slope field plot

Verification of solutions

$$y = (e^{x} + c_1)\, e^{-2x}$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$y' + 2y = e^{-x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + e^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = e^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + 2y) = \mu(x)\, e^{-x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+2y)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=2\mu(x)$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{2x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int \mu(x)\,\mathrm{e}^{-x}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int \mu(x)\,\mathrm{e}^{-x}dx+c_1$

- Solve for y

  $y=\frac{\int \mu(x)\mathrm{e}^{-x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{2x}$

  $y=\frac{\int \mathrm{e}^{-x}\mathrm{e}^{2x}dx+c_1}{\mathrm{e}^{2x}}$

- Evaluate the integrals on the rhs

  $y=\frac{\mathrm{e}^{x}+c_1}{\mathrm{e}^{2x}}$

- Simplify

  $y=(\mathrm{e}^{x}+c_1)\,\mathrm{e}^{-2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+2*y(x)=exp(-x),y(x), singsol=all)
```

$$y(x) = (e^x + c_1)\, e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 17

```
DSolve[y'[x]+2*y[x]==Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(e^x + c_1)$$

6.2 problem 126

Internal problem ID [15028]
Internal file name [`OUTPUT/15028_Sunday_April_21_2024_01_20_56_PM_64219149/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 126.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y'x - y = -x^2}$$

With initial conditions

$$[y(1) = 0]$$

6.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = x$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

6.2.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(xy) &= (x)(x) \\ \mathrm{d}(xy) &= x^2\, \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} xy &= \int x^2\, \mathrm{d}x \\ xy &= \frac{x^3}{3} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{x^2}{3} + \frac{c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + \frac{1}{3}$$

$$c_1 = -\frac{1}{3}$$

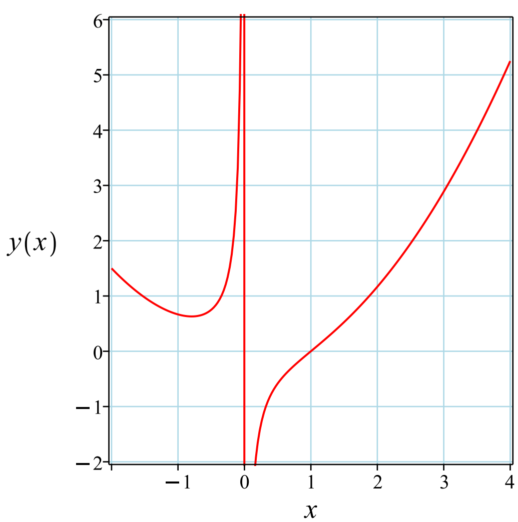
Substituting c_1 found above in the general solution gives

$$y = \frac{x^3 - 1}{3x}$$

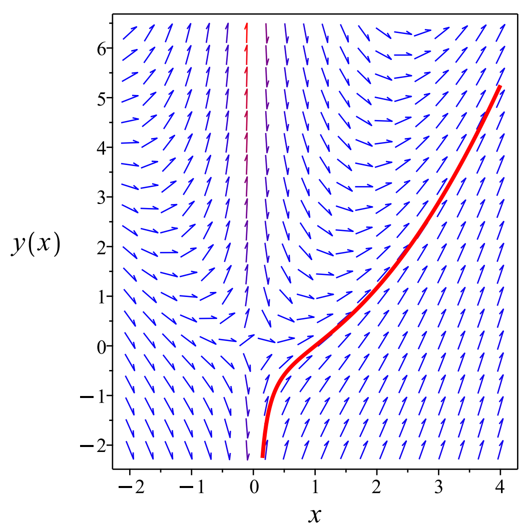
Summary

The solution(s) found are the following

$$y = \frac{x^3 - 1}{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^3 - 1}{3x}$$

Verified OK.

6.2.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{-x^2 + y}{x} \tag{1}$$

Which becomes

$$0 = (-x)\, dy + \left(x^2 - y\right) dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + \left(x^2 - y\right) dx = d\left(\frac{1}{3}x^3 - xy\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{3}x^3 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^3 + 3c_1}{3x} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 2c_1 + \frac{1}{3}$$

$$c_1 = -\frac{1}{6}$$

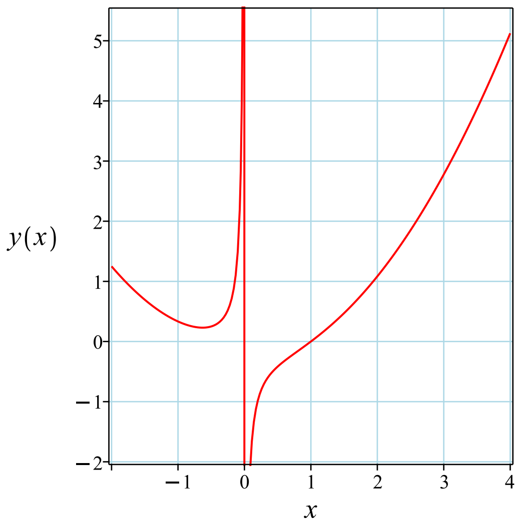
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^3 - x - 1}{6x}$$

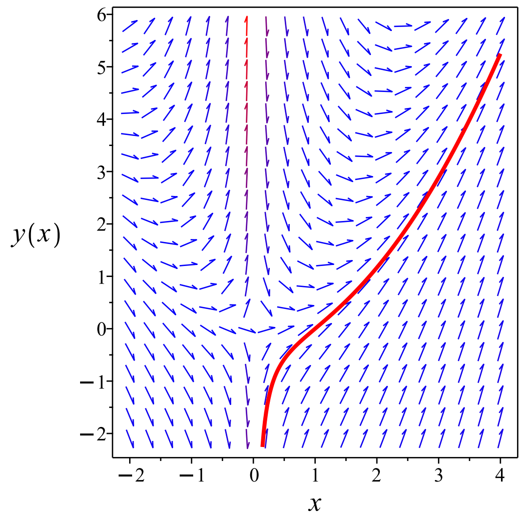
Summary

The solution(s) found are the following

$$y = \frac{2x^3 - x - 1}{6x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2x^3 - x - 1}{6x}$$

Warning, solution could not be verified

6.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^2 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 156: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^2 + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = \frac{x^3}{3} + c_1$$

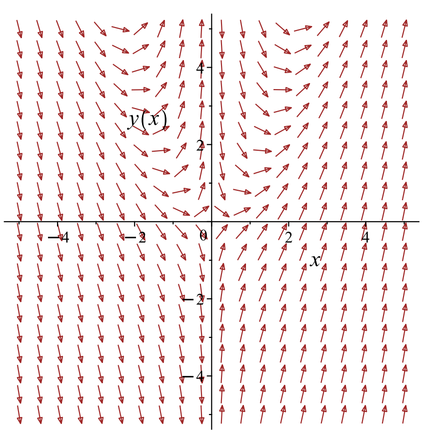
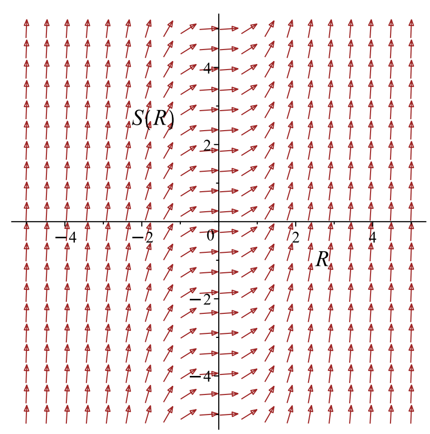
Which simplifies to

$$yx = \frac{x^3}{3} + c_1$$

Which gives

$$y = \frac{x^3 + 3c_1}{3x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = R^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + \frac{1}{3}$$

$$c_1 = -\frac{1}{3}$$

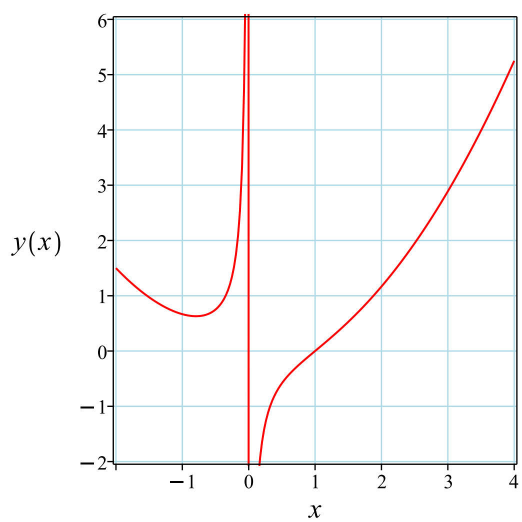
Substituting c_1 found above in the general solution gives

$$y = \frac{x^3 - 1}{3x}$$

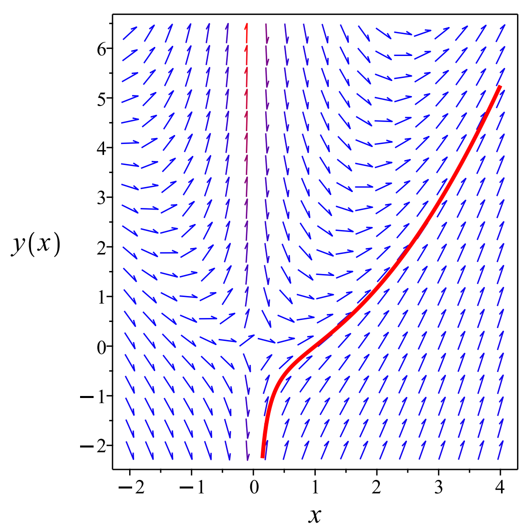
Summary

The solution(s) found are the following

$$y = \frac{x^3 - 1}{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^3 - 1}{3x}$$

Verified OK.

6.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= \left(-x^2 + y\right)\mathrm{d}x\\ \left(x^2 - y\right)\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 - y\\ N(x,y) &= -x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 - y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x^2 - y\,\mathrm{d}x\\ \phi &= \frac{1}{3}x^3 - xy + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x$. Therefore equation (4) becomes

$$-x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 - xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 - xy$$

The solution becomes

$$y = -\frac{-x^3 + 3c_1}{3x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{3} - c_1$$

$$c_1 = \frac{1}{3}$$

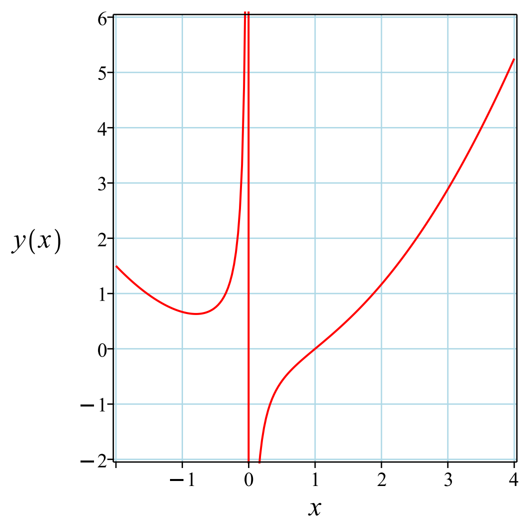
Substituting c_1 found above in the general solution gives

$$y = \frac{x^3 - 1}{3x}$$

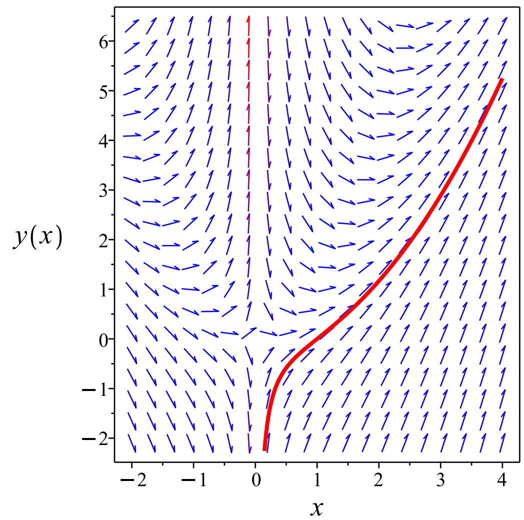
Summary

The solution(s) found are the following

$$y = \frac{x^3 - 1}{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^3 - 1}{3x}$$

Verified OK.

6.2.6 Maple step by step solution

Let's solve

$[-y'x - y = -x^2, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) x dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x$

  $y = \frac{\int x^2 dx + c_1}{x}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^3}{3} + c_1}{x}$
- Simplify

  $y = \frac{x^3 + 3c_1}{3x}$
- Use initial condition $y(1) = 0$

  $0 = c_1 + \frac{1}{3}$
- Solve for c_1

  $c_1 = -\frac{1}{3}$
- Substitute $c_1 = -\frac{1}{3}$ into general solution and simplify

  $y = \frac{x^3 - 1}{3x}$
- Solution to the IVP

  $y = \frac{x^3 - 1}{3x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([x^2-x*diff(y(x),x)=y(x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = \frac{x^3 - 1}{3x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 17

```
DSolve[{x^2-x*y'[x]==y[x],{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3 - 1}{3x}$$

6.3 problem 127

Internal problem ID [15029]
Internal file name [`OUTPUT/15029_Sunday_April_21_2024_01_20_57_PM_57167490/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 127.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - 2yx = 2x\,e^{x^2}}$$

6.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 2x\,e^{x^2} \end{aligned}$$

Hence the ode is

$$y' - 2yx = 2x\,e^{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -2x dx} \\ &= e^{-x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(2x\, e^{x^2}\right) \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= \left(e^{-x^2}\right)\left(2x\, e^{x^2}\right) \\ d\left(e^{-x^2} y\right) &= (2x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x^2} y &= \int 2x\, dx \\ e^{-x^2} y &= x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = x^2 e^{x^2} + c_1 e^{x^2}$$

which simplifies to

$$y = e^{x^2}\left(x^2 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{x^2}\left(x^2 + c_1\right) \tag{1}$$

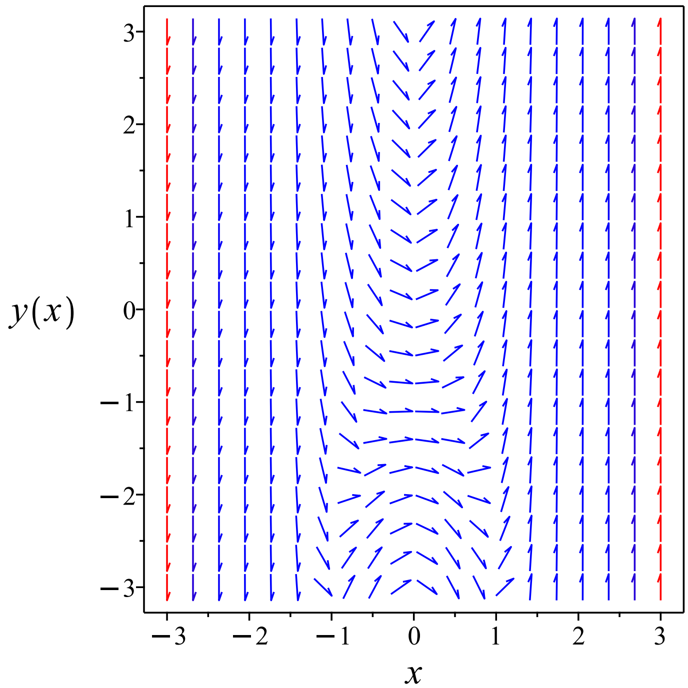


Figure 204: Slope field plot

Verification of solutions

$$y = e^{x^2}(x^2 + c_1)$$

Verified OK.

6.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2xy + 2x\,e^{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 159: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^2}} dy \end{aligned}$$

Which results in

$$S = e^{-x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2xy + 2x e^{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2x e^{-x^2} y \\ S_y &= e^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^2} y = x^2 + c_1$$

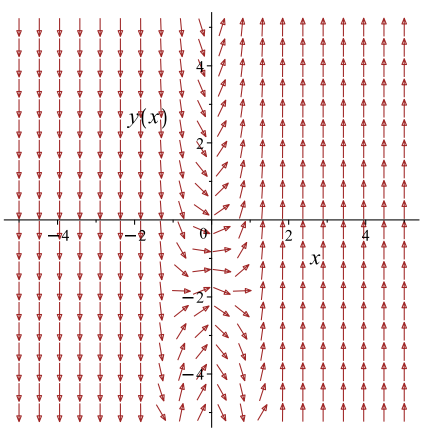
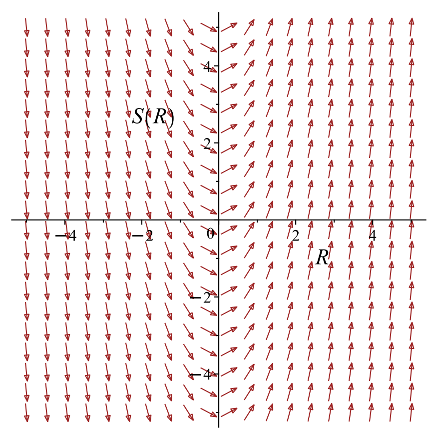
Which simplifies to

$$e^{-x^2} y = x^2 + c_1$$

Which gives

$$y = e^{x^2} \left(x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2xy + 2x\,e^{x^2}$ 	$R = x$ $S = e^{-x^2} y$	$\frac{dS}{dR} = 2R$ 

Summary

The solution(s) found are the following

$$y = e^{x^2} \left(x^2 + c_1\right) \tag{1}$$

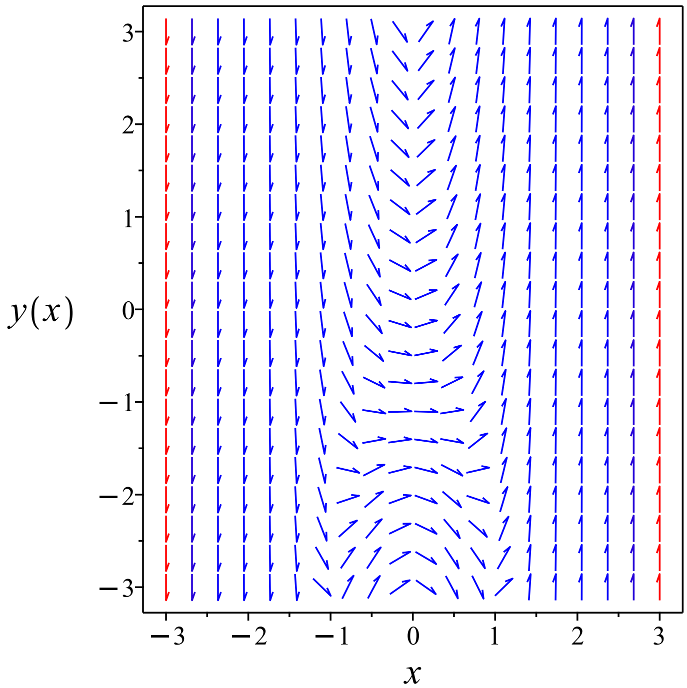


Figure 205: Slope field plot

Verification of solutions

$$y = e^{x^2}(x^2 + c_1)$$

Verified OK.

6.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(2xy + 2x\,\mathrm{e}^{x^2}\right)\mathrm{d}x$$
$$\left(-2xy - 2x\,\mathrm{e}^{x^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2xy - 2x\,\mathrm{e}^{x^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2xy - 2x\,\mathrm{e}^{x^2}\right)\\ &= -2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-2x) - (0)) \\ &= -2x \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -2x \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x^2} \\ &= \mathrm{e}^{-x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x^2}\left(-2xy - 2x\,\mathrm{e}^{x^2}\right) \\ &= -2x\left(1 + \mathrm{e}^{-x^2} y\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x^2}(1) \\ &= \mathrm{e}^{-x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2x\left(1 + \mathrm{e}^{-x^2} y\right)\right) + \left(\mathrm{e}^{-x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x\left(1 + \mathrm{e}^{-x^2} y\right) \mathrm{d}x$$

$$\phi = -x^2 + \mathrm{e}^{-x^2} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x^2} = \mathrm{e}^{-x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + \mathrm{e}^{-x^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + \mathrm{e}^{-x^2} y$$

The solution becomes

$$y = e^{x^2}(x^2 + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{x^2}(x^2 + c_1) \tag{1}$$

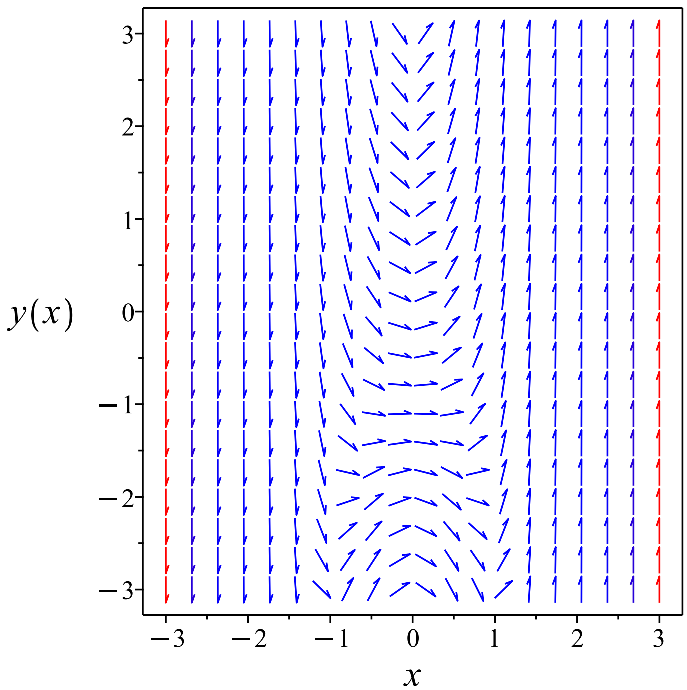


Figure 206: Slope field plot

Verification of solutions

$$y = e^{x^2}(x^2 + c_1)$$

Verified OK.

6.3.4 Maple step by step solution

Let's solve

$y' - 2yx = 2x\,e^{x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2yx + 2x\,e^{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2yx = 2x\,e^{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - 2yx) = 2\mu(x)\,x\,e^{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - 2yx) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = e^{-x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x\,e^{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,x\,e^{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)x\,e^{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-x^2}$

  $y = \frac{\int 2x\,e^{x^2} e^{-x^2} dx + c_1}{e^{-x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{e^{-x^2}}$

- Simplify

$$y = e^{x^2}(x^2 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)-2*x*y(x)=2*x*exp(x^2),y(x), singsol=all)
```

$$y(x) = (x^2 + c_1)\, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 17

```
DSolve[y'[x]-2*x*y[x]==2*x*Exp[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(x^2 + c_1)$$

6.4 problem 128

Internal problem ID [15030]
Internal file name [`OUTPUT/15030_Sunday_April_21_2024_01_20_57_PM_9281443/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 128.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2yx = e^{-x^2}}$$

6.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = e^{-x^2}$$

Hence the ode is

$$y' + 2yx = e^{-x^2}$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{-x^2}\right)$$
$$\frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)\left(e^{-x^2}\right)$$
$$d\left(e^{x^2} y\right) = dx$$

Integrating gives

$$e^{x^2} y = \int dx$$
$$e^{x^2} y = x + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = x\,e^{-x^2} + c_1 e^{-x^2}$$

which simplifies to

$$y = e^{-x^2}(x + c_1)$$

Summary
The solution(s) found are the following

$$y = e^{-x^2}(x + c_1) \tag{1}$$

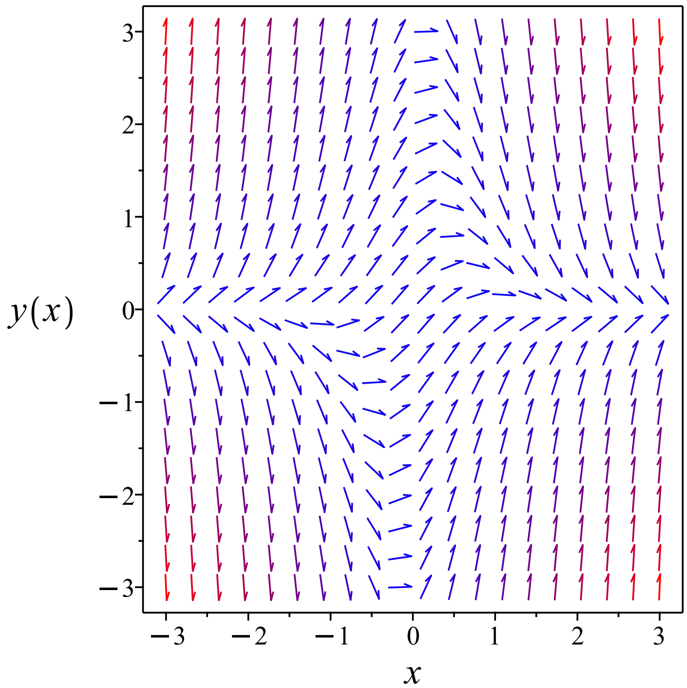


Figure 207: Slope field plot

Verification of solutions

$$y = e^{-x^2}(x + c_1)$$

Verified OK.

6.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2xy + e^{-x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 162: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2xy + e^{-x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\, e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x^2} = x + c_1$$

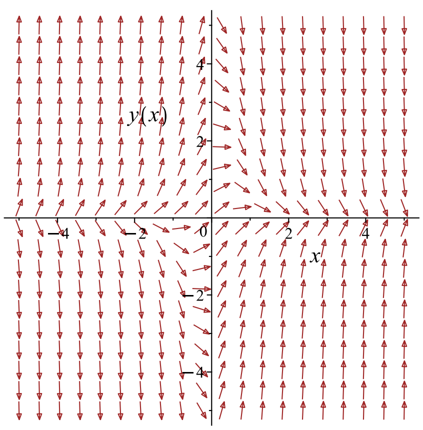
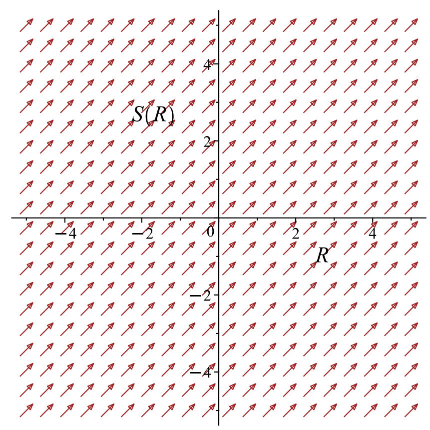
Which simplifies to

$$y\,\mathrm{e}^{x^2} = x + c_1$$

Which gives

$$y = \mathrm{e}^{-x^2}(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2xy + \mathrm{e}^{-x^2}$ 	$R = x$ $S = \mathrm{e}^{x^2} y$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-x^2}(x + c_1) \tag{1}$$

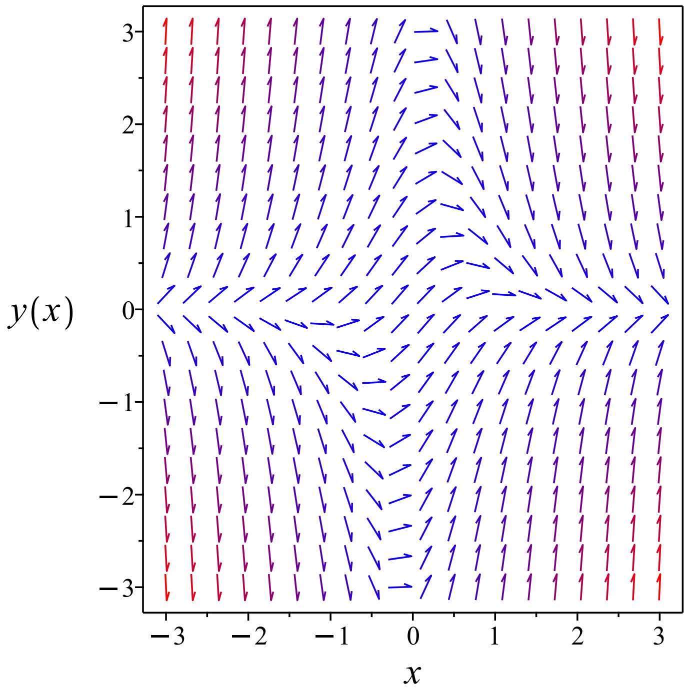


Figure 208: Slope field plot

Verification of solutions

$$y = e^{-x^2}(x + c_1)$$

Verified OK.

6.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-2xy + \mathrm{e}^{-x^2}\right) \mathrm{d}x$$
$$\left(2xy - \mathrm{e}^{-x^2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2xy - \mathrm{e}^{-x^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy - \mathrm{e}^{-x^2}\right)\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((2x) - (0)) \\
&= 2x
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 2x \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x^2} \\
&= \mathrm{e}^{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x^2}\left(2xy - \mathrm{e}^{-x^2}\right) \\
&= 2x\,\mathrm{e}^{x^2} y - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x^2}(1) \\
&= \mathrm{e}^{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(2x\,\mathrm{e}^{x^2} y - 1\right) + \left(\mathrm{e}^{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int 2x\,\mathrm{e}^{x^2}y - 1\,\mathrm{d}x$$

$$\phi = -x + \mathrm{e}^{x^2}y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{x^2} = \mathrm{e}^{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \mathrm{e}^{x^2}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \mathrm{e}^{x^2}y$$

The solution becomes

$$y = e^{-x^2}(x + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}(x + c_1) \tag{1}$$

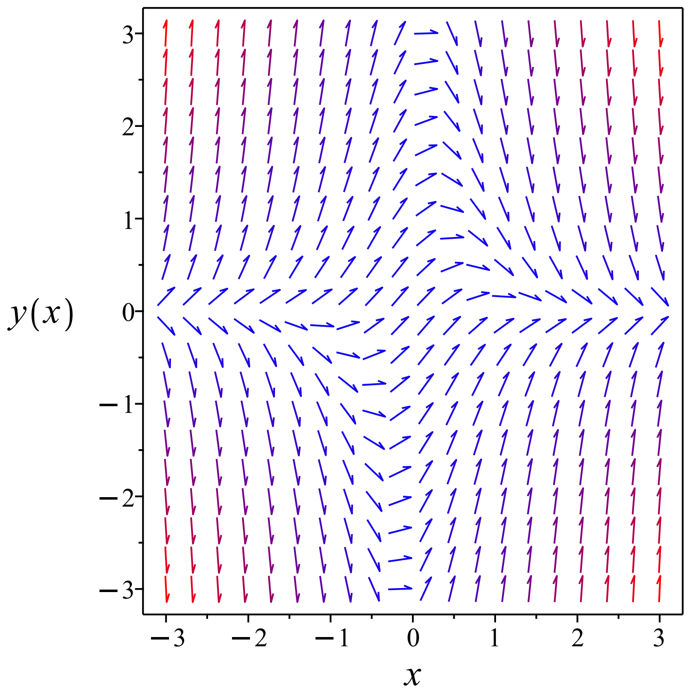


Figure 209: Slope field plot

Verification of solutions

$$y = e^{-x^2}(x + c_1)$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$y' + 2yx = \mathrm{e}^{-x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2yx + \mathrm{e}^{-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2yx = \mathrm{e}^{-x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + 2yx) = \mu(x)\,\mathrm{e}^{-x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + 2yx) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{-x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{-x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{-x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x^2}$

  $y = \frac{\int \mathrm{e}^{-x^2}\mathrm{e}^{x^2} dx + c_1}{\mathrm{e}^{x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{\mathrm{e}^{x^2}}$

- Simplify

$$y = e^{-x^2}(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+2*x*y(x)=exp(-x^2),y(x), singsol=all)
```

$$y(x) = (x + c_1)\,e^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 17

```
DSolve[y'[x]+2*x*y[x]==Exp[-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(x + c_1)$$

6.5 problem 129

Internal problem ID [15031]
Internal file name [`OUTPUT/15031_Sunday_April_21_2024_01_20_58_PM_34699866/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 129.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x)\, y' - \sin(x)\, y = 2x}$$

With initial conditions

$$[y(0) = 0]$$

6.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\tan(x)$$
$$q(x) = 2x\sec(x)$$

Hence the ode is

$$y' - \tan(x)\,y = 2x\sec(x)$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z815 \lor \frac{1}{2}\pi + \pi\_Z815 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2x\sec(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z815 \lor \frac{1}{2}\pi + \pi\_Z815 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.5.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\tan(x)dx}\\ &= \cos(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(2x\sec(x))\\ \frac{d}{dx}(\cos(x)\,y) &= (\cos(x))(2x\sec(x))\\ d(\cos(x)\,y) &= (2x)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\cos(x)\,y &= \int 2x\,dx\\ \cos(x)\,y &= x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \sec(x)\,x^2 + c_1\sec(x)$$

which simplifies to

$$y = \sec(x)\left(x^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

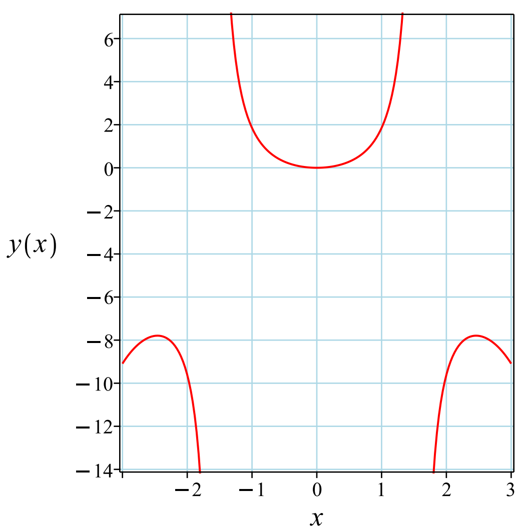
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\, x^2$$

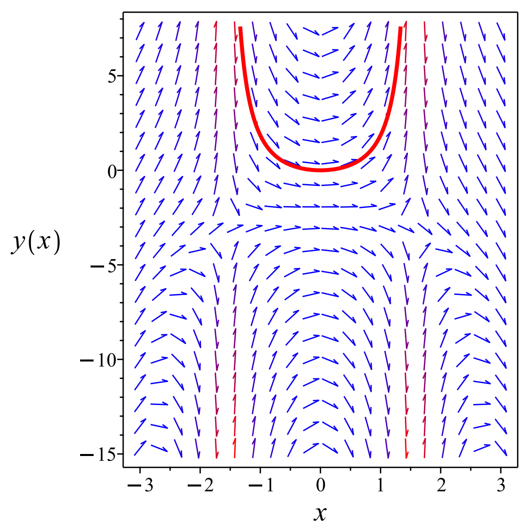
Summary

The solution(s) found are the following

$$y = \sec(x)\, x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x)\, x^2$$

Verified OK.

6.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(x)\,y + 2x}{\cos(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 165: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{\cos(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\cos(x)}} dy
\end{aligned}
$$

Which results in

$$
S = \cos(x)\, y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sin(x)\, y + 2x}{\cos(x)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\sin(x)\,y \\
S_y &= \cos(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 2x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 2R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R^2 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y\cos(x) = x^2 + c_1
$$

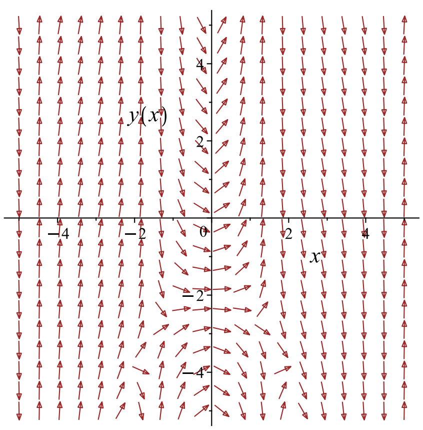
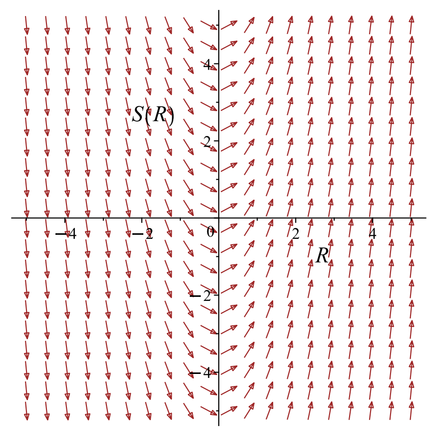
Which simplifies to

$$
y\cos(x) = x^2 + c_1
$$

Which gives

$$
y = \frac{x^2 + c_1}{\cos(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(x)y+2x}{\cos(x)}$ 	$R = x$ $S = \cos(x)\, y$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

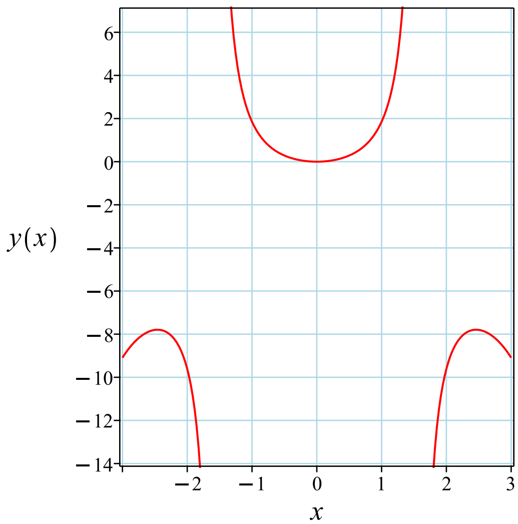
Substituting c_1 found above in the general solution gives

$$y = \frac{x^2}{\cos(x)}$$

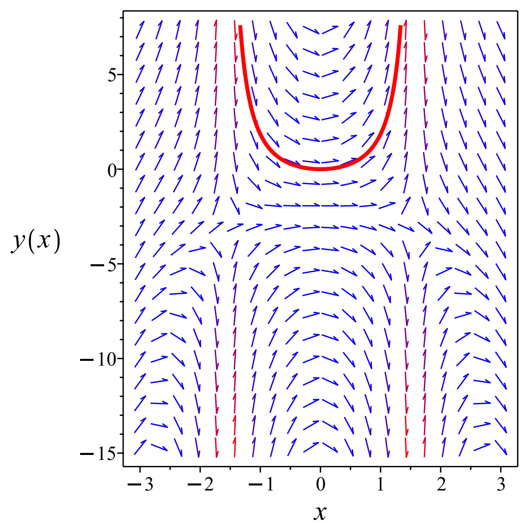
Summary

The solution(s) found are the following

$$y = \frac{x^2}{\cos(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2}{\cos(x)}$$

Verified OK.

6.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (\cos(x)) \, \mathrm{d}y &= (\sin(x) \, y + 2x) \, \mathrm{d}x \\ (-\sin(x) \, y - 2x) \, \mathrm{d}x + (\cos(x)) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\sin(x) \, y - 2x \\ N(x, y) &= \cos(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x) \, y - 2x) \\ &= -\sin(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x)) \\ &= -\sin(x) \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\sin(x) y - 2x \, \mathrm{d}x$$
$$\phi = -x^2 + \cos(x) y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + \cos(x) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + \cos(x) y$$

The solution becomes

$$y = \frac{x^2 + c_1}{\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

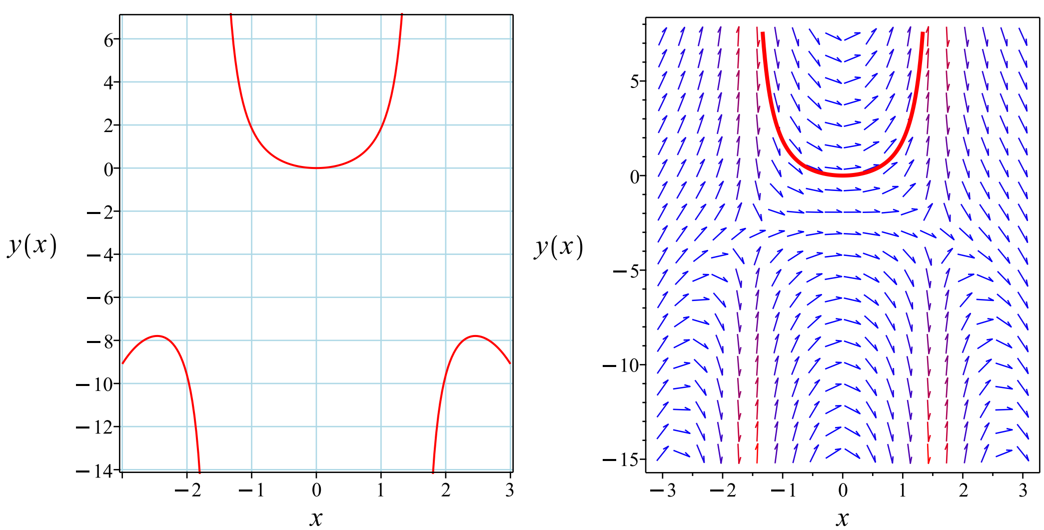
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^2}{\cos(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{\cos(x)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{x^2}{\cos(x)}$$

Verified OK.

6.5.5 Maple step by step solution

Let's solve

$[\cos(x) y' - \sin(x) y = 2x, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{\sin(x)y}{\cos(x)} + \frac{2x}{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{\sin(x)y}{\cos(x)} = \frac{2x}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{\sin(x)y}{\cos(x)}\right) = \frac{2\mu(x)x}{\cos(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) \left(y' - \frac{\sin(x)y}{\cos(x)}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)\sin(x)}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x) = \cos(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \frac{2\mu(x)x}{\cos(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \frac{2\mu(x)x}{\cos(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)x}{\cos(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cos(x)$

  $y = \frac{\int 2x dx + c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{\cos(x)}$

- Simplify

  $y = \sec(x)(x^2 + c_1)$
- Use initial condition $y(0) = 0$

  $0 = c_1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \sec(x)\, x^2$
- Solution to the IVP

  $y = \sec(x)\, x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)*cos(x)-y(x)*sin(x)=2*x,y(0) = 0],y(x), singsol=all)
```

$$y(x) = x^2 \sec(x)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 11

```
DSolve[{y'[x]*Cos[x]-y[x]*Sin[x]==2*x,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 \sec(x)$$

6.6 problem 130

Internal problem ID [15032]
Internal file name [`OUTPUT/15032_Sunday_April_21_2024_01_20_59_PM_79478299/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 130.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - 2y = \cos(x)\, x^3}$$

6.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \cos(x)\, x^2$$

Hence the ode is

$$y' - \frac{2y}{x} = \cos(x)\, x^2$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\cos(x)\, x^2\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\cos(x)\, x^2\right)$$
$$d\left(\frac{y}{x^2}\right) = \cos(x)\, dx$$

Integrating gives

$$\frac{y}{x^2} = \int \cos(x)\, dx$$
$$\frac{y}{x^2} = \sin(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = \sin(x)\, x^2 + c_1 x^2$$

which simplifies to

$$y = x^2(\sin(x) + c_1)$$

Summary
The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

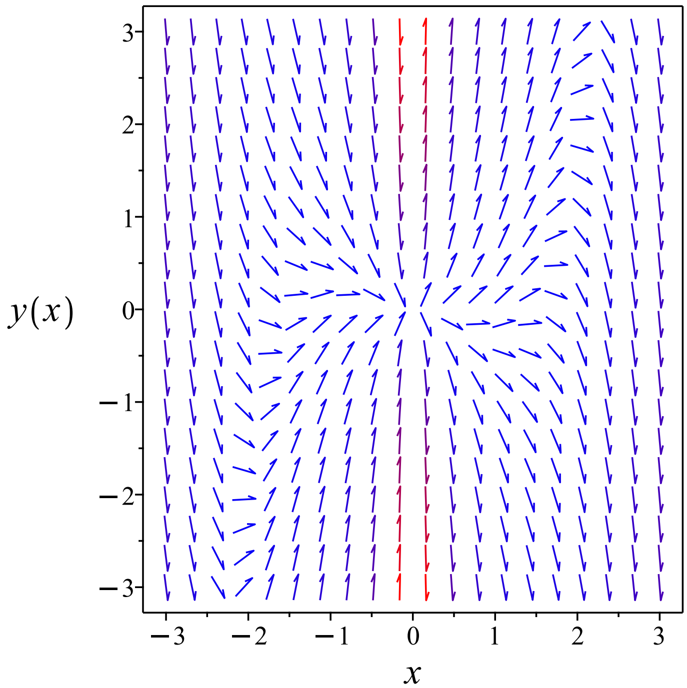


Figure 213: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

6.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2y + \cos(x)\, x^3}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 168: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y + \cos(x)\, x^3}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = \sin(x) + c_1$$

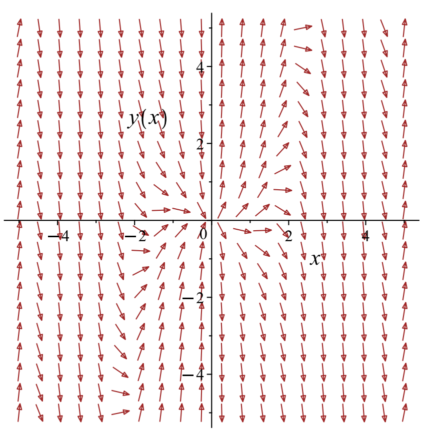
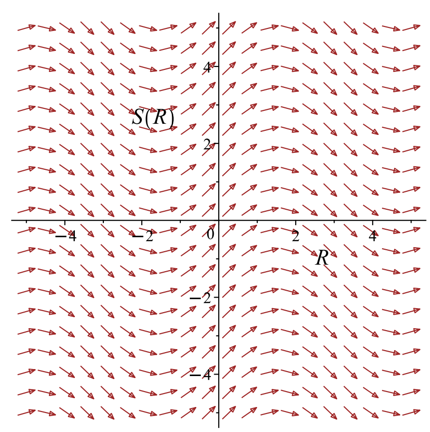
Which simplifies to

$$\frac{y}{x^2} = \sin(x) + c_1$$

Which gives

$$y = x^2(\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y+\cos(x)x^3}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = \cos(R)$ 

Summary

The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

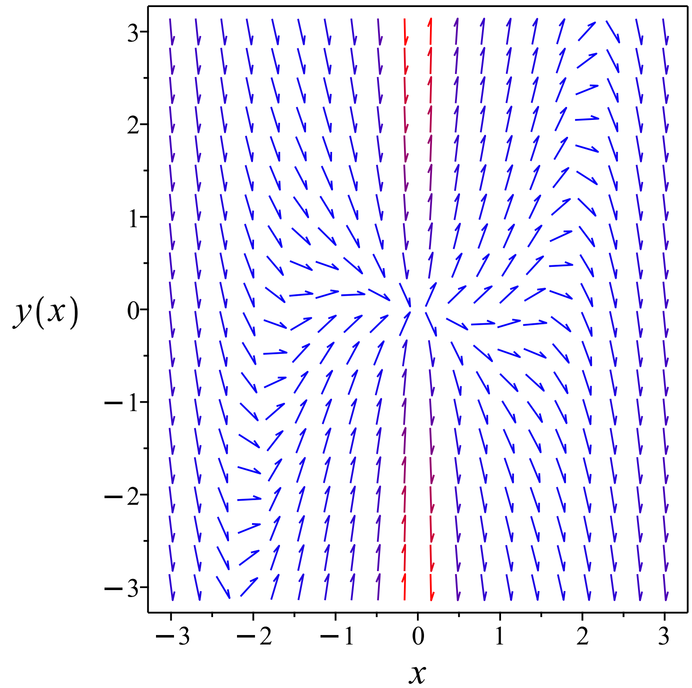


Figure 214: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

6.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(2y + \cos(x)\,x^3\right)\mathrm{d}x$$
$$\left(-2y - \cos(x)\,x^3\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2y - \cos(x)\,x^3$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-2y - \cos(x)\,x^3\right)$$
$$= -2$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x)$$
$$= 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-2) - (1)) \\
&= -\frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}\left(-2y - \cos(x)\, x^3\right) \\
&= \frac{-2y - \cos(x)\, x^3}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(x) \\
&= \frac{1}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-2y - \cos(x)\, x^3}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-2y - \cos(x)\, x^3}{x^3} \mathrm{d}x$$

$$\phi = -\sin(x) + \frac{y}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) + \frac{y}{x^2}$$

The solution becomes

$$y = x^2(\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

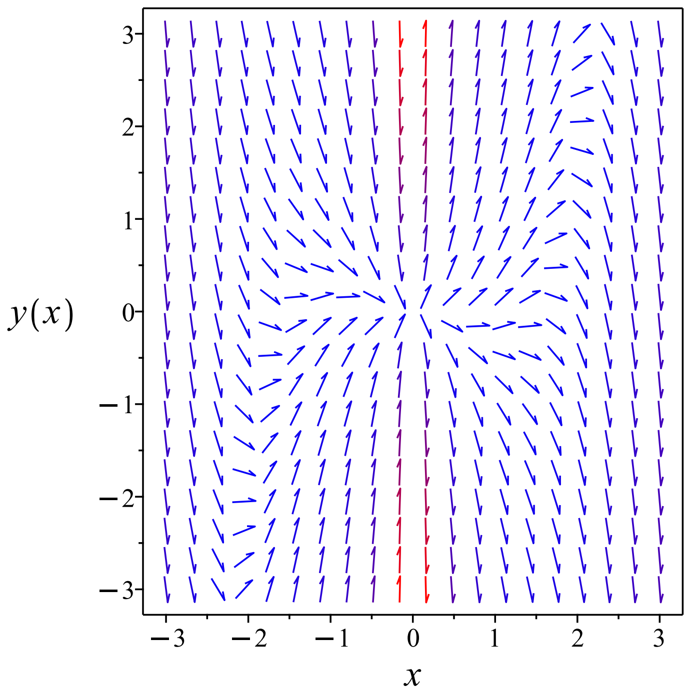


Figure 215: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

6.6.4 Maple step by step solution

Let's solve

$y'x - 2y = \cos(x)\, x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + \cos(x)\, x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = \cos(x)\, x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{2y}{x}\right) = \mu(x) \cos(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \cos(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \cos(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) \cos(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2 \left(\int \cos(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2 (\sin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)-2*y(x)=x^3*cos(x),y(x), singsol=all)
```

$$y(x) = (\sin(x) + c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 14

```
DSolve[x*y'[x]-2*y[x]==x^3*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(\sin(x) + c_1)$$

6.7 problem 131

Internal problem ID [15033]
Internal file name [`OUTPUT/15033_Sunday_April_21_2024_01_21_00_PM_81956911/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 131.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \tan(x)\, y = \frac{1}{\cos(x)^3}}$$

With initial conditions

$$[y(0) = 0]$$

6.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\tan(x)\\ q(x) &= \sec(x)^3\end{aligned}$$

Hence the ode is

$$y' - \tan(x)\, y = \sec(x)^3$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z815 \vee \frac{1}{2}\pi + \pi\_Z815 < x\right\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \sec(x)^3$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z815 \vee \frac{1}{2}\pi + \pi\_Z815 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.7.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\tan(x)dx}\\ &= \cos(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sec(x)^3\right)\\ \frac{d}{dx}(\cos(x)\, y) &= (\cos(x))\left(\sec(x)^3\right)\\ d(\cos(x)\, y) &= \sec(x)^2\, dx\end{aligned}$$

Integrating gives

$$\cos(x) y = \int \sec(x)^2 \, dx$$

$$\cos(x) y = \tan(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \sec(x)\tan(x) + c_1 \sec(x)$$

which simplifies to

$$y = \sec(x)(\tan(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

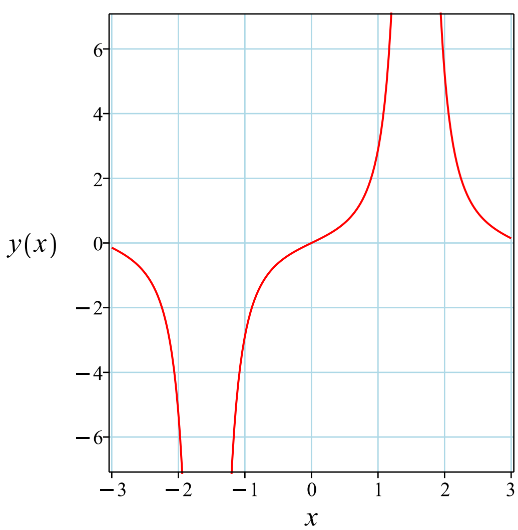
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\tan(x)$$

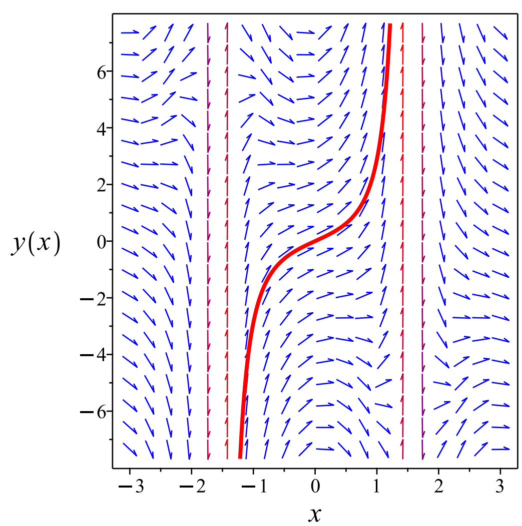
Summary

The solution(s) found are the following

$$y = \sec(x)\tan(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x)\tan(x)$$

Verified OK.

6.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\tan(x)\, y \cos(x)^3 + 1}{\cos(x)^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 171: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = \cos(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\tan(x)\, y \cos(x)^3 + 1}{\cos(x)^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\sin(x)\, y \\ S_y &= \cos(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\cos(x) = \tan(x) + c_1$$

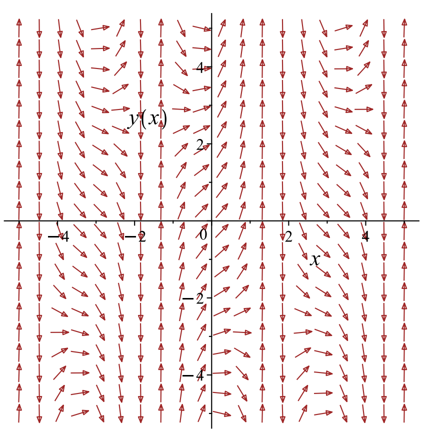
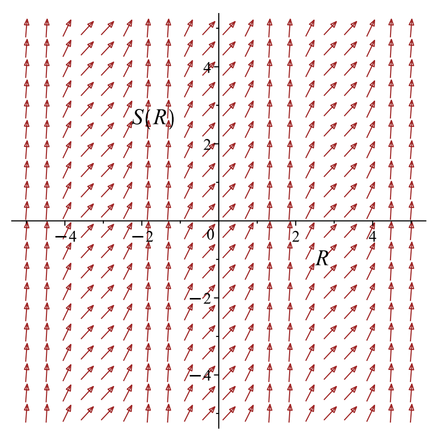
Which simplifies to

$$y\cos(x) = \tan(x) + c_1$$

Which gives

$$y = \frac{\tan(x) + c_1}{\cos(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\tan(x)y\cos(x)^3+1}{\cos(x)^3}$ 	$R = x$ $S = \cos(x)\, y$	$\frac{dS}{dR} = \sec(R)^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

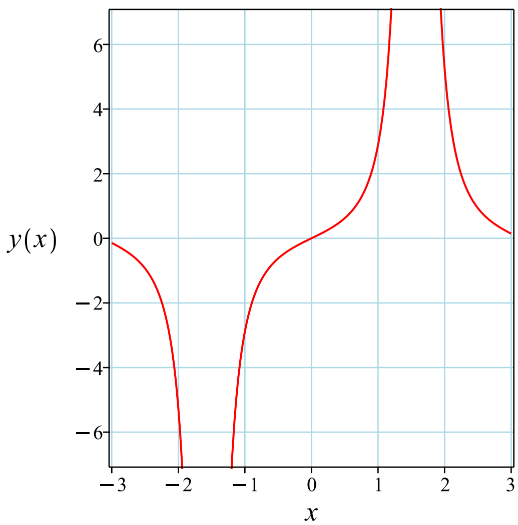
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\tan(x)$$

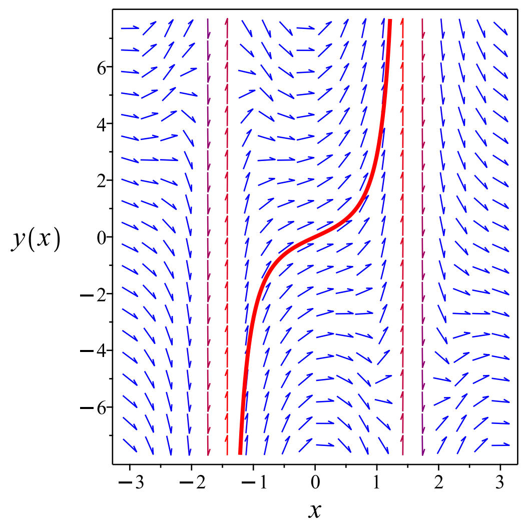
Summary

The solution(s) found are the following

$$y = \sec(x)\tan(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x)\tan(x)$$

Verified OK.

6.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\mathrm{d} y=\left(\tan (x) y+\frac{1}{\cos (x)^3}\right) \mathrm{d} x$$

$$\left(-\tan (x) y-\frac{1}{\cos (x)^3}\right) \mathrm{d} x+\mathrm{d} y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\tan (x) y-\frac{1}{\cos (x)^3} \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\tan (x) y-\frac{1}{\cos (x)^3}\right) \\ &= -\tan (x) \end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-\tan(x)) - (0)) \\
&= -\tan(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\tan(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cos(x))} \\
&= \cos(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(x)\left(-\tan(x)\, y - \frac{1}{\cos(x)^3}\right) \\
&= -\sin(x)\, y - \sec(x)^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(x)(1) \\
&= \cos(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\sin(x)\, y - \sec(x)^2\right) + (\cos(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\sin(x)\, y - \sec(x)^2 \, \mathrm{d}x$$

$$\phi = -\tan(x) + \cos(x)\, y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\tan(x) + \cos(x)\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\tan(x) + \cos(x)\, y$$

The solution becomes

$$y = \frac{\tan(x) + c_1}{\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

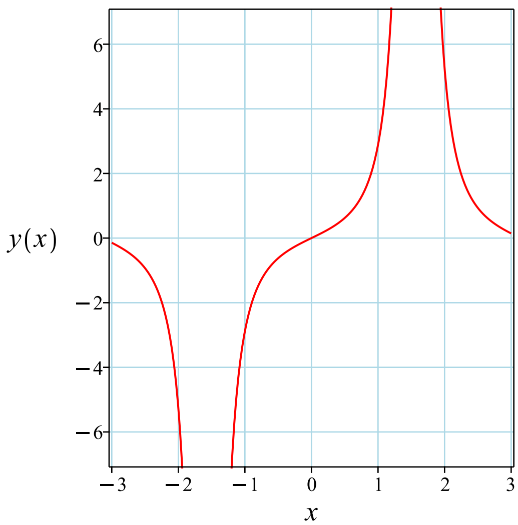
Substituting c_1 found above in the general solution gives

$$y = \sec(x) \tan(x)$$

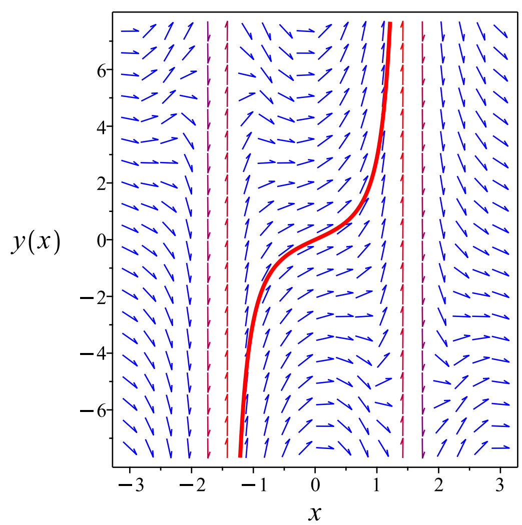
Summary

The solution(s) found are the following

$$y = \sec(x) \tan(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x) \tan(x)$$

Verified OK.

6.7.5 Maple step by step solution

Let's solve

$\left[y' - \tan(x)\, y = \frac{1}{\cos(x)^3}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \tan(x)\, y + \frac{1}{\cos(x)^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \tan(x)\, y = \frac{1}{\cos(x)^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \tan(x)\, y\right) = \frac{\mu(x)}{\cos(x)^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \tan(x)\, y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\tan(x)$

- Solve to find the integrating factor

  $\mu(x) = \cos(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{\cos(x)^3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{\cos(x)^3} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{\cos(x)^3} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cos(x)$

  $y = \frac{\int \frac{1}{\cos(x)^2} dx + c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{\tan(x) + c_1}{\cos(x)}$

- Simplify

  $y = \sec(x)(\tan(x) + c_1)$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \sec(x)\tan(x)$

- Solution to the IVP

  $y = \sec(x)\tan(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)-y(x)*tan(x)=1/cos(x)^3,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \sec(x)\tan(x)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 10

```
DSolve[{y'[x]-y[x]*Tan[x]==1/Cos[x]^3,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(x)\sec(x)$$

6.8 problem 132

Internal problem ID [15034]
Internal file name [`OUTPUT/15034_Sunday_April_21_2024_01_21_08_PM_94159032/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 132.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x \ln(x) y' - y = 3x^3 \ln(x)^2}$$

6.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x \ln(x)}$$
$$q(x) = 3x^2 \ln(x)$$

Hence the ode is

$$y' - \frac{y}{x \ln(x)} = 3x^2 \ln(x)$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x \ln(x)}dx}$$
$$= \frac{1}{\ln(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(3x^2 \ln(x)\right)$$
$$\frac{d}{dx}\left(\frac{y}{\ln(x)}\right) = \left(\frac{1}{\ln(x)}\right)\left(3x^2 \ln(x)\right)$$
$$d\left(\frac{y}{\ln(x)}\right) = \left(3x^2\right) dx$$

Integrating gives

$$\frac{y}{\ln(x)} = \int 3x^2 \, dx$$
$$\frac{y}{\ln(x)} = x^3 + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\ln(x)}$ results in

$$y = x^3 \ln(x) + c_1 \ln(x)$$

which simplifies to

$$y = \ln(x)\left(x^3 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \ln(x)\left(x^3 + c_1\right) \tag{1}$$

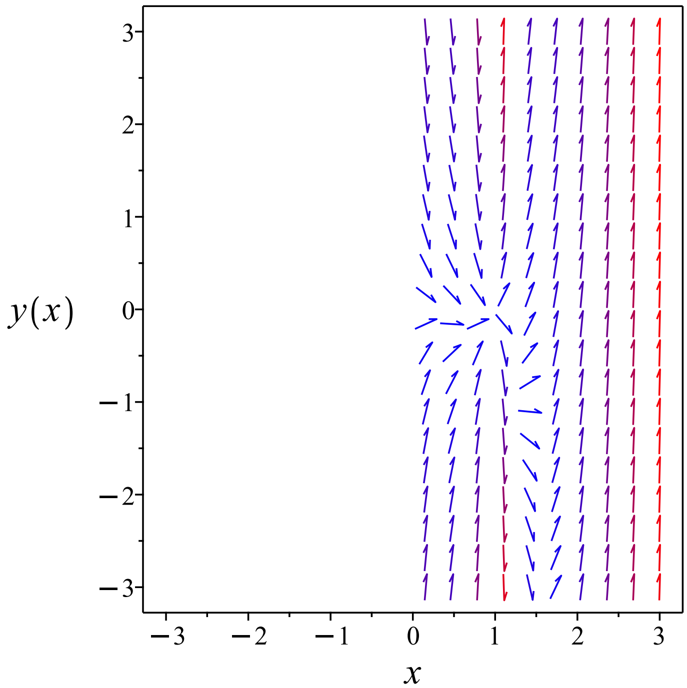


Figure 219: Slope field plot

Verification of solutions

$$y = \ln(x)\left(x^3 + c_1\right)$$

Verified OK.

6.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + 3x^3 \ln(x)^2}{x \ln(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 174: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \ln(x) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\ln(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\ln(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + 3x^3 \ln(x)^2}{x \ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{\ln(x)^2 x} \\ S_y &= \frac{1}{\ln(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\ln(x)} = x^3 + c_1$$

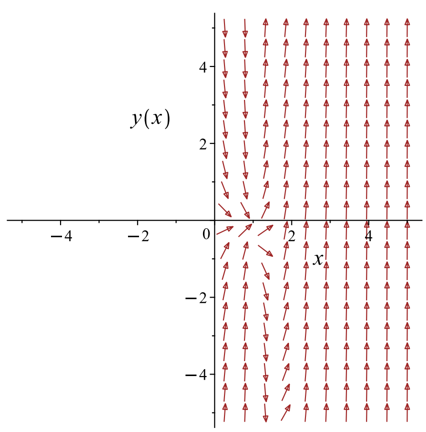
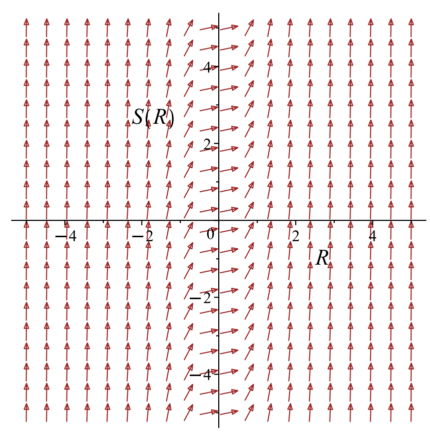
Which simplifies to

$$\frac{y}{\ln(x)} = x^3 + c_1$$

Which gives

$$y = x^3 \ln(x) + c_1 \ln(x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+3x^3\ln(x)^2}{x\ln(x)}$ 	$R = x$ $S = \frac{y}{\ln(x)}$	$\frac{dS}{dR} = 3R^2$ 

Summary
The solution(s) found are the following

$$y = x^3 \ln(x) + c_1 \ln(x) \tag{1}$$

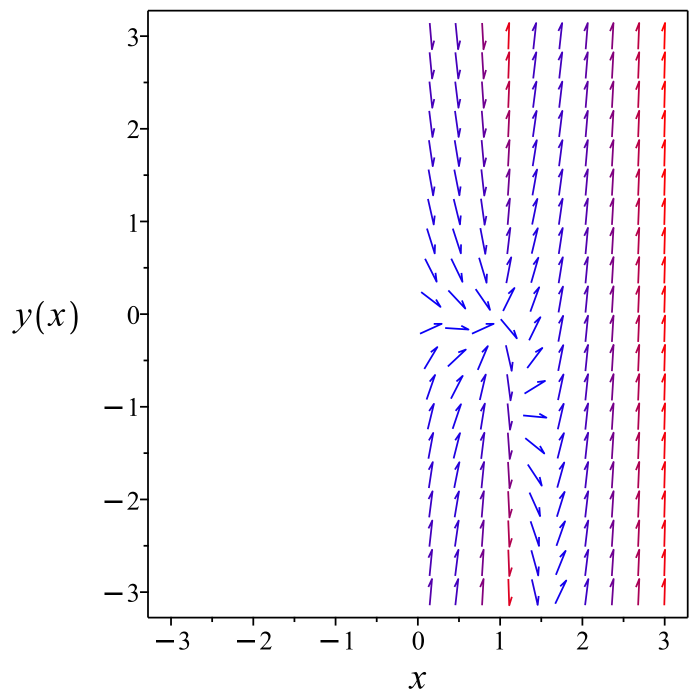


Figure 220: Slope field plot

Verification of solutions

$$y = x^3 \ln(x) + c_1 \ln(x)$$

Verified OK.

6.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x \ln (x))\,\mathrm{d}y &= \left(y + 3x^3 \ln (x)^2\right)\mathrm{d}x\\ \left(-y - 3x^3 \ln (x)^2\right)\mathrm{d}x + (x \ln (x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y - 3x^3 \ln (x)^2\\ N(x, y) &= x \ln (x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - 3x^3 \ln (x)^2\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x \ln (x))\\ &= \ln (x) + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\ln(x)}((-1) - (\ln(x) + 1)) \\ &= \frac{-2 - \ln(x)}{x\ln(x)} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{-2-\ln(x)}{x\ln(x)}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(x) - 2\ln(\ln(x))} \\ &= \frac{1}{\ln(x)^2 x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{\ln(x)^2 x}\left(-y - 3x^3\ln(x)^2\right) \\ &= \frac{-y - 3x^3\ln(x)^2}{\ln(x)^2 x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{\ln(x)^2 x}(x\ln(x)) \\ &= \frac{1}{\ln(x)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-y - 3x^3\ln(x)^2}{\ln(x)^2 x}\right) + \left(\frac{1}{\ln(x)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-y - 3x^3\ln(x)^2}{\ln(x)^2 x}\,\mathrm{d}x$$
$$\phi = -x^3 + \frac{y}{\ln(x)} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{\ln(x)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{\ln(x)}$. Therefore equation (4) becomes

$$\frac{1}{\ln(x)} = \frac{1}{\ln(x)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 + \frac{y}{\ln(x)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 + \frac{y}{\ln(x)}$$

The solution becomes

$$y = x^3 \ln(x) + c_1 \ln(x)$$

Summary

The solution(s) found are the following

$$y = x^3 \ln(x) + c_1 \ln(x) \tag{1}$$

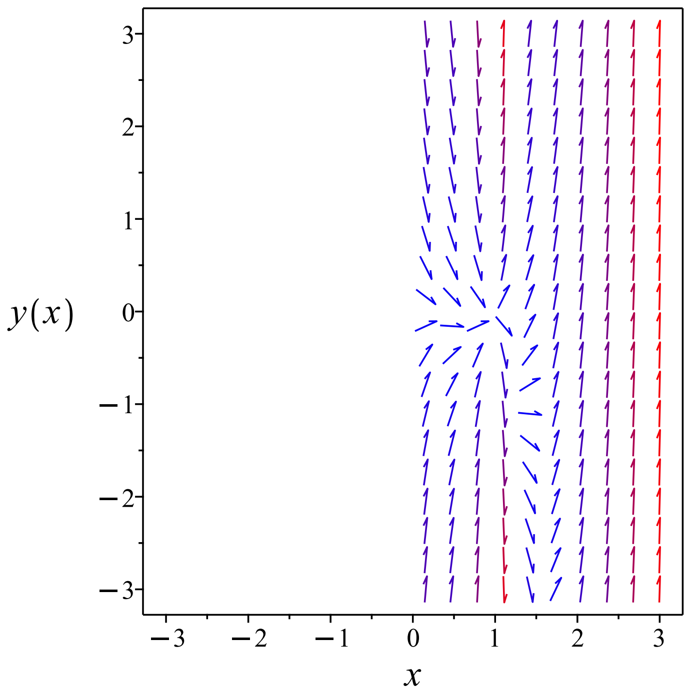


Figure 221: Slope field plot

Verification of solutions

$$y = x^3 \ln(x) + c_1 \ln(x)$$

Verified OK.

6.8.4 Maple step by step solution

Let's solve

$x \ln (x) y' - y = 3x^3 \ln (x)^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x \ln(x)} + 3x^2 \ln (x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x \ln(x)} = 3x^2 \ln (x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{y}{x \ln(x)}\right) = 3\mu(x) x^2 \ln (x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) \left(y' - \frac{y}{x \ln(x)}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x \ln(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\ln(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int 3\mu(x) x^2 \ln (x) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int 3\mu(x) x^2 \ln (x) dx + c_1$

- Solve for y

  $y = \frac{\int 3\mu(x)x^2 \ln(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\ln(x)}$

  $y = \ln (x) \left(\int 3x^2 dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \ln (x) (x^3 + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)*x*ln(x)-y(x)=3*x^3*(ln(x))^2,y(x), singsol=all)
```

$$y(x) = (x^3 + c_1)\ln(x)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 14

```
DSolve[y'[x]*x*Log[x]-y[x]==3*x^3*(Log[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x^3 + c_1)\log(x)$$

6.9 problem 133

Internal problem ID [15035]
Internal file name [`OUTPUT/15035_Sunday_April_21_2024_01_21_08_PM_63877819/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 133.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{(-y^2 + 2x)\, y' - 2y = 0}$$

6.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y}{y^2 - 2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{2y(b_3 - a_2)}{y^2 - 2x} - \frac{4y^2 a_3}{(y^2 - 2x)^2} + \frac{4y(x a_2 + y a_3 + a_1)}{(y^2 - 2x)^2} - \left(-\frac{2}{y^2 - 2x} + \frac{4y^2}{(y^2 - 2x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-y^4 b_2 + 6x\, y^2 b_2 - 2y^3 a_2 + 4y^3 b_3 + 2y^2 b_1 + 4x b_1 - 4y a_1}{(-y^2 + 2x)^2} = 0$$

Setting the numerator to zero gives

$$y^4 b_2 - 6x\, y^2 b_2 + 2y^3 a_2 - 4y^3 b_3 - 2y^2 b_1 - 4x b_1 + 4y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_2^4 + 2a_2 v_2^3 - 6b_2 v_1 v_2^2 - 4b_3 v_2^3 - 2b_1 v_2^2 + 4a_1 v_2 - 4b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-6b_2 v_1 v_2^2 - 4b_1 v_1 + b_2 v_2^4 + (2a_2 - 4b_3)\, v_2^3 - 2b_1 v_2^2 + 4a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
4a_1 &= 0 \\
-4b_1 &= 0 \\
-2b_1 &= 0 \\
-6b_2 &= 0 \\
2a_2 - 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 0 - \left(-\frac{2y}{y^2 - 2x}\right)(y) \\
&= -\frac{2y^2}{-y^2 + 2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{2y^2}{-y^2+2x}} dy
\end{aligned}$$

Which results in

$$S = \frac{y}{2} + \frac{x}{y}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y}{y^2-2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{y} \\
S_y &= \frac{1}{2} - \frac{x}{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

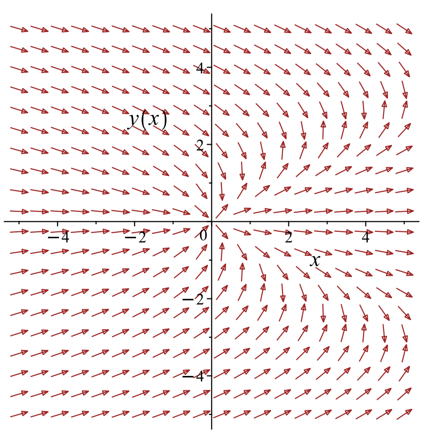
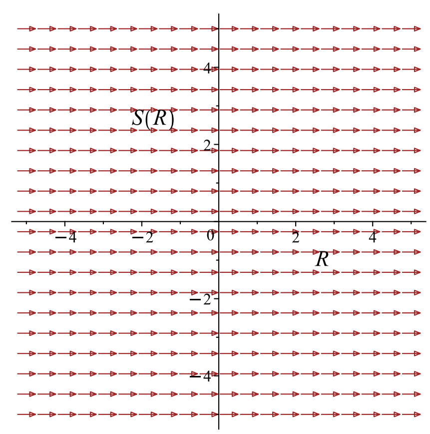
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{2} + \frac{x}{y} = c_1$$

Which simplifies to

$$\frac{y}{2} + \frac{x}{y} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y}{y^2-2x}$ 	$R = x$ $S = \frac{y}{2} + \frac{x}{y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{y}{2} + \frac{x}{y} = c_1 \tag{1}$$

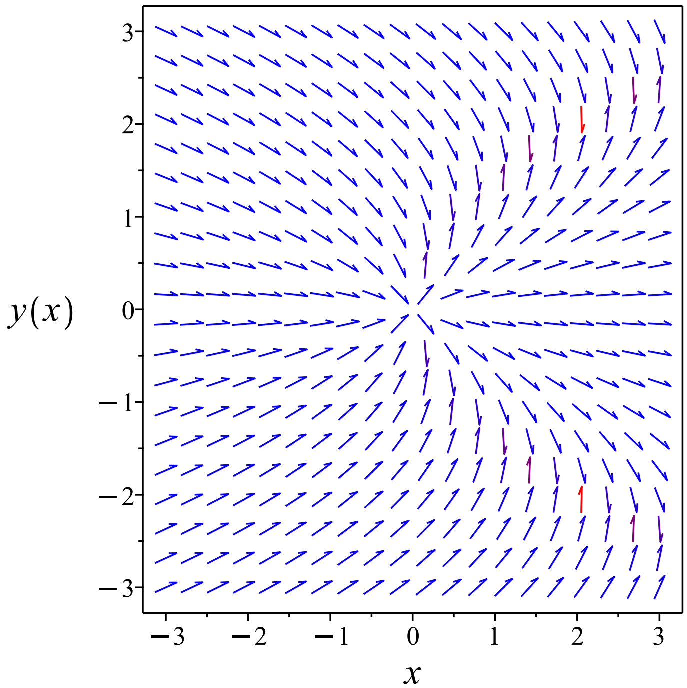


Figure 222: Slope field plot

Verification of solutions

$$\frac{y}{2}+\frac{x}{y}=c_1$$

Verified OK.

6.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-y^2 + 2x\right) \mathrm{d}y = (2y) \, \mathrm{d}x$$
$$(-2y) \, \mathrm{d}x + \left(-y^2 + 2x\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2y$$
$$N(x, y) = -y^2 + 2x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-2y)$$
$$= -2$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-y^2 + 2x\right)$$
$$= 2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-y^2 + 2x}((-2) - (2)) \\
&= -\frac{4}{-y^2 + 2x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{2y}((2) - (-2)) \\
&= -\frac{2}{y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}(-2y) \\
&= -\frac{2}{y}
\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(-y^2 + 2x) \\
&= \frac{-y^2 + 2x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{2}{y}\right) + \left(\frac{-y^2 + 2x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{2}{y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{2x}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-y^2+2x}{y^2}$. Therefore equation (4) becomes

$$
\frac{-y^2 + 2x}{y^2} = \frac{2x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{2x}{y} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2x}{y} - y$$

Summary

The solution(s) found are the following

$$-\frac{2x}{y} - y = c_1 \tag{1}$$

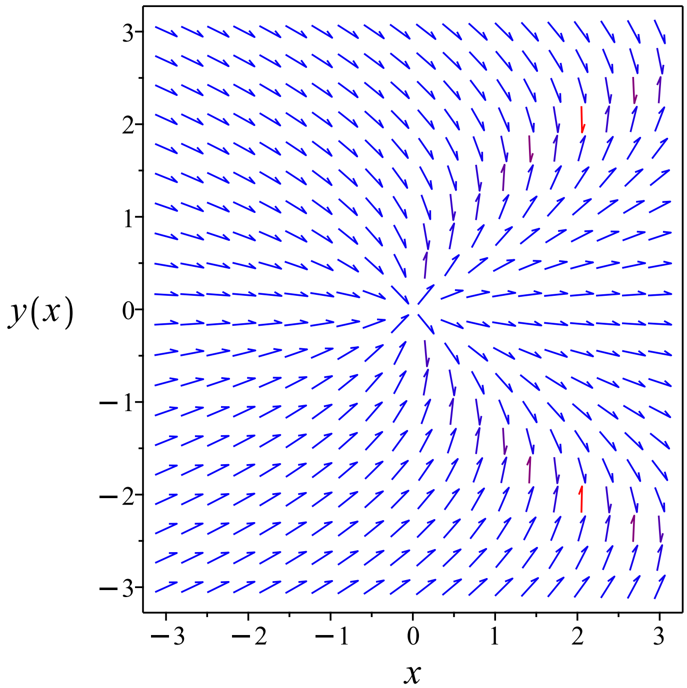


Figure 223: Slope field plot

Verification of solutions

$$-\frac{2x}{y} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve((2*x-y(x)^2)*diff(y(x),x)=2*y(x),y(x), singsol=all)
```

$$y(x) = c_1 - \sqrt{c_1^2 - 2x}$$
$$y(x) = c_1 + \sqrt{c_1^2 - 2x}$$

✓ Solution by Mathematica

Time used: 0.253 (sec). Leaf size: 46

```
DSolve[(2*x-y[x]^2)*y'[x]==2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 - \sqrt{-2x + c_1{}^2}$$
$$y(x) \to \sqrt{-2x + c_1{}^2} + c_1$$
$$y(x) \to 0$$

6.10 problem 134

Internal problem ID [15036]
Internal file name [`OUTPUT/15036_Sunday_April_21_2024_01_21_09_PM_67743928/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 134.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + y\cos(x) = \cos(x)}$$

With initial conditions

$$[y(0) = 1]$$

6.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cos(x) \\ q(x) &= \cos(x) \end{aligned}$$

Hence the ode is

$$y' + y\cos(x) = \cos(x)$$

The domain of $p(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.10.2 Solving as separable ode

In canonical form the ODE is

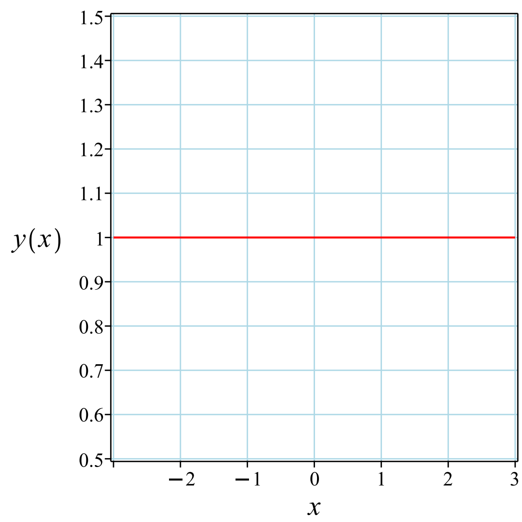
$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \cos(x)(1 - y) \end{aligned}$$

Where $f(x) = \cos(x)$ and $g(y) = 1 - y$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 1$ is zero, then the solution is

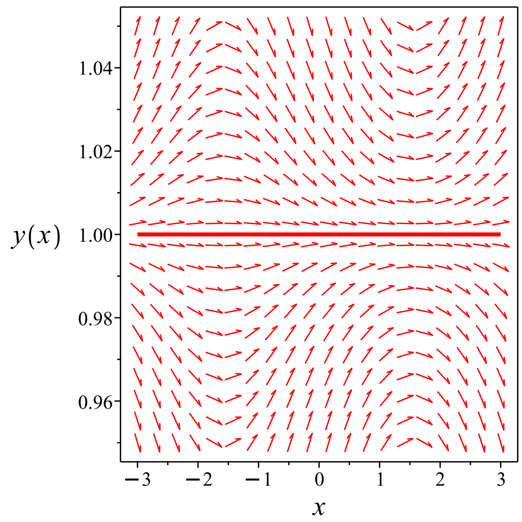
$$\begin{aligned} y &= y_0 \\ &= 1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 1$$

Verified OK.

6.10.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \cos(x)dx} \\ &= e^{\sin(x)}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\ \frac{d}{dx}\left(e^{\sin(x)}y\right) &= \left(e^{\sin(x)}\right)(\cos(x)) \\ d\left(e^{\sin(x)}y\right) &= \left(\cos(x)\, e^{\sin(x)}\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\sin(x)}y &= \int \cos(x)\, e^{\sin(x)}\, dx \\ e^{\sin(x)}y &= e^{\sin(x)} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)} e^{\sin(x)} + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = 1 + c_1 e^{-\sin(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

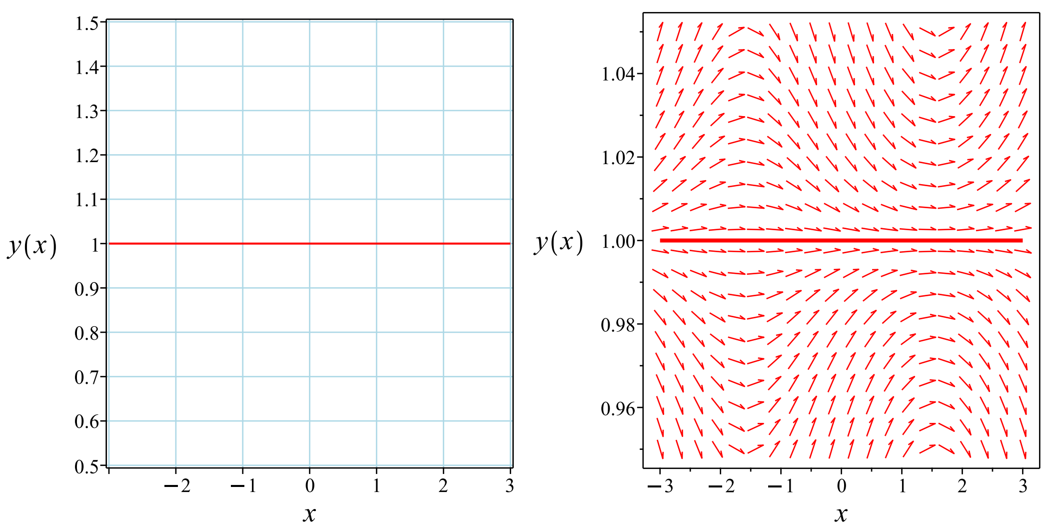
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 1$$

Summary
The solution(s) found are the following

$$y = 1 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

6.10.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\cos(x)\, y+\cos(x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 177: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{-\sin(x)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-\sin(x)}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\cos(x)\, y + \cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= \cos(x)\, \mathrm{e}^{\sin(x)} y\\ S_y &= \mathrm{e}^{\sin(x)}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x)\, e^{\sin(x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)\, e^{\sin(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{\sin(R)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)} y = e^{\sin(x)} + c_1$$

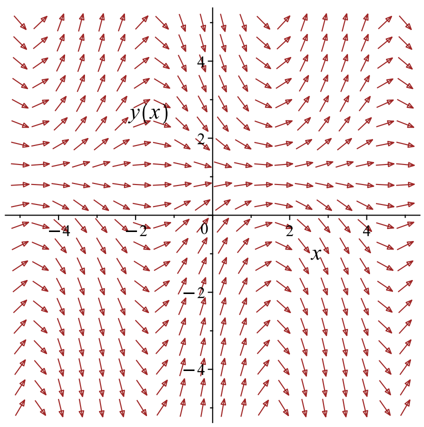
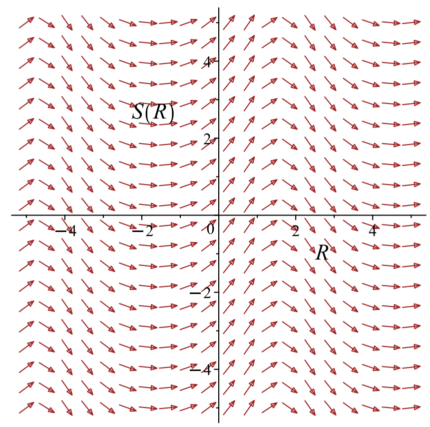
Which simplifies to

$$e^{\sin(x)} y = e^{\sin(x)} + c_1$$

Which gives

$$y = e^{-\sin(x)} \left(e^{\sin(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cos(x)\, y + \cos(x)$ 	$R = x$ $S = \mathrm{e}^{\sin(x)} y$	$\frac{dS}{dR} = \cos(R)\, \mathrm{e}^{\sin(R)}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

$$c_1 = 0$$

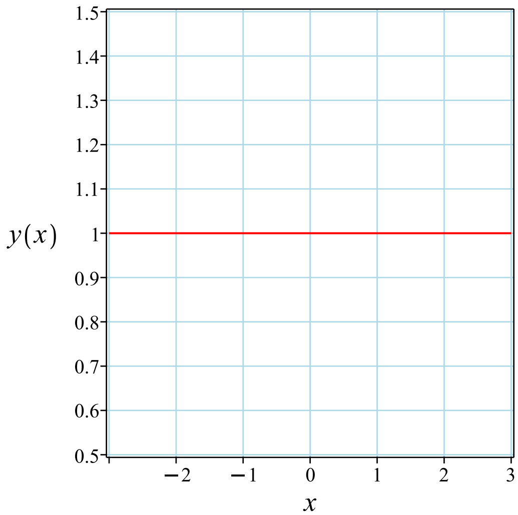
Substituting c_1 found above in the general solution gives

$$y = 1$$

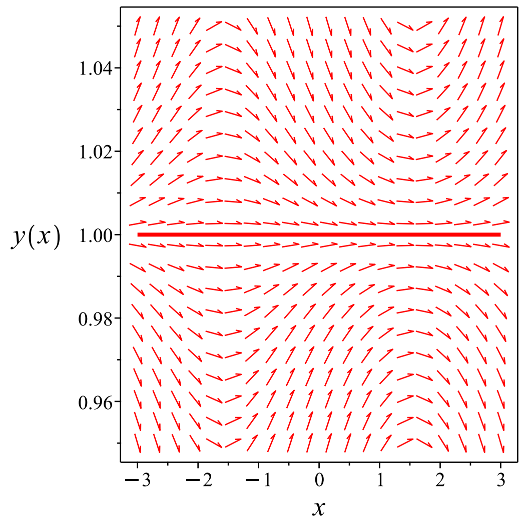
Summary

The solution(s) found are the following

$$y = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 1$$

Verified OK.

6.10.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-y}\right)\mathrm{d}y &= (\cos(x))\,\mathrm{d}x\\ (-\cos(x))\,\mathrm{d}x + \left(\frac{1}{1-y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\cos(x)\\ N(x, y) &= \frac{1}{1-y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x))\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\cos(x) \mathrm{d}x$$
$$\phi = -\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-y}$. Therefore equation (4) becomes

$$\frac{1}{1-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{1}{y-1}\right) \mathrm{d}y$$
$$f(y) = -\ln(y-1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) - \ln(y - 1)$$

The solution becomes

$$y = e^{-\sin(x) - c_1} + 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

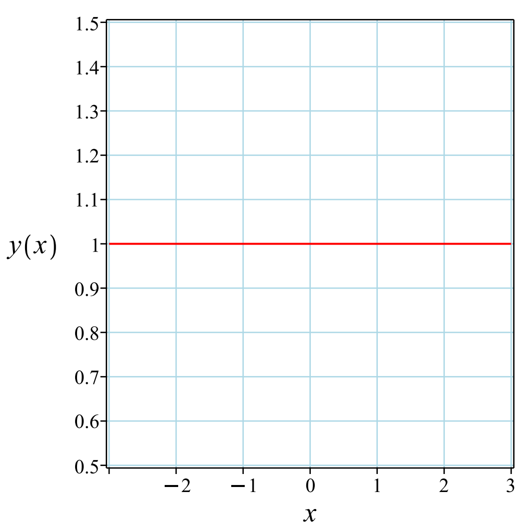
$$1 = e^{-c_1} + 1$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = e^{-\sin(x) - c_1} + 1 = y = 1$ and this result satisfies the given initial condition.

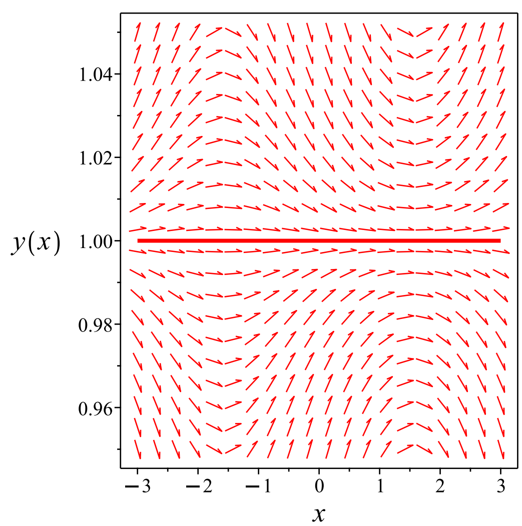
Summary

The solution(s) found are the following

$$y = 1$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

6.10.6 Maple step by step solution

Let's solve

$[y' + y\cos(x) = \cos(x), y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = -\cos(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1}dx = \int -\cos(x)\,dx + c_1$

- Evaluate integral

  $\ln(y-1) = -\sin(x) + c_1$

- Solve for y

  $y = e^{-\sin(x)+c_1} + 1$

- Use initial condition $y(0) = 1$

  $1 = 1 + e^{c_1}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)+y(x)*cos(x)=cos(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 6

```
DSolve[{y'[x]+y[x]*Cos[x]==Cos[x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1$$

6.11 problem 135

Internal problem ID [15037]
Internal file name [`OUTPUT/15037_Sunday_April_21_2024_01_21_10_PM_63517066/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 135.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y' - \frac{y}{2\ln(y)\, y + y - x} = 0}$$

6.11.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{y}{2\ln(y)\, y + y - x} \tag{1}$$

Which becomes

$$(2\ln(y)\, y + y)\, dy = (x)\, dy + (y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(x)\, dy + (y)\, dx = d(xy)$$

Hence (2) becomes

$$(2\ln(y)\, y + y)\, dy = d(xy)$$

Integrating both sides gives gives the solution as

$$\ln(y)\, y^2 = yx + c_1$$

Summary

The solution(s) found are the following

$$\ln(y)\, y^2 = yx + c_1 \tag{1}$$

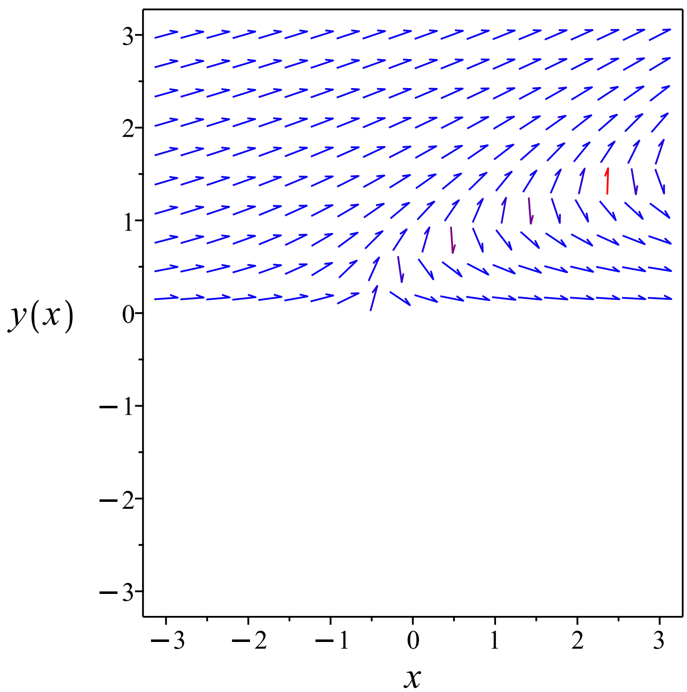


Figure 228: Slope field plot

Verification of solutions

$$\ln(y)\, y^2 = yx + c_1$$

Verified OK.

6.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y}{2\ln(y)\,y + y - x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 + \frac{y(b_3 - a_2)}{2\ln(y)\,y + y - x} - \frac{y^2 a_3}{(2\ln(y)\,y + y - x)^2} - \frac{y(x a_2 + y a_3 + a_1)}{(2\ln(y)\,y + y - x)^2} \\
- \left(\frac{1}{2\ln(y)\,y + y - x} - \frac{y(3 + 2\ln(y))}{(2\ln(y)\,y + y - x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{4\ln(y)^2 y^2 b_2 - 4\ln(y)\,x y b_2 - 2\ln(y)\,y^2 a_2 + 4\ln(y)\,y^2 b_2 + 2\ln(y)\,y^2 b_3 + 2x^2 b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + 3}{(2\ln(y)\,y + y - x)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
4\ln(y)^2 y^2 b_2 - 4\ln(y)\,x y b_2 - 2\ln(y)\,y^2 a_2 + 4\ln(y)\,y^2 b_2 + 2\ln(y)\,y^2 b_3 \\
+ 2x^2 b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + 3y^2 b_3 + x b_1 - y a_1 + 2y b_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(y) = v_3\}$$

The above PDE (6E) now becomes

$$4v_3^2v_2^2b_2 - 2v_3v_2^2a_2 - 4v_3v_1v_2b_2 + 4v_3v_2^2b_2 + 2v_3v_2^2b_3 - v_2^2a_2 - 2v_2^2a_3 + 2v_1^2b_2 + v_2^2b_2 + 3v_2^2b_3 - v_2a_1 + v_1b_1 + 2v_2b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$2v_1^2b_2 - 4v_3v_1v_2b_2 + v_1b_1 + 4v_3^2v_2^2b_2 + (-2a_2 + 4b_2 + 2b_3)\,v_2^2v_3 + (-a_2 - 2a_3 + b_2 + 3b_3)\,v_2^2 + (-a_1 + 2b_1)\,v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -4b_2 &= 0 \\ 2b_2 &= 0 \\ 4b_2 &= 0 \\ -a_1 + 2b_1 &= 0 \\ -2a_2 + 4b_2 + 2b_3 &= 0 \\ -a_2 - 2a_3 + b_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= b_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = y + x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y}{2\ln(y)\,y + y - x}\right)(y + x) \\
&= \frac{2\ln(y)\,y^2 - 2xy}{2\ln(y)\,y + y - x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2\ln(y)y^2 - 2xy}{2\ln(y)y + y - x}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(\ln(y)\,y^2 - xy)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{2\ln(y)\, y + y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{-2\ln(y)\, y + 2x} \\ S_y &= \frac{2\ln(y)\, y + y - x}{2y\,(\ln(y)\, y - x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

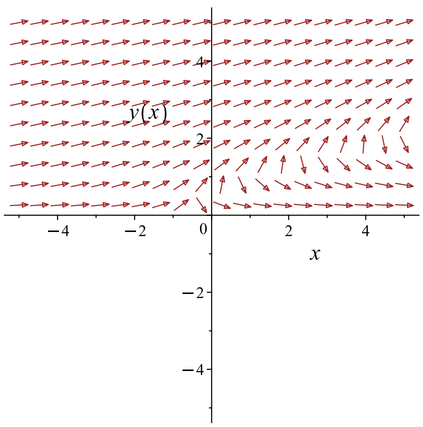
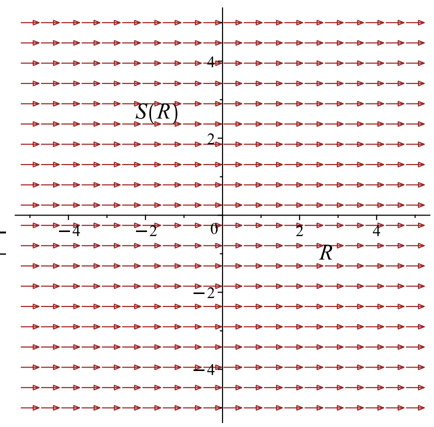
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(\ln(y)\, y - x)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\ln(\ln(y)\, y - x)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{2\ln(y)y+y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(\ln(y)y -}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{\ln(\ln(y)\, y - x)}{2} = c_1 \tag{1}$$

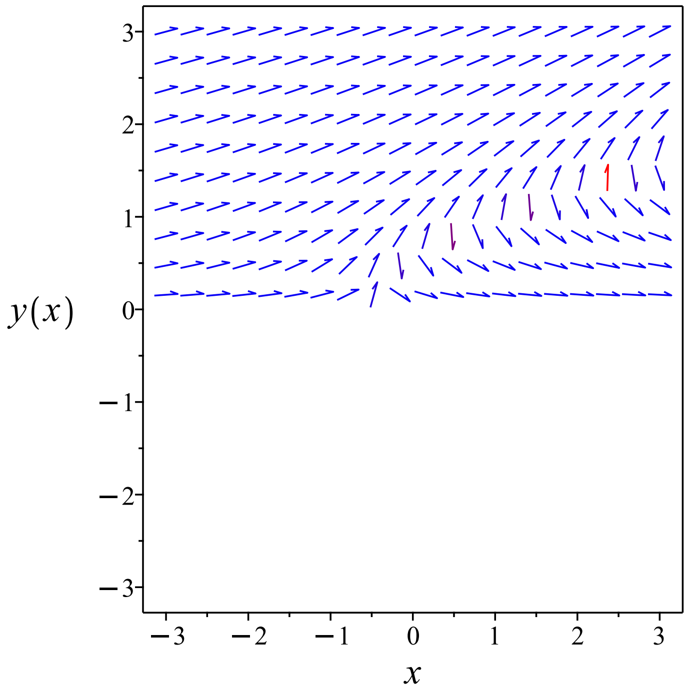


Figure 229: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{2}+\frac{\ln(\ln(y)\,y-x)}{2}=c_1$$

Verified OK.

6.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2 \ln (y) y + y - x) \,\mathrm{d}y &= (y) \,\mathrm{d}x \\
(-y) \,\mathrm{d}x + (2 \ln (y) y + y - x) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -y \\
N(x, y) &= 2 \ln (y) y + y - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2 \ln (y) y + y - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -y \,\mathrm{d}x$$
$$\phi = -xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\ln(y)\,y + y - x$. Therefore equation (4) becomes

$$2\ln(y)\,y + y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2\ln(y)\,y + y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int ((2\ln(y) + 1)\,y) \,\mathrm{d}y$$
$$f(y) = \ln(y)\,y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y)\,y^2 - xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y)\, y^2 - xy$$

Summary
The solution(s) found are the following

$$\ln(y)\, y^2 - yx = c_1 \tag{1}$$

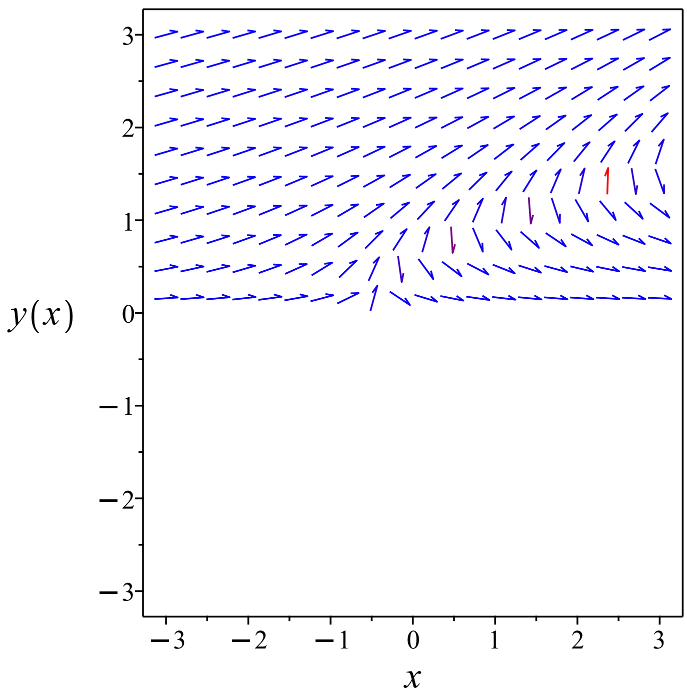


Figure 230: Slope field plot

Verification of solutions

$$\ln(y)\, y^2 - yx = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)=y(x)/(2*y(x)*ln(y(x))+y(x)-x),y(x), singsol=all)
```

$$y(x) = e^{\mathrm{RootOf}\left(\_Z e^{2\_Z} - x\,e^{\_Z} + c_1\right)}$$

✓ Solution by Mathematica

Time used: 0.141 (sec). Leaf size: 19

```
DSolve[y'[x]==y[x]/(2*y[x]*Log[y[x]]+y[x]-x),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[x = y(x)\log(y(x)) + \frac{c_1}{y(x)}, y(x)\right]$$

6.12 problem 136

Internal problem ID [15038]
Internal file name [OUTPUT/15038_Sunday_April_21_2024_01_21_11_PM_52278851/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 136.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\left(\frac{e^{-y^2}}{2} - yx\right) y' = 1}$$

6.12.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{\mathrm{e}^{-y^2}}{2} - xy\right)\mathrm{d}y &= \mathrm{d}x\\ -\mathrm{d}x + \left(\frac{\mathrm{e}^{-y^2}}{2} - xy\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -1\\ N(x, y) &= \frac{\mathrm{e}^{-y^2}}{2} - xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{-y^2}}{2} - xy\right) \\ &= -y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{2}{-2xy + \mathrm{e}^{-y^2}}((0) - (-y)) \\ &= \frac{2y}{-2xy + \mathrm{e}^{-y^2}}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -1((-y) - (0)) \\ &= y\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int y \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{y^2}{2}} \\ &= \mathrm{e}^{\frac{y^2}{2}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{\frac{y^2}{2}}(-1) \\ &= -\mathrm{e}^{\frac{y^2}{2}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{y^2}{2}}\left(\frac{\mathrm{e}^{-y^2}}{2} - xy\right) \\
&= -y\,\mathrm{e}^{\frac{y^2}{2}} x + \frac{\mathrm{e}^{-\frac{y^2}{2}}}{2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{\frac{y^2}{2}}\right) + \left(-y\,\mathrm{e}^{\frac{y^2}{2}} x + \frac{\mathrm{e}^{-\frac{y^2}{2}}}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{\frac{y^2}{2}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\mathrm{e}^{\frac{y^2}{2}} x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -y\,\mathrm{e}^{\frac{y^2}{2}} x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y\,\mathrm{e}^{\frac{y^2}{2}} x + \frac{\mathrm{e}^{-\frac{y^2}{2}}}{2}$. Therefore equation (4) becomes

$$
-y\,\mathrm{e}^{\frac{y^2}{2}} x + \frac{\mathrm{e}^{-\frac{y^2}{2}}}{2} = -y\,\mathrm{e}^{\frac{y^2}{2}} x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{e^{-\frac{y^2}{2}}}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,dy = \int \left(\frac{e^{-\frac{y^2}{2}}}{2}\right) dy$$

$$f(y) = \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{y\sqrt{2}}{2}\right)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -e^{\frac{y^2}{2}} x + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{y\sqrt{2}}{2}\right)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{\frac{y^2}{2}} x + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{y\sqrt{2}}{2}\right)}{4}$$

Summary

The solution(s) found are the following

$$-e^{\frac{y^2}{2}} x + \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{y\sqrt{2}}{2}\right)}{4} = c_1 \tag{1}$$

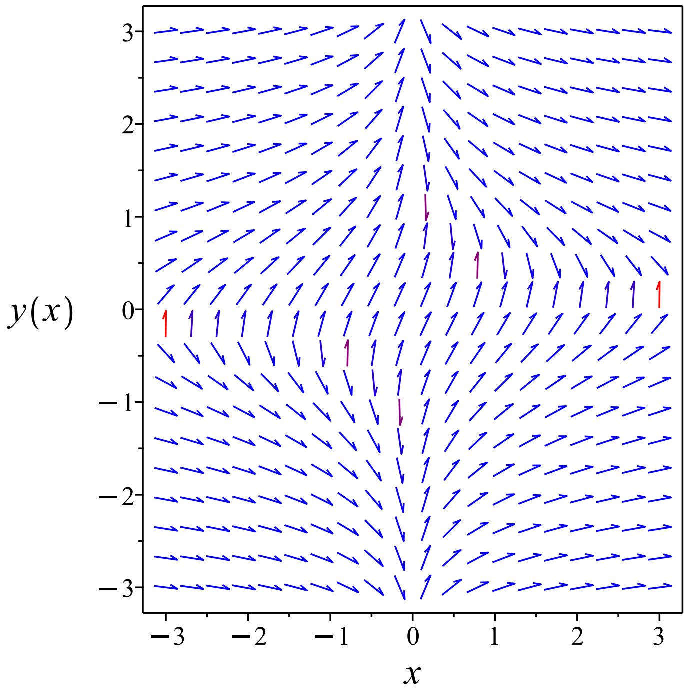


Figure 231: Slope field plot

Verification of solutions

$$-e^{\frac{y^2}{2}}x+\frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{y\sqrt{2}}{2}\right)}{4}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 34

```
dsolve((exp(-(y(x)^2))/2-x*y(x))*diff(y(x),x)-1=0,y(x), singsol=all)
```

$$\frac{\left(-\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,y(x)}{2}\right)-4c_1\right)\mathrm{e}^{-\frac{y(x)^2}{2}}}{4}+x=0$$

✓ Solution by Mathematica

Time used: 0.211 (sec). Leaf size: 32

```
DSolve[(Exp[-(y[x]^2)/2]-x*y[x])*y'[x]-1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x=e^{-\frac{1}{2}y(x)^2}y(x)+c_1e^{-\frac{1}{2}y(x)^2},y(x)\right]$$

6.13 problem 137

Internal problem ID [15039]
Internal file name [`OUTPUT/15039_Sunday_April_21_2024_01_21_13_PM_215885/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 137.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\,e^{x} = 2x\,e^{e^{x}}}$$

6.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -e^{x} \\ q(x) &= 2x\,e^{e^{x}} \end{aligned}$$

Hence the ode is

$$y' - y\,e^{x} = 2x\,e^{e^{x}}$$

The integrating factor μ is

$$\mu = e^{\int -e^x dx}$$
$$= e^{-e^x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2x\, e^{e^x}\right)$$
$$\frac{d}{dx}\left(e^{-e^x} y\right) = \left(e^{-e^x}\right)\left(2x\, e^{e^x}\right)$$
$$d\left(e^{-e^x} y\right) = (2x)\, dx$$

Integrating gives

$$e^{-e^x} y = \int 2x\, dx$$
$$e^{-e^x} y = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-e^x}$ results in

$$y = e^{e^x} x^2 + c_1 e^{e^x}$$

which simplifies to

$$y = e^{e^x}\left(x^2 + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{e^x}\left(x^2 + c_1\right) \tag{1}$$

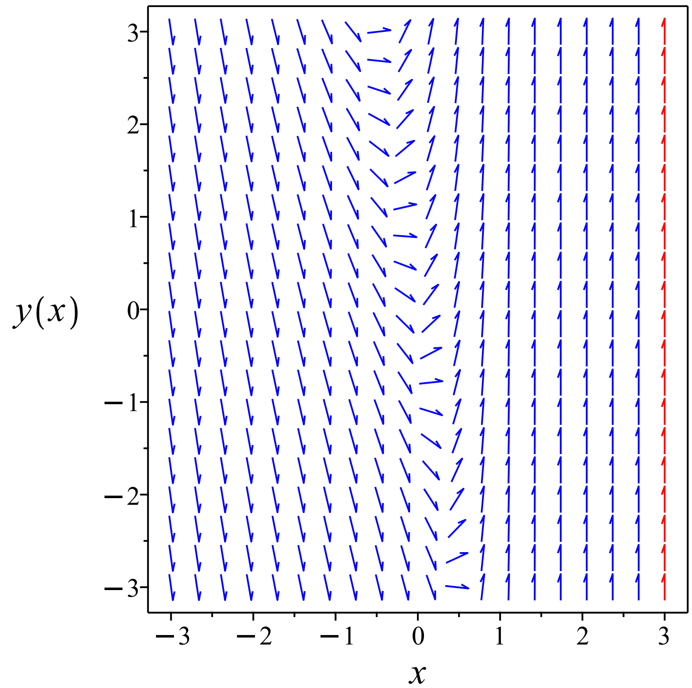


Figure 232: Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{\mathrm{e}^{x}} \left(x^{2}+c_{1}\right)$$

Verified OK.

6.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \mathrm{e}^{x} y + 2x\,\mathrm{e}^{\mathrm{e}^{x}} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 180: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{\mathrm{e}^{x}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\mathrm{e}^x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\mathrm{e}^x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^x y + 2x\,\mathrm{e}^{\mathrm{e}^x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y\,\mathrm{e}^{x-\mathrm{e}^x} \\ S_y &= \mathrm{e}^{-\mathrm{e}^x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-e^x} y = x^2 + c_1$$

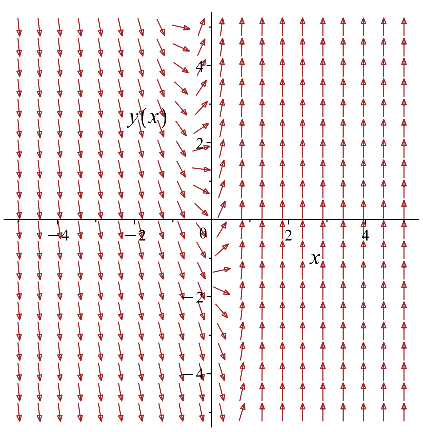
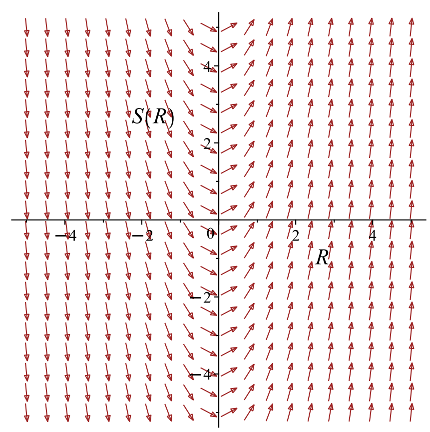
Which simplifies to

$$e^{-e^x} y = x^2 + c_1$$

Which gives

$$y = e^{e^x} (x^2 + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^x y + 2x\,e^{e^x}$ 	$R = x$ $S = e^{-e^x} y$	$\frac{dS}{dR} = 2R$ 

Summary
The solution(s) found are the following

$$y = e^{e^x} (x^2 + c_1) \tag{1}$$

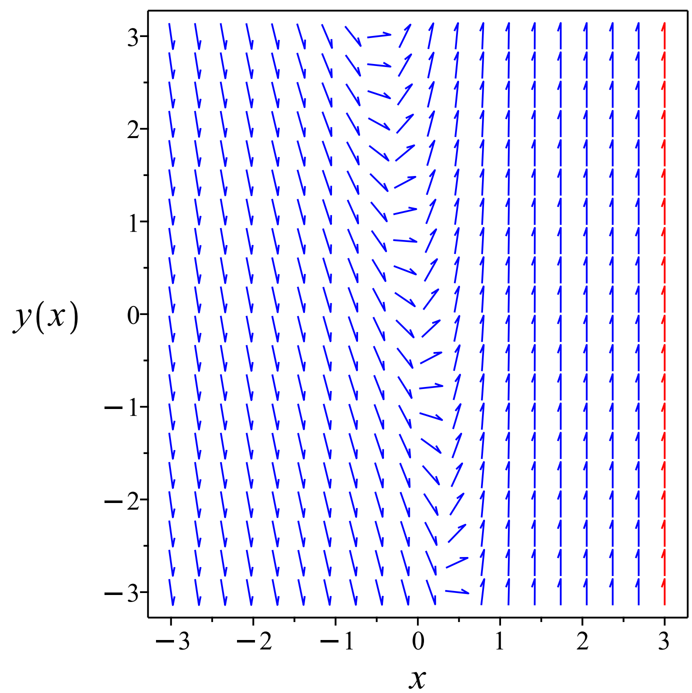


Figure 233: Slope field plot

Verification of solutions

$$y = e^{e^x}(x^2 + c_1)$$

Verified OK.

6.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\mathrm{e}^x y + 2x\,\mathrm{e}^{\mathrm{e}^x}\right) \mathrm{d}x$$
$$\left(-\mathrm{e}^x y - 2x\,\mathrm{e}^{\mathrm{e}^x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\mathrm{e}^x y - 2x\,\mathrm{e}^{\mathrm{e}^x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\mathrm{e}^x y - 2x\,\mathrm{e}^{\mathrm{e}^x}\right)$$
$$= -\mathrm{e}^x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\mathrm{e}^{x}) - (0)) \\ &= -\mathrm{e}^{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -\mathrm{e}^{x} \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\mathrm{e}^{x}} \\ &= \mathrm{e}^{-\mathrm{e}^{x}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-\mathrm{e}^{x}}\left(-\mathrm{e}^{x} y - 2x\,\mathrm{e}^{\mathrm{e}^{x}}\right) \\ &= -y\,\mathrm{e}^{x-\mathrm{e}^{x}} - 2x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-\mathrm{e}^{x}}(1) \\ &= \mathrm{e}^{-\mathrm{e}^{x}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-y\,\mathrm{e}^{x-\mathrm{e}^{x}} - 2x\right) + \left(\mathrm{e}^{-\mathrm{e}^{x}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -y \, \mathrm{e}^{x-\mathrm{e}^{x}} - 2x \, \mathrm{d}x$$
$$\phi = -x^2 + \mathrm{e}^{-\mathrm{e}^{x}} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\mathrm{e}^{x}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\mathrm{e}^{x}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\mathrm{e}^{x}} = \mathrm{e}^{-\mathrm{e}^{x}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 + \mathrm{e}^{-\mathrm{e}^{x}} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 + \mathrm{e}^{-\mathrm{e}^{x}} y$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{e}^{x}} (x^2 + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{e}^{x}} (x^2 + c_1) \tag{1}$$

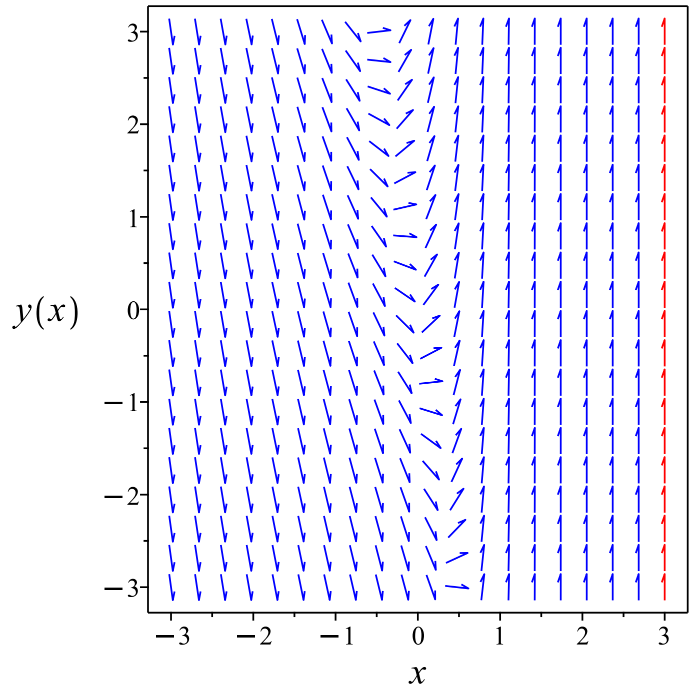


Figure 234: Slope field plot

Verification of solutions

$$y = e^{e^x}(x^2 + c_1)$$

Verified OK.

6.13.4 Maple step by step solution

Let's solve

$y' - y\,e^x = 2x\,e^{e^x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\,e^x + 2x\,e^{e^x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\,e^x = 2x\,e^{e^x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y\,e^x) = 2\mu(x)\,x\,e^{e^x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-y\,\mathrm{e}^{x})=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)\,\mathrm{e}^{x}$
- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{-\mathrm{e}^{x}}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 2\mu(x)\,x\,\mathrm{e}^{\mathrm{e}^{x}}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 2\mu(x)\,x\,\mathrm{e}^{\mathrm{e}^{x}}dx+c_1$
- Solve for y

  $y=\frac{\int 2\mu(x)x\,\mathrm{e}^{\mathrm{e}^{x}}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\mathrm{e}^{-\mathrm{e}^{x}}$

  $y=\frac{\int 2x\,\mathrm{e}^{\mathrm{e}^{x}}\mathrm{e}^{-\mathrm{e}^{x}}dx+c_1}{\mathrm{e}^{-\mathrm{e}^{x}}}$
- Evaluate the integrals on the rhs

  $y=\frac{x^2+c_1}{\mathrm{e}^{-\mathrm{e}^{x}}}$
- Simplify

  $y=\mathrm{e}^{\mathrm{e}^{x}}(x^2+c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)-y(x)*exp(x)=2*x*exp( exp(x) ),y(x), singsol=all)
```

$$y(x) = (x^2 + c_1)\, e^{e^x}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 17

```
DSolve[y'[x]-y[x]*Exp[x]==2*x*Exp[ Exp[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{e^x}(x^2 + c_1)$$

6.14 problem 138

Internal problem ID [15040]
Internal file name [`OUTPUT/15040_Sunday_April_21_2024_01_21_13_PM_3168034/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 138.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + yx\,\mathrm{e}^{x} = \mathrm{e}^{(1-x)\mathrm{e}^{x}}}$$

6.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x\,\mathrm{e}^{x}$$
$$q(x) = \mathrm{e}^{-\mathrm{e}^{x}(x-1)}$$

Hence the ode is

$$y' + yx\,\mathrm{e}^{x} = \mathrm{e}^{-\mathrm{e}^{x}(x-1)}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int x\,e^{x}dx} \\
&= e^{e^{x}(x-1)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{-e^{x}(x-1)}\right) \\
\frac{d}{dx}\left(e^{e^{x}(x-1)}y\right) &= \left(e^{e^{x}(x-1)}\right)\left(e^{-e^{x}(x-1)}\right) \\
d\left(e^{e^{x}(x-1)}y\right) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{e^{x}(x-1)}y &= \int dx \\
e^{e^{x}(x-1)}y &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{e^{x}(x-1)}$ results in

$$
y = e^{-e^{x}(x-1)}x + c_1 e^{-e^{x}(x-1)}
$$

which simplifies to

$$
y = e^{-e^{x}(x-1)}(x + c_1)
$$

Summary
The solution(s) found are the following

$$
y = e^{-e^{x}(x-1)}(x + c_1) \tag{1}
$$

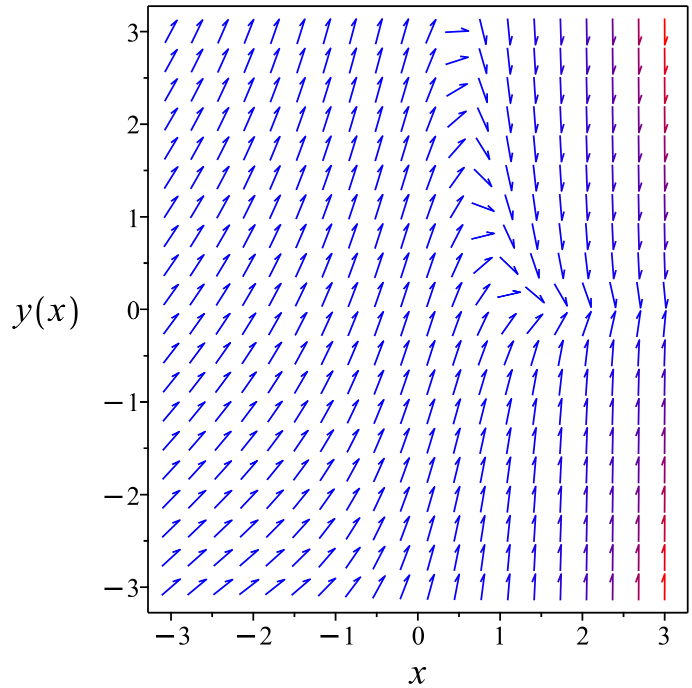


Figure 235: Slope field plot

Verification of solutions

$$y = e^{-e^x(x-1)}(x + c_1)$$

Verified OK.

6.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -x e^x y + e^{(1-x)e^x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 183: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\mathrm{e}^{x}(x-1)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\mathrm{e}^{x}(x-1)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\mathrm{e}^{x}(x-1)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -x\,\mathrm{e}^{x} y + \mathrm{e}^{(1-x)\mathrm{e}^{x}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= xy\,\mathrm{e}^{x+\mathrm{e}^{x}(x-1)} \\ S_y &= \mathrm{e}^{\mathrm{e}^{x}(x-1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{e^x(x-1)} y = x + c_1$$

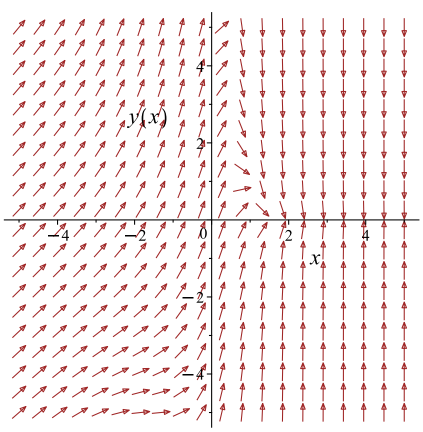
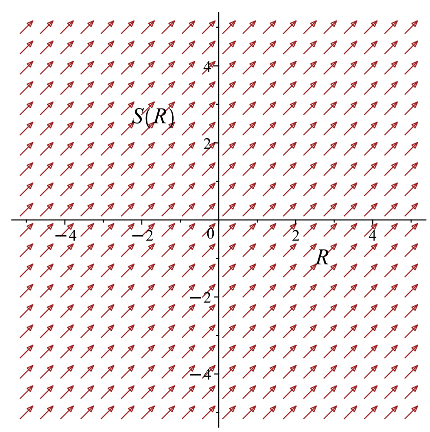
Which simplifies to

$$e^{e^x(x-1)} y = x + c_1$$

Which gives

$$y = e^{-e^x(x-1)}(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x\,e^x y + e^{(1-x)e^x}$ 	$R = x$ $S = e^{e^x(x-1)} y$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = e^{-e^x(x-1)}(x + c_1) \tag{1}$$

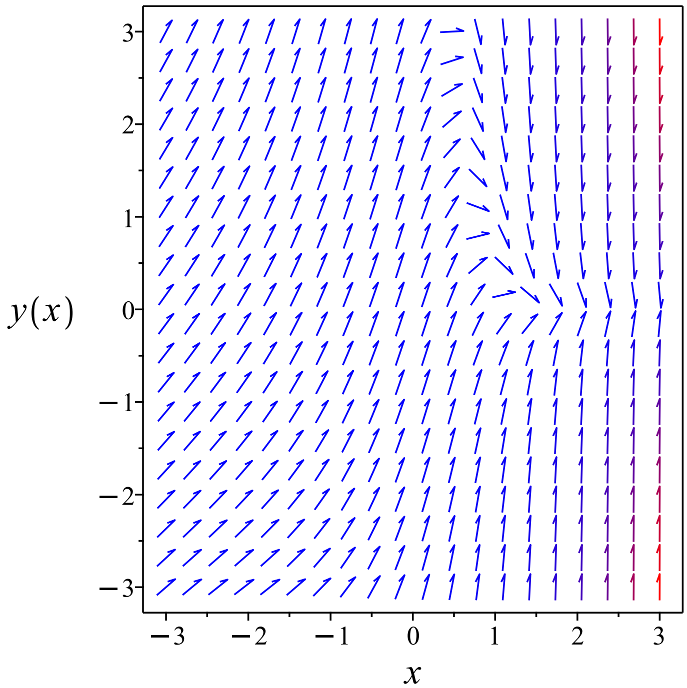


Figure 236: Slope field plot

Verification of solutions

$$y = e^{-e^x(x-1)}(x + c_1)$$

Verified OK.

6.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(-x\,\mathrm{e}^{x} y + \mathrm{e}^{(1-x)\mathrm{e}^{x}}\right) \mathrm{d}x$$
$$\left(x\,\mathrm{e}^{x} y - \mathrm{e}^{(1-x)\mathrm{e}^{x}}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x\,\mathrm{e}^{x} y - \mathrm{e}^{(1-x)\mathrm{e}^{x}}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(x\,\mathrm{e}^{x} y - \mathrm{e}^{(1-x)\mathrm{e}^{x}}\right)$$
$$= x\,\mathrm{e}^{x}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((x\,\mathrm{e}^{x}) - (0)) \\ &= x\,\mathrm{e}^{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int x\,\mathrm{e}^{x}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\mathrm{e}^{x}(x-1)} \\ &= \mathrm{e}^{\mathrm{e}^{x}(x-1)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{\mathrm{e}^{x}(x-1)}\left(x\,\mathrm{e}^{x}y - \mathrm{e}^{(1-x)\mathrm{e}^{x}}\right) \\ &= xy\,\mathrm{e}^{x+\mathrm{e}^{x}(x-1)} - 1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{\mathrm{e}^{x}(x-1)}(1) \\ &= \mathrm{e}^{\mathrm{e}^{x}(x-1)} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(xy\,\mathrm{e}^{x+\mathrm{e}^{x}(x-1)} - 1\right) + \left(\mathrm{e}^{\mathrm{e}^{x}(x-1)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int xy \, \mathrm{e}^{x+\mathrm{e}^x(x-1)} - 1 \mathrm{d}x \\
\phi &= -x + \mathrm{e}^{\mathrm{e}^x(x-1)} y + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{\mathrm{e}^x(x-1)} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\mathrm{e}^x(x-1)}$. Therefore equation (4) becomes

$$
\mathrm{e}^{\mathrm{e}^x(x-1)} = \mathrm{e}^{\mathrm{e}^x(x-1)} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -x + \mathrm{e}^{\mathrm{e}^x(x-1)} y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -x + \mathrm{e}^{\mathrm{e}^x(x-1)} y
$$

The solution becomes

$$
y = \mathrm{e}^{-\mathrm{e}^x(x-1)}(x + c_1)
$$

Summary
The solution(s) found are the following

$$y = e^{-e^x(x-1)}(x + c_1) \tag{1}$$

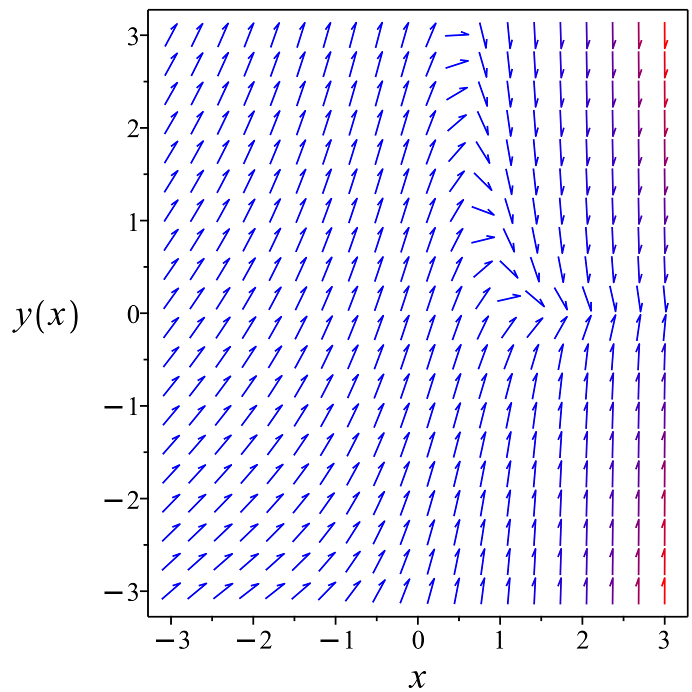


Figure 237: Slope field plot

Verification of solutions

$$y = e^{-e^x(x-1)}(x + c_1)$$

Verified OK.

6.14.4 Maple step by step solution

Let's solve

$y' + yx\,e^x = e^{(1-x)e^x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -yx\,e^x + e^{-e^x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + yx\,\mathrm{e}^{x} = \mathrm{e}^{-\mathrm{e}^{x}(x-1)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + yx\,\mathrm{e}^{x}\right) = \mu(x)\,\mathrm{e}^{-\mathrm{e}^{x}(x-1)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + yx\,\mathrm{e}^{x}\right) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\,x\,\mathrm{e}^{x}$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\mathrm{e}^{x}(x-1)}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{-\mathrm{e}^{x}(x-1)} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{-\mathrm{e}^{x}(x-1)} dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{-\mathrm{e}^{x}(x-1)} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{\mathrm{e}^{x}(x-1)}$

  $y = \frac{\int \mathrm{e}^{-\mathrm{e}^{x}(x-1)}\mathrm{e}^{\mathrm{e}^{x}(x-1)} dx + c_1}{\mathrm{e}^{\mathrm{e}^{x}(x-1)}}$
- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{\mathrm{e}^{\mathrm{e}^{x}(x-1)}}$
- Simplify

  $y = \mathrm{e}^{-\mathrm{e}^{x}(x-1)}(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+x*y(x)*exp(x)=exp( (1-x)*exp(x) ),y(x), singsol=all)
```

$$y(x) = (x + c_1)\,e^{-(-1+x)e^x}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 20

```
DSolve[y'[x]+x*y[x]*Exp[x]==Exp[ (1-x)*Exp[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-e^x(x-1)}(x + c_1)$$

6.15 problem 148

Internal problem ID [15041]
Internal file name [`OUTPUT/15041_Sunday_April_21_2024_01_21_14_PM_9870188/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 148.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y\ln(2) = 2^{\sin(x)}(\cos(x) - 1)\ln(2)}$$

6.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\ln(2) \\ q(x) &= 2^{\sin(x)}(\cos(x) - 1)\ln(2) \end{aligned}$$

Hence the ode is

$$y' - y\ln(2) = 2^{\sin(x)}(\cos(x) - 1)\ln(2)$$

The integrating factor μ is

$$\mu = e^{\int -\ln(2)dx}$$
$$= e^{-\ln(2)x}$$

Which simplifies to

$$\mu = 2^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2^{\sin(x)}(\cos(x)-1)\ln(2)\right)$$
$$\frac{d}{dx}\left(2^{-x}y\right) = \left(2^{-x}\right)\left(2^{\sin(x)}(\cos(x)-1)\ln(2)\right)$$
$$d\left(2^{-x}y\right) = \left(2^{\sin(x)-x}(\cos(x)-1)\ln(2)\right)\, dx$$

Integrating gives

$$2^{-x}y = \int 2^{\sin(x)-x}(\cos(x)-1)\ln(2)\, dx$$
$$2^{-x}y = \frac{\tan\left(\frac{x}{2}\right)^2 e^{\left(\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2}-x\right)\ln(2)} + e^{\left(\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2}-x\right)\ln(2)}}{1+\tan\left(\frac{x}{2}\right)^2} + c_1$$

Dividing both sides by the integrating factor $\mu = 2^{-x}$ results in

$$y = \frac{2^x\left(\tan\left(\frac{x}{2}\right)^2 e^{\left(\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2}-x\right)\ln(2)} + e^{\left(\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)^2}-x\right)\ln(2)}\right)}{1+\tan\left(\frac{x}{2}\right)^2} + c_1 2^x$$

which simplifies to

$$y = c_1 2^x + 2^{\sin(x)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 2^x + 2^{\sin(x)} \tag{1}$$

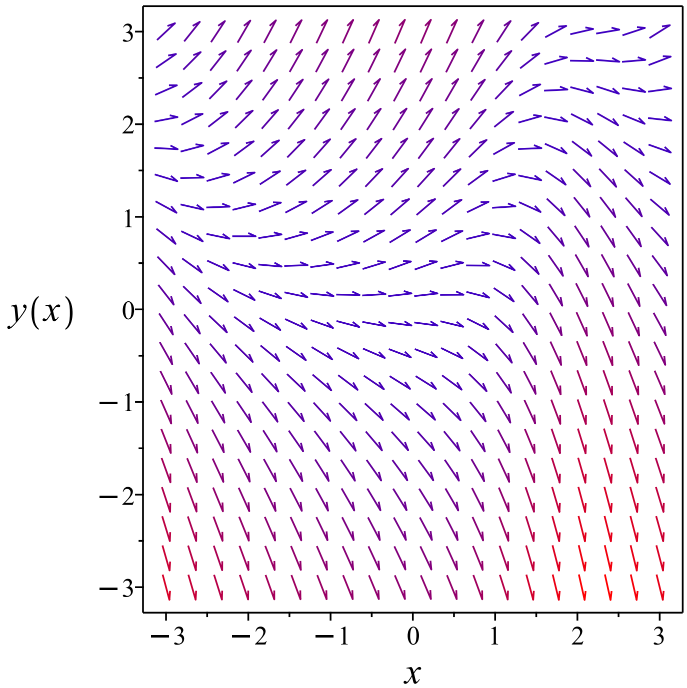


Figure 238: Slope field plot

Verification of solutions

$$y = c_1 2^x + 2^{\sin(x)}$$

Verified OK.

6.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2^{\sin(x)} \ln(2) \cos(x) - 2^{\sin(x)} \ln(2) + y \ln(2)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 186: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{\ln(2)x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\ln(2)x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\ln(2)x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2^{\sin(x)} \ln(2) \cos(x) - 2^{\sin(x)} \ln(2) + y \ln(2)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\ln(2)\, 2^{-x} y \\ S_y &= 2^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2^{\sin(x)-x}(\cos(x) - 1)\ln(2) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2^{\sin(R)-R}(\cos(R) - 1)\ln(2)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + 2^{-\frac{\tan\left(\frac{R}{2}\right)^2 R - 2\tan\left(\frac{R}{2}\right) + R}{1+\tan\left(\frac{R}{2}\right)^2}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2^{-x} y = c_1 + 2^{-\frac{\tan\left(\frac{x}{2}\right)^2 x - 2\tan\left(\frac{x}{2}\right) + x}{1+\tan\left(\frac{x}{2}\right)^2}}$$

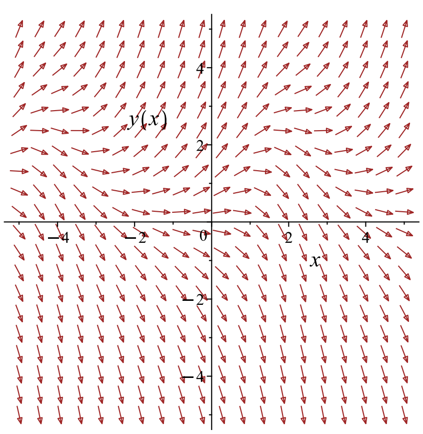
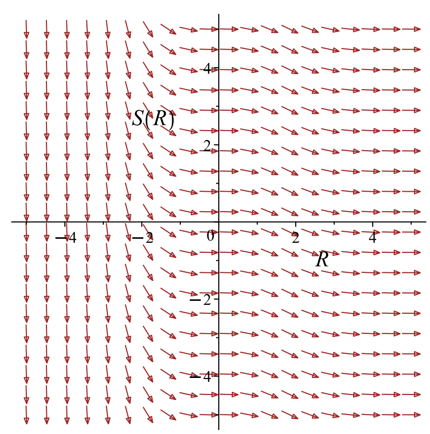
Which simplifies to

$$2^{-x} y - c_1 - 2^{\sin(x) - x} = 0$$

Which gives

$$y = 2^x \left(2^{\sin(x) - x} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2^{\sin(x)} \ln(2) \cos(x) - 2^{\sin(x)} \ln(2) + y \ln(2)$ 	$R = x$ $S = 2^{-x} y$	$\frac{dS}{dR} = 2^{\sin(R) - R}(\cos(R) - 1) \ln(2)$ 

Summary
The solution(s) found are the following

$$y = 2^x \left(2^{\sin(x) - x} + c_1\right) \tag{1}$$

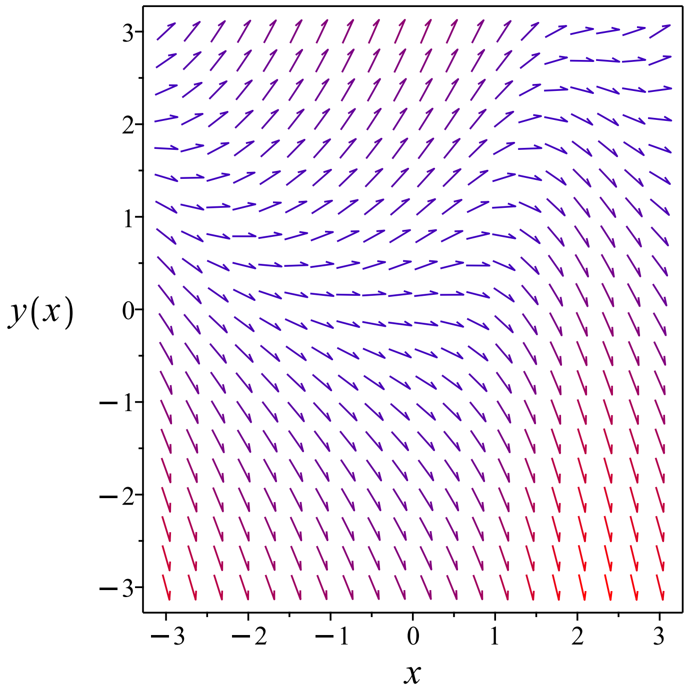


Figure 239: Slope field plot

Verification of solutions

$$y = 2^x \left(2^{\sin(x)-x} + c_1\right)$$

Verified OK.

6.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(y \ln(2) + 2^{\sin(x)}(\cos(x) - 1)\ln(2)\right)\mathrm{d}x$$
$$\left(-y \ln(2) - 2^{\sin(x)}(\cos(x) - 1)\ln(2)\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y \ln(2) - 2^{\sin(x)}(\cos(x) - 1)\ln(2)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y \ln(2) - 2^{\sin(x)}(\cos(x) - 1)\ln(2)\right) \\ &= -\ln(2)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-\ln(2)) - (0)) \\ &= -\ln(2) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\ln(2) \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(2)x} \\ &= 2^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= 2^{-x}\left(-y\ln(2) - 2^{\sin(x)}(\cos(x) - 1)\ln(2)\right) \\ &= -\ln(2)\left(2^{\sin(x)}(\cos(x) - 1) + y\right)2^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= 2^{-x}(1) \\ &= 2^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\ln(2)\left(2^{\sin(x)}(\cos(x) - 1) + y\right)2^{-x}\right) + \left(2^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\ln(2) \left(2^{\sin(x)}(\cos(x) - 1) + y\right) 2^{-x} \, \mathrm{d}x$$

$$\phi = 2^{-x}\left(-2^{\sin(x)} + y\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2^{-x}$. Therefore equation (4) becomes

$$2^{-x} = 2^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2^{-x}\left(-2^{\sin(x)} + y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2^{-x}\left(-2^{\sin(x)} + y\right)$$

The solution becomes

$$y = \left(2^{\sin(x)} 2^{-x} + c_1\right) 2^{x}$$

Summary

The solution(s) found are the following

$$y = \left(2^{\sin(x)}2^{-x} + c_1\right) 2^x \tag{1}$$

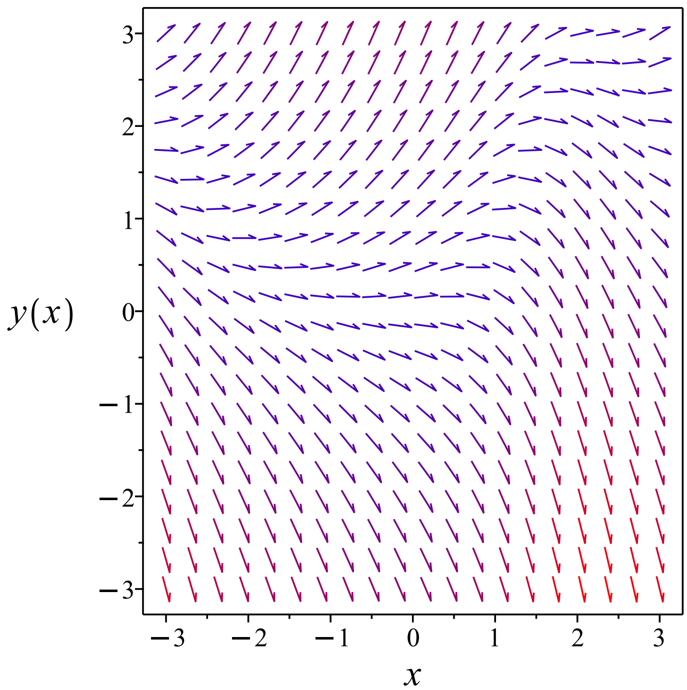


Figure 240: Slope field plot

Verification of solutions

$$y = \left(2^{\sin(x)}2^{-x} + c_1\right) 2^x$$

Verified OK.

6.15.4 Maple step by step solution

Let's solve

$y' - y\ln(2) = 2^{\sin(x)}(\cos(x) - 1)\ln(2)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\ln(2) + 2^{\sin(x)}(\cos(x) - 1)\ln(2)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\ln(2) = 2^{\sin(x)}(\cos(x) - 1)\ln(2)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - y\ln(2)\right) = \mu(x)\,2^{\sin(x)}(\cos(x) - 1)\ln(2)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - y\ln(2)\right) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\ln(2)$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{2^x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\,2^{\sin(x)}(\cos(x) - 1)\ln(2)\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,2^{\sin(x)}(\cos(x) - 1)\ln(2)\,dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)2^{\sin(x)}(\cos(x)-1)\ln(2)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{2^x}$

  $y = 2^x\left(\int \frac{2^{\sin(x)}(\cos(x)-1)\ln(2)}{2^x}dx + c_1\right)$
- Evaluate the integrals on the rhs

  $$y = 2^x\left(\frac{\tan\left(\frac{x}{2}\right)^2 \mathrm{e}^{\frac{2\tan\left(\frac{x}{2}\right)\ln(2)}{1+\tan\left(\frac{x}{2}\right)^2}} + \mathrm{e}^{\frac{2\tan\left(\frac{x}{2}\right)\ln(2)}{1+\tan\left(\frac{x}{2}\right)^2}}}{\mathrm{e}^{\ln(2)x}\left(1+\tan\left(\frac{x}{2}\right)^2\right)} + c_1\right)$$
- Simplify

  $y = c_1 2^x + 2^{\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)-y(x)*ln(2)=2^(sin(x))*(cos(x)-1)*ln(2),y(x), singsol=all)
```

$$y(x) = 2^x c_1 + 2^{\sin(x)}$$

✓ Solution by Mathematica

Time used: 0.368 (sec). Leaf size: 16

```
DSolve[y'[x]-y[x]*Log[2]==2^(Sin[x])*(Cos[x]-1)*Log[2],y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to 2^{\sin(x)} + c_1 2^x$$

6.16 problem 149

Internal problem ID [15042]
Internal file name [`OUTPUT/15042_Sunday_April_21_2024_01_21_15_PM_82957208/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 149.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = -2\,\mathrm{e}^{-x}}$$

With initial conditions

$$[y(\infty) = 0]$$

6.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = -2\,\mathrm{e}^{-x}$$

Hence the ode is

$$y' - y = -2\,e^{-x}$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = \infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

6.16.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-2\,e^{-x}\right) \\ \frac{d}{dx}\left(e^{-x}y\right) &= \left(e^{-x}\right)\left(-2\,e^{-x}\right) \\ d\left(e^{-x}y\right) &= \left(-2\,e^{-2x}\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x}y &= \int -2\,e^{-2x}\,dx \\ e^{-x}y &= e^{-2x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{-2x}e^{x} + e^{x}c_1$$

which simplifies to

$$y = e^{-x} + e^{x}c_1$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \text{signum}\,(c_1)\,\infty$$

Solving for c_1 gives

$$c_1 = -\left(-y + \mathrm{e}^{-x}\right) \mathrm{e}^{-x}$$

Using given initial conditions results in $c_1 = 0$ Hence the solution is

$$y = \mathrm{e}^{-x}$$

Therefore the solution is

$$y = \mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-x} \tag{1}$$

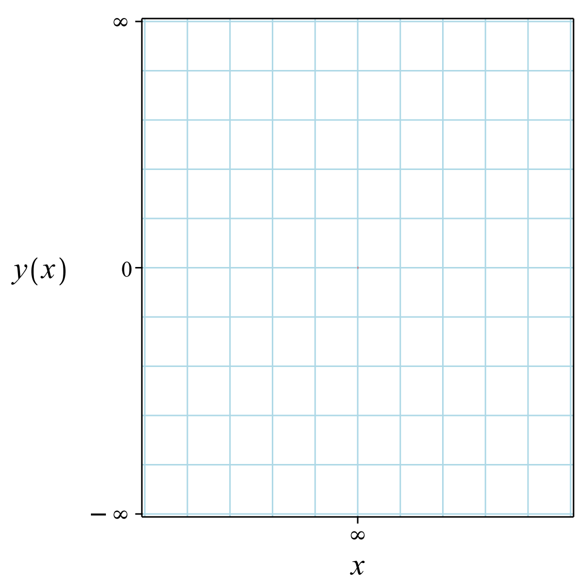


Figure 241: Solution plot

Verification of solutions

$$y = \mathrm{e}^{-x}$$

Verified OK.

6.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y - 2\,\mathrm{e}^{-x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 189: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{x}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-x} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = y - 2\,e^{-x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -e^{-x} y \\
S_y &= e^{-x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2\,\mathrm{e}^{-2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2\,\mathrm{e}^{-2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \mathrm{e}^{-2R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = \mathrm{e}^{-2x} + c_1$$

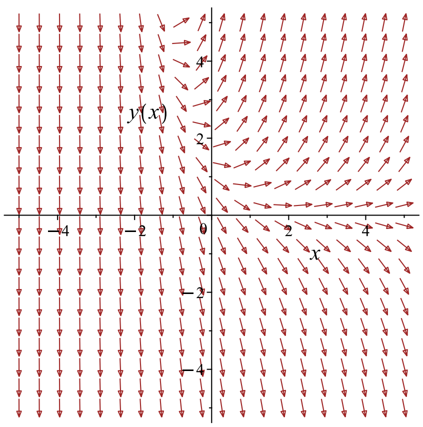
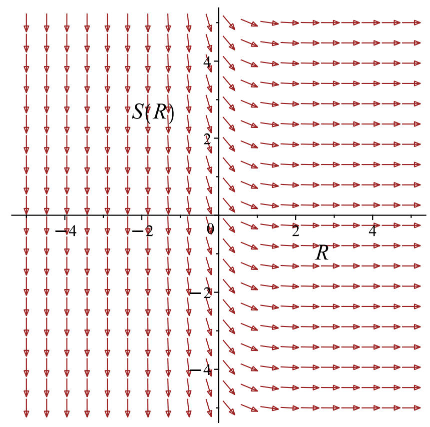
Which simplifies to

$$\mathrm{e}^{-x} y = \mathrm{e}^{-2x} + c_1$$

Which gives

$$y = \left(\mathrm{e}^{-2x} + c_1\right) \mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = y - 2\,\mathrm{e}^{-x}$

</td><td>$R = x$
$S = \mathrm{e}^{-x} y$</td><td>$\frac{dS}{dR} = -2\,\mathrm{e}^{-2R}$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{signum}\,(c_1)\,\infty$$

Solving for c_1 gives

$$c_1 = \mathrm{e}^{-x} y - \mathrm{e}^{-2x}$$

Using given initial conditions results in $c_1 = 0$ Hence the solution is

$$y = \mathrm{e}^{-2x} \mathrm{e}^{x}$$

Therefore the solution is

$$y = \mathrm{e}^{-2x} \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x} \mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-2x} \mathrm{e}^{x}$$

Verified OK.

6.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(y - 2\,\mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(-y + 2\,\mathrm{e}^{-x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y + 2\,\mathrm{e}^{-x}\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y + 2\,\mathrm{e}^{-x}\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x}\\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-x}\left(-y + 2\,\mathrm{e}^{-x}\right)\\ &= -\mathrm{e}^{-x}\left(y - 2\,\mathrm{e}^{-x}\right)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x}\left(y - 2\,\mathrm{e}^{-x}\right)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{-x}\left(y - 2\,\mathrm{e}^{-x}\right)\mathrm{d}x \\
\phi &= \mathrm{e}^{-x}y - \mathrm{e}^{-2x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{-x}y - e^{-2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{-x}y - e^{-2x}$$

The solution becomes

$$y = \left(e^{-2x} + c_1\right) e^{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \operatorname{signum}(c_1) \infty$$

Solving for c_1 gives

$$c_1 = e^{-x}y - e^{-2x}$$

Using given initial conditions results in $c_1 = 0$ Hence the solution is

$$y = e^{-2x} e^{x}$$

Therefore the solution is

$$y = e^{-2x} e^{x}$$

Summary
The solution(s) found are the following

$$y = e^{-2x} e^{x} \tag{1}$$

Verification of solutions

$$y = e^{-2x} e^{x}$$

Verified OK.

6.16.5 Maple step by step solution

Let's solve

$[y' - y = -2\,\mathrm{e}^{-x}, y(\infty) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y - 2\,\mathrm{e}^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = -2\,\mathrm{e}^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = -2\mu(x)\,\mathrm{e}^{-x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -2\mu(x)\,\mathrm{e}^{-x}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -2\mu(x)\,\mathrm{e}^{-x}dx + c_1$

- Solve for y

  $y = \frac{\int -2\mu(x)\mathrm{e}^{-x}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int -2\left(\mathrm{e}^{-x}\right)^2dx + c_1}{\mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\left(\mathrm{e}^{-x}\right)^2 + c_1}{\mathrm{e}^{-x}}$

- Simplify

  $y = \mathrm{e}^{-x} + \mathrm{e}^{x}c_1$

- Use initial condition $y(\infty) = 0$

  $$0 = c_1 \infty$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 8

```
dsolve([diff(y(x),x)-y(x)=-2*exp(-x),y(infinity) = 0],y(x), singsol=all)
```

$$y(x) = e^{-x}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 10

```
DSolve[{y'[x]-y[x]==-2*Exp[-x],{y[Infinity]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}$$

6.17 problem 150

Internal problem ID [15043]
Internal file name [`OUTPUT/15043_Sunday_April_21_2024_01_21_16_PM_2029673/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 150.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

Unable to solve or complete the solution.

$$\boxed{y' \sin(x) - y \cos(x) = -\frac{\sin(x)^2}{x^2}}$$

With initial conditions

$$[y(\infty) = 0]$$

6.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\cot(x) \\ q(x) &= -\frac{\sin(x)}{x^2} \end{aligned}$$

Hence the ode is

$$y' - y\cot(x) = -\frac{\sin(x)}{x^2}$$

The domain of $p(x) = -\cot(x)$ is

$$\{x < \pi\_Z796 \lor \pi\_Z796 < x\}$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = -\frac{\sin(x)}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

6.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)} \end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{\sin(x)}{x^2}\right) \\ \frac{d}{dx}(\csc(x)\, y) &= (\csc(x))\left(-\frac{\sin(x)}{x^2}\right) \\ d(\csc(x)\, y) &= \left(-\frac{1}{x^2}\right) dx \end{aligned}$$

Integrating gives

$$\csc(x)\, y = \int -\frac{1}{x^2}\, \mathrm{d}x$$
$$\csc(x)\, y = \frac{1}{x} + c_1$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = \frac{\sin(x)}{x} + \sin(x)\, c_1$$

which simplifies to

$$y = \sin(x)\left(\frac{1}{x} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -|c_1|..|c_1|$$

Unable to solve for constant of integration. Verification of solutions N/A

6.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-\cos(x)\, y\, x^2 + \sin(x)^2}{\sin(x)\, x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 192: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \sin(x) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-\cos(x)\, y\, x^2 + \sin(x)^2}{\sin(x)\, x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x)\cot(x)\, y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = \frac{1}{x} + c_1$$

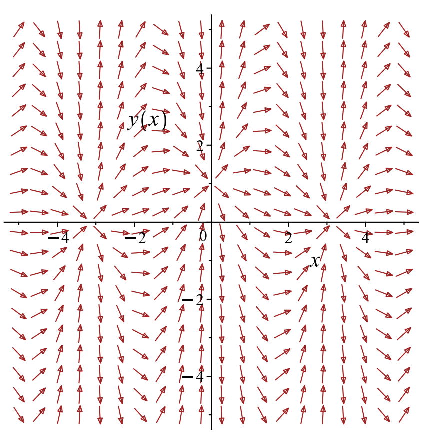
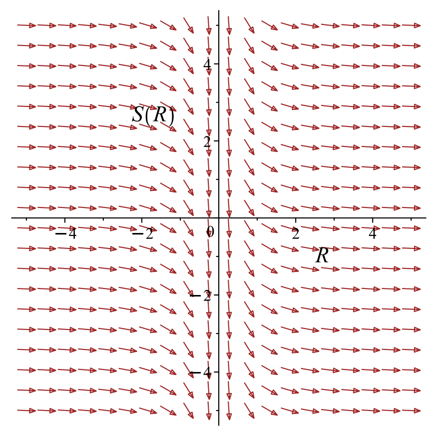
Which simplifies to

$$\csc(x)\, y = \frac{1}{x} + c_1$$

Which gives

$$y = \frac{c_1 x + 1}{\csc(x)\, x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-\cos(x) y\, x^2 + \sin(x)^2}{\sin(x) x^2}$ 	$R = x$ $S = \csc(x)\, y$	$\frac{dS}{dR} = -\frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = -|c_1|..|c_1|
$$

Unable to solve for constant of integration. Verification of solutions N/A

6.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$(\sin(x))\,\mathrm{d}y = \left(\cos(x)\,y - \frac{\sin(x)^2}{x^2}\right)\mathrm{d}x$$

$$\left(-\cos(x)\,y + \frac{\sin(x)^2}{x^2}\right)\mathrm{d}x + (\sin(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\cos(x)\,y + \frac{\sin(x)^2}{x^2} \\ N(x,y) &= \sin(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\cos(x)\,y + \frac{\sin(x)^2}{x^2}\right) \\ &= -\cos(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)) \\ &= \cos(x) \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \csc(x)\left((-\cos(x)) - (\cos(x))\right) \\ &= -2\cot(x) \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -2\cot(x)\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\sin(x))}\\ &= \csc(x)^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \csc(x)^2\left(-\cos(x)\,y+\frac{\sin(x)^2}{x^2}\right)\\ &= -\csc(x)\cot(x)\,y+\frac{1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \csc(x)^2(\sin(x))\\ &= \csc(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\csc(x)\cot(x)\,y+\frac{1}{x^2}\right)+(\csc(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int-\csc(x)\cot(x)\,y+\frac{1}{x^2}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{1}{x}+\csc(x)\,y+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \csc(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc(x)$. Therefore equation (4) becomes

$$\csc(x) = \csc(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{x} + \csc(x)\,y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{x} + \csc(x)\,y$$

The solution becomes

$$y = \frac{c_1 x + 1}{\csc(x)\,x}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -|c_1|..|c_1|$$

Unable to solve for constant of integration. Verification of solutions N/A

6.17.5 Maple step by step solution

Let's solve

$\left[y' \sin(x) - y\cos(x) = -\frac{\sin(x)^2}{x^2}, y(\infty) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y\cos(x)}{\sin(x)} - \frac{\sin(x)}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y\cos(x)}{\sin(x)} = -\frac{\sin(x)}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y\cos(x)}{\sin(x)}\right) = -\frac{\mu(x)\sin(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y\cos(x)}{\sin(x)}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)\cos(x)}{\sin(x)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int -\frac{\mu(x)\sin(x)}{x^2}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)\sin(x)}{x^2}dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)\sin(x)}{x^2}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\sin(x)}$

  $y = \sin(x)\left(\int -\frac{1}{x^2}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \sin(x)\left(\frac{1}{x} + c_1\right)$

- Simplify

  $y = \frac{\sin(x)(c_1x+1)}{x}$
- Use initial condition $y(\infty) = 0$

  $0 = \mathit{undefined}(c_1\infty + 1)$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)*sin(x)-y(x)*cos(x)=-sin(x)^2/x^2,y(infinity) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)}{x}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 19

```
DSolve[{y'[x]*Sin[x]-y[x]*Cos[x]==-Sin[x]^2/x^2,{y[Infinity]==0}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sin(x)\left(\text{Interval}[\{0, \text{Indeterminate}\}, \{\text{Indeterminate}, 0\}] + \frac{1}{x}\right)$$

6.18 problem 151

Internal problem ID [15044]
Internal file name [`OUTPUT/15044_Sunday_April_21_2024_01_21_19_PM_5426682/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 151.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^2y'\cos\left(\frac{1}{x}\right)-y\sin\left(\frac{1}{x}\right)=-1}$$

With initial conditions

$$[y(\infty)=1]$$

6.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{\tan\left(\frac{1}{x}\right)}{x^2}\\ q(x) &= -\frac{\sec\left(\frac{1}{x}\right)}{x^2}\end{aligned}$$

Hence the ode is

$$y' - \frac{\tan\left(\frac{1}{x}\right)y}{x^2} = -\frac{\sec\left(\frac{1}{x}\right)}{x^2}$$

The domain of $p(x) = -\frac{\tan\left(\frac{1}{x}\right)}{x^2}$ is

$$\left\{-\infty \leq x < 0, 0 < x < \frac{2}{\pi\left(1+2\_Z818\right)}, \frac{2}{\pi\left(1+2\_Z818\right)} < x \leq \infty\right\}$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = -\frac{\sec\left(\frac{1}{x}\right)}{x^2}$ is

$$\left\{-\infty \leq x < 0, 0 < x < \frac{2}{\pi\left(1+2\_Z818\right)}, \frac{2}{\pi\left(1+2\_Z818\right)} < x \leq \infty\right\}$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

6.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{\tan\left(\frac{1}{x}\right)}{x^2}dx}\\ &= \frac{1}{\cos\left(\frac{1}{x}\right)}\end{aligned}$$

Which simplifies to

$$\mu = \sec\left(\frac{1}{x}\right)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{\sec\left(\frac{1}{x}\right)}{x^2}\right)$$

$$\frac{d}{dx}\left(\sec\left(\frac{1}{x}\right)y\right) = \left(\sec\left(\frac{1}{x}\right)\right)\left(-\frac{\sec\left(\frac{1}{x}\right)}{x^2}\right)$$

$$d\left(\sec\left(\frac{1}{x}\right)y\right) = \left(-\frac{\sec\left(\frac{1}{x}\right)^2}{x^2}\right) dx$$

Integrating gives

$$\sec\left(\frac{1}{x}\right)y = \int -\frac{\sec\left(\frac{1}{x}\right)^2}{x^2}\, dx$$

$$\sec\left(\frac{1}{x}\right)y = \tan\left(\frac{1}{x}\right) + c_1$$

Dividing both sides by the integrating factor $\mu = \sec\left(\frac{1}{x}\right)$ results in

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + c_1\cos\left(\frac{1}{x}\right)$$

which simplifies to

$$y = c_1\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) \tag{1}$$

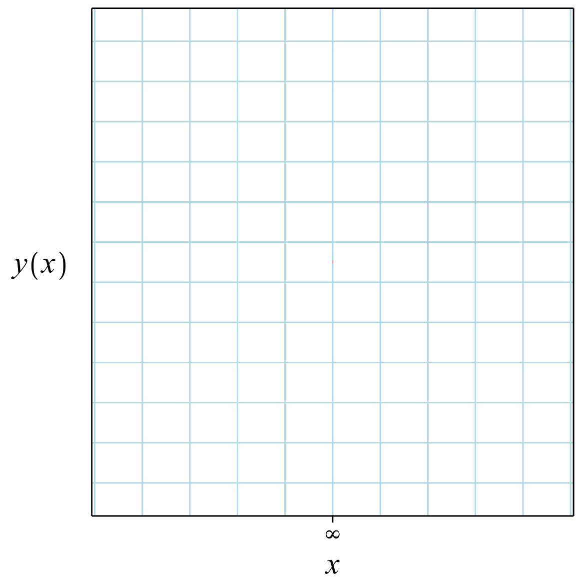


Figure 242: Solution plot

Verification of solutions

$$y = \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

Verified OK.

6.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y\sin\left(\frac{1}{x}\right) - 1}{x^2\cos\left(\frac{1}{x}\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 195: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \cos\left(\frac{1}{x}\right) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos\left(\frac{1}{x}\right)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos\left(\frac{1}{x}\right)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y \sin\left(\frac{1}{x}\right) - 1}{x^2 \cos\left(\frac{1}{x}\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) y}{x^2} \\ S_y &= \sec\left(\frac{1}{x}\right) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sec\left(\frac{1}{x}\right)^2}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\sec\left(\frac{1}{R}\right)^2}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan\left(\frac{1}{R}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sec\left(\frac{1}{x}\right) y = \tan\left(\frac{1}{x}\right) + c_1$$

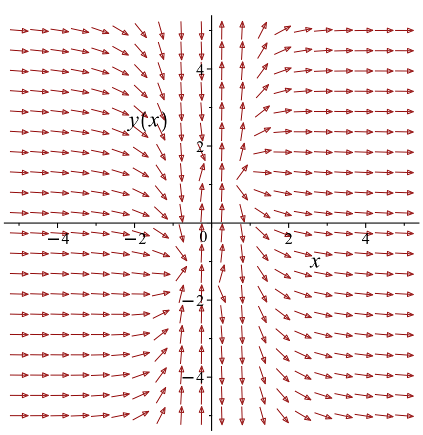
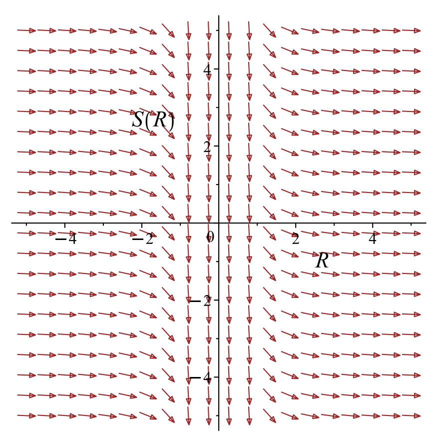
Which simplifies to

$$\sec\left(\frac{1}{x}\right) y = \tan\left(\frac{1}{x}\right) + c_1$$

Which gives

$$y = \frac{\tan\left(\frac{1}{x}\right) + c_1}{\sec\left(\frac{1}{x}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y \sin\left(\frac{1}{x}\right) - 1}{x^2 \cos\left(\frac{1}{x}\right)}$ 	$R = x$ $S = \sec\left(\frac{1}{x}\right) y$	$\frac{dS}{dR} = -\frac{\sec\left(\frac{1}{R}\right)^2}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) \tag{1}$$

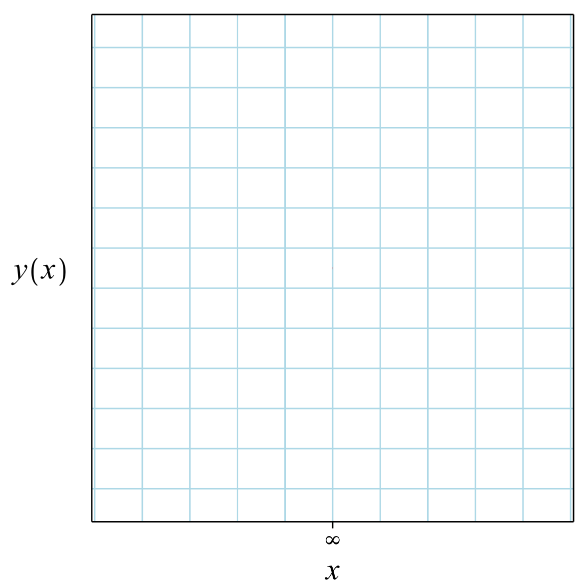


Figure 243: Solution plot

Verification of solutions

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

Verified OK.

6.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2\cos\left(\frac{1}{x}\right)\right)\mathrm{d}y&=\left(y\sin\left(\frac{1}{x}\right)-1\right)\mathrm{d}x\\ \left(-y\sin\left(\frac{1}{x}\right)+1\right)\mathrm{d}x+\left(x^2\cos\left(\frac{1}{x}\right)\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -y\sin\left(\frac{1}{x}\right) + 1$$
$$N(x,y) = x^2\cos\left(\frac{1}{x}\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\sin\left(\frac{1}{x}\right) + 1\right)\\ &= -\sin\left(\frac{1}{x}\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\cos\left(\frac{1}{x}\right)\right)\\ &= 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\sec\left(\frac{1}{x}\right)}{x^2}\left(\left(-\sin\left(\frac{1}{x}\right)\right) - \left(2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)\right)\right)\\ &= \frac{-2\tan\left(\frac{1}{x}\right) - 2x}{x^2}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{-2\tan\left(\frac{1}{x}\right)-2x}{x^2}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln\left(\frac{1}{x}\right)-2\ln\left(\cos\left(\frac{1}{x}\right)\right)} \\
&= \frac{1}{\cos\left(\frac{1}{x}\right)^{2} x^{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\cos\left(\frac{1}{x}\right)^{2} x^{2}}\left(-y\sin\left(\frac{1}{x}\right)+1\right) \\
&= \frac{\sec\left(\frac{1}{x}\right)^{2}\left(-y\sin\left(\frac{1}{x}\right)+1\right)}{x^{2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\cos\left(\frac{1}{x}\right)^{2} x^{2}}\left(x^{2}\cos\left(\frac{1}{x}\right)\right) \\
&= \sec\left(\frac{1}{x}\right)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\sec\left(\frac{1}{x}\right)^{2}\left(-y\sin\left(\frac{1}{x}\right)+1\right)}{x^{2}}\right) + \left(\sec\left(\frac{1}{x}\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x
$$

$$
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{\sec\left(\frac{1}{x}\right)^{2}\left(-y\sin\left(\frac{1}{x}\right)+1\right)}{x^{2}}\,\mathrm{d}x
$$

$$
\phi = -\tan\left(\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right)y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sec\left(\frac{1}{x}\right) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec\left(\frac{1}{x}\right)$. Therefore equation (4) becomes

$$\sec\left(\frac{1}{x}\right) = \sec\left(\frac{1}{x}\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\tan\left(\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right) y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\tan\left(\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right) y$$

The solution becomes

$$y = \frac{\tan\left(\frac{1}{x}\right) + c_1}{\sec\left(\frac{1}{x}\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) \tag{1}$$

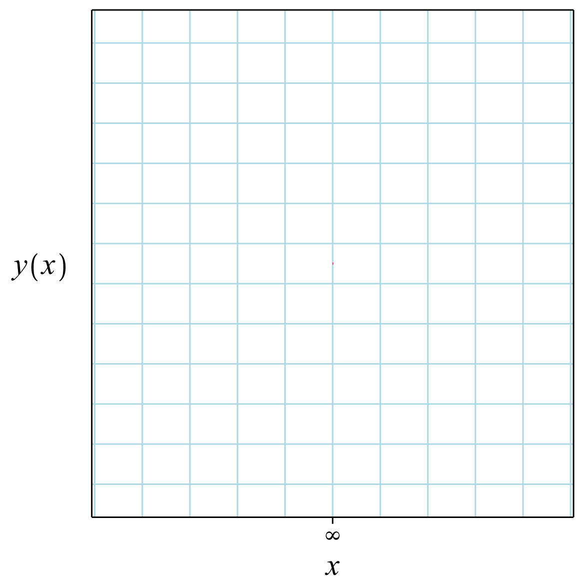


Figure 244: Solution plot

Verification of solutions

$$y = \cos\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

Verified OK.

6.18.5 Maple step by step solution

Let's solve

$\left[x^2 y' \cos\left(\frac{1}{x}\right) - y \sin\left(\frac{1}{x}\right) = -1, y(\infty) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = \frac{\sin\left(\frac{1}{x}\right) y}{x^2 \cos\left(\frac{1}{x}\right)} - \frac{1}{x^2 \cos\left(\frac{1}{x}\right)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{\sin\left(\frac{1}{x}\right) y}{x^2 \cos\left(\frac{1}{x}\right)} = -\frac{1}{x^2 \cos\left(\frac{1}{x}\right)}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{\sin\left(\frac{1}{x}\right) y}{x^2 \cos\left(\frac{1}{x}\right)}\right) = -\frac{\mu(x)}{x^2 \cos\left(\frac{1}{x}\right)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x) \left(y' - \frac{\sin\left(\frac{1}{x}\right) y}{x^2 \cos\left(\frac{1}{x}\right)}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x) \sin\left(\frac{1}{x}\right)}{x^2 \cos\left(\frac{1}{x}\right)}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\cos\left(\frac{1}{x}\right)}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{\mu(x)}{x^2 \cos\left(\frac{1}{x}\right)} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int -\frac{\mu(x)}{x^2 \cos\left(\frac{1}{x}\right)} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x)}{x^2 \cos\left(\frac{1}{x}\right)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\cos\left(\frac{1}{x}\right)}$

$$y = \cos\left(\frac{1}{x}\right) \left(\int -\frac{1}{\cos\left(\frac{1}{x}\right)^2 x^2} dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = \cos\left(\frac{1}{x}\right) \left(\tan\left(\frac{1}{x}\right) + c_1\right)$$

- Simplify

$$y = c_1 \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

- Use initial condition $y(\infty) = 1$

$$1 = c_1$$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$

- Solution to the IVP

  $y = \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x)*cos(1/x)-y(x)*sin(1/x)=-1,y(infinity) = 1],y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 14

```
DSolve[{x^2*y'[x]*Cos[1/x]-y[x]*Sin[1/x]==-1,{y[Infinity]==1}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

6.19 problem 152

Internal problem ID [15045]
Internal file name [`OUTPUT/15045_Sunday_April_21_2024_01_21_21_PM_53792717/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 152.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

Unable to solve or complete the solution.

$$\boxed{2y'x - y = 1 - \frac{2}{\sqrt{x}}}$$

With initial conditions

$$[y(\infty) = -1]$$

6.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= \frac{\sqrt{x}-2}{2x^{\frac{3}{2}}} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{2x} = \frac{\sqrt{x}-2}{2x^{\frac{3}{2}}}$$

The domain of $p(x) = -\frac{1}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = \frac{\sqrt{x}-2}{2x^{\frac{3}{2}}}$ is

$$\{0 < x\}$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

6.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{2x}dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sqrt{x}-2}{2x^{\frac{3}{2}}}\right) \\ \frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{1}{\sqrt{x}}\right)\left(\frac{\sqrt{x}-2}{2x^{\frac{3}{2}}}\right) \\ d\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{\sqrt{x}-2}{2x^2}\right) dx \end{aligned}$$

Integrating gives

$$\frac{y}{\sqrt{x}} = \int \frac{\sqrt{x}-2}{2x^2}\, \mathrm{d}x$$
$$\frac{y}{\sqrt{x}} = -\frac{1}{\sqrt{x}} + \frac{1}{x} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = \sqrt{x}\left(-\frac{1}{\sqrt{x}} + \frac{1}{x}\right) + c_1\sqrt{x}$$

which simplifies to

$$y = \frac{c_1 x^{\frac{3}{2}} + \sqrt{x} - x}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \operatorname{signum}(c_1)\,\infty$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

6.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y\sqrt{x} + \sqrt{x} - 2}{2x^{\frac{3}{2}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 198: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sqrt{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\sqrt{x} + \sqrt{x} - 2}{2x^{\frac{3}{2}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2x^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x - 2\sqrt{x}}{2x^{\frac{5}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R - 2\sqrt{R}}{2R^{\frac{5}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{\sqrt{R}} + \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x}} = -\frac{1}{\sqrt{x}} + \frac{1}{x} + c_1$$

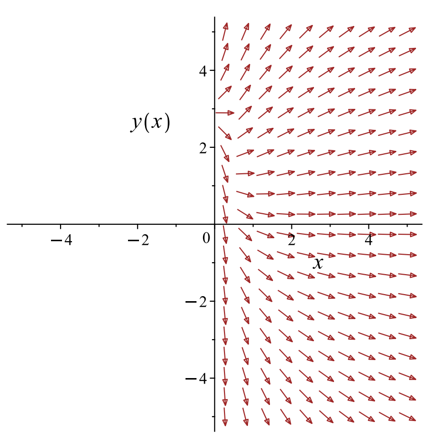
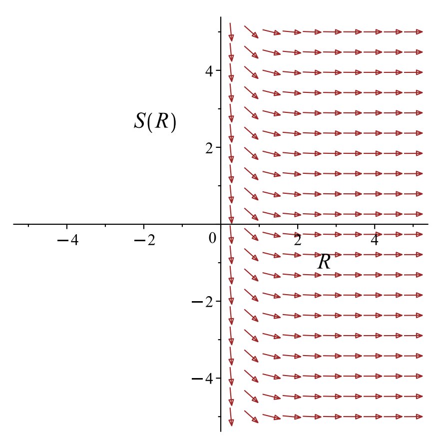
Which simplifies to

$$\frac{y}{\sqrt{x}} = -\frac{1}{\sqrt{x}} + \frac{1}{x} + c_1$$

Which gives

$$y = \frac{c_1 x^{\frac{3}{2}} + \sqrt{x} - x}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\sqrt{x}+\sqrt{x}-2}{2x^{\frac{3}{2}}}$ 	$R = x$ $S = \frac{y}{\sqrt{x}}$	$\frac{dS}{dR} = \frac{R-2\sqrt{R}}{2R^{\frac{5}{2}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \operatorname{signum}(c_1)\,\infty$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

6.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(2x)\, \mathrm{d}y &= \left(y + 1 - \frac{2}{\sqrt{x}}\right) \mathrm{d}x \\
\left(-y - 1 + \frac{2}{\sqrt{x}}\right) \mathrm{d}x + (2x)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -y - 1 + \frac{2}{\sqrt{x}} \\
N(x,y) &= 2x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - 1 + \frac{2}{\sqrt{x}}\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x) \\
&= 2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2x}((-1) - (2)) \\
&= -\frac{3}{2x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{3}{2x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(x)}{2}} \\ &= \frac{1}{x^{\frac{3}{2}}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^{\frac{3}{2}}}\left(-y-1+\frac{2}{\sqrt{x}}\right) \\ &= \frac{2+(-y-1)\sqrt{x}}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^{\frac{3}{2}}}(2x) \\ &= \frac{2}{\sqrt{x}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2+(-y-1)\sqrt{x}}{x^2}\right)+\left(\frac{2}{\sqrt{x}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{2+(-y-1)\sqrt{x}}{x^2}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{(2y+2)\,x-2\sqrt{x}}{x^{\frac{3}{2}}}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2}{\sqrt{x}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{\sqrt{x}}$. Therefore equation (4) becomes

$$\frac{2}{\sqrt{x}} = \frac{2}{\sqrt{x}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2y+2)\,x - 2\sqrt{x}}{x^{\frac{3}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2y+2)\,x - 2\sqrt{x}}{x^{\frac{3}{2}}}$$

The solution becomes

$$y = \frac{c_1 x^{\frac{3}{2}} + 2\sqrt{x} - 2x}{2x}$$

Initial conditions are used to solve for c_1. Substituting $x = \infty$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \operatorname{signum}(c_1)\,\infty$$

Warning: Unable to solve for constant of integration.

Verification of solutions N/A

6.19.5 Maple step by step solution

Let's solve

$\left[2y'x-y=1-\frac{2}{\sqrt{x}}, y(\infty)=-1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{y}{2x}+\frac{\sqrt{x}-2}{2x^{\frac{3}{2}}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{y}{2x}=\frac{\sqrt{x}-2}{2x^{\frac{3}{2}}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-\frac{y}{2x}\right)=\frac{\mu(x)(\sqrt{x}-2)}{2x^{\frac{3}{2}}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{y}{2x}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x)=\frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)(\sqrt{x}-2)}{2x^{\frac{3}{2}}}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)(\sqrt{x}-2)}{2x^{\frac{3}{2}}}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)(\sqrt{x}-2)}{2x^{\frac{3}{2}}}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{\sqrt{x}}$

  $y=\sqrt{x}\left(\int\frac{\sqrt{x}-2}{2x^2}dx+c_1\right)$

- Evaluate the integrals on the rhs

$$y = \sqrt{x}\left(-\frac{1}{\sqrt{x}} + \frac{1}{x} + c_1\right)$$

- Simplify

$$y = \frac{c_1 x^{\frac{3}{2}} + \sqrt{x} - x}{x}$$

- Use initial condition $y(\infty) = -1$

$$-1 = 0$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([2*x*diff(y(x),x)-y(x)=1-2/sqrt(x),y(infinity) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{x} - 1}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 12

```
DSolve[{2*x*y'[x]-y[x]==1-2/Sqrt[x],{y[Infinity]==-1}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{\sqrt{x}} - 1$$

6.20 problem 153

Internal problem ID [15046]
Internal file name [`OUTPUT/15046_Sunday_April_21_2024_01_21_22_PM_75243278/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 153.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y'x + y = (x^2 + 1)\,e^x}$$

With initial conditions

$$[y(-\infty) = 1]$$

6.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{e^x(x^2 + 1)}{2x}$$

Hence the ode is

$$y' + \frac{y}{2x} = \frac{e^{x}(x^2+1)}{2x}$$

The domain of $p(x) = \frac{1}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -\infty$ is inside this domain. The domain of $q(x) = \frac{e^{x}(x^2+1)}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -\infty$ is also inside this domain. Hence solution exists and is unique.

6.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2x}dx} \\ &= \sqrt{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{e^{x}(x^2+1)}{2x}\right) \\ \frac{d}{dx}(y\sqrt{x}) &= (\sqrt{x})\left(\frac{e^{x}(x^2+1)}{2x}\right) \\ d(y\sqrt{x}) &= \left(\frac{e^{x}(x^2+1)}{2\sqrt{x}}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\sqrt{x} &= \int \frac{e^{x}(x^2+1)}{2\sqrt{x}} dx \\ y\sqrt{x} &= \frac{x^{\frac{3}{2}}e^{x}}{2} - \frac{3\sqrt{x}\,e^{x}}{4} + \frac{7\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{8} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$y = \frac{\frac{x^{\frac{3}{2}}e^{x}}{2} - \frac{3\sqrt{x}\,e^{x}}{4} + \frac{7\sqrt{\pi}\,\operatorname{erfi}(\sqrt{x})}{8}}{\sqrt{x}} + \frac{c_1}{\sqrt{x}}$$

which simplifies to

$$y = \frac{4x^{\frac{3}{2}} \mathrm{e}^{x} - 6\sqrt{x}\, \mathrm{e}^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 8c_1}{8\sqrt{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = -\infty$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 0$$

This shows that no solution exist.

Summary

The solution(s) found are the following

$$y = \frac{4x^{\frac{3}{2}} \mathrm{e}^{x} - 6\sqrt{x}\, \mathrm{e}^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 8c_1}{8\sqrt{x}}$$

Verification of solutions

$$y = \frac{4x^{\frac{3}{2}} \mathrm{e}^{x} - 6\sqrt{x}\, \mathrm{e}^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 8c_1}{8\sqrt{x}}$$

Warning, solution could not be verified

6.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x^2 \mathrm{e}^{x} + \mathrm{e}^{x} - y}{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 201: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{\sqrt{x}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x}}} dy \end{aligned}$$

Which results in

$$S = y\sqrt{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 e^x + e^x - y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{2\sqrt{x}} \\ S_y &= \sqrt{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^x (x^2 + 1)}{2\sqrt{x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^R (R^2 + 1)}{2\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\mathrm{e}^{R} R^{\frac{3}{2}}}{2} - \frac{3\,\mathrm{e}^{R}\sqrt{R}}{4} + \frac{7\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{R}\right)}{8} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sqrt{x} = \frac{x^{\frac{3}{2}}\mathrm{e}^{x}}{2} - \frac{3\sqrt{x}\,\mathrm{e}^{x}}{4} + \frac{7\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{8} + c_1$$

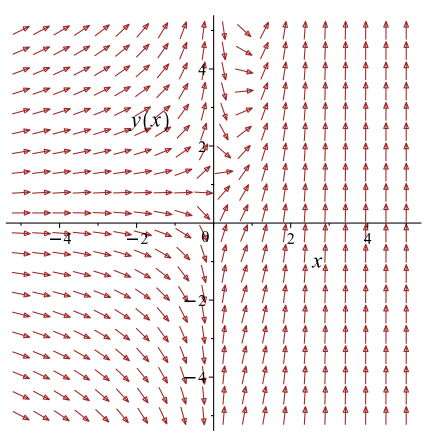
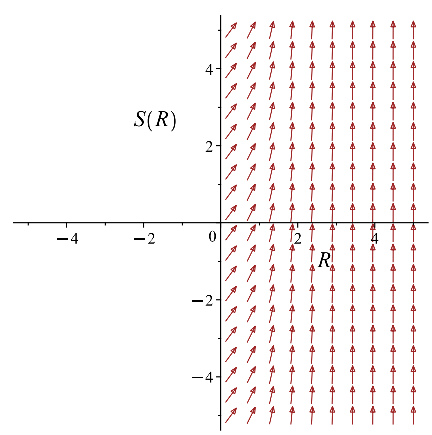
Which simplifies to

$$y\sqrt{x} = \frac{x^{\frac{3}{2}}\mathrm{e}^{x}}{2} - \frac{3\sqrt{x}\,\mathrm{e}^{x}}{4} + \frac{7\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right)}{8} + c_1$$

Which gives

$$y = \frac{4x^{\frac{3}{2}}\mathrm{e}^{x} - 6\sqrt{x}\,\mathrm{e}^{x} + 7\sqrt{\pi}\,\operatorname{erfi}\left(\sqrt{x}\right) + 8c_1}{8\sqrt{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2\mathrm{e}^{x}+\mathrm{e}^{x}-y}{2x}$ 	$R = x$ $S = y\sqrt{x}$	$\frac{dS}{dR} = \frac{\mathrm{e}^{R}(R^2+1)}{2\sqrt{R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -\infty$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 0$$

This shows that no solution exist.

Summary

The solution(s) found are the following

$$y = \frac{4x^{\frac{3}{2}} e^{x} - 6\sqrt{x}\, e^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 8c_1}{8\sqrt{x}}$$

Verification of solutions

$$y = \frac{4x^{\frac{3}{2}} e^{x} - 6\sqrt{x}\, e^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 8c_1}{8\sqrt{x}}$$

Warning, solution could not be verified

6.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x)\,\mathrm{d}y &= \left(-y+\left(x^2+1\right)\mathrm{e}^x\right)\mathrm{d}x\\ \left(y-\left(x^2+1\right)\mathrm{e}^x\right)\mathrm{d}x+(2x)\,\mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-\left(x^2+1\right)\mathrm{e}^x\\ N(x,y) &= 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y-\left(x^2+1\right)\mathrm{e}^x\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x}((1)-(2))\\ &= -\frac{1}{2x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{1}{2x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(x)}{2}}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{x}}\left(y - \left(x^2+1\right)\mathrm{e}^x\right)\\ &= -\frac{x^2\mathrm{e}^x + \mathrm{e}^x - y}{\sqrt{x}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{\sqrt{x}}(2x)\\ &= 2\sqrt{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{x^2\mathrm{e}^x + \mathrm{e}^x - y}{\sqrt{x}}\right) + \left(2\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x^2 \mathrm{e}^x + \mathrm{e}^x - y}{\sqrt{x}} \, \mathrm{d}x$$

$$\phi = -x^{\frac{3}{2}} \mathrm{e}^x - \frac{7\sqrt{\pi}\, \mathrm{erfi}\left(\sqrt{x}\right)}{4} + 2\left(y + \frac{3\,\mathrm{e}^x}{4}\right)\sqrt{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\sqrt{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\sqrt{x}$. Therefore equation (4) becomes

$$2\sqrt{x} = 2\sqrt{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^{\frac{3}{2}} \mathrm{e}^x - \frac{7\sqrt{\pi}\, \mathrm{erfi}\left(\sqrt{x}\right)}{4} + 2\left(y + \frac{3\,\mathrm{e}^x}{4}\right)\sqrt{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^{\frac{3}{2}} \mathrm{e}^x - \frac{7\sqrt{\pi}\, \mathrm{erfi}\left(\sqrt{x}\right)}{4} + 2\left(y + \frac{3\,\mathrm{e}^x}{4}\right)\sqrt{x}$$

The solution becomes

$$y = \frac{4x^{\frac{3}{2}} \mathrm{e}^x - 6\sqrt{x}\, \mathrm{e}^x + 7\sqrt{\pi}\, \mathrm{erfi}\left(\sqrt{x}\right) + 4c_1}{8\sqrt{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = -\infty$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 0$$

This shows that no solution exist.

Summary

The solution(s) found are the following

$$y = \frac{4x^{\frac{3}{2}} e^{x} - 6\sqrt{x}\, e^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 4c_1}{8\sqrt{x}}$$

Verification of solutions

$$y = \frac{4x^{\frac{3}{2}} e^{x} - 6\sqrt{x}\, e^{x} + 7\sqrt{\pi}\, \operatorname{erfi}\left(\sqrt{x}\right) + 4c_1}{8\sqrt{x}}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 11

```
dsolve([2*x*diff(y(x),x)+y(x)=(x^2+1)*exp(x),y(-infinity) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\infty i}{\sqrt{\operatorname{signum}(x)}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{2*x*y'[x]+y[x]==(x^2+1)*Exp[x],{y[-Infinity]==1}},y[x],x,IncludeSingularSolutions ->
```

{}

6.21 problem 154

Internal problem ID [15047]
Internal file name [`OUTPUT/15047_Sunday_April_21_2024_01_21_23_PM_4449053/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 154.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x + y = 2x}$$

6.21.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = 2$$

Hence the ode is

$$y' + \frac{y}{x} = 2$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2) \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= (x)(2) \\
\mathrm{d}(xy) &= (2x)\ \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
xy &= \int 2x\,\mathrm{d}x \\
xy &= x^2 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = x + \frac{c_1}{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x + \frac{c_1}{x} \tag{1}$$

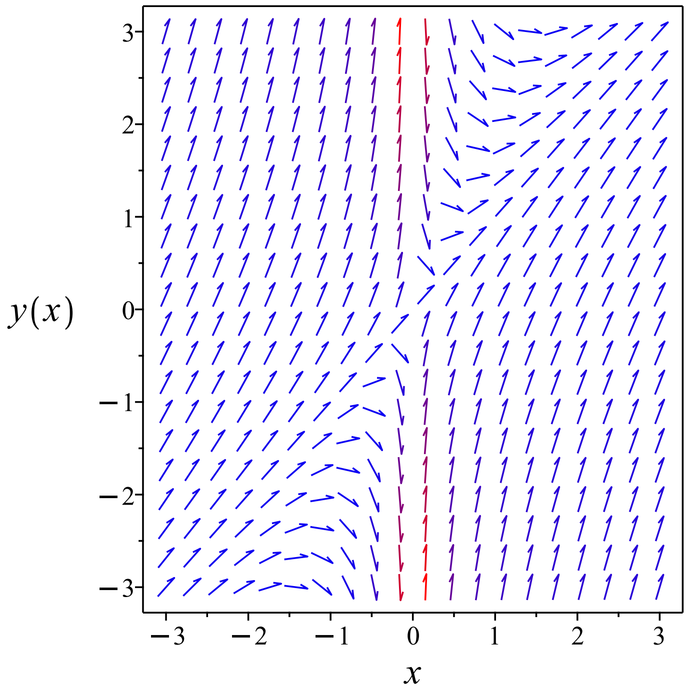


Figure 245: Slope field plot

Verification of solutions

$$y = x + \frac{c_1}{x}$$

Verified OK.

6.21.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x + u(x)\,x = 2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-2u+2}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -2u + 2$. Integrating both sides gives

$$\frac{1}{-2u+2}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{-2u+2}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{\ln(u-1)}{2} = \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{u-1}} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{\sqrt{u-1}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \frac{\left(c_3^2 \mathrm{e}^{2c_2} x^2 + 1\right) \mathrm{e}^{-2c_2}}{x\, c_3^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_3^2 \mathrm{e}^{2c_2} x^2 + 1\right) \mathrm{e}^{-2c_2}}{x\, c_3^2} \tag{1}$$

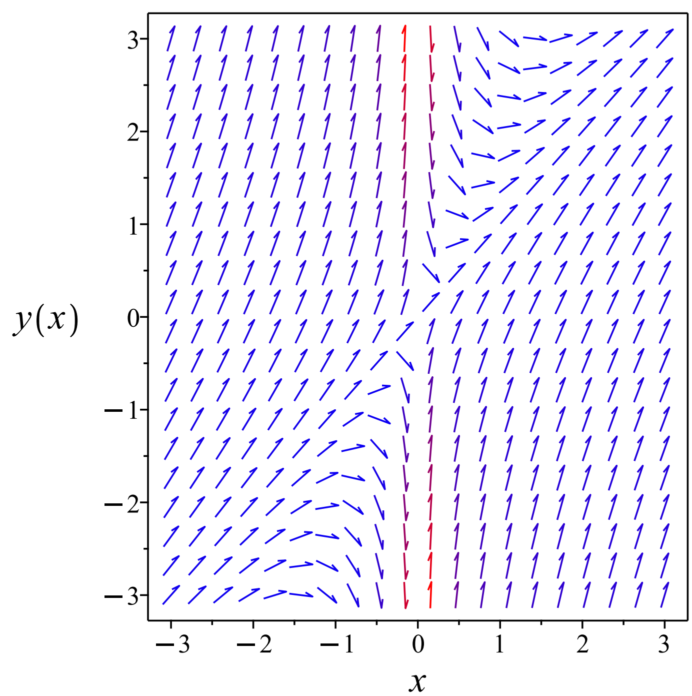


Figure 246: Slope field plot

Verification of solutions

$$y = \frac{(c_3^2 e^{2c_2} x^2 + 1) e^{-2c_2}}{x c_3^2}$$

Verified OK.

6.21.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x - y}{x} \tag{1}$$

Which becomes

$$0 = (-x) dy + (2x - y) dx \tag{2}$$

But the RHS is complete differential because

$$(-x) dy + (2x - y) dx = d(x^2 - xy)$$

Hence (2) becomes

$$0 = d(x^2 - xy)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + c_1}{x} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 + c_1}{x} + c_1 \tag{1}$$

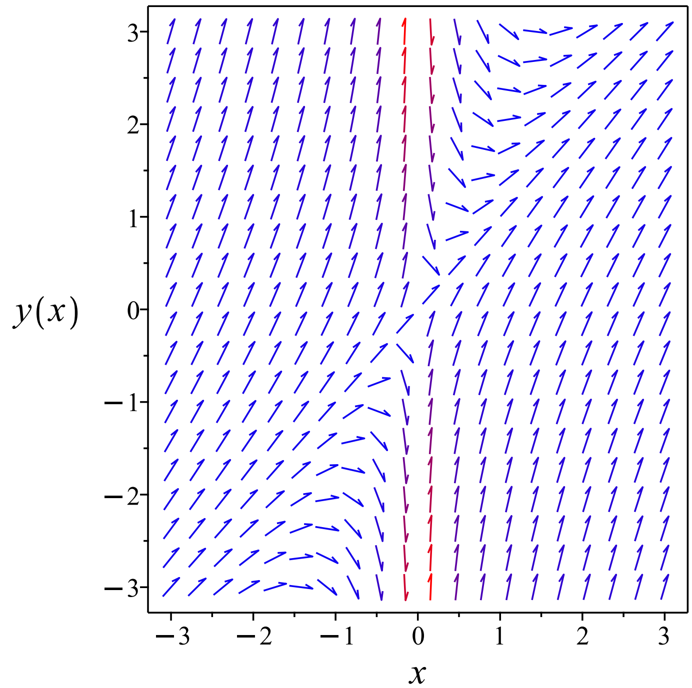


Figure 247: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{x} + c_1$$

Verified OK.

6.21.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 203: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = x^2 + c_1$$

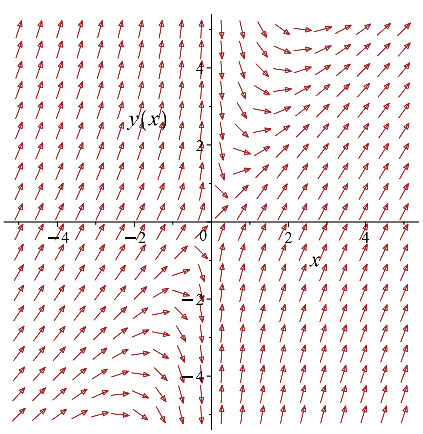
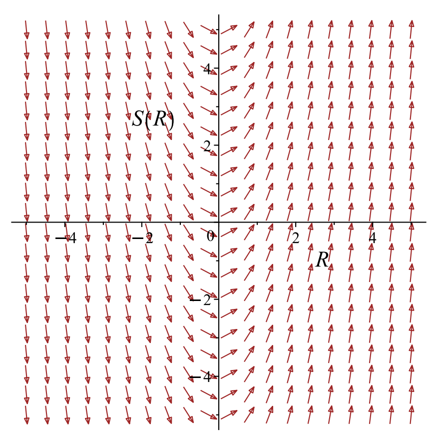
Which simplifies to

$$yx = x^2 + c_1$$

Which gives

$$y = \frac{x^2 + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-2x+y}{x}$
</td><td>$R = x$
$S = xy$</td><td>$\frac{dS}{dR} = 2R$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{x^2 + c_1}{x} \tag{1}$$

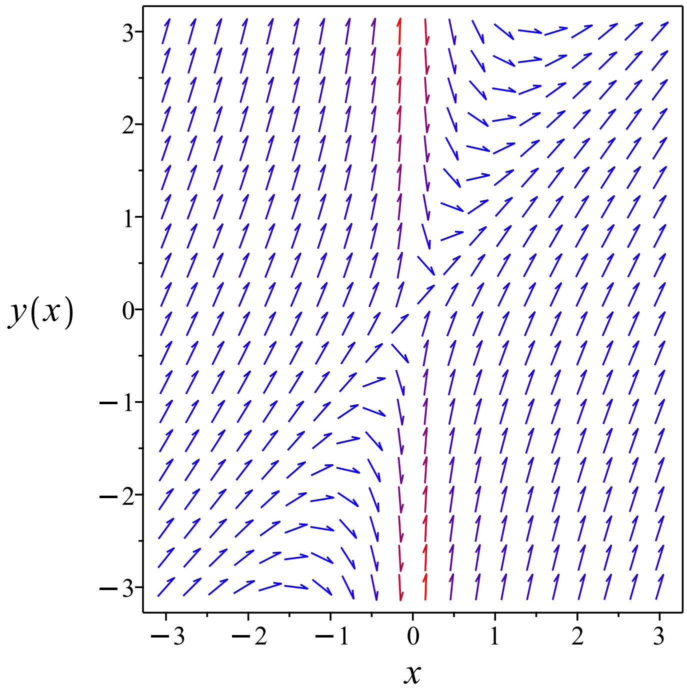


Figure 248: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{x}$$

Verified OK.

6.21.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (2x-y)\,\mathrm{d}x\\ (-2x+y)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2x+y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x+y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x + y \, \mathrm{d}x$$

$$\phi = -x(-y + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(-y + x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(-y + x)$$

The solution becomes

$$y = \frac{x^2 + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + c_1}{x} \tag{1}$$

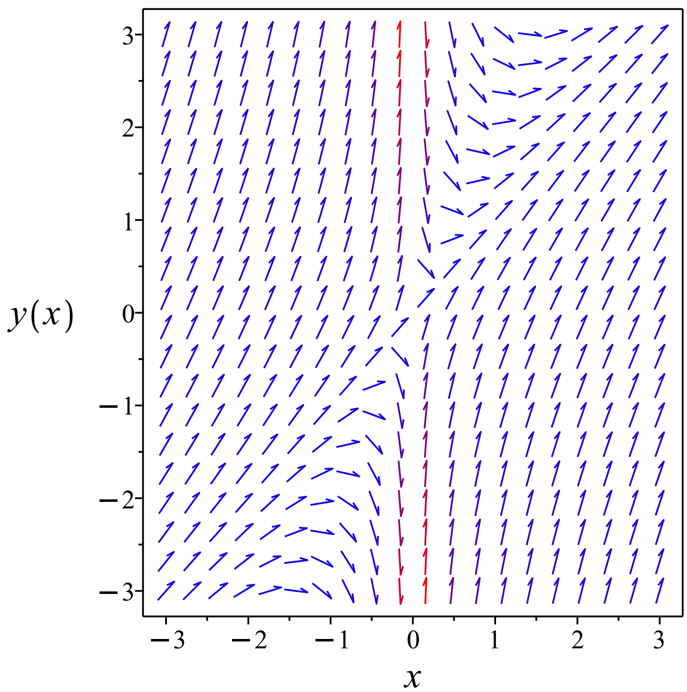


Figure 249: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1}{x}$$

Verified OK.

6.21.6 Maple step by step solution

Let's solve

$y'x + y = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2 - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = 2\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int 2xdx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)+y(x)=2*x,y(x), singsol=all)
```

$$y(x) = x + \frac{c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 13

```
DSolve[x*y'[x]+y[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{c_1}{x}$$

6.22 problem 155

Internal problem ID [15048]
Internal file name [`OUTPUT/15048_Sunday_April_21_2024_01_21_24_PM_15614469/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 155.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \sin (x) + y \cos (x) = 1}$$

6.22.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot (x)$$
$$q(x) = \csc (x)$$

Hence the ode is

$$y' + y \cot (x) = \csc (x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \cot(x)dx} \\
&= \sin(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\csc(x)) \\
\frac{d}{dx}(\sin(x)\, y) &= (\sin(x))(\csc(x)) \\
d(\sin(x)\, y) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin(x)\, y &= \int dx \\
\sin(x)\, y &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$
y = \csc(x)\, x + c_1 \csc(x)
$$

which simplifies to

$$
y = \csc(x)(x + c_1)
$$

Summary

The solution(s) found are the following

$$
y = \csc(x)(x + c_1) \tag{1}
$$

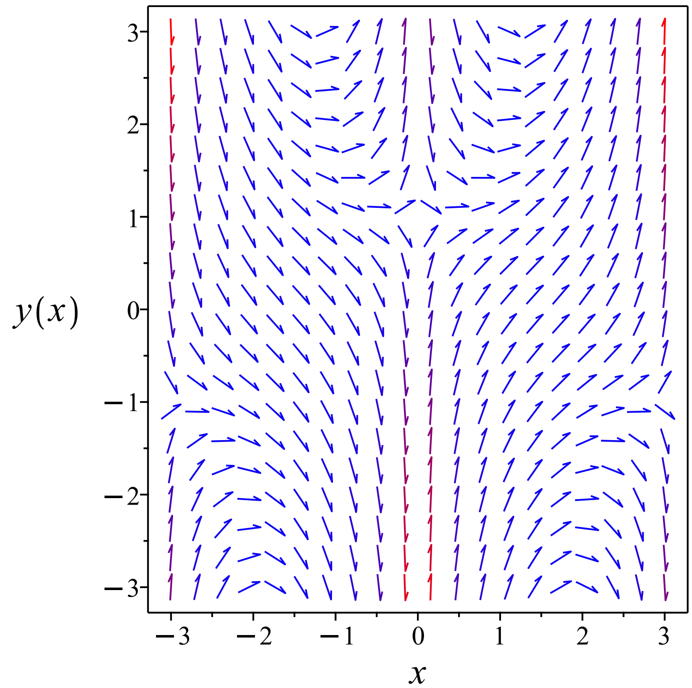


Figure 250: Slope field plot

Verification of solutions

$$y = \csc(x)(x + c_1)$$

Verified OK.

6.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\cos(x)\, y - 1}{\sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\cos(x)\, y - 1}{\sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x) y = x + c_1$$

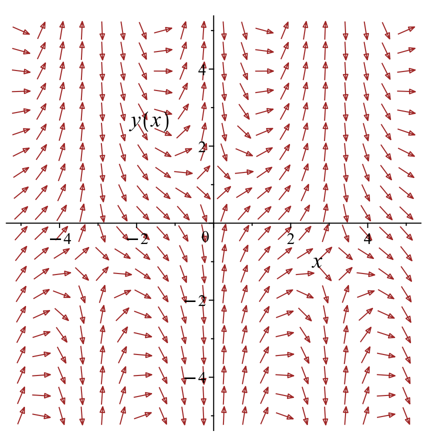
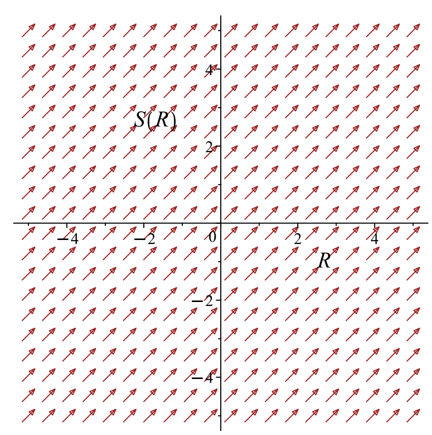
Which simplifies to

$$\sin(x) y = x + c_1$$

Which gives

$$y = \frac{x + c_1}{\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\cos(x)y-1}{\sin(x)}$ 	$R = x$ $S = \sin(x) y$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{\sin(x)} \tag{1}$$

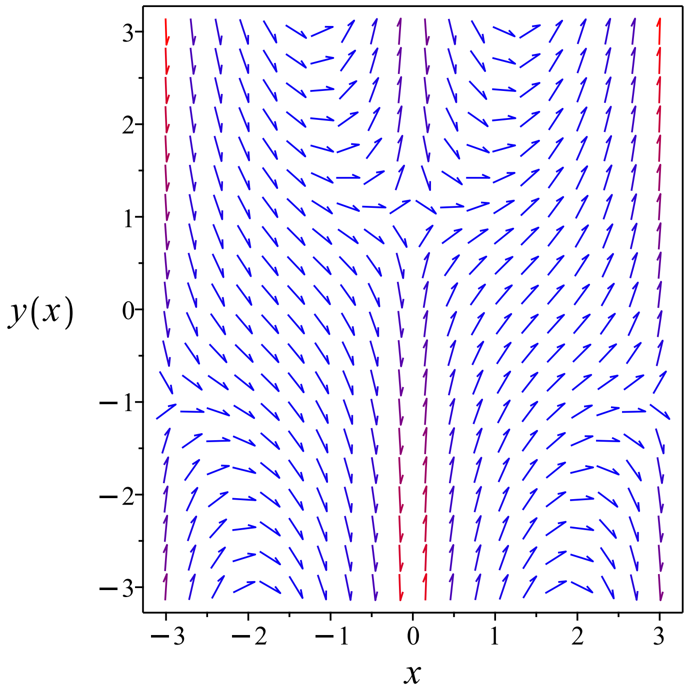


Figure 251: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\sin(x)}$$

Verified OK.

6.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\sin(x))\,\mathrm{d}y = (-\cos(x)\,y + 1)\,\mathrm{d}x$$
$$(\cos(x)\,y - 1)\,\mathrm{d}x + (\sin(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \cos(x)\,y - 1$$
$$N(x, y) = \sin(x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\,y - 1) \\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)) \\ &= \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \cos(x)\,y-1\,\mathrm{d}x$$
$$\phi=-x+\sin(x)\,y+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\sin(x)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\sin(x)$. Therefore equation (4) becomes

$$\sin(x)=\sin(x)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-x+\sin(x)\,y+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=-x+\sin(x)\,y$$

The solution becomes

$$y = \frac{x + c_1}{\sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x + c_1}{\sin(x)} \tag{1}$$

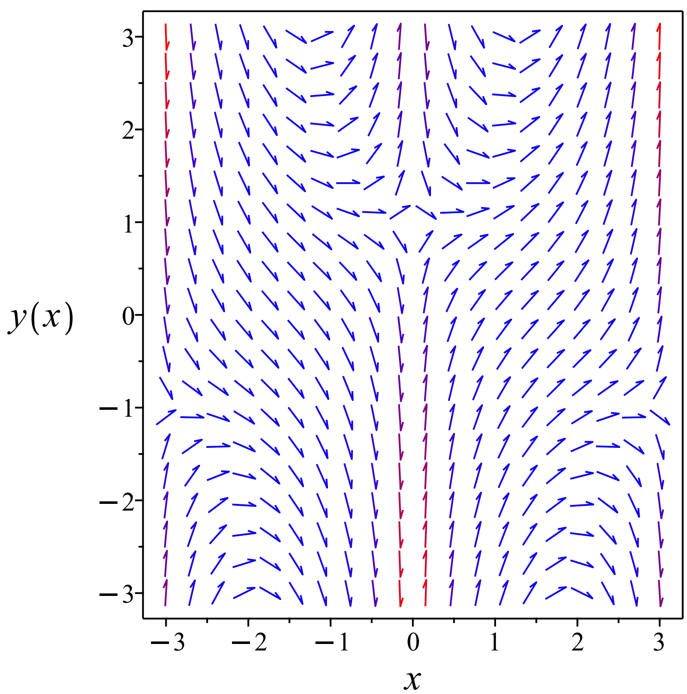


Figure 252: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{\sin(x)}$$

Verified OK.

6.22.4 Maple step by step solution

Let's solve

$y' \sin(x) + y \cos(x) = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y \cos(x)}{\sin(x)} + \frac{1}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y \cos(x)}{\sin(x)} = \frac{1}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y \cos(x)}{\sin(x)}\right) = \frac{\mu(x)}{\sin(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y \cos(x)}{\sin(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x) \cos(x)}{\sin(x)}$

- Solve to find the integrating factor

  $\mu(x) = \sin(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{\sin(x)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{\sin(x)} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{\sin(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

  $y = \frac{\int 1 dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{\sin(x)}$

- Simplify

  $y = \csc(x)(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(sin(x)*diff(y(x),x)+y(x)*cos(x)=1,y(x), singsol=all)
```

$$y(x) = (x + c_1)\csc(x)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 12

```
DSolve[Sin[x]*y'[x]+y[x]*Cos[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (x + c_1)\csc(x)$$

6.23 problem 156

Internal problem ID [15049]
Internal file name [`OUTPUT/15049_Sunday_April_21_2024_01_21_25_PM_1233703/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 156.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x)\,y' - \sin(x)\,y = -\sin(2x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 0\right]$$

6.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\tan(x)$$
$$q(x) = -2\sin(x)$$

Hence the ode is

$$y' - \tan(x) y = -2 \sin(x)$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z815 \vee \frac{1}{2}\pi + \pi\_Z815 < x\right\}$$

But the point $x_0 = \frac{\pi}{2}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

6.23.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\tan(x) dx} \\ &= \cos(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-2\sin(x)) \\ \frac{d}{dx}(\cos(x) y) &= (\cos(x))(-2\sin(x)) \\ d(\cos(x) y) &= (-\sin(2x))\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\cos(x) y &= \int -\sin(2x)\, dx \\ \cos(x) y &= \frac{\cos(2x)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \frac{\sec(x)\cos(2x)}{2} + c_1 \sec(x)$$

which simplifies to

$$y = \cos(x) - \frac{\sec(x)}{2} + c_1 \sec(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Solving for c_1 gives

$$c_1 = -\frac{2\cos(x) - \sec(x) - 2y}{2\sec(x)}$$

Using given initial conditions results in $c_1 = \frac{1}{2}$ Hence the solution is

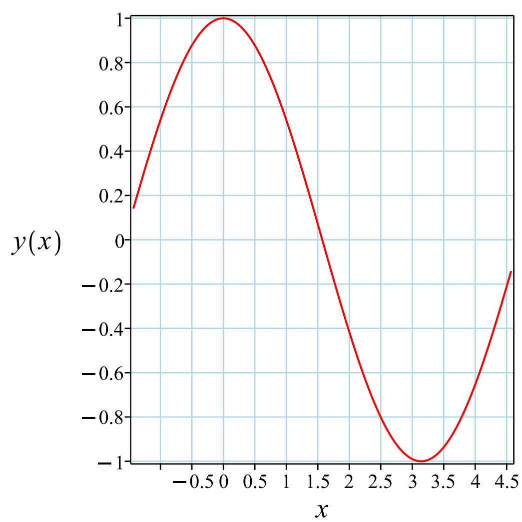
$$y = \cos(x)$$

Therefore the solution is

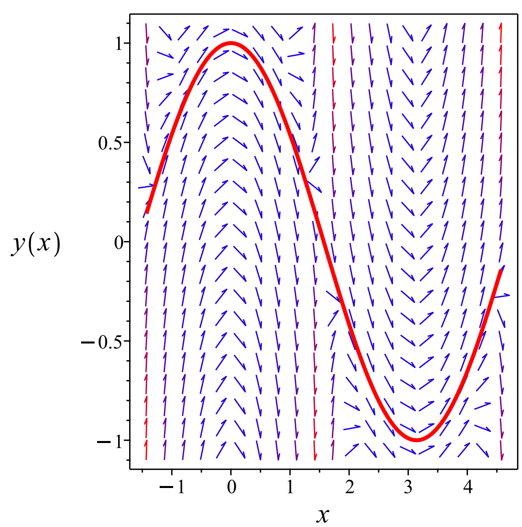
$$y = \cos(x)$$

Summary
The solution(s) found are the following

$$y = \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(x)$$

Verified OK.

6.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-\sin(x)\, y + \sin(2x)}{\cos(x)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 209: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\cos(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cos(x)}} dy \end{aligned}$$

Which results in

$$S = \cos(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-\sin(x)\, y + \sin(2x)}{\cos(x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\sin(x)\, y \\
S_y &= \cos(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\sin(2x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\sin(2R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\cos(2R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y\cos(x) = \frac{\cos(2x)}{2} + c_1
$$

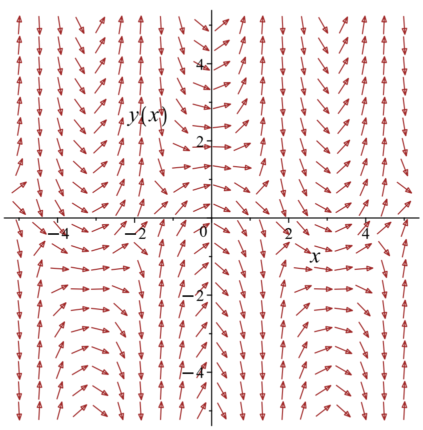
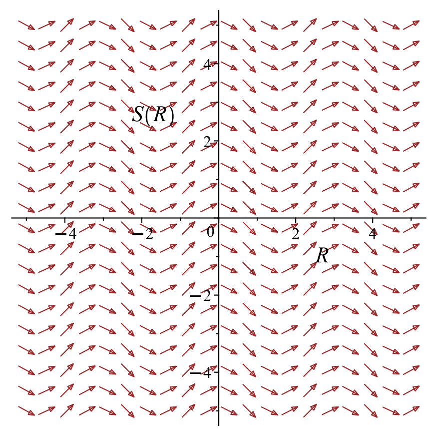
Which simplifies to

$$
y\cos(x) = \frac{\cos(2x)}{2} + c_1
$$

Which gives

$$
y = \frac{\cos(2x) + 2c_1}{2\cos(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-\sin(x)y+\sin(2x)}{\cos(x)}$  	$R = x$ $S = \cos(x)\, y$	$\frac{dS}{dR} = -\sin(2R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Solving for c_1 gives

$$c_1 = -\cos(x)^2 + \cos(x)\, y + \frac{1}{2}$$

Using given initial conditions results in $c_1 = \frac{1}{2}$ Hence the solution is

$$y = \frac{1 + \cos(2x)}{2\cos(x)}$$

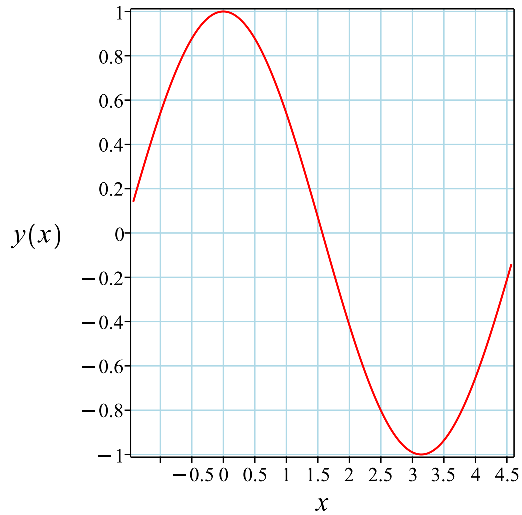
Therefore the solution is

$$y = \frac{1 + \cos(2x)}{2\cos(x)}$$

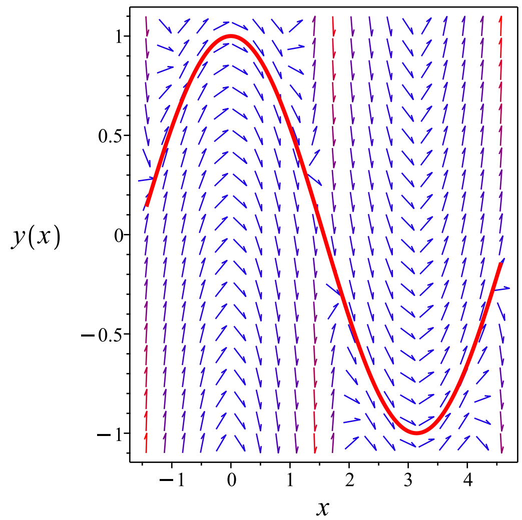
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{1 + \cos(2x)}{2\cos(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1 + \cos(2x)}{2\cos(x)}$$

Verified OK.

6.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(x))\,\mathrm{d}y &= (\sin(x)\,y - \sin(2x))\,\mathrm{d}x\\ (-\sin(x)\,y + \sin(2x))\,\mathrm{d}x + (\cos(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\sin(x)\,y + \sin(2x)\\ N(x, y) &= \cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x)\,y + \sin(2x))\\ &= -\sin(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x))\\ &= -\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\sin(x) y + \sin(2x) \mathrm{d}x$$

$$\phi = -\cos(x)^2 + \cos(x) y + \frac{1}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\cos(x)^2 + \cos(x) y + \frac{1}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\cos(x)^2 + \cos(x) y + \frac{1}{2}$$

The solution becomes

$$y = \frac{2\cos(x)^2 - 1 + 2c_1}{2\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Solving for c_1 gives

$$c_1 = -\cos(x)^2 + \cos(x) y + \frac{1}{2}$$

Using given initial conditions results in $c_1 = \frac{1}{2}$ Hence the solution is

$$y = \cos(x)$$

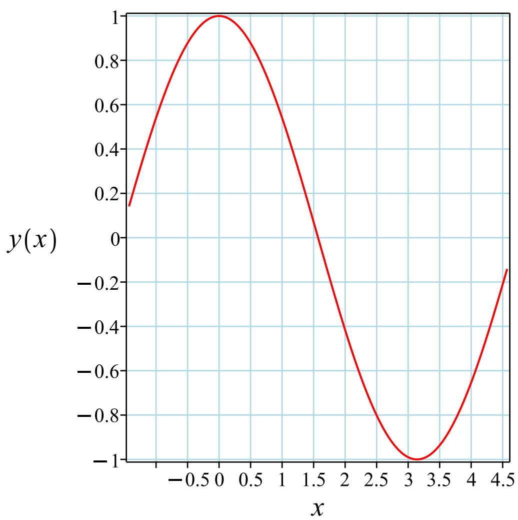
Therefore the solution is

$$y = \cos(x)$$

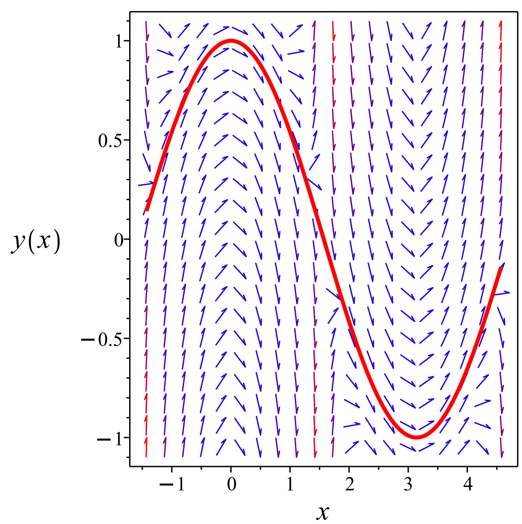
Summary

The solution(s) found are the following

$$y = \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(x)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 6

```
dsolve([cos(x)*diff(y(x),x)-y(x)*sin(x)=-sin(2*x),y(1/2*Pi) = 0],y(x), singsol=all)
```

$$y(x) = \cos(x)$$

✓ Solution by Mathematica

Time used: 0.133 (sec). Leaf size: 7

```
DSolve[{Cos[x]*y'[x]-y[x]*Sin[x]==-Sin[2*x],{y[Pi/2]==0}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \cos(x)$$

6.24 problem 157

Internal problem ID [15050]
Internal file name [`OUTPUT/15050_Sunday_April_21_2024_01_21_26_PM_15551205/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 157.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 2yx - 2y^2x = 0}$$

6.24.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 2xy(y-1) \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y(y-1)$. Integrating both sides gives

$$\frac{1}{y\,(y-1)}\,dy = 2x\,dx$$

$$\int \frac{1}{y\,(y-1)}\,dy = \int 2x\,dx$$
$$\ln\left(y-1\right)-\ln\left(y\right) = x^2 + c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = e^{x^2+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = c_2 e^{x^2}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{-1+c_2 e^{x^2}} \tag{1}$$

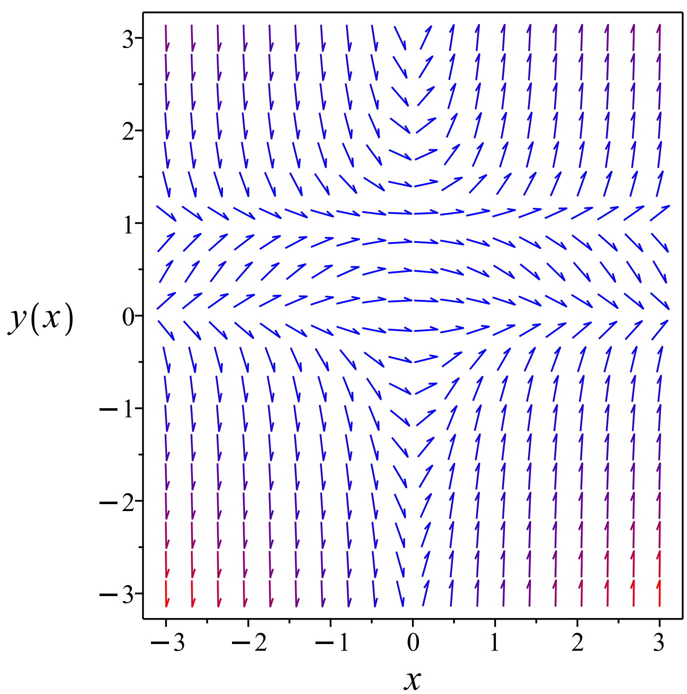


Figure 256: Slope field plot

Verification of solutions

$$y = -\frac{1}{-1+c_2 e^{x^2}}$$

Verified OK.

6.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= 2y^2x - 2xy \\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 211: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{2x}} dx
\end{aligned}
$$

Which results in

$$
S = x^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = 2y^2 x - 2xy
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 2x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\,(y-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\,(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R-1) - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 = \ln(y-1) - \ln(y) + c_1$$

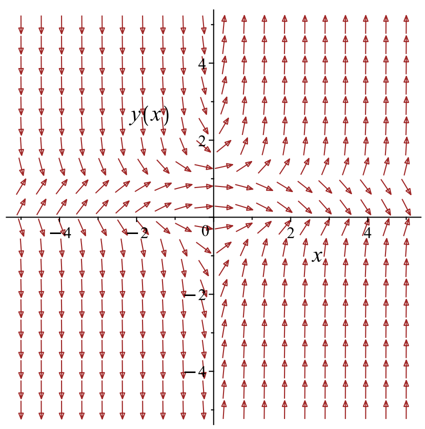
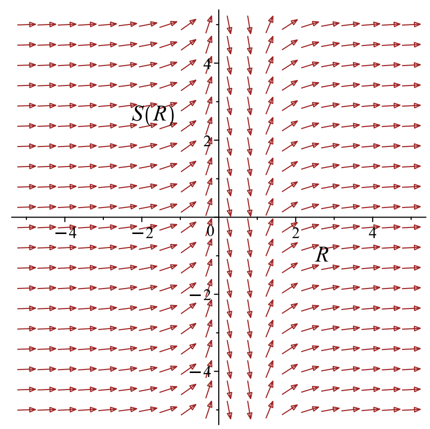
Which simplifies to

$$x^2 = \ln(y-1) - \ln(y) + c_1$$

Which gives

$$y = \frac{e^{-x^2+c_1}}{-1+e^{-x^2+c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2y^2x - 2xy$ 	$R = y$ $S = x^2$	$\frac{dS}{dR} = \frac{1}{R(R-1)}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{-x^2+c_1}}{-1 + e^{-x^2+c_1}} \tag{1}$$

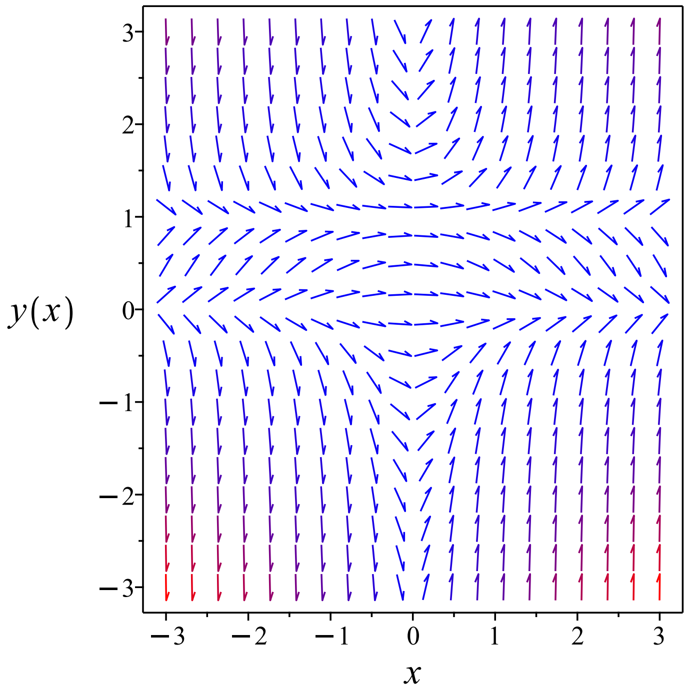


Figure 257: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2+c_1}}{-1+e^{-x^2+c_1}}$$

Verified OK.

6.24.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= 2y^2x - 2xy \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -2xy + 2xy^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -2x \\ f_1(x) &= 2x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = -\frac{2x}{y} + 2x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -2w(x)\, x + 2x \\ w' &= 2xw - 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= -2x \end{aligned}$$

Hence the ode is

$$w'(x) - 2w(x)\, x = -2x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -2x dx} \\
&= e^{-x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\
\frac{d}{dx}\left(e^{-x^2} w\right) &= \left(e^{-x^2}\right)(-2x) \\
d\left(e^{-x^2} w\right) &= \left(-2x\, e^{-x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x^2} w &= \int -2x\, e^{-x^2}\, dx \\
e^{-x^2} w &= e^{-x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$
w(x) = e^{-x^2} e^{x^2} + c_1 e^{x^2}
$$

which simplifies to

$$
w(x) = 1 + c_1 e^{x^2}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = 1 + c_1 e^{x^2}
$$

Or

$$
y = \frac{1}{1 + c_1 e^{x^2}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{1 + c_1 e^{x^2}} \tag{1}
$$

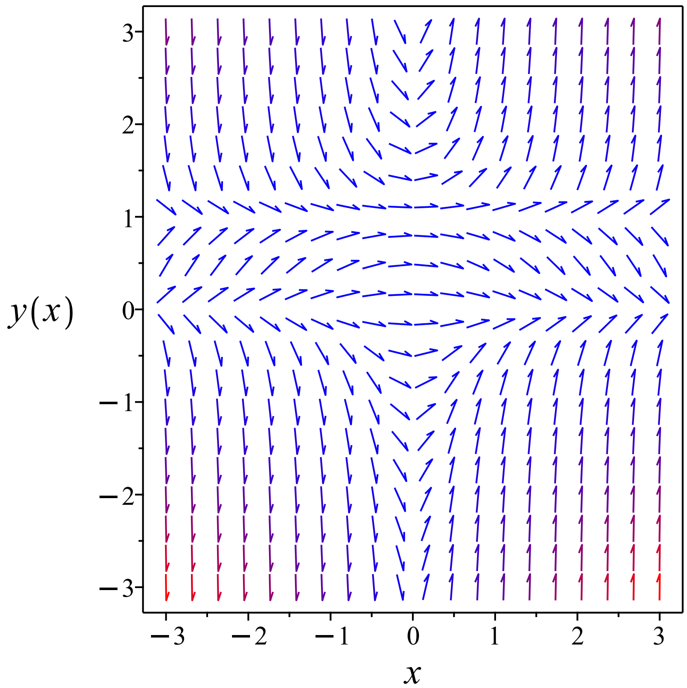


Figure 258: Slope field plot

Verification of solutions

$$y = \frac{1}{1 + c_1 e^{x^2}}$$

Verified OK.

6.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2y\,(y-1)}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(\frac{1}{2y\,(y-1)}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{2y\,(y-1)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y\left(y-1\right)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -x\, \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y(y-1)}$. Therefore equation (4) becomes

$$\frac{1}{2y\left(y-1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y\left(y-1\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y\,(y-1)}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y-1)}{2} - \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y-1)}{2} - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y-1)}{2} - \frac{\ln(y)}{2}$$

The solution becomes

$$y = -\frac{1}{\mathrm{e}^{x^2+2c_1} - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\mathrm{e}^{x^2+2c_1} - 1} \tag{1}$$

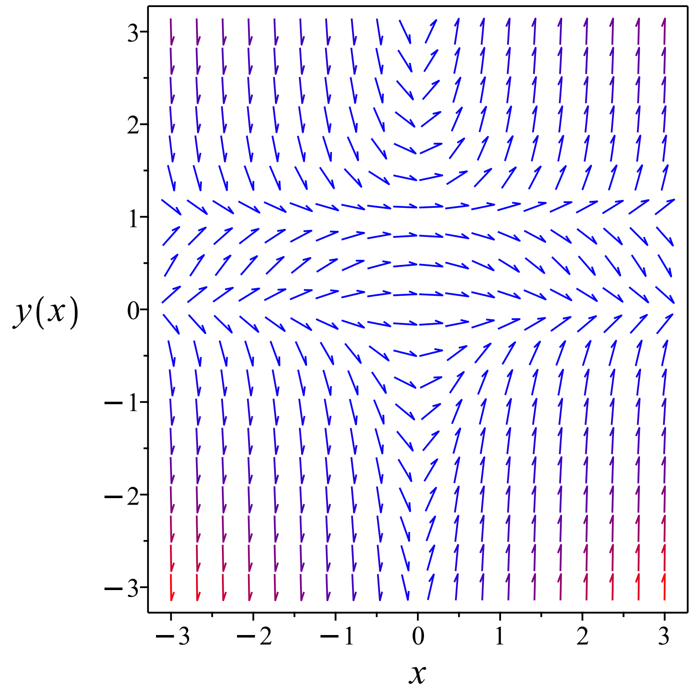


Figure 259: Slope field plot

Verification of solutions

$$y = -\frac{1}{e^{x^2+2c_1} - 1}$$

Verified OK.

6.24.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2y^2x - 2xy \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2y^2x - 2xy$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -2x$ and $f_2(x) = 2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{2xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \\ f_1 f_2 &= -4x^2 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$2xu''(x) - \left(-4x^2 + 2\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \mathrm{e}^{-x^2}$$

The above shows that

$$u'(x) = -2c_2 x \, \mathrm{e}^{-x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 \mathrm{e}^{-x^2}}{c_1 + c_2 \mathrm{e}^{-x^2}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\mathrm{e}^{-x^2}}{c_3 + \mathrm{e}^{-x^2}}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-x^2}}{c_3 + e^{-x^2}} \tag{1}$$

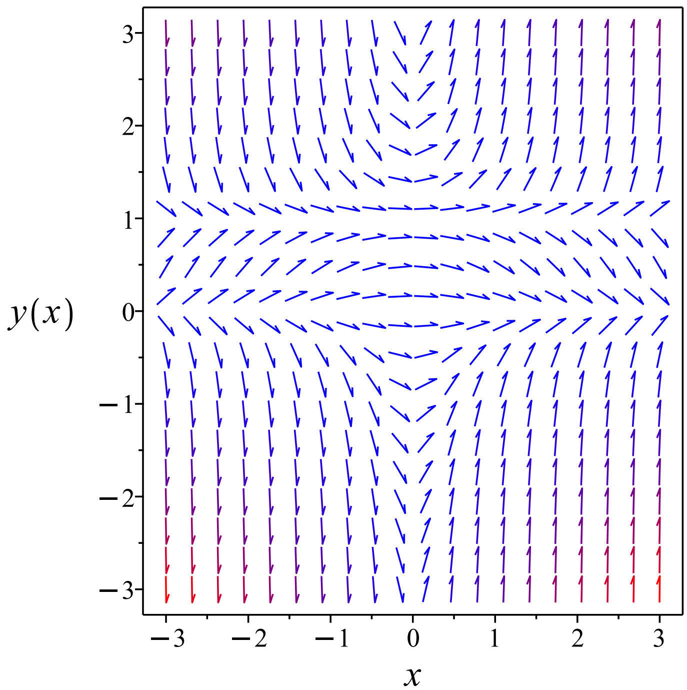


Figure 260: Slope field plot

Verification of solutions

$$y = \frac{e^{-x^2}}{c_3 + e^{-x^2}}$$

Verified OK.

6.24.6 Maple step by step solution

Let's solve

$y' + 2yx - 2y^2x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y(y-1)} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y-1)}dx = \int 2x dx + c_1$

- Evaluate integral

  $\ln(y-1) - \ln(y) = x^2 + c_1$

- Solve for y

  $y = -\frac{1}{e^{x^2+c_1}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+2*x*y(x)=2*x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{1 + e^{x^2} c_1}$$

✓ Solution by Mathematica

Time used: 0.204 (sec). Leaf size: 27

```
DSolve[y'[x]+2*x*y[x]==2*x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{1 + e^{x^2+c_1}}$$
$$y(x) \to 0$$
$$y(x) \to 1$$

6.25 problem 158

Internal problem ID [15051]
Internal file name [`OUTPUT/15051_Sunday_April_21_2024_01_21_27_PM_47992157/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 158.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{3y^2xy' - 2y^3 = x^3}$$

6.25.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)^2\,x^3(u'(x)\,x + u(x)) - 2u(x)^3\,x^3 = x^3$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^3 - 1}{3u^2x}
\end{aligned}$$

Where $f(x) = -\frac{1}{3x}$ and $g(u) = \frac{u^3-1}{u^2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^3-1}{u^2}}\,du &= -\frac{1}{3x}\,dx \\
\int \frac{1}{\frac{u^3-1}{u^2}}\,du &= \int -\frac{1}{3x}\,dx \\
\frac{\ln(u^3-1)}{3} &= -\frac{\ln(x)}{3} + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
(u^3-1)^{\frac{1}{3}} = \mathrm{e}^{-\frac{\ln(x)}{3}+c_2}
$$

Which simplifies to

$$
(u^3-1)^{\frac{1}{3}} = \frac{c_3}{x^{\frac{1}{3}}}
$$

Which simplifies to

$$
\left(u(x)^3-1\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}
$$

The solution is

$$
\left(u(x)^3-1\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{y^3}{x^3}-1\right)^{\frac{1}{3}} &= \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}} \\
\left(\frac{y^3-x^3}{x^3}\right)^{\frac{1}{3}} &= \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\left(\frac{y^3-x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}} \tag{1}
$$

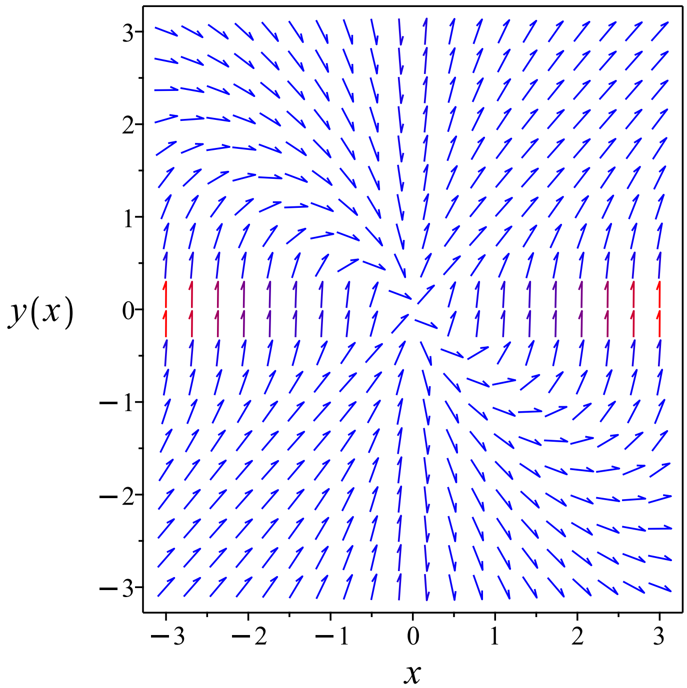


Figure 261: Slope field plot

Verification of solutions

$$\left(\frac{y^3 - x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}}$$

Verified OK.

6.25.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + 2y^3}{3y^2 x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 214: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^2}{y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 + 2y^3}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y^3}{3x^3} \\ S_y &= \frac{y^2}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{3} + c_1 \tag{4}$$

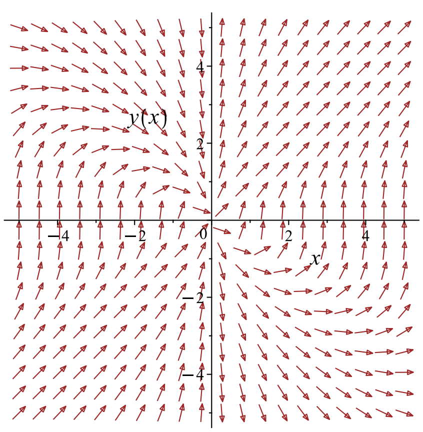
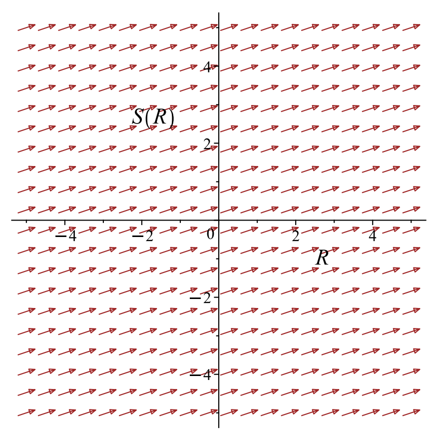
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x^2} = \frac{x}{3} + c_1$$

Which simplifies to

$$\frac{y^3}{3x^2} = \frac{x}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+2y^3}{3y^2x}$ 	$R = x$ $S = \frac{y^3}{3x^2}$	$\frac{dS}{dR} = \frac{1}{3}$ 

Summary
The solution(s) found are the following

$$\frac{y^3}{3x^2} = \frac{x}{3} + c_1 \tag{1}$$

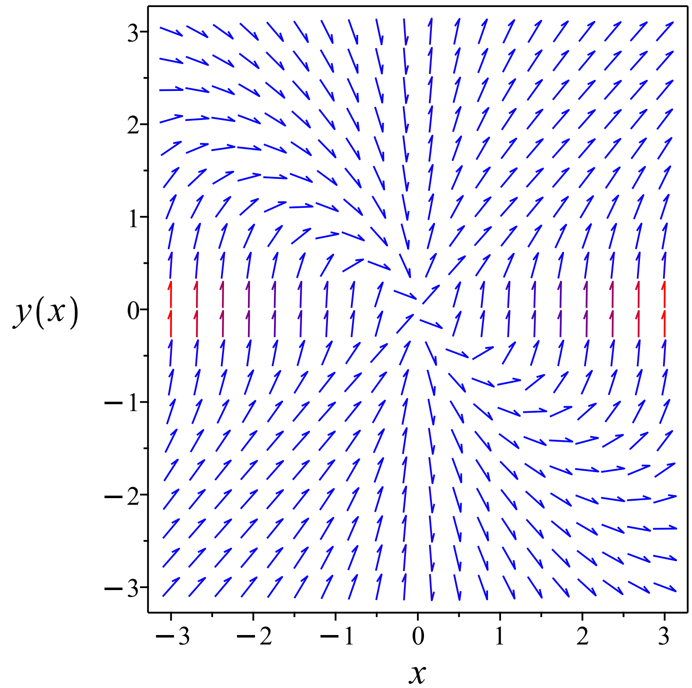


Figure 262: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^2} = \frac{x}{3} + c_1$$

Verified OK.

6.25.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^3+2y^3}{3y^2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{3x}y + \frac{x^2}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{3x} \\ f_1(x) &= \frac{x^2}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{2y^3}{3x} + \frac{x^2}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{2w(x)}{3x} + \frac{x^2}{3} \\ w' &= \frac{2w}{x} + x^2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= x^2 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(x^2\right) \\ d\left(\frac{w}{x^2}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int dx \\ \frac{w}{x^2} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = c_1 x^2 + x^3$$

which simplifies to

$$w(x) = x^2(x + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x^2(x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \left(x^2(x + c_1)\right)^{\frac{1}{3}} \\ y(x) &= \frac{\left(x^2(x + c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2} \\ y(x) &= -\frac{\left(x^2(x + c_1)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(x^2(x + c_1)\right)^{\frac{1}{3}} \tag{1}$$

$$y = \frac{\left(x^2(x + c_1)\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2} \tag{2}$$

$$y = -\frac{\left(x^2(x + c_1)\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

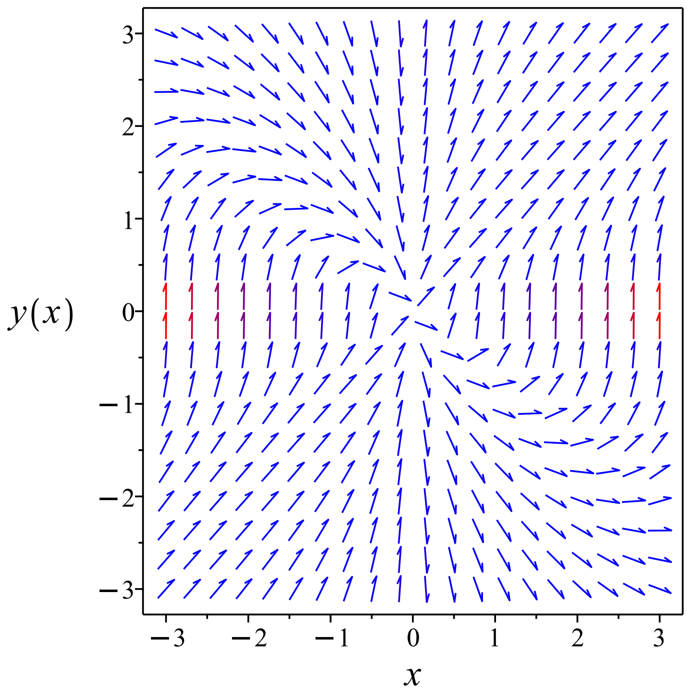


Figure 263: Slope field plot

Verification of solutions

$$y = \left(x^2(x+c_1)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(x^2(x+c_1)\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(x^2(x+c_1)\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{2}$$

Verified OK.

6.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3y^2x\right)\mathrm{d}y &= \left(x^3+2y^3\right)\mathrm{d}x\\ \left(-x^3-2y^3\right)\mathrm{d}x + \left(3y^2x\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^3-2y^3\\ N(x,y) &= 3y^2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3-2y^3\right)\\ &= -6y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3y^2x\right)\\ &= 3y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{3x\,y^2}\left(\left(-6y^2\right)-\left(3y^2\right)\right)\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}\left(-x^3 - 2y^3\right) \\
&= \frac{-x^3 - 2y^3}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}\left(3y^2 x\right) \\
&= \frac{3y^2}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^3 - 2y^3}{x^3}\right) + \left(\frac{3y^2}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^3 - 2y^3}{x^3} \mathrm{d}x$$
$$\phi = -x + \frac{y^3}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2}{x^2}$. Therefore equation (4) becomes

$$\frac{3y^2}{x^2} = \frac{3y^2}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{y^3}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \frac{y^3}{x^2}$$

Summary

The solution(s) found are the following

$$-x + \frac{y^3}{x^2} = c_1 \tag{1}$$

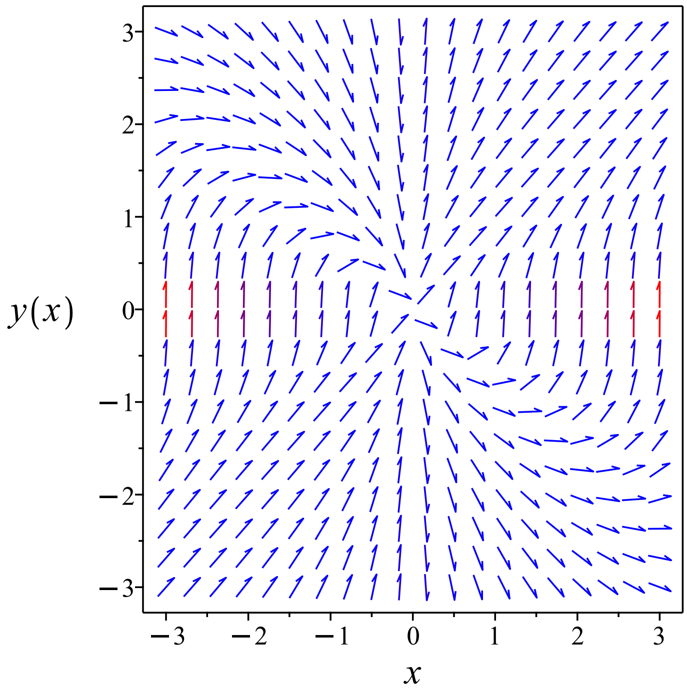


Figure 264: Slope field plot

Verification of solutions

$$-x+\frac{y^3}{x^2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(3*x*y(x)^2*diff(y(x),x)-2*y(x)^3=x^3,y(x), singsol=all)
```

$$y(x) = \left(\left(x + c_1\right) x^2\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left(\left(x + c_1\right) x^2\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{\left(\left(x + c_1\right) x^2\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.18 (sec). Leaf size: 66

```
DSolve[3*x*y[x]^2*y'[x]-2*y[x]^3==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x^{2/3} \sqrt[3]{x + c_1}$$
$$y(x) \rightarrow -\sqrt[3]{-1} x^{2/3} \sqrt[3]{x + c_1}$$
$$y(x) \rightarrow (-1)^{2/3} x^{2/3} \sqrt[3]{x + c_1}$$

6.26 problem 159

Internal problem ID [15052]
Internal file name [`OUTPUT/15052_Sunday_April_21_2024_01_21_34_PM_19957448/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 159.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{(x^3 + e^y)\, y' = 3x^2}$$

6.26.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{3x^2}{x^3 + e^y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{3x^2(b_3 - a_2)}{x^3 + e^y} - \frac{9x^4 a_3}{(x^3 + e^y)^2} - \left(\frac{6x}{x^3 + e^y} - \frac{9x^4}{(x^3 + e^y)^2}\right)(xa_2 + ya_3 + a_1) + \frac{3x^2 e^y (xb_2 + yb_3 + b_1)}{(x^3 + e^y)^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^6 b_2 + 3x^5 b_3 + 3x^4 y a_3 + 5 e^y x^3 b_2 + 3 e^y x^2 y b_3 + 3x^4 a_1 - 9x^4 a_3 - 9 e^y x^2 a_2 + 3 e^y x^2 b_1 + 3 e^y x^2 b_3 - 6 e^y x y a_3}{(x^3 + e^y)^2} = 0$$

Setting the numerator to zero gives

$$x^6 b_2 + 3x^5 b_3 + 3x^4 y a_3 + 5 e^y x^3 b_2 + 3 e^y x^2 y b_3 + 3x^4 a_1 - 9x^4 a_3 - 9 e^y x^2 a_2 + 3 e^y x^2 b_1 + 3 e^y x^2 b_3 - 6 e^y x y a_3 + e^{2y} b_2 - 6 e^y x a_1 = 0 \tag{6E}$$

Simplifying the above gives

$$x^6 b_2 + 3x^5 b_3 + 3x^4 y a_3 + 5 e^y x^3 b_2 + 3 e^y x^2 y b_3 + 3x^4 a_1 - 9x^4 a_3 - 9 e^y x^2 a_2 + 3 e^y x^2 b_1 + 3 e^y x^2 b_3 - 6 e^y x y a_3 + e^{2y} b_2 - 6 e^y x a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^y, e^{2y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^y = v_3, e^{2y} = v_4\}$$

The above PDE (6E) now becomes

$$v_1^6 b_2 + 3v_1^4 v_2 a_3 + 3v_1^5 b_3 + 3v_1^4 a_1 - 9v_1^4 a_3 + 5v_3 v_1^3 b_2 + 3v_3 v_1^2 v_2 b_3 - 9v_3 v_1^2 a_2 - 6v_3 v_1 v_2 a_3 + 3v_3 v_1^2 b_1 + 3v_3 v_1^2 b_3 - 6v_3 v_1 a_1 + v_4 b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$v_1^6 b_2 + 3v_1^5 b_3 + 3v_1^4 v_2 a_3 + (3a_1 - 9a_3) v_1^4 + 5v_3 v_1^3 b_2 + 3v_3 v_1^2 v_2 b_3 + (-9a_2 + 3b_1 + 3b_3) v_1^2 v_3 - 6v_3 v_1 v_2 a_3 - 6v_3 v_1 a_1 + v_4 b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -6a_1 &= 0 \\ -6a_3 &= 0 \\ 3a_3 &= 0 \\ 5b_2 &= 0 \\ 3b_3 &= 0 \\ 3a_1 - 9a_3 &= 0 \\ -9a_2 + 3b_1 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 3a_2 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 3 - \left(\frac{3x^2}{x^3 + \mathrm{e}^y}\right)(x) \\
&= \frac{3\,\mathrm{e}^y}{x^3 + \mathrm{e}^y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3\,\mathrm{e}^y}{x^3 + \mathrm{e}^y}} dy
\end{aligned}$$

Which results in

$$S = -\frac{x^3 \mathrm{e}^{-y}}{3} + \frac{\ln(\mathrm{e}^y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x^2}{x^3 + \mathrm{e}^y}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -x^2 \mathrm{e}^{-y} \\
S_y &= \frac{x^3 \mathrm{e}^{-y}}{3} + \frac{1}{3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x^3 \mathrm{e}^{-y}}{3} + \frac{y}{3} = c_1
$$

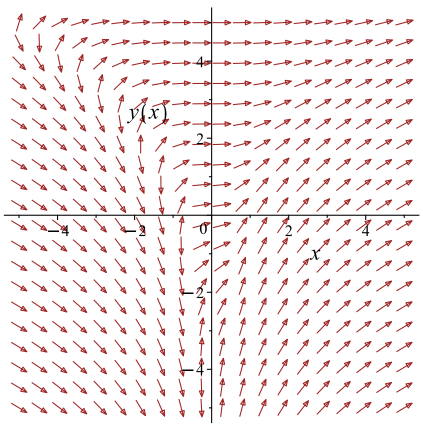
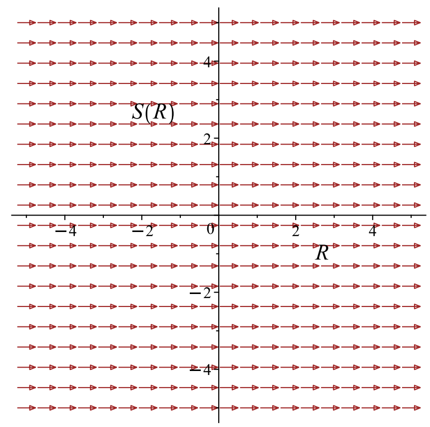
Which simplifies to

$$
-\frac{x^3 \mathrm{e}^{-y}}{3} + \frac{y}{3} = c_1
$$

Which gives

$$
y = \mathrm{LambertW}\left(x^3 \mathrm{e}^{-3c_1}\right) + 3c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2}{x^3+e^y}$ 	$R = x$ $S = -\frac{x^3 e^{-y}}{3} + \frac{y}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(x^3 e^{-3c_1}\right) + 3c_1 \tag{1}$$

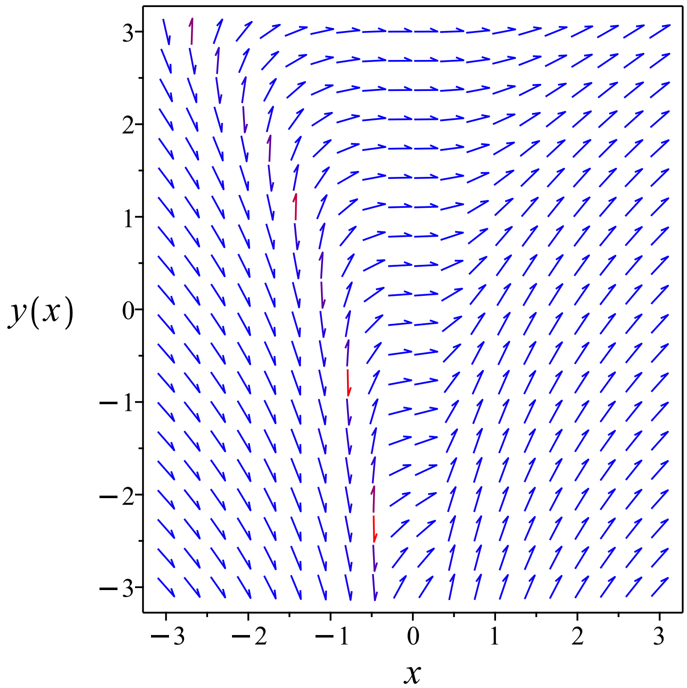


Figure 265: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(x^3 e^{-3c_1}\right) + 3c_1$$

Verified OK.

6.26.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(x^3+\mathrm{e}^y\right)\mathrm{d}y=\left(3x^2\right)\mathrm{d}x$$
$$\left(-3x^2\right)\mathrm{d}x+\left(x^3+\mathrm{e}^y\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=-3x^2$$
$$N(x,y)=x^3+\mathrm{e}^y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-3x^2\right)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x^3+\mathrm{e}^y\right)\\&=3x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^3 + \mathrm{e}^y}\left((0) - \left(3x^2\right)\right) \\
&= -\frac{3x^2}{x^3 + \mathrm{e}^y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{3x^2}\left(\left(3x^2\right) - (0)\right) \\
&= -1
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -1 \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-y} \\
&= \mathrm{e}^{-y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-y}\left(-3x^2\right) \\
&= -3x^2\mathrm{e}^{-y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-y}\left(x^3 + \mathrm{e}^y\right) \\
&= 1 + x^3\mathrm{e}^{-y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-3x^2\mathrm{e}^{-y}\right) + \left(1 + x^3\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -3x^2\mathrm{e}^{-y}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x^3\mathrm{e}^{-y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^3\mathrm{e}^{-y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1 + x^3\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$
1 + x^3\mathrm{e}^{-y} = x^3\mathrm{e}^{-y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (1)\,\mathrm{d}y \\
f(y) &= y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 e^{-y} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 e^{-y} + y$$

The solution becomes

$$y = \text{LambertW}\left(x^3 e^{-c_1}\right) + c_1$$

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(x^3 e^{-c_1}\right) + c_1 \tag{1}$$

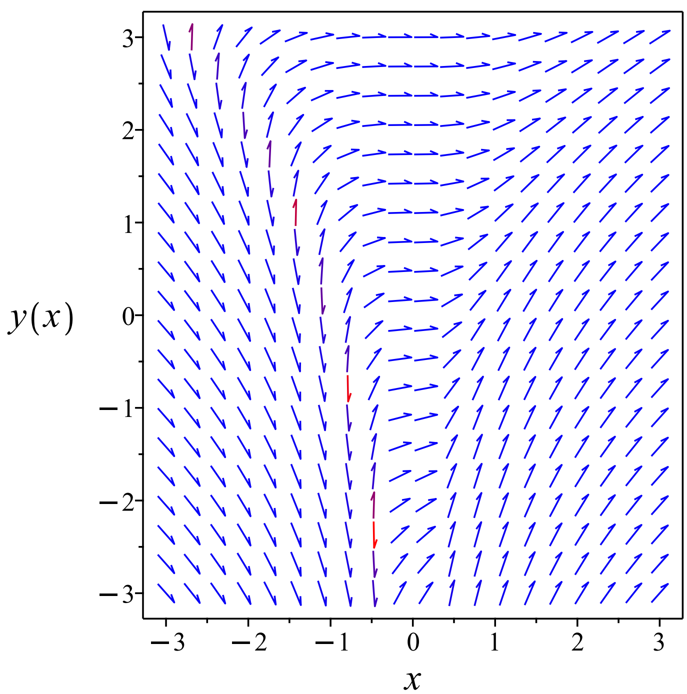


Figure 266: Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(x^3 e^{-c_1}\right) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 3/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((x^3+exp(y(x)))*diff(y(x),x)=3*x^2,y(x), singsol=all)
```

$$y(x) = \ln\left(\frac{x^3}{\operatorname{LambertW}\left(\frac{x^3}{c_1}\right)}\right)$$

✓ Solution by Mathematica

Time used: 3.536 (sec). Leaf size: 19

```
DSolve[(x^3+Exp[y[x]])*y'[x]==3*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to W\left(e^{-c_1}x^3\right) + c_1$$

6.27 problem 160

Internal problem ID [15053]
Internal file name [`OUTPUT/15053_Sunday_April_21_2024_01_21_35_PM_32505698/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 160.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 3yx - y\,\mathrm{e}^{x^2} = 0}$$

6.27.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= y\left(\mathrm{e}^{x^2} - 3x\right) \end{aligned}$$

Where $f(x) = e^{x^2} - 3x$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y} dy &= e^{x^2} - 3x\, dx \\
\int \frac{1}{y} dy &= \int e^{x^2} - 3x\, dx \\
\ln(y) &= -\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_1 \\
y &= e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_1} \\
&= c_1 e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}} \tag{1}
$$

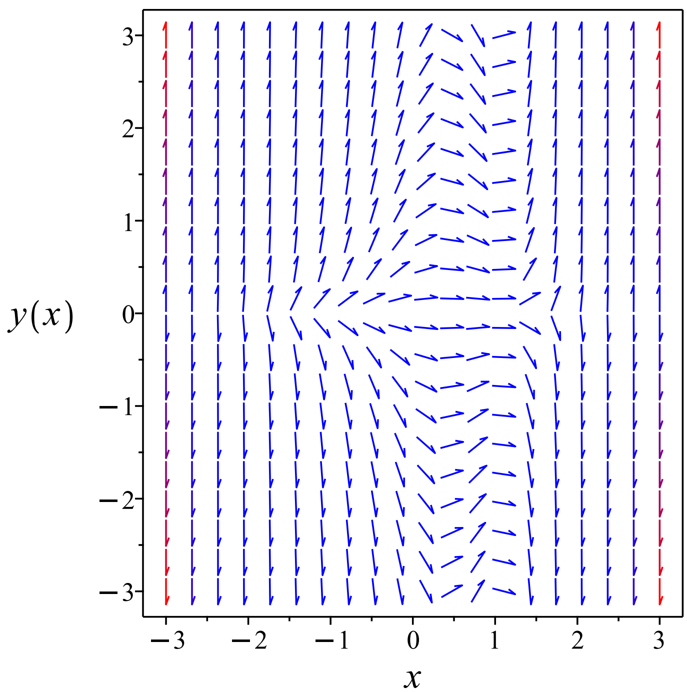


Figure 267: Slope field plot

Verification of solutions

$$
y = c_1 e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}}
$$

Verified OK.

6.27.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\mathrm{e}^{x^2} + 3x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + y\left(-\mathrm{e}^{x^2} + 3x\right) = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \left(-\mathrm{e}^{x^2}+3x\right)dx} \\ &= \mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}}$ results in

$$y = c_1 \mathrm{e}^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} \tag{1}$$

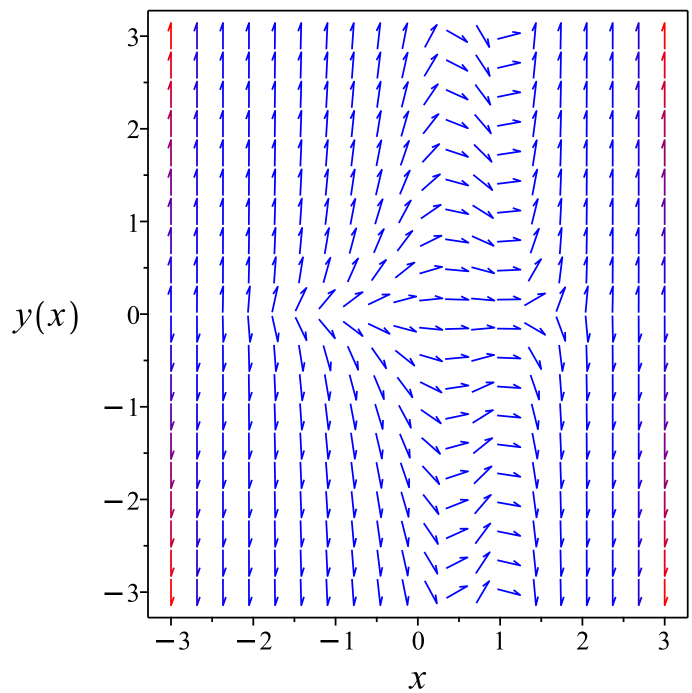


Figure 268: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2}}$$

Verified OK.

6.27.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + 3u(x)\,x^2 - u(x)\,x\,e^{x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u\left(x\,e^{x^2} - 3x^2 - 1\right)}{x} \end{aligned}$$

Where $f(x) = \frac{x\,e^{x^2}-3x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x\,e^{x^2}-3x^2-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x\,e^{x^2}-3x^2-1}{x}\,dx \\
\ln(u) &= -\frac{3x^2}{2} - \ln(x) + \frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2} + c_2 \\
u &= e^{-\frac{3x^2}{2}-\ln(x)+\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2}+c_2} \\
&= c_2 e^{-\frac{3x^2}{2}-\ln(x)+\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2}}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x c_2 e^{-\frac{3x^2}{2}-\ln(x)+\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x c_2 e^{-\frac{3x^2}{2}-\ln(x)+\frac{\sqrt{\pi}\,\operatorname{erfi}(x)}{2}} \tag{1}
$$

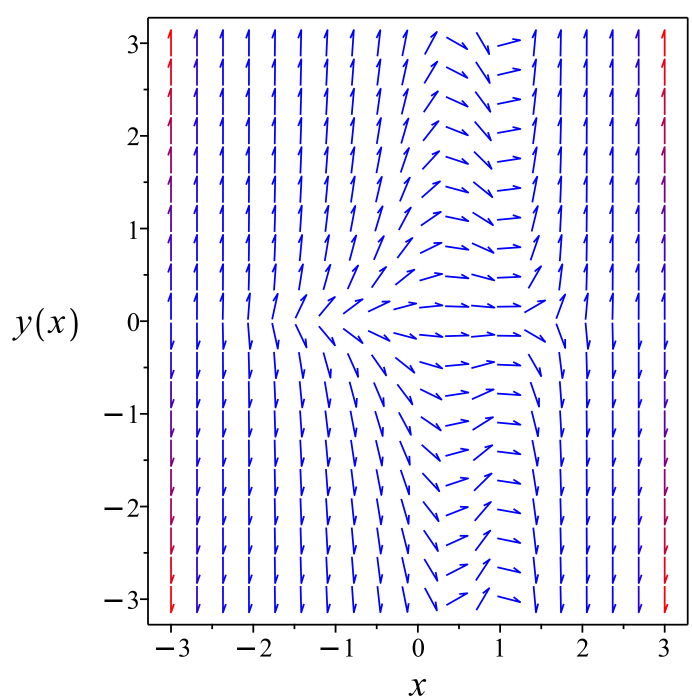


Figure 269: Slope field plot

Verification of solutions

$$y = xc_2 \mathrm{e}^{-\frac{3x^2}{2} - \ln(x) + \frac{\sqrt{\pi}\, \mathrm{erfi}(x)}{2}}$$

Verified OK.

6.27.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -3xy + \mathrm{e}^{x^2} y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 216: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{3x^2}{2} - \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -3xy + e^{x^2} y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \left(-e^{x^2} + 3x\right) e^{\frac{3x^2}{2} - \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}} y \\ S_y &= e^{\frac{3x^2}{2} - \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} y = c_1$$

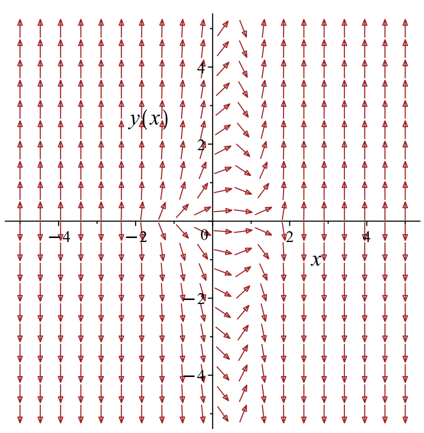
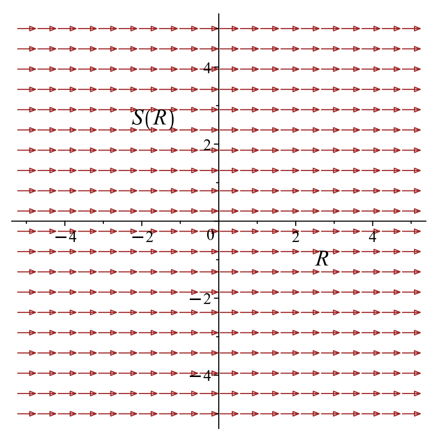
Which simplifies to

$$\mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} y = c_1$$

Which gives

$$y = c_1 \mathrm{e}^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -3xy + \mathrm{e}^{x^2} y$ 	$R = x$ $S = \mathrm{e}^{\frac{3x^2}{2}-\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}} \tag{1}$$

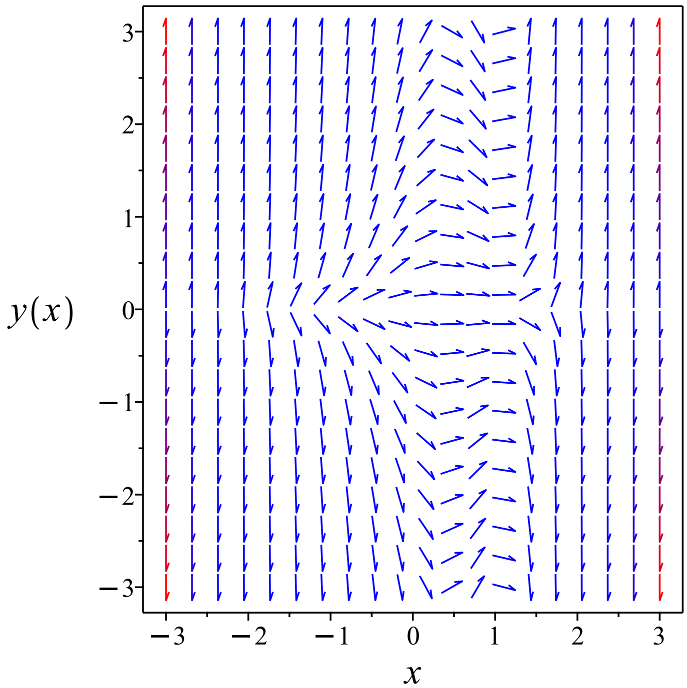


Figure 270: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}}$$

Verified OK.

6.27.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y=\left(\mathrm{e}^{x^2}-3x\right)\mathrm{d}x$$
$$\left(-\mathrm{e}^{x^2}+3x\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=-\mathrm{e}^{x^2}+3x$$
$$N(x,y)=\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(-\mathrm{e}^{x^2}+3x\right)$$
$$=0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\mathrm{e}^{x^2} + 3x \,\mathrm{d}x \\
\phi &= \frac{3x^2}{2} - \frac{\sqrt{\pi}\,\mathrm{erfi}\,(x)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3x^2}{2} - \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3x^2}{2} - \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + \ln(y)$$

The solution becomes

$$y = e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{3x^2}{2} + \frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_1} \tag{1}$$

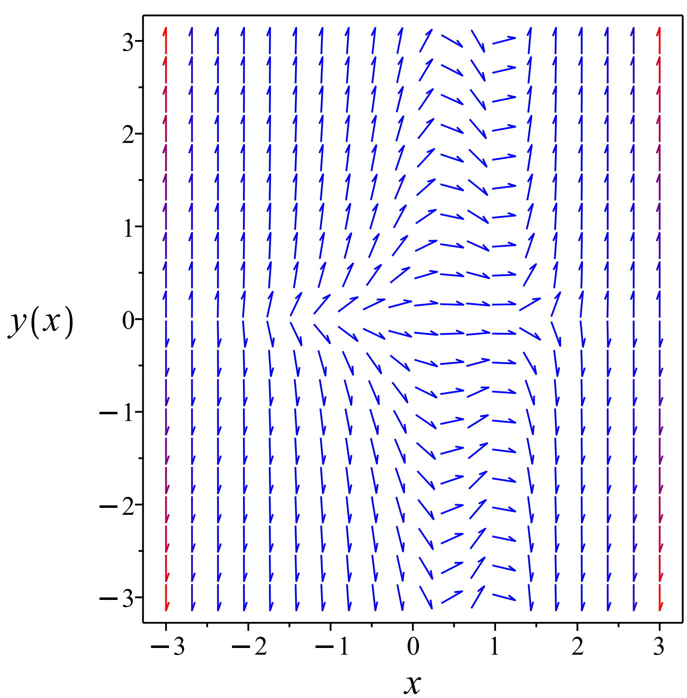


Figure 271: Slope field plot

Verification of solutions

$$y = e^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}+c_1}$$

Verified OK.

6.27.6 Maple step by step solution

Let's solve

$y' + 3yx - y\,e^{x^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = e^{x^2} - 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \left(e^{x^2} - 3x\right)dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{3x^2}{2} + \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2} + c_1$

- Solve for y

  $y = e^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)+3*x*y(x)=y(x)*exp(x^2),y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{3x^2}{2}+\frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2}}$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 33

```
DSolve[y'[x]+3*x*y[x]==y[x]*Exp[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{1}{2}\left(\sqrt{\pi}\operatorname{erfi}(x)-3x^2\right)}$$
$$y(x) \to 0$$

6.28 problem 161

Internal problem ID [15054]
Internal file name [OUTPUT/15054_Sunday_April_21_2024_01_21_36_PM_25578689/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 161.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{y' - 2y\,\mathrm{e}^{x} - 2\sqrt{y\,\mathrm{e}^{x}} = 0}$$

Unable to determine ODE type.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(y(x), x), x)+2*y(x)*exp(2*x)-y(x)*exp(x)-3*(diff
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   trying symmetries linear in x and y(x)
   -> Try solving first the homogeneous part of the ODE
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
            A Liouvillian solution exists
            Reducible group (found an exponential solution)
            Group is reducible, not completely reducible
            Solution has integrals. Trying a special function solution free of integrals...
            -> Trying a solution in terms of special functions:
               -> Bessel
               -> elliptic
               -> Legendre
               <- Kummer successful
            <- special function solution successful
               -> Trying to convert hypergeometric functions to elementary form...
               <- elementary form is not straightforward to achieve - returning special func
            <- Kovacics algorithm successful
         Change of variables used:
            [x = ln(t)]
         Linear ODE actually solved:
            (4*t-2)*u(t)+(-6*t+1)*diff(u(t),t)+2*t*diff(diff(u(t),t),t) = 0
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 53

```
dsolve(diff(y(x),x)-2*y(x)*exp(x)=2*sqrt(y(x)*exp(x)),y(x), singsol=all)
```

$$\frac{y(x)\,\mathrm{e}^{\frac{x}{2}-\mathrm{e}^{x}}-\left(\int \mathrm{e}^{\frac{x}{2}-\mathrm{e}^{x}}dx\right)\sqrt{y\left(x\right)\mathrm{e}^{x}}+c_1\sqrt{y\left(x\right)\mathrm{e}^{x}}}{\sqrt{y\left(x\right)\mathrm{e}^{x}}}=0$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 56

```
DSolve[y'[x]-2*y[x]*Exp[x]==2*Sqrt[y[x]*Exp[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2\left(\sqrt{\pi}\sqrt{y(x)}\text{erf}\left(\frac{\sqrt{e^x y(x)}}{\sqrt{y(x)}}\right)-e^{-e^x}y(x)\right)}{\sqrt{y(x)}}=c_1,y(x)\right]$$

6.29 problem 162

Internal problem ID [15055]
Internal file name [`OUTPUT/15055_Sunday_April_21_2024_01_21_39_PM_80471787/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 162.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{2y' \ln (x) + \frac{y}{x} - \frac{\cos (x)}{y} = 0}$$

6.29.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y^2 + \cos (x)\, x}{2 \ln (x)\, xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y \ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x)\, y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y^2 + \cos(x)\, x}{2\ln(x)\, xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2x} \\ S_y &= y\ln(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos(x)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cos(R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\sin(R)}{2} + c_1 \tag{4}$$

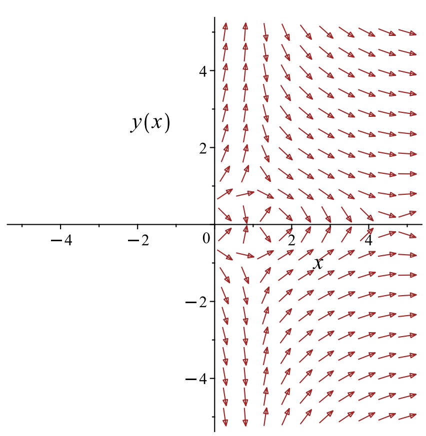
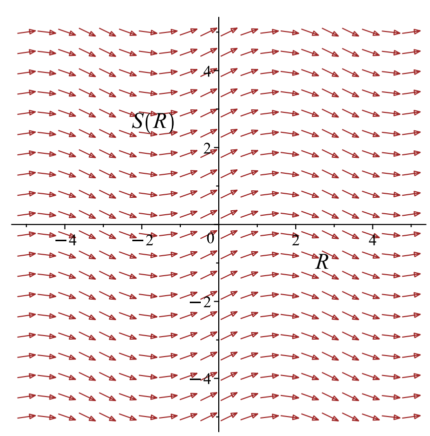
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 \ln(x)}{2} = \frac{\sin(x)}{2} + c_1$$

Which simplifies to

$$\frac{y^2 \ln(x)}{2} = \frac{\sin(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y^2+\cos(x)x}{2\ln(x)xy}$ 	$R = x$ $S = \frac{\ln(x)\, y^2}{2}$	$\frac{dS}{dR} = \frac{\cos(R)}{2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2 \ln(x)}{2} = \frac{\sin(x)}{2} + c_1 \tag{1}$$

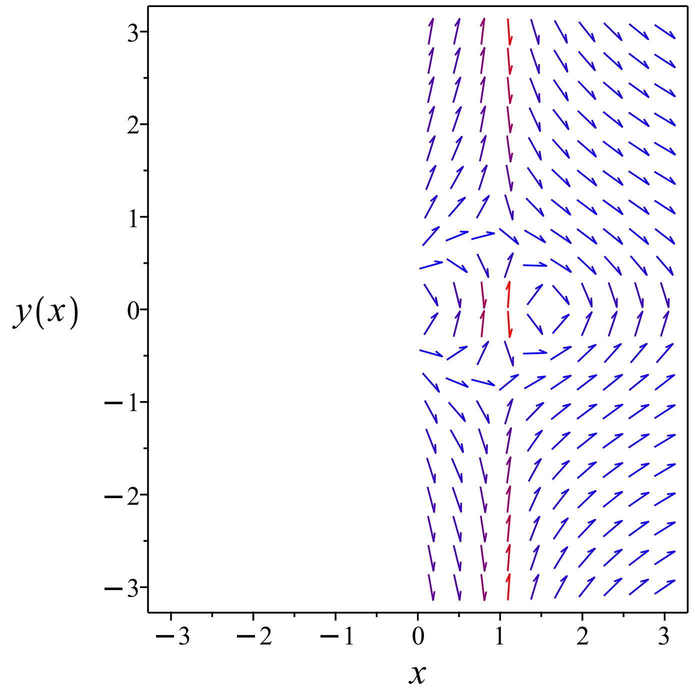


Figure 272: Slope field plot

Verification of solutions

$$\frac{y^2 \ln(x)}{2} = \frac{\sin(x)}{2} + c_1$$

Verified OK.

6.29.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-y^2 + \cos(x) x}{2 \ln(x) x y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x \ln(x)} y + \frac{\cos(x)}{2 \ln(x)} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2x\ln(x)} \\ f_1(x) &= \frac{\cos(x)}{2\ln(x)} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2x\ln(x)} + \frac{\cos(x)}{2\ln(x)} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{2x\ln(x)} + \frac{\cos(x)}{2\ln(x)} \\ w' &= -\frac{w}{x\ln(x)} + \frac{\cos(x)}{\ln(x)} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x \ln(x)} \\
q(x) &= \frac{\cos(x)}{\ln(x)}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{w(x)}{x \ln(x)} = \frac{\cos(x)}{\ln(x)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x \ln(x)} dx} \\
&= \ln(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu) \left(\frac{\cos(x)}{\ln(x)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x) w) &= (\ln(x)) \left(\frac{\cos(x)}{\ln(x)}\right) \\
\mathrm{d}(\ln(x) w) &= \cos(x) \, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\ln(x) w &= \int \cos(x) \, \mathrm{d}x \\
\ln(x) w &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$
w(x) = \frac{\sin(x)}{\ln(x)} + \frac{c_1}{\ln(x)}
$$

which simplifies to

$$
w(x) = \frac{\sin(x) + c_1}{\ln(x)}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = \frac{\sin(x) + c_1}{\ln(x)}
$$

Solving for y gives

$$y(x) = \frac{\sqrt{\ln(x)(\sin(x)+c_1)}}{\ln(x)}$$
$$y(x) = -\frac{\sqrt{\ln(x)(\sin(x)+c_1)}}{\ln(x)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{\ln(x)(\sin(x)+c_1)}}{\ln(x)} \tag{1}$$
$$y = -\frac{\sqrt{\ln(x)(\sin(x)+c_1)}}{\ln(x)} \tag{2}$$

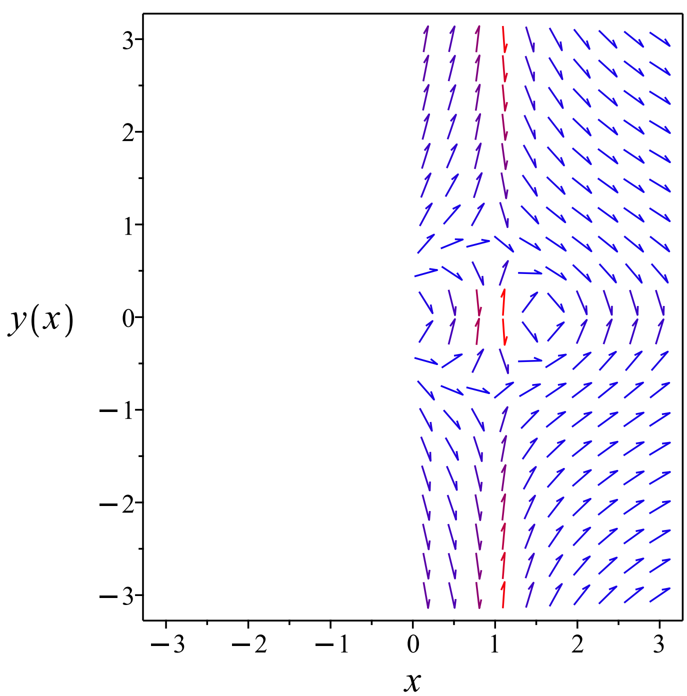


Figure 273: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\ln (x) (\sin (x) + c_1)}}{\ln (x)}$$

Verified OK.

$$y = -\frac{\sqrt{\ln (x) (\sin (x) + c_1)}}{\ln (x)}$$

Verified OK.

6.29.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(2\ln(x))\,\mathrm{d}y = \left(-\frac{y}{x} + \frac{\cos(x)}{y}\right)\mathrm{d}x$$

$$\left(\frac{y}{x} - \frac{\cos(x)}{y}\right)\mathrm{d}x + (2\ln(x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{y}{x} - \frac{\cos(x)}{y} \\ N(x, y) &= 2\ln(x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{x} - \frac{\cos(x)}{y}\right) \\ &= \frac{1}{x} + \frac{\cos(x)}{y^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2\ln(x)) \\ &= \frac{2}{x} \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2\ln(x)}\left(\left(\frac{1}{x} + \frac{\cos(x)}{y^2}\right) - \left(\frac{2}{x}\right)\right) \\ &= \frac{-\frac{1}{x} + \frac{\cos(x)}{y^2}}{2\ln(x)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{xy}{-y^2 + \cos(x)\,x}\left(\left(\frac{2}{x}\right) - \left(\frac{1}{x} + \frac{\cos(x)}{y^2}\right)\right) \\
&= \frac{1}{y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1}{y} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(y)} \\
&= y
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= y\left(\frac{y}{x} - \frac{\cos(x)}{y}\right) \\
&= \frac{y^2 - \cos(x)\,x}{x}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= y(2\ln(x)) \\
&= 2y\ln(x)
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^2 - \cos(x)\,x}{x}\right) + (2y\ln(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{y^2 - \cos(x)\, x}{x} \mathrm{d}x$$

$$\phi = -\sin(x) + \ln(x)\, y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y \ln(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y \ln(x)$. Therefore equation (4) becomes

$$2y \ln(x) = 2y \ln(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \ln(x)\, y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) + \ln(x)\, y^2$$

Summary

The solution(s) found are the following

$$y^2 \ln(x) - \sin(x) = c_1 \tag{1}$$

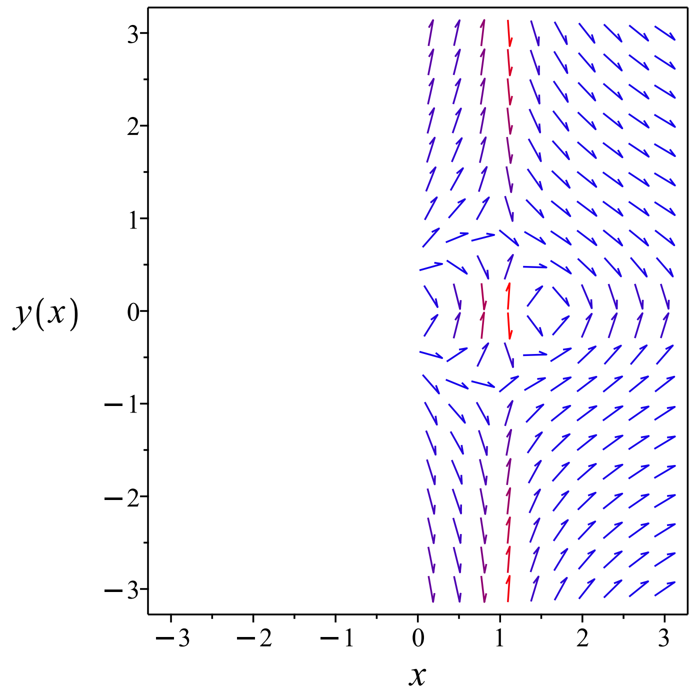


Figure 274: Slope field plot

Verification of solutions

$$y^2 \ln(x) - \sin(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(2*diff(y(x),x)*ln(x)+y(x)/x=cos(x)/y(x),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\ln(x)(\sin(x) + c_1)}}{\ln(x)}$$
$$y(x) = -\frac{\sqrt{\ln(x)(\sin(x) + c_1)}}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.311 (sec). Leaf size: 42

```
DSolve[2*y'[x]*Log[x]+y[x]/x==Cos[x]/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\sin(x) + c_1}}{\sqrt{\log(x)}}$$
$$y(x) \to \frac{\sqrt{\sin(x) + c_1}}{\sqrt{\log(x)}}$$

6.30 problem 163

Internal problem ID [15056]
Internal file name [`OUTPUT/15056_Sunday_April_21_2024_01_21_43_PM_90483149/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 163.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{2y' \sin(x) + y \cos(x) - \sin(x)^2 y^3 = 0}$$

6.30.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y\left(-\sin(x)^2 y^2 + \cos(x)\right)}{2\sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 221: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sin(x)\, y^3
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)\, y^3} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2\sin(x)\, y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y\left(-\sin(x)^2 y^2 + \cos(x)\right)}{2\sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\csc(x)\cot(x)}{2y^2} \\ S_y &= \frac{\csc(x)}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

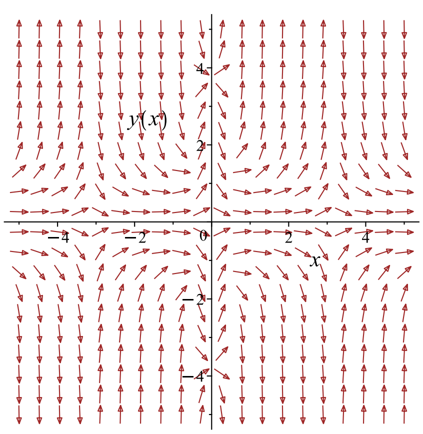
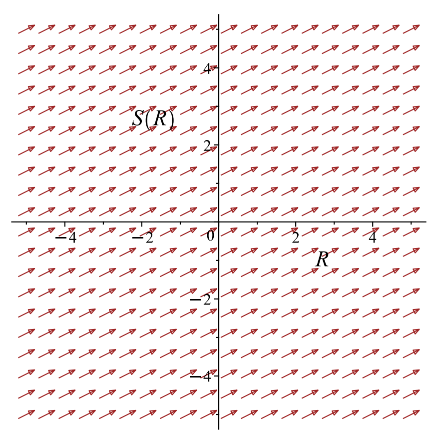
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\csc(x)}{2y^2} = \frac{x}{2} + c_1$$

Which simplifies to

$$-\frac{\csc(x)}{2y^2} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\left(-\sin(x)^2 y^2+\cos(x)\right)}{2\sin(x)}$ 	$R = x$ $S = -\frac{\csc(x)}{2y^2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary

The solution(s) found are the following

$$-\frac{\csc(x)}{2y^2} = \frac{x}{2} + c_1 \tag{1}$$

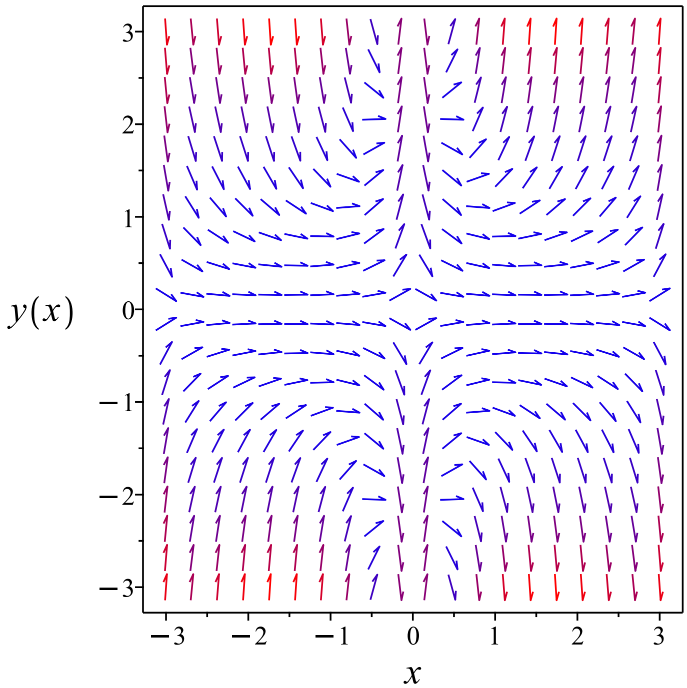


Figure 275: Slope field plot

Verification of solutions

$$-\frac{\csc(x)}{2y^2} = \frac{x}{2} + c_1$$

Verified OK.

6.30.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-\sin(x)^2 y^2 + \cos(x))}{2\sin(x)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{\cos(x)}{2\sin(x)}y + \frac{\sin(x)}{2}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{\cos(x)}{2\sin(x)} \\
f_1(x) &= \frac{\sin(x)}{2} \\
n &= 3
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{\cos(x)}{2\sin(x)\,y^2} + \frac{\sin(x)}{2} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{\cos(x)\,w(x)}{2\sin(x)} + \frac{\sin(x)}{2} \\
w' &= \frac{\cos(x)\,w}{\sin(x)} - \sin(x)
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\cot(x) \\
q(x) &= -\sin(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \cot(x)\, w(x) = -\sin(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\cot(x)dx} \\
&= \frac{1}{\sin(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \csc(x)
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-\sin(x)) \\
\frac{d}{dx}(\csc(x)\, w) &= (\csc(x))(-\sin(x)) \\
d(\csc(x)\, w) &= -1\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\csc(x)\, w &= \int -1\, dx \\
\csc(x)\, w &= -x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$
w(x) = -x\sin(x) + \sin(x)\, c_1
$$

which simplifies to

$$
w(x) = \sin(x)(-x + c_1)
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = \sin(x)(-x + c_1)
$$

Solving for y gives

$$y(x) = \frac{1}{\sqrt{\sin(x)(-x+c_1)}}$$
$$y(x) = -\frac{1}{\sqrt{\sin(x)(-x+c_1)}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{\sin(x)(-x+c_1)}} \tag{1}$$
$$y = -\frac{1}{\sqrt{\sin(x)(-x+c_1)}} \tag{2}$$

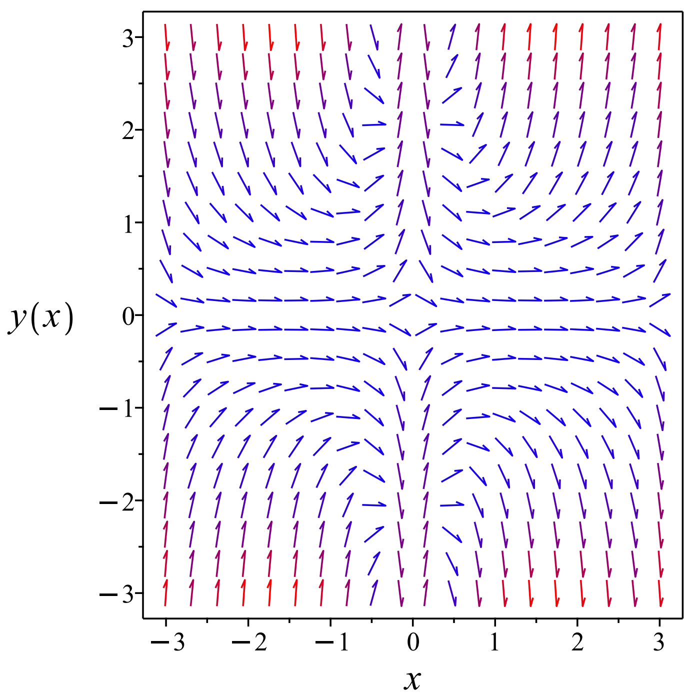


Figure 276: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{\sin(x)(-x+c_1)}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{\sin(x)(-x+c_1)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(2*diff(y(x),x)*sin(x)+y(x)*cos(x)=y(x)^3*sin(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{(-x+c_1)\sin(x)}}$$
$$y(x) = -\frac{1}{\sqrt{(-x+c_1)\sin(x)}}$$

✓ Solution by Mathematica

Time used: 0.516 (sec). Leaf size: 43

```
DSolve[2*y'[x]*Sin[x]+y[x]*Cos[x]==y[x]^3*Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{(-x+c_1)\sin(x)}}$$
$$y(x) \to \frac{1}{\sqrt{-((x-c_1)\sin(x))}}$$
$$y(x) \to 0$$

6.31 problem 164

Internal problem ID [15057]
Internal file name [OUTPUT/15057_Sunday_April_21_2024_01_21_51_PM_92906130/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 164.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\left(1+x^{2}+y^{2}\right) y'+yx=0}$$

6.31.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+y^2+1\right)\mathrm{d}y&=(-xy)\,\mathrm{d}x\\(xy)\,\mathrm{d}x+\left(x^2+y^2+1\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=xy\\N(x,y)&=x^2+y^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(xy)\\&=x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x^2+y^2+1\right)\\&=2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\frac{1}{x^2+y^2+1}((x)-(2x))\\&=-\frac{x}{x^2+y^2+1}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{xy}((2x) - (x)) \\ &= \frac{1}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{1}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(y)} \\ &= y \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= y(xy) \\ &= y^2 x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= y\left(x^2 + y^2 + 1\right) \\ &= y\left(x^2 + y^2 + 1\right) \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(y^2 x\right) + \left(y\left(x^2 + y^2 + 1\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int y^2 x\,\mathrm{d}x$$
$$\phi = \frac{y^2x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = x^2y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = y(x^2+y^2+1)$. Therefore equation (4) becomes

$$y(x^2+y^2+1) = x^2y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3 + y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^3+y)\,\mathrm{d}y$$
$$f(y) = \frac{(y^2+1)^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^2x^2}{2} + \frac{(y^2+1)^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2 x^2}{2} + \frac{(y^2+1)^2}{4}$$

Summary

The solution(s) found are the following

$$\frac{x^2 y^2}{2} + \frac{(1+y^2)^2}{4} = c_1 \tag{1}$$

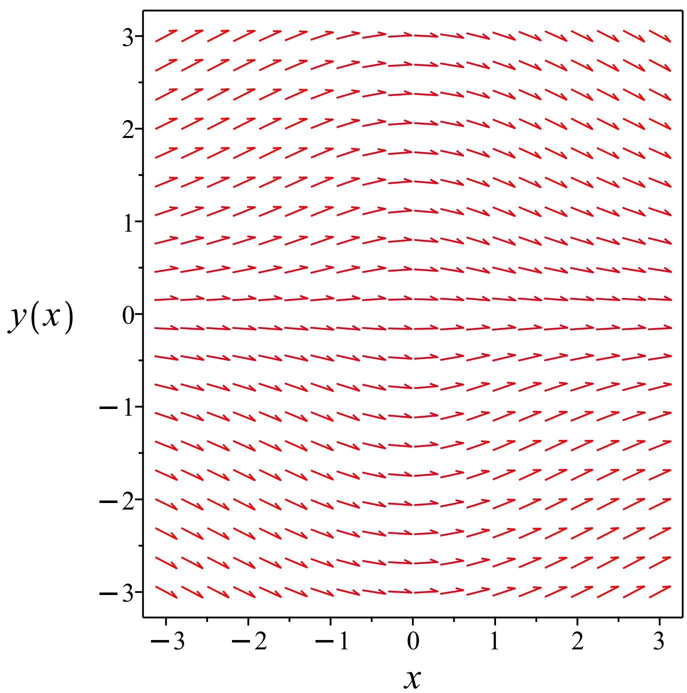


Figure 277: Slope field plot

Verification of solutions

$$\frac{x^2 y^2}{2} + \frac{(1+y^2)^2}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 113

```
dsolve((x^2+y(x)^2+1)*diff(y(x),x)+x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2 - 1 - \sqrt{x^4 + 2x^2 - 4c_1}}$$
$$y(x) = \sqrt{-x^2 - 1 + \sqrt{x^4 + 2x^2 - 4c_1}}$$
$$y(x) = -\sqrt{-x^2 - 1 - \sqrt{x^4 + 2x^2 - 4c_1}}$$
$$y(x) = -\sqrt{-x^2 - 1 + \sqrt{x^4 + 2x^2 - 4c_1}}$$

✓ Solution by Mathematica

Time used: 2.437 (sec). Leaf size: 146

```
DSolve[(x^2+y[x]^2+1)*y'[x]+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 - \sqrt{x^4 + 2x^2 + 1 + 4c_1} - 1}$$
$$y(x) \to \sqrt{-x^2 - \sqrt{x^4 + 2x^2 + 1 + 4c_1} - 1}$$
$$y(x) \to -\sqrt{-x^2 + \sqrt{x^4 + 2x^2 + 1 + 4c_1} - 1}$$
$$y(x) \to \sqrt{-x^2 + \sqrt{x^4 + 2x^2 + 1 + 4c_1} - 1}$$
$$y(x) \to 0$$

6.32 problem 165

Internal problem ID [15058]
Internal file name [`OUTPUT/15058_Sunday_April_21_2024_01_21_52_PM_77885278/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 165.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - y\cos(x) - y^2\cos(x) = 0}$$

6.32.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \cos(x)\left(y^2+y\right) \end{aligned}$$

Where $f(x) = \cos(x)$ and $g(y) = y^2 + y$. Integrating both sides gives

$$\frac{1}{y^2+y}\,dy = \cos(x)\,dx$$

$$\int \frac{1}{y^2+y}\,dy = \int \cos(x)\,dx$$
$$-\ln(y+1)+\ln(y) = \sin(x)+c_1$$

Raising both side to exponential gives

$$e^{-\ln(y+1)+\ln(y)} = e^{\sin(x)+c_1}$$

Which simplifies to

$$\frac{y}{y+1} = c_2 e^{\sin(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_2 e^{\sin(x)}}{-1+c_2 e^{\sin(x)}} \tag{1}$$

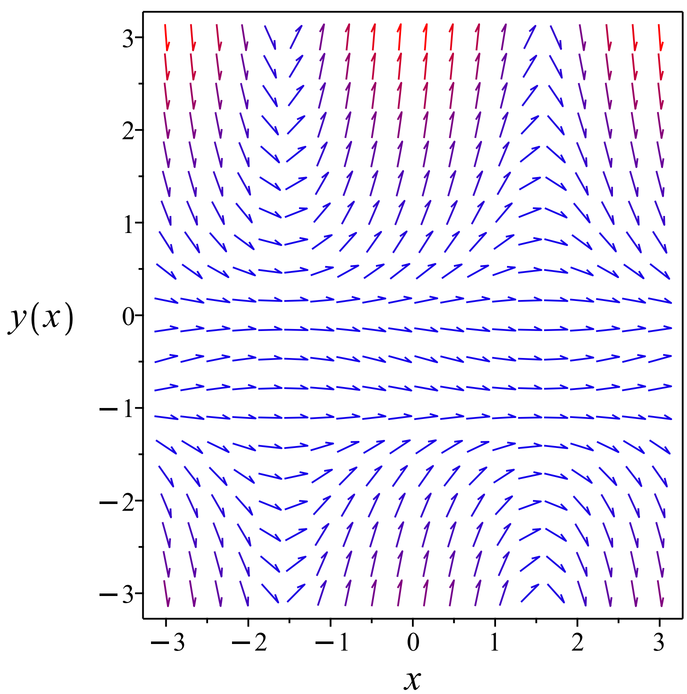


Figure 278: Slope field plot

Verification of solutions

$$y = -\frac{c_2 e^{\sin(x)}}{-1+c_2 e^{\sin(x)}}$$

Verified OK.

6.32.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \cos(x)\, y + \cos(x)\, y^2 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 223: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{\cos(x)} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{\cos(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \sin(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \cos(x)\, y + \cos(x)\, y^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \cos(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y(y+1)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R(R+1)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R+1) + \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sin(x) = -\ln(y+1) + \ln(y) + c_1
$$

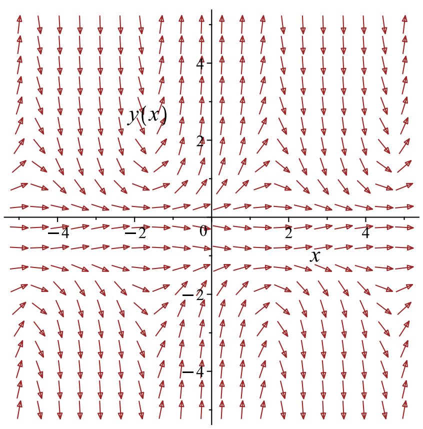
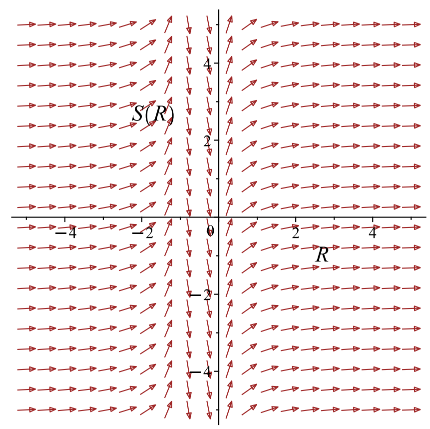
Which simplifies to

$$
\sin(x) = -\ln(y+1) + \ln(y) + c_1
$$

Which gives

$$
y = \frac{1}{e^{-\sin(x)+c_1} - 1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(x)\, y + \cos(x)\, y^2$ 	$R = y$ $S = \sin(x)$	$\frac{dS}{dR} = \frac{1}{R(R+1)}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{e^{-\sin(x)+c_1} - 1} \tag{1}$$

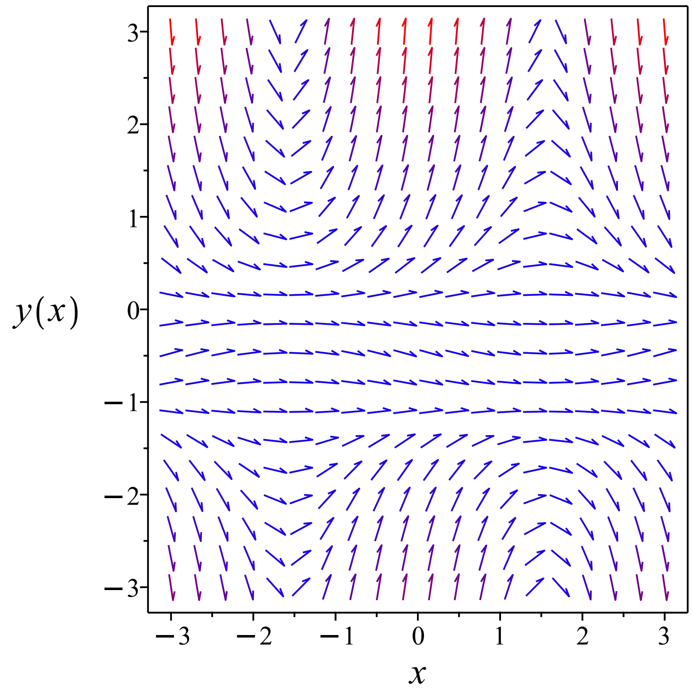


Figure 279: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{\mathrm{e}^{-\sin(x)+c_1} - 1}$$

Verified OK.

6.32.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \cos\left(x\right) y+\cos\left(x\right) y^{2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \cos\left(x\right) y+\cos\left(x\right) y^{2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \cos(x) \\
f_1(x) &= \cos(x) \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{\cos(x)}{y} + \cos(x) \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \cos(x)\, w(x) + \cos(x) \\
w' &= -\cos(x)\, w - \cos(x)
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \cos(x) \\
q(x) &= -\cos(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \cos(x)\, w(x) = -\cos(x)
$$

The integrating factor μ is

$$\mu = e^{\int \cos(x)dx}$$
$$= e^{\sin(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(-\cos(x))$$
$$\frac{d}{dx}\left(e^{\sin(x)}w\right) = \left(e^{\sin(x)}\right)(-\cos(x))$$
$$d\left(e^{\sin(x)}w\right) = \left(-\cos(x)\,e^{\sin(x)}\right)\,dx$$

Integrating gives

$$e^{\sin(x)}w = \int -\cos(x)\,e^{\sin(x)}\,dx$$
$$e^{\sin(x)}w = -e^{\sin(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$w(x) = -e^{-\sin(x)}e^{\sin(x)} + c_1e^{-\sin(x)}$$

which simplifies to

$$w(x) = -1 + c_1e^{-\sin(x)}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = -1 + c_1e^{-\sin(x)}$$

Or

$$y = \frac{1}{-1 + c_1e^{-\sin(x)}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{-1 + c_1e^{-\sin(x)}} \tag{1}$$

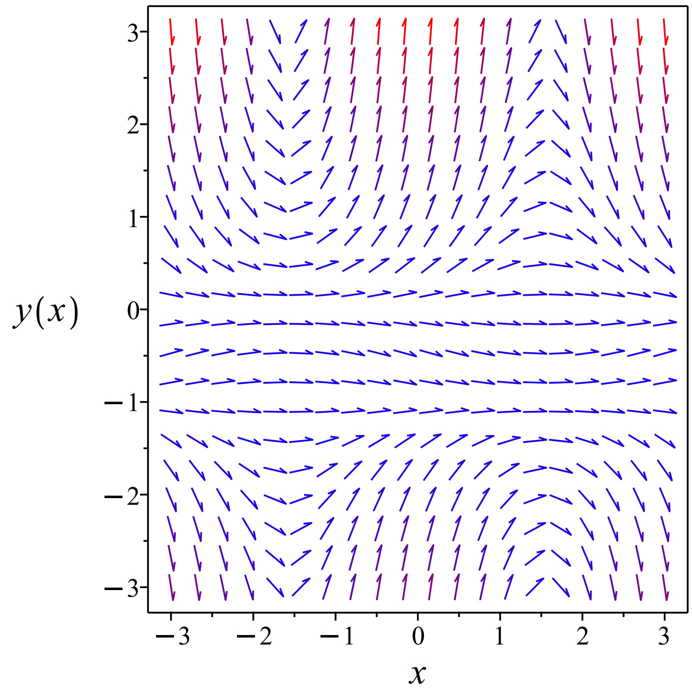


Figure 280: Slope field plot

Verification of solutions

$$y = \frac{1}{-1 + c_1 e^{-\sin(x)}}$$

Verified OK.

6.32.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2 + y}\right) \mathrm{d}y = (\cos(x))\,\mathrm{d}x$$
$$(-\cos(x))\,\mathrm{d}x + \left(\frac{1}{y^2 + y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\cos(x)$$
$$N(x, y) = \frac{1}{y^2 + y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x))\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^2+y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\cos(x)\,\mathrm{d}x$$
$$\phi = -\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+y}$. Therefore equation (4) becomes

$$\frac{1}{y^2+y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y(y+1)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y(y+1)}\right)\mathrm{d}y$$
$$f(y) = -\ln(y+1) + \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) - \ln(y+1) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) - \ln(y+1) + \ln(y)$$

The solution becomes

$$y = -\frac{e^{\sin(x)+c_1}}{-1 + e^{\sin(x)+c_1}}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{\sin(x)+c_1}}{-1 + e^{\sin(x)+c_1}} \tag{1}$$

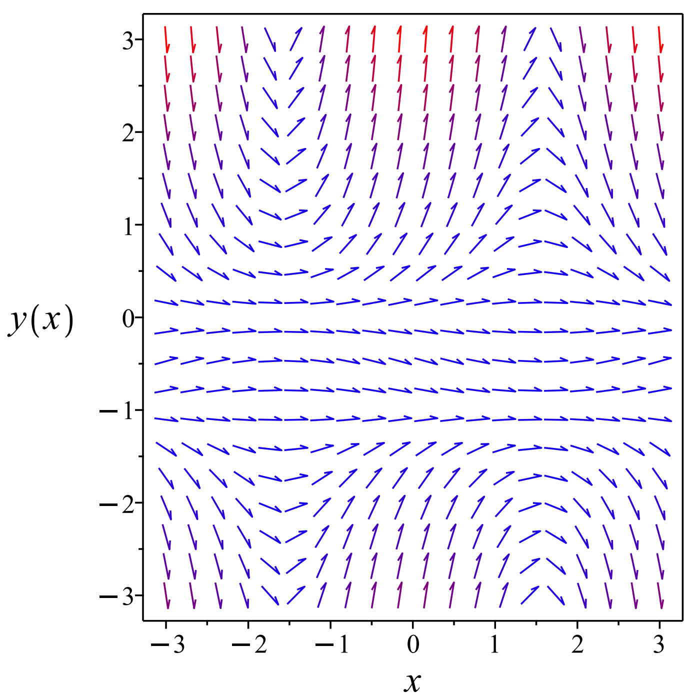


Figure 281: Slope field plot

Verification of solutions

$$y = -\frac{e^{\sin(x)+c_1}}{-1 + e^{\sin(x)+c_1}}$$

Verified OK.

6.32.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \cos(x) y + \cos(x) y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \cos(x) y + \cos(x) y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \cos(x)$ and $f_2(x) = \cos(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\cos(x) u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\sin(x) \\ f_1 f_2 &= \cos(x)^2 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\cos(x) u''(x) - \left(\cos(x)^2 - \sin(x)\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 \mathrm{e}^{\sin(x)}$$

The above shows that

$$u'(x) = c_2 \cos(x) \mathrm{e}^{\sin(x)}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 \mathrm{e}^{\sin(x)}}{c_1 + c_2 \mathrm{e}^{\sin(x)}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{\mathrm{e}^{\sin(x)}}{c_3 + \mathrm{e}^{\sin(x)}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{\sin(x)}}{c_3 + \mathrm{e}^{\sin(x)}} \tag{1}$$

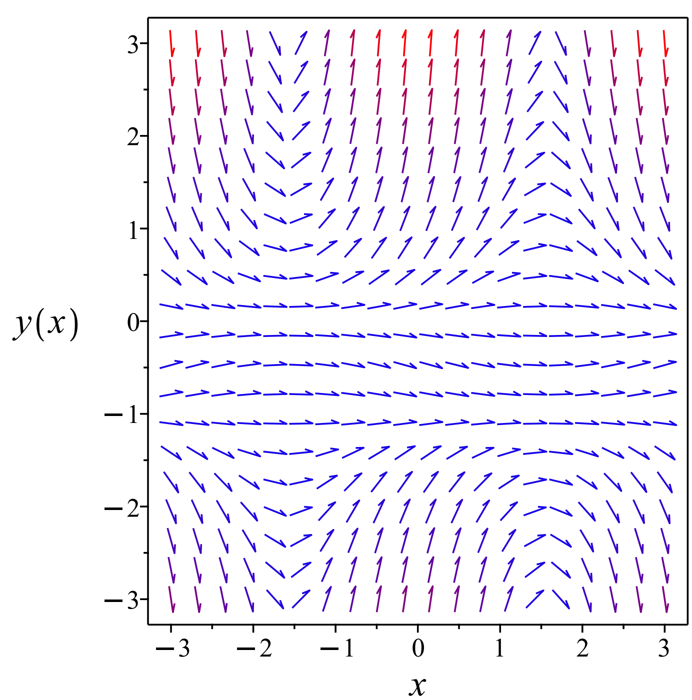


Figure 282: Slope field plot

Verification of solutions

$$y = -\frac{e^{\sin(x)}}{c_3 + e^{\sin(x)}}$$

Verified OK.

6.32.6 Maple step by step solution

Let's solve

$y' - y\cos(x) - y^2\cos(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y+1)} = \cos(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y+1)}dx = \int \cos(x)\,dx + c_1$

- Evaluate integral

  $-\ln(y+1) + \ln(y) = \sin(x) + c_1$

- Solve for y

  $y = -\frac{e^{\sin(x)+c_1}}{-1+e^{\sin(x)+c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)-y(x)*cos(x)=y(x)^2*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{1}{\mathrm{e}^{-\sin(x)} c_1 - 1}$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 35

```
DSolve[y'[x]-y[x]*Cos[x]==y[x]^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{\sin(x)+c_1}}{-1 + e^{\sin(x)+c_1}}$$
$$y(x) \to -1$$
$$y(x) \to 0$$

6.33 problem 166

Internal problem ID [15059]
Internal file name [OUTPUT/15059_Sunday_April_21_2024_01_21_53_PM_88960118/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 166.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - \tan(y) - \frac{e^{x}}{\cos(y)} = 0}$$

6.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y))\,\mathrm{d}y &= (\tan(y)\cos(y) + \mathrm{e}^x)\,\mathrm{d}x \\ (-\tan(y)\cos(y) - \mathrm{e}^x)\,\mathrm{d}x + (\cos(y))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\tan(y)\cos(y) - \mathrm{e}^x \\ N(x, y) &= \cos(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\tan(y)\cos(y) - \mathrm{e}^x) \\ &= -\cos(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \sec(y)\left(\left(-\left(\tan(y)^2 + 1\right)\cos(y) + \tan(y)\sin(y)\right) - (0)\right) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-\tan(y)\cos(y) - \mathrm{e}^{x}) \\
&= -\sin(y)\,\mathrm{e}^{-x} - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(\cos(y)) \\
&= \cos(y)\,\mathrm{e}^{-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\sin(y)\,\mathrm{e}^{-x} - 1\right) + \left(\cos(y)\,\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\sin(y)\,\mathrm{e}^{-x} - 1 \,\mathrm{d}x$$

$$\phi = -x + \sin(y)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(y)\,\mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y)\,\mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\cos(y)\,\mathrm{e}^{-x} = \cos(y)\,\mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \sin(y)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \sin(y)\,\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$-x + \sin(y)\,\mathrm{e}^{-x} = c_1 \tag{1}$$

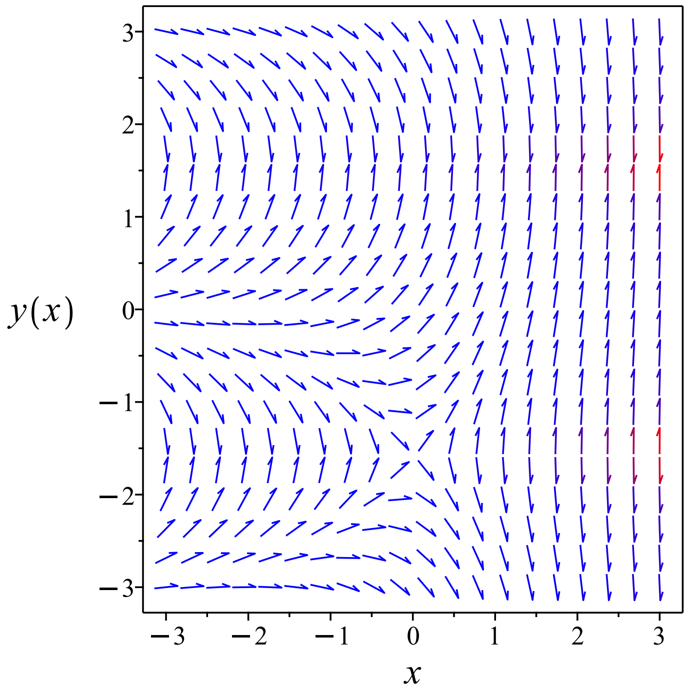


Figure 283: Slope field plot

<u>Verification of solutions</u>

$$-x + \sin(y)\,\mathrm{e}^{-x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)+(y(x)*exp(x)+1)/exp(x), y(x)`      *** Su
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)*sin(x)/cos(x), y(x)`      *** Subl
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)-tan(y(x))=exp(x)/cos(y(x)),y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 11.451 (sec). Leaf size: 14

```
DSolve[y'[x]-Tan[y[x]]==Exp[x]/Cos[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(e^x(x + c_1)\right)$$

6.34 problem 167

Internal problem ID [15060]
Internal file name [`OUTPUT/15060_Sunday_April_21_2024_01_21_56_PM_98789621/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 167.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{y' - y(\mathrm{e}^{x} + \ln(y)) = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5`[0, y*exp(x)], [0, -exp(x)*x*y+y*ln(y)]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)=y(x)*(exp(x)+ln(y(x))),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{e}^{x}(x+c_1)}$$

✓ Solution by Mathematica

Time used: 0.372 (sec). Leaf size: 15

```
DSolve[y'[x]==y[x]*(Exp[x]+Log[y[x]]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{e^x(x+c_1)}$$

6.35 problem 168

Internal problem ID [15061]
Internal file name [`OUTPUT/15061_Sunday_April_21_2024_01_21_57_PM_92896486/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 168.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]`

$$\boxed{\cos(y)\, y' + \sin(y) = x + 1}$$

6.35.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y))\,\mathrm{d}y &= (-\sin(y) + x + 1)\,\mathrm{d}x\\ (\sin(y) - x - 1)\,\mathrm{d}x + (\cos(y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \sin(y) - x - 1\\ N(x, y) &= \cos(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(y) - x - 1)\\ &= \cos(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \sec(y)\,((\cos(y)) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int 1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}(\sin(y) - x - 1) \\ &= (\sin(y) - x - 1)\,\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(\cos(y)) \\ &= \mathrm{e}^{x}\cos(y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((\sin(y) - x - 1)\,\mathrm{e}^{x}) + (\mathrm{e}^{x}\cos(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int (\sin(y) - x - 1)\,\mathrm{e}^{x}\,\mathrm{d}x \\ \phi &= (\sin(y) - x)\,\mathrm{e}^{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^x \cos(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^x \cos(y)$. Therefore equation (4) becomes

$$e^x \cos(y) = e^x \cos(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (\sin(y) - x) e^x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (\sin(y) - x) e^x$$

<u>Summary</u>

The solution(s) found are the following

$$(\sin(y) - x) e^x = c_1 \tag{1}$$

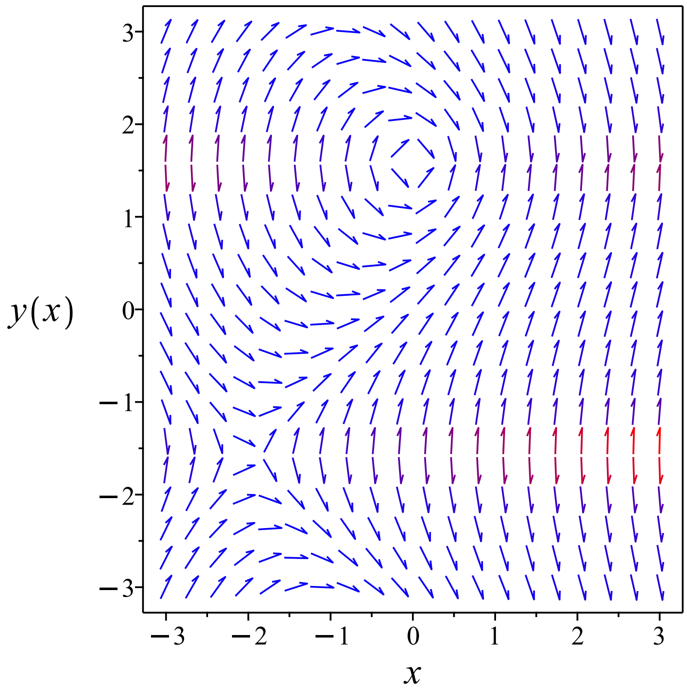


Figure 284: Slope field plot

Verification of solutions

$$(\sin(y) - x) e^{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)*cos(y(x))+sin(y(x))=x+1,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(-x + c_1 \mathrm{e}^{-x}\right)$$

✓ Solution by Mathematica

Time used: 13.261 (sec). Leaf size: 17

```
DSolve[y'[x]*Cos[y[x]]+Sin[y[x]]==x+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(x - c_1 e^{-x}\right)$$

6.36 problem 169

Internal problem ID [15062]
Internal file name [`OUTPUT/15062_Sunday_April_21_2024_01_22_00_PM_87799186/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 169.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{yy' - (x-1)\,\mathrm{e}^{-\frac{y^2}{2}} = -1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5`[0, (exp(-1/2*y^2)*x-2*exp(-1/2*y^2)-1)/y]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(y(x)*diff(y(x),x)+1=(x-1)*exp(-y(x)^2/2),y(x), singsol=all)
```

$$y(x) = \sqrt{2}\,\sqrt{\ln\left(-c_1 e^{-x} + x - 2\right)}$$
$$y(x) = -\sqrt{2}\,\sqrt{\ln\left(-c_1 e^{-x} + x - 2\right)}$$

✓ Solution by Mathematica

Time used: 7.375 (sec). Leaf size: 60

```
DSolve[y[x]*y'[x]+1==(x-1)*Exp[-y[x]^2/2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{-x + \log\left(e^x(x-2) + c_1\right)}$$
$$y(x) \to \sqrt{2}\sqrt{-x + \log\left(e^x(x-2) + c_1\right)}$$

6.37 problem 170

Internal problem ID [15063]
Internal file name [`OUTPUT/15063_Sunday_April_21_2024_01_22_02_PM_29534322/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 6. Linear equations of the first order. The Bernoulli equation. Exercises page 54
Problem number: 170.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' + x\sin(2y) - 2x\,\mathrm{e}^{-x^2}\cos(y)^2 = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5`[0, exp(-x^2)*(1+cos(2*y))], [0, exp(-x^2)*cos(2*
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)+x*sin(2*y(x))=2*x*exp(-x^2)*cos(y(x))^2,y(x), singsol=all)
```

$$y(x) = \arctan\left(\left(x^2 + 2c_1\right) e^{-x^2}\right)$$

✓ Solution by Mathematica

Time used: 10.038 (sec). Leaf size: 70

```
DSolve[y'[x]+x*Sin[2*y[x]]==2*x*Exp[-x^2]*Cos[y[x]]^2,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \arctan\left(e^{-x^2}\left(x^2 + c_1\right)\right)$$
$$y(x) \to -\frac{1}{2}\pi e^{x^2}\sqrt{e^{-2x^2}}$$
$$y(x) \to \frac{1}{2}\pi e^{x^2}\sqrt{e^{-2x^2}}$$

7 Section 7, Total differential equations. The integrating factor. Exercises page 61

7.1 problem 175

Internal problem ID [15064]
Internal file name [`OUTPUT/15064_Sunday_April_21_2024_01_22_05_PM_20451675/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 175.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{x(2x^2+y^2)+y(x^2+2y^2)\,y'=0}$$

7.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\left(2x^2+u(x)^2\,x^2\right)+u(x)\,x\left(x^2+2u(x)^2\,x^2\right)\left(u'(x)\,x+u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2(u^4+u^2+1)}{x\,(2u^3+u)}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^4+u^2+1}{2u^3+u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^4+u^2+1}{2u^3+u}}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{\frac{u^4+u^2+1}{2u^3+u}}\,du &= \int -\frac{2}{x}\,dx \\
\frac{\ln\left(u^4+u^2+1\right)}{2} &= -2\ln\left(x\right)+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{u^4+u^2+1} = \mathrm{e}^{-2\ln(x)+c_2}
$$

Which simplifies to

$$
\sqrt{u^4+u^2+1} = \frac{c_3}{x^2}
$$

Which simplifies to

$$
\sqrt{u\left(x\right)^4+u\left(x\right)^2+1} = \frac{c_3\mathrm{e}^{c_2}}{x^2}
$$

The solution is

$$
\sqrt{u\left(x\right)^4+u\left(x\right)^2+1} = \frac{c_3\mathrm{e}^{c_2}}{x^2}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\sqrt{\frac{y^4}{x^4}+\frac{y^2}{x^2}+1} &= \frac{c_3\mathrm{e}^{c_2}}{x^2} \\
\sqrt{\frac{y^4+x^2y^2+x^4}{x^4}} &= \frac{c_3\mathrm{e}^{c_2}}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\sqrt{\frac{y^4+x^2y^2+x^4}{x^4}} = \frac{c_3\mathrm{e}^{c_2}}{x^2} \tag{1}
$$

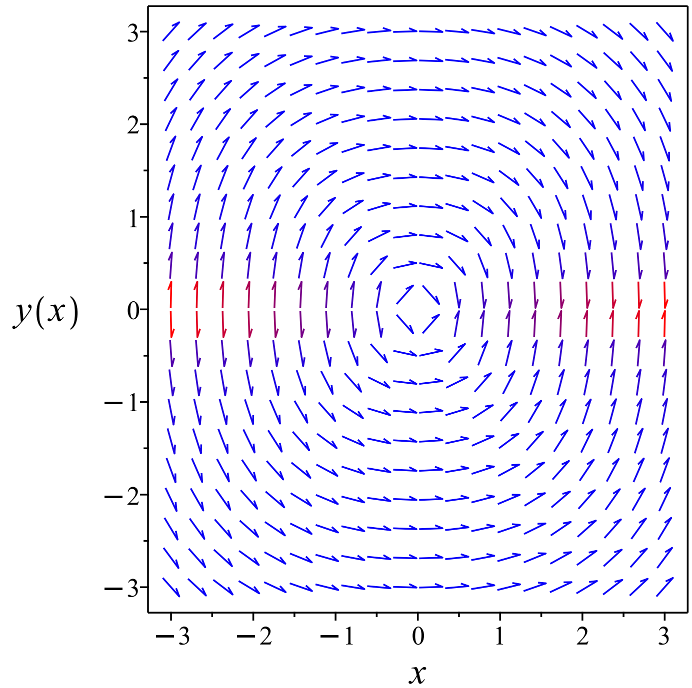


Figure 285: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^4+x^2y^2+x^4}{x^4}}=\frac{c_3\mathrm{e}^{c_2}}{x^2}$$

Verified OK.

7.1.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x(2x^2+y^2)}{y\,(x^2+2y^2)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{x(2x^2+y^2)(b_3-a_2)}{y(x^2+2y^2)} - \frac{x^2(2x^2+y^2)^2 a_3}{y^2(x^2+2y^2)^2} \\ & - \left(-\frac{2x^2+y^2}{y(x^2+2y^2)} - \frac{4x^2}{y(x^2+2y^2)} + \frac{2x^2(2x^2+y^2)}{y(x^2+2y^2)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{2x}{x^2+2y^2} + \frac{x(2x^2+y^2)}{y^2(x^2+2y^2)} + \frac{4x(2x^2+y^2)}{(x^2+2y^2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^6a_3+2x^6b_2-4x^5ya_2+4x^5yb_3+2x^4y^2a_3+10x^4y^2b_2-16x^3y^3a_2+16x^3y^3b_3-10x^2y^4a_3-2x^2y^4b_2-}{y^2(x^2+2y}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & -4x^6a_3-2x^6b_2+4x^5ya_2-4x^5yb_3-2x^4y^2a_3-10x^4y^2b_2+16x^3y^3a_2 \\ & -16x^3y^3b_3+10x^2y^4a_3+2x^2y^4b_2+4x\,y^5a_2-4x\,y^5b_3+2y^6a_3+4y^6b_2 \\ & -2x^5b_1+2x^4ya_1-11x^3y^2b_1+11x^2y^3a_1-2x\,y^4b_1+2y^5a_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 4a_2v_1^5v_2+16a_2v_1^3v_2^3+4a_2v_1v_2^5-4a_3v_1^6-2a_3v_1^4v_2^2+10a_3v_1^2v_2^4+2a_3v_2^6 \\ & -2b_2v_1^6-10b_2v_1^4v_2^2+2b_2v_1^2v_2^4+4b_2v_2^6-4b_3v_1^5v_2-16b_3v_1^3v_2^3-4b_3v_1v_2^5 \\ & +2a_1v_1^4v_2+11a_1v_1^2v_2^3+2a_1v_2^5-2b_1v_1^5-11b_1v_1^3v_2^2-2b_1v_1v_2^4=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-4a_3 - 2b_2)\, v_1^6 + (4a_2 - 4b_3)\, v_1^5 v_2 - 2b_1 v_1^5 + (-2a_3 - 10b_2)\, v_1^4 v_2^2 \\
&+ 2a_1 v_1^4 v_2 + (16a_2 - 16b_3)\, v_1^3 v_2^3 - 11 b_1 v_1^3 v_2^2 + (10a_3 + 2b_2)\, v_1^2 v_2^4 \\
&+ 11 a_1 v_1^2 v_2^3 + (4a_2 - 4b_3)\, v_1 v_2^5 - 2b_1 v_1 v_2^4 + (2a_3 + 4b_2)\, v_2^6 + 2a_1 v_2^5 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
11a_1 &= 0 \\
-11b_1 &= 0 \\
-2b_1 &= 0 \\
4a_2 - 4b_3 &= 0 \\
16a_2 - 16b_3 &= 0 \\
-4a_3 - 2b_2 &= 0 \\
-2a_3 - 10b_2 &= 0 \\
2a_3 + 4b_2 &= 0 \\
10a_3 + 2b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{x(2x^2+y^2)}{y\,(x^2+2y^2)}\right)(x) \\
&= \frac{2x^4+2y^2x^2+2y^4}{x^2y+2y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^4+2y^2x^2+2y^4}{x^2y+2y^3}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^4+y^2x^2+y^4)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(2x^2+y^2)}{y\,(x^2+2y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x(2x^2+y^2)}{2\left(x^2+xy+y^2\right)\left(x^2-xy+y^2\right)} \\
S_y &= \frac{y(x^2+2y^2)}{2\left(x^2+xy+y^2\right)\left(x^2-xy+y^2\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

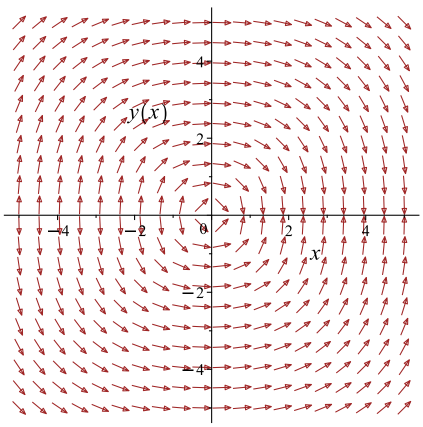
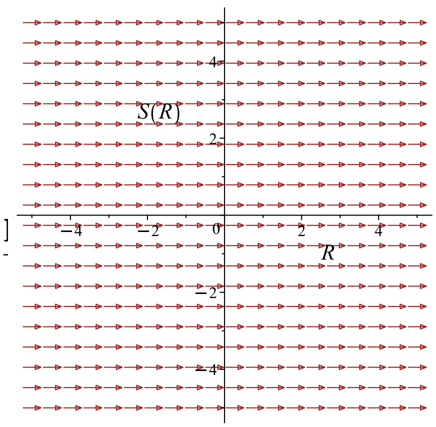
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(yx+y^2+x^2\right)}{4} + \frac{\ln\left(y^2-yx+x^2\right)}{4} = c_1
$$

Which simplifies to

$$
\frac{\ln\left(yx+y^2+x^2\right)}{4} + \frac{\ln\left(y^2-yx+x^2\right)}{4} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(2x^2+y^2)}{y(x^2+2y^2)}$ 	$R = x$ $S = \frac{\ln(x^2+xy+y^2)}{4} +$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(yx + y^2 + x^2)}{4} + \frac{\ln(y^2 - yx + x^2)}{4} = c_1 \tag{1}$$

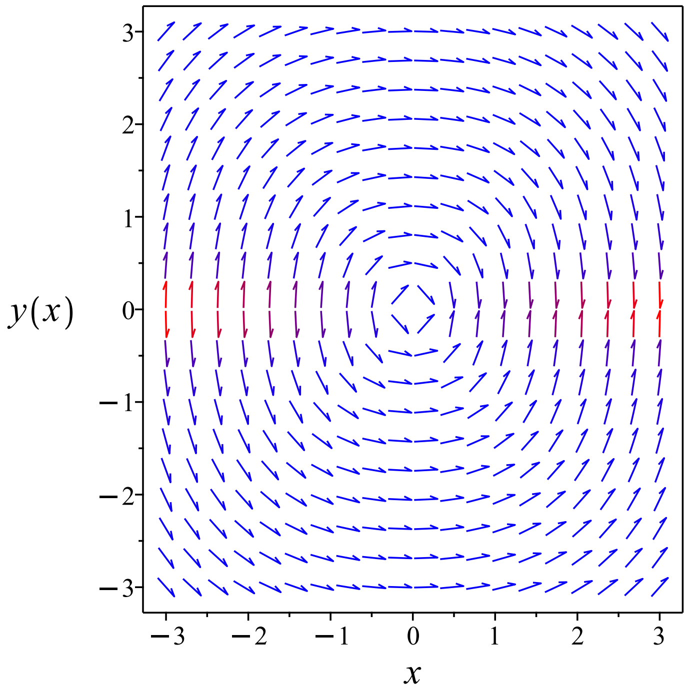


Figure 286: Slope field plot

Verification of solutions

$$\frac{\ln\left(yx+y^2+x^2\right)}{4}+\frac{\ln\left(y^2-yx+x^2\right)}{4}=c_1$$

Verified OK.

7.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y\left(x^2 + 2y^2\right)\right) \mathrm{d}y = \left(-x\left(2x^2 + y^2\right)\right) \mathrm{d}x$$
$$\left(x\left(2x^2 + y^2\right)\right) \mathrm{d}x + \left(y\left(x^2 + 2y^2\right)\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x\left(2x^2 + y^2\right)$$
$$N(x, y) = y\left(x^2 + 2y^2\right)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\left(2x^2 + y^2\right)\right) \\ &= 2xy\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\left(x^2 + 2y^2\right)\right) \\ &= 2xy\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x(2x^2 + y^2) \, \mathrm{d}x$$

$$\phi = \frac{(2x^2 + y^2)^2}{8} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{(2x^2 + y^2)\, y}{2} + f'(y) \tag{4}$$

$$= x^2 y + \frac{1}{2} y^3 + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(x^2 + 2y^2)$. Therefore equation (4) becomes

$$y(x^2 + 2y^2) = x^2 y + \frac{1}{2} y^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3y^3}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{3y^3}{2}\right) \mathrm{d}y$$

$$f(y) = \frac{3y^4}{8} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2x^2+y^2)^2}{8} + \frac{3y^4}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2x^2+y^2)^2}{8} + \frac{3y^4}{8}$$

Summary
The solution(s) found are the following

$$\frac{(2x^2+y^2)^2}{8} + \frac{3y^4}{8} = c_1 \tag{1}$$

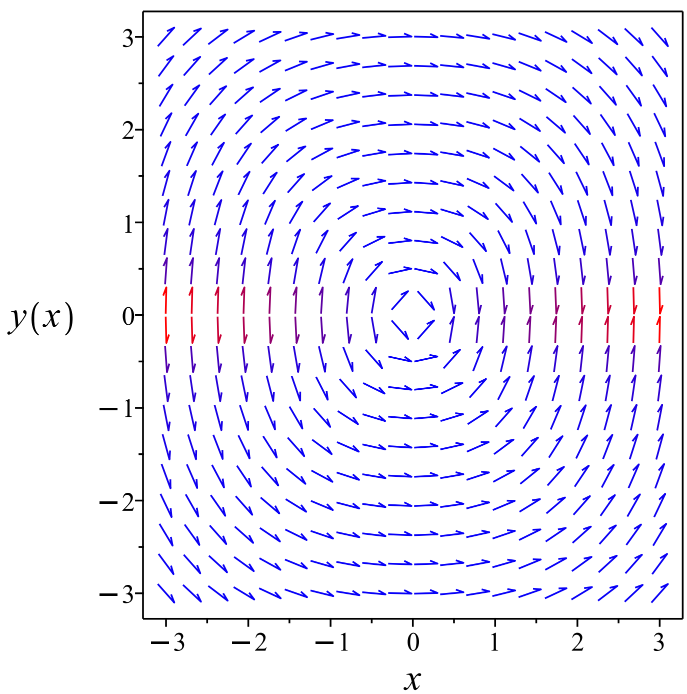


Figure 287: Slope field plot

Verification of solutions

$$\frac{(2x^2+y^2)^2}{8} + \frac{3y^4}{8} = c_1$$

Verified OK.

7.1.4 Maple step by step solution

Let's solve

$x(2x^2 + y^2) + y(x^2 + 2y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2xy = 2xy$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int x(2x^2 + y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{(2x^2+y^2)^2}{8} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $y(x^2 + 2y^2) = \frac{(2x^2+y^2)y}{2} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y(x^2 + 2y^2) - \frac{(2x^2+y^2)y}{2}$

- Solve for $f_1(y)$

  $f_1(y) = \frac{3y^4}{8}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{(2x^2+y^2)^2}{8} + \frac{3y^4}{8}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{(2x^2+y^2)^2}{8} + \frac{3y^4}{8} = c_1$$

- Solve for y

$$\left\{y = -\frac{\sqrt{-2x^2-2\sqrt{-3x^4+8c_1}}}{2}, y = \frac{\sqrt{-2x^2-2\sqrt{-3x^4+8c_1}}}{2}, y = -\frac{\sqrt{-2x^2+2\sqrt{-3x^4+8c_1}}}{2}, y = \frac{\sqrt{-2x^2+2\sqrt{-3x^4+8c_1}}}{2}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 125

```
dsolve(x*(2*x^2+y(x)^2)+y(x)*(x^2+2*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2c_1x^2 - 2\sqrt{-3c_1^2x^4+4}}}{2\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{-2c_1x^2 - 2\sqrt{-3c_1^2x^4+4}}}{2\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{-2c_1x^2 + 2\sqrt{-3c_1^2x^4+4}}}{2\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{-2c_1x^2 + 2\sqrt{-3c_1^2x^4+4}}}{2\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 23.583 (sec). Leaf size: 303

```
DSolve[x*(2*x^2+y[x]^2)+y[x]*(x^2+2*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{\sqrt{-x^2 - \sqrt{-3x^4 + 4e^{2c_1}}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-x^2 - \sqrt{-3x^4 + 4e^{2c_1}}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-x^2 + \sqrt{-3x^4 + 4e^{2c_1}}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-x^2 + \sqrt{-3x^4 + 4e^{2c_1}}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-\sqrt{3}\sqrt{-x^4} - x^2}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-\sqrt{3}\sqrt{-x^4} - x^2}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{\sqrt{3}\sqrt{-x^4} - x^2}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{\sqrt{3}\sqrt{-x^4} - x^2}}{\sqrt{2}}$$

7.2 problem 176

Internal problem ID [15065]
Internal file name [OUTPUT/15065_Sunday_April_21_2024_01_23_12_PM_58516233/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 176.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact, _rational]`

$$\boxed{6y^2x + (6x^2y + 4y^3)\, y' = -3x^2}$$

7.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(6x^2y+4y^3\right)\mathrm{d}y &= \left(-6y^2x-3x^2\right)\mathrm{d}x\\ \left(6y^2x+3x^2\right)\mathrm{d}x + \left(6x^2y+4y^3\right)\mathrm{d}y &= 0 \qquad (2\text{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 6y^2x+3x^2\\ N(x,y) &= 6x^2y+4y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(6y^2x+3x^2\right)\\ &= 12xy\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(6x^2y+4y^3\right)\\ &= 12xy\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 6y^2x + 3x^2 \, \mathrm{d}x$$
$$\phi = x^2(3y^2 + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 6x^2y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 6x^2y + 4y^3$. Therefore equation (4) becomes

$$6x^2y + 4y^3 = 6x^2y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (4y^3) \, \mathrm{d}y$$
$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2(3y^2 + x) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2(3y^2 + x) + y^4$$

Summary

The solution(s) found are the following

$$x^2(3y^2+x)+y^4=c_1 \tag{1}$$

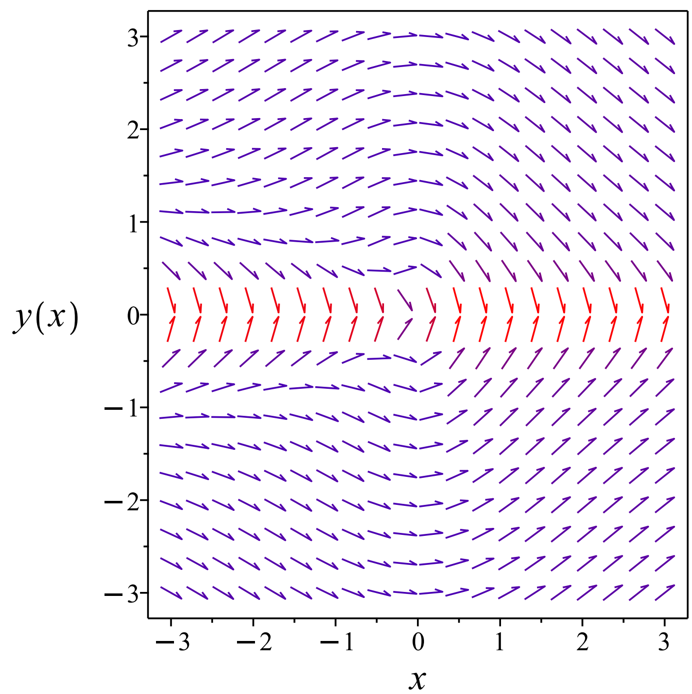


Figure 288: Slope field plot

Verification of solutions

$$x^2(3y^2+x)+y^4=c_1$$

Verified OK.

7.2.2 Maple step by step solution

Let's solve

$6y^2x+(6x^2y+4y^3)\,y'=-3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $12xy = 12xy$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (6y^2x + 3x^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = 3y^2x^2 + x^3 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $6x^2y + 4y^3 = 6x^2y + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 4y^3$

- Solve for $f_1(y)$

  $f_1(y) = y^4$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = 3y^2x^2 + y^4 + x^3$

- Substitute $F(x, y)$ into the solution of the ODE

  $3y^2x^2 + y^4 + x^3 = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{-6x^2-2\sqrt{9x^4-4x^3+4c_1}}}{2}, y = \frac{\sqrt{-6x^2-2\sqrt{9x^4-4x^3+4c_1}}}{2}, y = -\frac{\sqrt{-6x^2+2\sqrt{9x^4-4x^3+4c_1}}}{2}, y = \frac{\sqrt{-6x^2+2\sqrt{}}}{}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 125

```
dsolve((3*x^2+6*x*y(x)^2)+(6*x^2*y(x)+4*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-6x^2 - 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = \frac{\sqrt{-6x^2 - 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = -\frac{\sqrt{-6x^2 + 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = \frac{\sqrt{-6x^2 + 2\sqrt{9x^4 - 4x^3 - 4c_1}}}{2}$$

✓ Solution by Mathematica

Time used: 5.982 (sec). Leaf size: 163

```
DSolve[(3*x^2+6*x*y[x]^2)+(6*x^2*y[x]+4*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{\sqrt{-3x^2 - \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-3x^2 - \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-3x^2 + \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-3x^2 + \sqrt{9x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$

7.3 problem 177

Internal problem ID [15066]
Internal file name [`OUTPUT/15066_Sunday_April_21_2024_01_23_13_PM_71038155/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 177.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{x}{\sqrt{x^2+y^2}}+\frac{1}{y}+\left(\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2}\right)y'=-\frac{1}{x}}$$

7.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}\right)\mathrm{d}y = \left(-\frac{x}{\sqrt{x^2+y^2}} - \frac{1}{x} - \frac{1}{y}\right)\mathrm{d}x$$
$$\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}\right)\mathrm{d}x + \left(\frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}$$
$$N(x,y) = \frac{y}{\sqrt{x^2+y^2}} + \frac{1}{y} - \frac{x}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{x} + \frac{1}{y}\right)\\ &= -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} - \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2}\right)\\ &= -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}-\frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x}{\sqrt{x^2+y^2}}+\frac{1}{x}+\frac{1}{y}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{\sqrt{x^2+y^2}\,y+y\ln(x)+x}{y}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\frac{y^2}{\sqrt{x^2+y^2}}+\sqrt{x^2+y^2}+\ln(x)}{y}-\frac{\sqrt{x^2+y^2}\,y+y\ln(x)+x}{y^2}+f'(y) \tag{4}$$

$$= \frac{y^3-x\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}}+f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2} = \frac{y^3-x\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\sqrt{x^2+y^2}\,y + y\ln(x) + x}{y} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\sqrt{x^2+y^2}\,y + y\ln(x) + x}{y} + \ln(y)$$

Summary

The solution(s) found are the following

$$\frac{y\sqrt{x^2+y^2} + y\ln(x) + x}{y} + \ln(y) = c_1 \tag{1}$$

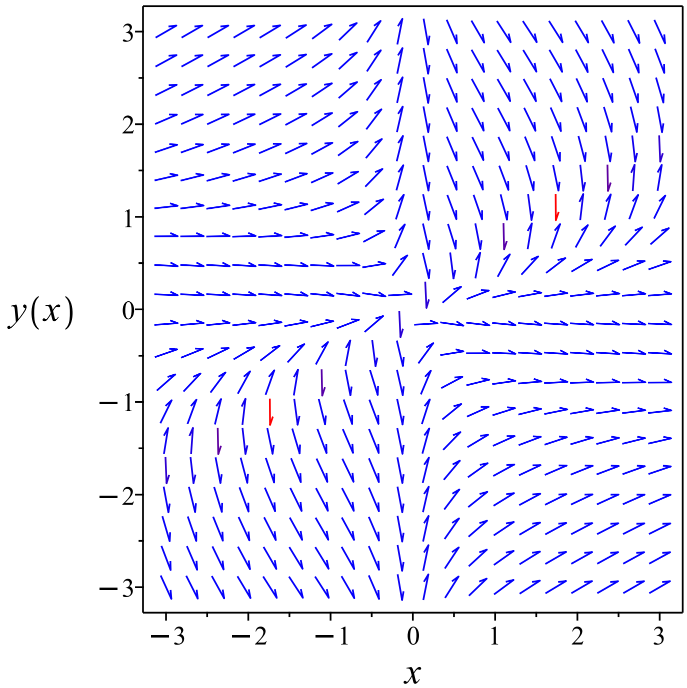


Figure 289: Slope field plot

Verification of solutions

$$\frac{y\sqrt{x^2+y^2}+y\ln(x)+x}{y}+\ln(y)=c_1$$

Verified OK.

7.3.2 Maple step by step solution

Let's solve

$\frac{x}{\sqrt{x^2+y^2}}+\frac{1}{y}+\left(\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2}\right)y'=-\frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

- ◦ Compute derivative of lhs

$$F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$$

- Evaluate derivatives

  $$-\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}-\frac{1}{y^2}=-\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}-\frac{1}{y^2}$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $$\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $$F(x,y)=\int\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{1}{x}+\frac{1}{y}\right)dx+f_1(y)$$

- Evaluate integral

  $$F(x,y)=\ln(x)+\frac{x}{y}+\sqrt{x^2+y^2}+f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

  $$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

  $$\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{y}-\frac{x}{y^2}=-\frac{x}{y^2}+\frac{y}{\sqrt{x^2+y^2}}+\frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

  $$\frac{d}{dy}f_1(y)=\frac{1}{y}$$

- Solve for $f_1(y)$

  $$f_1(y)=\ln(y)$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $$F(x,y)=\ln(x)+\frac{x}{y}+\sqrt{x^2+y^2}+\ln(y)$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\ln(x)+\frac{x}{y}+\sqrt{x^2+y^2}+\ln(y)=c_1$$

- Solve for y

  $$y=e^{RootOf\left(-(e^{\_Z})^4+(e^{\_Z})^2\ln(x)^2-2(e^{\_Z})^2\ln(x)c_1+2(e^{\_Z})^2\ln(x)\_Z+(e^{\_Z})^2c_1^2-2(e^{\_Z})^2c_1\_Z+(e^{\_Z})^2\_Z^2-x^2(e^{\_Z})^2+2e\right.}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve((x/sqrt(x^2+y(x)^2)+1/x+1/y(x))+(y(x)/sqrt(x^2+y(x)^2)+1/y(x)-x/y(x)^2)*diff(y(x),x)=
```

$$\frac{y(x)\ln\left(y(x)\right)+\left(\sqrt{x^2+y\left(x\right)^2}+c_1+\ln\left(x\right)\right)y(x)+x}{y\left(x\right)}=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x/Sqrt[x^2+y[x]^2]+1/x+1/y[x])+(y[x]/Sqrt[x^2+y[x]^2]+1/y[x]-x/y[x]^2)*y'[x]==0,y[x]
```

Not solved

7.4 problem 178

Internal problem ID [15067]
Internal file name [OUTPUT/15067_Sunday_April_21_2024_01_23_45_PM_19432001/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 178.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{3x^2\tan(y) - \frac{2y^3}{x^3} + \left(x^3\sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}\right)y' = 0}$$

7.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}\right)\mathrm{d}y = \left(-3x^2 \tan(y) + \frac{2y^3}{x^3}\right)\mathrm{d}x$$
$$\left(3x^2 \tan(y) - \frac{2y^3}{x^3}\right)\mathrm{d}x + \left(x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 3x^2 \tan(y) - \frac{2y^3}{x^3}$$
$$N(x,y) = x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 \tan(y) - \frac{2y^3}{x^3}\right)\\ &= \frac{3x^5 \sec(y)^2 - 6y^2}{x^3}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}\right) \\ &= 3x^2 \sec(y)^2 - \frac{6y^2}{x^3}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int 3x^2 \tan(y) - \frac{2y^3}{x^3} \,\mathrm{d}x\end{aligned}$$

$$\phi = \tan(y)\, x^3 + \frac{y^3}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \left(\tan(y)^2 + 1\right) x^3 + \frac{3y^2}{x^2} + f'(y) \tag{4}$$

$$= \frac{x^5 \sec(y)^2 + 3y^2}{x^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}$. Therefore equation (4) becomes

$$x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2} = \frac{x^5 \sec(y)^2 + 3y^2}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y^3)\,\mathrm{d}y$$

$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \tan(y)\,x^3 + \frac{y^3}{x^2} + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \tan(y)\,x^3 + \frac{y^3}{x^2} + y^4$$

Summary

The solution(s) found are the following

$$\tan(y)\,x^3 + \frac{y^3}{x^2} + y^4 = c_1 \tag{1}$$

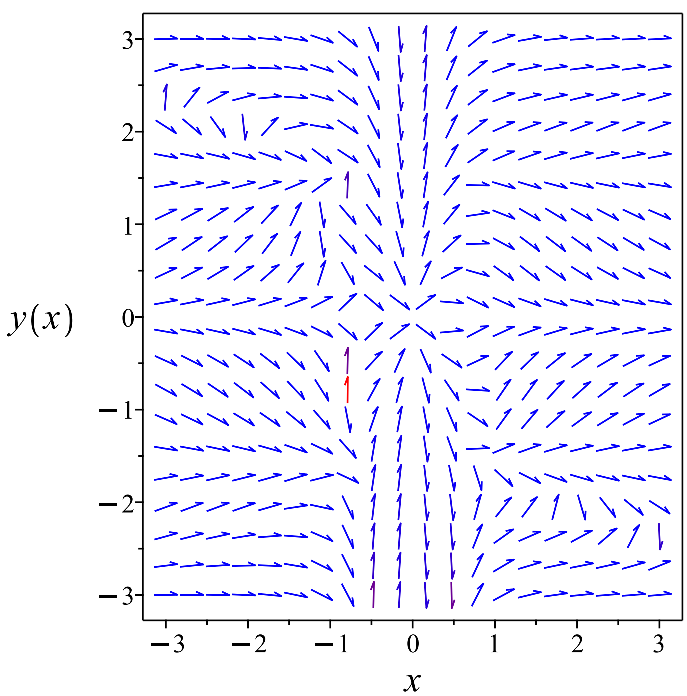


Figure 290: Slope field plot

Verification of solutions

$$\tan(y)\,x^3 + \frac{y^3}{x^2} + y^4 = c_1$$

Verified OK.

7.4.2 Maple step by step solution

Let's solve

$3x^2\tan(y) - \frac{2y^3}{x^3} + \left(x^3\sec(y)^2 + 4y^3 + \frac{3y^2}{x^2}\right)y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$
  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$
  - Evaluate derivatives

    $3x^2\left(\tan(y)^2 + 1\right) - \frac{6y^2}{x^3} = 3x^2\sec(y)^2 - \frac{6y^2}{x^3}$
  - Simplify

    $\frac{3x^5\sec(y)^2 - 6y^2}{x^3} = 3x^2\sec(y)^2 - \frac{6y^2}{x^3}$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int\left(3x^2\tan(y) - \frac{2y^3}{x^3}\right)dx + f_1(y)$
- Evaluate integral

  $F(x,y) = \tan(y)\,x^3 + \frac{y^3}{x^2} + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^3 \sec(y)^2 + 4y^3 + \frac{3y^2}{x^2} = (\tan(y)^2 + 1)\, x^3 + \frac{3y^2}{x^2} + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = x^3 \sec(y)^2 + 4y^3 - (\tan(y)^2 + 1)\, x^3$
- Solve for $f_1(y)$

  $f_1(y) = y^4$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \tan(y)\, x^3 + \frac{y^3}{x^2} + y^4$
- Substitute $F(x, y)$ into the solution of the ODE

  $\tan(y)\, x^3 + \frac{y^3}{x^2} + y^4 = c_1$
- Solve for y

  $y = \mathit{RootOf}\left(-\tan(\_Z)\, x^5 - x^2\_Z^4 + c_1 x^2 - \_Z^3\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((3*x^2*tan(y(x))-2*y(x)^3/x^3 )+( x^3*sec(y(x))^2+4*y(x)^3+ 3*y(x)^2/x^2 )*diff(y(x)
```

$$x^3 \tan(y(x)) + \frac{y(x)^3}{x^2} + y(x)^4 + c_1 = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(3*x^2*Tan[y[x]]-2*y[x]^3/x^3  )+( x^3*Sec[y[x]]^2+4*y[x]^3+ 3*y[x]^2/x^2 )*y'[x]==0,
```

Not solved

7.5 problem 179

Internal problem ID [15068]
Internal file name [`OUTPUT/15068_Sunday_April_21_2024_01_24_34_PM_85038681/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 179.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _exact, _rational]
```

$$\boxed{\frac{x^2+y^2}{x^2y}-\frac{(x^2+y^2)\,y'}{y^2x}=-2x}$$

7.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x^2+u(x)^2\,x^2}{x^3u\,(x)}-\frac{\left(x^2+u(x)^2\,x^2\right)\left(u'(x)\,x+u(x)\right)}{u\,(x)^2\,x^3}=-2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{2u^2x}{u^2+1} \end{aligned}$$

Where $f(x) = 2x$ and $g(u) = \frac{u^2}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u^2+1}}\,du = 2x\,dx$$

$$\int \frac{1}{\frac{u^2}{u^2+1}}\,du = \int 2x\,dx$$

$$u - \frac{1}{u} = x^2 + c_2$$

The solution is

$$u(x) - \frac{1}{u(x)} - x^2 - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} - \frac{x}{y} - x^2 - c_2 = 0$$

$$\frac{y}{x} - \frac{x}{y} - x^2 - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} - \frac{x}{y} - x^2 - c_2 = 0 \tag{1}$$

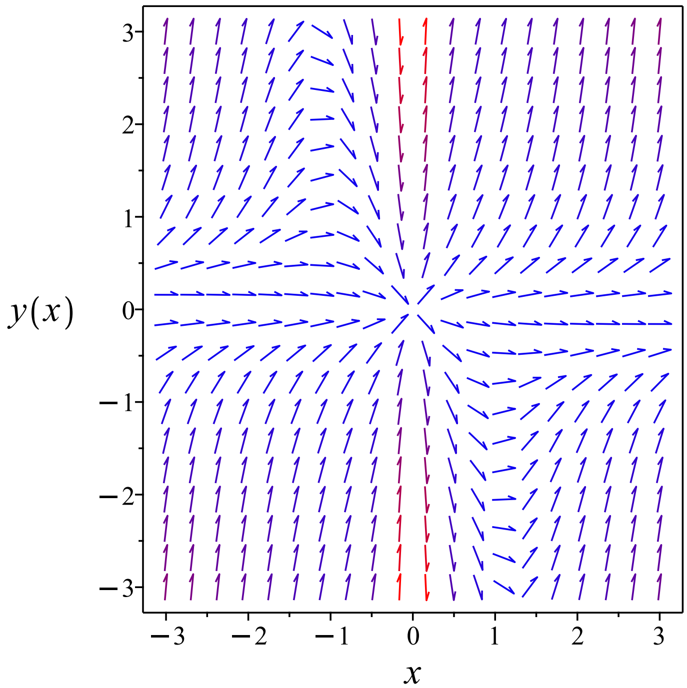


Figure 291: Slope field plot

Verification of solutions

$$\frac{y}{x}-\frac{x}{y}-x^2-c_2=0$$

Verified OK.

7.5.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{x^2+y^2}{y^2 x}\right) \mathrm{d}y &= \left(-2x - \frac{x^2+y^2}{x^2 y}\right) \mathrm{d}x \\
\left(2x + \frac{x^2+y^2}{x^2 y}\right) \mathrm{d}x + \left(-\frac{x^2+y^2}{y^2 x}\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 2x + \frac{x^2+y^2}{x^2 y} \\
N(x,y) &= -\frac{x^2+y^2}{y^2 x}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x + \frac{x^2+y^2}{x^2 y}\right) \\
&= \frac{-x^2+y^2}{y^2 x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x^2+y^2}{y^2 x}\right) \\
&= \frac{-x^2+y^2}{y^2 x^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 2x + \frac{x^2+y^2}{x^2 y} \, \mathrm{d}x \\
\phi &= \frac{x^2-y^2}{yx} + x^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{2}{x} - \frac{x^2-y^2}{y^2 x} + f'(y) \qquad (4) \\
&= \frac{-x^2-y^2}{y^2 x} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x^2+y^2}{y^2 x}$. Therefore equation (4) becomes

$$
-\frac{x^2+y^2}{y^2 x} = \frac{-x^2-y^2}{y^2 x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 - y^2}{yx} + x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 - y^2}{yx} + x^2$$

Summary

The solution(s) found are the following

$$\frac{x^2 - y^2}{yx} + x^2 = c_1 \tag{1}$$

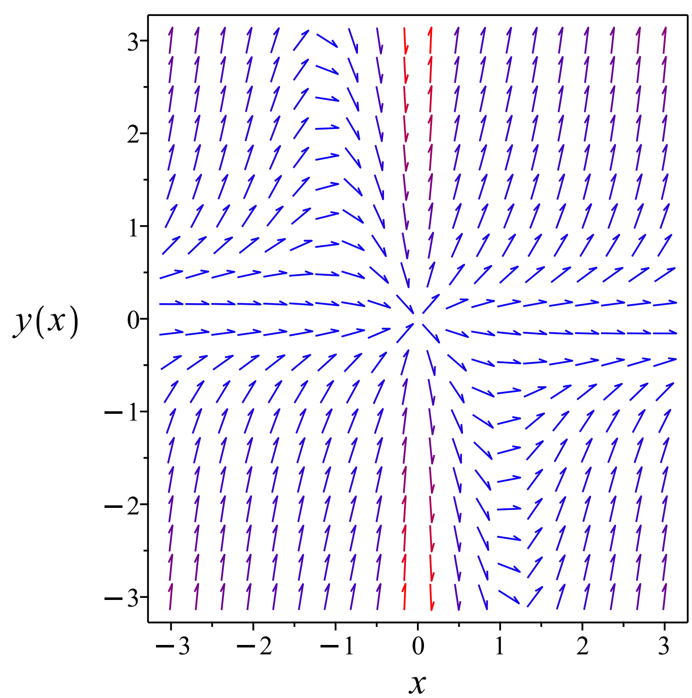


Figure 292: Slope field plot

Verification of solutions

$$\frac{x^2 - y^2}{yx} + x^2 = c_1$$

Verified OK.

7.5.3 Maple step by step solution

Let's solve

$\frac{x^2+y^2}{x^2 y} - \frac{(x^2+y^2)y'}{y^2 x} = -2x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $\frac{2}{x^2} - \frac{x^2+y^2}{x^2 y^2} = -\frac{2}{y^2} + \frac{x^2+y^2}{x^2 y^2}$

  - Simplify

    $\frac{-x^2+y^2}{y^2 x^2} = \frac{-x^2+y^2}{y^2 x^2}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(2x + \frac{x^2+y^2}{x^2 y}\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x - \frac{y^2}{x}}{y} + x^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-\frac{x^2+y^2}{y^2x} = -\frac{x-\frac{y^2}{x}}{y^2} - \frac{2}{x} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -\frac{x^2+y^2}{y^2x} + \frac{x-\frac{y^2}{x}}{y^2} + \frac{2}{x}$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{x-\frac{y^2}{x}}{y} + x^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{x-\frac{y^2}{x}}{y} + x^2 = c_1$

- Solve for y

  $\left\{y = \left(\frac{x^2}{2} - \frac{c_1}{2} - \frac{\sqrt{x^4-2c_1x^2+c_1^2+4}}{2}\right)x, y = \left(\frac{x^2}{2} - \frac{c_1}{2} + \frac{\sqrt{x^4-2c_1x^2+c_1^2+4}}{2}\right)x\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve((2*x+ (x^2+y(x)^2)/(x^2*y(x))  )=( (x^2+y(x)^2)/(x*y(x)^2) )*diff(y(x),x),y(x), sings
```

$$y(x) = -\frac{\left(-x^2+\sqrt{x^4+4c_1x^2+4c_1^2+4}-2c_1\right)x}{2}$$
$$y(x) = \frac{\left(x^2+2c_1+\sqrt{x^4+4c_1x^2+4c_1^2+4}\right)x}{2}$$

✓ Solution by Mathematica

Time used: 0.364 (sec). Leaf size: 78

```
DSolve[(2*x+ (x^2+y[x]^2)/(x^2*y[x])  )==( (x^2+y[x]^2)/(x*y[x]^2) )*y'[x],y[x],x,IncludeSin
```

$$y(x) \to \frac{1}{2}x\left(x^2-\sqrt{x^4+2c_1x^2+4+c_1{}^2}+c_1\right)$$
$$y(x) \to \frac{1}{2}x\left(x^2+\sqrt{x^4+2c_1x^2+4+c_1{}^2}+c_1\right)$$
$$y(x) \to 0$$

7.6 problem 180

Internal problem ID [15069]
Internal file name [`OUTPUT/15069_Sunday_April_21_2024_01_24_36_PM_57153371/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 180.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{\sin(2x)}{y}+\left(y-\frac{\sin(x)^2}{y^2}\right)y'=-x}$$

7.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y - \frac{\sin(x)^2}{y^2}\right) \mathrm{d}y = \left(-\frac{\sin(2x)}{y} - x\right) \mathrm{d}x$$

$$\left(\frac{\sin(2x)}{y} + x\right) \mathrm{d}x + \left(y - \frac{\sin(x)^2}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{\sin(2x)}{y} + x$$
$$N(x, y) = y - \frac{\sin(x)^2}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sin(2x)}{y} + x\right) \\ &= -\frac{\sin(2x)}{y^2}\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y - \frac{\sin(x)^2}{y^2}\right) \\
&= -\frac{\sin(2x)}{y^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{\sin(2x)}{y} + x \,\mathrm{d}x
\end{aligned}$$

$$\phi = \frac{x^2}{2} - \frac{\cos(2x)}{2y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\cos(2x)}{2y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - \frac{\sin(x)^2}{y^2}$. Therefore equation (4) becomes

$$y - \frac{\sin(x)^2}{y^2} = \frac{\cos(2x)}{2y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}
f'(y) &= -\frac{-2y^3 + 2\sin(x)^2 + \cos(2x)}{2y^2} \\
&= \frac{2y^3 - 1}{2y^2}
\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2y^3-1}{2y^2}\right)\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + \frac{1}{2y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} - \frac{\cos(2x)}{2y} + \frac{y^2}{2} + \frac{1}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} - \frac{\cos(2x)}{2y} + \frac{y^2}{2} + \frac{1}{2y}$$

Summary

The solution(s) found are the following

$$\frac{x^2}{2} - \frac{\cos(2x)}{2y} + \frac{y^2}{2} + \frac{1}{2y} = c_1 \tag{1}$$

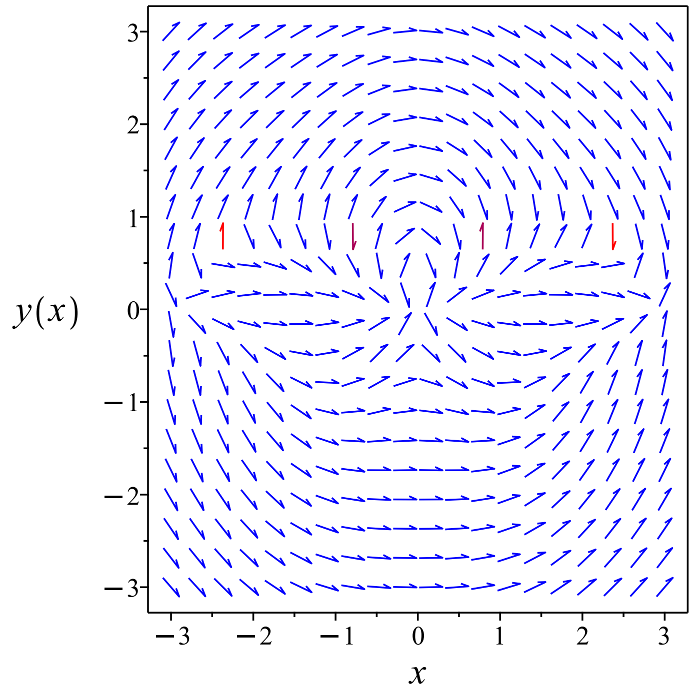


Figure 293: Slope field plot

Verification of solutions

$$\frac{x^2}{2} - \frac{\cos(2x)}{2y} + \frac{y^2}{2} + \frac{1}{2y} = c_1$$

Verified OK.

7.6.2 Maple step by step solution

Let's solve

$\frac{\sin(2x)}{y} + \left(y - \frac{\sin(x)^2}{y^2}\right) y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ◦ Compute derivative of lhs

$$F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$$

- Evaluate derivatives

  $$-\frac{\sin(2x)}{y^2} = -\frac{2\sin(x)\cos(x)}{y^2}$$

- Simplify

  $$-\frac{\sin(2x)}{y^2} = -\frac{\sin(2x)}{y^2}$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $$\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $$F(x,y) = \int \left(\frac{\sin(2x)}{y} + x\right) dx + f_1(y)$$

- Evaluate integral

  $$F(x,y) = \frac{x^2}{2} - \frac{\cos(2x)}{2y} + f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

  $$N(x,y) = \frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

  $$y - \frac{\sin(x)^2}{y^2} = \frac{\cos(2x)}{2y^2} + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

  $$\frac{d}{dy}f_1(y) = y - \frac{\sin(x)^2}{y^2} - \frac{\cos(2x)}{2y^2}$$

- Solve for $f_1(y)$

  $$f_1(y) = \frac{y^2}{2} + \frac{\cos(2x)}{2y} + \frac{\sin(x)^2}{y}$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $$F(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{\sin(x)^2}{y}$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{x^2}{2} + \frac{y^2}{2} + \frac{\sin(x)^2}{y} = c_1$$

- Solve for y

$$\left\{ y = \frac{\left(-27\sin(x)^2+3\sqrt{3x^6-18c_1x^4+36c_1^2x^2-24c_1^3+81\sin(x)^4}\right)^{\frac{1}{3}}}{3} - \frac{3\left(\frac{x^2}{3}-\frac{2c_1}{3}\right)}{\left(-27\sin(x)^2+3\sqrt{3x^6-18c_1x^4+36c_1^2x^2-24c_1^3+81\sin(x)^4}\right.}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 401

```
dsolve(( sin(2*x)/y(x)+x  )+( y(x)-sin(x)^2/y(x)^2 )*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-108+108\cos(2x)+12\sqrt{12x^6+72c_1x^4+144x^2c_1^2+96c_1^3+81-162\cos(2x)+81\cos(2x)^2}\right)^{\frac{2}{3}} - 1 \cdots}{6\left(-108+108\cos(2x)+12\sqrt{12x^6+72c_1x^4+144x^2c_1^2+96c_1^3+81-162\cos(2x)+81\cos(2x) \cdots}\right.}$$

$$y(x) = -\frac{\left(\frac{i\sqrt{3}}{12}+\frac{1}{12}\right)\left(-108+108\cos(2x)+12\sqrt{12x^6+72c_1x^4+144x^2c_1^2+96c_1^3+81-162\cos(2x)+81\,\mathrm{co}\cdots}\right.}{\left(-108+108\cos(2x)+12\sqrt{12x^6+72c_1x^4+144x^2c_1^2+96c_1^3+81-162\cos(2x)+\cdots}\right.}$$

$$y(x) = \frac{\frac{\left(-108+108\cos(2x)+12\sqrt{12x^6+72c_1x^4+144x^2c_1^2+96c_1^3+81-162\cos(2x)+81\cos(2x)^2}\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)}{12}+\left(x^2+2c_1\right)\left(1+i\sqrt{3}\right)}{\left(-108+108\cos(2x)+12\sqrt{12x^6+72c_1x^4+144x^2c_1^2+96c_1^3+81-162\cos(2x)+81\cos(2x)^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 6.373 (sec). Leaf size: 394

```
DSolve[( Sin[2*x]/y[x]+x  )+( y[x]-Sin[x]^2/y[x]^2 )*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{\sqrt[3]{2}\left(2\sqrt{3}\sqrt{27\sin^4(x)+(x^2-c_1)^3}+9\cos(2x)-9\right)^{2/3}-2\sqrt[3]{3}(x^2-c_1)}{6^{2/3}\sqrt[3]{2\sqrt{3}\sqrt{27\sin^4(x)+(x^2-c_1)^3}+9\cos(2x)-9}}$$

$$y(x) \to \frac{6\sqrt[3]{2}(1+i\sqrt{3})(x^2-c_1)+i6^{2/3}(\sqrt{3}+i)\left(2\sqrt{3}\sqrt{27\sin^4(x)+(x^2-c_1)^3}+9\cos(2x)-9\right)^{2/3}}{12\sqrt[3]{3}\sqrt[3]{2\sqrt{3}\sqrt{27\sin^4(x)+(x^2-c_1)^3}+9\cos(2x)-9}}$$

$$y(x) \to \frac{6\sqrt[3]{2}(1-i\sqrt{3})(x^2-c_1)-6^{2/3}(1+i\sqrt{3})\left(2\sqrt{3}\sqrt{27\sin^4(x)+(x^2-c_1)^3}+9\cos(2x)-9\right)^{2/3}}{12\sqrt[3]{3}\sqrt[3]{2\sqrt{3}\sqrt{27\sin^4(x)+(x^2-c_1)^3}+9\cos(2x)-9}}$$

$$y(x) \to 0$$

7.7 problem 181

Internal problem ID [15070]
Internal file name [`OUTPUT/15070_Sunday_April_21_2024_01_24_44_PM_2008027/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 181.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{-y+\left(2y-x+3y^{2}\right)y'=-3x^{2}+2x}$$

7.7.1 Solving as differentialType ode

Writing the ode as

$$y'=\frac{-3x^{2}+2x+y}{2y-x+3y^{2}} \tag{1}$$

Which becomes

$$\left(-3y^{2}-2y\right)dy=(-x)\,dy+\left(3x^{2}-2x-y\right)dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy+\left(3x^{2}-2x-y\right)dx=d\left(x^{3}-x^{2}-xy\right)$$

Hence (2) becomes

$$\left(-3y^{2}-2y\right)dy=d\left(x^{3}-x^{2}-xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81}\right.}{6}$$

$$y = -\frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 8}\right.}{12}$$

$$y = -\frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 8}\right.}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81c_1^2}\right.}{6} - \frac{6\left(-\frac{x}{3}-\frac{1}{9}\right)}{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81}\right.} - \frac{1}{3} + c_1 \tag{1}$$

$$y = -\frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81}\right.}{12} + \frac{-x-\frac{1}{3}}{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81}\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81c_1^2+54c_1x-15x^2+12c_1}\right)^{\frac{1}{3}}}{6} + \frac{}{\left(-36x\right.}\right.}{2} + c_1 \tag{2}$$

$$y = -\frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81}\right.}{12} + \frac{-x-\frac{1}{3}}{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81}\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81c_1^2+54c_1x-15x^2+12c_1}\right)^{\frac{1}{3}}}{6} + \frac{}{\left(-36x\right.}\right.}{2} + c_1 \tag{3}$$

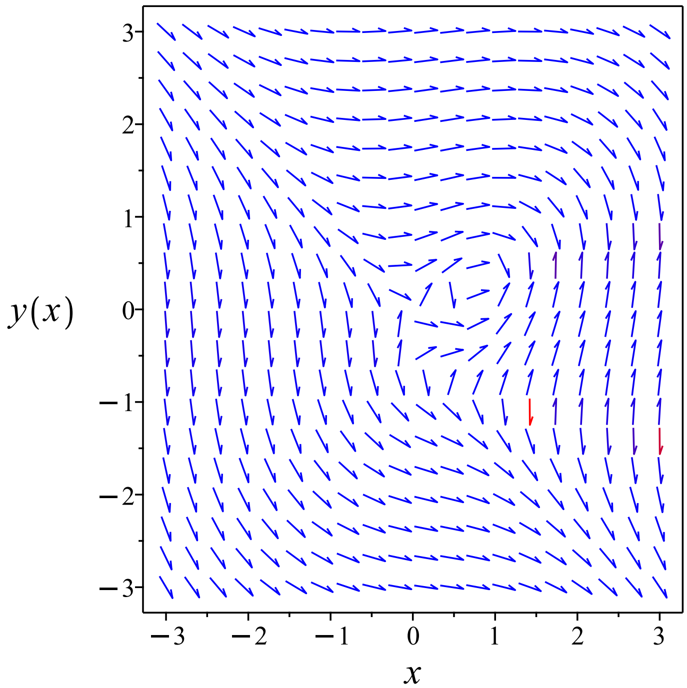


Figure 294: Slope field plot

Verification of solutions

$$y = \frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81c_1^2 \cdots}\right.}{6} - \frac{6\left(-\frac{x}{3} - \frac{1}{9}\right)}{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81 \cdots}\right.} - \frac{1}{3} + c_1$$

Verified OK.

$$y = -\frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81 \cdots}\right.}{12} + \frac{-x - \frac{1}{3}}{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81 \cdots}\right.} - \frac{1}{3} + \frac{i\sqrt{3}\left(\frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81c_1^2+54c_1x-15x^2+12c_1}\right)^{\frac{1}{3}}}{6} + \frac{\cdots}{\left(-36x \cdots\right.}\right.}{2 \cdots} + c_1$$

Verified OK.

$$y = -\frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81 \cdots}\right.}{12} + \frac{-x - \frac{1}{3}}{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81 \cdots}\right.} - \frac{1}{3} - \frac{i\sqrt{3}\left(\frac{\left(-36x-108x^3+108x^2-108c_1-8+12\sqrt{81x^6-162x^5+162c_1x^3+135x^4-162c_1x^2-54x^3+81c_1^2+54c_1x-15x^2+12c_1}\right)^{\frac{1}{3}}}{6} + \frac{\cdots}{\left(-36x \cdots\right.}\right.}{2 \cdots} + c_1$$

Verified OK.

7.7.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3y^2-x+2y\right)\mathrm{d}y&=\left(-3x^2+2x+y\right)\mathrm{d}x\\ \left(3x^2-2x-y\right)\mathrm{d}x+\left(3y^2-x+2y\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^2-2x-y\\ N(x,y)&=3y^2-x+2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2 - 2x - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2 - x + 2y) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3x^2 - 2x - y \, \mathrm{d}x$$

$$\phi = x(x^2 - x - y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 - x + 2y$. Therefore equation (4) becomes

$$3y^2 - x + 2y = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y^2 + 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (3y^2 + 2y) \,\mathrm{d}y$$
$$f(y) = y^3 + y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x^2 - x - y) + y^3 + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x^2 - x - y) + y^3 + y^2$$

Summary

The solution(s) found are the following

$$x(x^2 - x - y) + y^3 + y^2 = c_1 \tag{1}$$

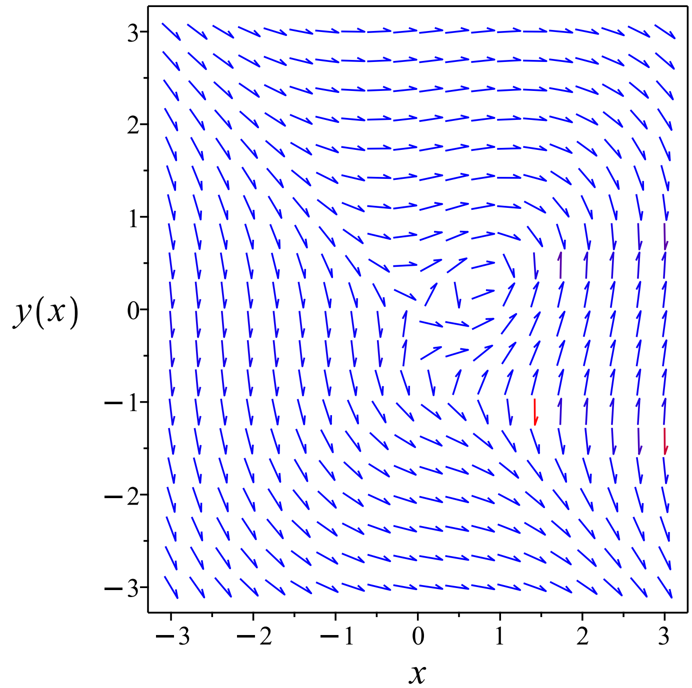


Figure 295: Slope field plot

Verification of solutions

$$x(x^2 - x - y) + y^3 + y^2 = c_1$$

Verified OK.

7.7.3 Maple step by step solution

Let's solve

$-y + (2y - x + 3y^2)\, y' = -3x^2 + 2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

- Evaluate derivatives

  $-1 = -1$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (3x^2 - 2x - y)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x^3 - x^2 - xy + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3y^2 - x + 2y = -x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 3y^2 + 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^3 + y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x^3 + y^3 - x^2 - xy + y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^3 + y^3 - x^2 - xy + y^2 = c_1$

- Solve for y

$$\left\{ y = \frac{\left(-36x-108x^3+108x^2+108c_1-8+12\sqrt{81x^6-162x^5-162c_1x^3+135x^4+162c_1x^2-54x^3+81c_1^2-54c_1x-15x^2-12c_1}\right)^{\frac{1}{3}}}{6} - \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 637

```
dsolve(( 3*x^2-2*x-y(x)  )+( 2*y(x)-x+3*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81c_1^2}\right.}{6} + \frac{2x + \frac{2}{3}}{\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 162c_1x^3 + 135x^4 - 162c_1x^2 - 54x^3 + 81\right.}} - \frac{1}{3}$$

$$y(x) = \frac{i\left(4 - \left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 135x^4 + (162c_1 - 54)x^3 + (-162c_1\right.\right.}}{12\left(-36x - 108x^3 + \right.}$$

$$y(x) = \frac{i\left(\left(-36x - 108x^3 + 108x^2 - 108c_1 - 8 + 12\sqrt{81x^6 - 162x^5 + 135x^4 + (162c_1 - 54)x^3 + (-162c_1 - 1\right.\right.}}{12\left(-36x - 108x^3 + \right.}$$

✓ Solution by Mathematica

Time used: 5.636 (sec). Leaf size: 478

```
DSolve[( 3*x^2-2*x-y[x]  )+( 2*y[x]-x+3*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{6}\left(-\frac{2\sqrt[3]{2}(3x+1)}{\sqrt[3]{27x^3-27x^2+\sqrt{-4(3x+1)^3+(27x^3-27x^2+9x+2+27c_1)^2}+9x+2+27c_1}} - 2^{2/3}\sqrt[3]{27x^3-27x^2+\sqrt{-4(3x+1)^3+(27x^3-27x^2+9x+2+27c_1)^2}+9x+2+27c_1} - 2\right)$$

$$y(x) \to \frac{1}{12}\left(\frac{2\sqrt[3]{2}(1+i\sqrt{3})(3x+1)}{\sqrt[3]{27x^3-27x^2+\sqrt{-4(3x+1)^3+(27x^3-27x^2+9x+2+27c_1)^2}+9x+2+27c_1}} + 2^{2/3}\left(1-i\sqrt{3}\right)\sqrt[3]{27x^3-27x^2+\sqrt{-4(3x+1)^3+(27x^3-27x^2+9x+2+27c_1)^2}+9x+2+27c_1} - 4\right)$$

$$y(x) \to \frac{1}{12}\left(\frac{2\sqrt[3]{2}(1-i\sqrt{3})(3x+1)}{\sqrt[3]{27x^3-27x^2+\sqrt{-4(3x+1)^3+(27x^3-27x^2+9x+2+27c_1)^2}+9x+2+27c_1}} + 2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{27x^3-27x^2+\sqrt{-4(3x+1)^3+(27x^3-27x^2+9x+2+27c_1)^2}+9x+2+27c_1} - 4\right)$$

7.8 problem 182

Internal problem ID [15071]
Internal file name [`OUTPUT/15071_Sunday_April_21_2024_01_24_51_PM_7277892/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 182.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{xy}{\sqrt{x^2+1}}+2yx-\frac{y}{x}+\left(\sqrt{x^2+1}+x^2-\ln\left(x\right)\right)y'=0}$$

7.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y\left(2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}\right)}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)} \end{aligned}$$

Where $f(x) = \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}\,dx\\
\int \frac{1}{y}\,dy &= \int \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}\,dx\\
\ln\left(y\right) &= \int \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}dx + c_1\\
y &= \mathrm{e}^{\int \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx+c_1}\\
&= c_1\mathrm{e}^{\int \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{\int \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx} \tag{1}$$

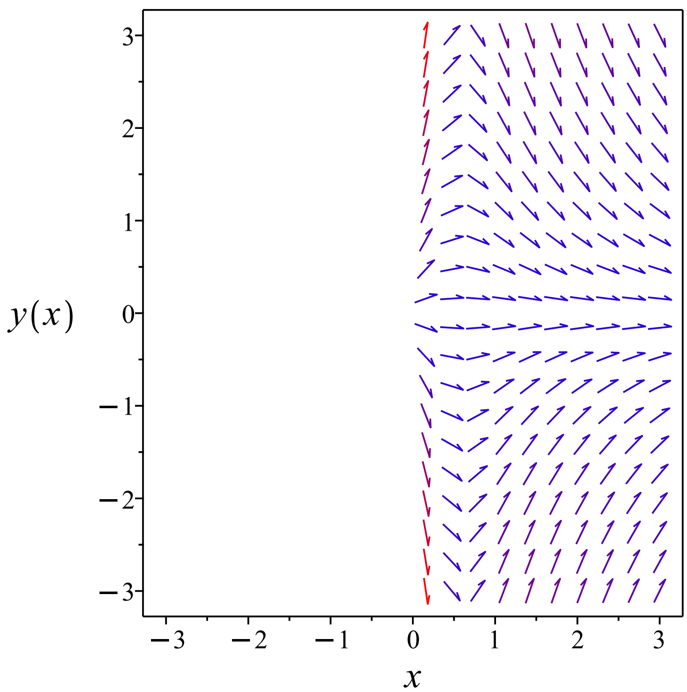


Figure 296: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\int \frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx}$$

Verified OK.

7.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln\left(x\right)\right)\sqrt{x^2+1}+x^2+1\right)}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{\left(-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}\right)y}{x\left(\left(x^2-\ln\left(x\right)\right)\sqrt{x^2+1}+x^2+1\right)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx} y\right) = 0$$

Integrating gives

$$\mathrm{e}^{\int -\frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx}$ results in

$$y = c_1 \mathrm{e}^{\int \frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\int \frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx} \tag{1}$$

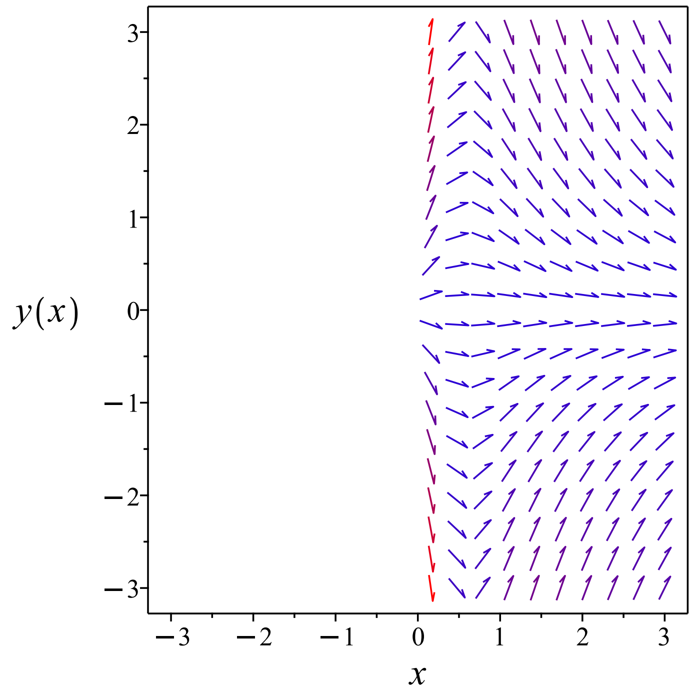


Figure 297: Slope field plot

Verification of solutions

$$y = c_1 e^{\int \frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx}$$

Verified OK.

7.8.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{x^2 u(x)}{\sqrt{x^2+1}} + 2u(x)\,x^2 - u(x) + \left(\sqrt{x^2+1} + x^2 - \ln(x)\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u\left(-3x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1\right)}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}
\end{aligned}$$

Where $f(x) = -\frac{-3x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{-3x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}\,dx\\
\int\frac{1}{u}\,du &= \int -\frac{-3x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}\,dx\\
\ln\left(u\right) &= \int -\frac{-3x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}dx+c_2\\
u &= \mathrm{e}^{\int -\frac{-3x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx+c_2}\\
&= c_2\mathrm{e}^{\int -\frac{-3x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux\\
&= xc_2\mathrm{e}^{\int -\frac{-3x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = xc_2\mathrm{e}^{\int -\frac{-3x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx} \tag{1}$$

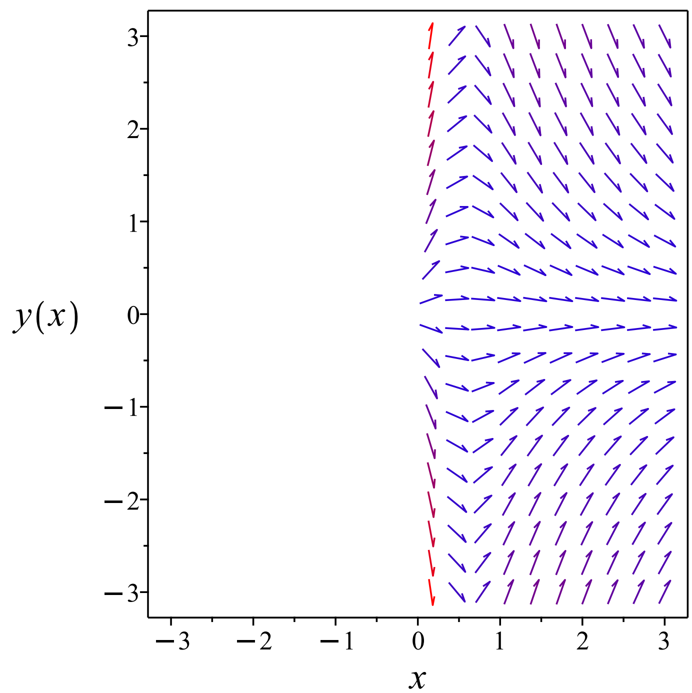


Figure 298: Slope field plot

Verification of solutions

$$
y = x c_2 \mathrm{e}^{\int -\frac{-3x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-2x^2+\sqrt{x^2+1}-1}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}dx}
$$

Verified OK.

7.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = \frac{y\left(2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}\right)}{x\left(-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1\right)}
$$
$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 233: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\int \frac{-2x^2\sqrt{x^2+1} - x^2 + \sqrt{x^2+1}}{x\left(\left(x^2 - \ln(x)\right)\sqrt{x^2+1} + x^2 + 1\right)} dx}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\int \frac{-2x^2\sqrt{x^2+1}-x^2+\sqrt{x^2+1}}{x\left(\left(x^2-\ln(x)\right)\sqrt{x^2+1}+x^2+1\right)}dx}} dy \end{aligned}$$

7.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might

or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right) \mathrm{d}y = \left(\frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2\right.}\right.$$

$$\left(-\frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -\frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)} \\
N(x,y) &= \frac{1}{y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x^2\sqrt{x^2+1}+x^2-\sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1}+\ln\left(x\right)\sqrt{x^2+1}-x^2-1\right)}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{2x^2\sqrt{x^2+1} + x^2 - \sqrt{x^2+1}}{x\left(-x^2\sqrt{x^2+1} + \ln(x)\sqrt{x^2+1} - x^2 - 1\right)} \mathrm{d}x$$

$$\phi = \int^x -\frac{2\_a^2\sqrt{\_a^2+1} + \_a^2 - \sqrt{\_a^2+1}}{\_a\left(-\_a^2\sqrt{\_a^2+1} + \ln(\_a)\sqrt{\_a^2+1} - \_a^2 - 1\right)} d\_a + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x -\frac{2\_a^2\sqrt{\_a^2+1} + \_a^2 - \sqrt{\_a^2+1}}{\_a\left(-\_a^2\sqrt{\_a^2+1} + \ln(\_a)\sqrt{\_a^2+1} - \_a^2 - 1\right)} d\_a + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x -\frac{2\_a^2\sqrt{\_a^2+1}+\_a^2-\sqrt{\_a^2+1}}{\_a\left(-\_a^2\sqrt{\_a^2+1}+\ln\left(\_a\right)\sqrt{\_a^2+1}-\_a^2-1\right)}d\_a + \ln\left(y\right)$$

The solution becomes

$$y = \mathrm{e}^{-\left(\int^x -\frac{2\_a^2\sqrt{\_a^2+1}+\_a^2-\sqrt{\_a^2+1}}{\_a\left(-\_a^2\sqrt{\_a^2+1}+\ln(\_a)\sqrt{\_a^2+1}-\_a^2-1\right)}d\_a\right)+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\left(\int^x -\frac{2\_a^2\sqrt{\_a^2+1}+\_a^2-\sqrt{\_a^2+1}}{\_a\left(-\_a^2\sqrt{\_a^2+1}+\ln(\_a)\sqrt{\_a^2+1}-\_a^2-1\right)}d\_a\right)+c_1} \tag{1}$$

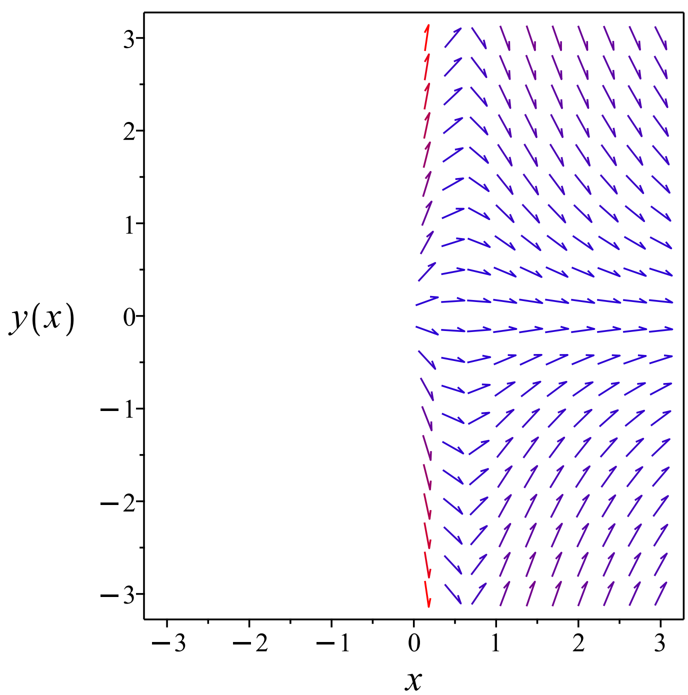


Figure 299: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\left(\int^{x} -\frac{2\_a^{2}\sqrt{\_a^{2}+1}+\_a^{2}-\sqrt{\_a^{2}+1}}{\_a\left(-\_a^{2}\sqrt{\_a^{2}+1}+\ln(\_a)\sqrt{\_a^{2}+1}-\_a^{2}-1\right)}d\_a\right)+c_1}$$

Verified OK.

7.8.6 Maple step by step solution

Let's solve

$\frac{xy}{\sqrt{x^2+1}} + 2yx - \frac{y}{x} + \left(\sqrt{x^2+1} + x^2 - \ln(x)\right) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(\frac{xy}{\sqrt{x^2+1}} + 2yx - \frac{y}{x} + \left(\sqrt{x^2+1} + x^2 - \ln(x)\right) y'\right) dx = \int 0 dx + c_1$

- Evaluate integral

  $-\frac{\left(-\sqrt{x^2+1}\, x^3 + \sqrt{x^2+1}\, \ln(x) x - x^3 - x\right) y}{\sqrt{x^2+1}\, x} = c_1$

- Solve for y

  $y = -\frac{c_1\sqrt{x^2+1}}{-x^2\sqrt{x^2+1}+\ln(x)\sqrt{x^2+1}-x^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve(( x*y(x)/sqrt(1+x^2) + 2*x*y(x) -y(x)/x  )+(  sqrt(1+x^2) + x^2-ln(x) )*diff(y(x),x)=
```

$$y(x) = c_1 \mathrm{e}^{-\left(\int \frac{2\sqrt{x^2+1}\, x^2 + x^2 - \sqrt{x^2+1}}{\sqrt{x^2+1}\, x\left(\sqrt{x^2+1}+x^2-\ln(x)\right)} dx\right)}$$

✓ Solution by Mathematica

Time used: 7.409 (sec). Leaf size: 94

```
DSolve[( x*y[x]/Sqrt[1+x^2] + 2*x*y[x] -y[x]/x  )+(  Sqrt[1+x^2] + x^2-Log[x] )*y'[x]==0,y[x
```

$$y(x) \to c_1 \exp\left(\int_1^x \frac{\sqrt{K[1]^2+1} - K[1]^2\left(2\sqrt{K[1]^2+1}+1\right)}{K[1]\left(\left(\sqrt{K[1]^2+1}+1\right)K[1]^2 - \sqrt{K[1]^2+1}\log(K[1])+1\right)}dK[1]\right)$$

$$y(x) \to 0$$

7.9 problem 184

Internal problem ID [15072]
Internal file name [`OUTPUT/15072_Sunday_April_21_2024_01_26_05_PM_13350514/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 184.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\sin(y) + \sin(x)\, y + \left(\cos(y)\, x - \cos(x) + \frac{1}{y}\right) y' = -\frac{1}{x}}$$

7.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\cos(y)\, x - \cos(x) + \frac{1}{y}\right) \mathrm{d}y = \left(-\sin(y) - \sin(x)\, y - \frac{1}{x}\right) \mathrm{d}x$$
$$\left(\sin(y) + \sin(x)\, y + \frac{1}{x}\right) \mathrm{d}x + \left(\cos(y)\, x - \cos(x) + \frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \sin(y) + \sin(x)\, y + \frac{1}{x}$$
$$N(x, y) = \cos(y)\, x - \cos(x) + \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sin(y) + \sin(x)\, y + \frac{1}{x}\right) \\ &= \cos(y) + \sin(x) \end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\cos(y)\, x - \cos(x) + \frac{1}{y}\right)$$
$$= \cos(y) + \sin(x)$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, dx = \int M\, dx$$
$$\int \frac{\partial \phi}{\partial x}\, dx = \int \sin(y) + \sin(x)\, y + \frac{1}{x}\, dx$$
$$\phi = \ln(x) - \cos(x)\, y + \sin(y)\, x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(y)\, x - \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y)\, x - \cos(x) + \frac{1}{y}$. Therefore equation (4) becomes

$$\cos(y)\, x - \cos(x) + \frac{1}{y} = \cos(y)\, x - \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int \left(\frac{1}{y}\right) dy$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \cos(x)\, y + \sin(y)\, x + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \cos(x)\, y + \sin(y)\, x + \ln(y)$$

Summary

The solution(s) found are the following

$$\ln(x) - y\cos(x) + \sin(y)\, x + \ln(y) = c_1 \tag{1}$$

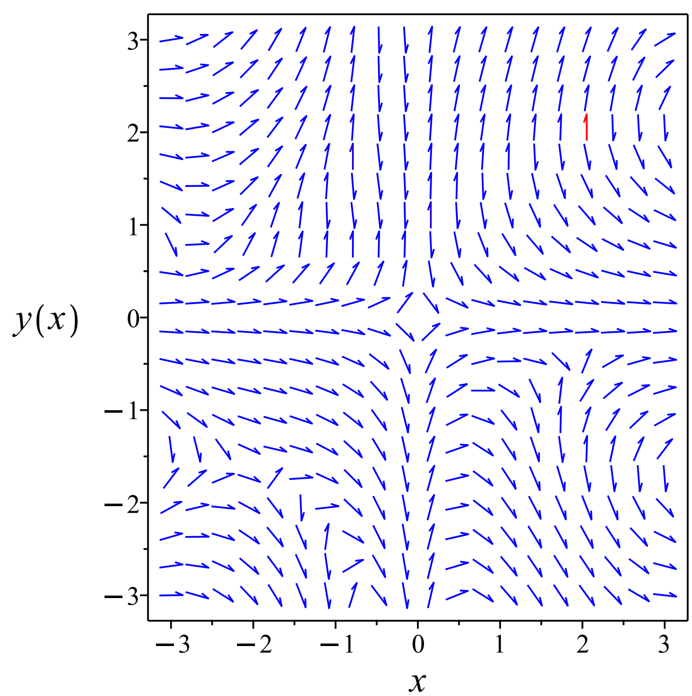


Figure 300: Slope field plot

Verification of solutions

$$\ln(x) - y\cos(x) + \sin(y)\, x + \ln(y) = c_1$$

Verified OK.

7.9.2 Maple step by step solution

Let's solve

$\sin(y)+\sin(x)\,y+\left(\cos(y)\,x-\cos(x)+\frac{1}{y}\right)y'=-\frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $\cos(y)+\sin(x)=\cos(y)+\sin(x)$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(\sin(y)+\sin(x)\,y+\frac{1}{x}\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\ln(x)-\cos(x)\,y+\sin(y)\,x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\cos(y)\,x-\cos(x)+\frac{1}{y}=-\cos(x)+\cos(y)\,x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=\frac{1}{y}$

- Solve for $f_1(y)$

  $f_1(y)=\ln(y)$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \ln(x) - \cos(x)\, y + \sin(y)\, x + \ln(y)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\ln(x) - \cos(x)\, y + \sin(y)\, x + \ln(y) = c_1$$

- Solve for y

$$y = \mathit{RootOf}(-\sin(\_Z)\, x + \cos(x)\_Z - \ln(\_Zx) + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve(( sin(y(x))+y(x)*sin(x)+1/x )+( x*cos(y(x))-cos(x)+1/y(x) )*diff(y(x),x)=0,y(x), sing
```

$$-y(x)\cos(x) + \sin(y(x))\, x + \ln(x) + \ln(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.332 (sec). Leaf size: 23

```
DSolve[( Sin[y[x]]+y[x]*Sin[x]+1/x )+( x*Cos[y[x]]-Cos[x]+1/y[x] )*y'[x]==0,y[x],x,IncludeSi
```

$$\text{Solve}[\log(y(x)) + x\sin(y(x)) - y(x)\cos(x) + \log(x) = c_1, y(x)]$$

7.10 problem 185

Internal problem ID [15073]
Internal file name [`OUTPUT/15073_Sunday_April_21_2024_01_27_21_PM_57886284/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 185.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{y+\sin(x)\cos(yx)^2}{\cos(yx)^2}+\left(\frac{x}{\cos(yx)^2}+\sin(y)\right)y'=0}$$

7.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x}{\cos(xy)^2} + \sin(y)\right)\mathrm{d}y = \left(-\frac{y + \sin(x)\cos(xy)^2}{\cos(xy)^2}\right)\mathrm{d}x$$
$$\left(\frac{y + \sin(x)\cos(xy)^2}{\cos(xy)^2}\right)\mathrm{d}x + \left(\frac{x}{\cos(xy)^2} + \sin(y)\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y + \sin(x)\cos(xy)^2}{\cos(xy)^2}$$
$$N(x,y) = \frac{x}{\cos(xy)^2} + \sin(y)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y + \sin(x)\cos(xy)^2}{\cos(xy)^2}\right)\\ &= \sec(xy)^2\,(1 + 2\tan(xy)\,xy)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{\cos(xy)^2} + \sin(y)\right)\\ &= \sec(xy)^2 (1 + 2\tan(xy)\, xy)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y + \sin(x)\cos(xy)^2}{\cos(xy)^2}\,\mathrm{d}x\end{aligned}$$

$$\phi = \tan(xy) - \cos(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\left(1 + \tan(xy)^2\right) + f'(y) \tag{4}$$

$$= \sec(xy)^2\, x + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{\cos(xy)^2} + \sin(y)$. Therefore equation (4) becomes

$$\frac{x}{\cos(xy)^2} + \sin(y) = \sec(xy)^2\, x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\frac{\sec(xy)^2\, x\cos(xy)^2 - \sin(y)\cos(xy)^2 - x}{\cos(xy)^2}\\ &= \sin(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\sin(y))\,\mathrm{d}y$$
$$f(y) = -\cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \tan(xy) - \cos(x) - \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \tan(xy) - \cos(x) - \cos(y)$$

Summary
The solution(s) found are the following

$$\tan(yx) - \cos(x) - \cos(y) = c_1 \tag{1}$$

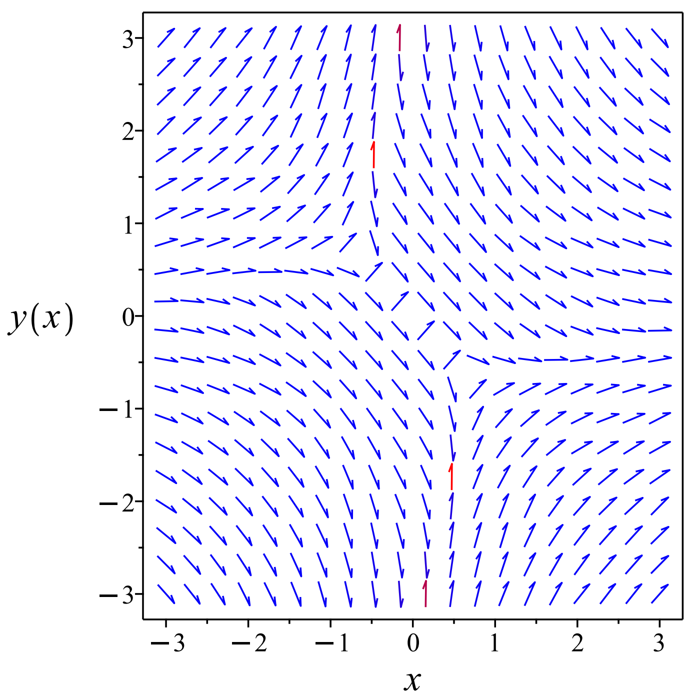


Figure 301: Slope field plot

Verification of solutions

$$\tan(yx) - \cos(x) - \cos(y) = c_1$$

Verified OK.

7.10.2 Maple step by step solution

Let's solve

$\frac{y+\sin(x)\cos(yx)^2}{\cos(yx)^2} + \left(\frac{x}{\cos(yx)^2} + \sin(y)\right) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  ◦ Evaluate derivatives

    $\frac{1-2\sin(x)\cos(xy)x\sin(xy)}{\cos(xy)^2} + \frac{2\left(y+\sin(x)\cos(xy)^2\right)x\sin(xy)}{\cos(xy)^3} = \frac{1}{\cos(xy)^2} + \frac{2x\sin(xy)y}{\cos(xy)^3}$

  ◦ Simplify

    $\sec(xy)^2(1+2\tan(xy)xy) = \sec(xy)^2(1+2\tan(xy)xy)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \frac{y+\sin(x)\cos(xy)^2}{\cos(xy)^2}dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \tan(xy) - \cos(x) + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

$$\frac{x}{\cos(xy)^2} + \sin(y) = x\left(1 + \tan(xy)^2\right) + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \frac{x}{\cos(xy)^2} + \sin(y) - x\left(1 + \tan(xy)^2\right)$$

- Solve for $f_1(y)$

$$f_1(y) = -\cos(y)$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \tan(xy) - \cos(x) - \cos(y)$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\tan(xy) - \cos(x) - \cos(y) = c_1$$

- Solve for y

$$y = \mathit{RootOf}(\_Zx - \arctan(\cos(\_Z) + c_1 + \cos(x)))$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(( (y(x)+sin(x)*cos(x*y(x))^2 )/cos(x*y(x))^2 )+( x/cos(x*y(x))^2+sin(y(x)) )*diff(y(x
```

$$\tan(xy(x)) - \cos(x) - \cos(y(x)) + c_1 = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[( (y[x]+Sin[x]*Cos[x*y[x]]^2 )/Cos[x*y[x]]^2 )+( x/Cos[x*y[x]]^2+Sin[y[x]] )*y'[x]==0
```

Not solved

7.11 problem 186

Internal problem ID [15074]
Internal file name [`OUTPUT/15074_Sunday_April_21_2024_01_29_01_PM_55889654/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 186.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\frac{2x}{y^{3}}+\frac{\left(y^{2}-3x^{2}\right) y'}{y^{4}}=0}$$

With initial conditions

$$[y(1)=1]$$

7.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{2}{x^{2}u\left(x\right)^{3}}+\frac{\left(u(x)^{2}\,x^{2}-3x^{2}\right)\left(u'(x)\,x+u(x)\right)}{u\left(x\right)^{4}x^{4}}=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^{3}-u}{x\left(u^{2}-3\right)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-u}{u^2-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-u}{u^2-3}} du = -\frac{1}{x} dx$$
$$\int \frac{1}{\frac{u^3-u}{u^2-3}} du = \int -\frac{1}{x} dx$$
$$-\ln(u-1) - \ln(u+1) + 3\ln(u) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(u-1)-\ln(u+1)+3\ln(u)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^3}{u^2-1} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^3}{u(x)^2-1} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^3}{x^3\left(\frac{y^2}{x^2}-1\right)} = \frac{c_3}{x}$$
$$\frac{y^3}{x(y^2-x^2)} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{y^3}{(-y+x)(y+x)} = c_3$$

Writing the solution as

$$-c_1 y^3 = (-y+x)(y+x)$$

Where $c_1 = \frac{1}{c_3}$ and solving for c_1 after applying initial conditions gives $c_1 = 0$. Hence the above solution becomes

$$0 = (-y+x)(y+x)$$

Summary
The solution(s) found are the following

$$0 = (-y + x)(y + x) \tag{1}$$

Verification of solutions

$$0 = (-y + x)(y + x)$$

Verified OK.

7.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2xy}{-3x^2 + y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{2xy(b_3 - a_2)}{-3x^2 + y^2} - \frac{4x^2 y^2 a_3}{(-3x^2 + y^2)^2} \\ & - \left(-\frac{2y}{-3x^2 + y^2} - \frac{12x^2 y}{(-3x^2 + y^2)^2} \right) (x a_2 + y a_3 + a_1) \\ & - \left(-\frac{2x}{-3x^2 + y^2} + \frac{4x\, y^2}{(-3x^2 + y^2)^2} \right) (x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^4b_2 + 2x^2y^2a_3 - 8x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 - 6x^3b_1 + 6x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1}{(3x^2 - y^2)^2} = 0$$

Setting the numerator to zero gives

$$3x^4b_2 + 2x^2y^2a_3 - 8x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 + y^4b_2 - 6x^3b_1 + 6x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$4a_2v_1v_2^3 + 2a_3v_1^2v_2^2 + 2a_3v_2^4 + 3b_2v_1^4 - 8b_2v_1^2v_2^2 + b_2v_2^4 - 4b_3v_1v_2^3 + 6a_1v_1^2v_2 + 2a_1v_2^3 - 6b_1v_1^3 - 2b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1^4 - 6b_1v_1^3 + (2a_3 - 8b_2)\,v_1^2v_2^2 + 6a_1v_1^2v_2 + (4a_2 - 4b_3)\,v_1v_2^3 - 2b_1v_1v_2^2 + (2a_3 + b_2)\,v_2^4 + 2a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
2a_1 &= 0 \\
6a_1 &= 0 \\
-6b_1 &= 0 \\
-2b_1 &= 0 \\
3b_2 &= 0 \\
4a_2 - 4b_3 &= 0 \\
2a_3 - 8b_2 &= 0 \\
2a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2xy}{-3x^2 + y^2}\right)(x) \\
&= \frac{x^2y - y^3}{3x^2 - y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2y-y^3}{3x^2-y^2}} dy \end{aligned}$$

Which results in

$$S = -\ln(y+x) - \ln(y-x) + 3\ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2xy}{-3x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x}{x^2 - y^2} \\ S_y &= -\frac{1}{y+x} + \frac{1}{-y+x} + \frac{3}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

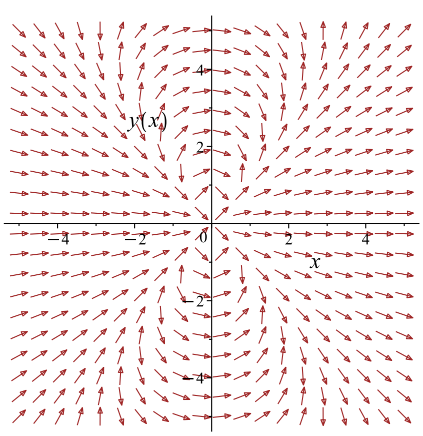
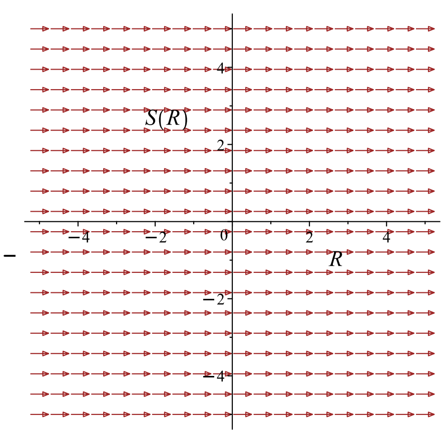
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y+x) - \ln(y-x) + 3\ln(y) = c_1$$

Which simplifies to

$$-\ln(y+x) - \ln(y-x) + 3\ln(y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2xy}{-3x^2+y^2}$ 	$R = x$ $S = -\ln(y+x) - \ln(y -$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\infty = c_1$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

7.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-3x^2+y^2}{y^4}\right)\mathrm{d}y = \left(-\frac{2x}{y^3}\right)\mathrm{d}x$$

$$\left(\frac{2x}{y^3}\right)\mathrm{d}x + \left(\frac{-3x^2+y^2}{y^4}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{2x}{y^3}$$
$$N(x,y) = \frac{-3x^2+y^2}{y^4}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{y^3}\right)\\ &= -\frac{6x}{y^4}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-3x^2+y^2}{y^4}\right)\\ &= -\frac{6x}{y^4}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\mathrm{d}x = \int \frac{2x}{y^3}\mathrm{d}x$$
$$\phi = \frac{x^2}{y^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{3x^2}{y^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-3x^2+y^2}{y^4}$. Therefore equation (4) becomes

$$\frac{-3x^2 + y^2}{y^4} = -\frac{3x^2}{y^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{y^3} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{y^3} - \frac{1}{y}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2}{y^3} - \frac{1}{y} = 0$$

The above simplifies to

$$x^2 - y^2 = 0$$

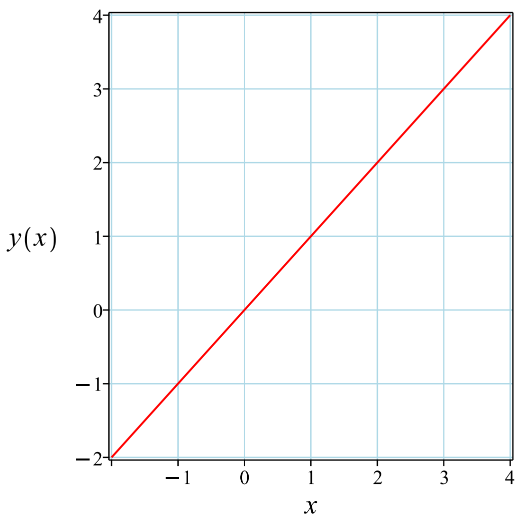
Solving for y from the above gives

$$y = x$$

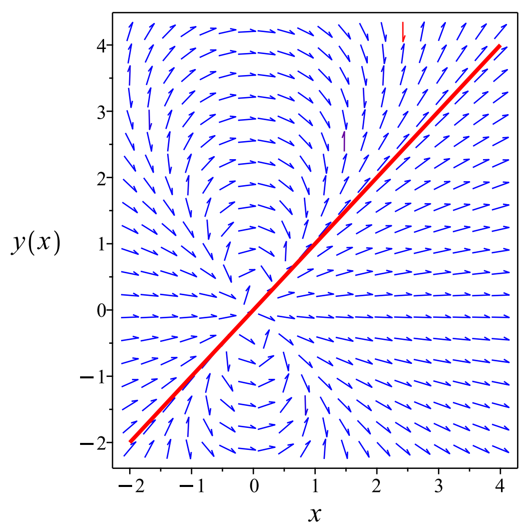
Summary

The solution(s) found are the following

$$y = x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.5 (sec). Leaf size: 5

```
dsolve([( 2*x/y(x)^3)+( (y(x)^2-3*x^2)/y(x)^4 )*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{( 2*x/y[x]^3)+( (y[x]^2-3*x^2)/y[x]^4 )*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularSo
```

Timed out

7.12 problem 187

Internal problem ID [15075]
Internal file name [`OUTPUT/15075_Sunday_April_21_2024_01_29_04_PM_39592413/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 187.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{y(x^2+y^2+a^2)\,y'+x(y^2-a^2+x^2)=0}$$

7.12.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(y\left(a^2+x^2+y^2\right)\right)\mathrm{d}y&=\left(-x\left(-a^2+x^2+y^2\right)\right)\mathrm{d}x\\ \left(x\left(-a^2+x^2+y^2\right)\right)\mathrm{d}x+\left(y\left(a^2+x^2+y^2\right)\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x\left(-a^2+x^2+y^2\right)\\ N(x,y)&=y\left(a^2+x^2+y^2\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x\left(-a^2+x^2+y^2\right)\right)\\ &=2xy\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(y\left(a^2+x^2+y^2\right)\right)\\ &=2xy\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x\left(-a^2 + x^2 + y^2\right) \mathrm{d}x$$

$$\phi = \frac{\left(a^2 - x^2 - y^2\right)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\left(a^2 - x^2 - y^2\right) y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y(a^2 + x^2 + y^2)$. Therefore equation (4) becomes

$$y\left(a^2 + x^2 + y^2\right) = -\left(a^2 - x^2 - y^2\right) y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2a^2 y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(2a^2 y\right) \mathrm{d}y$$

$$f(y) = y^2 a^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(a^2 - x^2 - y^2\right)^2}{4} + y^2 a^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(a^2 - x^2 - y^2\right)^2}{4} + y^2 a^2$$

Summary

The solution(s) found are the following

$$\frac{\left(a^2-x^2-y^2\right)^2}{4}+y^2a^2=c_1 \tag{1}$$

Verification of solutions

$$\frac{\left(a^2-x^2-y^2\right)^2}{4}+y^2a^2=c_1$$

Verified OK.

7.12.2 Maple step by step solution

Let's solve

$y(x^2+y^2+a^2)\,y'+x(y^2-a^2+x^2)=0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $2xy=2xy$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int x(-a^2+x^2+y^2)\,dx+f_1(y)$
- Evaluate integral

  $F(x,y)=\frac{(-a^2+x^2+y^2)^2}{4}+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

$N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y(a^2+x^2+y^2) = (-a^2+x^2+y^2)\,y + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = y(a^2+x^2+y^2) - (-a^2+x^2+y^2)\,y$

- Solve for $f_1(y)$

  $f_1(y) = y^2a^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{(-a^2+x^2+y^2)^2}{4} + y^2a^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{(-a^2+x^2+y^2)^2}{4} + y^2a^2 = c_1$

- Solve for y

  $\left\{y = \sqrt{-a^2-x^2-2\sqrt{a^2x^2+c_1}}, y = \sqrt{-a^2-x^2+2\sqrt{a^2x^2+c_1}}, y = -\sqrt{-a^2-x^2-2\sqrt{a^2x^2-}}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 129

```
dsolve(( y(x)*(x^2+y(x)^2+a^2))*diff(y(x),x)+x*(x^2+y(x)^2-a^2)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-a^2 - x^2 - 2\sqrt{a^2x^2 - c_1}}$$
$$y(x) = \sqrt{-a^2 - x^2 + 2\sqrt{a^2x^2 - c_1}}$$
$$y(x) = -\sqrt{-a^2 - x^2 - 2\sqrt{a^2x^2 - c_1}}$$
$$y(x) = -\sqrt{-a^2 - x^2 + 2\sqrt{a^2x^2 - c_1}}$$

✓ Solution by Mathematica

Time used: 2.374 (sec). Leaf size: 165

```
DSolve[( y[x]*(x^2+y[x]^2+a^2))*y'[x]+x*(x^2+y[x]^2-a^2)==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\sqrt{-a^2 - \sqrt{a^4 + 4a^2x^2 + 4c_1} - x^2}$$
$$y(x) \to \sqrt{-a^2 - \sqrt{a^4 + 4a^2x^2 + 4c_1} - x^2}$$
$$y(x) \to -\sqrt{-a^2 + \sqrt{a^4 + 4a^2x^2 + 4c_1} - x^2}$$
$$y(x) \to \sqrt{-a^2 + \sqrt{a^4 + 4a^2x^2 + 4c_1} - x^2}$$

7.13 problem 188

Internal problem ID [15076]
Internal file name [`OUTPUT/15076_Sunday_April_21_2024_01_29_06_PM_53337072/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 188.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{3x^2y + y^3 + (x^3 + 3y^2x)\, y' = 0}$$

7.13.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$3x^3u(x) + u(x)^3\, x^3 + \left(x^3 + 3u(x)^2\, x^3\right) \left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{4u(u^2+1)}{x\,(3u^2+1)} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = \frac{(u^2+1)u}{3u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{(u^2+1)u}{3u^2+1}} \, du = -\frac{4}{x} \, dx$$

$$\int \frac{1}{\frac{(u^2+1)u}{3u^2+1}} \, du = \int -\frac{4}{x} \, dx$$

$$\ln\left(u(u^2+1)\right) = -4\ln(x) + c_2$$

Raising both side to exponential gives

$$u(u^2+1) = e^{-4\ln(x)+c_2}$$

Which simplifies to

$$u(u^2+1) = \frac{c_3}{x^4}$$

Which simplifies to

$$u(x)\left(u(x)^2+1\right) = \frac{c_3 e^{c_2}}{x^4}$$

The solution is

$$u(x)\left(u(x)^2+1\right) = \frac{c_3 e^{c_2}}{x^4}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y\left(\frac{y^2}{x^2}+1\right)}{x} = \frac{c_3 e^{c_2}}{x^4}$$

$$\frac{y(x^2+y^2)}{x^3} = \frac{c_3 e^{c_2}}{x^4}$$

Which simplifies to

$$y(x^2+y^2) = \frac{c_3 e^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$y(x^2+y^2) = \frac{c_3 e^{c_2}}{x} \tag{1}$$

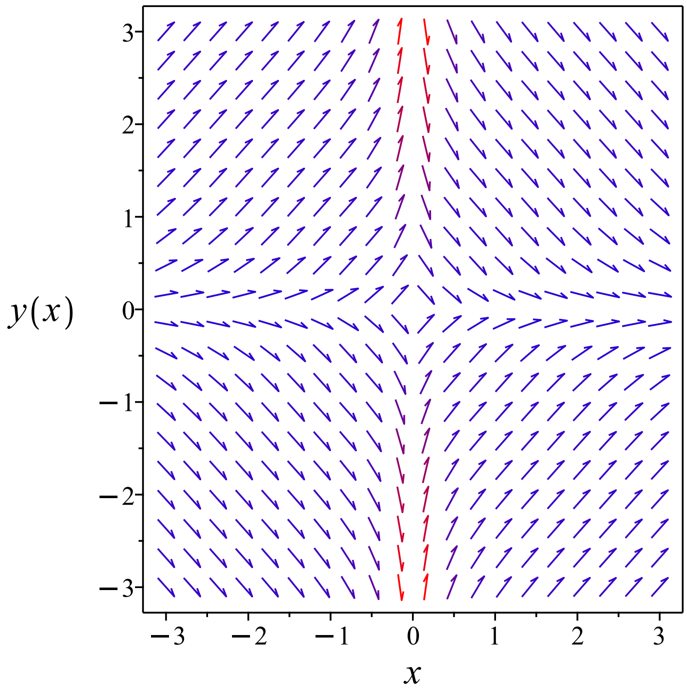


Figure 303: Slope field plot

Verification of solutions

$$y(x^2+y^2) = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

7.13.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(3x^2+y^2)}{x(x^2+3y^2)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y(3x^2+y^2)(b_3-a_2)}{x(x^2+3y^2)} - \frac{y^2(3x^2+y^2)^2 a_3}{x^2(x^2+3y^2)^2} \\ & - \left(-\frac{6y}{x^2+3y^2} + \frac{y(3x^2+y^2)}{x^2(x^2+3y^2)} + \frac{2y(3x^2+y^2)}{(x^2+3y^2)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{3x^2+y^2}{x(x^2+3y^2)} - \frac{2y^2}{x(x^2+3y^2)} + \frac{6y^2(3x^2+y^2)}{x(x^2+3y^2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^6b_2 - 12x^4y^2a_3 + 16x^3y^3a_2 - 16x^3y^3b_3 + 12x^2y^4b_2 - 4y^6a_3 + 3x^5b_1 - 3x^4ya_1 - 6x^3y^2b_1 + 6x^2y^3a_1 + 3x}{(x^2+3y^2)^2 x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4x^6b_2 - 12x^4y^2a_3 + 16x^3y^3a_2 - 16x^3y^3b_3 + 12x^2y^4b_2 - 4y^6a_3 \\ & + 3x^5b_1 - 3x^4ya_1 - 6x^3y^2b_1 + 6x^2y^3a_1 + 3x\,y^4b_1 - 3y^5a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 16a_2v_1^3v_2^3 - 12a_3v_1^4v_2^2 - 4a_3v_2^6 + 4b_2v_1^6 + 12b_2v_1^2v_2^4 - 16b_3v_1^3v_2^3 \\ & - 3a_1v_1^4v_2 + 6a_1v_1^2v_2^3 - 3a_1v_2^5 + 3b_1v_1^5 - 6b_1v_1^3v_2^2 + 3b_1v_1v_2^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^6 + 3b_1v_1^5 - 12a_3v_1^4v_2^2 - 3a_1v_1^4v_2 + (16a_2 - 16b_3)\,v_1^3v_2^3 \\ - 6b_1v_1^3v_2^2 + 12b_2v_1^2v_2^4 + 6a_1v_1^2v_2^3 + 3b_1v_1v_2^4 - 4a_3v_2^6 - 3a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -3a_1 &= 0 \\ 6a_1 &= 0 \\ -12a_3 &= 0 \\ -4a_3 &= 0 \\ -6b_1 &= 0 \\ 3b_1 &= 0 \\ 4b_2 &= 0 \\ 12b_2 &= 0 \\ 16a_2 - 16b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(3x^2+y^2)}{x\,(x^2+3y^2)}\right)(x) \\
&= \frac{4x^2y+4y^3}{x^2+3y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x^2y+4y^3}{x^2+3y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(y(x^2+y^2)\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(3x^2+y^2)}{x\,(x^2+3y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{2x^2 + 2y^2} \\
S_y &= \frac{1}{4y} + \frac{y}{2x^2 + 2y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{4x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{4R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{4} + c_1 \tag{4}
$$

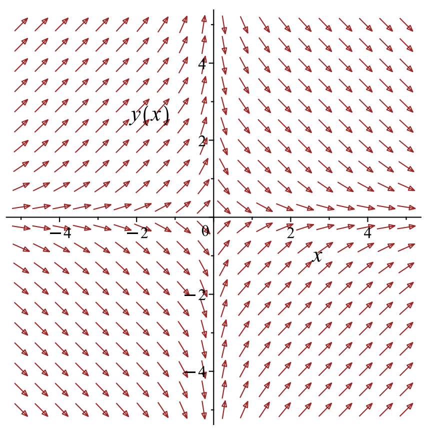
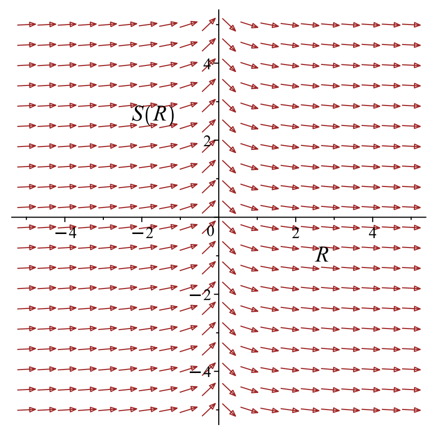
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{4} + \frac{\ln(x^2 + y^2)}{4} = -\frac{\ln(x)}{4} + c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{4} + \frac{\ln(x^2 + y^2)}{4} = -\frac{\ln(x)}{4} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(3x^2+y^2)}{x(x^2+3y^2)}$
</td><td>$R = x$
$S = \frac{\ln(y)}{4} + \frac{\ln(x^2+y^2)}{4}$</td><td>$\frac{dS}{dR} = -\frac{1}{4R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{4} + \frac{\ln(x^2+y^2)}{4} = -\frac{\ln(x)}{4} + c_1 \tag{1}$$

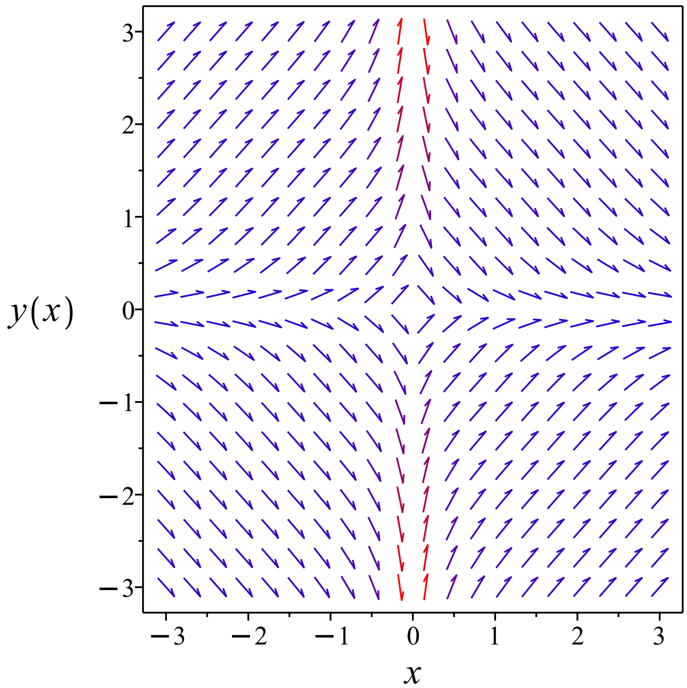


Figure 304: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{4}+\frac{\ln(x^2+y^2)}{4}=-\frac{\ln(x)}{4}+c_1$$

Verified OK.

7.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x^3 + 3y^2 x) \,\mathrm{d}y &= (-3x^2 y - y^3) \,\mathrm{d}x \\
(3x^2 y + y^3) \,\mathrm{d}x + (x^3 + 3y^2 x) \,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3x^2 y + y^3 \\
N(x, y) &= x^3 + 3y^2 x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2 y + y^3) \\
&= 3x^2 + 3y^2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3 + 3y^2 x) \\
&= 3x^2 + 3y^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3x^2y + y^3 \, \mathrm{d}x$$

$$\phi = xy(x^2 + y^2) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (x^2 + y^2)\, x + 2y^2x + f'(y) \tag{4}$$

$$= x^3 + 3y^2x + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 + 3y^2x$. Therefore equation (4) becomes

$$x^3 + 3y^2x = x^3 + 3y^2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy(x^2 + y^2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy(x^2 + y^2)$$

Summary

The solution(s) found are the following

$$y(x^2 + y^2)\, x = c_1 \tag{1}$$

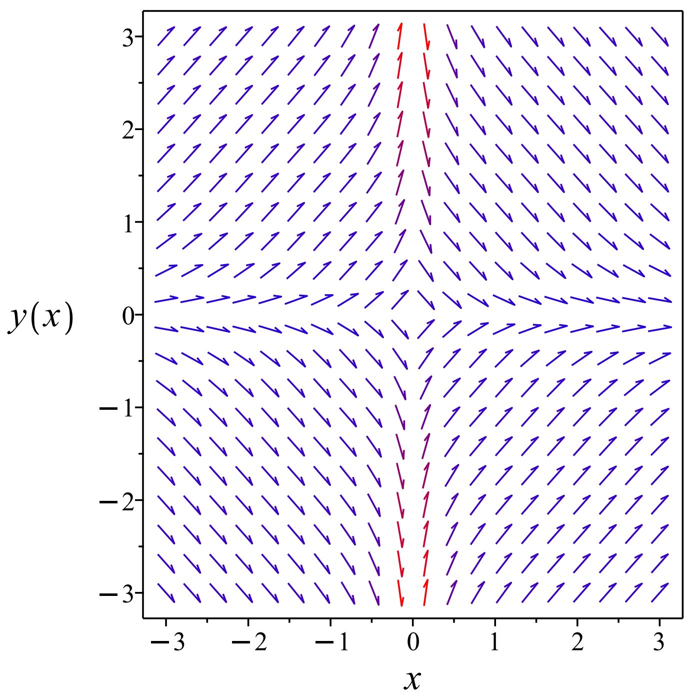


Figure 305: Slope field plot

Verification of solutions

$$y(x^2 + y^2)\, x = c_1$$

Verified OK.

7.13.4 Maple step by step solution

Let's solve

$3x^2y + y^3 + (x^3 + 3y^2x)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- ◦ Evaluate derivatives

  $3x^2 + 3y^2 = 3x^2 + 3y^2$

- ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x^2 y + y^3)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y(x^3 + y^2 x) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x^3 + 3y^2 x = x^3 + 3y^2 x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y(x^3 + y^2 x)$

- Substitute $F(x, y)$ into the solution of the ODE

  $y(x^3 + y^2 x) = c_1$

- Solve for y

$$\left\{y = \frac{\left(\left(12\sqrt{3}\,\sqrt{4x^8+27c_1^2}+108c_1\right)x^2\right)^{\frac{1}{3}}}{6x} - \frac{2x^3}{\left(\left(12\sqrt{3}\,\sqrt{4x^8+27c_1^2}+108c_1\right)x^2\right)^{\frac{1}{3}}}, y = -\frac{\left(\left(12\sqrt{3}\,\sqrt{4x^8+27c_1^2}+108c_1\right)x^2\right)}{12x}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 274

```
dsolve(( 3*x^2*y(x)+y(x)^3)+(x^3+3*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{12^{\frac{1}{3}}\left(x^4c_1^2 12^{\frac{1}{3}} - \left(\left(\sqrt{3}\,\sqrt{4c_1^4x^8+27}+9\right)x^2c_1\right)^{\frac{2}{3}}\right)}{6c_1x\left(\left(\sqrt{3}\,\sqrt{4c_1^4x^8+27}+9\right)x^2c_1\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{3^{\frac{1}{3}}2^{\frac{2}{3}}\left(\left(1+i\sqrt{3}\right)\left(\left(\sqrt{3}\,\sqrt{4c_1^4x^8+27}+9\right)x^2c_1\right)^{\frac{2}{3}} + c_1^2 2^{\frac{2}{3}}x^4\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)\right)}{12\left(\left(\sqrt{3}\,\sqrt{4c_1^4x^8+27}+9\right)x^2c_1\right)^{\frac{1}{3}}xc_1}$$

$$y(x) = \frac{\left(\left(i\sqrt{3}-1\right)\left(\left(\sqrt{3}\,\sqrt{4c_1^4x^8+27}+9\right)x^2c_1\right)^{\frac{2}{3}} + c_1^2\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)2^{\frac{2}{3}}x^4\right)3^{\frac{1}{3}}2^{\frac{2}{3}}}{12\left(\left(\sqrt{3}\,\sqrt{4c_1^4x^8+27}+9\right)x^2c_1\right)^{\frac{1}{3}}xc_1}$$

✓ Solution by Mathematica

Time used: 60.214 (sec). Leaf size: 338

```
DSolve[( 3*x^2*y[x]+y[x]^3)+(x^3+3*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{-2\sqrt[3]{3}x^2 + \sqrt[3]{2}\left(\frac{\sqrt{12x^8+81e^{2c_1}}+9e^{c_1}}{x}\right)^{2/3}}{6^{2/3}\sqrt[3]{\frac{\sqrt{12x^8+81e^{2c_1}}+9e^{c_1}}{x}}}$$

$$y(x) \to \frac{i2^{2/3}\sqrt[3]{3}\left(\sqrt{3}+i\right)\left(\frac{\sqrt{12x^8+81e^{2c_1}}+9e^{c_1}}{x}\right)^{2/3} + 2\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}+3i\right)x^2}{12\sqrt[3]{\frac{\sqrt{12x^8+81e^{2c_1}}+9e^{c_1}}{x}}}$$

$$y(x) \to \frac{2\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}-3i\right)x^2 - i2^{2/3}\sqrt[3]{3}\left(\sqrt{3}-i\right)\left(\frac{\sqrt{12x^8+81e^{2c_1}}+9e^{c_1}}{x}\right)^{2/3}}{12\sqrt[3]{\frac{\sqrt{12x^8+81e^{2c_1}}+9e^{c_1}}{x}}}$$

7.14 problem 189

Internal problem ID [15077]
Internal file name [OUTPUT/15077_Sunday_April_21_2024_01_29_14_PM_69843454/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 189.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel,
    `2nd type`, `class B`]]
```

$$\boxed{-x^2y + x^2(y - x)\, y' = -1}$$

7.14.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x^2(y-x)\right)\mathrm{d}y &= \left(x^2y-1\right)\mathrm{d}x \\
\left(-x^2y+1\right)\mathrm{d}x + \left(x^2(y-x)\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -x^2y+1 \\
N(x,y) &= x^2(y-x)
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2y+1\right) \\
&= -x^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2(y-x)\right) \\
&= -3x^2+2xy
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2\,(y-x)}\left(\left(-x^2\right) - \left(2x(y-x)-x^2\right)\right) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(-x^2 y + 1\right)\\ &= \frac{-x^2 y + 1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}\left(x^2(y - x)\right)\\ &= y - x\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x^2 y + 1}{x^2}\right) + (y - x)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^2y+1}{x^2} \mathrm{d}x$$
$$\phi = -xy - \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x$. Therefore equation (4) becomes

$$y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (y) \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -xy - \frac{1}{x} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy - \frac{1}{x} + \frac{y^2}{2}$$

Summary
The solution(s) found are the following

$$-yx - \frac{1}{x} + \frac{y^2}{2} = c_1 \tag{1}$$

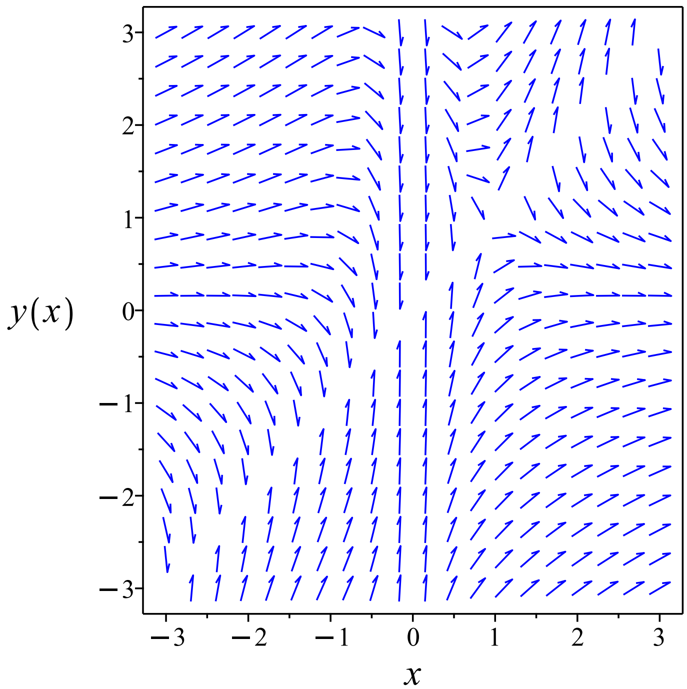


Figure 306: Slope field plot

Verification of solutions

$$-yx - \frac{1}{x} + \frac{y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(( 1-x^2*y(x))+x^2*(y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2 + \sqrt{x (x^3 - 2c_1 x + 2)}}{x}$$
$$y(x) = \frac{x^2 - \sqrt{x (x^3 - 2c_1 x + 2)}}{x}$$

✓ Solution by Mathematica

Time used: 0.493 (sec). Leaf size: 66

```
DSolve[( 1-x^2*y[x])+x^2*(y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \sqrt{-\frac{1}{x^2}}\sqrt{-x (x^3 + c_1 x + 2)}$$
$$y(x) \to x - \sqrt{-\frac{1}{x^2}}\sqrt{-x (x^3 + c_1 x + 2)}$$

7.15 problem 190

Internal problem ID [15078]
Internal file name [`OUTPUT/15078_Sunday_April_21_2024_01_29_15_PM_52623879/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 190.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y - y'x = -x^2}$$

7.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = x$$

Hence the ode is

$$y' - \frac{y}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ d\left(\frac{y}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int dx \\ \frac{y}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x + x^2$$

which simplifies to

$$y = x(x + c_1)$$

Summary

The solution(s) found are the following

$$y = x(x + c_1) \tag{1}$$

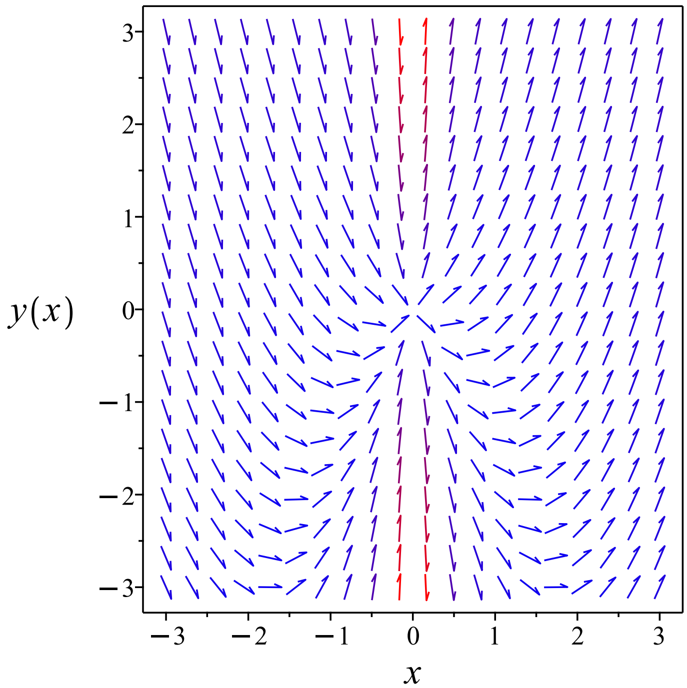


Figure 307: Slope field plot

Verification of solutions

$$y = x(x + c_1)$$

Verified OK.

7.15.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - (u'(x)\,x + u(x))\,x = -x^2$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 1\, \mathrm{d}x \\ &= x + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(x + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(x + c_2) \tag{1}$$

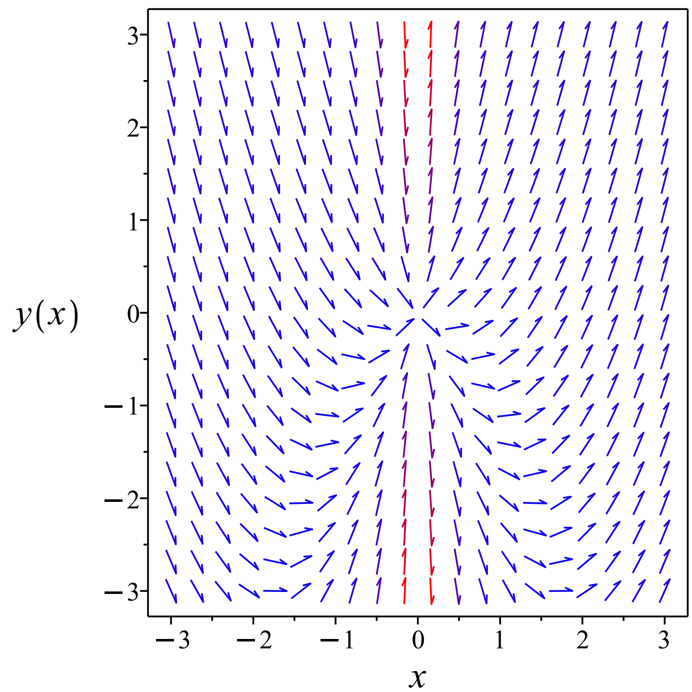


Figure 308: Slope field plot

Verification of solutions

$$y = x(x + c_2)$$

Verified OK.

7.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 240: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = x + c_1$$

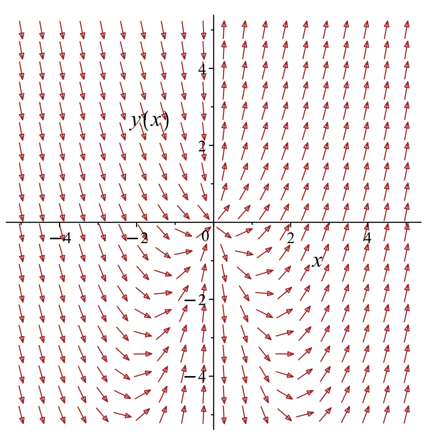
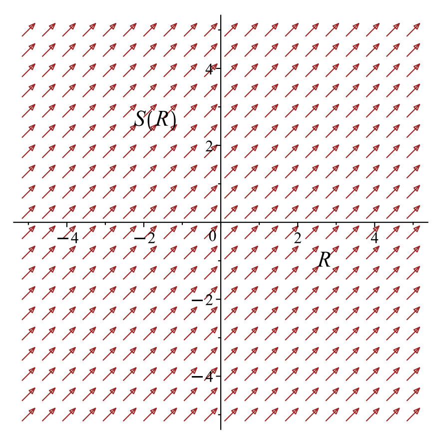
Which simplifies to

$$\frac{y}{x} = x + c_1$$

Which gives

$$y = x(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = x(x + c_1) \tag{1}$$

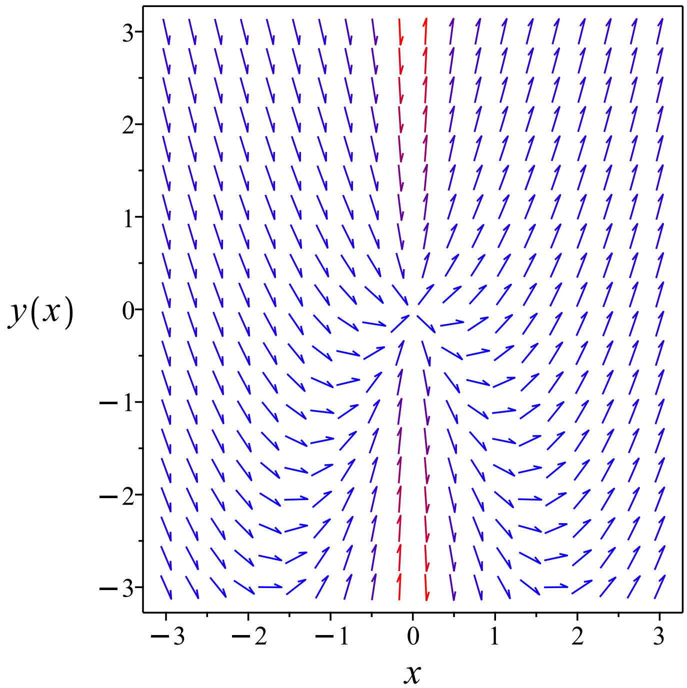


Figure 309: Slope field plot

Verification of solutions

$$y = x(x + c_1)$$

Verified OK.

7.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(-x) \,\mathrm{d}y = \left(-x^2 - y\right) \mathrm{d}x$$
$$\left(x^2 + y\right) \mathrm{d}x + (-x) \,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x^2 + y$$
$$N(x, y) = -x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\ &= -1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x}((1) - (-1)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2}(x^2 + y) \\ &= \frac{x^2 + y}{x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2}(-x) \\ &= -\frac{1}{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x^2 + y}{x^2}\right) + \left(-\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{x^2+y}{x^2}\,\mathrm{d}x$$
$$\phi = x - \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y}{x}$$

The solution becomes

$$y = -(-x + c_1)\, x$$

Summary

The solution(s) found are the following

$$y = -(-x + c_1)\, x \tag{1}$$

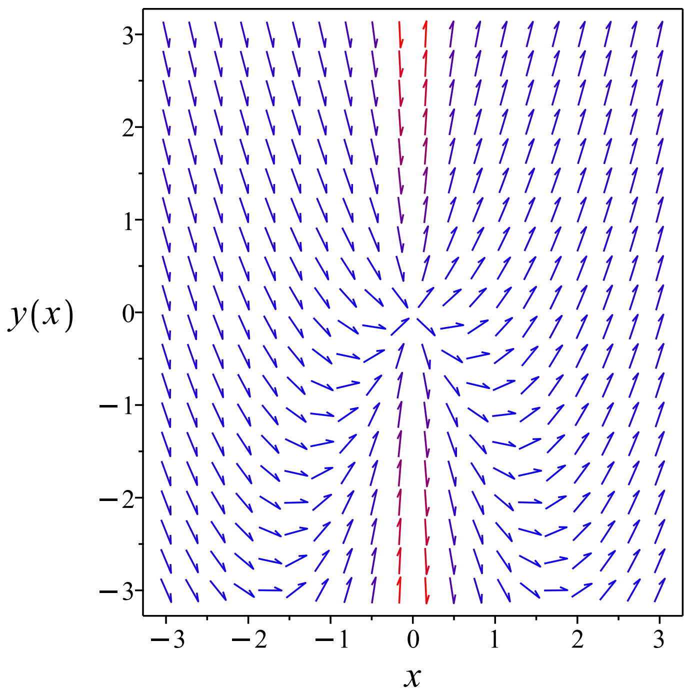


Figure 310: Slope field plot

Verification of solutions

$$y = -(-x + c_1)\, x$$

Verified OK.

7.15.5 Maple step by step solution

Let's solve

$y - y'x = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int 1 dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(( x^2+y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (x + c_1)\,x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 11

```
DSolve[( x^2+y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x + c_1)$$

7.16 problem 191

Internal problem ID [15079]
Internal file name [`OUTPUT/15079_Sunday_April_21_2024_01_29_16_PM_45472149/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 191.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x}$$

7.16.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 + x}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 243: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2 + x}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}$$

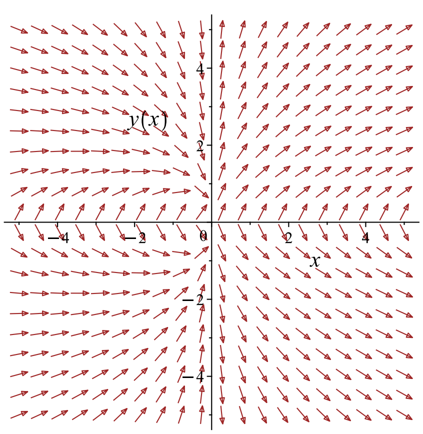
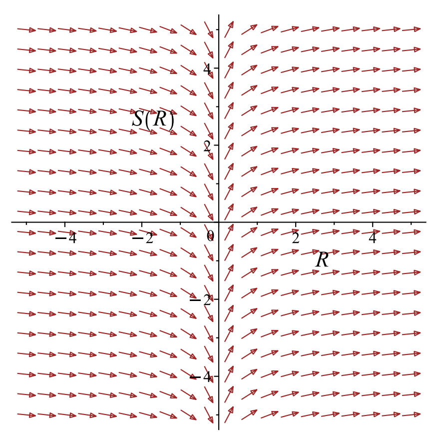
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+x}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x} = \frac{\ln(x)}{2} + c_1 \tag{1}$$

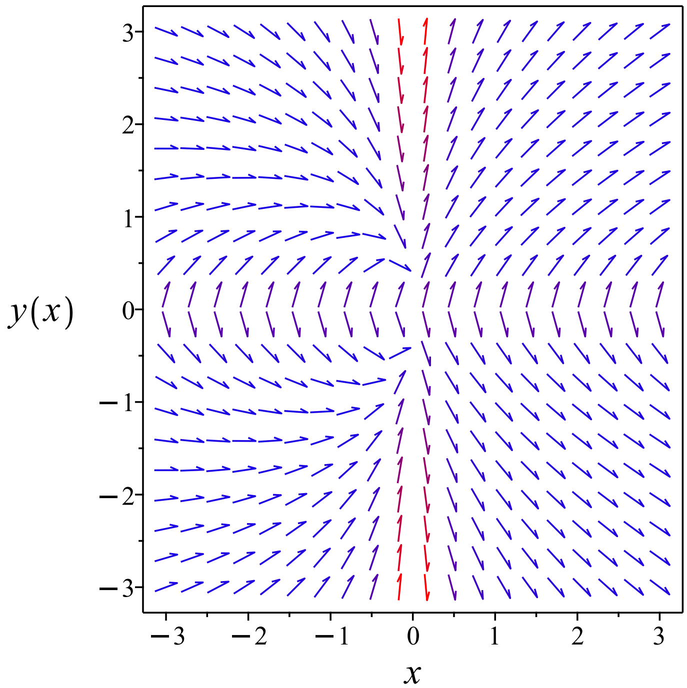


Figure 311: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = \frac{\ln(x)}{2} + c_1$$

Verified OK.

7.16.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 + x}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{1}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= \frac{1}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} + \frac{1}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{1}{2} \\ w' &= \frac{w}{x} + 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 1 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = 1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= \mu \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \frac{1}{x} \\ d\left(\frac{w}{x}\right) &= \frac{1}{x}dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \frac{1}{x}\, dx \\ \frac{w}{x} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\ln(x) + c_1 x$$

which simplifies to

$$w(x) = x(\ln(x) + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x(\ln(x) + c_1)} \\ y(x) &= -\sqrt{x(\ln(x) + c_1)}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x (\ln (x) + c_1)} \tag{1}$$
$$y = -\sqrt{x (\ln (x) + c_1)} \tag{2}$$

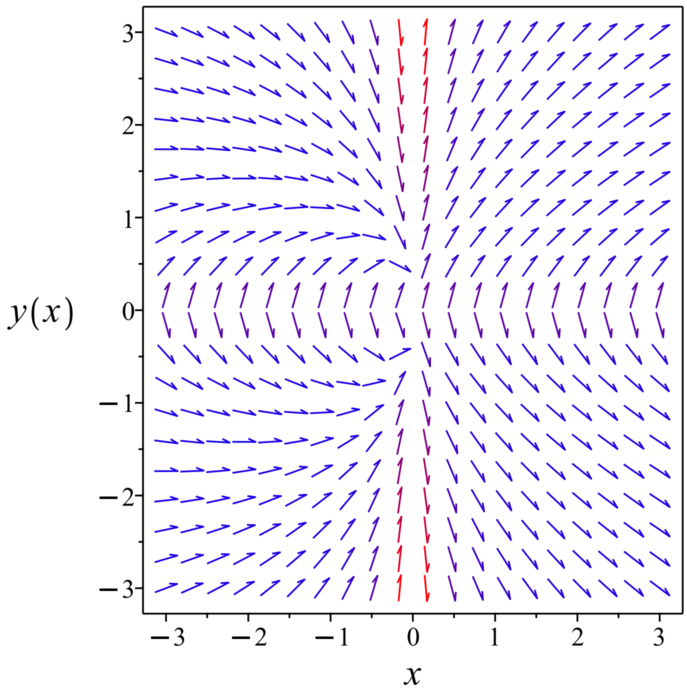


Figure 312: Slope field plot

Verification of solutions

$$y = \sqrt{x (\ln (x) + c_1)}$$

Verified OK.

$$y = -\sqrt{x (\ln (x) + c_1)}$$

Verified OK.

7.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y&=\left(-y^2-x\right)\mathrm{d}x\\ \left(y^2+x\right)\mathrm{d}x+(-2xy)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2+x\\ N(x,y)&=-2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2 + x) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy) \\ &= -2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{2xy}((2y) - (-2y)) \\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(y^2 + x) \\ &= \frac{y^2 + x}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^2+x}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y^2+x}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{y^2}{x} + \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{x} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{x} + \ln(x)$$

Summary

The solution(s) found are the following

$$-\frac{y^2}{x} + \ln(x) = c_1 \tag{1}$$

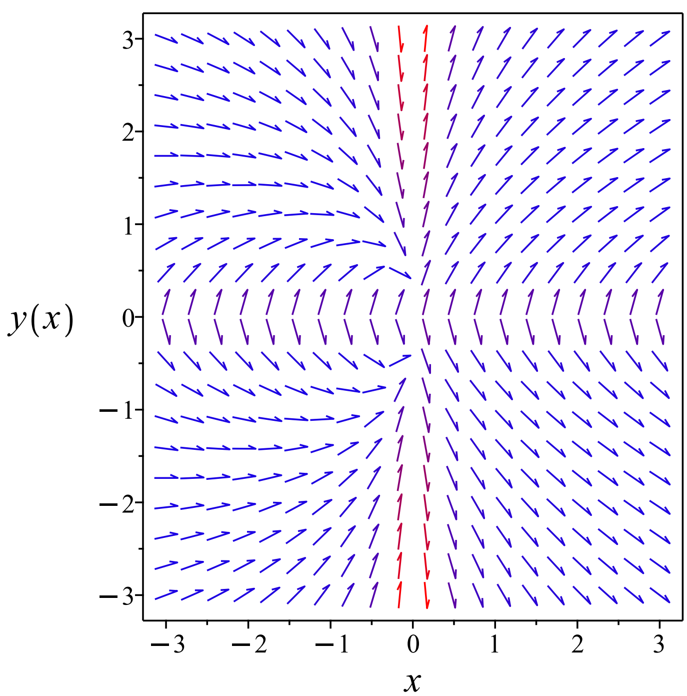


Figure 313: Slope field plot

Verification of solutions

$$-\frac{y^2}{x} + \ln(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(( x+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(c_1 + \ln(x))\, x}$$
$$y(x) = -\sqrt{(c_1 + \ln(x))\, x}$$

✓ Solution by Mathematica

Time used: 0.178 (sec). Leaf size: 40

```
DSolve[( x+y[x]^2)-2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{\log(x) + c_1}$$
$$y(x) \to \sqrt{x}\sqrt{\log(x) + c_1}$$

7.17 problem 192

Internal problem ID [15080]
Internal file name [`OUTPUT/15080_Sunday_April_21_2024_01_29_17_PM_89720108/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 192.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2x^2y + 2y + (2x^3 + 2x)\, y' = -5}$$

7.17.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{5}{2x\,(x^2+1)}$$

Hence the ode is

$$y' + \frac{y}{x} = -\frac{5}{2x\,(x^2+1)}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{5}{2x\left(x^{2}+1\right)}\right) \\
\frac{d}{dx}(xy) &= (x)\left(-\frac{5}{2x\left(x^{2}+1\right)}\right) \\
d(xy) &= \left(-\frac{5}{2x^{2}+2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int -\frac{5}{2x^{2}+2}\, dx \\
xy &= -\frac{5\arctan\left(x\right)}{2}+c_{1}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = -\frac{5\arctan\left(x\right)}{2x}+\frac{c_{1}}{x}
$$

Summary
The solution(s) found are the following

$$
y = -\frac{5\arctan\left(x\right)}{2x}+\frac{c_{1}}{x} \tag{1}
$$

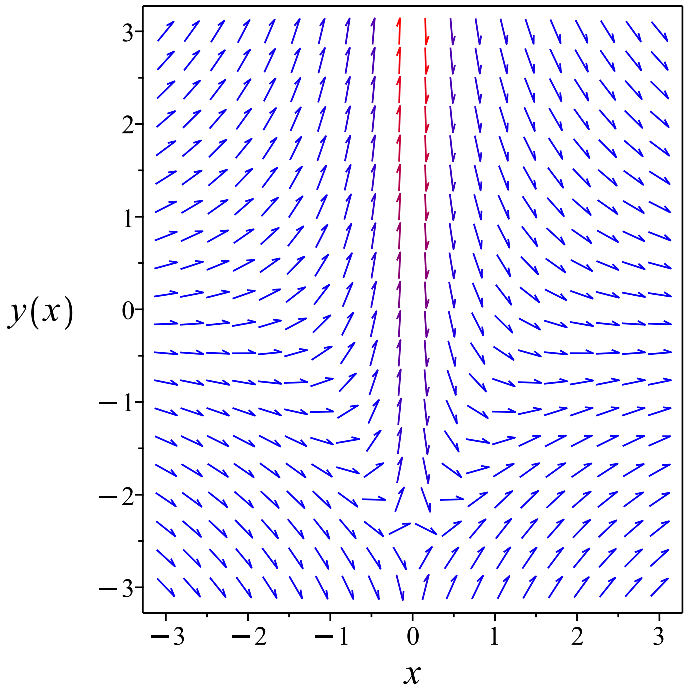


Figure 314: Slope field plot

Verification of solutions

$$y = -\frac{5 \arctan(x)}{2x} + \frac{c_1}{x}$$

Verified OK.

7.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2x^2y + 2y + 5}{2x(x^2 + 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 245: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy\end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x^2 y + 2y + 5}{2x\,(x^2+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{5}{2x^2 + 2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{5}{2R^2 + 2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{5 \arctan (R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = -\frac{5 \arctan (x)}{2} + c_1$$

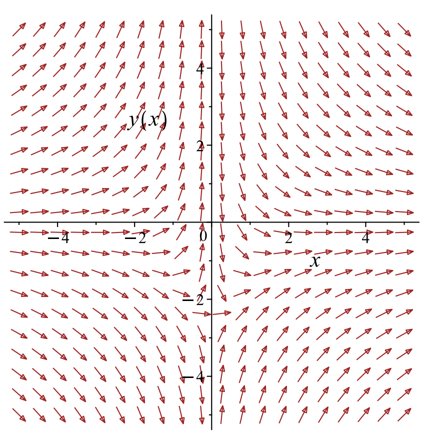
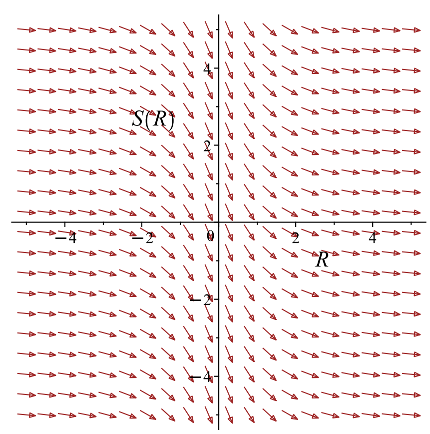
Which simplifies to

$$yx = -\frac{5 \arctan (x)}{2} + c_1$$

Which gives

$$y = -\frac{5 \arctan (x) - 2c_1}{2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x^2y+2y+5}{2x(x^2+1)}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = -\frac{5}{2R^2+2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{5\arctan(x) - 2c_1}{2x} \tag{1}$$

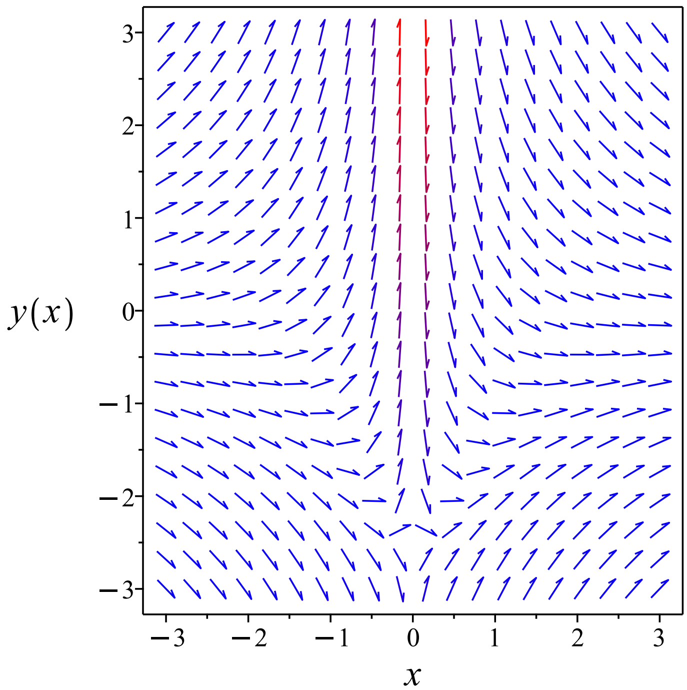


Figure 315: Slope field plot

Verification of solutions

$$y = -\frac{5\arctan(x) - 2c_1}{2x}$$

Verified OK.

7.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x^3 + 2x)\,\mathrm{d}y &= (-2x^2y - 2y - 5)\,\mathrm{d}x \\ (2x^2y + 2y + 5)\,\mathrm{d}x + (2x^3 + 2x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2x^2y + 2y + 5 \\ N(x, y) &= 2x^3 + 2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x^2y + 2y + 5) \\ &= 2x^2 + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x^3+2x)\\ &= 6x^2+2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x^3+2x}\left(\left(2x^2+2\right)-\left(6x^2+2\right)\right)\\ &= -\frac{2x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{2x}{x^2+1} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln\left(x^2+1\right)}\\ &= \frac{1}{x^2+1}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2+1}\left(2x^2y+2y+5\right)\\ &= \frac{2x^2y+2y+5}{x^2+1}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2+1}\left(2x^3+2x\right)\\ &= 2x\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{2x^2y + 2y + 5}{x^2 + 1}\right) + (2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{2x^2y + 2y + 5}{x^2 + 1}\,\mathrm{d}x$$
$$\phi = 2xy + 5\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = 2x$. Therefore equation (4) becomes

$$2x = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2xy + 5\arctan(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2xy + 5\arctan(x)$$

Summary

The solution(s) found are the following

$$2yx + 5\arctan(x) = c_1 \tag{1}$$

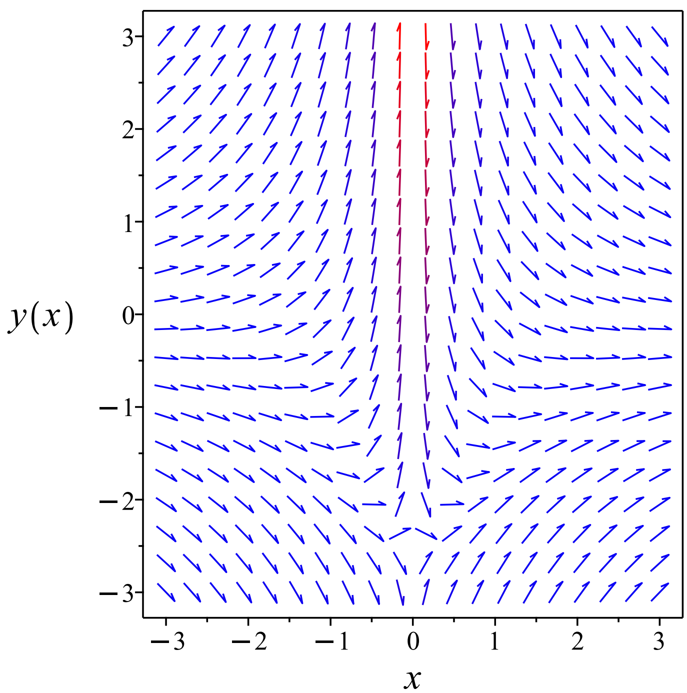


Figure 316: Slope field plot

Verification of solutions

$$2yx + 5\arctan(x) = c_1$$

Verified OK.

7.17.4 Maple step by step solution

Let's solve

$2x^2y + 2y + (2x^3 + 2x)\, y' = -5$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} - \frac{5}{2x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = -\frac{5}{2x(x^2+1)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = -\frac{5\mu(x)}{2x(x^2+1)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{5\mu(x)}{2x(x^2+1)} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{5\mu(x)}{2x(x^2+1)} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{5\mu(x)}{2x\left(x^2+1\right)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int -\frac{5}{2\left(x^2+1\right)} dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{5\arctan(x)}{2} + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(( 2*x^2*y(x)+2*y(x)+5)+(2*x^3+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\frac{5\arctan(x)}{2} + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 21

```
DSolve[( 2*x^2*y[x]+2*y[x]+5)+(2*x^3+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-5\arctan(x) + 2c_1}{2x}$$

7.18 problem 193

Internal problem ID [15081]
Internal file name [`OUTPUT/15081_Sunday_April_21_2024_01_29_18_PM_42802337/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 193.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{-2y^3x+3y'y^2x^2=-x^4\ln(x)}$$

7.18.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^3\ln(x)+2y^3}{3y^2x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 248: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0\\
\eta(x,y) &= \frac{x^2}{y^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x^3 \ln(x) + 2y^3}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y^3}{3x^3} \\ S_y &= \frac{y^2}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\ln(x)}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{\ln(R)}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R \ln(R)}{3} + \frac{R}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x^2} = -\frac{x \ln(x)}{3} + \frac{x}{3} + c_1$$

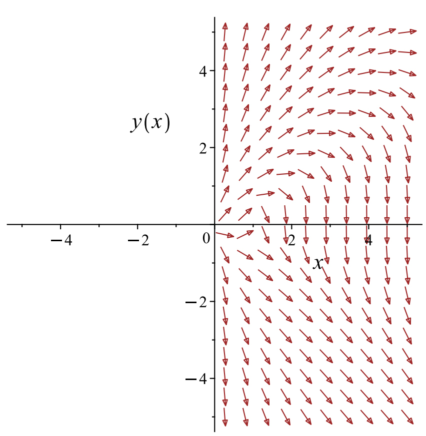
Which simplifies to

$$\frac{y^3}{3x^2} = -\frac{x \ln(x)}{3} + \frac{x}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{-x^3 \ln(x)+2y^3}{3y^2 x}$



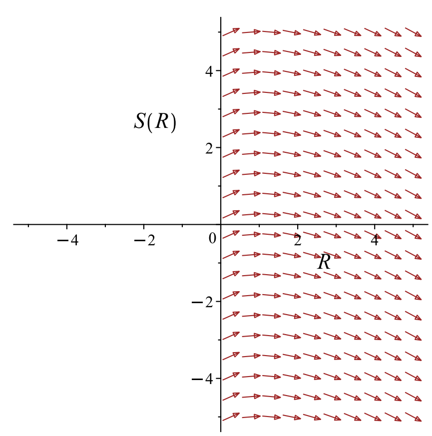
</td><td>

$R = x$

$S = \frac{y^3}{3x^2}$

</td><td>

$\frac{dS}{dR} = -\frac{\ln(R)}{3}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{y^3}{3x^2} = -\frac{x \ln(x)}{3} + \frac{x}{3} + c_1 \tag{1}$$

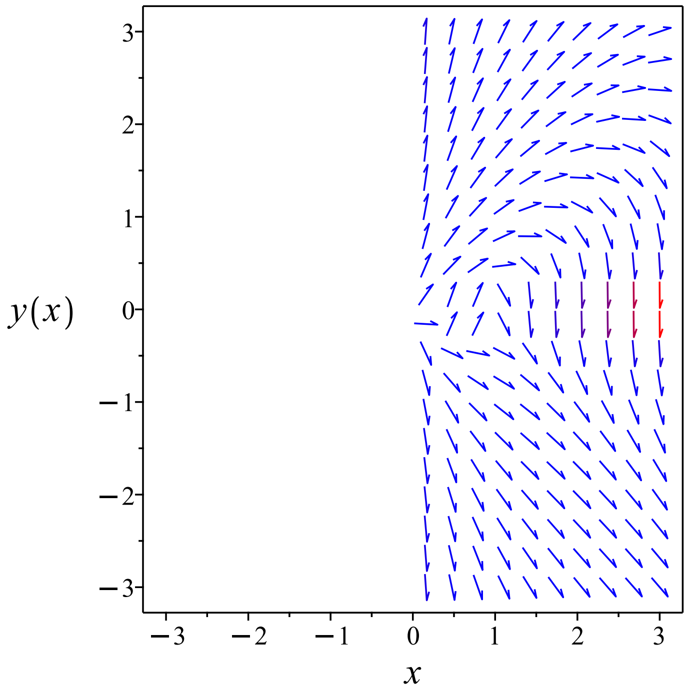


Figure 317: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^2} = -\frac{x\ln(x)}{3} + \frac{x}{3} + c_1$$

Verified OK.

7.18.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x^3\ln(x) + 2y^3}{3y^2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{3x}y - \frac{x^2\ln(x)}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{3x} \\ f_1(x) &= -\frac{x^2 \ln(x)}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{2y^3}{3x} - \frac{x^2 \ln(x)}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{2w(x)}{3x} - \frac{x^2 \ln(x)}{3} \\ w' &= \frac{2w}{x} - x^2 \ln(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -x^2 \ln(x) \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -x^2 \ln(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-x^2 \ln(x)\right) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-x^2 \ln(x)\right) \\ d\left(\frac{w}{x^2}\right) &= (-\ln(x))\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -\ln(x)\ dx \\ \frac{w}{x^2} &= -x\ln(x) + x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = x^2(-x\ln(x) + x) + c_1 x^2$$

which simplifies to

$$w(x) = -x^2(x\ln(x) - c_1 - x)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = -x^2(x\ln(x) - c_1 - x)$$

Solving for y gives

$$\begin{aligned}y(x) &= \left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}} \\ y(x) &= -\frac{\left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2} \\ y(x) &= \frac{\left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}} \tag{1}$$

$$y = -\frac{\left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2} \tag{2}$$

$$y = \frac{\left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2} \tag{3}$$

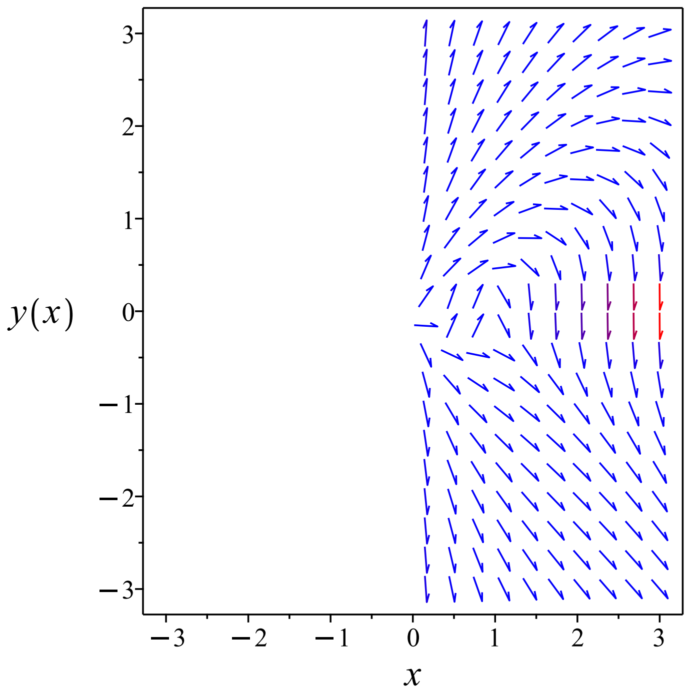


Figure 318: Slope field plot

Verification of solutions

$$y = \left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = -\frac{\left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Verified OK.

$$y = \frac{\left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

Verified OK.

7.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3y^2x^2)\,\mathrm{d}y &= \left(-x^4 \ln(x) + 2x\,y^3\right)\mathrm{d}x \\ \left(x^4 \ln(x) - 2x\,y^3\right)\mathrm{d}x + (3y^2x^2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^4 \ln(x) - 2x\,y^3 \\ N(x,y) &= 3y^2x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^4 \ln(x) - 2x\,y^3\right) \\ &= -6y^2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2x^2) \\ &= 6y^2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{3x^2y^2}\left((-6y^2x) - (6y^2x)\right) \\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(x^4 \ln(x) - 2x\,y^3\right) \\ &= \frac{x^3 \ln(x) - 2y^3}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(3y^2x^2\right) \\ &= \frac{3y^2}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x^3 \ln(x) - 2y^3}{x^3}\right) + \left(\frac{3y^2}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x^3 \ln(x) - 2y^3}{x^3} \mathrm{d}x$$

$$\phi = x \ln(x) - x + \frac{y^3}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2}{x^2}$. Therefore equation (4) becomes

$$\frac{3y^2}{x^2} = \frac{3y^2}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x \ln(x) - x + \frac{y^3}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x \ln(x) - x + \frac{y^3}{x^2}$$

Summary

The solution(s) found are the following

$$x \ln(x) - x + \frac{y^3}{x^2} = c_1 \tag{1}$$

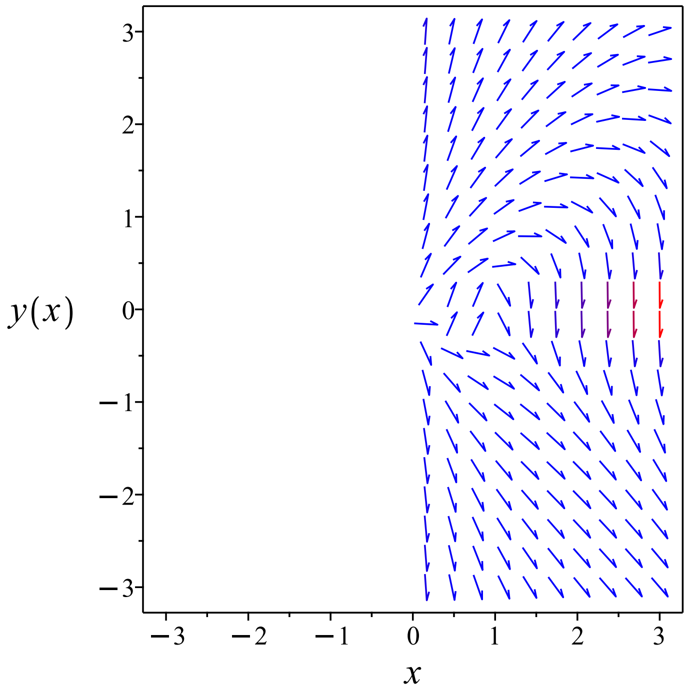


Figure 319: Slope field plot

Verification of solutions

$$x\ln(x) - x + \frac{y^3}{x^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 84

```
dsolve(( x^4*ln(x)-2*x*y(x)^3)+(3*x^2*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(-x^2(x\ln(x) - c_1 - x)\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{(-x^2(x\ln(x) - c_1 - x))^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{(-x^2(x\ln(x) - c_1 - x))^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.485 (sec). Leaf size: 77

```
DSolve[( x^4*Log[x]-2*x*y[x]^3)+(3*x^2*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \sqrt[3]{x^2(x + x(-\log(x)) + c_1)}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{x^2(x + x(-\log(x)) + c_1)}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{x^2(x + x(-\log(x)) + c_1)}$$

7.19 problem 194

Internal problem ID [15082]
Internal file name [`OUTPUT/15082_Sunday_April_21_2024_01_29_19_PM_71309339/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 194.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\cos(y)\,y' + \sin(y) = -\sin(x) - x}$$

7.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y))\,\mathrm{d}y &= (-x - \sin(x) - \sin(y))\,\mathrm{d}x\\ (x + \sin(x) + \sin(y))\,\mathrm{d}x + (\cos(y))\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + \sin(x) + \sin(y)\\ N(x, y) &= \cos(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + \sin(x) + \sin(y))\\ &= \cos(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y))\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \sec(y)\,((\cos(y)) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}(x + \sin(x) + \sin(y)) \\ &= (x + \sin(x) + \sin(y))\, \mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(\cos(y)) \\ &= \mathrm{e}^{x} \cos(y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((x + \sin(x) + \sin(y))\, \mathrm{e}^{x}) + (\mathrm{e}^{x} \cos(y))\, \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int (x + \sin(x) + \sin(y))\, \mathrm{e}^{x} \, \mathrm{d}x\end{aligned}$$

$$\phi = \frac{\mathrm{e}^{x}(-2 + 2x - \cos(x) + \sin(x) + 2\sin(y))}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{x} \cos(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{x} \cos(y)$. Therefore equation (4) becomes

$$e^{x} \cos(y) = e^{x} \cos(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{e^{x}(-2 + 2x - \cos(x) + \sin(x) + 2\sin(y))}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{x}(-2 + 2x - \cos(x) + \sin(x) + 2\sin(y))}{2}$$

Summary

The solution(s) found are the following

$$\frac{e^{x}(-2 + 2x - \cos(x) + \sin(x) + 2\sin(y))}{2} = c_1 \tag{1}$$

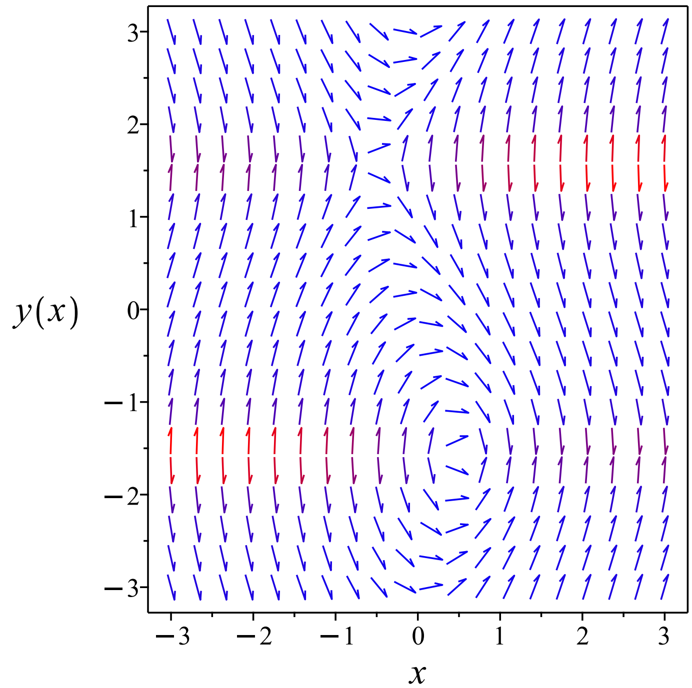


Figure 320: Slope field plot

Verification of solutions

$$\frac{e^x(-2 + 2x - \cos(x) + \sin(x) + 2\sin(y))}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(( x+sin(x)+sin(y(x)))+( cos(y(x)) )*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(x + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} - 1 + c_1 e^{-x}\right)$$

✓ Solution by Mathematica

Time used: 33.179 (sec). Leaf size: 61

```
DSolve[( x+Sin[x]+Sin[y[x]])+( Cos[y[x]] )*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{1}{2}(-2x - \sin(x) + \cos(x) + 2c_1 e^{-x} + 2)\right)$$
$$y(x) \to -\arcsin\left(\frac{1}{2}(2x + \sin(x) - \cos(x) - 2c_1 e^{-x} - 2)\right)$$

7.20 problem 195

Internal problem ID [15083]
Internal file name [`OUTPUT/15083_Sunday_April_21_2024_01_29_23_PM_18233425/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 195.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{2y^2x - 3y^3 + (7 - 3y^2x)\, y' = 0}$$

7.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-3y^2x+7\right)\mathrm{d}y&=\left(-2y^2x+3y^3\right)\mathrm{d}x\\ \left(2y^2x-3y^3\right)\mathrm{d}x+\left(-3y^2x+7\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2y^2x-3y^3\\ N(x,y)&=-3y^2x+7\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(2y^2x-3y^3\right)\\ &=y(4x-9y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(-3y^2x+7\right)\\ &=-3y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{-3y^2x+7}\left(\left(4xy-9y^2\right)-\left(-3y^2\right)\right)\\ &=\frac{-4xy+6y^2}{3y^2x-7}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y^2 (2x-3y)}\left(\left(-3y^2\right) - \left(4xy - 9y^2\right)\right) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int -\frac{2}{y} \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}\left(2y^2x - 3y^3\right) \\ &= 2x - 3y \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2}\left(-3y^2x + 7\right) \\ &= \frac{-3y^2x + 7}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ (2x-3y) + \left(\frac{-3y^2x + 7}{y^2}\right)\frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int 2x - 3y\,\mathrm{d}x$$
$$\phi = x(x-3y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-3y^2x+7}{y^2}$. Therefore equation (4) becomes

$$\frac{-3y^2x+7}{y^2} = -3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{7}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{7}{y^2}\right)\mathrm{d}y$$
$$f(y) = -\frac{7}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x-3y) - \frac{7}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x - 3y) - \frac{7}{y}$$

Summary

The solution(s) found are the following

$$x(x - 3y) - \frac{7}{y} = c_1 \tag{1}$$

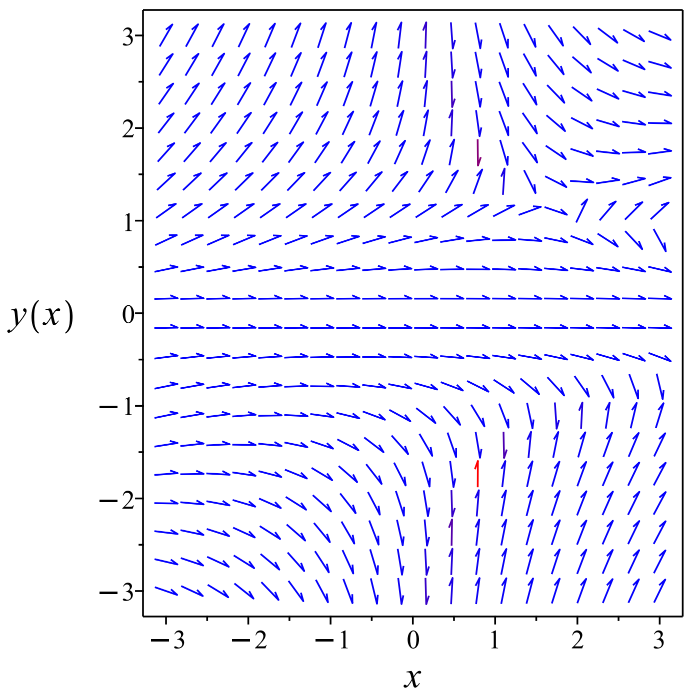


Figure 321: Slope field plot

Verification of solutions

$$x(x - 3y) - \frac{7}{y} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 65

```
dsolve(( 2*x*y(x)^2-3*y(x)^3)+( 7-3*x*y(x)^2 )*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2 + c_1 + \sqrt{x^4 + 2c_1x^2 + c_1^2 - 84x}}{6x}$$
$$y(x) = \frac{x^2 - \sqrt{x^4 + 2c_1x^2 + c_1^2 - 84x} + c_1}{6x}$$

✓ Solution by Mathematica

Time used: 0.406 (sec). Leaf size: 86

```
DSolve[( 2*x*y[x]^2-3*y[x]^3)+( 7-3*x*y[x]^2 )*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{x^2 - \sqrt{x^4 + 2c_1x^2 - 84x + c_1{}^2} + c_1}{6x}$$
$$y(x) \to \frac{x^2 + \sqrt{x^4 + 2c_1x^2 - 84x + c_1{}^2} + c_1}{6x}$$
$$y(x) \to 0$$

7.21 problem 196

Internal problem ID [15084]
Internal file name [OUTPUT/15084_Sunday_April_21_2024_01_29_24_PM_99646652/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 196.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{3y^2 + \left(2y^3 - 6yx\right) y' = x}$$

7.21.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{3y^2 - x}{2y\left(y^2 - 3x\right)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2-\frac{(3y^2-x)(b_3-a_2)}{2y(y^2-3x)}-\frac{(3y^2-x)^2 a_3}{4y^2(y^2-3x)^2} \\ & -\left(\frac{1}{2y(y^2-3x)}-\frac{3(3y^2-x)}{2y(y^2-3x)^2}\right)(xa_2+ya_3+a_1) \\ & -\left(-\frac{3}{y^2-3x}+\frac{3y^2-x}{2y^2(y^2-3x)}+\frac{3y^2-x}{(y^2-3x)^2}\right)(xb_2+yb_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4y^6b_2+30x\,y^4b_2-6y^5a_2+12y^5b_3-24x^2y^2b_2+4x\,y^3a_2-8x\,y^3b_3-7y^4a_3+6y^4b_1+6x^3b_2-6x^2ya_2}{4y^2(-y^2+3x)^2}=0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4y^6b_2-30x\,y^4b_2+6y^5a_2-12y^5b_3+24x^2y^2b_2-4x\,y^3a_2+8x\,y^3b_3+7y^4a_3-6y^4b_1 \\ & -6x^3b_2+6x^2ya_2-12x^2yb_3+6x\,y^2a_3-12x\,y^2b_1+16y^3a_1-x^2a_3-6x^2b_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 4b_2v_2^6+6a_2v_2^5-30b_2v_1v_2^4-12b_3v_2^5-4a_2v_1v_2^3+7a_3v_2^4-6b_1v_2^4+24b_2v_1^2v_2^2 \\ & +8b_3v_1v_2^3+16a_1v_2^3+6a_2v_1^2v_2+6a_3v_1v_2^2-12b_1v_1v_2^2-6b_2v_1^3-12b_3v_1^2v_2-a_3v_1^2 \\ & -6b_1v_1^2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&-6b_2v_1^3 + 24b_2v_1^2v_2^2 + (6a_2 - 12b_3)\, v_1^2v_2 + (-a_3 - 6b_1)\, v_1^2 \\ &- 30b_2v_1v_2^4 + (-4a_2 + 8b_3)\, v_1v_2^3 + (6a_3 - 12b_1)\, v_1v_2^2 \\ &+ 4b_2v_2^6 + (6a_2 - 12b_3)\, v_2^5 + (7a_3 - 6b_1)\, v_2^4 + 16a_1v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 16a_1 &= 0 \\ -30b_2 &= 0 \\ -6b_2 &= 0 \\ 4b_2 &= 0 \\ 24b_2 &= 0 \\ -4a_2 + 8b_3 &= 0 \\ 6a_2 - 12b_3 &= 0 \\ -a_3 - 6b_1 &= 0 \\ 6a_3 - 12b_1 &= 0 \\ 7a_3 - 6b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{3y^2 - x}{2y\,(y^2 - 3x)}\right)(2x) \\
&= \frac{-y^4 + x^2}{-y^3 + 3xy} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^4+x^2}{-y^3+3xy}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(y^2 + x\right) - \frac{\ln\left(y^2 - x\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y^2 - x}{2y\,(y^2 - 3x)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-3y^2 + x}{-2y^4 + 2x^2} \\
S_y &= \frac{-y^3 + 3xy}{-y^4 + x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

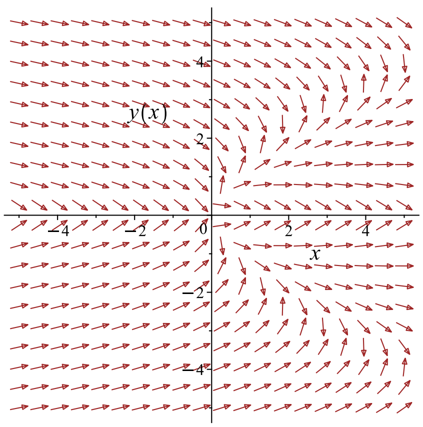
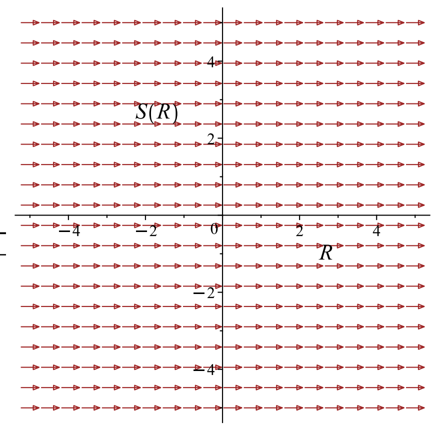
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln\left(x + y^2\right) - \frac{\ln\left(-x + y^2\right)}{2} = c_1
$$

Which simplifies to

$$
\ln\left(x + y^2\right) - \frac{\ln\left(-x + y^2\right)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y^2-x}{2y(y^2-3x)}$ 	$R = x$ $S = \ln(y^2 + x) - \frac{\ln(y^2 -}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(x + y^2) - \frac{\ln(-x + y^2)}{2} = c_1 \tag{1}$$

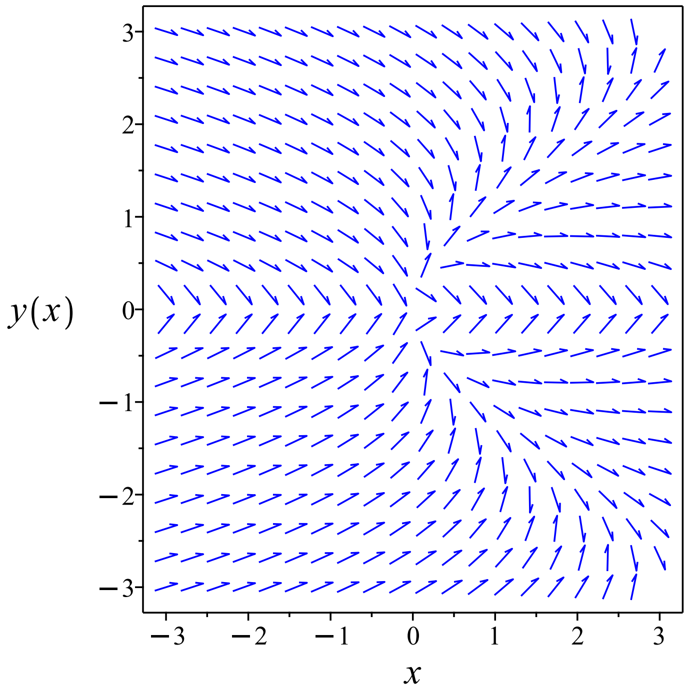


Figure 322: Slope field plot

Verification of solutions

$$\ln \left(x+y^{2}\right)-\frac{\ln \left(-x+y^{2}\right)}{2}=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 101

```
dsolve(( 3*y(x)^2-x)+( 2*y(x)^3-6*x*y(x) )*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2\sqrt{c_1\left(c_1-8x\right)}+2c_1-4x}}{2}$$
$$y(x) = \frac{\sqrt{-2\sqrt{c_1\left(c_1-8x\right)}+2c_1-4x}}{2}$$
$$y(x) = -\frac{\sqrt{2\sqrt{c_1\left(c_1-8x\right)}+2c_1-4x}}{2}$$
$$y(x) = \frac{\sqrt{2\sqrt{c_1\left(c_1-8x\right)}+2c_1-4x}}{2}$$

✓ Solution by Mathematica

Time used: 11.553 (sec). Leaf size: 185

```
DSolve[( 3*y[x]^2-x)+( 2*y[x]^3-6*x*y[x] )*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-2x-e^{\frac{c_1}{2}}\sqrt{8x+e^{c_1}}-e^{c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-2x-e^{\frac{c_1}{2}}\sqrt{8x+e^{c_1}}-e^{c_1}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-2x+e^{\frac{c_1}{2}}\sqrt{8x+e^{c_1}}-e^{c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-2x+e^{\frac{c_1}{2}}\sqrt{8x+e^{c_1}}-e^{c_1}}}{\sqrt{2}}$$

7.22 problem 197

Internal problem ID [15085]
Internal file name [`OUTPUT/15085_Sunday_April_21_2024_01_29_28_PM_34099516/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 197.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x^2 - 1}$$

7.22.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2 + 1}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 250: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + y^2 + 1}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2 + 1}{2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2 + 1}{2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} - \frac{1}{2R} + c_1 \tag{4}$$

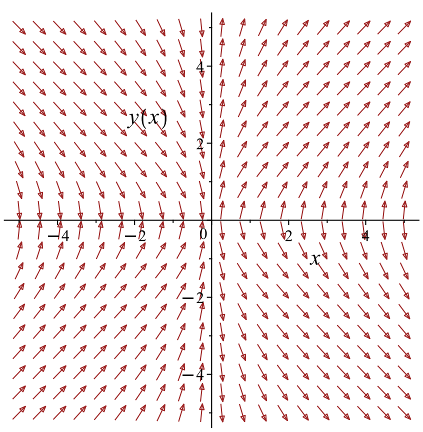
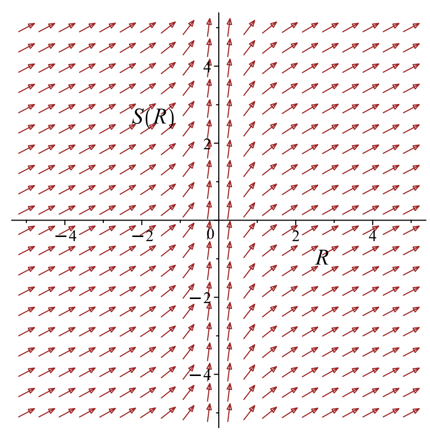
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x}{2} - \frac{1}{2x} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x}{2} - \frac{1}{2x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{R^2+1}{2R^2}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} = \frac{x}{2} - \frac{1}{2x} + c_1 \tag{1}$$

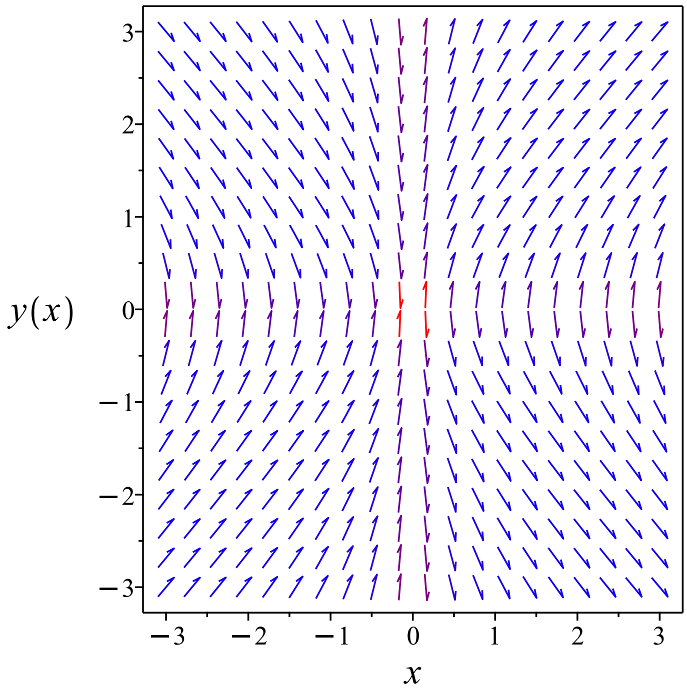


Figure 323: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = \frac{x}{2} - \frac{1}{2x} + c_1$$

Verified OK.

7.22.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2 + 1}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x^2 + 1}{2x}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= \frac{x^2+1}{2x} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{2x} + \frac{x^2+1}{2x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x^2+1}{2x} \\
w' &= \frac{w}{x} + \frac{x^2+1}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{x^2+1}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{x^2+1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{x^2+1}{x}\right)\\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{x^2+1}{x}\right)\\ d\left(\frac{w}{x}\right) &= \left(\frac{x^2+1}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \frac{x^2+1}{x^2}\, dx\\ \frac{w}{x} &= x - \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(x - \frac{1}{x}\right) + c_1 x$$

which simplifies to

$$w(x) = c_1 x + x^2 - 1$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x + x^2 - 1$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{c_1 x + x^2 - 1}\\ y(x) &= -\sqrt{c_1 x + x^2 - 1}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 x + x^2 - 1} \tag{1}$$

$$y = -\sqrt{c_1 x + x^2 - 1} \tag{2}$$

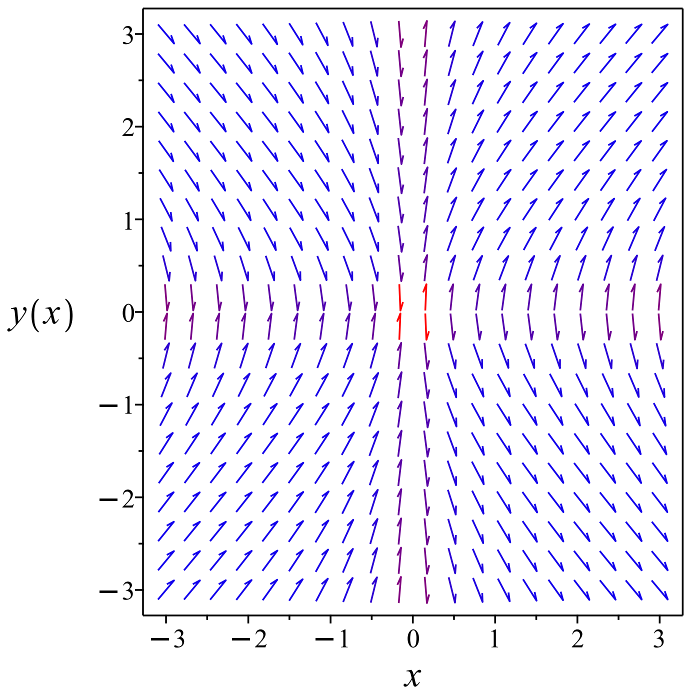


Figure 324: Slope field plot

Verification of solutions

$$y = \sqrt{c_1 x + x^2 - 1}$$

Verified OK.

$$y = -\sqrt{c_1 x + x^2 - 1}$$

Verified OK.

7.22.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y&=\left(-x^2-y^2-1\right)\mathrm{d}x\\ \left(x^2+y^2+1\right)\mathrm{d}x+(-2xy)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+y^2+1\\ N(x,y)&=-2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2 + 1\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy)\\ &= -2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{2xy}((2y) - (-2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(x^2 + y^2 + 1\right)\\ &= \frac{x^2 + y^2 + 1}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2+1}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2+1}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2-y^2-1}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 - y^2 - 1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 - y^2 - 1}{x}$$

Summary

The solution(s) found are the following

$$\frac{-y^2 + x^2 - 1}{x} = c_1 \tag{1}$$

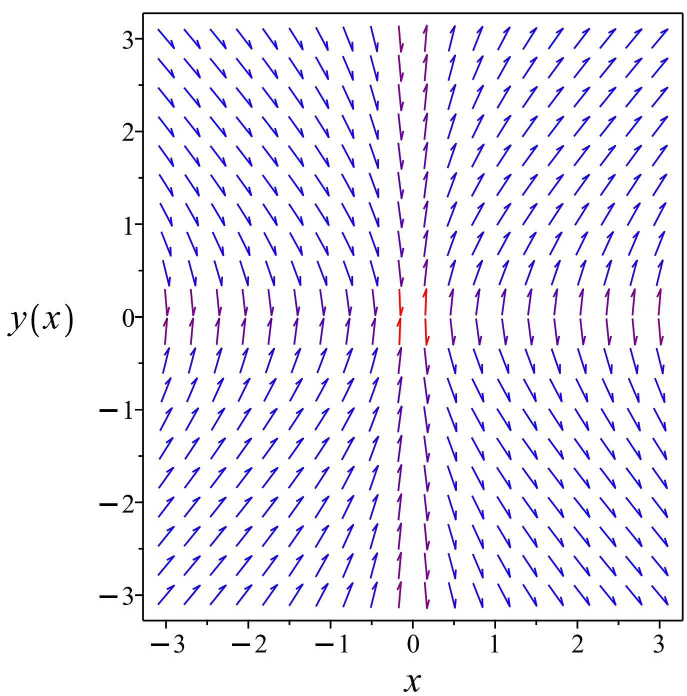


Figure 325: Slope field plot

Verification of solutions

$$\frac{-y^2+x^2-1}{x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(( x^2+y(x)^2+1)-( 2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\sqrt{c_1x+x^2-1}$$
$$y(x)=-\sqrt{c_1x+x^2-1}$$

✓ Solution by Mathematica

Time used: 0.26 (sec). Leaf size: 37

```
DSolve[( x^2+y[x]^2+1)-( 2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\sqrt{x^2+c_1x-1}$$
$$y(x)\to \sqrt{x^2+c_1x-1}$$

7.23 problem 198

Internal problem ID [15086]
Internal file name [`OUTPUT/15086_Sunday_April_21_2024_01_29_29_PM_9442229/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 7, Total differential equations. The integrating factor. Exercises page 61
Problem number: 198.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [
   _Abel, `2nd type`, `class A`]]
```

$$\boxed{-yx + (x^2 + y)\, y' = -x}$$

7.23.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{(y-1)\, x}{x^2 + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + xya_5 + y^2 a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^2 b_4 + xyb_5 + y^2 b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 + \frac{(y-1)\,x(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{x^2+y} \\
&- \frac{(y-1)^2\,x^2(xa_5 + 2ya_6 + a_3)}{(x^2+y)^2} \\
&- \left(\frac{y-1}{x^2+y} - \frac{2(y-1)\,x^2}{(x^2+y)^2}\right)(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1) \\
&- \left(\frac{x}{x^2+y} - \frac{(y-1)\,x}{(x^2+y)^2}\right)(x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^5b_4 - x^4ya_4 + x^4yb_5 - x^3y^2a_5 + x^3y^2b_6 - x^2y^3a_6 + x^4a_4 - x^4b_5 + 2x^3ya_5 + 4x^3yb_4 - 2x^3yb_6 - 3x^2y^2a_4 +}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&x^5b_4 - x^4ya_4 + x^4yb_5 - x^3y^2a_5 + x^3y^2b_6 - x^2y^3a_6 + x^4a_4 - x^4b_5 + 2x^3ya_5 \\
&+4x^3yb_4 - 2x^3yb_6 - 3x^2y^2a_4 + 3x^2y^2a_6 + 3x^2y^2b_5 - 2x\,y^3a_5 + 2x\,y^3b_6 - y^4a_6 \\
&- x^3a_5 - x^3b_1 - x^3b_3 - x^3b_4 + x^2ya_1 + x^2ya_3 + 3x^2ya_4 - 2x^2ya_6 + 2x^2yb_2 \\
&- 2x^2yb_5 - 2x\,y^2a_2 + 2x\,y^2a_5 + x\,y^2b_3 + 2x\,y^2b_4 - 3x\,y^2b_6 - y^3a_3 + y^3a_6 \\
&+y^3b_5 - x^2a_1 - x^2a_3 - x^2b_2 + 2xya_2 - 2xyb_3 - y^2a_1 + y^2a_3 + y^2b_2 - xb_1 + ya_1 \\
&= 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_4 v_1^4 v_2 - a_5 v_1^3 v_2^2 - a_6 v_1^2 v_2^3 + b_4 v_1^5 + b_5 v_1^4 v_2 + b_6 v_1^3 v_2^2 + a_4 v_1^4 - 3a_4 v_1^2 v_2^2 \\
&+ 2a_5 v_1^3 v_2 - 2a_5 v_1 v_2^3 + 3a_6 v_1^2 v_2^2 - a_6 v_2^4 + 4b_4 v_1^3 v_2 - b_5 v_1^4 + 3b_5 v_1^2 v_2^2 \\
&- 2b_6 v_1^3 v_2 + 2b_6 v_1 v_2^3 + a_1 v_1^2 v_2 - 2a_2 v_1 v_2^2 + a_3 v_1^2 v_2 - a_3 v_2^3 + 3a_4 v_1^2 v_2 \\
&- a_5 v_1^3 + 2a_5 v_1 v_2^2 - 2a_6 v_1^2 v_2 + a_6 v_2^3 - b_1 v_1^3 + 2b_2 v_1^2 v_2 - b_3 v_1^3 \\
&+ b_3 v_1 v_2^2 - b_4 v_1^3 + 2b_4 v_1 v_2^2 - 2b_5 v_1^2 v_2 + b_5 v_2^3 - 3b_6 v_1 v_2^2 - a_1 v_1^2 - a_1 v_2^2 \\
&+ 2a_2 v_1 v_2 - a_3 v_1^2 + a_3 v_2^2 - b_2 v_1^2 + b_2 v_2^2 - 2b_3 v_1 v_2 + a_1 v_2 - b_1 v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&b_4 v_1^5 + (-a_4 + b_5) v_1^4 v_2 + (a_4 - b_5) v_1^4 + (-a_5 + b_6) v_1^3 v_2^2 \\
&+ (2a_5 + 4b_4 - 2b_6) v_1^3 v_2 + (-a_5 - b_1 - b_3 - b_4) v_1^3 - a_6 v_1^2 v_2^3 \\
&+ (-3a_4 + 3a_6 + 3b_5) v_1^2 v_2^2 + (a_1 + a_3 + 3a_4 - 2a_6 + 2b_2 - 2b_5) v_1^2 v_2 \\
&+ (-a_1 - a_3 - b_2) v_1^2 + (-2a_5 + 2b_6) v_1 v_2^3 \\
&+ (-2a_2 + 2a_5 + b_3 + 2b_4 - 3b_6) v_1 v_2^2 + (2a_2 - 2b_3) v_1 v_2 - b_1 v_1 \\
&- a_6 v_2^4 + (-a_3 + a_6 + b_5) v_2^3 + (-a_1 + a_3 + b_2) v_2^2 + a_1 v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_4 &= 0 \\
-a_6 &= 0 \\
-b_1 &= 0 \\
2a_2 - 2b_3 &= 0 \\
-a_4 + b_5 &= 0 \\
a_4 - b_5 &= 0 \\
-2a_5 + 2b_6 &= 0 \\
-a_5 + b_6 &= 0 \\
-a_1 - a_3 - b_2 &= 0 \\
-a_1 + a_3 + b_2 &= 0 \\
-a_3 + a_6 + b_5 &= 0 \\
-3a_4 + 3a_6 + 3b_5 &= 0 \\
2a_5 + 4b_4 - 2b_6 &= 0 \\
-a_5 - b_1 - b_3 - b_4 &= 0 \\
-2a_2 + 2a_5 + b_3 + 2b_4 - 3b_6 &= 0 \\
a_1 + a_3 + 3a_4 - 2a_6 + 2b_2 - 2b_5 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_6 \\
a_3 &= b_5 \\
a_4 &= b_5 \\
a_5 &= b_6 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= -b_5 \\
b_3 &= -b_6 \\
b_4 &= 0 \\
b_5 &= b_5 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = xy - x$$
$$\eta = y^2 - y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y^2 - y - \left(\frac{(y-1)\,x}{x^2+y}\right)(xy - x) \\
&= \frac{x^2y + y^3 - x^2 - y^2}{x^2+y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2y+y^3-x^2-y^2}{x^2+y}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(x^2+y^2)}{2} + \ln(y-1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{(y-1)\,x}{x^2+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{x^2+y^2} \\
S_y &= -\frac{y}{x^2+y^2} + \frac{1}{y-1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

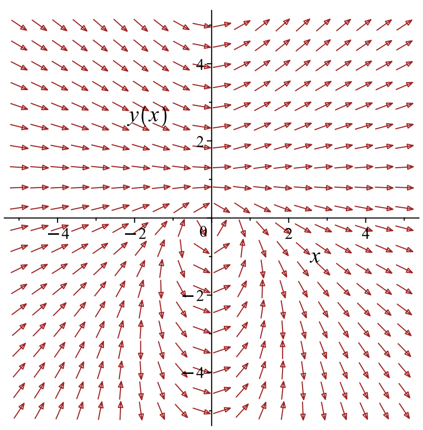
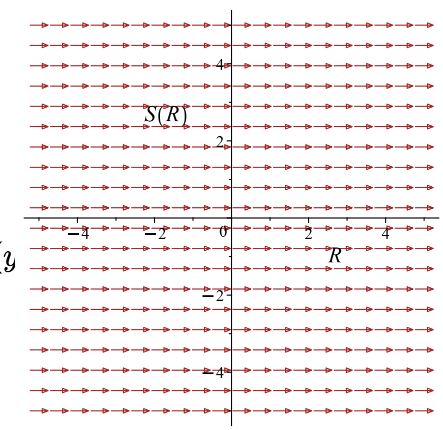
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln\left(x^2+y^2\right)}{2} + \ln\left(y-1\right) = c_1$$

Which simplifies to

$$-\frac{\ln\left(x^2+y^2\right)}{2} + \ln\left(y-1\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(y-1)x}{x^2+y}$ 	$R = x$ $S = -\frac{\ln(x^2+y^2)}{2} + \ln(y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+y^2)}{2} + \ln(y-1) = c_1 \tag{1}$$

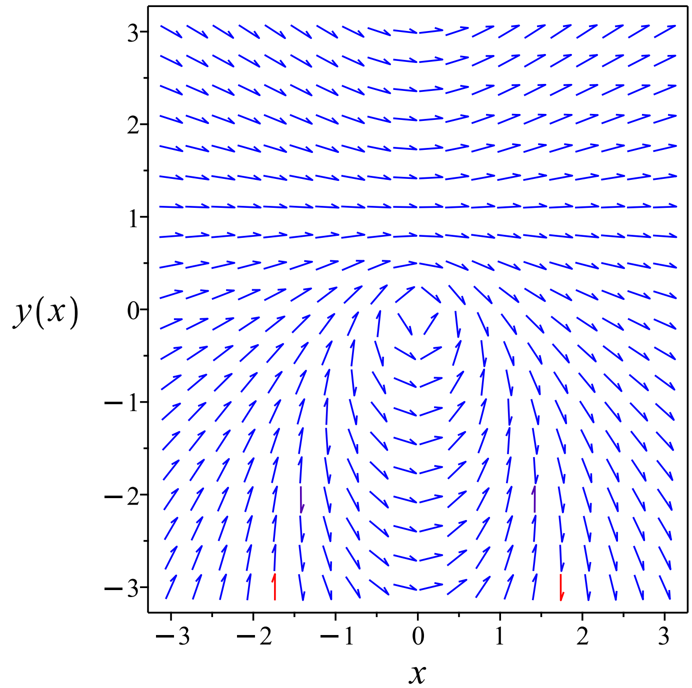


Figure 326: Slope field plot

Verification of solutions

$$-\frac{\ln(x^2+y^2)}{2}+\ln(y-1)=c_1$$

Verified OK.

7.23.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y)\,\mathrm{d}y &= (xy - x)\,\mathrm{d}x \\ (-xy + x)\,\mathrm{d}x + (x^2 + y)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -xy + x \\ N(x, y) &= x^2 + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy + x) \\ &= -x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2+y}((-x) - (2x)) \\ &= -\frac{3x}{x^2+y} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{(y-1)\,x}((2x) - (-x)) \\ &= -\frac{3}{y-1} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{3}{y-1} \,\mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(y-1)} \\ &= \frac{1}{(y-1)^3} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{(y-1)^3}(-xy + x) \\ &= -\frac{x}{(y-1)^2} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(y-1)^3}\left(x^2+y\right) \\
&= \frac{x^2+y}{(y-1)^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{x}{(y-1)^2}\right) + \left(\frac{x^2+y}{(y-1)^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{(y-1)^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2\,(y-1)^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^2}{(y-1)^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+y}{(y-1)^3}$. Therefore equation (4) becomes

$$
\frac{x^2+y}{(y-1)^3} = \frac{x^2}{(y-1)^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{(y-1)^3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{(y-1)^3}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{y-1} - \frac{1}{2(y-1)^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2(y-1)^2} - \frac{1}{y-1} - \frac{1}{2(y-1)^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2(y-1)^2} - \frac{1}{y-1} - \frac{1}{2(y-1)^2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2(y-1)^2} - \frac{1}{y-1} - \frac{1}{2(y-1)^2} = c_1 \tag{1}$$

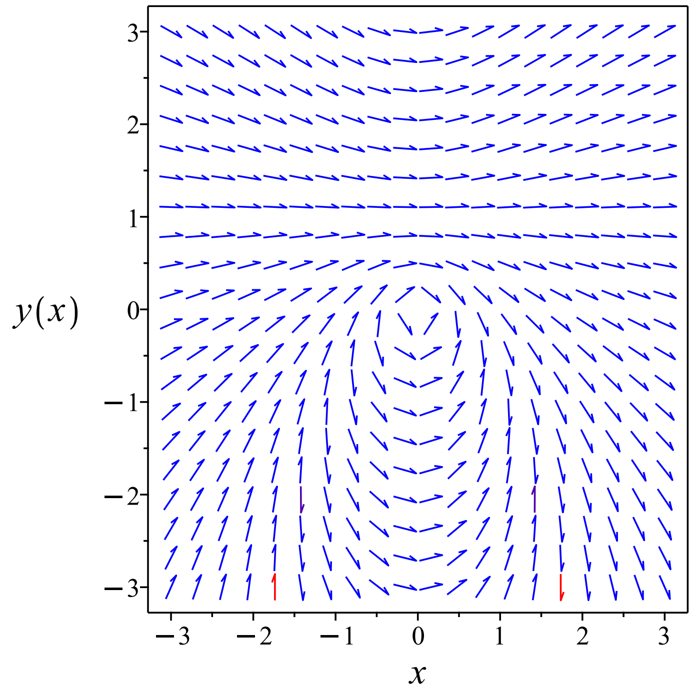


Figure 327: Slope field plot

Verification of solutions

$$-\frac{x^2}{2\left(y-1\right)^2}-\frac{1}{y-1}-\frac{1}{2\left(y-1\right)^2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(( x -x*y(x) )+( y(x)+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2c_1 + 1 - \sqrt{2c_1x^2 + 2c_1 + 1}}{2c_1}$$
$$y(x) = \frac{2c_1 + 1 + \sqrt{2c_1x^2 + 2c_1 + 1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 4.513 (sec). Leaf size: 295

```
DSolve[( x -x*y[x] )+( y[x]+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + \frac{1}{\frac{1}{x^2+1} - \frac{1+i}{(x^2+1)\sqrt{-2(x^2+1)\cosh\left(\frac{2c_1}{9}\right) - 2(x^2+1)\sinh\left(\frac{2c_1}{9}\right) + 2i}}}$$
$$y(x) \to -x^2 + \frac{1}{\frac{1}{x^2+1} + \frac{1+i}{(x^2+1)\sqrt{-2(x^2+1)\cosh\left(\frac{2c_1}{9}\right) - 2(x^2+1)\sinh\left(\frac{2c_1}{9}\right) + 2i}}}$$
$$y(x) \to -x^2 + \frac{1}{\frac{1}{x^2+1} - \frac{1+i}{\sqrt{2}(x^2+1)\sqrt{(x^2+1)\cosh\left(\frac{2c_1}{9}\right) + (x^2+1)\sinh\left(\frac{2c_1}{9}\right) + i}}}$$
$$y(x) \to -x^2 + \frac{1}{\frac{1}{x^2+1} + \frac{1+i}{\sqrt{2}(x^2+1)\sqrt{(x^2+1)\cosh\left(\frac{2c_1}{9}\right) + (x^2+1)\sinh\left(\frac{2c_1}{9}\right) + i}}}$$
$$y(x) \to 1$$
$$y(x) \to \frac{1}{2}(1 - x^2)$$

8 Section 8. First order not solved for the derivative. Exercises page 67

8.1 problem 199

Internal problem ID [15087]
Internal file name [OUTPUT/15087_Sunday_April_21_2024_01_29_31_PM_59751946/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 199.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{4y'^2 = 9x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{3\sqrt{x}}{2} \tag{1}$$
$$y' = -\frac{3\sqrt{x}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{3\sqrt{x}}{2}\, \mathrm{d}x \\ &= x^{\frac{3}{2}} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^{\frac{3}{2}} + c_1 \tag{1}$$

Verification of solutions

$$y = x^{\frac{3}{2}} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{3\sqrt{x}}{2}\, \mathrm{d}x \\ &= -x^{\frac{3}{2}} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x^{\frac{3}{2}} + c_2 \tag{1}$$

Verification of solutions

$$y = -x^{\frac{3}{2}} + c_2$$

Verified OK.

8.1.1 Maple step by step solution

Let's solve

$4y'^2 = 9x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int 4y'^2 dx = \int 9x dx + c_1$

- Cannot compute integral

  $\int 4y'^2 dx = \frac{9x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(4*diff(y(x),x)^2-9*x=0,y(x), singsol=all)
```

$$y(x) = -x^{\frac{3}{2}} + c_1$$
$$y(x) = x^{\frac{3}{2}} + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 27

```
DSolve[4*y'[x]^2-9*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^{3/2} + c_1$$
$$y(x) \to x^{3/2} + c_1$$

8.2 problem 200

Internal problem ID [15088]
Internal file name [`OUTPUT/15088_Sunday_April_21_2024_01_29_32_PM_25756279/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 200.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'^2 - 2yy' - y^2(e^{2x} - 1) = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = (e^x + 1)\, y \tag{1}$$
$$y' = (1 - e^x)\, y \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= (e^x + 1)\, y \end{aligned}$$

Where $f(x) = e^x + 1$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= e^x + 1\, dx \\
\int \frac{1}{y}\, dy &= \int e^x + 1\, dx \\
\ln(y) &= x + e^x + c_1 \\
y &= e^{x + e^x + c_1} \\
&= c_1 e^{x + e^x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x + e^x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x + e^x}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x) g(y) \\
&= (1 - e^x)\, y
\end{aligned}$$

Where $f(x) = 1 - e^x$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= 1 - e^x\, dx \\
\int \frac{1}{y}\, dy &= \int 1 - e^x\, dx \\
\ln(y) &= x - e^x + c_2 \\
y &= e^{x - e^x + c_2} \\
&= c_2 e^{x - e^x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{x - e^x} \tag{1}$$

Verification of solutions

$$y = c_2 \mathrm{e}^{x-\mathrm{e}^x}$$

Verified OK.

8.2.1 Maple step by step solution

Let's solve

$y'^2 - 2yy' - y^2(\mathrm{e}^{2x} - 1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \mathrm{e}^x + 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int (\mathrm{e}^x + 1) \, dx + c_1$

- Evaluate integral

  $\ln(y) = x + \mathrm{e}^x + c_1$

- Solve for y

  $y = \mathrm{e}^{x+\mathrm{e}^x+c_1}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)^2-2*y(x)*diff(y(x),x)=y(x)^2*(exp(2*x)-1),y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = c_1 \mathrm{e}^{x-\mathrm{e}^x}$$
$$y(x) = c_1 \mathrm{e}^{x+\mathrm{e}^x}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 36

```
DSolve[y'[x]^2-2*y[x]*y'[x]==y[x]^2*(Exp[2*x]-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x-e^x}$$
$$y(x) \to c_1 e^{x+e^x}$$
$$y(x) \to 0$$

8.3 problem 201

Internal problem ID [15089]
Internal file name [`OUTPUT/15089_Sunday_April_21_2024_01_29_33_PM_83060937/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 201.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'^2 - 2y'x = 8x^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -2x \tag{1}$$
$$y' = 4x \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int -2x \, \mathrm{d}x \\ &= -x^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -x^2 + c_1 \tag{1}$$

Verification of solutions

$$y = -x^2 + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int 4x \, \mathrm{d}x \\ &= 2x^2 + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = 2x^2 + c_2 \tag{1}$$

Verification of solutions

$$y = 2x^2 + c_2$$

Verified OK.

8.3.1 Maple step by step solution

Let's solve

$y'^2 - 2y'x = 8x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(y'^2 - 2y'x\right) dx = \int 8x^2 dx + c_1$

- Cannot compute integral

  $\int \left(y'^2 - 2y'x\right) dx = \frac{8x^3}{3} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)^2-2*x*diff(y(x),x)-8*x^2=0,y(x), singsol=all)
```

$$y(x) = 2x^2 + c_1$$
$$y(x) = -x^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[y'[x]^2-2*x*y'[x]-8*x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + c_1$$
$$y(x) \to 2x^2 + c_1$$

8.4 problem 202

Internal problem ID [15090]
Internal file name [`OUTPUT/15090_Sunday_April_21_2024_01_29_33_PM_59533299/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 202.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2{y'}^2 + 3xyy' + 2y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{y}{x} \tag{1}$$

$$y' = -\frac{2y}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{1}{x}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{1}{x}\, dx \\
\ln(y) &= -\ln(x) + c_1 \\
y &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x}
$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Where $f(x) = -\frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{2}{x}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{2}{x}\, dx \\
\ln(y) &= -2\ln(x) + c_2 \\
y &= e^{-2\ln(x)+c_2} \\
&= \frac{c_2}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x^2} \tag{1}
$$

Verification of solutions

$$y = \frac{c_2}{x^2}$$

Verified OK.

8.4.1 Maple step by step solution

Let's solve

$x^2 y'^2 + 3xyy' + 2y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)^2+3*x*y(x)*diff(y(x),x)+2*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x}$$
$$y(x) = \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 26

```
DSolve[x^2*y'[x]^2+3*x*y[x]*y'[x]+2*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2}$$
$$y(x) \to \frac{c_1}{x}$$
$$y(x) \to 0$$

8.5 problem 203

Internal problem ID [15091]
Internal file name [OUTPUT/15091_Sunday_April_21_2024_01_29_34_PM_85331927/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 203.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{{y'}^2 - (2x+y)\,y' + yx = -x^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = x \tag{1}$$
$$y' = y + x \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int x \, \mathrm{d}x \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{x^2}{2} + c_1$$

Verified OK.

Solving equation (2)

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = x$$

Hence the ode is

$$y' - y = x$$

The integrating factor μ is

$$\mu = e^{\int(-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x)$$
$$\frac{d}{dx}(e^{-x}y) = (e^{-x})(x)$$
$$d(e^{-x}y) = (x e^{-x})\, dx$$

Integrating gives

$$e^{-x}y = \int x e^{-x}\, dx$$
$$e^{-x}y = -(x+1)e^{-x} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}(x+1)e^{-x} + c_2 e^{x}$$

which simplifies to

$$y = -x - 1 + c_2 e^{x}$$

Summary

The solution(s) found are the following

$$y = -x - 1 + c_2 e^x \tag{1}$$

Verification of solutions

$$y = -x - 1 + c_2 e^x$$

Verified OK.

8.5.1 Maple step by step solution

Let's solve

$y'^2 - (2x + y) y' + yx = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(y'^2 - (2x + y) y' + yx\right) dx = \int -x^2 dx + c_1$

- Cannot compute integral

  $\int \left(y'^2 - (2x + y) y' + yx\right) dx = -\frac{x^3}{3} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)^2-(2*x+y(x))*diff(y(x),x)+x^2+x*y(x)=0,y(x), singsol=all)
```

$$
y(x) = \frac{x^2}{2} + c_1
$$
$$
y(x) = -x - 1 + \mathrm{e}^x c_1
$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 30

```
DSolve[y'[x]^2-(2*x+y[x])*y'[x]+x^2+x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{x^2}{2} + c_1
$$
$$
y(x) \to -x + c_1 e^x - 1
$$

8.6 problem 204

Internal problem ID [15092]
Internal file name [`OUTPUT/15092_Sunday_April_21_2024_01_29_35_PM_19635907/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 204.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{{y'}^{3}+(x+2)\,\mathrm{e}^{y}=0}$$

8.6.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}}=f(x)g(y) \tag{1}$$

Where $n=3, m=1, f=-x-2, g=\mathrm{e}^{y}$. Hence the ode is

$$(y')^{3}=\mathrm{e}^{y}(-x-2)$$

Solving for y' from (1) gives

$$\begin{aligned}
y' &= (fg)^{\frac{1}{3}} \\
y' &= -\frac{(fg)^{\frac{1}{3}}}{2}+\frac{i\sqrt{3}\,(fg)^{\frac{1}{3}}}{2} \\
y' &= -\frac{(fg)^{\frac{1}{3}}}{2}-\frac{i\sqrt{3}\,(fg)^{\frac{1}{3}}}{2}
\end{aligned}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$
\begin{aligned}
-x - 2 &> 0 \\
e^y &> 0
\end{aligned}
$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= f^{\frac{1}{3}} g^{\frac{1}{3}} \\
y' &= \frac{f^{\frac{1}{3}} g^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{2} \\
y' &= -\frac{f^{\frac{1}{3}} g^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{g^{\frac{1}{3}}}\, dy &= \left(f^{\frac{1}{3}}\right)\, dx \\
\frac{2}{g^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}\, dy &= \left(f^{\frac{1}{3}}\right)\, dx \\
-\frac{2}{g^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}\, dy &= \left(f^{\frac{1}{3}}\right)\, dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\left(e^y\right)^{\frac{1}{3}}}\, dy &= \left((-x-2)^{\frac{1}{3}}\right)\, dx \\
\frac{2}{\left(e^y\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}\, dy &= \left((-x-2)^{\frac{1}{3}}\right)\, dx \\
-\frac{2}{\left(e^y\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}\, dy &= \left((-x-2)^{\frac{1}{3}}\right)\, dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\left(e^y\right)^{\frac{1}{3}}} dy &= \int (-x-2)^{\frac{1}{3}}\, dx + c_1 \\
\int \frac{2}{\left(e^y\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)} dy &= \int (-x-2)^{\frac{1}{3}}\, dx + c_1 \\
\int -\frac{2}{\left(e^y\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)} dy &= \int (-x-2)^{\frac{1}{3}}\, dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
-\frac{3}{\left(e^{y}\right)^{\frac{1}{3}}} &= -\frac{3(-x-2)^{\frac{4}{3}}}{4}+c_1 \\
-\frac{6}{\left(e^{y}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)} &= -\frac{3(-x-2)^{\frac{4}{3}}}{4}+c_1 \\
\frac{6}{\left(e^{y}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)} &= -\frac{3(-x-2)^{\frac{4}{3}}}{4}+c_1
\end{aligned}
$$

Therefore

$$
y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^{3}}\right)
$$

$$
y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^{3}}\right)
$$

$$
y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^{3}}\right)
$$

Summary

The solution(s) found are the following

$$
y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^{3}}\right) \tag{1}
$$

$$
y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^{3}}\right) \tag{2}
$$

$$
y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^{3}}\right) \tag{3}
$$

Verification of solutions

$$y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^3}\right)$$

Verified OK. `{0 < -x-2, 0 < exp(y)}`

$$y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^3}\right)$$

Verified OK. `{0 < -x-2, 0 < exp(y)}`

$$y = \ln\left(\frac{1728}{\left(3\left(-x-2\right)^{\frac{4}{3}}-4c_1\right)^3}\right)$$

Verified OK. `{0 < -x-2, 0 < exp(y)}`

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   <- exact successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   <- exact successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   <- exact successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 87

```
dsolve(diff(y(x),x)^3+(x+2)*exp(y(x))=0,y(x), singsol=all)
```

$$y(x) = 3\ln(12) - 3\ln\left((6+3x)(2+x)^{\frac{1}{3}} + 4c_1\right)$$
$$y(x) = 3\ln(24) - 3\ln\left(-3\left(1+i\sqrt{3}\right)(2+x)^{\frac{4}{3}} + 8c_1\right)$$
$$y(x) = 3\ln(24) - 3\ln\left(3\left(i\sqrt{3}-1\right)(2+x)^{\frac{4}{3}} + 8c_1\right)$$

✓ Solution by Mathematica

Time used: 6.699 (sec). Leaf size: 126

```
DSolve[y'[x]^3+(x+2)*Exp[y[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3\log\left(\frac{1}{12}\left(3\sqrt[3]{x+2}x + 6\sqrt[3]{x+2} - 4c_1\right)\right)$$
$$y(x) \to -3\log\left(\frac{1}{12}\left(-3\sqrt[3]{-1}\sqrt[3]{x+2}x - 6\sqrt[3]{-1}\sqrt[3]{x+2} - 4c_1\right)\right)$$
$$y(x) \to -3\log\left(\frac{1}{12}\left(3(-1)^{2/3}\sqrt[3]{x+2}x + 6(-1)^{2/3}\sqrt[3]{x+2} - 4c_1\right)\right)$$

8.7 problem 205

Internal problem ID [15093]
Internal file name [`OUTPUT/15093_Sunday_April_21_2024_01_29_36_PM_40189821/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 205.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^3 - yy'^2 + x^2y' - x^2y = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = ix \tag{1}$$
$$y' = -ix \tag{2}$$
$$y' = y \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int ix \, \mathrm{d}x \\ &= \frac{ix^2}{2} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{ix^2}{2} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{i x^2}{2} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -ix \, \mathrm{d}x \\ &= -\frac{i x^2}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{i x^2}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{i x^2}{2} + c_2$$

Verified OK.

Solving equation (3)

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{y} dy &= c_3 + x \\ \ln(y) &= c_3 + x \\ y &= \mathrm{e}^{c_3 + x} \\ y &= c_3 \mathrm{e}^{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3 \mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = c_3 \mathrm{e}^{x}$$

Verified OK.

8.7.1 Maple step by step solution

Let's solve

$y'^3 - yy'^2 + x^2y' - x^2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int 1dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = e^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)^3=y(x)*diff(y(x),x)^2-x^2*diff(y(x),x)+x^2*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{ix^2}{2} + c_1$$
$$y(x) = \frac{ix^2}{2} + c_1$$
$$y(x) = \mathrm{e}^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 43

```
DSolve[y'[x]^3==y[x]*y'[x]^2-x^2*y'[x]+x^2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to c_1 - \frac{ix^2}{2}$$
$$y(x) \to \frac{ix^2}{2} + c_1$$

8.8 problem 206

Internal problem ID [15094]
Internal file name [`OUTPUT/15094_Sunday_April_21_2024_01_29_37_PM_38332231/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 206.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{{y'}^2 - yy' = -\mathrm{e}^x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2} \tag{1}$$

$$y' = \frac{y}{2} - \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(\frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right)(b_3 - a_2) - \left(\frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right)^2 a_3 + \frac{\mathrm{e}^x (xa_2 + ya_3 + a_1)}{\sqrt{y^2 - 4\,\mathrm{e}^x}} - \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2 - 4\,\mathrm{e}^x}}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{-(y^2 - 4\,\mathrm{e}^x)^{\frac{3}{2}} a_3 - \sqrt{y^2 - 4\,\mathrm{e}^x}\, y^2 a_3 - 2y^3 a_3 + 4\,\mathrm{e}^x x a_2 + 12\,\mathrm{e}^x y a_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, x b_2 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, y a_2 - 2}{4\sqrt{y^2 - 4\,\mathrm{e}^x}}$$

$$= 0$$

Setting the numerator to zero gives

$$-(y^2 - 4\,\mathrm{e}^x)^{\frac{3}{2}} a_3 - \sqrt{y^2 - 4\,\mathrm{e}^x}\, y^2 a_3 - 2y^3 a_3 + 4\,\mathrm{e}^x x a_2 + 12\,\mathrm{e}^x y a_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, x b_2 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, y a_2 - 2xyb_2 - 2y^2 a_2 + 4\,\mathrm{e}^x a_1 + 8\,\mathrm{e}^x a_2 - 8\,\mathrm{e}^x b_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, b_1 + 4b_2\sqrt{y^2 - 4\,\mathrm{e}^x} - 2yb_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(y^2 - 4\,\mathrm{e}^x)^{\frac{3}{2}} a_3 - 2(y^2 - 4\,\mathrm{e}^x)\, y a_3 - \sqrt{y^2 - 4\,\mathrm{e}^x}\, y^2 a_3 + 4\,\mathrm{e}^x x a_2 + 4\,\mathrm{e}^x y a_3 - 2(y^2 - 4\,\mathrm{e}^x)\, a_2 + 2(y^2 - 4\,\mathrm{e}^x)\, b_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, x b_2 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, y a_2 - 2xyb_2 - 2y^2 b_3 + 4\,\mathrm{e}^x a_1 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, b_1 + 4b_2\sqrt{y^2 - 4\,\mathrm{e}^x} - 2yb_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
-2\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3 - 2y^3a_3 + 4\,\mathrm{e}^x x a_2 + 4\,\mathrm{e}^x\sqrt{y^2-4\,\mathrm{e}^x}\,a_3 + 12\,\mathrm{e}^x y a_3 \\
- 2\sqrt{y^2-4\,\mathrm{e}^x}\,x b_2 - 2xyb_2 - 2\sqrt{y^2-4\,\mathrm{e}^x}\,y a_2 - 2y^2a_2 + 4\,\mathrm{e}^x a_1 \\
+ 8\,\mathrm{e}^x a_2 - 8\,\mathrm{e}^x b_3 - 2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1 + 4b_2\sqrt{y^2-4\,\mathrm{e}^x} - 2yb_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y^2-4\,\mathrm{e}^x}, \mathrm{e}^x\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{y^2-4\,\mathrm{e}^x} = v_3, \mathrm{e}^x = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-2v_2^3a_3 - 2v_3v_2^2a_3 + 4v_4v_1a_2 - 2v_2^2a_2 - 2v_3v_2a_2 + 12v_4v_2a_3 + 4v_4v_3a_3 \\
- 2v_1v_2b_2 - 2v_3v_1b_2 + 4v_4a_1 + 8v_4a_2 - 2v_2b_1 - 2v_3b_1 + 4b_2v_3 - 8v_4b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
-2v_1v_2b_2 - 2v_3v_1b_2 + 4v_4v_1a_2 - 2v_2^3a_3 - 2v_3v_2^2a_3 - 2v_2^2a_2 - 2v_3v_2a_2 \\
+ 12v_4v_2a_3 - 2v_2b_1 + 4v_4v_3a_3 + (-2b_1 + 4b_2)\,v_3 + (4a_1 + 8a_2 - 8b_3)\,v_4 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_2 &= 0 \\
4a_2 &= 0 \\
-2a_3 &= 0 \\
4a_3 &= 0 \\
12a_3 &= 0 \\
-2b_1 &= 0 \\
-2b_2 &= 0 \\
-2b_1 + 4b_2 &= 0 \\
4a_1 + 8a_2 - 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 2b_3\\ a_2 &= 0\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 2\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= y - \left(\frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right)(2)\\ &= -\sqrt{y^2 - 4\,\mathrm{e}^x}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{-\sqrt{y^2 - 4\,\mathrm{e}^x}} dy\end{aligned}$$

Which results in

$$S = -\ln\left(y + \sqrt{y^2 - 4\,e^x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{2} + \frac{\sqrt{y^2 - 4\,e^x}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2\,e^x}{\sqrt{y^2 - 4\,e^x}\,\left(y + \sqrt{y^2 - 4\,e^x}\right)} \\
S_y &= -\frac{1}{\sqrt{y^2 - 4\,e^x}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y\sqrt{y^2 - 4\,e^x} + y^2 - 4\,e^x}{\sqrt{y^2 - 4\,e^x}\,\left(y + \sqrt{y^2 - 4\,e^x}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{y^2 - 4\,e^x}\right) = c_1$$

Which simplifies to

$$-\ln\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right) = c_1$$

Which gives

$$y = \frac{\left(4\,\mathrm{e}^{x} \mathrm{e}^{2 c_1}+1\right) \mathrm{e}^{-c_1}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4\,\mathrm{e}^{x} \mathrm{e}^{2 c_1}+1\right) \mathrm{e}^{-c_1}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4\,\mathrm{e}^{x} \mathrm{e}^{2 c_1}+1\right) \mathrm{e}^{-c_1}}{2}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{y}{2}-\frac{\sqrt{y^{2}-4\,\mathrm{e}^{x}}}{2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\left(\frac{y}{2}-\frac{\sqrt{y^2-4\,\mathrm{e}^x}}{2}\right)(b_3-a_2)-\left(\frac{y}{2}-\frac{\sqrt{y^2-4\,\mathrm{e}^x}}{2}\right)^2 a_3 \\
&-\frac{\mathrm{e}^x(xa_2+ya_3+a_1)}{\sqrt{y^2-4\,\mathrm{e}^x}}-\left(\frac{1}{2}-\frac{y}{2\sqrt{y^2-4\,\mathrm{e}^x}}\right)(xb_2+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(y^2-4\,\mathrm{e}^x)^{\frac{3}{2}}\,a_3+\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3-2y^3a_3+4\,\mathrm{e}^x xa_2+12\,\mathrm{e}^x ya_3+2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2+2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2-2}{4\sqrt{y^2-4\,\mathrm{e}^x}}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-(y^2-4\,\mathrm{e}^x)^{\frac{3}{2}}\,a_3-\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3+2y^3a_3-4\,\mathrm{e}^x xa_2-12\,\mathrm{e}^x ya_3 \\
&-2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2+2xyb_2+2y^2a_2-4\,\mathrm{e}^x a_1 \\
&-8\,\mathrm{e}^x a_2+8\,\mathrm{e}^x b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}+2yb_1=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-(y^2-4\,\mathrm{e}^x)^{\frac{3}{2}}\,a_3+2(y^2-4\,\mathrm{e}^x)\,ya_3-\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3-4\,\mathrm{e}^x xa_2-4\,\mathrm{e}^x ya_3 \\
&+2(y^2-4\,\mathrm{e}^x)\,a_2-2(y^2-4\,\mathrm{e}^x)\,b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2 \\
&+2xyb_2+2y^2b_3-4\,\mathrm{e}^x a_1-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}+2yb_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3+2y^3a_3-4\,\mathrm{e}^x xa_2+4\,\mathrm{e}^x\sqrt{y^2-4\,\mathrm{e}^x}\,a_3-12\,\mathrm{e}^x ya_3 \\
&-2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2+2xyb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2+2y^2a_2-4\,\mathrm{e}^x a_1 \\
&-8\,\mathrm{e}^x a_2+8\,\mathrm{e}^x b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}+2yb_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{y^2-4\,\mathrm{e}^x},\mathrm{e}^x\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^2 - 4\,\mathrm{e}^x} = v_3, \mathrm{e}^x = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &2v_2^3a_3 - 2v_3v_2^2a_3 - 4v_4v_1a_2 + 2v_2^2a_2 - 2v_3v_2a_2 - 12v_4v_2a_3 + 4v_4v_3a_3 \\ &+ 2v_1v_2b_2 - 2v_3v_1b_2 - 4v_4a_1 - 8v_4a_2 + 2v_2b_1 - 2v_3b_1 + 4b_2v_3 + 8v_4b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &2v_1v_2b_2 - 2v_3v_1b_2 - 4v_4v_1a_2 + 2v_2^3a_3 - 2v_3v_2^2a_3 + 2v_2^2a_2 - 2v_3v_2a_2 - 12v_4v_2a_3 \\ &+ 2v_2b_1 + 4v_4v_3a_3 + (-2b_1 + 4b_2)\,v_3 + (-4a_1 - 8a_2 + 8b_3)\,v_4 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_2 &= 0 \\ -2a_2 &= 0 \\ 2a_2 &= 0 \\ -12a_3 &= 0 \\ -2a_3 &= 0 \\ 2a_3 &= 0 \\ 4a_3 &= 0 \\ 2b_1 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ -2b_1 + 4b_2 &= 0 \\ -4a_1 - 8a_2 + 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 2b_3 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2 \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y}{2} - \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right)(2) \\ &= \sqrt{y^2 - 4\,\mathrm{e}^x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{y^2 - 4\,\mathrm{e}^x}} dy \end{aligned}$$

Which results in

$$S = \ln\left(y + \sqrt{y^2 - 4\,e^x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{2} - \frac{\sqrt{y^2 - 4\,e^x}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2\,e^x}{\sqrt{y^2 - 4\,e^x}\,\left(y + \sqrt{y^2 - 4\,e^x}\right)} \\
S_y &= \frac{1}{\sqrt{y^2 - 4\,e^x}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(y + \sqrt{y^2 - 4\,e^x}\right) = c_1$$

Which simplifies to

$$\ln\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right) = c_1$$

Which gives

$$y = \frac{\left(\mathrm{e}^{2c_1}+4\,\mathrm{e}^{x}\right)\mathrm{e}^{-c_1}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{2c_1}+4\,\mathrm{e}^{x}\right)\mathrm{e}^{-c_1}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\mathrm{e}^{2c_1}+4\,\mathrm{e}^{x}\right)\mathrm{e}^{-c_1}}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(y(x), x), x)-(diff(y(x), x)), y(x)`
         Methods for second order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         checking if the LODE has constant coefficients
         <- constant coefficients successful
      <- 1st order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 34

```
dsolve(diff(y(x),x)^2-y(x)*diff(y(x),x)+exp(x)=0,y(x), singsol=all)
```

$$y(x) = -2\,\mathrm{e}^{\frac{x}{2}}$$
$$y(x) = 2\,\mathrm{e}^{\frac{x}{2}}$$
$$y(x) = \frac{\mathrm{e}^{x} c_1^2 + 1}{c_1}$$

✓ Solution by Mathematica

Time used: 60.203 (sec). Leaf size: 59

```
DSolve[y'[x]^2-y[x]*y'[x]+Exp[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-e^{-c_1}\left(-e^{x} + e^{c_1}\right)^2}$$
$$y(x) \to \sqrt{-e^{-c_1}\left(e^{x} - e^{c_1}\right)^2}$$

8.9 problem 207

Internal problem ID [15095]
Internal file name [`OUTPUT/15095_Sunday_April_21_2024_01_29_42_PM_49773837/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 207.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y'^2 - 4y'x + 2y = -2x^2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2x + \sqrt{2x^2 - 2y} \tag{1}$$
$$y' = 2x - \sqrt{2x^2 - 2y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = 2x + \sqrt{2x^2 - 2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(2x + \sqrt{2x^2 - 2y}\right)(b_3 - a_2) - \left(2x + \sqrt{2x^2 - 2y}\right)^2 a_3 - \left(2 + \frac{2x}{\sqrt{2x^2 - 2y}}\right)(xa_2 + ya_3 + a_1) + \frac{xb_2 + yb_3 + b_1}{\sqrt{2x^2 - 2y}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(2x^2 - 2y)^{\frac{3}{2}} a_3 + 4\sqrt{2x^2 - 2y}\, x^2 a_3 + 8x^3 a_3 + 4\sqrt{2x^2 - 2y}\, xa_2 - 2\sqrt{2x^2 - 2y}\, xb_3 + 2\sqrt{2x^2 - 2y}\, ya_3 + 4}{\sqrt{2x^2 - 2y}} = 0$$

Setting the numerator to zero gives

$$-(2x^2 - 2y)^{\frac{3}{2}} a_3 - 4\sqrt{2x^2 - 2y}\, x^2 a_3 - 8x^3 a_3 - 4\sqrt{2x^2 - 2y}\, xa_2 + 2\sqrt{2x^2 - 2y}\, xb_3 - 2\sqrt{2x^2 - 2y}\, ya_3 - 4x^2 a_2 + 2x^2 b_3 + 6xya_3 - 2\sqrt{2x^2 - 2y}\, a_1 + b_2\sqrt{2x^2 - 2y} - 2xa_1 + xb_2 + 2ya_2 - yb_3 + b_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(2x^2 - 2y)^{\frac{3}{2}} a_3 - 4(2x^2 - 2y)\, xa_3 - 4\sqrt{2x^2 - 2y}\, x^2 a_3 - (2x^2 - 2y)\, a_2 + (2x^2 - 2y)\, b_3 - 4\sqrt{2x^2 - 2y}\, xa_2 + 2\sqrt{2x^2 - 2y}\, xb_3 - 2\sqrt{2x^2 - 2y}\, ya_3 - 2x^2 a_2 - 2xya_3 - 2\sqrt{2x^2 - 2y}\, a_1 + b_2\sqrt{2x^2 - 2y} - 2xa_1 + xb_2 + yb_3 + b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-6\sqrt{2x^2-2y}\,x^2a_3-8x^3a_3-4\sqrt{2x^2-2y}\,xa_2+2\sqrt{2x^2-2y}\,xb_3-4x^2a_2+2x^2b_3+6xya_3-2\sqrt{2x^2-2y}\,a_1+b_2\sqrt{2x^2-2y}-2xa_1+xb_2+2ya_2-yb_3+b_1=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{2x^2-2y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{2x^2-2y}=v_3\right\}$$

The above PDE (6E) now becomes

$$-8v_1^3a_3-6v_3v_1^2a_3-4v_1^2a_2-4v_3v_1a_2+6v_1v_2a_3+2v_1^2b_3+2v_3v_1b_3-2v_1a_1-2v_3a_1+2v_2a_2+v_1b_2+b_2v_3-v_2b_3+b_1=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$-8v_1^3a_3-6v_3v_1^2a_3+(-4a_2+2b_3)\,v_1^2+6v_1v_2a_3+(-4a_2+2b_3)\,v_1v_3+(-2a_1+b_2)\,v_1+(2a_2-b_3)\,v_2+(-2a_1+b_2)\,v_3+b_1=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}b_1&=0\\-8a_3&=0\\-6a_3&=0\\6a_3&=0\\-2a_1+b_2&=0\\-4a_2+2b_3&=0\\2a_2-b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 2a_1 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 2x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 2x - \left(2x + \sqrt{2x^2 - 2y}\right)(1) \\
&= -\sqrt{2x^2 - 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{2x^2 - 2y}} dy
\end{aligned}
$$

Which results in

$$S = \sqrt{2x^2 - 2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2x + \sqrt{2x^2 - 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{\sqrt{2x^2 - 2y}} \\ S_y &= -\frac{1}{\sqrt{2x^2 - 2y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{2x^2 - 2y} = -x + c_1$$

Which simplifies to

$$\sqrt{2x^2-2y} = -x+c_1$$

Which gives

$$y = -\frac{1}{2}c_1^2 + c_1 x + \frac{1}{2}x^2$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2}c_1^2 + c_1 x + \frac{1}{2}x^2 \tag{1}$$

Verification of solutions

$$y = -\frac{1}{2}c_1^2 + c_1 x + \frac{1}{2}x^2$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned} y' &= 2x - \sqrt{2x^2-2y} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \left(2x - \sqrt{2x^2-2y}\right)(b_3 - a_2) - \left(2x - \sqrt{2x^2-2y}\right)^2 a_3 \\ - \left(2 - \frac{2x}{\sqrt{2x^2-2y}}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{\sqrt{2x^2-2y}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(2x^2-2y\right)^{\frac{3}{2}} a_3+4\sqrt{2x^2-2y}\, x^2 a_3-8x^3 a_3+4\sqrt{2x^2-2y}\, x a_2-2\sqrt{2x^2-2y}\, x b_3+2\sqrt{2x^2-2y}\, y a_3-4\cdots}{\sqrt{2x^2-2y}}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\left(2x^2-2y\right)^{\frac{3}{2}} a_3-4\sqrt{2x^2-2y}\, x^2 a_3+8x^3 a_3-4\sqrt{2x^2-2y}\, x a_2 \\ &+2\sqrt{2x^2-2y}\, x b_3-2\sqrt{2x^2-2y}\, y a_3+4x^2 a_2-2x^2 b_3-6xy a_3 \\ &-2\sqrt{2x^2-2y}\, a_1+b_2\sqrt{2x^2-2y}+2x a_1-x b_2-2y a_2+y b_3-b_1=0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\left(2x^2-2y\right)^{\frac{3}{2}} a_3+4\left(2x^2-2y\right) x a_3-4\sqrt{2x^2-2y}\, x^2 a_3+\left(2x^2-2y\right) a_2 \\ &-\left(2x^2-2y\right) b_3-4\sqrt{2x^2-2y}\, x a_2+2\sqrt{2x^2-2y}\, x b_3-2\sqrt{2x^2-2y}\, y a_3 \\ &+2x^2 a_2+2xy a_3-2\sqrt{2x^2-2y}\, a_1+b_2\sqrt{2x^2-2y}+2x a_1-x b_2-y b_3-b_1=0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &8x^3 a_3-6\sqrt{2x^2-2y}\, x^2 a_3+4x^2 a_2-2x^2 b_3-4\sqrt{2x^2-2y}\, x a_2+2\sqrt{2x^2-2y}\, x b_3 \\ &-6xy a_3+2x a_1-x b_2-2\sqrt{2x^2-2y}\, a_1+b_2\sqrt{2x^2-2y}-2y a_2+y b_3-b_1=0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{2x^2-2y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{2x^2-2y}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &8v_1^3 a_3-6v_3 v_1^2 a_3+4v_1^2 a_2-4v_3 v_1 a_2-6v_1 v_2 a_3-2v_1^2 b_3+2v_3 v_1 b_3 \\ &+2v_1 a_1-2v_3 a_1-2v_2 a_2-v_1 b_2+b_2 v_3+v_2 b_3-b_1=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$8v_1^3a_3 - 6v_3v_1^2a_3 + (4a_2 - 2b_3)v_1^2 - 6v_1v_2a_3 + (-4a_2 + 2b_3)v_1v_3 \\ + (2a_1 - b_2)v_1 + (-2a_2 + b_3)v_2 + (-2a_1 + b_2)v_3 - b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -6a_3 &= 0 \\ 8a_3 &= 0 \\ -b_1 &= 0 \\ -2a_1 + b_2 &= 0 \\ 2a_1 - b_2 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ -2a_2 + b_3 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 2a_1 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 2x \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= 2x - \left(2x - \sqrt{2x^2 - 2y}\right)(1) \\ &= \sqrt{2x^2 - 2y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{2x^2 - 2y}} dy \end{aligned}$$

Which results in

$$S = -\sqrt{2x^2 - 2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2x - \sqrt{2x^2 - 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x}{\sqrt{2x^2 - 2y}} \\ S_y &= \frac{1}{\sqrt{2x^2 - 2y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\sqrt{2x^2 - 2y} = -x + c_1$$

Which simplifies to

$$-\sqrt{2x^2 - 2y} = -x + c_1$$

Which gives

$$y = -\frac{1}{2}c_1^2 + c_1 x + \frac{1}{2}x^2$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2}c_1^2 + c_1 x + \frac{1}{2}x^2 \tag{1}$$

Verification of solutions

$$y = -\frac{1}{2}c_1^2 + c_1 x + \frac{1}{2}x^2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 77

```
dsolve(diff(y(x),x)^2-4*x*diff(y(x),x)+2*y(x)+2*x^2=0,y(x), singsol=all)
```

$$y(x) = x^2$$
$$y(x) = \frac{1}{2}x^2 + c_1x - \frac{1}{2}c_1^2$$
$$y(x) = \frac{1}{2}x^2 - c_1x - \frac{1}{2}c_1^2$$
$$y(x) = \frac{1}{2}x^2 - c_1x - \frac{1}{2}c_1^2$$
$$y(x) = \frac{1}{2}x^2 + c_1x - \frac{1}{2}c_1^2$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]^2-4*x*y'[x]+2*y[x]+2*x^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

8.10 problem 208

Internal problem ID [15096]
Internal file name [`OUTPUT/15096_Sunday_April_21_2024_01_29_44_PM_52719553/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 208.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y - {y'}^{2} \mathrm{e}^{y'} = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\,\mathrm{LambertW}\left(\frac{\sqrt{y}}{2}\right) \tag{1}$$

$$y' = 2\,\mathrm{LambertW}\left(-\frac{\sqrt{y}}{2}\right) \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{2\,\mathrm{LambertW}\left(\frac{\sqrt{y}}{2}\right)} dy = \int dx$$

$$\frac{y}{4\,\mathrm{LambertW}\left(\frac{\sqrt{y}}{2}\right)^{2}} + \frac{y}{2\,\mathrm{LambertW}\left(\frac{\sqrt{y}}{2}\right)} = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{y}{4 \operatorname{LambertW}\left(\frac{\sqrt{y}}{2}\right)^{2}}+\frac{y}{2 \operatorname{LambertW}\left(\frac{\sqrt{y}}{2}\right)}=x+c_{1} \tag{1}$$

Verification of solutions

$$\frac{y}{4 \operatorname{LambertW}\left(\frac{\sqrt{y}}{2}\right)^{2}}+\frac{y}{2 \operatorname{LambertW}\left(\frac{\sqrt{y}}{2}\right)}=x+c_{1}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{2 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)} d y=\int d x$$

$$\frac{y}{4 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)^{2}}+\frac{y}{2 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)}=x+c_{2}$$

Summary
The solution(s) found are the following

$$\frac{y}{4 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)^{2}}+\frac{y}{2 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)}=x+c_{2} \tag{1}$$

Verification of solutions

$$\frac{y}{4 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)^{2}}+\frac{y}{2 \operatorname{LambertW}\left(-\frac{\sqrt{y}}{2}\right)}=x+c_{2}$$

Verified OK.

8.10.1 Maple step by step solution

Let's solve

$y - {y'}^2 e^{y'} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\mathit{LambertW}\left(\frac{\sqrt{y}}{2}\right)} = 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\mathit{LambertW}\left(\frac{\sqrt{y}}{2}\right)}dx = \int 2dx + c_1$

- Evaluate integral

  $\frac{y}{2\mathit{LambertW}\left(\frac{\sqrt{y}}{2}\right)^2} + \frac{y}{\mathit{LambertW}\left(\frac{\sqrt{y}}{2}\right)} = 2x + c_1$

- Solve for y

  $y = \left(\mathit{LambertW}\left(\frac{(2x+c_1)\mathrm{e}}{2}\right) - 1\right)^2 \left(\mathrm{e}^{\frac{\mathit{LambertW}\left(\frac{(2x+c_1)\mathrm{e}}{2}\right)}{2} - \frac{1}{2}}\right)^2$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 38

```
dsolve(y(x)=diff(y(x),x)^2*exp(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{(x - c_1)\left(\operatorname{LambertW}\left((x - c_1)\,\mathrm{e}\right) - 1\right)^2}{\operatorname{LambertW}\left((x - c_1)\,\mathrm{e}\right)}$$

✓ Solution by Mathematica

Time used: 0.283 (sec). Leaf size: 102

```
DSolve[y[x]==y'[x]^2*Exp[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{\#1}{W\left(-\frac{\sqrt{\#1}}{2}\right)} + \frac{\#1}{2W\left(-\frac{\sqrt{\#1}}{2}\right)^2}\&\right][2x + c_1]$$

$$y(x) \to \text{InverseFunction}\left[\frac{\#1}{W\left(\frac{\sqrt{\#1}}{2}\right)} + \frac{\#1}{2W\left(\frac{\sqrt{\#1}}{2}\right)^2}\&\right][2x + c_1]$$

$$y(x) \to 0$$

8.11 problem 209

Internal problem ID [15097]
Internal file name [`OUTPUT/15097_Sunday_April_21_2024_01_29_44_PM_13723187/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 209.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \mathrm{e}^{\frac{y'}{y}}}$$

8.11.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{y\,\mathrm{LambertW}\left(-\frac{1}{y}\right)}dy = x + c_1$$

$$-\frac{1}{\mathrm{LambertW}\left(-\frac{1}{y}\right)} + \ln\left(\mathrm{LambertW}\left(-\frac{1}{y}\right)\right) = x + c_1$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = -\mathrm{LambertW}\left(\mathrm{e}^{-x-c_1}\right)\mathrm{e}^{-\frac{1}{\mathrm{LambertW}\left(\mathrm{e}^{-x-c_1}\right)}} \tag{1}$$

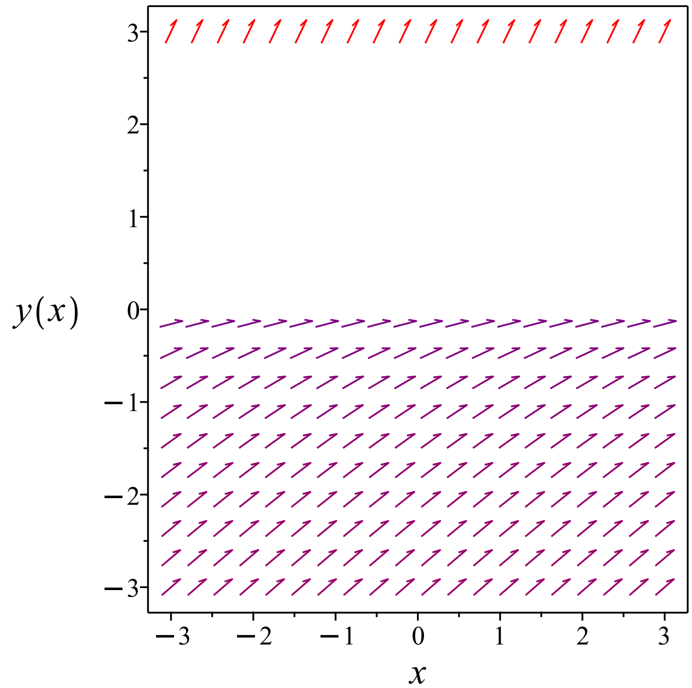


Figure 328: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(e^{-x-c_1}\right) e^{-\frac{1}{\text{LambertW}\left(e^{-x-c_1}\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)=exp(diff(y(x),x)/y(x)),y(x), singsol=all)
```

$$y(x) = -\mathrm{LambertW}\left(c_1 e^{-x}\right) e^{-\frac{1}{\mathrm{LambertW}\left(c_1 e^{-x}\right)}}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 33

```
DSolve[y'[x]==Exp[y'[x]/y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{W\left(-\frac{1}{\#1}\right)} - \log\left(W\left(-\frac{1}{\#1}\right)\right) \&\right][-x + c_1]$$

8.12 problem 210

Internal problem ID [15098]
Internal file name [`OUTPUT/15098_Sunday_April_21_2024_01_29_45_PM_67800578/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 210.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-\ln(y') - \sin(y') = -x}$$

8.12.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \operatorname{RootOf}\left(\_Z - \mathrm{e}^{-\sin(\_Z)+x}\right) \mathrm{d}x \\ &= \int \operatorname{RootOf}\left(\_Z - \mathrm{e}^{-\sin(\_Z)+x}\right) dx + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \int \operatorname{RootOf}\left(\_Z - \mathrm{e}^{-\sin(\_Z)+x}\right) dx + c_1 \tag{1}$$

Verification of solutions

$$y = \int \operatorname{RootOf}\left(\_Z - \mathrm{e}^{-\sin(\_Z)+x}\right) dx + c_1$$

Warning, solution could not be verified

8.12.2 Maple step by step solution

Let's solve

$-\ln(y') - \sin(y') = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (-\ln(y') - \sin(y'))\,dx = \int -x dx + c_1$

- Cannot compute integral

  $\int (-\ln(y') - \sin(y'))\,dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x=ln(diff(y(x),x))+sin(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \int \mathrm{RootOf}\left(-x + \ln\left(\_Z\right) + \sin\left(\_Z\right)\right) dx + c_1$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 33

```
DSolve[x==Log[y'[x]]+Sin[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}[\{y(x) = K[1] + K[1]\sin(K[1]) + \cos(K[1]) + c_1, x = \log(K[1]) + \sin(K[1])\}, \{y(x), K[1]\}]$$

8.13 problem 211

Internal problem ID [15099]
Internal file name [`OUTPUT/15099_Sunday_April_21_2024_01_29_46_PM_37466647/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 211.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-{y'}^{2}+2y' = -x+2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 1+\sqrt{x-1} \tag{1}$$
$$y' = 1-\sqrt{x-1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 1+\sqrt{x-1}\,\mathrm{d}x \\ &= x+\frac{2(x-1)^{\frac{3}{2}}}{3}+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x+\frac{2(x-1)^{\frac{3}{2}}}{3}+c_1 \tag{1}$$

Verification of solutions

$$y = x + \frac{2(x-1)^{\frac{3}{2}}}{3} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int 1 - \sqrt{x-1}\, \mathrm{d}x \\ &= x - \frac{2(x-1)^{\frac{3}{2}}}{3} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x - \frac{2(x-1)^{\frac{3}{2}}}{3} + c_2 \tag{1}$$

Verification of solutions

$$y = x - \frac{2(x-1)^{\frac{3}{2}}}{3} + c_2$$

Verified OK.

8.13.1 Maple step by step solution

Let's solve

$-y'^2 + 2y' = -x + 2$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(-y'^2 + 2y'\right) dx = \int (-x + 2)\, dx + c_1$

- Cannot compute integral

  $\int \left(-y'^2 + 2y'\right) dx = -\frac{1}{2}x^2 + 2x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(x=diff(y(x),x)^2-2*diff(y(x),x)+2,y(x), singsol=all)
```

$$y(x) = \frac{(-2x+2)\sqrt{-1+x}}{3} + x + c_1$$
$$y(x) = \frac{(2x-2)\sqrt{-1+x}}{3} + x + c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 39

```
DSolve[x==y'[x]^2-2*y'[x]+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{3}(x-1)^{3/2} + x + c_1$$
$$y(x) \to \frac{2}{3}(x-1)^{3/2} + x + c_1$$

8.14 problem 212

Internal problem ID [15100]
Internal file name [`OUTPUT/15100_Sunday_April_21_2024_01_29_47_PM_93339442/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 212.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y - y' \ln(y') = 0}$$

8.14.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{\text{LambertW}(y)}{y} dy = x + c_1$$
$$\frac{\text{LambertW}(y)^2}{2} + \text{LambertW}(y) = x + c_1$$

Solving for y gives these solutions

Summary
The solution(s) found are the following

$$y = \left(-1 - \sqrt{1 + 2c_1 + 2x}\right) \mathrm{e}^{-1-\sqrt{1+2c_1+2x}} \tag{1}$$
$$y = \left(-1 + \sqrt{1 + 2c_1 + 2x}\right) \mathrm{e}^{-1+\sqrt{1+2c_1+2x}} \tag{2}$$

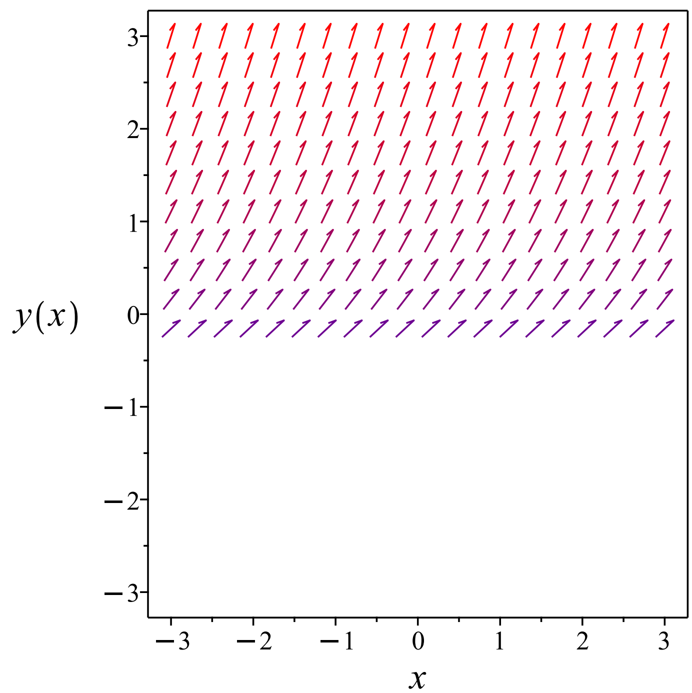


Figure 329: Slope field plot

Verification of solutions

$$y = \left(-1-\sqrt{1+2c_1+2x}\right) e^{-1-\sqrt{1+2c_1+2x}}$$

Verified OK.

$$y = \left(-1+\sqrt{1+2c_1+2x}\right) e^{-1+\sqrt{1+2c_1+2x}}$$

Verified OK.

8.14.2 Maple step by step solution

Let's solve

$y - y' \ln(y') = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \mathit{LambertW}(y)}{y} = 1$

- Integrate both sides with respect to x

$$\int \frac{y' \mathit{LambertW}(y)}{y} dx = \int 1 dx + c_1$$

- Evaluate integral

$$\frac{\mathit{LambertW}(y)^2}{2} + \mathit{LambertW}(y) = x + c_1$$

- Solve for y

$$\left\{y = \left(-1 - \sqrt{1 + 2c_1 + 2x}\right) e^{-1-\sqrt{1+2c_1+2x}}, y = \left(-1 + \sqrt{1 + 2c_1 + 2x}\right) e^{-1+\sqrt{1+2c_1+2x}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 63

```
dsolve(y(x)=diff(y(x),x)*ln(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \left(-1 - \sqrt{1 - 2c_1 + 2x}\right) e^{-1-\sqrt{1-2c_1+2x}}$$
$$y(x) = \left(-1 + \sqrt{1 - 2c_1 + 2x}\right) e^{-1+\sqrt{1-2c_1+2x}}$$

✓ Solution by Mathematica

Time used: 4.166 (sec). Leaf size: 83

```
DSolve[y[x]==y'[x]*Log[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-1-\sqrt{2x+1+2c_1}}\left(1 + \sqrt{2x + 1 + 2c_1}\right)$$
$$y(x) \to e^{-1+\sqrt{2x+1+2c_1}}\left(-1 + \sqrt{2x + 1 + 2c_1}\right)$$
$$y(x) \to 0$$

8.15 problem 213

Internal problem ID [15101]
Internal file name [`OUTPUT/15101_Sunday_April_21_2024_01_29_48_PM_31706959/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 213.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y-(y'-1)\,\mathrm{e}^{y'}=0}$$

8.15.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{\mathrm{LambertW}\left(\mathrm{e}^{-1}y\right)+1}dy &= x+c_1\\ \frac{y}{\mathrm{LambertW}\left(\mathrm{e}^{-1}y\right)} &= x+c_1\end{aligned}$$

Solving for y gives these solutions

$$y_1 = c_1\ln\left(x+c_1\right)+x\ln\left(x+c_1\right)-c_1-x$$

Summary
The solution(s) found are the following

$$y = c_1\ln\left(x+c_1\right)+x\ln\left(x+c_1\right)-c_1-x \tag{1}$$

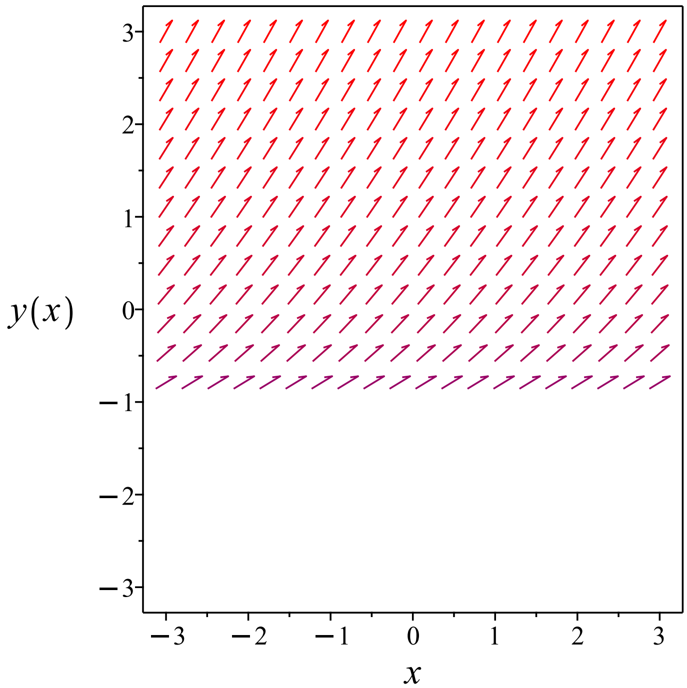


Figure 330: Slope field plot

Verification of solutions

$$y = c_1 \ln(x + c_1) + x \ln(x + c_1) - c_1 - x$$

Verified OK.

8.15.2 Maple step by step solution

Let's solve

$y - (y' - 1)\,\mathrm{e}^{y'} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\mathit{LambertW}(y\,\mathrm{e}^{-1})+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\mathit{LambertW}(y\,\mathrm{e}^{-1})+1} dx = \int 1 dx + c_1$

- Evaluate integral

$$\frac{y}{LambertW\left(y\,e^{-1}\right)} = x + c_1$$

- Solve for y

$$y = c_1 \ln\left(x + c_1\right) + x \ln\left(x + c_1\right) - c_1 - x$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
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trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(y(x)=(diff(y(x),x)-1)*exp(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = \left(\ln\left(x - c_1\right) - 1\right)\left(x - c_1\right)$$

✓ Solution by Mathematica

Time used: 0.435 (sec). Leaf size: 22

```
DSolve[y[x]==(y'[x]-1)*Exp[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_1)(-1 + \log(x + c_1))$$
$$y(x) \to -1$$

8.16 problem 214

Internal problem ID [15102]
Internal file name [`OUTPUT/15102_Sunday_April_21_2024_01_29_48_PM_92731794/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 214.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x{y'}^{2}-{\mathrm e}^{\frac{1}{y'}}=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1}{2\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)} \tag{1}$$

$$y' = \frac{1}{2\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{1}{2\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)}\,\mathrm{d}x \\
&= \frac{x}{4\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)^{2}} + \frac{x}{2\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)} + c_1
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{4\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)^2} + \frac{x}{2\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{x}{4\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)^2} + \frac{x}{2\,\mathrm{LambertW}\left(\frac{\sqrt{x}}{2}\right)} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{2\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)}\,\mathrm{d}x \\ &= \frac{x}{4\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)^2} + \frac{x}{2\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{4\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)^2} + \frac{x}{2\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{x}{4\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)^2} + \frac{x}{2\,\mathrm{LambertW}\left(-\frac{\sqrt{x}}{2}\right)} + c_2$$

Verified OK.

8.16.1 Maple step by step solution

Let's solve

$x{y'}^2 - \mathrm{e}^{\frac{1}{y'}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

$$\int \left(x{y'}^{2}-e^{\frac{1}{y'}}\right)dx = \int 0 dx + c_1$$

- Cannot compute integral

$$\int \left(x{y'}^{2}-e^{\frac{1}{y'}}\right)dx = c_1$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(diff(y(x),x)^2*x=exp(1/diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \frac{4c_1 \operatorname{LambertW}\left(-\frac{\sqrt{x}}{2}\right)^2 + 2x \operatorname{LambertW}\left(-\frac{\sqrt{x}}{2}\right) + x}{4 \operatorname{LambertW}\left(-\frac{\sqrt{x}}{2}\right)^2}$$

$$y(x) = \frac{4c_1 \operatorname{LambertW}\left(\frac{\sqrt{x}}{2}\right)^2 + 2x \operatorname{LambertW}\left(\frac{\sqrt{x}}{2}\right) + x}{4 \operatorname{LambertW}\left(\frac{\sqrt{x}}{2}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 67

```
DSolve[y'[x]^2*x==Exp[1/y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \int_1^x \frac{1}{2W\left(-\frac{1}{2\sqrt{\frac{1}{K[1]}}}\right)} dK[1] + c_1$$

$$y(x) \to \int_1^x \frac{1}{2W\left(\frac{1}{2\sqrt{\frac{1}{K[2]}}}\right)} dK[2] + c_1$$

8.17 problem 215

Internal problem ID [15103]
Internal file name [`OUTPUT/15103_Sunday_April_21_2024_01_29_49_PM_90012658/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 215.
ODE order: 1.
ODE degree: 6.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{x\left(y'^2+1\right)^{\frac{3}{2}} = a}$$

Solving the given ode for y' results in 6 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{\left(a\,x^2\right)^{\frac{2}{3}}-x^2}}{x} \tag{1}$$

$$y' = -\frac{\sqrt{\left(a\,x^2\right)^{\frac{2}{3}}-x^2}}{x} \tag{2}$$

$$y' = \frac{\sqrt{-2i\left(a\,x^2\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^2\right)^{\frac{2}{3}}-4x^2}}{2x} \tag{3}$$

$$y' = -\frac{\sqrt{-2i\left(a\,x^2\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^2\right)^{\frac{2}{3}}-4x^2}}{2x} \tag{4}$$

$$y' = \frac{\sqrt{2}\,\sqrt{i\left(a\,x^2\right)^{\frac{2}{3}}\sqrt{3}-\left(a\,x^2\right)^{\frac{2}{3}}-2x^2}}{2x} \tag{5}$$

$$y' = -\frac{\sqrt{2}\,\sqrt{i\left(a\,x^2\right)^{\frac{2}{3}}\sqrt{3}-\left(a\,x^2\right)^{\frac{2}{3}}-2x^2}}{2x} \tag{6}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$y = \int \frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}\,\mathrm{d}x$$
$$= \int \frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_1$$

Summary
The solution(s) found are the following

$$y = \int \frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_1 \tag{1}$$

Verification of solutions

$$y = \int \frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$y = \int -\frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}\,\mathrm{d}x$$
$$= \int -\frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_2$$

Summary
The solution(s) found are the following

$$y = \int -\frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_2 \tag{1}$$

Verification of solutions

$$y = \int -\frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_2$$

Verified OK.

Solving equation (3)

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}\,\mathrm{d}x \\
&= \int \frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}dx + c_3
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \int \frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}dx + c_3 \tag{1}
$$

Verification of solutions

$$
y = \int \frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}dx + c_3
$$

Verified OK.

Solving equation (4)

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}\,\mathrm{d}x \\
&= \int -\frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}dx + c_4
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \int -\frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}dx + c_4 \tag{1}
$$

Verification of solutions

$$
y = \int -\frac{\sqrt{-2i\left(a\,x^{2}\right)^{\frac{2}{3}}\sqrt{3}-2\left(a\,x^{2}\right)^{\frac{2}{3}}-4x^{2}}}{2x}dx + c_4
$$

Verified OK.

Solving equation (5)

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}\,\mathrm{d}x \\ &= \int \frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}dx + c_5 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \int \frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}dx + c_5 \tag{1}$$

Verification of solutions

$$y = \int \frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}dx + c_5$$

Verified OK.

Solving equation (6)

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}\,\mathrm{d}x \\ &= \int -\frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}dx + c_6 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \int -\frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}dx + c_6 \tag{1}$$

Verification of solutions

$$y = \int -\frac{\sqrt{2}\,\sqrt{i\,(a\,x^2)^{\frac{2}{3}}\,\sqrt{3}-(a\,x^2)^{\frac{2}{3}}-2x^2}}{2x}dx + c_6$$

Verified OK.

8.17.1 Maple step by step solution

Let's solve

$x\left(y'^2+1\right)^{\frac{3}{2}}=a$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int x\left(y'^2+1\right)^{\frac{3}{2}}dx=\int adx+c_1$

- Cannot compute integral

  $\int x\left(y'^2+1\right)^{\frac{3}{2}}dx=ax+c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 233

```
dsolve(x*(1+diff(y(x),x)^2)^(3/2)=a,y(x), singsol=all)
```

$$y(x) = \int \frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx + c_1$$

$$y(x) = -\frac{\left(\int \frac{\sqrt{-2i\sqrt{3}\,(a\,x^2)^{\frac{2}{3}} - 2(a\,x^2)^{\frac{2}{3}} - 4x^2}}{x}dx\right)}{2} + c_1$$

$$y(x) = \frac{\left(\int \frac{\sqrt{-2i\sqrt{3}\,(a\,x^2)^{\frac{2}{3}} - 2(a\,x^2)^{\frac{2}{3}} - 4x^2}}{x}dx\right)}{2} + c_1$$

$$y(x) = -\left(\int \frac{\sqrt{(a\,x^2)^{\frac{2}{3}} - x^2}}{x}dx\right) + c_1$$

$$y(x) = -\frac{\sqrt{2}\left(\int \frac{\sqrt{i\sqrt{3}\,(a\,x^2)^{\frac{2}{3}} - (a\,x^2)^{\frac{2}{3}} - 2x^2}}{x}dx\right)}{2} + c_1$$

$$y(x) = \frac{\sqrt{2}\left(\int \frac{\sqrt{i\sqrt{3}\,(a\,x^2)^{\frac{2}{3}} - (a\,x^2)^{\frac{2}{3}} - 2x^2}}{x}dx\right)}{2} + c_1$$

✓ Solution by Mathematica

Time used: 19.313 (sec). Leaf size: 375

```
DSolve[x*(1+y'[x]^2)^(3/2)==a,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{x}\sqrt{\frac{a^{2/3}}{x^{2/3}}-1}\left(x^{2/3}-a^{2/3}\right)+c_1$$

$$y(x) \to \sqrt[3]{x}\sqrt{\frac{a^{2/3}}{x^{2/3}}-1}\left(a^{2/3}-x^{2/3}\right)+c_1$$

$$y(x) \to c_1-\frac{1}{2}\sqrt[3]{x}\sqrt{-1+\frac{i\left(\sqrt{3}+i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1-i\sqrt{3}\right)a^{2/3}\right)$$

$$y(x) \to \frac{1}{2}\sqrt[3]{x}\sqrt{-1+\frac{i\left(\sqrt{3}+i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1-i\sqrt{3}\right)a^{2/3}\right)+c_1$$

$$y(x) \to c_1-\frac{1}{2}\sqrt[3]{x}\sqrt{-1-\frac{i\left(\sqrt{3}-i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1+i\sqrt{3}\right)a^{2/3}\right)$$

$$y(x) \to \frac{1}{2}\sqrt[3]{x}\sqrt{-1-\frac{i\left(\sqrt{3}-i\right)a^{2/3}}{2x^{2/3}}}\left(2x^{2/3}+\left(1+i\sqrt{3}\right)a^{2/3}\right)+c_1$$

8.18 problem 216

Internal problem ID [15104]
Internal file name [`OUTPUT/15104_Sunday_April_21_2024_01_29_51_PM_25456079/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 216.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y^{\frac{2}{5}}+{y'}^{\frac{2}{5}}=a^{\frac{2}{5}}}$$

8.18.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{5}{2}}}dy = \int dx$$

$$\frac{5y^{\frac{3}{5}}}{3\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{3}{2}}}-\frac{5y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}+5\arctan\left(\frac{y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}\right)=x+c_1$$

Summary

The solution(s) found are the following

$$\frac{5y^{\frac{3}{5}}}{3\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{3}{2}}}-\frac{5y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}+5\arctan\left(\frac{y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}\right)=x+c_1 \tag{1}$$

Verification of solutions

$$\frac{5y^{\frac{3}{5}}}{3\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{3}{2}}}-\frac{5y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}+5\arctan\left(\frac{y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}\right)=x+c_1$$

Verified OK.

8.18.2 Maple step by step solution

Let's solve

$y^{\frac{2}{5}}+y'^{\frac{2}{5}}=a^{\frac{2}{5}}$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{5}{2}}}=1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{5}{2}}}dx=\int 1dx+c_1$

- Evaluate integral

  $\frac{5y^{\frac{3}{5}}}{3\left(-y^{\frac{2}{5}}+a^{\frac{2}{5}}\right)^{\frac{3}{2}}}-\frac{5y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}+5\arctan\left(\frac{y^{\frac{1}{5}}}{\sqrt{-y^{\frac{2}{5}}+a^{\frac{2}{5}}}}\right)=x+c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 26

```
dsolve(y(x)^(2/5)+diff(y(x),x)^(2/5)=a^(2/5),y(x), singsol=all)
```

$$x - \left(\int^{y(x)} \frac{1}{\left(a^{\frac{2}{5}} - \_a^{\frac{2}{5}}\right)^{\frac{5}{2}}} d\_a\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.746 (sec). Leaf size: 89

```
DSolve[y[x]^(2/5)+y'[x]^(2/5)==a^(2/5),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[5\arctan\left(\frac{\sqrt[5]{\#1}}{\sqrt{a^{2/5} - \#1^{2/5}}}\right) + \frac{5\sqrt[5]{\#1}\left(4\#1^{2/5} - 3a^{2/5}\right)}{3\left(a^{2/5} - \#1^{2/5}\right)^{3/2}}\&\right][x + c_1]$$

$$y(x) \to a$$

8.19 problem 217

Internal problem ID [15105]
Internal file name [`OUTPUT/15105_Sunday_April_21_2024_01_29_53_PM_8617977/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 217.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{-y' - \sin(y') = -x}$$

8.19.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \operatorname{RootOf}(-x + \_Z + \sin(\_Z))\, \mathrm{d}x \\ &= \int \operatorname{RootOf}(-x + \_Z + \sin(\_Z))\, dx + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \int \operatorname{RootOf}(-x + \_Z + \sin(\_Z))\, dx + c_1 \tag{1}$$

Verification of solutions

$$y = \int \operatorname{RootOf}(-x + \_Z + \sin(\_Z))\, dx + c_1$$

Verified OK.

8.19.2 Maple step by step solution

Let's solve

$-y' - \sin(y') = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (-y' - \sin(y'))\,dx = \int -x dx + c_1$

- Cannot compute integral

  $\int (-y' - \sin(y'))\,dx = -\frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(x=diff(y(x),x)+sin(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = \int \mathrm{RootOf}\left(-x+\_Z+\sin\left(\_Z\right)\right)dx + c_1$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 38

```
DSolve[x==y'[x]+Sin[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\left\{x = K[1]+\sin(K[1]), y(x) = \frac{K[1]^2}{2} + K[1]\sin(K[1]) + \cos(K[1]) + c_1\right\}, \{y(x), K[1]\}\right]$$

8.20 problem 218

Internal problem ID [15106]
Internal file name [`OUTPUT/15106_Sunday_April_21_2024_01_29_53_PM_38661339/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 218.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y - y'(1 + y' \cos(y')) = 0}$$

8.20.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\operatorname{RootOf}\left(\_Z^2 \cos(\_Z) - y + \_Z\right)} dy = \int dx$$
$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 \cos(\_Z) - \_a + \_Z\right)} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 \cos(\_Z) - \_a + \_Z\right)} d\_a = x + c_1 \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 \cos(\_Z) - \_a + \_Z\right)} d\_a = x + c_1$$

Verified OK.

8.20.2 Maple step by step solution

Let's solve

$y - y'(1 + y' \cos(y')) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{RootOf\left(\_Z^2 \cos\left(\_Z\right) - y + \_Z\right)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{RootOf\left(\_Z^2 \cos\left(\_Z\right) - y + \_Z\right)} dx = \int 1 dx + c_1$

- Cannot compute integral

  $\int \frac{y'}{RootOf\left(\_Z^2 \cos\left(\_Z\right) - y + \_Z\right)} dx = x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(y(x)=diff(y(x),x)*(1+diff(y(x),x)*cos(diff(y(x),x))),y(x), singsol=all)
```

$$y(x) = 0$$

$$x - \left(\int^{y(x)} \frac{1}{\mathrm{RootOf}\left(\cos\left(\_Z\right)\_Z^2 - \_a + \_Z\right)} d\_a\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.154 (sec). Leaf size: 38

```
DSolve[y[x]==y'[x]*(1+y'[x]*Cos[y'[x]]),y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\left\{x = \log(K[1]) + \sin(K[1]) + K[1]\cos(K[1]) + c_1, y(x) = K[1] + K[1]^2\cos(K[1])\right\}, \{y(x), K[1]\}\right]$$

8.21 problem 219

Internal problem ID [15107]
Internal file name [`OUTPUT/15107_Sunday_April_21_2024_01_29_54_PM_16189521/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8. First order not solved for the derivative. Exercises page 67
Problem number: 219.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y - \arcsin\left(y'\right) - \ln\left({y'}^2 + 1\right) = 0}$$

8.21.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sin\left(\mathrm{RootOf}\left(-y + \_Z + \ln\left(\sin\left(\_Z\right)^2 + 1\right)\right)\right)} dy = \int dx$$
$$\int^{y} \frac{1}{\sin\left(\mathrm{RootOf}\left(-\_a + \_Z + \ln\left(\sin\left(\_Z\right)^2 + 1\right)\right)\right)} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{\sin\left(\mathrm{RootOf}\left(-\_a + \_Z + \ln\left(\sin\left(\_Z\right)^2 + 1\right)\right)\right)} d\_a = x + c_1 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{1}{\sin\left(\mathrm{RootOf}\left(-\_a + \_Z + \ln\left(\sin\left(\_Z\right)^2 + 1\right)\right)\right)} d\_a = x + c_1$$

Warning, solution could not be verified

8.21.2 Maple step by step solution

Let's solve

$y - \arcsin(y') - \ln(y'^2 + 1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sin\left(\mathit{RootOf}\left(-y + \_Z + \ln\left(\sin(\_Z)^2 + 1\right)\right)\right)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sin\left(\mathit{RootOf}\left(-y + \_Z + \ln\left(\sin(\_Z)^2 + 1\right)\right)\right)} dx = \int 1 dx + c_1$

- Cannot compute integral

  $\int \frac{y'}{\sin\left(\mathit{RootOf}\left(-y + \_Z + \ln\left(\sin(\_Z)^2 + 1\right)\right)\right)} dx = x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(y(x)=arcsin(diff(y(x),x))+ln(1+diff(y(x),x)^2),y(x), singsol=all)
```

$$x-\left(\int^{y(x)}\csc\left(\mathrm{RootOf}\left(-\_a+\_Z+\ln\left(2-\cos\left(\_Z\right)^{2}\right)\right)\right)d\_a\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.245 (sec). Leaf size: 46

```
DSolve[y[x]==ArcSin[y'[x]]+Log[1+y'[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[\left\{x=2\arctan(K[1])-\mathrm{arctanh}\left(\sqrt{1-K[1]^2}\right)+c_1,y(x)=\arcsin(K[1])+\log\left(K[1]^2+1\right)\right\},\{y(x),K[1]\}\right]$$

9 Section 8.3. The Lagrange and Clairaut equations. Exercises page 72

9.1 problem 220

Internal problem ID [15108]
Internal file name [`OUTPUT/15108_Sunday_April_21_2024_01_29_55_PM_99143670/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 220.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y-2y'x-\ln(y')=0}$$

9.1.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$y-2px-\ln(p)=0$$

Solving for y from the above results in

$$y=2px+\ln(p) \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f+(xf'+g')\frac{dp}{dx} \\ p-f &= (xf'+g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 2p$$
$$g = \ln(p)$$

Hence (2) becomes

$$-p = \left(2x + \frac{1}{p}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = -\infty$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x + \frac{1}{p(x)}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) + \frac{1}{p}}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p}$$
$$q(p) = -\frac{1}{p^2}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = -\frac{1}{p^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp}\\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dp}(\mu x) &= (\mu)\left(-\frac{1}{p^2}\right)\\ \frac{d}{dp}(p^2 x) &= (p^2)\left(-\frac{1}{p^2}\right)\\ d(p^2 x) &= -1\, dp\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int -1\, dp\\ p^2 x &= -p + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{1}{p} + \frac{c_1}{p^2}$$

which simplifies to

$$x(p) = \frac{-p + c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = e^{-\operatorname{LambertW}(2x\, e^{y}) + y}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{2(-2c_1 x + \operatorname{LambertW}(2x\, e^{y}))\, x}{\operatorname{LambertW}(2x\, e^{y})^2}$$

Summary

The solution(s) found are the following

$$y = -\infty \tag{1}$$

$$x = -\frac{2(-2c_1 x + \text{LambertW}(2x\,e^y))\,x}{\text{LambertW}(2x\,e^y)^2} \tag{2}$$

Verification of solutions

$$y = -\infty$$

Warning, solution could not be verified

$$x = -\frac{2(-2c_1 x + \text{LambertW}(2x\,e^y))\,x}{\text{LambertW}(2x\,e^y)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
dsolve(y(x)=2*x*diff(y(x),x)+ln(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = -1 + \sqrt{4c_1 x + 1} - \ln(2) + \ln\left(\frac{-1 + \sqrt{4c_1 x + 1}}{x}\right)$$

$$y(x) = -1 - \sqrt{4c_1 x + 1} - \ln(2) + \ln\left(\frac{-1 - \sqrt{4c_1 x + 1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 32

```
DSolve[y[x]==2*x*y'[x]+Log[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[W\left(2xe^{y(x)}\right)-\log\left(W\left(2xe^{y(x)}\right)+2\right)-y(x)=c_1,y(x)\right]$$

9.2 problem 221

Internal problem ID [15109]
Internal file name [`OUTPUT/15109_Sunday_April_21_2024_01_29_56_PM_39610767/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 221.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y - x(1 + y') - {y'}^2 = 0}$$

9.2.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - x(1 + p) - p^2 = 0$$

Solving for y from the above results in

$$y = x(1 + p) + p^2 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 1 + p \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
-1 = (x + 2p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-1 = 0
$$

No singular solution are found

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{1}{x + 2p\,(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -x(p) - 2p \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= 1 \\
q(p) &= -2p
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) + x(p) = -2p
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dp} \\
&= \mathrm{e}^{p}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dp}(\mu x) = (\mu)(-2p)$$
$$\frac{d}{dp}(e^p x) = (e^p)(-2p)$$
$$d(e^p x) = (-2p\,e^p)\,dp$$

Integrating gives

$$e^p x = \int -2p\,e^p\,dp$$
$$e^p x = -2(p-1)\,e^p + c_1$$

Dividing both sides by the integrating factor $\mu = e^p$ results in

$$x(p) = -2\,e^{-p}(p-1)\,e^p + c_1 e^{-p}$$

which simplifies to

$$x(p) = -2p + 2 + c_1 e^{-p}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = -\frac{x}{2} + \frac{\sqrt{x^2+4y-4x}}{2}$$
$$p = -\frac{x}{2} - \frac{\sqrt{x^2+4y-4x}}{2}$$

Substituting the above in the solution for x found above gives

$$x = x - \sqrt{x^2+4y-4x} + 2 + c_1 e^{\frac{x}{2} - \frac{\sqrt{x^2+4y-4x}}{2}}$$
$$x = x + \sqrt{x^2+4y-4x} + 2 + c_1 e^{\frac{x}{2} + \frac{\sqrt{x^2+4y-4x}}{2}}$$

Summary

The solution(s) found are the following

$$x = x - \sqrt{x^2+4y-4x} + 2 + c_1 e^{\frac{x}{2} - \frac{\sqrt{x^2+4y-4x}}{2}} \tag{1}$$
$$x = x + \sqrt{x^2+4y-4x} + 2 + c_1 e^{\frac{x}{2} + \frac{\sqrt{x^2+4y-4x}}{2}} \tag{2}$$

Verification of solutions

$$x = x - \sqrt{x^2 + 4y - 4x} + 2 + c_1 \mathrm{e}^{\frac{x}{2} - \frac{\sqrt{x^2+4y-4x}}{2}}$$

Verified OK.

$$x = x + \sqrt{x^2 + 4y - 4x} + 2 + c_1 \mathrm{e}^{\frac{x}{2} + \frac{\sqrt{x^2+4y-4x}}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve(y(x)=x*(1+diff(y(x),x))+diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = x - \frac{x^2}{4} + \operatorname{LambertW}\left(\frac{c_1 \mathrm{e}^{-1+\frac{x}{2}}}{2}\right)^2 + 2 \operatorname{LambertW}\left(\frac{c_1 \mathrm{e}^{-1+\frac{x}{2}}}{2}\right) + 1$$

✓ Solution by Mathematica

Time used: 2.322 (sec). Leaf size: 177

```
DSolve[y[x]==x*(1+y'[x])+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\sqrt{x^2+4y(x)-4x}+2\log\left(\sqrt{x^2+4y(x)-4x}-x+2\right)-2\log\left(-x\sqrt{x^2+4y(x)-4x}+x^2+4y(x)-2x-4\right)+x=c_1,y(x)\right]$$

$$\text{Solve}\left[-4\text{arctanh}\left(\frac{(x-5)\sqrt{x^2+4y(x)-4x}-x^2-4y(x)+7x-6}{(x-3)\sqrt{x^2+4y(x)-4x}-x^2-4y(x)+5x-2}\right)+\sqrt{x^2+4y(x)-4x}+x=c_1,y(x)\right]$$

9.3 problem 222

Internal problem ID [15110]
Internal file name [`OUTPUT/15110_Sunday_April_21_2024_01_29_57_PM_99480851/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 222.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{y - 2y'x - \sin(y') = 0}$$

9.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - 2px - \sin(p) = 0$$

Solving for y from the above results in

$$y = 2px + \sin(p) \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 2p$$
$$g = \sin(p)$$

Hence (2) becomes

$$-p = (2x + \cos(p))\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x + \cos(p(x))} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) + \cos(p)}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p}$$
$$q(p) = -\frac{\cos(p)}{p}$$

Hence the ode is

$$\frac{d}{dp}x(p)+\frac{2x(p)}{p}=-\frac{\cos\left(p\right)}{p}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp}\\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{\cos\left(p\right)}{p}\right)\\ \frac{\mathrm{d}}{\mathrm{d}p}\left(p^2 x\right) &= \left(p^2\right)\left(-\frac{\cos\left(p\right)}{p}\right)\\ \mathrm{d}\left(p^2 x\right) &= \left(-p\cos\left(p\right)\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int -p\cos\left(p\right)\,\mathrm{d}p\\ p^2 x &= -\cos\left(p\right)-p\sin\left(p\right)+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=p^2$ results in

$$x(p)=\frac{-\cos\left(p\right)-p\sin\left(p\right)}{p^2}+\frac{c_1}{p^2}$$

which simplifies to

$$x(p)=\frac{-\cos\left(p\right)-p\sin\left(p\right)+c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p=\mathrm{RootOf}\left(2\_Zx+\sin\left(\_Z\right)-y\right)$$

Substituting the above in the solution for x found above gives

$$x = \frac{-\cos\left(\mathrm{RootOf}\left(2\_Zx+\sin\left(\_Z\right)-y\right)\right)-\mathrm{RootOf}\left(2\_Zx+\sin\left(\_Z\right)-y\right)\sin\left(\mathrm{RootOf}\left(2\_Zx+\sin\left(\_Z\right)\right.\right.}{\mathrm{RootOf}\left(2\_Zx+\sin\left(\_Z\right)-y\right)^2}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{-\cos(\mathrm{RootOf}(2\_Zx + \sin(\_Z) - y)) - \mathrm{RootOf}(2\_Zx + \sin(\_Z) - y)\sin(\mathrm{RootOf}(2\_Zx + \sin(\_Z)}{\mathrm{RootOf}(2\_Zx + \sin(\_Z) - y)^2} \tag{2}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{-\cos(\mathrm{RootOf}(2\_Zx + \sin(\_Z) - y)) - \mathrm{RootOf}(2\_Zx + \sin(\_Z) - y)\sin(\mathrm{RootOf}(2\_Zx + \sin(\_Z)}{\mathrm{RootOf}(2\_Zx + \sin(\_Z) - y)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple
Time used: 0.11 (sec). Leaf size: 44

```
dsolve(y(x)=2*x*diff(y(x),x)+sin(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = 0$$

$$\left[x(\_T) = \frac{-\_T\sin(\_T) - \cos(\_T) + c_1}{\_T^2}, y(\_T) = \frac{-\_T\sin(\_T) - 2\cos(\_T) + 2c_1}{\_T}\right]$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 47

```
DSolve[y[x]==2*x*y'[x]+Sin[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\left\{x=\frac{-K[1]\sin(K[1])-\cos(K[1])}{K[1]^2}+\frac{c_1}{K[1]^2},y(x)=2xK[1]+\sin(K[1])\right\},\{y(x),K[1]\}\right]$$

9.4 problem 223

Internal problem ID [15111]
Internal file name [`OUTPUT/15111_Sunday_April_21_2024_01_29_58_PM_53052229/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 223.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{y - x{y'}^2 + \frac{1}{y'} = 0}$$

9.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - x\,p^2 + \frac{1}{p} = 0$$

Solving for y from the above results in

$$y = x\,p^2 - \frac{1}{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2$$
$$g = -\frac{1}{p}$$

Hence (2) becomes

$$-p^2 + p = \left(2xp + \frac{1}{p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 1$$

Removing solutions for p which leads to undefined results and substituting these in (1A) gives

$$y = x - 1$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\,x + \frac{1}{p(x)^2}} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{2x(p)\,p + \frac{1}{p^2}}{-p^2 + p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p-1}$$
$$q(p) = -\frac{1}{p^3 (p-1)}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p-1} = -\frac{1}{p^3 (p-1)}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{p-1}dp}$$
$$= (p-1)^2$$

The ode becomes

$$\frac{d}{dp}(\mu x) = (\mu)\left(-\frac{1}{p^3 (p-1)}\right)$$
$$\frac{d}{dp}\left((p-1)^2 x\right) = \left((p-1)^2\right)\left(-\frac{1}{p^3 (p-1)}\right)$$
$$d\left((p-1)^2 x\right) = \left(\frac{-p+1}{p^3}\right) dp$$

Integrating gives

$$(p-1)^2 x = \int \frac{-p+1}{p^3} dp$$
$$(p-1)^2 x = -\frac{1}{2p^2} + \frac{1}{p} + c_1$$

Dividing both sides by the integrating factor $\mu = (p-1)^2$ results in

$$x(p) = \frac{-\frac{1}{2p^2} + \frac{1}{p}}{(p-1)^2} + \frac{c_1}{(p-1)^2}$$

which simplifies to

$$x(p) = \frac{2c_1 p^2 + 2p - 1}{2(p-1)^2 p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = \frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{6x} + \frac{2y}{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}$$

$$p = -\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{12x} - \frac{y}{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+}{x}}\right.\right.}{6x}\right.}{}$$

$$p = -\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{12x} - \frac{y}{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(\left(108+12\sqrt{3}\,\sqrt{\frac{-4y^3+}{x}}\right.\right.}{6x}\right.}{}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{54\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}} 3^{\frac{1}{3}} 2^{\frac{2}{3}} x^3 \left(-\frac{2^{\frac{1}{3}}\left(\sqrt{\frac{-4y^3+27x}{x}}\, c_1 3^{\frac{1}{6}} + 2\left(y+\frac{3c_1}{2}\right)3^{\frac{2}{3}}\right)x\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{3} - \frac{2^{\frac{2}{3}}}{}\right.}{\left(2^{\frac{2}{3}}3^{\frac{1}{3}}xy + \left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}\right)^2 \left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}}yx + 2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(9+\sqrt{3}\right.\right.\right.}$$

$$x = -\frac{36\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}\left(\left(\frac{8yc_1}{9}-\frac{2x}{3}\right)\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}} + x\left(-\frac{2^{\frac{1}{3}}\left(\left(i3^{\frac{2}{3}}+3^{\frac{1}{6}}\right)c_1\sqrt{\frac{-4}{}}\right.}{}\right.\right.}{\left(-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}}{6} + x\left(2\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}\right.\right.}$$

$$x = \frac{36\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}} 3^{\frac{1}{3}} 2^{\frac{2}{3}} \left(\left(-\frac{8yc_{1}}{9}+\frac{2x}{3}\right)\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}} + x\left(-\frac{2^{\frac{1}{3}}\left(\left(i3^{\frac{2}{3}}-3^{\frac{1}{6}}\right)c_{1}\cdots\right.}{\cdots}\right.\right.}{\left(-\frac{\left(3^{\frac{1}{3}}+i3^{\frac{5}{6}}\right)2^{\frac{2}{3}}\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}}}{6} + x\left(-2\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{1}{3}}\cdots\right.\right.}$$

<u>Summary</u>

The solution(s) found are the following

$$y = x-1 \tag{1}$$

$$x = -\frac{54\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}} 3^{\frac{1}{3}} 2^{\frac{2}{3}} x^{3}\left(-\frac{2^{\frac{1}{3}}\left(\sqrt{\frac{-4y^{3}+27x}{x}}\,c_{1}3^{\frac{1}{6}}+2\left(y+\frac{3c_{1}}{2}\right)3^{\frac{2}{3}}\right)x\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{1}{3}}}{3} - \frac{2^{\frac{2}{3}}\cdots}{\cdots}\right.}{\left(2^{\frac{2}{3}}3^{\frac{1}{3}}xy+\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}}\right)^{2}\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}}yx+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(9+\sqrt{3}\cdots\right.\right.\right.} \tag{2}$$

$$x = -\frac{36\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}}\left(\left(\frac{8yc_{1}}{9}-\frac{2x}{3}\right)\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}} + x\left(-\frac{2^{\frac{1}{3}}\left(\left(i3^{\frac{2}{3}}+3^{\frac{1}{6}}\right)c_{1}\sqrt{\frac{-4\cdots}{\cdots}}\right.}{\cdots}\right.\right.}{\left(-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}}}{6} + x\left(2\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{1}{3}}\cdots\right.\right.} \tag{3}$$

$$x = \frac{36\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}} 3^{\frac{1}{3}} 2^{\frac{2}{3}} \left(\left(-\frac{8yc_{1}}{9}+\frac{2x}{3}\right)\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}} + x\left(-\frac{2^{\frac{1}{3}}\left(\left(i3^{\frac{2}{3}}-3^{\frac{1}{6}}\right)c_{1}\cdots\right.}{\cdots}\right.\right.}{\left(-\frac{\left(3^{\frac{1}{3}}+i3^{\frac{5}{6}}\right)2^{\frac{2}{3}}\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{2}{3}}}{6} + x\left(-2\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^{3}+27x}{x}}\right)x^{2}\right)^{\frac{1}{3}}\cdots\right.\right.} \tag{4}$$

Verification of solutions

$$y = x - 1$$

Verified OK.

$$x = -\frac{54\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}} 3^{\frac{1}{3}} 2^{\frac{2}{3}} x^3 \left(-\frac{2^{\frac{1}{3}}\left(\sqrt{\frac{-4y^3+27x}{x}}\,c_1 3^{\frac{1}{6}}+2\left(y+\frac{3c_1}{2}\right)3^{\frac{2}{3}}\right)x\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}}{3} - 2^{\frac{2}{3}}\right.}{\left(2^{\frac{2}{3}}3^{\frac{1}{3}}xy+\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}\right)^2\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}}yx+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(9+\sqrt{3}\right.\right.\right.}$$

Warning, solution could not be verified

$$x = -\frac{36\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}\left(\left(\frac{8yc_1}{9}-\frac{2x}{3}\right)\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}+x\left(-\frac{2^{\frac{1}{3}}\left(\left(i3^{\frac{2}{3}}+3^{\frac{1}{6}}\right)c_1\sqrt{\frac{-4}{}}\right.}{}\right.\right.}{\left(-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}}{6}+x\left(2\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}\right.\right.}$$

Warning, solution could not be verified

$$x = \frac{36\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}} 3^{\frac{1}{3}} 2^{\frac{2}{3}}\left(\left(-\frac{8yc_1}{9}+\frac{2x}{3}\right)\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}+x\left(-\frac{2^{\frac{1}{3}}\left(\left(i3^{\frac{2}{3}}-3^{\frac{1}{6}}\right)c_1\right.}{}\right.\right.}{\left(-\frac{\left(3^{\frac{1}{3}}+i3^{\frac{5}{6}}\right)2^{\frac{2}{3}}\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{2}{3}}}{6}+x\left(-2\left(\left(9+\sqrt{3}\,\sqrt{\frac{-4y^3+27x}{x}}\right)x^2\right)^{\frac{1}{3}}\right.\right.}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 1835

```
dsolve(y(x)=x*diff(y(x),x)^2-1/diff(y(x),x),y(x), singsol=all)
```

$$\frac{12x^3\left(2\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}y(x)+x\left(\frac{2^{\frac{1}{3}}\left(3^{\frac{1}{6}}\sqrt{\frac{-4y(x)^3+27x}{x}}+3\,3^{\frac{2}{3}}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}}{2}+2^{\frac{2}{3}}3^{\frac{1}{3}}\right.\right.}{\left(2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}+2x\left(2^{\frac{1}{3}}3^{\frac{2}{3}}y(x)-3\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}\right)\right)^2\left(y(x)\right.}$$

$$+x$$

$$-\frac{18x^4\left(2^{\frac{2}{3}}3^{\frac{5}{6}}\sqrt{\frac{-4y(x)^3+27x}{x}}\,x+2y(x)\,3^{\frac{2}{3}}2^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}+9\,3^{\frac{1}{3}}2^{\frac{2}{3}}x-3\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3}{x}}\right.\right.\right.}{\left(2y(x)\,3^{\frac{2}{3}}2^{\frac{1}{3}}x+2^{\frac{2}{3}}3^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}-6x\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}\right)^2\left(\right.}$$

$$=0$$

$$-\frac{3x^3\left(\frac{8\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}y(x)}{9}+x\left(\left(\left(\frac{i3^{\frac{2}{3}}}{9}-\frac{3^{\frac{1}{6}}}{9}\right)\sqrt{\frac{-4y(x)^3+27x}{x}}+i3^{\frac{1}{6}}-\frac{3^{\frac{2}{3}}}{3}\right)2^{\frac{1}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3}{x}}\right.\right.\right.\right.}{2\left(\left(i-\sqrt{3}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+\left(i3^{\frac{1}{3}}+3^{\frac{5}{6}}\right)2^{\frac{2}{3}}y(x)\,x\right)^2\left(-\frac{\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+\right.\right.}{6}\right.}$$

$$+x$$

$$+\frac{216x^4\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}3^{\frac{1}{3}}2^{\frac{2}{3}}\left(-\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+y(x)\left(i3^{\frac{1}{6}}-\frac{3^{\frac{2}{3}}}{3}\right)2^{\frac{1}{3}}\left(\right.\right.}{\left(\left(1+i\sqrt{3}\right)\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+\left(-i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)xy(x)\,2^{\frac{2}{3}}\right)^2\left(\frac{\left(-3^{\frac{5}{6}}+i3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}\right.\right.}{2}\right.}$$

$$=0$$

$$\frac{3x^3\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}c_1\left(-\frac{8\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}y(x)}{9}+x\left(\left(\left(\frac{i3^{\frac{2}{3}}}{9}+\frac{3^{\frac{1}{6}}}{9}\right)\sqrt{\frac{-4y(x)^3+27x}{x}}\right.\right.\right.}{2\left(-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}}{6}+\left(2\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{1}{3}}+2^{\frac{1}{3}}y(x)\left(i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)\right)x\right.}$$

$$+x$$

$$216x^4\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)^2x^4\right)^{\frac{1}{3}}3^{\frac{1}{3}}2^{\frac{2}{3}}\left(\left(\left(\sqrt{3}\sqrt{\frac{-4y(x)^3+27x}{x}}+9\right)x^2\right)^{\frac{2}{3}}+y(x)\left(i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)2^{\frac{1}{3}}\left(\left(\right.\right.\right.$$

✓ Solution by Mathematica

Time used: 145.256 (sec). Leaf size: 19969

```
DSolve[y[x]==x*y'[x]^2-1/y'[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

9.5 problem 224

Internal problem ID [15112]
Internal file name [`OUTPUT/15112_Sunday_April_21_2024_01_32_38_PM_10092544/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 224.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[_dAlembert]`

$$\boxed{y - \frac{3xy'}{2} - e^{y'} = 0}$$

9.5.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - \frac{3xp}{2} - e^{p} = 0$$

Solving for y from the above results in

$$y = \frac{3xp}{2} + e^{p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{3p}{2}$$
$$g = e^{p}$$

Hence (2) becomes

$$-\frac{p}{2} = \left(\frac{3x}{2} + e^{p}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-\frac{p}{2} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 1$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2\left(\frac{3x}{2} + e^{p(x)}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2\left(\frac{3x(p)}{2} + e^{p}\right)}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{3}{p}$$
$$q(p) = -\frac{2\,e^{p}}{p}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{3x(p)}{p} = -\frac{2\,e^{p}}{p}$$

The integrating factor μ is

$$\mu = e^{\int \frac{3}{p}dp}$$
$$= p^3$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) = (\mu)\left(-\frac{2\,e^{p}}{p}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}p}(p^3 x) = (p^3)\left(-\frac{2\,e^{p}}{p}\right)$$
$$\mathrm{d}(p^3 x) = \left(-2p^2 e^{p}\right)\,\mathrm{d}p$$

Integrating gives

$$p^3 x = \int -2p^2 e^{p}\,\mathrm{d}p$$
$$p^3 x = -2\left(p^2 - 2p + 2\right)e^{p} + c_1$$

Dividing both sides by the integrating factor $\mu = p^3$ results in

$$x(p) = -\frac{2(p^2 - 2p + 2)\,e^{p}}{p^3} + \frac{c_1}{p^3}$$

which simplifies to

$$x(p) = \frac{(-2p^2 + 4p - 4)\,e^{p} + c_1}{p^3}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$p = -\operatorname{LambertW}\left(\frac{2\,e^{\frac{2y}{3x}}}{3x}\right) + \frac{2y}{3x}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{27\left(\left(-2\operatorname{LambertW}\left(\frac{2\,e^{\frac{2y}{3x}}}{3x}\right)^2 x^2 - 4x\left(x - \frac{2y}{3}\right)\operatorname{LambertW}\left(\frac{2\,e^{\frac{2y}{3x}}}{3x}\right) - 4x^2 + \frac{8yx}{3} - \frac{8y^2}{9}\right)e^{-\frac{3x\operatorname{LambertW}(}{3x}}}{\left(3x\operatorname{LambertW}\left(\frac{2\,e^{\frac{2y}{3x}}}{3x}\right) - 2y\right)^3}$$

Summary

The solution(s) found are the following

$$y = 1 \tag{1}$$

$$x = -\frac{27\left(\left(-2\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y}{3x}}}{3x}\right)^{2} x^{2} - 4x\left(x - \frac{2y}{3}\right)\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y}{3x}}}{3x}\right) - 4x^{2} + \frac{8yx}{3} - \frac{8y^{2}}{9}\right)\mathrm{e}^{-\frac{3x\,\mathrm{LambertW}\left(\right.}{3x}}}{\left(3x\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y}{3x}}}{3x}\right) - 2y\right)^{3}} \tag{2}$$

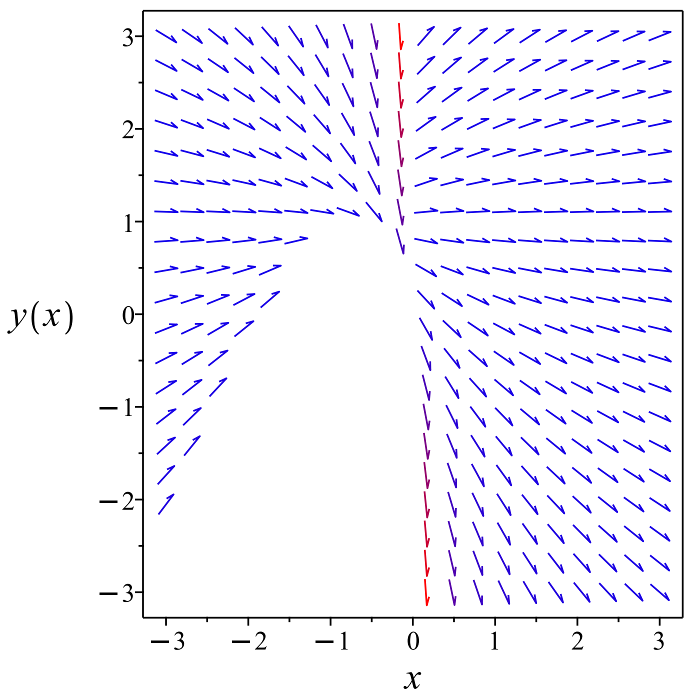


Figure 331: Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

$x =$

$$-\frac{27\left(\left(-2\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y}{3x}}}{3x}\right)^{2} x^{2}-4x\left(x-\frac{2y}{3}\right)\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y}{3x}}}{3x}\right)-4x^{2}+\frac{8yx}{3}-\frac{8y^{2}}{9}\right)\mathrm{e}^{-\frac{3x\,\mathrm{LambertW}\left(\right.}{3x}}}{\left(3x\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y}{3x}}}{3x}\right)-2y\right)^{3}}$$

Warning, solution could not be verified

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 201

```
dsolve(y(x)=3/2*x*diff(y(x),x)+exp(diff(y(x),x)),y(x), singsol=all)
```

$$y(x) = 1$$

$$\frac{27\left(\left(-2x^{2}\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y(x)}{3x}}}{3x}\right)^{2}-4\left(x-\frac{2y(x)}{3}\right)x\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{2y(x)}{3x}}}{3x}\right)-4x^{2}+\frac{8xy(x)}{3}-\frac{8y(x)^{2}}{9}\right)\mathrm{e}^{\frac{-3x\,\mathrm{La}}{}}}{\left(3x\,\mathrm{Lan}\right.}$$

$$= 0$$

✓ Solution by Mathematica

Time used: 0.567 (sec). Leaf size: 52

```
DSolve[y[x]==3/2*x*y'[x]+Exp[y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\left\{x=-\frac{2e^{K[1]}(K[1]^2-2K[1]+2)}{K[1]^3}+\frac{c_1}{K[1]^3},y(x)=\frac{3}{2}xK[1]+e^{K[1]}\right\},\{y(x),K[1]\}\right]$$

9.6 problem 225

Internal problem ID [15113]
Internal file name [`OUTPUT/15113_Sunday_April_21_2024_01_32_40_PM_42959583/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 225.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - \frac{a}{y'^2} = 0}$$

9.6.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - \frac{a}{p^2} = 0$$

Solving for y from the above results in

$$y = \frac{x\,p^3 + a}{p^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \frac{a}{p^2} \\ &= xp + \frac{a}{p^2} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \frac{a}{p^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{a}{c_1^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{a}{p^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{2a}{p^3} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{2^{\frac{1}{3}} \left(a\, x^{2}\right)^{\frac{1}{3}}}{x} \\
p_2 &= -\frac{2^{\frac{1}{3}} \left(a\, x^{2}\right)^{\frac{1}{3}}}{2 x}+\frac{i \sqrt{3}\, 2^{\frac{1}{3}} \left(a\, x^{2}\right)^{\frac{1}{3}}}{2 x} \\
p_3 &= -\frac{2^{\frac{1}{3}} \left(a\, x^{2}\right)^{\frac{1}{3}}}{2 x}-\frac{i \sqrt{3}\, 2^{\frac{1}{3}} \left(a\, x^{2}\right)^{\frac{1}{3}}}{2 x}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= \frac{3 a\, x^{2} 2^{\frac{1}{3}}}{2 \left(a\, x^{2}\right)^{\frac{2}{3}}} \\
y_2 &= -\frac{3 a\, x^{2} 2^{\frac{1}{3}}}{\left(a\, x^{2}\right)^{\frac{2}{3}} \left(1+i \sqrt{3}\right)} \\
y_3 &= \frac{3 a\, x^{2} 2^{\frac{1}{3}}}{\left(a\, x^{2}\right)^{\frac{2}{3}} \left(i \sqrt{3}-1\right)}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x +\frac{a}{c_1^{2}} \tag{1}
$$

$$
y = \frac{3 a\, x^{2} 2^{\frac{1}{3}}}{2 \left(a\, x^{2}\right)^{\frac{2}{3}}} \tag{2}
$$

$$
y = -\frac{3 a\, x^{2} 2^{\frac{1}{3}}}{\left(a\, x^{2}\right)^{\frac{2}{3}} \left(1+i \sqrt{3}\right)} \tag{3}
$$

$$
y = \frac{3 a\, x^{2} 2^{\frac{1}{3}}}{\left(a\, x^{2}\right)^{\frac{2}{3}} \left(i \sqrt{3}-1\right)} \tag{4}
$$

Verification of solutions

$$y = c_1 x + \frac{a}{c_1^2}$$

Verified OK.

$$y = \frac{3a\,x^2 2^{\frac{1}{3}}}{2\left(a\,x^2\right)^{\frac{2}{3}}}$$

Verified OK.

$$y = -\frac{3a\,x^2 2^{\frac{1}{3}}}{\left(a\,x^2\right)^{\frac{2}{3}}\left(1 + i\sqrt{3}\right)}$$

Verified OK.

$$y = \frac{3a\,x^2 2^{\frac{1}{3}}}{\left(a\,x^2\right)^{\frac{2}{3}}\left(i\sqrt{3} - 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 76

```
dsolve(y(x)=x*diff(y(x),x)+a/diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = \frac{3\, 2^{\frac{1}{3}} (a\, x^2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{3\, 2^{\frac{1}{3}} (a\, x^2)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{3\, 2^{\frac{1}{3}} (a\, x^2)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{4}$$
$$y(x) = \frac{c_1^3 x + a}{c_1^2}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 89

```
DSolve[y[x]==x*y'[x]+a/y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{a}{c_1{}^2} + c_1 x$$
$$y(x) \to \frac{3\sqrt[3]{a} x^{2/3}}{2^{2/3}}$$
$$y(x) \to -\frac{3\sqrt[3]{-1}\sqrt[3]{a} x^{2/3}}{2^{2/3}}$$
$$y(x) \to \frac{3(-1)^{2/3}\sqrt[3]{a} x^{2/3}}{2^{2/3}}$$

9.7 problem 226

Internal problem ID [15114]
Internal file name [`OUTPUT/15114_Sunday_April_21_2024_01_32_41_PM_71019088/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 226.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - y'^2 = 0}$$

9.7.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^2 + xp \\ &= p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x + 2p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = -\frac{x^2}{4}$$

Summary

The solution(s) found are the following

$$y = c_1^2 + c_1 x \tag{1}$$

$$y = -\frac{x^2}{4} \tag{2}$$

Verification of solutions

$$y = c_1^2 + c_1 x$$

Verified OK.

$$y = -\frac{x^2}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(y(x)=x*diff(y(x),x)+diff(y(x),x)^2,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -\frac{x^2}{4} \\
y(x) &= c_1(x + c_1)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 23

```
DSolve[y[x]==x*y'[x]+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to c_1(x + c_1) \\
y(x) &\to -\frac{x^2}{4}
\end{aligned}
$$

9.8 problem 227

Internal problem ID [15115]
Internal file name [`OUTPUT/15115_Sunday_April_21_2024_01_32_41_PM_11718234/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 227.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{xy'^2 - yy' - y' = -1}$$

9.8.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$x\,p^2 - yp - p = -1$$

Solving for y from the above results in

$$y = \frac{x\,p^2 - p + 1}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px - \frac{p-1}{p} \\ &= px - \frac{p-1}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\frac{p-1}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp+g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x+g')\frac{dp}{dx} \\ 0 &= (x+g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \frac{c_1 - 1}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\frac{p-1}{p}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{1}{p} + \frac{p-1}{p^2} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{1}{\sqrt{x}} \\
p_2 &= -\frac{1}{\sqrt{x}}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= 2\sqrt{x} - 1 \\
y_2 &= -1 - 2\sqrt{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x - \frac{c_1 - 1}{c_1} \tag{1}
$$

$$
y = 2\sqrt{x} - 1 \tag{2}
$$

$$
y = -1 - 2\sqrt{x} \tag{3}
$$

Verification of solutions

$$
y = c_1 x - \frac{c_1 - 1}{c_1}
$$

Verified OK.

$$
y = 2\sqrt{x} - 1
$$

Verified OK.

$$
y = -1 - 2\sqrt{x}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 38

```
dsolve(x*diff(y(x),x)^2-y(x)*diff(y(x),x)-diff(y(x),x)+1=0,y(x), singsol=all)
```

$$y(x) = -1 - 2\sqrt{x}$$
$$y(x) = -1 + 2\sqrt{x}$$
$$y(x) = \frac{c_1^2 x - c_1 + 1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 46

```
DSolve[x*y'[x]^2-y[x]*y'[x]-y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - 1 + \frac{1}{c_1}$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to -2\sqrt{x} - 1$$
$$y(x) \to 2\sqrt{x} - 1$$

9.9 problem 228

Internal problem ID [15116]
Internal file name [`OUTPUT/15116_Sunday_April_21_2024_01_32_42_PM_43690813/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 228.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y - xy' - a\sqrt{1 + y'^2} = 0}$$

9.9.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - a\sqrt{p^2 + 1} = 0$$

Solving for y from the above results in

$$y = a\sqrt{p^2 + 1} + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= a\sqrt{p^2 + 1} + xp \\ &= a\sqrt{p^2 + 1} + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = a\sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + a\sqrt{c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = a\sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{ap}{\sqrt{p^2 + 1}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{\sqrt{a^2 - x^2}}$$
$$p_2 = -\frac{x}{\sqrt{a^2 - x^2}}$$

Substituting the above back in (1) results in

$$y_1 = \frac{a\sqrt{\frac{a^2}{a^2-x^2}}\,\sqrt{a^2 - x^2} + x^2}{\sqrt{a^2 - x^2}}$$
$$y_2 = \frac{a\sqrt{\frac{a^2}{a^2-x^2}}\,\sqrt{a^2 - x^2} - x^2}{\sqrt{a^2 - x^2}}$$

Summary

The solution(s) found are the following

$$y = c_1 x + a\sqrt{c_1^2 + 1} \tag{1}$$
$$y = \frac{a\sqrt{\frac{a^2}{a^2-x^2}}\,\sqrt{a^2 - x^2} + x^2}{\sqrt{a^2 - x^2}} \tag{2}$$
$$y = \frac{a\sqrt{\frac{a^2}{a^2-x^2}}\,\sqrt{a^2 - x^2} - x^2}{\sqrt{a^2 - x^2}} \tag{3}$$

Verification of solutions

$$y = c_1 x + a\sqrt{c_1^2 + 1}$$

Verified OK.

$$y = \frac{a\sqrt{\frac{a^2}{a^2-x^2}}\,\sqrt{a^2 - x^2} + x^2}{\sqrt{a^2 - x^2}}$$

Verified OK.

$$y = \frac{a\sqrt{\frac{a^2}{a^2-x^2}}\,\sqrt{a^2 - x^2} - x^2}{\sqrt{a^2 - x^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 17

```
dsolve(y(x)=x*diff(y(x),x)+a*sqrt(1+diff(y(x),x)^2),y(x), singsol=all)
```

$$y(x) = c_1 x + a\sqrt{c_1^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 27

```
DSolve[y[x]==x*y'[x]+a*Sqrt[1+y'[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to a\sqrt{1 + c_1{}^2} + c_1 x$$
$$y(x) \to a$$

9.10 problem 229

Internal problem ID [15117]
Internal file name [`OUTPUT/15117_Sunday_April_21_2024_01_32_43_PM_88161217/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 8.3. The Lagrange and Clairaut equations. Exercises page 72
Problem number: 229.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_homogeneous, `class G`], _rational, _Clairaut]`

$$\boxed{-\frac{1}{y'^2} = -x + \frac{y}{y'}}$$

9.10.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-\frac{1}{p^2} = -x + \frac{y}{p}$$

Solving for y from the above results in

$$y = \frac{x\,p^2 - 1}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px - \frac{1}{p} \\ &= px - \frac{1}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = -\frac{1}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \frac{1}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\frac{1}{p}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{1}{p^2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = -\frac{1}{\sqrt{-x}}$$
$$p_2 = \frac{1}{\sqrt{-x}}$$

Substituting the above back in (1) results in

$$y_1 = 2\sqrt{-x}$$
$$y_2 = -2\sqrt{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 x - \frac{1}{c_1} \tag{1}$$
$$y = 2\sqrt{-x} \tag{2}$$
$$y = -2\sqrt{-x} \tag{3}$$

Verification of solutions

$$y = c_1 x - \frac{1}{c_1}$$

Verified OK.

$$y = 2\sqrt{-x}$$

Verified OK.

$$y = -2\sqrt{-x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(x=y(x)/diff(y(x),x)+1/diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -2\sqrt{-x}$$
$$y(x) = 2\sqrt{-x}$$
$$y(x) = c_1x - \frac{1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 47

```
DSolve[x==y[x]/y'[x]+1/y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1x - \frac{1}{c_1}$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to -2i\sqrt{x}$$
$$y(x) \to 2i\sqrt{x}$$

10 Section 9. The Riccati equation. Exercises page 75

10.1 problem 232

Internal problem ID [15118]
Internal file name [OUTPUT/15118_Sunday_April_21_2024_01_32_44_PM_32782612/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 9. The Riccati equation. Exercises page 75
Problem number: 232.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' \mathrm{e}^{-x} + y^2 - 2y\,\mathrm{e}^{x} = 1 - \mathrm{e}^{2x}}$$

10.1.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\left(y^2 - 2y\,\mathrm{e}^{x} - 1 + \mathrm{e}^{2x}\right)\mathrm{e}^{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2y\,\mathrm{e}^{2x} - \mathrm{e}^{x} y^2 - \mathrm{e}^{3x} + \mathrm{e}^{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -(-1 + \mathrm{e}^{2x})\,\mathrm{e}^{x}$, $f_1(x) = 2(\mathrm{e}^{x})^2$ and $f_2(x) = -\mathrm{e}^{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\mathrm{e}^{x} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -e^{x} \\ f_1 f_2 &= -2\, e^{x} e^{2x} \\ f_2^2 f_0 &= -e^{3x} \left(-1 + e^{2x}\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-e^{x} u''(x) - \left(-e^{x} - 2\, e^{x} e^{2x}\right) u'(x) - e^{3x} \left(-1 + e^{2x}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = e^{\frac{x}{2} + \frac{e^{2x}}{2}} \left(c_1 \sinh\left(\frac{x}{2}\right) + c_2 \cosh\left(\frac{x}{2}\right)\right)$$

The above shows that

$$u'(x) = \left(\left(e^{2x} c_2 + \frac{c_1}{2} + \frac{c_2}{2}\right) \cosh\left(\frac{x}{2}\right) + \sinh\left(\frac{x}{2}\right) \left(c_1 e^{2x} + \frac{c_1}{2} + \frac{c_2}{2}\right)\right) e^{\frac{x}{2} + \frac{e^{2x}}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(\left(e^{2x} c_2 + \frac{c_1}{2} + \frac{c_2}{2}\right) \cosh\left(\frac{x}{2}\right) + \sinh\left(\frac{x}{2}\right) \left(c_1 e^{2x} + \frac{c_1}{2} + \frac{c_2}{2}\right)\right) e^{-x}}{c_1 \sinh\left(\frac{x}{2}\right) + c_2 \cosh\left(\frac{x}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left((c_3 + 1)\, e^{-x} + 2\, e^{x}\right) \cosh\left(\frac{x}{2}\right) + 2 \sinh\left(\frac{x}{2}\right) \left(\left(\frac{c_3}{2} + \frac{1}{2}\right) e^{-x} + e^{x} c_3\right)}{2 c_3 \sinh\left(\frac{x}{2}\right) + 2 \cosh\left(\frac{x}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left((c_3 + 1)\, e^{-x} + 2\, e^{x}\right) \cosh\left(\frac{x}{2}\right) + 2 \sinh\left(\frac{x}{2}\right) \left(\left(\frac{c_3}{2} + \frac{1}{2}\right) e^{-x} + e^{x} c_3\right)}{2 c_3 \sinh\left(\frac{x}{2}\right) + 2 \cosh\left(\frac{x}{2}\right)} \tag{1}$$

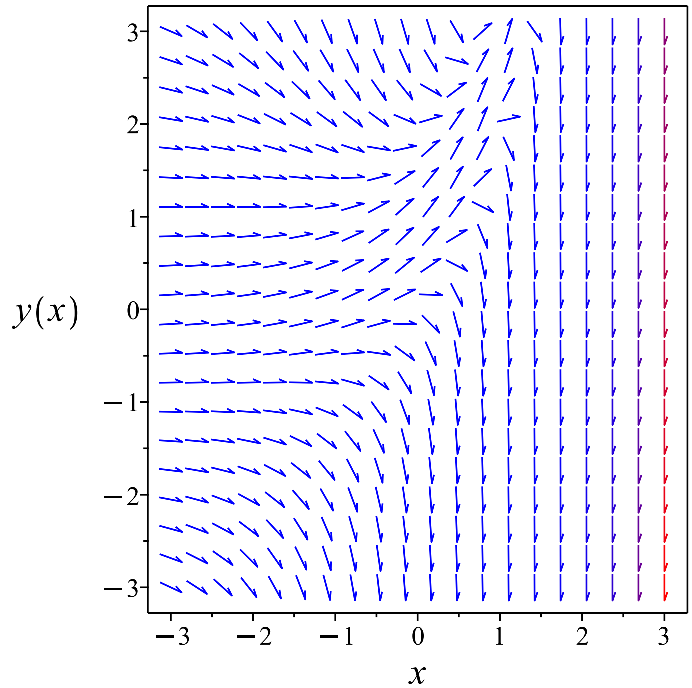


Figure 332: Slope field plot

Verification of solutions

$$y = \frac{\left(\left(c_3+1\right)\mathrm{e}^{-x}+2\,\mathrm{e}^{x}\right)\cosh\left(\frac{x}{2}\right)+2\sinh\left(\frac{x}{2}\right)\left(\left(\frac{c_3}{2}+\frac{1}{2}\right)\mathrm{e}^{-x}+\mathrm{e}^{x}c_3\right)}{2c_3\sinh\left(\frac{x}{2}\right)+2\cosh\left(\frac{x}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)*exp(-x)+y(x)^2-2*y(x)*exp(x)=1-exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x} + e^{2x}c_1 + c_1}{e^{x}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 0.311 (sec). Leaf size: 24

```
DSolve[y'[x]*Exp[-x]+y[x]^2-2*y[x]*Exp[x]==1-Exp[2*x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^x + \frac{1}{e^x + c_1}$$
$$y(x) \to e^x$$

10.2 problem 233

Internal problem ID [15119]
Internal file name [`OUTPUT/15119_Sunday_April_21_2024_01_32_46_PM_81085812/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 9. The Riccati equation. Exercises page 75
Problem number: 233.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]`

$$\boxed{y'+y^2-2y\sin(x)=-\sin(x)^2+\cos(x)}$$

10.2.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -y^2+2y\sin(x)-\sin(x)^2+\cos(x) \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2+2y\sin(x)-\sin(x)^2+\cos(x)$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=-\sin(x)^2+\cos(x)$, $f_1(x)=2\sin(x)$ and $f_2(x)=-1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= -2\sin(x) \\ f_2^2f_0 &= -\sin(x)^2 + \cos(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + 2\sin(x)\,u'(x) + \left(-\sin(x)^2 + \cos(x)\right)u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\cos(x)}(c_2x + c_1)$$

The above shows that

$$u'(x) = \mathrm{e}^{-\cos(x)}(\sin(x)\,(c_2x + c_1) + c_2)$$

Using the above in (1) gives the solution

$$y = \frac{\sin(x)\,(c_2x + c_1) + c_2}{c_2x + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sin(x)\,(c_3 + x) + 1}{c_3 + x}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)\,(c_3 + x) + 1}{c_3 + x} \tag{1}$$

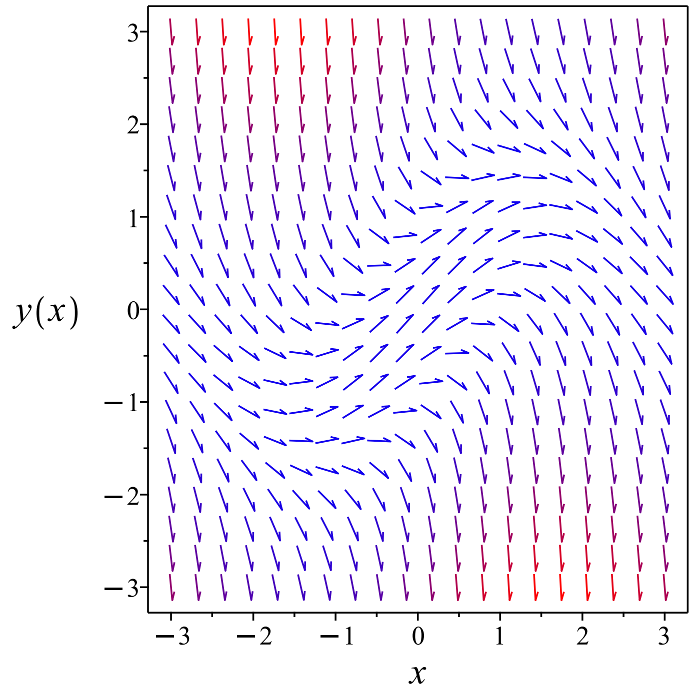


Figure 333: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sin(x)(c_3 + x) + 1}{c_3 + x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular polynomial solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+y(x)^2-2*y(x)*sin(x)+sin(x)^2-cos(x)=0,y(x), singsol=all)
```

$$y(x) = \sin(x) + \frac{1}{x - c_1}$$

✓ Solution by Mathematica

Time used: 0.257 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]^2-2*y[x]*Sin[x]+Sin[x]^2-Cos[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \sin(x) + \frac{1}{x + c_1}$$
$$y(x) \to \sin(x)$$

10.3 problem 234

Internal problem ID [15120]
Internal file name [`OUTPUT/15120_Sunday_April_21_2024_01_32_49_PM_73796204/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 9. The Riccati equation. Exercises page 75
Problem number: 234.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational, _Riccati]
```

$$\boxed{xy' - y^2 + (2x+1)\,y = x^2 + 2x}$$

10.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x^2 - 2xy + y^2 + 2x - y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(x^2-2xy+y^2+2x-y)(b_3-a_2)}{x} - \frac{(x^2-2xy+y^2+2x-y)^2 a_3}{x^2} \\
&- \left(\frac{2x-2y+2}{x} - \frac{x^2-2xy+y^2+2x-y}{x^2}\right)(xa_2+ya_3+a_1) \\
&- \frac{(2y-2x-1)(xb_2+yb_3+b_1)}{x} = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_3 - 4x^3ya_3 + 6x^2y^2a_3 - 4x\,y^3a_3 + y^4a_3 + 2x^3a_2 + 4x^3a_3 - 2x^3b_2 - x^3b_3 - 2x^2ya_2 - 9x^2ya_3 + 2x^2yb}{}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^4a_3 + 4x^3ya_3 - 6x^2y^2a_3 + 4x\,y^3a_3 - y^4a_3 - 2x^3a_2 - 4x^3a_3 + 2x^3b_2 + x^3b_3 \\
&+ 2x^2ya_2 + 9x^2ya_3 - 2x^2yb_2 - 8x\,y^2a_3 - x\,y^2b_3 + 3y^3a_3 - x^2a_1 - 2x^2a_2 - 4x^2a_3 \\
&+ 2x^2b_1 + 2b_2x^2 + 2x^2b_3 + 4xya_3 - 2xyb_1 + y^2a_1 - 2y^2a_3 + xb_1 - ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_3v_1^4 + 4a_3v_1^3v_2 - 6a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 - 2a_2v_1^3 + 2a_2v_1^2v_2 \\
&- 4a_3v_1^3 + 9a_3v_1^2v_2 - 8a_3v_1v_2^2 + 3a_3v_2^3 + 2b_2v_1^3 - 2b_2v_1^2v_2 \\
&+ b_3v_1^3 - b_3v_1v_2^2 - a_1v_1^2 + a_1v_2^2 - 2a_2v_1^2 - 4a_3v_1^2 + 4a_3v_1v_2 \\
&- 2a_3v_2^2 + 2b_1v_1^2 - 2b_1v_1v_2 + 2b_2v_1^2 + 2b_3v_1^2 - a_1v_2 + b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-a_3v_1^4+4a_3v_1^3v_2+(-2a_2-4a_3+2b_2+b_3)\,v_1^3-6a_3v_1^2v_2^2+(2a_2+9a_3-2b_2)\,v_1^2v_2 \\
&+(-a_1-2a_2-4a_3+2b_1+2b_2+2b_3)\,v_1^2+4a_3v_1v_2^3+(-8a_3-b_3)\,v_1v_2^2 \\
&+(4a_3-2b_1)\,v_1v_2+b_1v_1-a_3v_2^4+3a_3v_2^3+(a_1-2a_3)\,v_2^2-a_1v_2=0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
-a_1 &= 0 \\
-6a_3 &= 0 \\
-a_3 &= 0 \\
3a_3 &= 0 \\
4a_3 &= 0 \\
a_1-2a_3 &= 0 \\
-8a_3-b_3 &= 0 \\
4a_3-2b_1 &= 0 \\
2a_2+9a_3-2b_2 &= 0 \\
-2a_2-4a_3+2b_2+b_3 &= 0 \\
-a_1-2a_2-4a_3+2b_1+2b_2+2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= x
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= x - \left(\frac{x^2 - 2xy + y^2 + 2x - y}{x}\right)(x) \\ &= -x^2 + 2xy - y^2 - x + y \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^2 + 2xy - y^2 - x + y} dy\end{aligned}$$

Which results in

$$S = -\ln(-1 - x + y) + \ln(-x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 - 2xy + y^2 + 2x - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{(x-y+1)(x-y)} \\
S_y &= -\frac{1}{(x-y+1)(x-y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(-1-x+y) + \ln(-x+y) = -\ln(x) + c_1$$

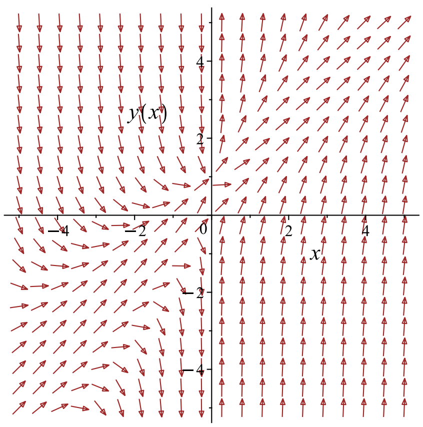
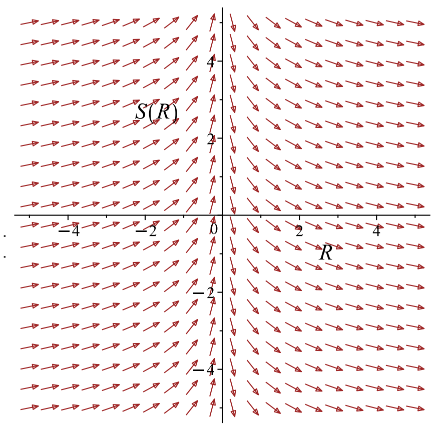
Which simplifies to

$$-\ln(-1-x+y) + \ln(-x+y) = -\ln(x) + c_1$$

Which gives

$$y = \frac{e^{c_1}x - x^2 + e^{c_1}}{e^{c_1} - x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2-2xy+y^2+2x-y}{x}$ 	$R = x$ $S = -\ln(-1-x+y) +$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{c_1} x - x^2 + e^{c_1}}{e^{c_1} - x} \tag{1}$$

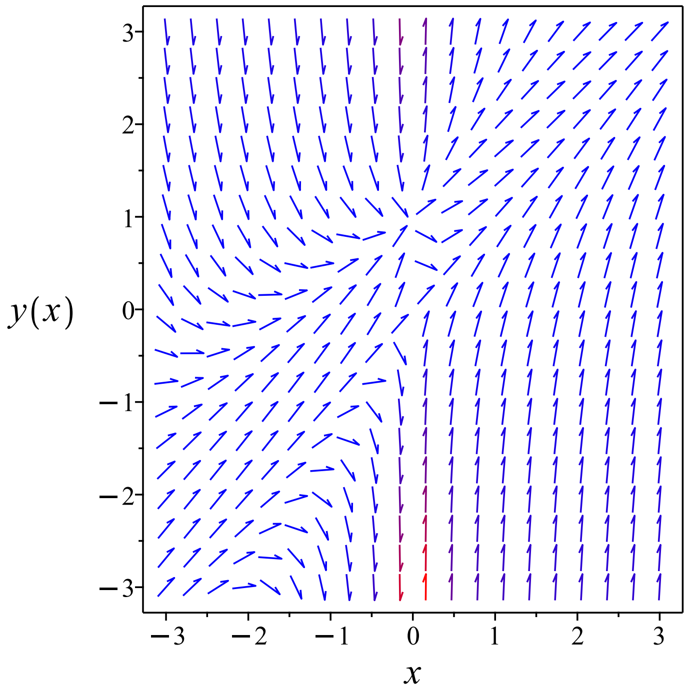


Figure 334: Slope field plot

Verification of solutions

$$y = \frac{e^{c_1} x - x^2 + e^{c_1}}{e^{c_1} - x}$$

Verified OK.

10.3.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 - 2xy + y^2 + 2x - y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x - 2y + \frac{y^2}{x} + 2 - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{x^2+2x}{x}$, $f_1(x) = \frac{-2x-1}{x}$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= \frac{-2x-1}{x^2} \\ f_2^2 f_0 &= \frac{x^2+2x}{x^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} - \left(-\frac{1}{x^2} + \frac{-2x-1}{x^2}\right) u'(x) + \frac{(x^2+2x)\, u(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{\mathrm{e}^{-x}(c_1 x + c_2)}{x}$$

The above shows that

$$u'(x) = -\frac{\mathrm{e}^{-x}(c_1 x^2 + c_2 x + c_2)}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_1 x^2 + c_2 x + c_2}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 x^2 + x + 1}{c_3 x + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{c_3 x^2 + x + 1}{c_3 x + 1} \tag{1}$$

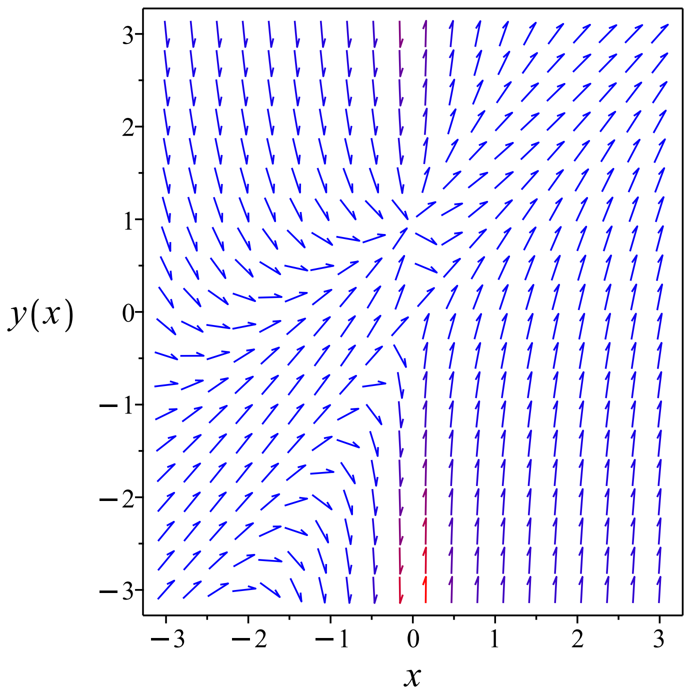


Figure 335: Slope field plot

Verification of solutions

$$y = \frac{c_3 x^2 + x + 1}{c_3 x + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x)-y(x)^2+(2*x+1)*y(x)=x^2+2*x,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 - x - 1}{c_1 x - 1}$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 34

```
DSolve[x*y'[x]-y[x]^2+(2*x+1)*y[x]==x^2+2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 - c_1 x - c_1}{x - c_1}$$
$$y(x) \to x + 1$$

10.4 problem 235

Internal problem ID [15121]
Internal file name [`OUTPUT/15121_Sunday_April_21_2024_01_32_50_PM_50767175/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 9. The Riccati equation. Exercises page 75
Problem number: 235.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{x^2y' - y^2x^2 - yx = 1}$$

10.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2y^2 + xy + 1}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(x^2y^2 + xy + 1)(b_3 - a_2)}{x^2} - \frac{(x^2y^2 + xy + 1)^2 a_3}{x^4} \\
&- \left(\frac{2x\, y^2 + y}{x^2} - \frac{2(x^2y^2 + xy + 1)}{x^3}\right)(xa_2 + ya_3 + a_1) \\
&- \frac{(2x^2y + x)(xb_2 + yb_3 + b_1)}{x^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^4y^4a_3 + 2x^5yb_2 + x^4y^2a_2 + x^4y^2b_3 + 2x^3y^3a_3 + 2x^4yb_1 + 2x^2y^2a_3 + x^3b_1 - x^2ya_1 - x^2a_2 - x^2b_3 - 2xa_1}{x^4}
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^4y^4a_3 - 2x^5yb_2 - x^4y^2a_2 - x^4y^2b_3 - 2x^3y^3a_3 - 2x^4yb_1 \\
&- 2x^2y^2a_3 - x^3b_1 + x^2ya_1 + x^2a_2 + x^2b_3 + 2xa_1 - a_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_3v_1^4v_2^4 - a_2v_1^4v_2^2 - 2a_3v_1^3v_2^3 - 2b_2v_1^5v_2 - b_3v_1^4v_2^2 - 2b_1v_1^4v_2 \\
&- 2a_3v_1^2v_2^2 + a_1v_1^2v_2 - b_1v_1^3 + a_2v_1^2 + b_3v_1^2 + 2a_1v_1 - a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} -2b_2v_1^5v_2 - a_3v_1^4v_2^4 + (-a_2 - b_3)\, v_1^4v_2^2 - 2b_1v_1^4v_2 - 2a_3v_1^3v_2^3 \\ - b_1v_1^3 - 2a_3v_1^2v_2^2 + a_1v_1^2v_2 + (a_2 + b_3)\, v_1^2 + 2a_1v_1 - a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -a_3 &= 0 \\ -2b_1 &= 0 \\ -b_1 &= 0 \\ -2b_2 &= 0 \\ -a_2 - b_3 &= 0 \\ a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x^2y^2 + xy + 1}{x^2}\right)(-x) \\
&= \frac{x^2y^2 + 2xy + 1}{x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2y^2 + 2xy + 1}{x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{xy + 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2y^2 + xy + 1}{x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(xy+1)^2} \\
S_y &= \frac{x}{(xy+1)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{1}{yx+1} = \ln(x) + c_1
$$

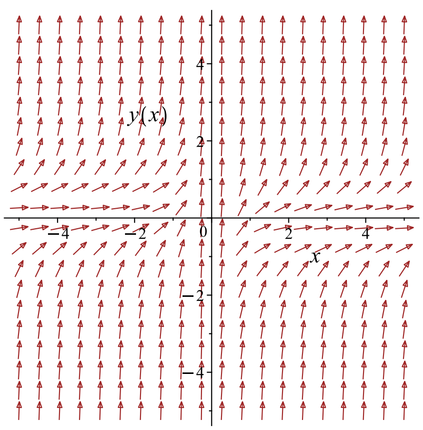
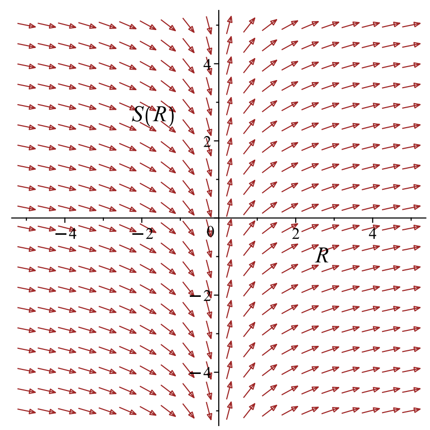
Which simplifies to

$$
-\frac{1}{yx+1} = \ln(x) + c_1
$$

Which gives

$$
y = -\frac{\ln(x) + c_1 + 1}{x(\ln(x) + c_1)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2y^2+xy+1}{x^2}$ 	$R = x$ $S = -\frac{1}{xy+1}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\ln(x) + c_1 + 1}{x(\ln(x) + c_1)} \tag{1}$$

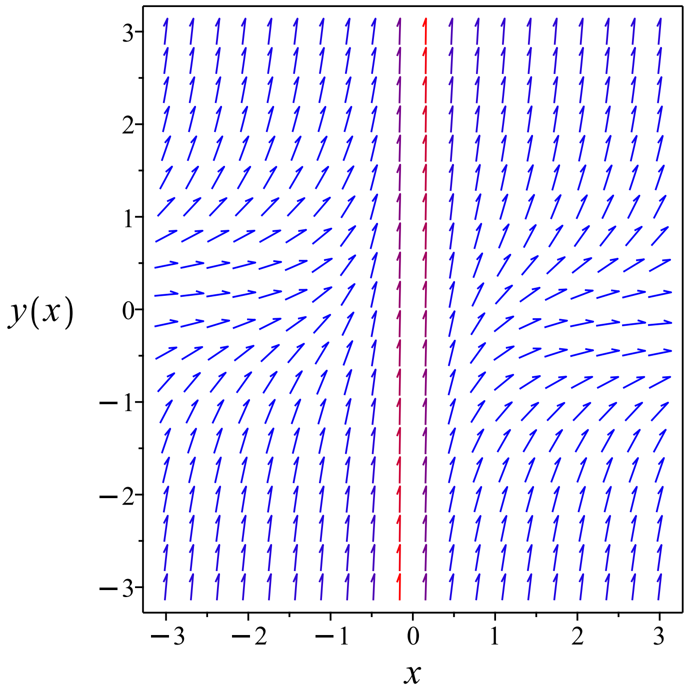


Figure 336: Slope field plot

Verification of solutions

$$y = -\frac{\ln(x) + c_1 + 1}{x(\ln(x) + c_1)}$$

Verified OK.

10.4.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x^2) \, \mathrm{d}y &= (x^2 y^2 + xy + 1) \, \mathrm{d}x \\
(-x^2 y^2 - xy - 1) \, \mathrm{d}x + (x^2) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 y^2 - xy - 1 \\
N(x, y) &= x^2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-x^2 y^2 - xy - 1) \\
&= -2x^2 y - x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x^2) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2}\left((-2x^2y - x) - (2x)\right) \\
&= \frac{-2xy - 3}{x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{x^2y^2 + xy + 1}\left((2x) - (-2x^2y - x)\right) \\
&= \frac{-2x^2y - 3x}{x^2y^2 + xy + 1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(2x) - (-2x^2y - x)}{x\left(-x^2y^2 - xy - 1\right) - y\left(x^2\right)} \\
&= \frac{-2xy - 3}{\left(xy + 1\right)^2}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t - 3}{\left(t + 1\right)^2}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-2t-3}{(t+1)^2}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{1}{t+1}-2\ln(t+1)}\\ &= \frac{\mathrm{e}^{\frac{1}{t+1}}}{(t+1)^2}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\left(-x^2y^2-xy-1\right)\\ &= -\frac{\left(x^2y^2+xy+1\right)\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\left(x^2\right)\\ &= \frac{x^2\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{\left(x^2y^2+xy+1\right)\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\right)+\left(\frac{x^2\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\frac{(x^2y^2+xy+1)\,\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}\,\mathrm{d}x$$

$$\phi = -x\,\mathrm{e}^{\frac{1}{xy+1}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{x^2\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{x^2\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2}$. Therefore equation (4) becomes

$$\frac{x^2\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2} = \frac{x^2\mathrm{e}^{\frac{1}{xy+1}}}{(xy+1)^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x\,\mathrm{e}^{\frac{1}{xy+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x\,\mathrm{e}^{\frac{1}{xy+1}}$$

The solution becomes

$$y = -\frac{\ln\left(-\frac{c_1}{x}\right) - 1}{x\ln\left(-\frac{c_1}{x}\right)}$$

Summary
The solution(s) found are the following

$$y = -\frac{\ln\left(-\frac{c_1}{x}\right) - 1}{x \ln\left(-\frac{c_1}{x}\right)} \tag{1}$$

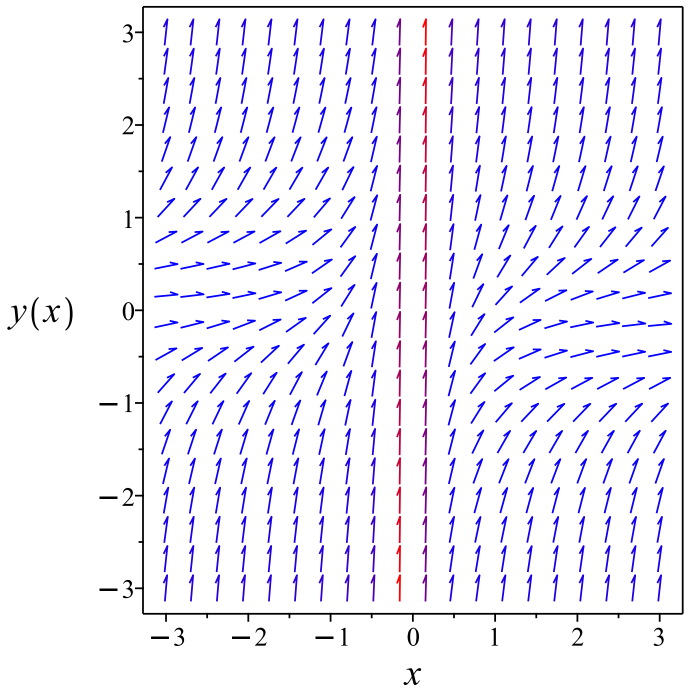


Figure 337: Slope field plot

Verification of solutions

$$y = -\frac{\ln\left(-\frac{c_1}{x}\right) - 1}{x \ln\left(-\frac{c_1}{x}\right)}$$

Verified OK.

10.4.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2y^2 + xy + 1}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{y}{x} + \frac{1}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2}$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{1}{x} \\ f_2^2 f_0 &= \frac{1}{x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{u'(x)}{x} + \frac{u(x)}{x^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = x(\ln(x)\, c_2 + c_1)$$

The above shows that

$$u'(x) = \ln(x)\, c_2 + c_1 + c_2$$

Using the above in (1) gives the solution

$$y = -\frac{\ln(x)\, c_2 + c_1 + c_2}{x(\ln(x)\, c_2 + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\ln(x) - c_3 - 1}{x(\ln(x) + c_3)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\ln(x) - c_3 - 1}{x(\ln(x) + c_3)} \tag{1}$$

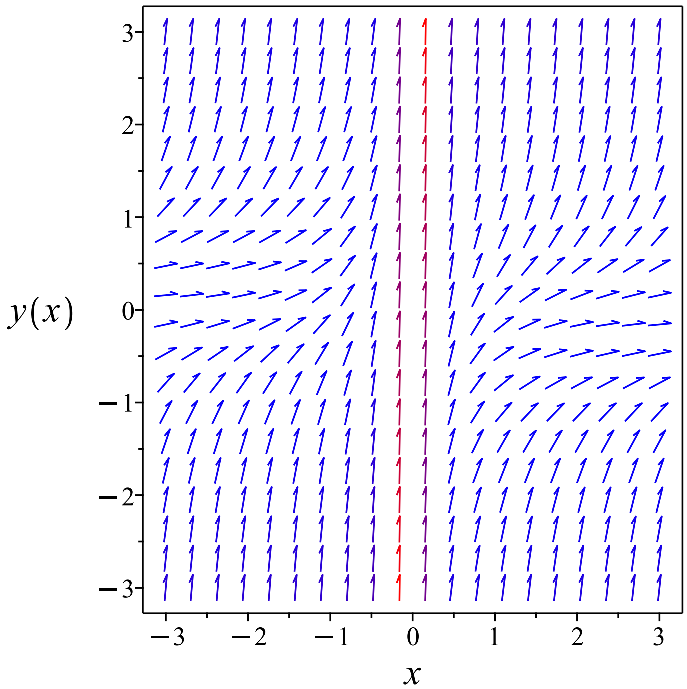


Figure 338: Slope field plot

Verification of solutions

$$y = \frac{-\ln(x) - c_3 - 1}{x(\ln(x) + c_3)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x)=x^2*y(x)^2+x*y(x)+1,y(x), singsol=all)
```

$$y(x) = \frac{-\ln(x) + c_1 - 1}{x(-c_1 + \ln(x))}$$

✓ Solution by Mathematica

Time used: 0.163 (sec). Leaf size: 33

```
DSolve[x^2*y'[x]==x^2*y[x]^2+x*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\log(x) + 1 + c_1}{x\log(x) + c_1 x}$$
$$y(x) \to -\frac{1}{x}$$

11 Section 11. Singular solutions of differential equations. Exercises page 92

11.1 problem 260

Internal problem ID [15122]
Internal file name [`OUTPUT/15122_Sunday_April_21_2024_01_32_51_PM_60173627/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 260.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

Unable to solve or complete the solution.

$$\boxed{\left(1+y'^2\right)y^2-4yy'=4x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{2+\sqrt{4-y^2+4x}}{y} \tag{1}$$

$$y'=\frac{2-\sqrt{4-y^2+4x}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4`[1, 2/y], [2+x, -(-y^2+2*x)/y], [1/2*x^2-1/2*y
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 71

```
dsolve((1+diff(y(x),x)^2)*y(x)^2-4*y(x)*diff(y(x),x)-4*x=0,y(x), singsol=all)
```

$$y(x) = -2\sqrt{x+1}$$
$$y(x) = 2\sqrt{x+1}$$
$$y(x) = \sqrt{-c_1^2 + 2c_1x - x^2 + 4x + 4}$$
$$y(x) = -\sqrt{-x^2 + (2c_1 + 4)\,x - c_1^2 + 4}$$

✓ Solution by Mathematica

Time used: 0.459 (sec). Leaf size: 65

```
DSolve[(1+y'[x]^2)*y[x]^2-4*y[x]*y'[x]-4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\sqrt{-4x^2 - 4(-4 + c_1)x + 16 - c_1{}^2}$$
$$y(x) \to \frac{1}{2}\sqrt{-4x^2 - 4(-4 + c_1)x + 16 - c_1{}^2}$$

11.2 problem 261

Internal problem ID [15123]
Internal file name [`OUTPUT/15123_Sunday_April_21_2024_01_32_56_PM_57658525/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 261.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 - 4y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\sqrt{y} \tag{1}$$
$$y' = -2\sqrt{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{2\sqrt{y}} dy = \int dx$$
$$\sqrt{y} = c_1 + x$$

Summary
The solution(s) found are the following

$$\sqrt{y} = c_1 + x \tag{1}$$

Verification of solutions

$$\sqrt{y} = c_1 + x$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{2\sqrt{y}}dy = \int dx$$
$$-\sqrt{y} = x + c_2$$

Summary

The solution(s) found are the following

$$-\sqrt{y} = x + c_2 \tag{1}$$

Verification of solutions

$$-\sqrt{y} = x + c_2$$

Verified OK.

11.2.1 Maple step by step solution

Let's solve

$y'^2 - 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int 2dx + c_1$

- Evaluate integral

  $2\sqrt{y} = 2x + c_1$

- Solve for y

  $y = x^2 + c_1x + \frac{1}{4}c_1^2$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)^2-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = (x - c_1)^2$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 38

```
DSolve[y'[x]^2-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(-2x + c_1)^2$$
$$y(x) \to \frac{1}{4}(2x + c_1)^2$$
$$y(x) \to 0$$

11.3 problem 262

Internal problem ID [15124]
Internal file name [`OUTPUT/15124_Sunday_April_21_2024_01_32_57_PM_74683976/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 262.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{{y'}^3 - 4xyy' + 8y^2 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{3} + \frac{4yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} - \frac{2yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} + i\sqrt{3}\left(\frac{\left(-108y^2 + 12\sqrt{\cdot}\right.}{}\right. \tag{2}$$

$$y' = -\frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}{6} - \frac{2yx}{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} - i\sqrt{3}\left(\frac{\left(-108y^2 + 12\sqrt{\cdot}\right.}{}\right. \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{align*}
y' &= \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{3\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}} \\
y' &= \omega(x, y)
\end{align*}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)(b_3-a_2)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}} - \frac{\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)^2 a_3}{9\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}} - \left(\frac{-\frac{144x^2y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}+12y}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}} + \frac{24\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)x^2y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}}\right)(xa_2+ya_3+a_1) - \left(\frac{\frac{-144y+\frac{2\left(-216x^3y^2+1944y^3\right)}{3\sqrt{-12y^3x^3+81y^4}}}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}+12x}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}} - \frac{\left(\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{9\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}}\right)(xb_2 + yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-72\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}\sqrt{-12y^3x^3+81y^4}\,xyb_2+8\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-}}{} = 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&72\bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{2}{3}} \sqrt{-12y^3x^3+81y^4}\, x y b_2 \\
&- 8\bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{4}{3}} \sqrt{-12y^3x^3+81y^4}\, x y a_3 \\
&- 48\bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{2}{3}} \sqrt{-12y^3x^3+81y^4}\, x^2 y^2 a_3 \\
&- 48\left(-12y^3x^3+81y^4\right)^{\frac{3}{2}} a_3 + 58320 y^6 a_3 \\
&- 11664 y^5 a_1 + 2592\sqrt{-12y^3x^3+81y^4}\, x\, y^3 a_2 \\
&+ 72\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x^4 y^2 b_2 \\
&+ 72\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x^3 y^3 a_2 \\
&+ 72\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x^3 y^3 b_3 \\
&+ 72\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x^2 y^4 a_3 \\
&+ 72\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x^3 y^2 b_1 \\
&+ 72\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x^2 y^3 a_1 \\
&- 648\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} x\, y^3 b_2 \\
&+ 72\bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{2}{3}} \sqrt{-12y^3x^3+81y^4}\, y^2 b_3 \\
&+ 432\sqrt{-12y^3x^3+81y^4}\, x^2 y^2 b_2 \\
&- 864\sqrt{-12y^3x^3+81y^4}\, x\, y^3 b_3 + 72\bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{2}{3}} \sqrt{-12y^3x^3+81y^4}\, y b_1 \\
&+ 432\sqrt{-12y^3x^3+81y^4}\, x\, y^2 b_1 - \bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{5}{3}} \sqrt{-12y^3x^3+81y^4}\, a_2 \\
&+ \bigg(-108y^2 \\
&+ 12\sqrt{-12y^3x^3+81y^4}\bigg)^{\frac{5}{3}} \sqrt{-12y^3x^3+81y^4}\, b_3 \\
&- 648\left(-108y^2 + 12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} y^4 b_3 \\
&+ 3b_2\bigg(-108y^2
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3 (4x^3-27y)}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3-27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3-27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3 (4x^3-27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3-27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3-27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
-36v_2\Big(&-48v_3\sqrt{3}\,v_1^3v_2^2a_3 - 6v_3v_5\sqrt{3}\,v_1b_2 - 9v_3v_5\sqrt{3}\,v_2a_2 \\
&+ 3v_3v_5\sqrt{3}\,v_2b_3 - 36v_3\sqrt{3}\,v_1^2v_2b_2 - 216v_3\sqrt{3}\,v_1v_2^2a_2 + 72v_3\sqrt{3}\,v_1v_2^2b_3 \\
&+ 27v_3v_4\sqrt{3}\,v_2b_2 - 36v_3\sqrt{3}\,v_1v_2b_1 - 4860v_2^5a_3 + 972v_2^4a_1 + 324v_2^3v_1b_1 \\
&- 72v_1^5v_2^2b_2 + 324v_2^3v_1^2b_2 + 1944v_2^4v_1a_2 - 216v_1^4v_2^3a_2 + 72v_1^4v_2^3b_3 \\
&+ 792v_1^3v_2^4a_3 - 72v_1^4v_2^2b_1 - 72v_1^3v_2^3a_1 - 648v_1v_2^4b_3 + 81v_5v_2^3a_2 - 27v_5v_2^3b_3 \\
&+ 54v_5v_2^2b_1 - 243v_4v_2^3b_2 + 4v_3v_5\sqrt{3}\,v_1^2v_2a_3 - 72v_3v_4\sqrt{3}\,v_1v_2^2a_3 \\
&- 96v_4v_1^4v_2^3a_3 - 6v_5v_1^4v_2b_2 - 18v_5v_1^3v_2^2a_2 + 6v_5v_1^3v_2^2b_3 - 6v_5v_1^2v_2^3a_3 \\
&- 6v_5v_1^3v_2b_1 - 6v_5v_1^2v_2^2a_1 + 36v_4v_1^3v_2^2b_2 + 648v_4v_1v_2^4a_3 \\
&+ 540v_3\sqrt{3}\,v_2^3a_3 + 54v_5v_1v_2^2b_2 - 6v_3v_5\sqrt{3}\,b_1 - 108v_3\sqrt{3}\,v_2^2a_1\Big) = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-972\sqrt{3}\,b_2v_3v_4v_2^2+216v_3v_5\sqrt{3}\,b_1v_2+1728\sqrt{3}\,a_3v_3v_1^3v_2^3\\
&+1296\sqrt{3}\,b_2v_3v_1^2v_2^2+1296\sqrt{3}\,b_1v_3v_1v_2^2+8748b_2v_4v_2^4\\
&+\left(-2916a_2+972b_3\right)v_5v_2^4-1944b_1v_5v_2^3-34992a_1v_2^5\\
&+174960a_3v_2^6-28512a_3v_1^3v_2^5+2592a_1v_1^3v_2^4-11664b_2v_1^2v_2^4\\
&-11664b_1v_1v_2^4+2592v_2^3b_2v_1^5+2592b_1v_1^4v_2^3-144\sqrt{3}\,a_3v_3v_5v_1^2v_2^2\\
&+2592\sqrt{3}\,a_3v_3v_4v_1v_2^3+216\sqrt{3}\,b_2v_3v_5v_1v_2+3456a_3v_4v_1^4v_2^4\\
&+\left(7776a_2-2592b_3\right)v_1^4v_2^4+216b_2v_5v_1^4v_2^2-1296b_2v_4v_1^3v_2^3\\
&+\left(648a_2-216b_3\right)v_5v_1^3v_2^3+216b_1v_5v_1^3v_2^2+216a_3v_5v_1^2v_2^4\\
&+216a_1v_5v_1^2v_2^3-23328a_3v_4v_1v_2^5+\left(-69984a_2+23328b_3\right)v_1v_2^5\\
&+\left(7776\sqrt{3}\,a_2-2592\sqrt{3}\,b_3\right)v_3v_1v_2^3-1944b_2v_5v_1v_2^3-19440\sqrt{3}\,a_3v_3v_2^4\\
&+3888\sqrt{3}\,a_1v_3v_2^3+\left(324\sqrt{3}\,a_2-108\sqrt{3}\,b_3\right)v_3v_5v_2^2=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-34992a_1 &= 0\\
216a_1 &= 0\\
2592a_1 &= 0\\
-28512a_3 &= 0\\
-23328a_3 &= 0\\
216a_3 &= 0\\
3456a_3 &= 0\\
174960a_3 &= 0\\
-11664b_1 &= 0\\
-1944b_1 &= 0\\
216b_1 &= 0\\
2592b_1 &= 0\\
-11664b_2 &= 0\\
-1944b_2 &= 0\\
-1296b_2 &= 0\\
216b_2 &= 0\\
2592b_2 &= 0\\
8748b_2 &= 0\\
3888\sqrt{3}\, a_1 &= 0\\
-19440\sqrt{3}\, a_3 &= 0\\
-144\sqrt{3}\, a_3 &= 0\\
1728\sqrt{3}\, a_3 &= 0\\
2592\sqrt{3}\, a_3 &= 0\\
216\sqrt{3}\, b_1 &= 0\\
1296\sqrt{3}\, b_1 &= 0\\
-972\sqrt{3}\, b_2 &= 0\\
216\sqrt{3}\, b_2 &= 0\\
1296\sqrt{3}\, b_2 &= 0\\
-69984a_2 + 23328b_3 &= 0\\
-2916a_2 + 972b_3 &= 0\\
648a_2 - 216b_3 &= 0\\
7776a_2 - 2592b_3 &= 0\\
324\sqrt{3}\, a_2 - 108\sqrt{3}\, b_3 &= 0\\
7776\sqrt{3}\, a_2 - 2592\sqrt{3}\, b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{x}{3}} \\
&= \frac{3y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^3
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^3}
$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{\frac{x}{3}}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= 3\ln(x)\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{3\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{3y}{x^4}\\ R_y &= \frac{1}{x^3}\\ S_x &= \frac{3}{x}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9x^3\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4y^3x^3 + 27y^4}\right)^{\frac{1}{3}}}{\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4y^3x^3 + 27y^4}\right)^{\frac{2}{3}} x + 12x^2y - 9y\left(-108y^2 + 12\sqrt{3}\,\sqrt{-4y^3x^3 + 27y^4}\right)^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{1}{3}}}{\sqrt{R}\left(12^{\frac{2}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{2}{3}} - 9\sqrt{R}\,12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R-4} - 9\sqrt{R}\right)^{\frac{1}{3}} + 12\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9\left(12\sqrt{81R-12}-108\sqrt{R}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81R-12}-9\sqrt{R}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{R}\,\left(12\sqrt{81R-12}-108\sqrt{R}\right)^{\frac{1}{3}}+12\right)\sqrt{R}}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a$$

Which simplifies to

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a$$

Summary

The solution(s) found are the following

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a + c_1 \tag{1}$$

Verification of solutions

$$3\ln(x) = \int^{\frac{y}{x^3}} \frac{9\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}}{\left(2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a-12}-9\sqrt{\_a}\right)^{2}\right)^{\frac{1}{3}}-9\sqrt{\_a}\,\left(12\sqrt{81\_a-12}-108\sqrt{\_a}\right)^{\frac{1}{3}}+12\right)\sqrt{\_a}}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\,yx - \left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}} - 12xy}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 \\
& + \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)(b_3 -}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}} \\
& - \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)^2 a_3}{36\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}} \\
& - \left(\frac{-\frac{144i\sqrt{3}\,x^2y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}-12i\sqrt{3}\,y+\frac{144x^2y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}-12y}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right. \\
& + \frac{12\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)x^2}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}} \\
& + ya_3 + a_1) \\
& - \left(\frac{\frac{2i\sqrt{3}\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-12i\sqrt{3}\,x-\frac{2\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-12x}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right. \\
& - \frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx-\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12xy\right)\left(-\right.}{18\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}} \\
& + yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3 (4x^3 - 27y)}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3 (4x^3 - 27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\,\sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
72v_2\Big(&72i\sqrt{3}\,v_1^4v_2^3b_3 + 792i\sqrt{3}\,v_1^3v_2^4a_3 - 72i\sqrt{3}\,v_1^4v_2^2b_1\\
&- 72i\sqrt{3}\,v_1^3v_2^3a_1 - 81i\sqrt{3}\,v_5v_2^3a_2 + 27i\sqrt{3}\,v_5v_2^3b_3\\
&+ 324i\sqrt{3}\,v_2^3v_1^2b_2 + 1944i\sqrt{3}\,v_2^4v_1a_2 - 648i\sqrt{3}\,v_1v_2^4b_3\\
&- 144iv_3v_1^3v_2^2a_3 - 54i\sqrt{3}\,v_5v_2^2b_1 + 324i\sqrt{3}\,v_2^3v_1b_1\\
&+ 18iv_5v_3v_1b_2 + 27iv_5v_3v_2a_2 - 9iv_5v_3v_2b_3 - 108iv_3v_1^2v_2b_2\\
&- 648iv_3v_1v_2^2a_2 + 216iv_3v_1v_2^2b_3 - 108iv_3v_1v_2b_1\\
&- 6v_5v_1^4v_2b_2 - 18v_5v_1^3v_2^2a_2 + 6v_5v_1^3v_2^2b_3 - 6v_5v_1^2v_2^3a_3\\
&- 6v_5v_1^3v_2b_1 - 6v_5v_1^2v_2^2a_1 - 72v_4v_1^3v_2^2b_2 - 1296v_4v_1v_2^4a_3\\
&+ 540\sqrt{3}\,v_3v_2^3a_3 + 54v_5v_1v_2^2b_2 - 6\sqrt{3}\,v_5v_3b_1 - 108\sqrt{3}\,v_3v_2^2a_1\\
&- 36\sqrt{3}\,v_3v_1^2v_2b_2 - 216\sqrt{3}\,v_3v_1v_2^2a_2 + 72\sqrt{3}\,v_3v_1v_2^2b_3\\
&- 54\sqrt{3}\,v_4v_3v_2b_2 - 36\sqrt{3}\,v_3v_1v_2b_1 - 72i\sqrt{3}\,v_1^5v_2^2b_2\\
&- 216i\sqrt{3}\,v_1^4v_2^3a_2 + 192v_4v_1^4v_2^3a_3 - 4860i\sqrt{3}\,v_2^5a_3\\
&+ 972i\sqrt{3}\,v_2^4a_1 + 1620iv_3v_2^3a_3 + 18iv_5v_3b_1 - 324iv_3v_2^2a_1\\
&- 48\sqrt{3}\,v_3v_1^3v_2^2a_3 - 6\sqrt{3}\,v_5v_3v_1b_2 - 9\sqrt{3}\,v_5v_3v_2a_2\\
&+ 3\sqrt{3}\,v_5v_3v_2b_3 + 324v_2^3v_1b_1 - 72v_1^5v_2^2b_2 + 1944v_2^4v_1a_2\\
&+ 324v_2^3v_1^2b_2 - 216v_1^4v_2^3a_2 + 72v_1^4v_2^3b_3 + 792v_1^3v_2^4a_3 - 72v_1^4v_2^2b_1\\
&- 72v_1^3v_2^3a_1 - 648v_1v_2^4b_3 + 81v_5v_2^3a_2 - 27v_5v_2^3b_3 + 54v_5v_2^2b_1\\
&+ 486v_4v_2^3b_2 - 4860v_2^5a_3 + 972v_2^4a_1 + 6i\sqrt{3}\,v_5v_1^3v_2b_1\\
&+ 6i\sqrt{3}\,v_5v_1^2v_2^2a_1 - 54i\sqrt{3}\,v_5v_1v_2^2b_2 - 12iv_5v_3v_1^2v_2a_3\\
&+ 144\sqrt{3}\,v_4v_3v_1v_2^2a_3 + 6i\sqrt{3}\,v_5v_1^4v_2b_2 + 18i\sqrt{3}\,v_5v_1^3v_2^2a_2\\
&- 6i\sqrt{3}\,v_5v_1^3v_2^2b_3 + 6i\sqrt{3}\,v_5v_1^2v_2^3a_3 + 4\sqrt{3}\,v_5v_3v_1^2v_2a_3\Big) = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&34992b_2v_4v_2^4 + \left(23328i\sqrt{3}\,b_1 + 23328b_1\right) v_1v_2^4 \\
&+ \left(23328i\sqrt{3}\,b_2 + 23328b_2\right) v_1^2v_2^4 - 5184b_2v_4v_1^3v_2^3 \\
&+ \left(38880\sqrt{3}\,a_3 + 116640ia_3\right) v_3v_2^4 \\
&+ \left(-5832i\sqrt{3}\,a_2 + 1944i\sqrt{3}\,b_3 + 5832a_2 - 1944b_3\right) v_5v_2^4 \\
&+ \left(-7776\sqrt{3}\,a_1 - 23328ia_1\right) v_3v_2^3 \\
&+ \left(-3888i\sqrt{3}\,b_1 + 3888b_1\right) v_5v_2^3 \\
&+ \left(57024i\sqrt{3}\,a_3 + 57024a_3\right) v_1^3v_2^5 \\
&+ \left(-5184i\sqrt{3}\,a_1 - 5184a_1\right) v_1^3v_2^4 \\
&+ \left(-5184i\sqrt{3}\,b_2 - 5184b_2\right) v_1^5v_2^3 \\
&+ \left(-15552i\sqrt{3}\,a_2 + 5184i\sqrt{3}\,b_3 - 15552a_2 + 5184b_3\right) v_1^4v_2^4 \\
&+ \left(-5184i\sqrt{3}\,b_1 - 5184b_1\right) v_1^4v_2^3 + \left(139968i\sqrt{3}\,a_2\right. \\
&\left. - 46656i\sqrt{3}\,b_3 + 139968a_2 - 46656b_3\right) v_1v_2^5 \\
&+ 13824a_3v_4v_1^4v_2^4 + \left(-3456\sqrt{3}\,a_3 - 10368ia_3\right) v_3v_1^3v_2^3 \\
&+ \left(1296i\sqrt{3}\,a_2 - 432i\sqrt{3}\,b_3 - 1296a_2 + 432b_3\right) v_5v_1^3v_2^3 \\
&+ \left(432i\sqrt{3}\,b_2 - 432b_2\right) v_5v_1^4v_2^2 + \left(-15552\sqrt{3}\,a_2\right. \\
&\left. + 5184\sqrt{3}\,b_3 - 46656ia_2 + 15552ib_3\right) v_3v_1v_2^3 \\
&+ \left(-3888i\sqrt{3}\,b_2 + 3888b_2\right) v_5v_1v_2^3 \\
&+ \left(-2592\sqrt{3}\,b_1 - 7776ib_1\right) v_3v_1v_2^2 \\
&+ \left(432i\sqrt{3}\,b_1 - 432b_1\right) v_5v_1^3v_2^2 \\
&+ \left(432i\sqrt{3}\,a_3 - 432a_3\right) v_5v_1^2v_2^4 \\
&+ \left(432i\sqrt{3}\,a_1 - 432a_1\right) v_5v_1^2v_2^3 \\
&+ \left(-2592\sqrt{3}\,b_2 - 7776ib_2\right) v_3v_1^2v_2^2 \\
&+ \left(-648\sqrt{3}\,a_2 + 216\sqrt{3}\,b_3 + 1944ia_2 - 648ib_3\right) v_3v_5v_2^2 \\
&+ \left(-432\sqrt{3}\,b_1 + 1296ib_1\right) v_3v_5v_2 - 93312a_3v_4v_1v_2^5 \\
&+ \left(-349920i\sqrt{3}\,a_3 - 349920a_3\right) v_2^6 \\
&+ \left(69984i\sqrt{3}\,a_1 + 69984a_1\right) v_2^5 - 3888\sqrt{3}\,b_2v_3v_4v_2^2 \\
&+ \left(288\sqrt{3}\,a_3 - 864ia_3\right) v_3v_5v_1^2v_2^2 \\
&+ \left(-432\sqrt{3}\,b_2 + 1296ib_2\right)
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-93312a_3 &= 0\\
13824a_3 &= 0\\
-5184b_2 &= 0\\
34992b_2 &= 0\\
10368\sqrt{3}\, a_3 &= 0\\
-3888\sqrt{3}\, b_2 &= 0\\
-7776\sqrt{3}\, a_1 - 23328ia_1 &= 0\\
-3456\sqrt{3}\, a_3 - 10368ia_3 &= 0\\
288\sqrt{3}\, a_3 - 864ia_3 &= 0\\
38880\sqrt{3}\, a_3 + 116640ia_3 &= 0\\
-2592\sqrt{3}\, b_1 - 7776ib_1 &= 0\\
-432\sqrt{3}\, b_1 + 1296ib_1 &= 0\\
-2592\sqrt{3}\, b_2 - 7776ib_2 &= 0\\
-432\sqrt{3}\, b_2 + 1296ib_2 &= 0\\
-349920i\sqrt{3}\, a_3 - 349920a_3 &= 0\\
-5184i\sqrt{3}\, a_1 - 5184a_1 &= 0\\
-5184i\sqrt{3}\, b_1 - 5184b_1 &= 0\\
-5184i\sqrt{3}\, b_2 - 5184b_2 &= 0\\
-3888i\sqrt{3}\, b_1 + 3888b_1 &= 0\\
-3888i\sqrt{3}\, b_2 + 3888b_2 &= 0\\
432i\sqrt{3}\, a_1 - 432a_1 &= 0\\
432i\sqrt{3}\, a_3 - 432a_3 &= 0\\
432i\sqrt{3}\, b_1 - 432b_1 &= 0\\
432i\sqrt{3}\, b_2 - 432b_2 &= 0\\
23328i\sqrt{3}\, b_1 + 23328b_1 &= 0\\
23328i\sqrt{3}\, b_2 + 23328b_2 &= 0\\
57024i\sqrt{3}\, a_3 + 57024a_3 &= 0\\
69984i\sqrt{3}\, a_1 + 69984a_1 &= 0\\
-15552\sqrt{3}\, a_2 + 5184\sqrt{3}\, b_3 - 46656ia_2 + 15552ib_3 &= 0\\
-648\sqrt{3}\, a_2 + 216\sqrt{3}\, b_3 + 1944ia_2 - 648ib_3 &= 0\\
-15552i\sqrt{3}\, a_2 + 5184i\sqrt{3}\, b_3 - 15552a_2 + 5184b_3 &= 0\\
-5832i\sqrt{3}\, a_2 + 1944i\sqrt{3}\, b_3 + 5832a_2 - 1944b_3 &= 0\\
1296i\sqrt{3}\, a_2 - 432i\sqrt{3}\, b_3 - 1296a_2 + 432b_3 &= 0\\
139968i\sqrt{3}\, a_2 - 46656i\sqrt{3}\, b_3 + 139968a_2 - 46656b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 3y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
y' = -\frac{i\sqrt{3}\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{2}{3}} - 12i\sqrt{3}\, yx + \left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{2}{3}} + 12xy}{6\left(-108y^2 + 12\sqrt{-12y^3x^3 + 81y^4}\right)^{\frac{1}{3}}}
$$

$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 \tag{5E}$$

$$-\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)(b_3}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}$$

$$-\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)^2 a_3}{36\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}}$$

$$-\left(-\frac{-\frac{144i\sqrt{3}\,x^2y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}-12i\sqrt{3}\,y-\frac{144x^2y^3}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}\sqrt{-12y^3x^3+81y^4}}+12y}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$-\frac{12\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)x}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}\sqrt{-12y^3x^3+81y^4}}$$

$$+ya_3+a_1)$$

$$-\left(-\frac{\frac{2i\sqrt{3}\left(-216y+\frac{-216x^3y^2+1944y^3}{\sqrt{-12y^3x^3+81y^4}}\right)}{3\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}-12i\sqrt{3}\,x+\frac{-144y+\frac{2\left(-216x^3y^2+1944y^3\right)}{3\sqrt{-12y^3x^3+81y^4}}}{\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}+12x}{6\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{1}{3}}}\right.$$

$$+\frac{\left(i\sqrt{3}\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}-12i\sqrt{3}\,yx+\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{2}{3}}+12xy\right)(-2}{18\left(-108y^2+12\sqrt{-12y^3x^3+81y^4}\right)^{\frac{4}{3}}}$$

$$+yb_3+b_1)=0$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{-y^3 (4x^3 - 27y)}, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}}, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{-y^3 (4x^3 - 27y)} = v_3, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{1}{3}} = v_4, \left(-108y^2 + 12\sqrt{3}\, \sqrt{-y^3 (4x^3 - 27y)}\right)^{\frac{2}{3}} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-72v_2\Big(-4\sqrt{3}\,v_3v_5v_1^2v_2a_3 - 144\sqrt{3}\,v_3v_4v_1v_2^2a_3 \\
&+ 6i\sqrt{3}\,v_5v_1^4v_2b_2 + 18i\sqrt{3}\,v_5v_1^3v_2^2a_2 - 6i\sqrt{3}\,v_5v_1^3v_2^2b_3 \\
&+ 6i\sqrt{3}\,v_5v_1^2v_2^3a_3 + 6i\sqrt{3}\,v_5v_1^3v_2b_1 + 6i\sqrt{3}\,v_5v_1^2v_2^2a_1 \\
&- 54i\sqrt{3}\,v_5v_1v_2^2b_2 + 648v_1v_2^4b_3 - 81v_5v_2^3a_2 + 27v_5v_2^3b_3 \\
&- 54v_5v_2^2b_1 - 486v_4v_2^3b_2 - 324v_2^3v_1^2b_2 - 1944v_2^4v_1a_2 \\
&- 324v_2^3v_1b_1 + 4860v_2^5a_3 - 972v_2^4a_1 + 72v_1^5v_2^2b_2 \\
&- 792v_1^3v_2^4a_3 + 216v_1^4v_2^3a_2 - 72v_1^4v_2^3b_3 + 72v_1^4v_2^2b_1 \\
&+ 72v_1^3v_2^3a_1 - 4860i\sqrt{3}\,v_2^5a_3 + 972i\sqrt{3}\,v_2^4a_1 + 1620iv_3v_2^3a_3 \\
&+ 18iv_3v_5b_1 - 324iv_3v_2^2a_1 + 48\sqrt{3}\,v_3v_1^3v_2^2a_3 + 6\sqrt{3}\,v_3v_5v_1b_2 \\
&+ 9\sqrt{3}\,v_3v_5v_2a_2 - 3\sqrt{3}\,v_3v_5v_2b_3 + 36\sqrt{3}\,v_3v_1^2v_2b_2 \\
&+ 216\sqrt{3}\,v_3v_1v_2^2a_2 - 72\sqrt{3}\,v_3v_1v_2^2b_3 + 54\sqrt{3}\,v_3v_4v_2b_2 \\
&+ 36\sqrt{3}\,v_3v_1v_2b_1 - 192v_4v_1^4v_2^3a_3 + 6v_5v_1^4v_2b_2 + 18v_5v_1^3v_2^2a_2 \\
&- 6v_5v_1^3v_2^2b_3 + 6v_5v_1^2v_2^3a_3 + 6v_5v_1^3v_2b_1 + 6v_5v_1^2v_2^2a_1 \\
&+ 72v_4v_1^3v_2^2b_2 + 1296v_4v_1v_2^4a_3 - 540\sqrt{3}\,v_3v_2^3a_3 \\
&- 54v_5v_1v_2^2b_2 + 6\sqrt{3}\,v_3v_5b_1 + 108\sqrt{3}\,v_3v_2^2a_1 \\
&- 72i\sqrt{3}\,v_1^3v_2^3a_1 - 81i\sqrt{3}\,v_5v_2^3a_2 + 27i\sqrt{3}\,v_5v_2^3b_3 \\
&+ 324i\sqrt{3}\,v_2^3v_1^2b_2 + 1944i\sqrt{3}\,v_2^4v_1a_2 - 648i\sqrt{3}\,v_1v_2^4b_3 \\
&- 144iv_3v_1^3v_2^2a_3 - 54i\sqrt{3}\,v_5v_2^2b_1 + 324i\sqrt{3}\,v_2^3v_1b_1 \\
&+ 18iv_3v_5v_1b_2 + 27iv_3v_5v_2a_2 - 9iv_3v_5v_2b_3 - 108iv_3v_1^2v_2b_2 \\
&- 648iv_3v_1v_2^2a_2 + 216iv_3v_1v_2^2b_3 - 108iv_3v_1v_2b_1 \\
&- 72i\sqrt{3}\,v_1^5v_2^2b_2 - 216i\sqrt{3}\,v_1^4v_2^3a_2 + 72i\sqrt{3}\,v_1^4v_2^3b_3 \\
&+ 792i\sqrt{3}\,v_1^3v_2^4a_3 - 72i\sqrt{3}\,v_1^4v_2^2b_1 - 12iv_3v_5v_1^2v_2a_3\Big) = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-93312a_3v_4v_1v_2^5 - 5184b_2v_4v_1^3v_2^3 + 13824a_3v_4v_1^4v_2^4 + \left(864ia_3 + 288\sqrt{3}\,a_3\right)v_3v_5v_1^2v_2^2 \\
&+ \left(-1296ib_2 - 432\sqrt{3}\,b_2\right)v_3v_5v_1v_2 - 3888\sqrt{3}\,b_2v_3v_4v_2^2 + \left(349920i\sqrt{3}\,a_3 - 349920a_3\right)v_2^6 \\
&+ \left(-69984i\sqrt{3}\,a_1 + 69984a_1\right)v_2^5 + 34992b_2v_4v_2^4 + 10368\sqrt{3}\,a_3v_3v_4v_1v_2^3 \\
&+ \left(-139968i\sqrt{3}\,a_2 + 46656i\sqrt{3}\,b_3 + 139968a_2 - 46656b_3\right)v_1v_2^5 \\
&+ \left(-23328i\sqrt{3}\,b_1 + 23328b_1\right)v_1v_2^4 + \left(-23328i\sqrt{3}\,b_2 + 23328b_2\right)v_1^2v_2^4 \\
&+ \left(15552i\sqrt{3}\,a_2 - 5184i\sqrt{3}\,b_3 - 15552a_2 + 5184b_3\right)v_1^4v_2^4 + \left(5184i\sqrt{3}\,b_1 - 5184b_1\right)v_1^4v_2^3 \\
&+ \left(-57024i\sqrt{3}\,a_3 + 57024a_3\right)v_1^3v_2^5 + \left(5184i\sqrt{3}\,a_1 - 5184a_1\right)v_1^3v_2^4 \\
&+ \left(5184i\sqrt{3}\,b_2 - 5184b_2\right)v_1^5v_2^3 + \left(-116640ia_3 + 38880\sqrt{3}\,a_3\right)v_3v_2^4 \\
&+ \left(5832i\sqrt{3}\,a_2 - 1944i\sqrt{3}\,b_3 + 5832a_2 - 1944b_3\right)v_5v_2^4 \\
&+ \left(23328ia_1 - 7776\sqrt{3}\,a_1\right)v_3v_2^3 + \left(3888i\sqrt{3}\,b_1 + 3888b_1\right)v_5v_2^3 \\
&+ \left(-432i\sqrt{3}\,a_1 - 432a_1\right)v_5v_1^2v_2^3 + \left(-432i\sqrt{3}\,b_2 - 432b_2\right)v_5v_1^4v_2^2 \\
&+ \left(-1944ia_2 + 648ib_3 - 648\sqrt{3}\,a_2 + 216\sqrt{3}\,b_3\right)v_3v_5v_2^2 + \left(-1296ib_1 - 432\sqrt{3}\,b_1\right)v_3v_5v_2 \\
&+ \left(46656ia_2 - 15552ib_3 - 15552\sqrt{3}\,a_2 + 5184\sqrt{3}\,b_3\right)v_3v_1v_2^3 \\
&+ \left(3888i\sqrt{3}\,b_2 + 3888b_2\right)v_5v_1v_2^3 + \left(7776ib_1 - 2592\sqrt{3}\,b_1\right)v_3v_1v_2^2 \\
&+ \left(7776ib_2 - 2592\sqrt{3}\,b_2\right)v_3v_1^2v_2^2 + \left(10368ia_3 - 3456\sqrt{3}\,a_3\right)v_3v_1^3v_2^3 \\
&+ \left(-1296i\sqrt{3}\,a_2 + 432i\sqrt{3}\,b_3 - 1296a_2 + 432b_3\right)v_5v_1^3v_2^3 \\
&+ \left(-432i\sqrt{3}\,b_1 - 432b_1\right)v_5v_1^3v_2^2 + \left(-432i\sqrt{3}\,a_3 - 432a_3\right)v_5v_1^2v_2^4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
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34992b_{2} &= 0 \\
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-3888\sqrt{3}\, b_{2} &= 0 \\
-116640i a_{3}+38880\sqrt{3}\, a_{3} &= 0 \\
-1296i b_{1}-432\sqrt{3}\, b_{1} &= 0 \\
-1296i b_{2}-432\sqrt{3}\, b_{2} &= 0 \\
864i a_{3}+288\sqrt{3}\, a_{3} &= 0 \\
7776i b_{1}-2592\sqrt{3}\, b_{1} &= 0 \\
7776i b_{2}-2592\sqrt{3}\, b_{2} &= 0 \\
10368i a_{3}-3456\sqrt{3}\, a_{3} &= 0 \\
23328i a_{1}-7776\sqrt{3}\, a_{1} &= 0 \\
-69984i\sqrt{3}\, a_{1}+69984a_{1} &= 0 \\
-57024i\sqrt{3}\, a_{3}+57024a_{3} &= 0 \\
-23328i\sqrt{3}\, b_{1}+23328b_{1} &= 0 \\
-23328i\sqrt{3}\, b_{2}+23328b_{2} &= 0 \\
-432i\sqrt{3}\, a_{1}-432a_{1} &= 0 \\
-432i\sqrt{3}\, a_{3}-432a_{3} &= 0 \\
-432i\sqrt{3}\, b_{1}-432b_{1} &= 0 \\
-432i\sqrt{3}\, b_{2}-432b_{2} &= 0 \\
3888i\sqrt{3}\, b_{1}+3888b_{1} &= 0 \\
3888i\sqrt{3}\, b_{2}+3888b_{2} &= 0 \\
5184i\sqrt{3}\, a_{1}-5184a_{1} &= 0 \\
5184i\sqrt{3}\, b_{1}-5184b_{1} &= 0 \\
5184i\sqrt{3}\, b_{2}-5184b_{2} &= 0 \\
349920i\sqrt{3}\, a_{3}-349920a_{3} &= 0 \\
-1944i a_{2}+648i b_{3}-648\sqrt{3}\, a_{2}+216\sqrt{3}\, b_{3} &= 0 \\
46656i a_{2}-15552i b_{3}-15552\sqrt{3}\, a_{2}+5184\sqrt{3}\, b_{3} &= 0 \\
-139968i\sqrt{3}\, a_{2}+46656i\sqrt{3}\, b_{3}+139968a_{2}-46656b_{3} &= 0 \\
-1296i\sqrt{3}\, a_{2}+432i\sqrt{3}\, b_{3}-1296a_{2}+432b_{3} &= 0 \\
5832i\sqrt{3}\, a_{2}-1944i\sqrt{3}\, b_{3}+5832a_{2}-1944b_{3} &= 0 \\
15552i\sqrt{3}\, a_{2}-5184i\sqrt{3}\, b_{3}-15552a_{2}+5184b_{3} &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
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\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (2*y(x)*x^3-8*y(x)^3)/(x^4-4*x*y(x)^2),
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)^3-4*x*y(x)*diff(y(x),x)+8*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = \frac{4x^3}{27}$$
$$y(x) = 0$$
$$y(x) = \frac{(4c_1x - 1)^2}{64c_1^3}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]^3-4*x*y[x]*y'[x]+8*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

11.4 problem 263

Internal problem ID [15125]
Internal file name [`OUTPUT/15125_Sunday_April_21_2024_01_34_28_PM_12755417/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 263.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 - y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -y \tag{1}$$
$$y' = y \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int -\frac{1}{y}dy = \int dx$$
$$-\ln(y) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{y} = c_2 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{c_2} \tag{1}$$

Verification of solutions

$$y = \frac{e^{-x}}{c_2}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y} dy &= c_3 + x \\ \ln(y) &= c_3 + x \\ y &= e^{c_3 + x} \\ y &= e^x c_3\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_3 \tag{1}$$

Verification of solutions

$$y = e^x c_3$$

Verified OK.

11.4.1 Maple step by step solution

Let's solve

$y'^2 - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = e^{x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)^2-y(x)^2=0,y(x), singsol=all)
```

$$y(x) = e^x c_1$$
$$y(x) = c_1 e^{-x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]^2-y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

11.5 problem 264

Internal problem ID [15126]
Internal file name [`OUTPUT/15126_Sunday_April_21_2024_01_34_29_PM_73397113/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 264.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y^{\frac{2}{3}} = a}$$

11.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{2}{3}} + a} dy = \int dx$$
$$\int^{y} \frac{1}{\_a^{\frac{2}{3}} + a} d\_a = x + c_1$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{1}{\_a^{\frac{2}{3}} + a} d\_a = x + c_1 \tag{1}$$

Verification of solutions

$$\int^{y} \frac{1}{\_a^{\frac{2}{3}} + a} d\_a = x + c_1$$

Verified OK.

11.5.2 Maple step by step solution

Let's solve

$y' - y^{\frac{2}{3}} = a$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{2}{3}}+a} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{2}{3}}+a} dx = \int 1 dx + c_1$

- Evaluate integral

  $3y^{\frac{1}{3}} + \sqrt{a}\,\arctan\left(\frac{\sqrt{3}\,\sqrt{a}-2y^{\frac{1}{3}}}{\sqrt{a}}\right) - \sqrt{a}\,\arctan\left(\frac{2y^{\frac{1}{3}}+\sqrt{3}\,\sqrt{a}}{\sqrt{a}}\right) - 2\sqrt{a}\,\arctan\left(\frac{y^{\frac{1}{3}}}{\sqrt{a}}\right) + \sqrt{a}\,\arctan\left(\frac{y}{a^{\frac{3}{2}}}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 85

```
dsolve(diff(y(x),x)=y(x)^(2/3)+a,y(x), singsol=all)
```

$$x-3y(x)^{\frac{1}{3}}+2\sqrt{a}\,\arctan\left(\frac{y(x)^{\frac{1}{3}}}{\sqrt{a}}\right)-\sqrt{a}\,\arctan\left(\frac{\sqrt{3}\,\sqrt{a}-2y(x)^{\frac{1}{3}}}{\sqrt{a}}\right)+\sqrt{a}\,\arctan\left(\frac{2y(x)^{\frac{1}{3}}+\sqrt{3}\,\sqrt{a}}{\sqrt{a}}\right)-\sqrt{a}\,\arctan\left(\frac{y(x)}{a^{\frac{3}{2}}}\right)+c_1=0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]=y[x]^(2/3)+a,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

11.6 problem 265

Internal problem ID [15127]
Internal file name [OUTPUT/15127_Sunday_April_21_2024_01_34_29_PM_2854590/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 265.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{(xy'+y)^2+3x^5(xy'-2y)=0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{-\frac{3x^5}{2}-y+\frac{3\sqrt{x^{10}+4yx^5}}{2}}{x} \tag{1}$$

$$y'=\frac{-\frac{3x^5}{2}-y-\frac{3\sqrt{x^{10}+4yx^5}}{2}}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= -\frac{3x^5+2y-3\sqrt{x^{10}+4y\,x^5}}{2x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(3x^5 + 2y - 3\sqrt{x^{10} + 4y\,x^5}\right)(b_3 - a_2)}{2x} \\ &- \frac{\left(3x^5 + 2y - 3\sqrt{x^{10} + 4y\,x^5}\right)^2 a_3}{4x^2} \\ &- \left(-\frac{15x^4 - \frac{3(10x^9+20x^4y)}{2\sqrt{x^{10}+4y\,x^5}}}{2x} + \frac{3x^5 + 2y - 3\sqrt{x^{10} + 4y\,x^5}}{2x^2}\right)(xa_2 + ya_3 + a_1) \\ &+ \frac{\left(2 - \frac{6x^5}{\sqrt{x^{10}+4y\,x^5}}\right)(xb_2 + yb_3 + b_1)}{2x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-18x^{15}a_3 + 9\sqrt{x^{10} + 4y\,x^5}\,x^{10}a_3 + 30x^{11}a_2 - 6x^{11}b_3 - 60x^{10}ya_3 + 24x^{10}a_1 - 30\sqrt{x^{10} + 4y\,x^5}\,x^6a_2 + 6\sqrt{}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &18x^{15}a_3 - 9\sqrt{x^{10} + 4y\,x^5}\,x^{10}a_3 - 30x^{11}a_2 + 6x^{11}b_3 + 60x^{10}ya_3 - 24x^{10}a_1 \\ &+ 30\sqrt{x^{10} + 4y\,x^5}\,x^6a_2 - 6\sqrt{x^{10} + 4y\,x^5}\,x^6b_3 + 12\sqrt{x^{10} + 4y\,x^5}\,x^5ya_3 \\ &- 12x^7b_2 - 60x^6ya_2 + 12x^6yb_3 + 12x^5y^2a_3 + 24\sqrt{x^{10} + 4y\,x^5}\,x^5a_1 \\ &- 12x^6b_1 - 36x^5ya_1 - 9\left(x^{10} + 4y\,x^5\right)^{\frac{3}{2}} a_3 + 8b_2\sqrt{x^{10} + 4y\,x^5}\,x^2 \\ &- 8\sqrt{x^{10} + 4y\,x^5}\,y^2a_3 + 4\sqrt{x^{10} + 4y\,x^5}\,xb_1 - 4\sqrt{x^{10} + 4y\,x^5}\,ya_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-9\sqrt{x^5\left(x^5+4y\right)}\,x^{10}a_3-30x^{11}a_2-30x^{10}ya_3-30x^{10}a_1\\
&+18\left(x^{10}+4y\,x^5\right)x^5a_3+30\sqrt{x^5\left(x^5+4y\right)}\,x^6a_2-6\sqrt{x^5\left(x^5+4y\right)}\,x^6b_3\\
&+12\sqrt{x^5\left(x^5+4y\right)}\,x^5ya_3-12x^7b_2-60x^6ya_2-12x^6yb_3\\
&-60x^5y^2a_3+24\sqrt{x^5\left(x^5+4y\right)}\,x^5a_1-12x^6b_1-60x^5ya_1\\
&-9\left(x^5\left(x^5+4y\right)\right)^{\frac{3}{2}}a_3+6\left(x^{10}+4y\,x^5\right)xb_3+18\left(x^{10}+4y\,x^5\right)ya_3\\
&+8b_2\sqrt{x^5\left(x^5+4y\right)}\,x^2-8\sqrt{x^5\left(x^5+4y\right)}\,y^2a_3+6\left(x^{10}+4y\,x^5\right)a_1\\
&+4\sqrt{x^5\left(x^5+4y\right)}\,xb_1-4\sqrt{x^5\left(x^5+4y\right)}\,ya_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&18x^{15}a_3-30x^{11}a_2+6x^{11}b_3-18\sqrt{x^5\left(x^5+4y\right)}\,x^{10}a_3+60x^{10}ya_3\\
&-24x^{10}a_1-12x^7b_2+30\sqrt{x^5\left(x^5+4y\right)}\,x^6a_2-6\sqrt{x^5\left(x^5+4y\right)}\,x^6b_3\\
&-60x^6ya_2+12x^6yb_3-24\sqrt{x^5\left(x^5+4y\right)}\,x^5ya_3+12x^5y^2a_3\\
&-12x^6b_1+24\sqrt{x^5\left(x^5+4y\right)}\,x^5a_1-36x^5ya_1+8b_2\sqrt{x^5\left(x^5+4y\right)}\,x^2\\
&-8\sqrt{x^5\left(x^5+4y\right)}\,y^2a_3+4\sqrt{x^5\left(x^5+4y\right)}\,xb_1-4\sqrt{x^5\left(x^5+4y\right)}\,ya_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{x^5\left(x^5+4y\right)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{x^5\left(x^5+4y\right)}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&18v_1^{15}a_3-30v_1^{11}a_2+60v_1^{10}v_2a_3-18v_3v_1^{10}a_3+6v_1^{11}b_3-24v_1^{10}a_1-60v_1^6v_2a_2\\
&+30v_3v_1^6a_2+12v_1^5v_2^2a_3-24v_3v_1^5v_2a_3-12v_1^7b_2+12v_1^6v_2b_3-6v_3v_1^6b_3\\
&-36v_1^5v_2a_1+24v_3v_1^5a_1-12v_1^6b_1-8v_3v_2^2a_3+8b_2v_3v_1^2-4v_3v_2a_1+4v_3v_1b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$18v_1^{15}a_3 + (-30a_2 + 6b_3)\, v_1^{11} + 60v_1^{10}v_2a_3 - 18v_3v_1^{10}a_3 - 24v_1^{10}a_1 - 12v_1^7b_2 + (-60a_2 + 12b_3)\, v_1^6v_2 + (30a_2 - 6b_3)\, v_1^6v_3 - 12v_1^6b_1 + 12v_1^5v_2^2a_3 - 24v_3v_1^5v_2a_3 - 36v_1^5v_2a_1 + 24v_3v_1^5a_1 + 8b_2v_3v_1^2 + 4v_3v_1b_1 - 8v_3v_2^2a_3 - 4v_3v_2a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-36a_1 &= 0\\
-24a_1 &= 0\\
-4a_1 &= 0\\
24a_1 &= 0\\
-24a_3 &= 0\\
-18a_3 &= 0\\
-8a_3 &= 0\\
12a_3 &= 0\\
18a_3 &= 0\\
60a_3 &= 0\\
-12b_1 &= 0\\
4b_1 &= 0\\
-12b_2 &= 0\\
8b_2 &= 0\\
-60a_2 + 12b_3 &= 0\\
-30a_2 + 6b_3 &= 0\\
30a_2 - 6b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 5a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = 5y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 5y - \left(-\frac{3x^5 + 2y - 3\sqrt{x^{10} + 4y\,x^5}}{2x}\right)(x) \\
&= 6y + \frac{3x^5}{2} - \frac{3\sqrt{x^{10} + 4y\,x^5}}{2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{6y + \frac{3x^5}{2} - \frac{3\sqrt{x^{10} + 4y\,x^5}}{2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{6} + \frac{\ln\left(-x^5 + \sqrt{x^{10} + 4y\,x^5}\right)}{6} - \frac{\ln\left(x^5 + \sqrt{x^{10} + 4y\,x^5}\right)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x^5 + 2y - 3\sqrt{x^{10} + 4y\,x^5}}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{5x^{\frac{3}{2}}}{6\sqrt{x^5 + 4y}} \\
S_y &= \frac{x^{\frac{5}{2}} + \sqrt{x^5 + 4y}}{6y\sqrt{x^5 + 4y}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-x^5 + \sqrt{x^5\,(x^5 + 4y)} - \frac{2y}{3}\right)\sqrt{x^5 + 4y} + \left(\sqrt{x^5\,(x^5 + 4y)} - 4y\right)x^{\frac{5}{2}} - x^{\frac{15}{2}}}{4\sqrt{x^5 + 4y}\,xy} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{6R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{6} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{6} + \frac{\ln\left(-x^5 + x^{\frac{5}{2}}\sqrt{x^5 + 4y}\right)}{6} - \frac{\ln\left(x^5 + x^{\frac{5}{2}}\sqrt{x^5 + 4y}\right)}{6} = -\frac{\ln(x)}{6} + c_1$$

Which simplifies to

$$\frac{\ln(y)}{6} + \frac{\ln\left(-x^5 + x^{\frac{5}{2}}\sqrt{x^5 + 4y}\right)}{6} - \frac{\ln\left(x^5 + x^{\frac{5}{2}}\sqrt{x^5 + 4y}\right)}{6} = -\frac{\ln(x)}{6} + c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{6}+\frac{\ln\left(-x^5+x^{\frac{5}{2}}\sqrt{x^5+4y}\right)}{6}-\frac{\ln\left(x^5+x^{\frac{5}{2}}\sqrt{x^5+4y}\right)}{6}=-\frac{\ln(x)}{6}+c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{6}+\frac{\ln\left(-x^5+x^{\frac{5}{2}}\sqrt{x^5+4y}\right)}{6}-\frac{\ln\left(x^5+x^{\frac{5}{2}}\sqrt{x^5+4y}\right)}{6}=-\frac{\ln(x)}{6}+c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned}y' &= -\frac{3x^5+3\sqrt{x^{10}+4y\,x^5}+2y}{2x}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2-\frac{\left(3x^5+3\sqrt{x^{10}+4y\,x^5}+2y\right)(b_3-a_2)}{2x}\\ &-\frac{\left(3x^5+3\sqrt{x^{10}+4y\,x^5}+2y\right)^2 a_3}{4x^2}\\ &-\left(-\frac{15x^4+\frac{15x^9+30x^4y}{\sqrt{x^{10}+4y\,x^5}}}{2x}+\frac{3x^5+3\sqrt{x^{10}+4y\,x^5}+2y}{2x^2}\right)(xa_2+ya_3+a_1)\\ &+\frac{\left(\frac{6x^5}{\sqrt{x^{10}+4y\,x^5}}+2\right)(xb_2+yb_3+b_1)}{2x}=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{18x^{15}a_3 + 9\sqrt{x^{10}+4y\,x^5}\,x^{10}a_3 - 30x^{11}a_2 + 6x^{11}b_3 + 60x^{10}ya_3 - 24x^{10}a_1 - 30\sqrt{x^{10}+4y\,x^5}\,x^6a_2 + 6\sqrt{x}}{}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned}&-18x^{15}a_3 - 9\sqrt{x^{10}+4y\,x^5}\,x^{10}a_3 + 30x^{11}a_2 - 6x^{11}b_3 - 60x^{10}ya_3 + 24x^{10}a_1\\&+ 30\sqrt{x^{10}+4y\,x^5}\,x^6a_2 - 6\sqrt{x^{10}+4y\,x^5}\,x^6b_3 + 12\sqrt{x^{10}+4y\,x^5}\,x^5ya_3\\&+ 12x^7b_2 + 60x^6ya_2 - 12x^6yb_3 - 12x^5y^2a_3 + 24\sqrt{x^{10}+4y\,x^5}\,x^5a_1\\&+ 12x^6b_1 + 36x^5ya_1 - 9\left(x^{10}+4y\,x^5\right)^{\frac{3}{2}}a_3 + 8b_2\sqrt{x^{10}+4y\,x^5}\,x^2\\&- 8\sqrt{x^{10}+4y\,x^5}\,y^2a_3 + 4\sqrt{x^{10}+4y\,x^5}\,xb_1 - 4\sqrt{x^{10}+4y\,x^5}\,ya_1 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&-9\sqrt{x^5\left(x^5+4y\right)}\,x^{10}a_3 + 30x^{11}a_2 + 30x^{10}ya_3 + 30x^{10}a_1\\&- 18\left(x^{10}+4y\,x^5\right)x^5a_3 + 30\sqrt{x^5\left(x^5+4y\right)}\,x^6a_2\\&- 6\sqrt{x^5\left(x^5+4y\right)}\,x^6b_3 + 12\sqrt{x^5\left(x^5+4y\right)}\,x^5ya_3 + 12x^7b_2\\&+ 60x^6ya_2 + 12x^6yb_3 + 60x^5y^2a_3 + 24\sqrt{x^5\left(x^5+4y\right)}\,x^5a_1\\&+ 12x^6b_1 + 60x^5ya_1 - 9\left(x^5\left(x^5+4y\right)\right)^{\frac{3}{2}}a_3 - 6\left(x^{10}+4y\,x^5\right)xb_3\\&- 18\left(x^{10}+4y\,x^5\right)ya_3 + 8b_2\sqrt{x^5\left(x^5+4y\right)}\,x^2 - 8\sqrt{x^5\left(x^5+4y\right)}\,y^2a_3\\&- 6\left(x^{10}+4y\,x^5\right)a_1 + 4\sqrt{x^5\left(x^5+4y\right)}\,xb_1 - 4\sqrt{x^5\left(x^5+4y\right)}\,ya_1 = 0\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}&-18x^{15}a_3 + 30x^{11}a_2 - 6x^{11}b_3 - 18\sqrt{x^5\left(x^5+4y\right)}\,x^{10}a_3 - 60x^{10}ya_3\\&+ 24x^{10}a_1 + 12x^7b_2 + 30\sqrt{x^5\left(x^5+4y\right)}\,x^6a_2 - 6\sqrt{x^5\left(x^5+4y\right)}\,x^6b_3\\&+ 60x^6ya_2 - 12x^6yb_3 - 24\sqrt{x^5\left(x^5+4y\right)}\,x^5ya_3 - 12x^5y^2a_3\\&+ 12x^6b_1 + 24\sqrt{x^5\left(x^5+4y\right)}\,x^5a_1 + 36x^5ya_1 + 8b_2\sqrt{x^5\left(x^5+4y\right)}\,x^2\\&- 8\sqrt{x^5\left(x^5+4y\right)}\,y^2a_3 + 4\sqrt{x^5\left(x^5+4y\right)}\,xb_1 - 4\sqrt{x^5\left(x^5+4y\right)}\,ya_1 = 0\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^5\left(x^5+4y\right)}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^5 \left(x^5 + 4y\right)} = v_3\right\}$$

The above PDE (6E) now becomes

$$-18v_1^{15}a_3 + 30v_1^{11}a_2 - 60v_1^{10}v_2a_3 - 18v_3v_1^{10}a_3 - 6v_1^{11}b_3 + 24v_1^{10}a_1 + 60v_1^6v_2a_2 + 30v_3v_1^6a_2 - 12v_1^5v_2^2a_3 - 24v_3v_1^5v_2a_3 + 12v_1^7b_2 - 12v_1^6v_2b_3 - 6v_3v_1^6b_3 + 36v_1^5v_2a_1 + 24v_3v_1^5a_1 + 12v_1^6b_1 - 8v_3v_2^2a_3 + 8b_2v_3v_1^2 - 4v_3v_2a_1 + 4v_3v_1b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-18v_1^{15}a_3 + (30a_2 - 6b_3)v_1^{11} - 60v_1^{10}v_2a_3 - 18v_3v_1^{10}a_3 + 24v_1^{10}a_1 + 12v_1^7b_2 + (60a_2 - 12b_3)v_1^6v_2 + (30a_2 - 6b_3)v_1^6v_3 + 12v_1^6b_1 - 12v_1^5v_2^2a_3 - 24v_3v_1^5v_2a_3 + 36v_1^5v_2a_1 + 24v_3v_1^5a_1 + 8b_2v_3v_1^2 + 4v_3v_1b_1 - 8v_3v_2^2a_3 - 4v_3v_2a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 &= 0 \\
24a_1 &= 0 \\
36a_1 &= 0 \\
-60a_3 &= 0 \\
-24a_3 &= 0 \\
-18a_3 &= 0 \\
-12a_3 &= 0 \\
-8a_3 &= 0 \\
4b_1 &= 0 \\
12b_1 &= 0 \\
8b_2 &= 0 \\
12b_2 &= 0 \\
30a_2 - 6b_3 &= 0 \\
60a_2 - 12b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 5a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 5y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 5y - \left(-\frac{3x^5 + 3\sqrt{x^{10} + 4y\,x^5} + 2y}{2x}\right)(x) \\
&= 6y + \frac{3x^5}{2} + \frac{3\sqrt{x^{10} + 4y\,x^5}}{2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{6y + \frac{3x^5}{2} + \frac{3\sqrt{x^{10}+4y\,x^5}}{2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(y)}{6} - \frac{\ln\left(-x^5 + \sqrt{x^{10} + 4y\,x^5}\right)}{6} + \frac{\ln\left(x^5 + \sqrt{x^{10} + 4y\,x^5}\right)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x^5 + 3\sqrt{x^{10} + 4y\,x^5} + 2y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{5x^{\frac{3}{2}}}{6\sqrt{x^5 + 4y}} \\ S_y &= \frac{-x^{\frac{5}{2}} + \sqrt{x^5 + 4y}}{6\sqrt{x^5 + 4y}\,y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-x^5 - \sqrt{x^5\left(x^5 + 4y\right)} - \frac{2y}{3}\right)\sqrt{x^5 + 4y} + \left(\sqrt{x^5\left(x^5 + 4y\right)} + 4y\right)x^{\frac{5}{2}} + x^{\frac{15}{2}}}{4\sqrt{x^5 + 4y}\,xy} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{6R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{6} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln \left(y\right)}{6}-\frac{\ln \left(-x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6}+\frac{\ln \left(x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6} = -\frac{\ln \left(x\right)}{6}+c_{1}$$

Which simplifies to

$$\frac{\ln \left(y\right)}{6}-\frac{\ln \left(-x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6}+\frac{\ln \left(x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6} = -\frac{\ln \left(x\right)}{6}+c_{1}$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(y\right)}{6}-\frac{\ln \left(-x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6}+\frac{\ln \left(x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6} = -\frac{\ln \left(x\right)}{6}+c_{1} \tag{1}$$

Verification of solutions

$$\frac{\ln \left(y\right)}{6}-\frac{\ln \left(-x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6}+\frac{\ln \left(x^{5}+x^{\frac{5}{2}} \sqrt{x^{5}+4 y}\right)}{6} = -\frac{\ln \left(x\right)}{6}+c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 65

```
dsolve((x*diff(y(x),x)+y(x))^2+3*x^5*(x*diff(y(x),x)-2*y(x))=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^5}{4}$$
$$y(x) = \frac{c_1(x^3 + c_1)}{x}$$
$$y(x) = \frac{c_1(-x^3 + c_1)}{x}$$
$$y(x) = \frac{c_1(-x^3 + c_1)}{x}$$
$$y(x) = \frac{c_1(x^3 + c_1)}{x}$$

✓ Solution by Mathematica

Time used: 1.645 (sec). Leaf size: 94

```
DSolve[(x*y'[x]+y[x])^2+3*x^5*(x*y'[x]-2*y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{i(\cosh(3c_1) + \sinh(3c_1))\,(x^3 - i\cosh(3c_1) - i\sinh(3c_1))}{x}$$
$$y(x) \to \frac{i(\cosh(3c_1) + \sinh(3c_1))\,(x^3 + i\cosh(3c_1) + i\sinh(3c_1))}{x}$$
$$y(x) \to 0$$

11.7 problem 266

Internal problem ID [15128]
Internal file name [`OUTPUT/15128_Sunday_April_21_2024_01_36_39_PM_84038794/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 266.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y(y - 2xy')^2 - 2y' = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2y^2x + \sqrt{4y^2x + 1} + 1}{4yx^2} \tag{1}$$

$$y' = \frac{2y^2x - \sqrt{4y^2x + 1} + 1}{4yx^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{2x\,y^2 + \sqrt{4x\,y^2 + 1} + 1}{4y\,x^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(2x\,y^2 + \sqrt{4x\,y^2+1}+1\right)(b_3 - a_2)}{4y\,x^2} - \frac{\left(2x\,y^2 + \sqrt{4x\,y^2+1}+1\right)^2 a_3}{16y^2x^4} - \left(\frac{2y^2 + \frac{2y^2}{\sqrt{4x\,y^2+1}}}{4y\,x^2} - \frac{2x\,y^2+\sqrt{4x\,y^2+1}+1}{2y\,x^3}\right)(xa_2 + ya_3 + a_1) - \left(\frac{4xy + \frac{4xy}{\sqrt{4x\,y^2+1}}}{4y\,x^2} - \frac{2x\,y^2+\sqrt{4x\,y^2+1}+1}{4y^2x^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-8b_2\sqrt{4x\,y^2+1}\,x^4y^2 - 4\sqrt{4x\,y^2+1}\,x^2y^4a_3 + 8\sqrt{4x\,y^2+1}\,x^3y^2b_1 - 8\sqrt{4x\,y^2+1}\,x^2y^3a_1 - 8x^3y^3a_2 - 16}{} = 0$$

Setting the numerator to zero gives

$$8b_2\sqrt{4x\,y^2+1}\,x^4y^2 + 4\sqrt{4x\,y^2+1}\,x^2y^4a_3 - 8\sqrt{4x\,y^2+1}\,x^3y^2b_1 + 8\sqrt{4x\,y^2+1}\,x^2y^3a_1 + 8x^3y^3a_2 + 16x^3y^3b_3 + 8x^2y^4a_3 + 24x^2y^3a_1 + 4\sqrt{4x\,y^2+1}\,x^3b_2 + 4\sqrt{4x\,y^2+1}\,x^2ya_2 + 8\sqrt{4x\,y^2+1}\,x^2yb_3 + 4\sqrt{4x\,y^2+1}\,x\,y^2a_3 - \left(4x\,y^2+1\right)^{\frac{3}{2}} a_3 + 4\sqrt{4x\,y^2+1}\,x^2b_1 + 8\sqrt{4x\,y^2+1}\,xya_1 + 4x^3b_2 + 4x^2ya_2 + 8x^2yb_3 - 4x\,y^2a_3 + 4x^2b_1 + 8xya_1 - a_3\sqrt{4x\,y^2+1} - 2a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-8x^{3}y^{3}a_{2}-8x^{2}y^{4}a_{3}-8x^{2}y^{3}a_{1}-16x^{4}y^{2}b_{2}-16x^{3}y^{3}b_{3} \\
&-16x^{3}y^{2}b_{1}+4\left(4x\,y^{2}+1\right)x^{3}b_{2}+4\left(4x\,y^{2}+1\right)x^{2}b_{1} \\
&+4\sqrt{4x\,y^{2}+1}\,x^{3}b_{2}+4\sqrt{4x\,y^{2}+1}\,x^{2}b_{1}+8b_{2}\sqrt{4x\,y^{2}+1}\,x^{4}y^{2} \\
&+4\sqrt{4x\,y^{2}+1}\,x^{2}y^{4}a_{3}-8\sqrt{4x\,y^{2}+1}\,x^{3}y^{2}b_{1} \\
&+8\sqrt{4x\,y^{2}+1}\,x^{2}y^{3}a_{1}+4\left(4x\,y^{2}+1\right)x^{2}ya_{2}+8\left(4x\,y^{2}+1\right)x^{2}yb_{3} \\
&+4\left(4x\,y^{2}+1\right)x\,y^{2}a_{3}+8\left(4x\,y^{2}+1\right)xya_{1}+4\sqrt{4x\,y^{2}+1}\,x^{2}ya_{2} \\
&+8\sqrt{4x\,y^{2}+1}\,x^{2}yb_{3}+4\sqrt{4x\,y^{2}+1}\,x\,y^{2}a_{3}+8\sqrt{4x\,y^{2}+1}\,xya_{1} \\
&-\left(4x\,y^{2}+1\right)^{\frac{3}{2}}a_{3}-2\left(4x\,y^{2}+1\right)a_{3}-a_{3}\sqrt{4x\,y^{2}+1}=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&8b_{2}\sqrt{4x\,y^{2}+1}\,x^{4}y^{2}+4\sqrt{4x\,y^{2}+1}\,x^{2}y^{4}a_{3}-8\sqrt{4x\,y^{2}+1}\,x^{3}y^{2}b_{1}+8x^{3}y^{3}a_{2} \\
&+16x^{3}y^{3}b_{3}+8\sqrt{4x\,y^{2}+1}\,x^{2}y^{3}a_{1}+8x^{2}y^{4}a_{3}+24x^{2}y^{3}a_{1}+4\sqrt{4x\,y^{2}+1}\,x^{3}b_{2} \\
&+4\sqrt{4x\,y^{2}+1}\,x^{2}ya_{2}+8\sqrt{4x\,y^{2}+1}\,x^{2}yb_{3}+4x^{3}b_{2}+4\sqrt{4x\,y^{2}+1}\,x^{2}b_{1}+4x^{2}ya_{2} \\
&+8x^{2}yb_{3}+8\sqrt{4x\,y^{2}+1}\,xya_{1}-4x\,y^{2}a_{3}+4x^{2}b_{1}+8xya_{1}-2a_{3}\sqrt{4x\,y^{2}+1}-2a_{3}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{4x\,y^{2}+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{4x\,y^{2}+1}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4v_{3}v_{1}^{2}v_{2}^{4}a_{3}+8b_{2}v_{3}v_{1}^{4}v_{2}^{2}+8v_{3}v_{1}^{2}v_{2}^{3}a_{1}+8v_{1}^{3}v_{2}^{3}a_{2}+8v_{1}^{2}v_{2}^{4}a_{3} \\
&-8v_{3}v_{1}^{3}v_{2}^{2}b_{1}+16v_{1}^{3}v_{2}^{3}b_{3}+24v_{1}^{2}v_{2}^{3}a_{1}+4v_{3}v_{1}^{2}v_{2}a_{2}+4v_{3}v_{1}^{3}b_{2} \\
&+8v_{3}v_{1}^{2}v_{2}b_{3}+8v_{3}v_{1}v_{2}a_{1}+4v_{1}^{2}v_{2}a_{2}-4v_{1}v_{2}^{2}a_{3}+4v_{3}v_{1}^{2}b_{1} \\
&+4v_{1}^{3}b_{2}+8v_{1}^{2}v_{2}b_{3}+8v_{1}v_{2}a_{1}+4v_{1}^{2}b_{1}-2a_{3}v_{3}-2a_{3}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}, v_{3}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&8b_2v_3v_1^4v_2^2 + (8a_2 + 16b_3)\, v_1^3v_2^3 - 8v_3v_1^3v_2^2b_1 + 4v_3v_1^3b_2 + 4v_1^3b_2 + 4v_3v_1^2v_2^4a_3 \\
&+ 8v_1^2v_2^4a_3 + 8v_3v_1^2v_2^3a_1 + 24v_1^2v_2^3a_1 + (4a_2 + 8b_3)\, v_1^2v_2v_3 + (4a_2 + 8b_3)\, v_1^2v_2 \\
&+ 4v_3v_1^2b_1 + 4v_1^2b_1 - 4v_1v_2^2a_3 + 8v_3v_1v_2a_1 + 8v_1v_2a_1 - 2a_3v_3 - 2a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
8a_1 &= 0 \\
24a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
4a_3 &= 0 \\
8a_3 &= 0 \\
-8b_1 &= 0 \\
4b_1 &= 0 \\
4b_2 &= 0 \\
8b_2 &= 0 \\
4a_2 + 8b_3 &= 0 \\
8a_2 + 16b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{2x\,y^2+\sqrt{4x\,y^2+1}+1}{4y\,x^2}\right)(-2x) \\
&= \frac{4x\,y^2+\sqrt{4x\,y^2+1}+1}{2xy} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x\,y^2+\sqrt{4x\,y^2+1}+1}{2xy}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} + \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4x\,y^2+1}}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x\,y^2+\sqrt{4x\,y^2+1}+1}{4y\,x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{1}{4x\sqrt{4x\,y^2+1}}\\
S_y &= \frac{1-\frac{1}{\sqrt{4x\,y^2+1}}}{2y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{4x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{4R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R)}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{2} + \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2} = \frac{\ln(x)}{4} + c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{2} + \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2} = \frac{\ln(x)}{4} + c_1
$$

Summary

The solution(s) found are the following

$$
\frac{\ln(y)}{2} + \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2} = \frac{\ln(x)}{4} + c_1 \tag{1}
$$

Verification of solutions

$$\frac{\ln(y)}{2}+\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2}=\frac{\ln(x)}{4}+c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y'=\frac{2x\,y^2-\sqrt{4x\,y^2+1}+1}{4y\,x^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2+\frac{\left(2x\,y^2-\sqrt{4x\,y^2+1}+1\right)(b_3-a_2)}{4y\,x^2}-\frac{\left(2x\,y^2-\sqrt{4x\,y^2+1}+1\right)^2a_3}{16y^2x^4}\\&-\left(\frac{2y^2-\frac{2y^2}{\sqrt{4x\,y^2+1}}}{4y\,x^2}-\frac{2x\,y^2-\sqrt{4x\,y^2+1}+1}{2y\,x^3}\right)(xa_2+ya_3+a_1)\\&-\left(\frac{4xy-\frac{4xy}{\sqrt{4x\,y^2+1}}}{4y\,x^2}-\frac{2x\,y^2-\sqrt{4x\,y^2+1}+1}{4y^2x^2}\right)(xb_2+yb_3+b_1)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-8b_2\sqrt{4x\,y^2+1}\,x^4y^2-4\sqrt{4x\,y^2+1}\,x^2y^4a_3+8\sqrt{4x\,y^2+1}\,x^3y^2b_1-8\sqrt{4x\,y^2+1}\,x^2y^3a_1+8x^3y^3a_2+16}{}$$
$$=0$$

Setting the numerator to zero gives

$$8b_2\sqrt{4x\,y^2+1}\,x^4y^2+4\sqrt{4x\,y^2+1}\,x^2y^4a_3-8\sqrt{4x\,y^2+1}\,x^3y^2b_1+8\sqrt{4x\,y^2+1}\,x^2y^3a_1-8x^3y^3a_2-16x^3y^3b_3-8x^2y^4a_3-24x^2y^3a_1+4\sqrt{4x\,y^2+1}\,x^3b_2+4\sqrt{4x\,y^2+1}\,x^2ya_2+8\sqrt{4x\,y^2+1}\,x^2yb_3+4\sqrt{4x\,y^2+1}\,x\,y^2a_3-\left(4x\,y^2+1\right)^{\frac{3}{2}}a_3+4\sqrt{4x\,y^2+1}\,x^2b_1+8\sqrt{4x\,y^2+1}\,xya_1-4x^3b_2-4x^2ya_2-8x^2yb_3+4x\,y^2a_3-4x^2b_1-8xya_1-a_3\sqrt{4x\,y^2+1}+2a_3=0 \tag{6E}$$

Simplifying the above gives

$$-\left(4x\,y^2+1\right)^{\frac{3}{2}}a_3+2\left(4x\,y^2+1\right)a_3-a_3\sqrt{4x\,y^2+1}+8b_2\sqrt{4x\,y^2+1}\,x^4y^2+4\sqrt{4x\,y^2+1}\,x^2y^4a_3-8\sqrt{4x\,y^2+1}\,x^3y^2b_1+8\sqrt{4x\,y^2+1}\,x^2y^3a_1-4\left(4x\,y^2+1\right)x^2ya_2-8\left(4x\,y^2+1\right)x^2yb_3-4\left(4x\,y^2+1\right)x\,y^2a_3-8\left(4x\,y^2+1\right)xya_1+4\sqrt{4x\,y^2+1}\,x^2ya_2+8\sqrt{4x\,y^2+1}\,x^2yb_3+4\sqrt{4x\,y^2+1}\,x\,y^2a_3+8\sqrt{4x\,y^2+1}\,xya_1+8x^3y^3a_2+8x^2y^4a_3+8x^2y^3a_1+16x^4y^2b_2+16x^3y^3b_3+16x^3y^2b_1-4\left(4x\,y^2+1\right)x^3b_2-4\left(4x\,y^2+1\right)x^2b_1+4\sqrt{4x\,y^2+1}\,x^3b_2+4\sqrt{4x\,y^2+1}\,x^2b_1=0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$8b_2\sqrt{4x\,y^2+1}\,x^4y^2+4\sqrt{4x\,y^2+1}\,x^2y^4a_3-8\sqrt{4x\,y^2+1}\,x^3y^2b_1-8x^3y^3a_2-16x^3y^3b_3+8\sqrt{4x\,y^2+1}\,x^2y^3a_1-8x^2y^4a_3-24x^2y^3a_1+4\sqrt{4x\,y^2+1}\,x^3b_2+4\sqrt{4x\,y^2+1}\,x^2ya_2+8\sqrt{4x\,y^2+1}\,x^2yb_3-4x^3b_2+4\sqrt{4x\,y^2+1}\,x^2b_1-4x^2ya_2-8x^2yb_3+8\sqrt{4x\,y^2+1}\,xya_1+4x\,y^2a_3-4x^2b_1-8xya_1-2a_3\sqrt{4x\,y^2+1}+2a_3=0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4x\,y^2+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \sqrt{4x\,y^2+1}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&4v_3v_1^2v_2^4a_3 + 8b_2v_3v_1^4v_2^2 + 8v_3v_1^2v_2^3a_1 - 8v_1^3v_2^3a_2 - 8v_1^2v_2^4a_3 \\
&- 8v_3v_1^3v_2^2b_1 - 16v_1^3v_2^3b_3 - 24v_1^2v_2^3a_1 + 4v_3v_1^2v_2a_2 + 4v_3v_1^3b_2 \\
&+ 8v_3v_1^2v_2b_3 + 8v_3v_1v_2a_1 - 4v_1^2v_2a_2 + 4v_1v_2^2a_3 + 4v_3v_1^2b_1 \\
&- 4v_1^3b_2 - 8v_1^2v_2b_3 - 8v_1v_2a_1 - 4v_1^2b_1 - 2a_3v_3 + 2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&8b_2v_3v_1^4v_2^2 + (-8a_2 - 16b_3)\, v_1^3v_2^3 - 8v_3v_1^3v_2^2b_1 + 4v_3v_1^3b_2 - 4v_1^3b_2 + 4v_3v_1^2v_2^4a_3 \\
&- 8v_1^2v_2^4a_3 + 8v_3v_1^2v_2^3a_1 - 24v_1^2v_2^3a_1 + (4a_2 + 8b_3)\, v_1^2v_2v_3 + (-4a_2 - 8b_3)\, v_1^2v_2 \\
&+ 4v_3v_1^2b_1 - 4v_1^2b_1 + 4v_1v_2^2a_3 + 8v_3v_1v_2a_1 - 8v_1v_2a_1 - 2a_3v_3 + 2a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-24a_1 &= 0 \\
-8a_1 &= 0 \\
8a_1 &= 0 \\
-8a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
4a_3 &= 0 \\
-8b_1 &= 0 \\
-4b_1 &= 0 \\
4b_1 &= 0 \\
-4b_2 &= 0 \\
4b_2 &= 0 \\
8b_2 &= 0 \\
-8a_2 - 16b_3 &= 0 \\
-4a_2 - 8b_3 &= 0 \\
4a_2 + 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{2x\, y^2 - \sqrt{4x\, y^2 + 1} + 1}{4y\, x^2}\right) (-2x) \\
&= \frac{4x\, y^2 - \sqrt{4x\, y^2 + 1} + 1}{2xy} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x\, y^2 - \sqrt{4x\, y^2 + 1} + 1}{2xy}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(y)}{2} - \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4x\,y^2+1}}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x\,y^2 - \sqrt{4x\,y^2+1}+1}{4y\,x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{4x\sqrt{4x\,y^2+1}} \\ S_y &= \frac{1+\frac{1}{\sqrt{4x\,y^2+1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{4x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{4R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2}-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2}=\frac{\ln(x)}{4}+c_1$$

Which simplifies to

$$\frac{\ln(y)}{2}-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2}=\frac{\ln(x)}{4}+c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2}-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2}=\frac{\ln(x)}{4}+c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{2}-\frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{4y^2x+1}}\right)}{2}=\frac{\ln(x)}{4}+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 99

```
dsolve(y(x)*(y(x)-2*x*diff(y(x),x))^2=2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\frac{1}{2\sqrt{-x}}$$
$$y(x) = \frac{1}{2\sqrt{-x}}$$
$$y(x) = 0$$
$$y(x) = \frac{\sqrt{(x+c_1)\,x}}{c_1\sqrt{x}}$$
$$y(x) = \frac{\sqrt{x\,(x-c_1)}}{c_1\sqrt{x}}$$
$$y(x) = -\frac{\sqrt{(x+c_1)\,x}}{c_1\sqrt{x}}$$
$$y(x) = -\frac{\sqrt{x\,(x-c_1)}}{c_1\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 1.935 (sec). Leaf size: 158

```
DSolve[y[x]*(y[x]-2*x*y'[x])^2==2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{e^{-2c_1}\,(2x-e^{c_1})}$$
$$y(x) \to \sqrt{2}\sqrt{e^{-2c_1}\,(2x-e^{c_1})}$$
$$y(x) \to -\sqrt{2}\sqrt{e^{-2c_1}\,(2x+e^{c_1})}$$
$$y(x) \to \sqrt{2}\sqrt{e^{-2c_1}\,(2x+e^{c_1})}$$
$$y(x) \to 0$$
$$y(x) \to -\frac{i}{2\sqrt{x}}$$
$$y(x) \to \frac{i}{2\sqrt{x}}$$

11.8 problem 267

Internal problem ID [15129]
Internal file name [`OUTPUT/15129_Sunday_April_21_2024_01_36_43_PM_91858825/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 267.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{8y'^3 - 12y'^2 - 27y = -27x}$$

11.8.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$8p^3 - 12p^2 - 27y = -27x$$

Solving for y from the above results in

$$y = \frac{8}{27}p^3 - \frac{4}{9}p^2 + x \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 1 \\
g &= \frac{8}{27}p^3 - \frac{4}{9}p^2
\end{aligned}
$$

Hence (2) becomes

$$
p - 1 = \left(\frac{8}{9}p^2 - \frac{8}{9}p\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - 1 = 0
$$

Solving for p from the above gives

$$
p = 1
$$

Substituting these in (1A) gives

$$
y = -\frac{4}{27} + x
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - 1}{\frac{8p(x)^2}{9} - \frac{8p(x)}{9}} \tag{3}
$$

This ODE is now solved for $p(x)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{8p}{9}dp &= x + c_1 \\
\frac{4p^2}{9} &= x + c_1
\end{aligned}
$$

Solving for p gives these solutions

$$
\begin{aligned}
p_1 &= -\frac{3\sqrt{x + c_1}}{2} \\
p_2 &= \frac{3\sqrt{x + c_1}}{2}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
y &= -(x + c_1)^{\frac{3}{2}} - c_1 \\
y &= (x + c_1)^{\frac{3}{2}} - c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = -\frac{4}{27} + x \tag{1}$$

$$y = -(x + c_1)^{\frac{3}{2}} - c_1 \tag{2}$$

$$y = (x + c_1)^{\frac{3}{2}} - c_1 \tag{3}$$

Verification of solutions

$$y = -\frac{4}{27} + x$$

Verified OK.

$$y = -(x + c_1)^{\frac{3}{2}} - c_1$$

Verified OK.

$$y = (x + c_1)^{\frac{3}{2}} - c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(8*diff(y(x),x)^3-12*diff(y(x),x)^2=27*(y(x)-x),y(x), singsol=all)
```

$$y(x) = x - \frac{4}{27}$$
$$y(x) = (-x + c_1)\sqrt{x - c_1} + c_1$$
$$y(x) = (x - c_1)^{\frac{3}{2}} + c_1$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[8*y'[x]^3-12*y'[x]^2==27*(y[x]-x),y[x],x,IncludeSingularSolutions -> True]
```

Timed out

11.9 problem 268

Internal problem ID [15130]
Internal file name [`OUTPUT/15130_Sunday_April_21_2024_01_36_45_PM_99181376/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 268.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{(y'-1)^2 - y^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -y + 1 \tag{1}$$
$$y' = y + 1 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{-y+1} dy = \int dx$$
$$-\ln(-y+1) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{-y+1} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{-y+1} = c_2 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{c_2} + 1 \tag{1}$$

Verification of solutions

$$y = -\frac{e^{-x}}{c_2} + 1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{y+1} dy = \int dx$$
$$\ln (y+1) = c_3 + x$$

Raising both side to exponential gives

$$y + 1 = e^{c_3 + x}$$

Which simplifies to

$$y + 1 = c_4 e^{x}$$

Summary

The solution(s) found are the following

$$y = c_4 e^{x} - 1 \tag{1}$$

Verification of solutions

$$y = c_4 e^{x} - 1$$

Verified OK.

11.9.1 Maple step by step solution

Let's solve

$(y'-1)^2 - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+1} dx = \int 1 dx + c_1$

- Evaluate integral

  $\ln(y+1) = x + c_1$

- Solve for y

  $y = e^{x+c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((diff(y(x),x)-1)^2=y(x)^2,y(x), singsol=all)
```

$$y(x) = -1 + e^x c_1$$
$$y(x) = 1 + c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 37

```
DSolve[(y'[x]-1)^2==y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + c_1 e^{-x}$$
$$y(x) \to -1 + c_1 e^{x}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

11.10 problem 269

Internal problem ID [15131]
Internal file name [`OUTPUT/15131_Sunday_April_21_2024_01_36_45_PM_42889222/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 269.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y - {y'}^2 + xy' = x}$$

11.10.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-p^2 + xp + y = x$$

Solving for y from the above results in

$$y = (-p+1)\,x + p^2 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -p + 1$$
$$g = p^2$$

Hence (2) becomes

$$2p - 1 = (-x + 2p)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$2p - 1 = 0$$

Solving for p from the above gives

$$p = \frac{1}{2}$$

Substituting these in (1A) gives

$$y = \frac{x}{2} + \frac{1}{4}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{2p(x) - 1}{-x + 2p\,(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-x(p) + 2p}{2p - 1} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{1}{2p - 1}$$
$$q(p) = \frac{2p}{2p - 1}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{2p-1} = \frac{2p}{2p-1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2p-1}dp} \\ &= \sqrt{2p-1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(\frac{2p}{2p-1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\sqrt{2p-1}\,x\right) &= \left(\sqrt{2p-1}\right)\left(\frac{2p}{2p-1}\right) \\ \mathrm{d}\left(\sqrt{2p-1}\,x\right) &= \left(\frac{2p}{\sqrt{2p-1}}\right)\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{2p-1}\,x &= \int \frac{2p}{\sqrt{2p-1}}\,\mathrm{d}p \\ \sqrt{2p-1}\,x &= \frac{2\sqrt{2p-1}\,(p+1)}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{2p-1}$ results in

$$x(p) = \frac{2p}{3} + \frac{2}{3} + \frac{c_1}{\sqrt{2p-1}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{x}{2} + \frac{\sqrt{x^2+4y-4x}}{2} \\ p &= \frac{x}{2} - \frac{\sqrt{x^2+4y-4x}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{x}{3} + \frac{\sqrt{x^2+4y-4x}}{3} + \frac{2}{3} + \frac{c_1}{\sqrt{x+\sqrt{x^2+4y-4x}-1}} \\ x &= \frac{x}{3} - \frac{\sqrt{x^2+4y-4x}}{3} + \frac{2}{3} + \frac{c_1}{\sqrt{x-\sqrt{x^2+4y-4x}-1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2}+\frac{1}{4} \tag{1}$$

$$x = \frac{x}{3}+\frac{\sqrt{x^{2}+4y-4x}}{3}+\frac{2}{3}+\frac{c_{1}}{\sqrt{x+\sqrt{x^{2}+4y-4x}-1}} \tag{2}$$

$$x = \frac{x}{3}-\frac{\sqrt{x^{2}+4y-4x}}{3}+\frac{2}{3}+\frac{c_{1}}{\sqrt{x-\sqrt{x^{2}+4y-4x}-1}} \tag{3}$$

Verification of solutions

$$y = \frac{x}{2}+\frac{1}{4}$$

Verified OK.

$$x = \frac{x}{3}+\frac{\sqrt{x^{2}+4y-4x}}{3}+\frac{2}{3}+\frac{c_{1}}{\sqrt{x+\sqrt{x^{2}+4y-4x}-1}}$$

Verified OK.

$$x = \frac{x}{3}-\frac{\sqrt{x^{2}+4y-4x}}{3}+\frac{2}{3}+\frac{c_{1}}{\sqrt{x-\sqrt{x^{2}+4y-4x}-1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 845

```
dsolve(y(x)=diff(y(x),x)^2-x*diff(y(x),x)+x^2/x,y(x), singsol=all)
```

$$y(x) = -2\left(\frac{-1+x}{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}} + \frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}{4} - \frac{1}{2}\right)\left(\frac{-1+x}{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}} + \frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}{4} + \frac{1}{2}\right)x + \frac{\left(1+\left(\frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}{2} - \frac{2(1-x)}{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}\right)^2\right)^2}{4}$$

$$y(x) = \frac{\left(1+\left(-\frac{i\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}{4} + \frac{i(1-x)}{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}} + \frac{\sqrt{3}\left(\frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2\right.}}{2}\right.}{}\right.\right.}{2} + \frac{\left(1+\left(\frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}{4} - \frac{1-x}{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x\right.}}{2}\right.}{}\right.\right.}{4}$$

$$y(x) = \frac{\left(1+\left(-\frac{i\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}}{4} + \frac{i(1-x)}{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2-12x+4}\right)^{\frac{1}{3}}} - \frac{\sqrt{3}\left(\frac{\left(-12c_1+4\sqrt{-4x^3+9c_1^2+12x^2\right.}}{2}\right.}{}\right.\right.}{2}$$

✓ Solution by Mathematica

Time used: 61.116 (sec). Leaf size: 2409

```
DSolve[y[x]==y'[x]^2-x*y'[x]+x^2/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\sqrt[3]{2}x^4 - 8\sqrt[3]{2}x^3 + 12\sqrt[3]{2}x^2 + 4x^2\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 - 10e^{3c_1}x^3 - 30x^2 + 30e^{3c_1}x^2 + 12x -}{}$$

$$y(x) \to \frac{1}{4}\left(2x^2 - 2x + 3\right) + \frac{\left(1 + i\sqrt{3}\right)(x-1)\left(-(x-1)^3 + 2e^{3c_1}\right)}{4\ 2^{2/3}\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 - 10e^{3c_1}x^3 - 30x^2 + 30e^{3c_1}x^2 + 12x - 30e^{3c_1}x + \sqrt{e^{3c_1}\left(4(x-1\right.}}} + \frac{i\left(\sqrt{3} + i\right)\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 - 10e^{3c_1}x^3 - 30x^2 + 30e^{3c_1}x^2 + 12x - 30e^{3c_1}x + \sqrt{e^{3c_1}\left(4(x\right.}}}{8\sqrt[3]{2}}$$

$$y(x) \to \frac{1}{4}\left(2x^2 - 2x + 3\right) + \frac{i\left(\sqrt{3} + i\right)(x-1)\left((x-1)^3 - 2e^{3c_1}\right)}{4\ 2^{2/3}\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 - 10e^{3c_1}x^3 - 30x^2 + 30e^{3c_1}x^2 + 12x - 30e^{3c_1}x + \sqrt{e^{3c_1}\left(4(x-1\right.}}} - \frac{i\left(\sqrt{3} - i\right)\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 - 10e^{3c_1}x^3 - 30x^2 + 30e^{3c_1}x^2 + 12x - 30e^{3c_1}x + \sqrt{e^{3c_1}\left(4(x\right.}}}{8\sqrt[3]{2}}$$

$$y(x) \to \frac{2\sqrt[3]{2}x^4 - 8\sqrt[3]{2}x^3 + 12\sqrt[3]{2}x^2 + 4x^2\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 + 10e^{3c_1}x^3 - 30x^2 - 30e^{3c_1}x^2 + 12x +}{}$$

$$y(x) \to \frac{1}{4}\left(2x^2 - 2x + 3\right) - \frac{i\left(\sqrt{3} - i\right)(x-1)\left((x-1)^3 + 2e^{3c_1}\right)}{4\ 2^{2/3}\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 + 10e^{3c_1}x^3 - 30x^2 - 30e^{3c_1}x^2 + 12x + 30e^{3c_1}x + \sqrt{e^{3c_1}\left(-4(x-\right.}}} + \frac{i\left(\sqrt{3} + i\right)\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 + 10e^{3c_1}x^3 - 30x^2 - 30e^{3c_1}x^2 + 12x + 30e^{3c_1}x + \sqrt{e^{3c_1}\left(-4\right.}}}{8\sqrt[3]{2}}$$

$$y(x) \to \frac{1}{4}\left(2x^2 - 2x + 3\right) + \frac{i\left(\sqrt{3} + i\right)(x-1)\left((x-1)^3 + 2e^{3c_1}\right)}{4\ 2^{2/3}\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 + 10e^{3c_1}x^3 - 30x^2 - 30e^{3c_1}x^2 + 12x + 30e^{3c_1}x + \sqrt{e^{3c_1}\left(-4(x-\right.}}} - \frac{i\left(\sqrt{3} - i\right)\sqrt[3]{-2x^6 + 12x^5 - 30x^4 + 40x^3 + 10e^{3c_1}x^3 - 30x^2 - 30e^{3c_1}x^2 + 12x + 30e^{3c_1}x + \sqrt{e^{3c_1}\left(-4\right.}}}{8\sqrt[3]{2}}$$

11.11 problem 270

Internal problem ID [15132]
Internal file name [`OUTPUT/15132_Sunday_April_21_2024_01_36_46_PM_29978615/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 270.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _dAlembert]`

$$\boxed{(xy' + y)^2 - y^2y' = 0}$$

11.11.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$(xp + y)^2 - y^2p = 0$$

Solving for y from the above results in

$$y = \frac{(1 + \sqrt{p})\, px}{p - 1} \tag{1A}$$

$$y = -\frac{(-1 + \sqrt{p})\, px}{p - 1} \tag{2A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{(1+\sqrt{p})\, p}{p-1}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{(1+\sqrt{p})\, p}{p-1} = x\left(\frac{\sqrt{p}}{2p-2} - \frac{(1+\sqrt{p})\, p}{(p-1)^2} + \frac{1+\sqrt{p}}{p-1}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{(1+\sqrt{p})\, p}{p-1} = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 4$$

Substituting these in (1A) gives

$$y = 0$$
$$y = 4x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{(1+\sqrt{p(x)})p(x)}{p(x)-1}}{x\left(\frac{\sqrt{p(x)}}{2p(x)-2} - \frac{(1+\sqrt{p(x)})p(x)}{(p(x)-1)^2} + \frac{1+\sqrt{p(x)}}{p(x)-1}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{\sqrt{p}}{2p-2} - \frac{(1+\sqrt{p})p}{(p-1)^2} + \frac{1+\sqrt{p}}{p-1}\right)}{p - \frac{(1+\sqrt{p})p}{p-1}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{p^{\frac{3}{2}} - 3\sqrt{p} - 2}{2p\left(p - 2 - \sqrt{p}\right)(p - 1)}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left(p^{\frac{3}{2}} - 3\sqrt{p} - 2\right)x(p)}{2p\left(p - 2 - \sqrt{p}\right)(p - 1)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{p^{\frac{3}{2}} - 3\sqrt{p} - 2}{2p(p - 2 - \sqrt{p})(p - 1)}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}p}\left(e^{\int -\frac{p^{\frac{3}{2}} - 3\sqrt{p} - 2}{2p(p - 2 - \sqrt{p})(p - 1)}dp}x\right) = 0$$

Integrating gives

$$e^{\int -\frac{p^{\frac{3}{2}} - 3\sqrt{p} - 2}{2p(p - 2 - \sqrt{p})(p - 1)}dp}x = c_3$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{p^{\frac{3}{2}} - 3\sqrt{p} - 2}{2p(p - 2 - \sqrt{p})(p - 1)}dp}$ results in

$$x(p) = c_3 e^{-\frac{\left(\int \frac{p^{\frac{3}{2}} - 3\sqrt{p} - 2}{p(-p + 2 + \sqrt{p})(p - 1)}dp\right)}{2}}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{\left(-1+\sqrt{p}\right) p}{p-1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p+\frac{\left(-1+\sqrt{p}\right) p}{p-1} = x\left(-\frac{\sqrt{p}}{2\left(p-1\right)}+\frac{\left(-1+\sqrt{p}\right) p}{\left(p-1\right)^{2}}-\frac{-1+\sqrt{p}}{p-1}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$
p+\frac{\left(-1+\sqrt{p}\right) p}{p-1} = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x)+\frac{\left(-1+\sqrt{p(x)}\right)p(x)}{p(x)-1}}{x\left(-\frac{\sqrt{p(x)}}{2(p(x)-1)}+\frac{\left(-1+\sqrt{p(x)}\right)p(x)}{(p(x)-1)^{2}}-\frac{-1+\sqrt{p(x)}}{p(x)-1}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(-\frac{\sqrt{p}}{2(p-1)}+\frac{(-1+\sqrt{p})p}{(p-1)^{2}}-\frac{-1+\sqrt{p}}{p-1}\right)}{p+\frac{(-1+\sqrt{p})p}{p-1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p)+p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{-p^{\frac{3}{2}} + 3\sqrt{p} - 2}{2p\left(p - 2 + \sqrt{p}\right)(p - 1)} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{\left(-p^{\frac{3}{2}} + 3\sqrt{p} - 2\right)x(p)}{2p\left(p - 2 + \sqrt{p}\right)(p - 1)} = 0
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-p^{\frac{3}{2}} + 3\sqrt{p} - 2}{2p(p-2+\sqrt{p})(p-1)}dp}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0
$$

$$
\frac{\mathrm{d}}{\mathrm{d}p}\left(\mathrm{e}^{\int -\frac{-p^{\frac{3}{2}} + 3\sqrt{p} - 2}{2p(p-2+\sqrt{p})(p-1)}dp} x\right) = 0
$$

Integrating gives

$$
\mathrm{e}^{\int -\frac{-p^{\frac{3}{2}} + 3\sqrt{p} - 2}{2p(p-2+\sqrt{p})(p-1)}dp} x = c_6
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-p^{\frac{3}{2}} + 3\sqrt{p} - 2}{2p(p-2+\sqrt{p})(p-1)}dp}$ results in

$$
x(p) = c_6 \mathrm{e}^{-\frac{\left(\int \frac{p^{\frac{3}{2}} - 3\sqrt{p} + 2}{p(p-2+\sqrt{p})(p-1)}dp\right)}{2}}
$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$
\begin{align*}
y &= 0 \tag{1} \\
y &= 4x \tag{2} \\
y &= 0 \tag{3}
\end{align*}
$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = 4x$$

Verified OK.

$$y = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 124

```
dsolve((x*diff(y(x),x)+y(x))^2=y(x)^2*diff(y(x),x),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 4x \\
y(x) &= 0 \\
y(x) &= -\frac{2c_1^2\left(-\sqrt{2}\,c_1 + x\right)}{-2c_1^2 + x^2} \\
y(x) &= -\frac{2c_1^2\left(\sqrt{2}\,c_1 + x\right)}{-2c_1^2 + x^2} \\
y(x) &= \frac{c_1^3\sqrt{2} - 2c_1^2 x}{-2c_1^2 + 4x^2} \\
y(x) &= \frac{c_1^2\left(\sqrt{2}\,c_1 + 2x\right)}{2c_1^2 - 4x^2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.635 (sec). Leaf size: 62

```
DSolve[(x*y'[x]+y[x])^2==y[x]^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\frac{4e^{-2c_1}}{2 + e^{2c_1}x} \\
y(x) &\to -\frac{e^{-2c_1}}{2 + 4e^{2c_1}x} \\
y(x) &\to 0 \\
y(x) &\to 4x
\end{aligned}
$$

11.12 problem 271

Internal problem ID [15133]
Internal file name [`OUTPUT/15133_Sunday_April_21_2024_01_36_48_PM_21693115/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 271.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y^2 {y'}^2 + y^2 = 1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{1-y^2}}{y} \tag{1}$$

$$y' = -\frac{\sqrt{1-y^2}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{\sqrt{-y^2+1}} dy = \int dx$$

$$\frac{(y-1)(y+1)}{\sqrt{1-y^2}} = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{(y-1)(y+1)}{\sqrt{1-y^2}} = x + c_1 \tag{1}$$

Verification of solutions

$$\frac{(y-1)(y+1)}{\sqrt{1-y^2}} = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{-y^2+1}}dy = \int dx$$
$$-\frac{(y-1)(y+1)}{\sqrt{1-y^2}} = x + c_2$$

Summary
The solution(s) found are the following

$$-\frac{(y-1)(y+1)}{\sqrt{1-y^2}} = x + c_2 \tag{1}$$

Verification of solutions

$$-\frac{(y-1)(y+1)}{\sqrt{1-y^2}} = x + c_2$$

Verified OK.

11.12.1 Maple step by step solution

Let's solve

$y^2{y'}^2 + y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{1-y^2}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{1-y^2}}dx = \int 1dx + c_1$

- Evaluate integral

  $-\sqrt{1-y^2} = x + c_1$

- Solve for y

  $\left\{y = \sqrt{-c_1^2 - 2c_1x - x^2 + 1}, y = -\sqrt{-c_1^2 - 2c_1x - x^2 + 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve(y(x)^2*diff(y(x),x)^2+y(x)^2=1,y(x), singsol=all)
```

$$y(x) = -1$$
$$y(x) = 1$$
$$y(x) = \sqrt{-c_1^2 + 2c_1x - x^2 + 1}$$
$$y(x) = -\sqrt{-(x - c_1 + 1)(x - c_1 - 1)}$$

✓ Solution by Mathematica

Time used: 0.185 (sec). Leaf size: 119

```
DSolve[y[x]^2*y'[x]^2+y[x]^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 - 2c_1x + 1 - c_1^2}$$
$$y(x) \to \sqrt{-x^2 - 2c_1x + 1 - c_1^2}$$
$$y(x) \to -\sqrt{-x^2 + 2c_1x + 1 - c_1^2}$$
$$y(x) \to \sqrt{-x^2 + 2c_1x + 1 - c_1^2}$$
$$y(x) \to -1$$
$$y(x) \to 1$$

11.13 problem 272

Internal problem ID [15134]
Internal file name [`OUTPUT/15134_Sunday_April_21_2024_01_36_48_PM_33329746/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 272.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y'^2 - yy' = -e^x}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{y}{2} + \frac{\sqrt{y^2 - 4\,e^x}}{2} \tag{1}$$
$$y' = \frac{y}{2} - \frac{\sqrt{y^2 - 4\,e^x}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{y}{2} + \frac{\sqrt{y^2 - 4\,e^x}}{2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(\frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right)(b_3 - a_2) - \left(\frac{y}{2} + \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right)^2 a_3 + \frac{\mathrm{e}^x (xa_2 + ya_3 + a_1)}{\sqrt{y^2 - 4\,\mathrm{e}^x}} - \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2 - 4\,\mathrm{e}^x}}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{-(y^2 - 4\,\mathrm{e}^x)^{\frac{3}{2}} a_3 - \sqrt{y^2 - 4\,\mathrm{e}^x}\, y^2 a_3 - 2y^3 a_3 + 4\,\mathrm{e}^x x a_2 + 12\,\mathrm{e}^x y a_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, x b_2 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, y a_2 - \cdots}{4\sqrt{y^2 - 4\,\mathrm{e}^x}} = 0$$

Setting the numerator to zero gives

$$-(y^2 - 4\,\mathrm{e}^x)^{\frac{3}{2}} a_3 - \sqrt{y^2 - 4\,\mathrm{e}^x}\, y^2 a_3 - 2y^3 a_3 + 4\,\mathrm{e}^x x a_2 + 12\,\mathrm{e}^x y a_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, x b_2 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, y a_2 - 2xyb_2 - 2y^2 a_2 + 4\,\mathrm{e}^x a_1 + 8\,\mathrm{e}^x a_2 - 8\,\mathrm{e}^x b_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, b_1 + 4b_2\sqrt{y^2 - 4\,\mathrm{e}^x} - 2yb_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(y^2 - 4\,\mathrm{e}^x)^{\frac{3}{2}} a_3 - 2(y^2 - 4\,\mathrm{e}^x)\, y a_3 - \sqrt{y^2 - 4\,\mathrm{e}^x}\, y^2 a_3 + 4\,\mathrm{e}^x x a_2 + 4\,\mathrm{e}^x y a_3 - 2(y^2 - 4\,\mathrm{e}^x)\, a_2 + 2(y^2 - 4\,\mathrm{e}^x)\, b_3 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, x b_2 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, y a_2 - 2xyb_2 - 2y^2 b_3 + 4\,\mathrm{e}^x a_1 - 2\sqrt{y^2 - 4\,\mathrm{e}^x}\, b_1 + 4b_2\sqrt{y^2 - 4\,\mathrm{e}^x} - 2yb_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-2\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3-2y^3a_3+4\,\mathrm{e}^x x a_2+4\,\mathrm{e}^x\sqrt{y^2-4\,\mathrm{e}^x}\,a_3+12\,\mathrm{e}^x y a_3 \\ -2\sqrt{y^2-4\,\mathrm{e}^x}\,x b_2-2xyb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,y a_2-2y^2a_2+4\,\mathrm{e}^x a_1 \\ +8\,\mathrm{e}^x a_2-8\,\mathrm{e}^x b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}-2yb_1=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{y^2-4\,\mathrm{e}^x},\mathrm{e}^x\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{y^2-4\,\mathrm{e}^x}=v_3,\mathrm{e}^x=v_4\right\}$$

The above PDE (6E) now becomes

$$-2v_2^3a_3-2v_3v_2^2a_3+4v_4v_1a_2-2v_2^2a_2-2v_3v_2a_2+12v_4v_2a_3+4v_4v_3a_3 \\ -2v_1v_2b_2-2v_3v_1b_2+4v_4a_1+8v_4a_2-2v_2b_1-2v_3b_1+4b_2v_3-8v_4b_3=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4\}$$

Equation (7E) now becomes

$$-2v_1v_2b_2-2v_3v_1b_2+4v_4v_1a_2-2v_2^3a_3-2v_3v_2^2a_3-2v_2^2a_2-2v_3v_2a_2 \\ +12v_4v_2a_3-2v_2b_1+4v_4v_3a_3+(-2b_1+4b_2)\,v_3+(4a_1+8a_2-8b_3)\,v_4=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_2&=0\\ 4a_2&=0\\ -2a_3&=0\\ 4a_3&=0\\ 12a_3&=0\\ -2b_1&=0\\ -2b_2&=0\\ -2b_1+4b_2&=0\\ 4a_1+8a_2-8b_3&=0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 2b_3 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 2 \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y}{2} + \frac{\sqrt{y^2-4\,\mathrm{e}^x}}{2}\right)(2) \\ &= -\sqrt{y^2-4\,\mathrm{e}^x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{y^2-4\,\mathrm{e}^x}} dy \end{aligned}$$

Which results in

$$S = -\ln \left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{2}+\frac{\sqrt{y^{2}-4\,\mathrm{e}^{x}}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{2\,\mathrm{e}^{x}}{\sqrt{y^{2}-4\,\mathrm{e}^{x}}\,\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right)}\\
S_y &= -\frac{1}{\sqrt{y^{2}-4\,\mathrm{e}^{x}}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y\sqrt{y^{2}-4\,\mathrm{e}^{x}}+y^{2}-4\,\mathrm{e}^{x}}{\sqrt{y^{2}-4\,\mathrm{e}^{x}}\,\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln \left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right) = c_1$$

Which simplifies to

$$-\ln\left(y+\sqrt{y^2-4\,\mathrm{e}^x}\right)=c_1$$

Which gives

$$y=\frac{\left(4\,\mathrm{e}^x\mathrm{e}^{2c_1}+1\right)\mathrm{e}^{-c_1}}{2}$$

Summary

The solution(s) found are the following

$$y=\frac{\left(4\,\mathrm{e}^x\mathrm{e}^{2c_1}+1\right)\mathrm{e}^{-c_1}}{2} \tag{1}$$

Verification of solutions

$$y=\frac{\left(4\,\mathrm{e}^x\mathrm{e}^{2c_1}+1\right)\mathrm{e}^{-c_1}}{2}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y'=\frac{y}{2}-\frac{\sqrt{y^2-4\,\mathrm{e}^x}}{2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\left(\frac{y}{2}-\frac{\sqrt{y^2-4\,\mathrm{e}^x}}{2}\right)(b_3-a_2)-\left(\frac{y}{2}-\frac{\sqrt{y^2-4\,\mathrm{e}^x}}{2}\right)^2 a_3 \\
&\quad-\frac{\mathrm{e}^x(xa_2+ya_3+a_1)}{\sqrt{y^2-4\,\mathrm{e}^x}}-\left(\frac{1}{2}-\frac{y}{2\sqrt{y^2-4\,\mathrm{e}^x}}\right)(xb_2+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(y^2-4\,\mathrm{e}^x)^{\frac{3}{2}}\,a_3+\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3-2y^3a_3+4\,\mathrm{e}^x xa_2+12\,\mathrm{e}^x ya_3+2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2+2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2-2}{4\sqrt{y^2-4\,\mathrm{e}^x}}
$$

$$=0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-(y^2-4\,\mathrm{e}^x)^{\frac{3}{2}}\,a_3-\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3+2y^3a_3-4\,\mathrm{e}^x xa_2-12\,\mathrm{e}^x ya_3 \\
&\quad-2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2+2xyb_2+2y^2a_2-4\,\mathrm{e}^x a_1 \\
&\quad-8\,\mathrm{e}^x a_2+8\,\mathrm{e}^x b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}+2yb_1=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-(y^2-4\,\mathrm{e}^x)^{\frac{3}{2}}\,a_3+2(y^2-4\,\mathrm{e}^x)\,ya_3-\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3-4\,\mathrm{e}^x xa_2-4\,\mathrm{e}^x ya_3 \\
&\quad+2(y^2-4\,\mathrm{e}^x)\,a_2-2(y^2-4\,\mathrm{e}^x)\,b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2 \\
&\quad+2xyb_2+2y^2b_3-4\,\mathrm{e}^x a_1-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}+2yb_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2\sqrt{y^2-4\,\mathrm{e}^x}\,y^2a_3+2y^3a_3-4\,\mathrm{e}^x xa_2+4\,\mathrm{e}^x\sqrt{y^2-4\,\mathrm{e}^x}\,a_3-12\,\mathrm{e}^x ya_3 \\
&\quad-2\sqrt{y^2-4\,\mathrm{e}^x}\,xb_2+2xyb_2-2\sqrt{y^2-4\,\mathrm{e}^x}\,ya_2+2y^2a_2-4\,\mathrm{e}^x a_1 \\
&\quad-8\,\mathrm{e}^x a_2+8\,\mathrm{e}^x b_3-2\sqrt{y^2-4\,\mathrm{e}^x}\,b_1+4b_2\sqrt{y^2-4\,\mathrm{e}^x}+2yb_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{y^2-4\,\mathrm{e}^x},\mathrm{e}^x\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y^2 - 4\,\mathrm{e}^x} = v_3, \mathrm{e}^x = v_4\right\}$$

The above PDE (6E) now becomes

$$2v_2^3 a_3 - 2v_3 v_2^2 a_3 - 4v_4 v_1 a_2 + 2v_2^2 a_2 - 2v_3 v_2 a_2 - 12v_4 v_2 a_3 + 4v_4 v_3 a_3 + 2v_1 v_2 b_2 - 2v_3 v_1 b_2 - 4v_4 a_1 - 8v_4 a_2 + 2v_2 b_1 - 2v_3 b_1 + 4b_2 v_3 + 8v_4 b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$2v_1 v_2 b_2 - 2v_3 v_1 b_2 - 4v_4 v_1 a_2 + 2v_2^3 a_3 - 2v_3 v_2^2 a_3 + 2v_2^2 a_2 - 2v_3 v_2 a_2 - 12v_4 v_2 a_3 + 2v_2 b_1 + 4v_4 v_3 a_3 + (-2b_1 + 4b_2)\, v_3 + (-4a_1 - 8a_2 + 8b_3)\, v_4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_2 &= 0 \\
-2a_2 &= 0 \\
2a_2 &= 0 \\
-12a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
4a_3 &= 0 \\
2b_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
-2b_1 + 4b_2 &= 0 \\
-4a_1 - 8a_2 + 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 2b_3 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2 \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{y}{2} - \frac{\sqrt{y^2 - 4\,\mathrm{e}^x}}{2}\right) (2) \\
&= \sqrt{y^2 - 4\,\mathrm{e}^x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sqrt{y^2 - 4\,\mathrm{e}^x}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{2}-\frac{\sqrt{y^{2}-4\,\mathrm{e}^{x}}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{2\,\mathrm{e}^{x}}{\sqrt{y^{2}-4\,\mathrm{e}^{x}}\,\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right)}\\
S_y &= \frac{1}{\sqrt{y^{2}-4\,\mathrm{e}^{x}}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right) = c_1$$

Which simplifies to

$$\ln\left(y+\sqrt{y^{2}-4\,\mathrm{e}^{x}}\right) = c_{1}$$

Which gives

$$y = \frac{\left(\mathrm{e}^{2c_{1}}+4\,\mathrm{e}^{x}\right)\mathrm{e}^{-c_{1}}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{2c_{1}}+4\,\mathrm{e}^{x}\right)\mathrm{e}^{-c_{1}}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(\mathrm{e}^{2c_{1}}+4\,\mathrm{e}^{x}\right)\mathrm{e}^{-c_{1}}}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
      <- 1st order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 34

```
dsolve(diff(y(x),x)^2-y(x)*diff(y(x),x)+exp(x)=0,y(x), singsol=all)
```

$$y(x) = -2\,\mathrm{e}^{\frac{x}{2}}$$
$$y(x) = 2\,\mathrm{e}^{\frac{x}{2}}$$
$$y(x) = \frac{\mathrm{e}^{x} c_1^2 + 1}{c_1}$$

✓ Solution by Mathematica

Time used: 60.179 (sec). Leaf size: 59

```
DSolve[y'[x]^2-y[x]*y'[x]+Exp[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-e^{-c_1} \left(-e^{x} + e^{c_1}\right)^2}$$
$$y(x) \to \sqrt{-e^{-c_1} \left(e^{x} - e^{c_1}\right)^2}$$

11.14 problem 273

Internal problem ID [15135]
Internal file name [`OUTPUT/15135_Sunday_April_21_2024_01_36_53_PM_85529182/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 273.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{3xy'^2 - 6yy' + 2y = -x}$$

11.14.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$3x\,p^2 - 6yp + 2y = -x$$

Solving for y from the above results in

$$y = \frac{x(3p^2 + 1)}{6p - 2} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{3p^2 + 1}{6p - 2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{3p^2 + 1}{6p - 2} = x\left(\frac{6p}{6p - 2} - \frac{6(3p^2 + 1)}{(6p - 2)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{3p^2 + 1}{6p - 2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 1 \\
p &= -\frac{1}{3}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= x \\
y &= -\frac{x}{3}
\end{aligned}
$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{3p(x)^2 + 1}{6p(x) - 2}}{x\left(\frac{6p(x)}{6p(x) - 2} - \frac{6\left(3p(x)^2 + 1\right)}{(6p(x) - 2)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{1}{3x}
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = -\frac{1}{3x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(-\frac{1}{3x}\right) \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{1}{3x}\right) \\ d\left(\frac{p}{x}\right) &= \left(-\frac{1}{3x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int -\frac{1}{3x^2} dx \\ \frac{p}{x} &= \frac{1}{3x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = \frac{1}{3} + c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{3\left(\frac{1}{3} + c_1 x\right)^2 + 1}{6c_1}$$

Summary

The solution(s) found are the following

$$y = x \tag{1}$$

$$y = -\frac{x}{3} \tag{2}$$

$$y = \frac{3\left(\frac{1}{3} + c_1 x\right)^2 + 1}{6c_1} \tag{3}$$

Verification of solutions

$$y = x$$

Verified OK.

$$y = -\frac{x}{3}$$

Verified OK.

$$y = \frac{3\left(\frac{1}{3} + c_1 x\right)^2 + 1}{6c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve(3*x*diff(y(x),x)^2-6*y(x)*diff(y(x),x)+x+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = x$$
$$y(x) = -\frac{x}{3}$$
$$y(x) = \frac{4c_1^2 + 2c_1 x + x^2}{6c_1}$$

✓ Solution by Mathematica

Time used: 0.307 (sec). Leaf size: 67

```
DSolve[3*x*y'[x]^2-6*y[x]*y'[x]+x+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3}x\left(-1 + 2\cosh\left(-\log(x) + \sqrt{3}c_1\right)\right)$$
$$y(x) \to -\frac{1}{3}x\left(-1 + 2\cosh\left(\log(x) + \sqrt{3}c_1\right)\right)$$
$$y(x) \to -\frac{x}{3}$$
$$y(x) \to x$$

11.15 problem 274

Internal problem ID [15136]
Internal file name [`OUTPUT/15136_Sunday_April_21_2024_01_36_54_PM_67669491/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 11. Singular solutions of differential equations. Exercises page 92
Problem number: 274.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

$$\boxed{y - xy' - \sqrt{a^2 y'^2 + b^2} = 0}$$

11.15.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y - xp - \sqrt{a^2 p^2 + b^2} = 0$$

Solving for y from the above results in

$$y = xp + \sqrt{a^2 p^2 + b^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \sqrt{a^2 p^2 + b^2} \\ &= xp + \sqrt{a^2 p^2 + b^2} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \sqrt{a^2p^2 + b^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1x + \sqrt{a^2c_1^2 + b^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{a^2p^2 + b^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{xb}{\sqrt{a^2 - x^2}\, a} \\
p_2 &= -\frac{xb}{\sqrt{a^2 - x^2}\, a}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= \frac{\sqrt{\frac{a^2 b^2}{a^2 - x^2}}\, \sqrt{a^2 - x^2}\, a + b\, x^2}{\sqrt{a^2 - x^2}\, a} \\
y_2 &= \frac{\sqrt{\frac{a^2 b^2}{a^2 - x^2}}\, \sqrt{a^2 - x^2}\, a - b\, x^2}{\sqrt{a^2 - x^2}\, a}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \sqrt{a^2 c_1^2 + b^2} \tag{1}
$$

$$
y = \frac{\sqrt{\frac{a^2 b^2}{a^2 - x^2}}\, \sqrt{a^2 - x^2}\, a + b\, x^2}{\sqrt{a^2 - x^2}\, a} \tag{2}
$$

$$
y = \frac{\sqrt{\frac{a^2 b^2}{a^2 - x^2}}\, \sqrt{a^2 - x^2}\, a - b\, x^2}{\sqrt{a^2 - x^2}\, a} \tag{3}
$$

Verification of solutions

$$
y = c_1 x + \sqrt{a^2 c_1^2 + b^2}
$$

Verified OK.

$$
y = \frac{\sqrt{\frac{a^2 b^2}{a^2 - x^2}}\, \sqrt{a^2 - x^2}\, a + b\, x^2}{\sqrt{a^2 - x^2}\, a}
$$

Verified OK.

$$
y = \frac{\sqrt{\frac{a^2 b^2}{a^2 - x^2}}\, \sqrt{a^2 - x^2}\, a - b\, x^2}{\sqrt{a^2 - x^2}\, a}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 21

```
dsolve(y(x)=x*diff(y(x),x)+sqrt(a^2*diff(y(x),x)^2+b^2),y(x), singsol=all)
```

$$y(x) = c_1 x + \sqrt{a^2 c_1^2 + b^2}$$

✓ Solution by Mathematica

Time used: 0.359 (sec). Leaf size: 37

```
DSolve[y[x]==x*y'[x]+Sqrt[a^2*y'[x]^2+b^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{b^2 + a^2 c_1{}^2} + c_1 x$$
$$y(x) \to \sqrt{b^2}$$

12 Section 12. Miscellaneous problems. Exercises page 93

12.1 problem 275

Internal problem ID [15137]
Internal file name [OUTPUT/15137_Sunday_April_21_2024_01_36_56_PM_25192221/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 275.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (-y + x)^2 = 1}$$

12.1.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = x^2 - 2xy + y^2 + 1$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + (x^2 - 2xy + y^2 + 1)(b_3 - a_2) - (x^2 - 2xy + y^2 + 1)^2 a_3 \\ &- (2x - 2y)(xa_2 + ya_3 + a_1) - (-2x + 2y)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}&-x^4a_3 + 4x^3ya_3 - 6x^2y^2a_3 + 4x\,y^3a_3 - y^4a_3 - 3x^2a_2 - 2x^2a_3 + 2x^2b_2 + x^2b_3 + 4xya_2 \\ &+ 2xya_3 - 2xyb_2 - y^2a_2 - y^2b_3 - 2xa_1 + 2xb_1 + 2ya_1 - 2yb_1 - a_2 - a_3 + b_2 + b_3 = 0\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}&-x^4a_3 + 4x^3ya_3 - 6x^2y^2a_3 + 4x\,y^3a_3 - y^4a_3 - 3x^2a_2 - 2x^2a_3 \\ &+ 2x^2b_2 + x^2b_3 + 4xya_2 + 2xya_3 - 2xyb_2 - y^2a_2 - y^2b_3 \\ &- 2xa_1 + 2xb_1 + 2ya_1 - 2yb_1 - a_2 - a_3 + b_2 + b_3 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-a_3v_1^4 + 4a_3v_1^3v_2 - 6a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 - 3a_2v_1^2 + 4a_2v_1v_2 \\ &- a_2v_2^2 - 2a_3v_1^2 + 2a_3v_1v_2 + 2b_2v_1^2 - 2b_2v_1v_2 + b_3v_1^2 - b_3v_2^2 \\ &- 2a_1v_1 + 2a_1v_2 + 2b_1v_1 - 2b_1v_2 - a_2 - a_3 + b_2 + b_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a_3v_1^4 + 4a_3v_1^3v_2 - 6a_3v_1^2v_2^2 + (-3a_2 - 2a_3 + 2b_2 + b_3)\,v_1^2 \\ + 4a_3v_1v_2^3 + (4a_2 + 2a_3 - 2b_2)\,v_1v_2 + (-2a_1 + 2b_1)\,v_1 - a_3v_2^4 \\ + (-a_2 - b_3)\,v_2^2 + (2a_1 - 2b_1)\,v_2 - a_2 - a_3 + b_2 + b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -6a_3 &= 0 \\ -a_3 &= 0 \\ 4a_3 &= 0 \\ -2a_1 + 2b_1 &= 0 \\ 2a_1 - 2b_1 &= 0 \\ -a_2 - b_3 &= 0 \\ 4a_2 + 2a_3 - 2b_2 &= 0 \\ -3a_2 - 2a_3 + 2b_2 + b_3 &= 0 \\ -a_2 - a_3 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= -2b_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left(x^2 - 2xy + y^2 + 1\right)(1) \\ &= -x^2 + 2xy - y^2 \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^2 + 2xy - y^2} dy \end{aligned}$$

Which results in

$$S = \frac{1}{-x + y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 - 2xy + y^2 + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{(x - y)^2} \\ S_y &= -\frac{1}{(x - y)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{-x+y} = -x + c_1$$

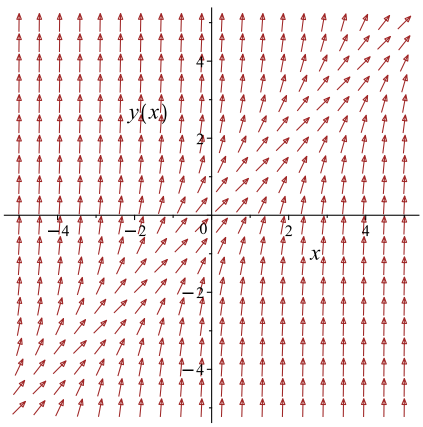
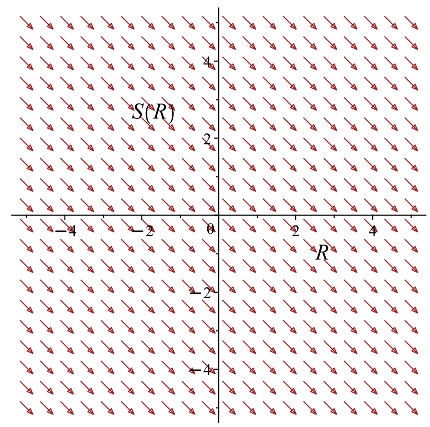
Which simplifies to

$$\frac{1}{-x+y} = -x + c_1$$

Which gives

$$y = \frac{c_1 x - x^2 + 1}{-x + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 - 2xy + y^2 + 1$ 	$R = x$ $S = \frac{1}{-x + y}$	$\frac{dS}{dR} = -1$ 

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1 x - x^2 + 1}{-x + c_1} \tag{1}$$

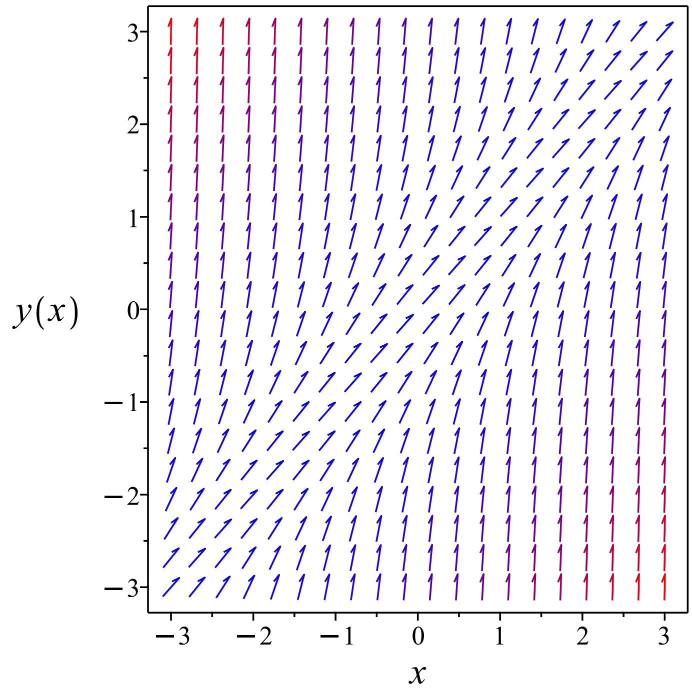


Figure 339: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - x^2 + 1}{-x + c_1}$$

Verified OK.

12.1.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 - 2xy + y^2 + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - 2xy + y^2 + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2 + 1$, $f_1(x) = -2x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -2x \\ f_2^2 f_0 &= x^2 + 1 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + 2xu'(x) + \left(x^2 + 1\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{-\frac{x^2}{2}} (c_2 x + c_1)$$

The above shows that

$$u'(x) = \mathrm{e}^{-\frac{x^2}{2}} \left(-c_2 x^2 - c_1 x + c_2\right)$$

Using the above in (1) gives the solution

$$y = -\frac{-c_2 x^2 - c_1 x + c_2}{c_2 x + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 x + x^2 - 1}{c_3 + x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 x + x^2 - 1}{c_3 + x} \tag{1}$$

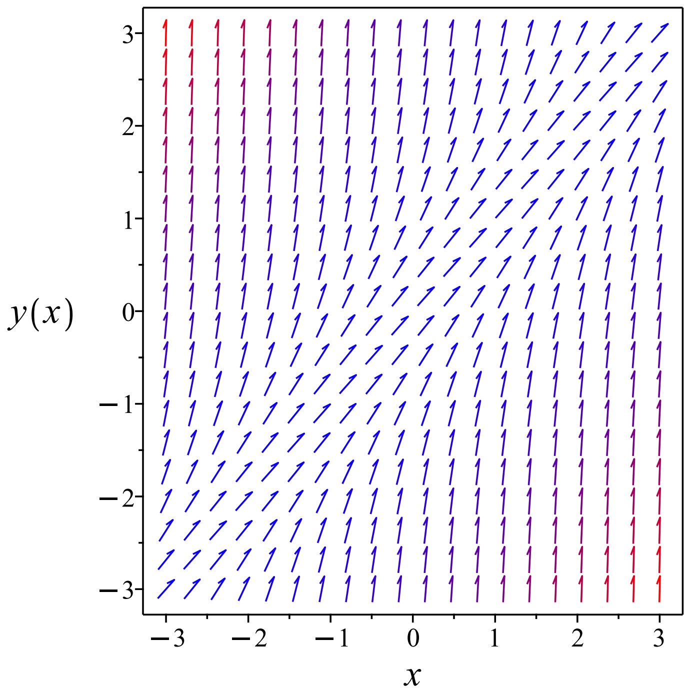


Figure 340: Slope field plot

Verification of solutions

$$y = \frac{c_3 x + x^2 - 1}{c_3 + x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)=(x-y(x))^2+1,y(x), singsol=all)
```

$$y(x) = \frac{c_1x + x^2 - 1}{x + c_1}$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 20

```
DSolve[y'[x]==(x-y[x])^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{1}{-x + c_1}$$
$$y(x) \to x$$

12.2 problem 276

Internal problem ID [15138]
Internal file name [`OUTPUT/15138_Sunday_April_21_2024_01_36_58_PM_79970788/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 276.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x \sin(x) y' + (\sin(x) - \cos(x) x) y = \sin(x) \cos(x) - x}$$

12.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{x \cot(x) - 1}{x}$$
$$q(x) = \frac{\cos(x) - x \csc(x)}{x}$$

Hence the ode is

$$y' - \frac{(x \cot(x) - 1) y}{x} = \frac{\cos(x) - x \csc(x)}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{x\cot(x)-1}{x}dx}$$
$$= e^{-\ln(\sin(x))+\ln(x)}$$

Which simplifies to

$$\mu = \frac{x}{\sin(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\cos(x) - x\csc(x)}{x}\right)$$
$$\frac{d}{dx}\left(\frac{xy}{\sin(x)}\right) = \left(\frac{x}{\sin(x)}\right)\left(\frac{\cos(x) - x\csc(x)}{x}\right)$$
$$d\left(\frac{xy}{\sin(x)}\right) = \left(\cot(x) - x\csc(x)^2\right)\,dx$$

Integrating gives

$$\frac{xy}{\sin(x)} = \int \cot(x) - x\csc(x)^2\,dx$$
$$\frac{xy}{\sin(x)} = x\cot(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{x}{\sin(x)}$ results in

$$y = \sin(x)\cot(x) + \frac{c_1\sin(x)}{x}$$

which simplifies to

$$y = \cos(x) + \frac{c_1\sin(x)}{x}$$

Summary

The solution(s) found are the following

$$y = \cos(x) + \frac{c_1\sin(x)}{x} \tag{1}$$

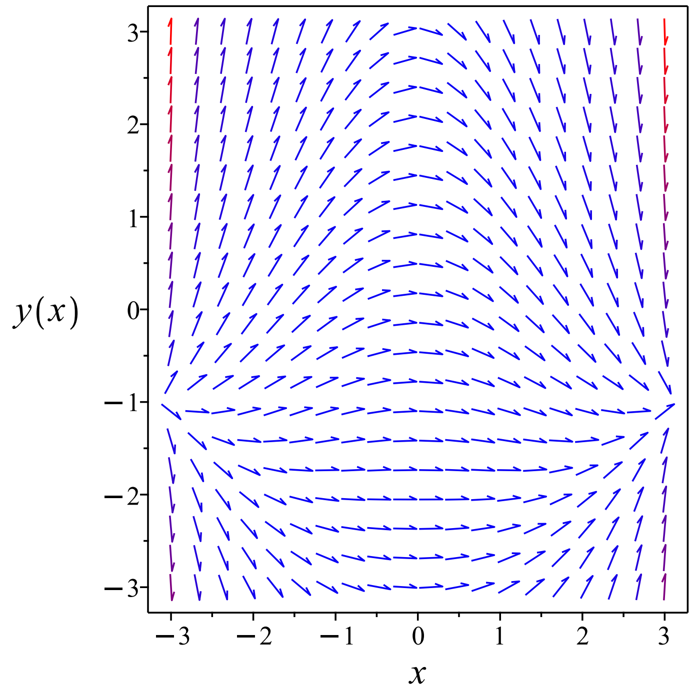


Figure 341: Slope field plot

Verification of solutions

$$y = \cos(x) + \frac{c_1 \sin(x)}{x}$$

Verified OK.

12.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\cos(x) xy + \sin(x) \cos(x) - y \sin(x) - x}{\sin(x) x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 274: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{\ln(\sin(x))-\ln(x)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\ln(\sin(x))-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{xy}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\cos(x)\,xy + \sin(x)\cos(x) - y\sin(x) - x}{\sin(x)\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y\csc(x)\,(x\cot(x) - 1) \\ S_y &= x\csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(x) - x\csc(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R) - R\csc(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R\cot(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{yx}{\sin(x)} = x\cot(x) + c_1$$

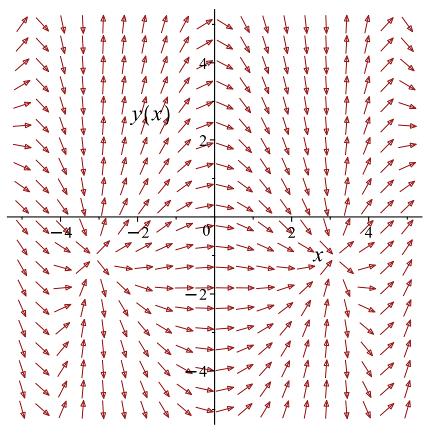
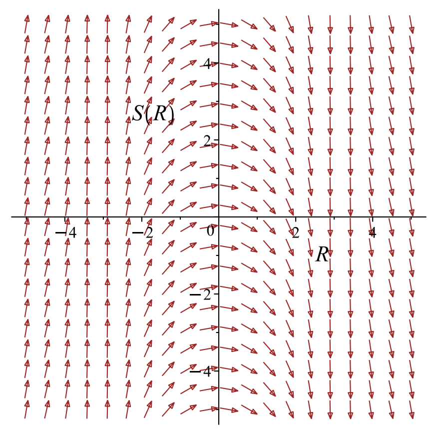
Which simplifies to

$$\frac{yx}{\sin(x)} = x\cot(x) + c_1$$

Which gives

$$y = \frac{\sin(x)\,(x\cot(x) + c_1)}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)xy+\sin(x)\cos(x)-y\sin(x)-x}{\sin(x)x}$ 	$R = x$ $S = \frac{xy}{\sin(x)}$	$\frac{dS}{dR} = \cot(R) - R\csc(R)^2$ 

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)(x\cot(x)+c_1)}{x} \tag{1}$$

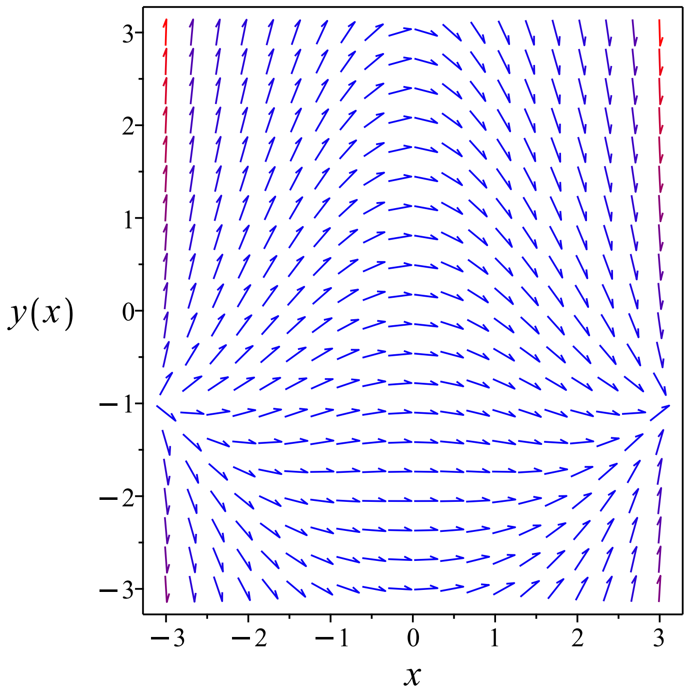


Figure 342: Slope field plot

Verification of solutions

$$y = \frac{\sin(x)(x\cot(x)+c_1)}{x}$$

Verified OK.

12.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(x\sin(x))\,\mathrm{d}y=(-(\sin(x)-\cos(x)\,x)\,y+\sin(x)\cos(x)$$

$$((\sin(x)-\cos(x)\,x)\,y-\sin(x)\cos(x)+x)\,\mathrm{d}x+(x\sin(x))\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=(\sin(x)-\cos(x)\,x)\,y-\sin(x)\cos(x)+x\\ N(x,y)&=x\sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}((\sin(x)-\cos(x)\,x)\,y-\sin(x)\cos(x)+x)\\ &=\sin(x)-\cos(x)\,x\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x \sin(x)) \\
&= \sin(x) + \cos(x) x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{\csc(x)}{x}((\sin(x) - \cos(x) x) - (\sin(x) + \cos(x) x)) \\
&= -2\cot(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -2\cot(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(\sin(x))} \\
&= \csc(x)^2
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \csc(x)^2 ((\sin(x) - \cos(x) x) y - \sin(x)\cos(x) + x) \\
&= x\csc(x)^2 + y(-x\cot(x) + 1)\csc(x) - \cot(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \csc(x)^2 (x \sin(x)) \\
&= x\csc(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(x\csc(x)^2 + y(-x\cot(x) + 1)\csc(x) - \cot(x)\right) + (x\csc(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x \csc(x)^2 + y(-x \cot(x) + 1) \csc(x) - \cot(x) \, \mathrm{d}x$$
$$\phi = x(\csc(x)\, y - \cot(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x \csc(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x \csc(x)$. Therefore equation (4) becomes

$$x \csc(x) = x \csc(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x(\csc(x)\, y - \cot(x)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(\csc(x)\, y - \cot(x))$$

The solution becomes

$$y = \frac{x \cot(x) + c_1}{x \csc(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x \cot(x) + c_1}{x \csc(x)} \tag{1}$$

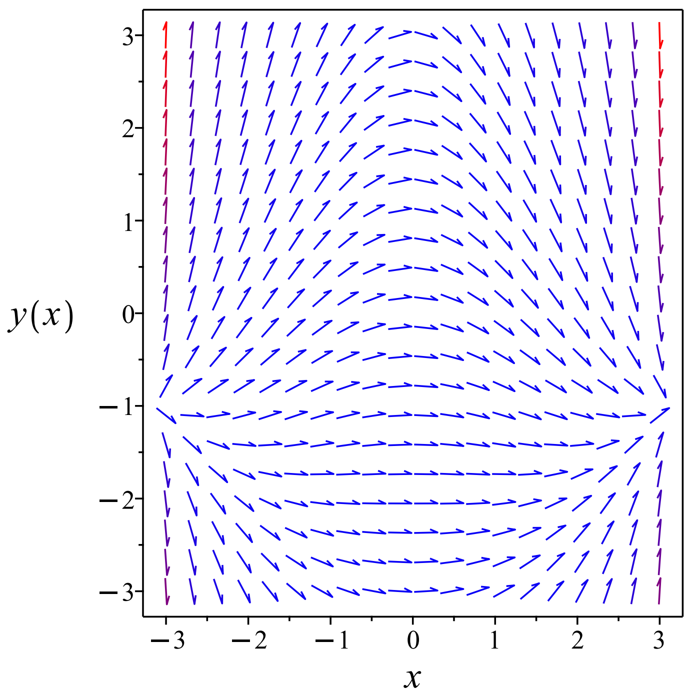


Figure 343: Slope field plot

Verification of solutions

$$y = \frac{x \cot(x) + c_1}{x \csc(x)}$$

Verified OK.

12.2.4 Maple step by step solution

Let's solve

$x \sin(x) y' + (\sin(x) - \cos(x) x) y = \sin(x) \cos(x) - x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{(\cos(x)x - \sin(x))y}{\sin(x)x} + \frac{\sin(x)\cos(x) - x}{\sin(x)x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{(\cos(x)x - \sin(x))y}{\sin(x)x} = \frac{\sin(x)\cos(x) - x}{\sin(x)x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{(\cos(x)x - \sin(x))y}{\sin(x)x}\right) = \frac{\mu(x)(\sin(x)\cos(x) - x)}{\sin(x)x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{(\cos(x)x - \sin(x))y}{\sin(x)x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)(\cos(x)x - \sin(x))}{\sin(x)x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{x}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(\sin(x)\cos(x) - x)}{\sin(x)x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(\sin(x)\cos(x) - x)}{\sin(x)x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)(\sin(x)\cos(x) - x)}{\sin(x)x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{x}{\sin(x)}$

  $y = \frac{\sin(x)\left(\int \frac{\sin(x)\cos(x) - x}{\sin(x)^2} dx + c_1\right)}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x)(x\cot(x) + c_1)}{x}$

- Simplify

  $y = \frac{\cos(x)x+\sin(x)c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(x*sin(x)*diff(y(x),x)+(sin(x)-x*cos(x))*y(x)=sin(x)*cos(x)-x,y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)\, c_1}{x} + \cos(x)$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 16

```
DSolve[x*Sin[x]*y'[x]+(Sin[x]-x*Cos[x])*y[x]==Sin[x]*Cos[x]-x,y[x],x,IncludeSingularSolution
```

$$y(x) \to \cos(x) + \frac{c_1 \sin(x)}{x}$$

12.3 problem 277

Internal problem ID [15139]
Internal file name [`OUTPUT/15139_Sunday_April_21_2024_01_37_04_PM_84217411/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 277.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + \cos(x)\, y - y^n \sin(2x) = 0}$$

12.3.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y\cos(x) + y^n \sin(2x) \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 277: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^n \mathrm{e}^{(n-1)\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^n \mathrm{e}^{(n-1)\sin(x)}} dy \end{aligned}$$

Which results in

$$S = -\frac{y\, y^{-n} \mathrm{e}^{-(n-1)\sin(x)}}{n-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y\cos(x) + y^n \sin(2x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^{-n+1} \cos(x)\, \mathrm{e}^{-(n-1)\sin(x)} \\ S_y &= y^{-n} \mathrm{e}^{-(n-1)\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\,\mathrm{e}^{-(n-1)\sin(x)} \cos(x)\sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\,\mathrm{e}^{-(n-1)\sin(R)} \cos(R)\sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1(n-1)^2 - 2\,\mathrm{e}^{-(n-1)\sin(R)}(1+(n-1)\sin(R))}{(n-1)^2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y^{-n+1}\mathrm{e}^{-(n-1)\sin(x)}}{n-1} = \frac{c_1(n-1)^2 - 2\,\mathrm{e}^{-(n-1)\sin(x)}(1+(n-1)\sin(x))}{(n-1)^2}$$

Which simplifies to

$$\frac{((-n+1)\,y^{-n+1} + 2 + (2n-2)\sin(x))\,\mathrm{e}^{-(n-1)\sin(x)} - c_1(n-1)^2}{(n-1)^2} = 0$$

Which gives

$$y = \mathrm{e}^{-\frac{\sin(x)n+\ln\left(\frac{2\sin(x)\mathrm{e}^{-(n-1)\sin(x)}n - c_1n^2 - 2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} + 2c_1n + 2\,\mathrm{e}^{-(n-1)\sin(x)} - c_1}{n-1}\right) - \sin(x)}{n-1}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\sin(x)n+\ln\left(\frac{2\sin(x)\mathrm{e}^{-(n-1)\sin(x)}n - c_1n^2 - 2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} + 2c_1n + 2\,\mathrm{e}^{-(n-1)\sin(x)} - c_1}{n-1}\right) - \sin(x)}{n-1}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{\sin(x)n+\ln\left(\frac{2\sin(x)\mathrm{e}^{-(n-1)\sin(x)}n - c_1n^2 - 2\sin(x)\mathrm{e}^{-(n-1)\sin(x)} + 2c_1n + 2\,\mathrm{e}^{-(n-1)\sin(x)} - c_1}{n-1}\right) - \sin(x)}{n-1}}$$

Verified OK.

12.3.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -y\cos(x) + y^n\sin(2x) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\cos(x)\,y + 2\sin(x)\cos(x)\,y^n \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\cos(x) \\ f_1(x) &= 2\sin(x)\cos(x) \\ n &= n \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^n$ gives

$$y'y^{-n} = -\cos(x)\, y^{-n+1} + 2\sin(x)\cos(x) \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{-n+1} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = (-n+1)\, y^{-n}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{-n+1} &= -\cos(x)\, w(x) + 2\sin(x)\cos(x) \\ w' &= -(-n+1)\cos(x)\, w + 2(-n+1)\sin(x)\cos(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -(n-1)\cos(x) \\
q(x) &= -(n-1)\sin(2x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) - (n-1)\cos(x)\, w(x) = -(n-1)\sin(2x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -(n-1)\cos(x)dx} \\
&= \mathrm{e}^{\sin(x)(-n+1)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \mathrm{e}^{-(n-1)\sin(x)}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-(n-1)\sin(2x)\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-(n-1)\sin(x)} w\right) &= \left(\mathrm{e}^{-(n-1)\sin(x)}\right)\left(-(n-1)\sin(2x)\right) \\
\mathrm{d}\left(\mathrm{e}^{-(n-1)\sin(x)} w\right) &= \left(-(n-1)\sin(2x)\,\mathrm{e}^{-(n-1)\sin(x)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-(n-1)\sin(x)} w &= \int -(n-1)\sin(2x)\,\mathrm{e}^{-(n-1)\sin(x)}\,\mathrm{d}x \\
\mathrm{e}^{-(n-1)\sin(x)} w &= \frac{2\,\mathrm{e}^{\sin(x)(-n+1)}(-n+1)\sin(x) - 2\,\mathrm{e}^{\sin(x)(-n+1)}}{-n+1} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-(n-1)\sin(x)}$ results in

$$
w(x) = \frac{2\,\mathrm{e}^{(n-1)\sin(x)}\left(\mathrm{e}^{\sin(x)(-n+1)}(-n+1)\sin(x) - \mathrm{e}^{\sin(x)(-n+1)}\right)}{-n+1} + c_1 \mathrm{e}^{(n-1)\sin(x)}
$$

which simplifies to

$$
w(x) = \frac{(n-1)\,c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1}
$$

Replacing w in the above by y^{-n+1} using equation (5) gives the final solution.

$$
y^{-n+1} = \frac{(n-1)\,c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1}
$$

Summary

The solution(s) found are the following

$$y^{-n+1} = \frac{(n-1)\, c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1} \tag{1}$$

Verification of solutions

$$y^{-n+1} = \frac{(n-1)\, c_1 \mathrm{e}^{(n-1)\sin(x)} + 2 + (2n-2)\sin(x)}{n-1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 49

```
dsolve(diff(y(x),x)+y(x)*cos(x)=y(x)^n*sin(2*x),y(x), singsol=all)
```

$$y(x) = \left(\frac{\mathrm{e}^{\sin(x)(n-1)} c_1 n - \mathrm{e}^{\sin(x)(n-1)} c_1 + 2\sin(x)\, n - 2\sin(x) + 2}{n-1}\right)^{-\frac{1}{n-1}}$$

✓ Solution by Mathematica

Time used: 6.877 (sec). Leaf size: 36

```
DSolve[y'[x]+y[x]*Cos[x]==y[x]^n*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(c_1 e^{(n-1)\sin(x)} + \frac{2}{n-1} + 2\sin(x)\right)^{\frac{1}{1-n}}$$

12.4 problem 278

Internal problem ID [15140]
Internal file name [`OUTPUT/15140_Sunday_April_21_2024_01_37_22_PM_78708226/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 278.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{-3y^2x + \left(y^3 - 3yx^2\right) y' = -x^3}$$

12.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-3u(x)^2\, x^3 + \left(u(x)^3\, x^3 - 3u(x)\, x^3\right) \left(u'(x)\, x + u(x)\right) = -x^3$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^4 - 6u^2 + 1}{ux\,(u^2 - 3)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^4-6u^2+1}{u(u^2-3)}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^4-6u^2+1}{u(u^2-3)}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{u^4-6u^2+1}{u(u^2-3)}}\,du &= \int -\frac{1}{x}\,dx \\
\frac{\ln\left(u^4-6u^2+1\right)}{4} &= -\ln\left(x\right)+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\left(u^4-6u^2+1\right)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\left(u^4-6u^2+1\right)^{\frac{1}{4}} = \frac{c_3}{x}
$$

Which simplifies to

$$
\left(u(x)^4-6u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}
$$

The solution is

$$
\left(u(x)^4-6u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{y^4}{x^4}-\frac{6y^2}{x^2}+1\right)^{\frac{1}{4}} &= \frac{c_3\mathrm{e}^{c_2}}{x} \\
\left(\frac{y^4-6y^2x^2+x^4}{x^4}\right)^{\frac{1}{4}} &= \frac{c_3\mathrm{e}^{c_2}}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\left(\frac{y^4-6y^2x^2+x^4}{x^4}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}
$$

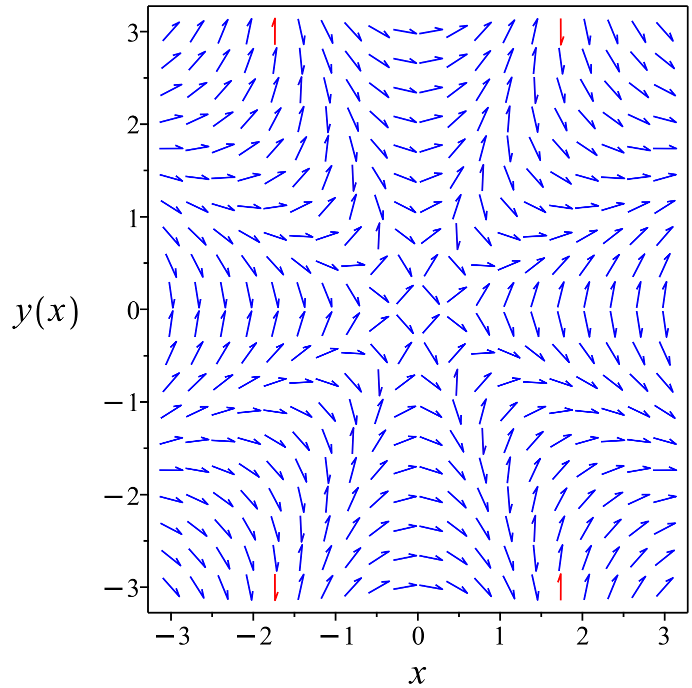


Figure 344: Slope field plot

Verification of solutions

$$\left(\frac{y^4-6y^2x^2+x^4}{x^4}\right)^{\frac{1}{4}}=\frac{c_3 e^{c_2}}{x}$$

Verified OK.

12.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x(-x^2+3y^2)}{y\,(-3x^2+y^2)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2+\frac{x(-x^2+3y^2)\,(b_3-a_2)}{y\,(-3x^2+y^2)}-\frac{x^2(-x^2+3y^2)^2\,a_3}{y^2\,(-3x^2+y^2)^2}\\
&-\left(\frac{-x^2+3y^2}{y\,(-3x^2+y^2)}-\frac{2x^2}{y\,(-3x^2+y^2)}+\frac{6x^2(-x^2+3y^2)}{y\,(-3x^2+y^2)^2}\right)(xa_2+ya_3+a_1)\\
&-\left(\frac{6x}{-3x^2+y^2}-\frac{x(-x^2+3y^2)}{y^2\,(-3x^2+y^2)}-\frac{2x(-x^2+3y^2)}{(-3x^2+y^2)^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^6a_3-3x^6b_2+6x^5ya_2-6x^5yb_3-3x^4y^2a_3-15x^4y^2b_2-4x^3y^3a_2+4x^3y^3b_3+15x^2y^4a_3+3x^2y^4b_2+6x}{y^2\,(3x^2-y^2)^2}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^6a_3+3x^6b_2-6x^5ya_2+6x^5yb_3+3x^4y^2a_3+15x^4y^2b_2+4x^3y^3a_2\\
&-4x^3y^3b_3-15x^2y^4a_3-3x^2y^4b_2-6x\,y^5a_2+6x\,y^5b_3-3y^6a_3+y^6b_2\\
&+3x^5b_1-3x^4ya_1+6x^3y^2b_1-6x^2y^3a_1+3x\,y^4b_1-3y^5a_1=0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-6a_2v_1^5v_2+4a_2v_1^3v_2^3-6a_2v_1v_2^5-a_3v_1^6+3a_3v_1^4v_2^2-15a_3v_1^2v_2^4-3a_3v_2^6\\
&+3b_2v_1^6+15b_2v_1^4v_2^2-3b_2v_1^2v_2^4+b_2v_2^6+6b_3v_1^5v_2-4b_3v_1^3v_2^3+6b_3v_1v_2^5\\
&-3a_1v_1^4v_2-6a_1v_1^2v_2^3-3a_1v_2^5+3b_1v_1^5+6b_1v_1^3v_2^2+3b_1v_1v_2^4=0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_3+3b_2)\,v_1^6+(-6a_2+6b_3)\,v_1^5v_2+3b_1v_1^5+(3a_3+15b_2)\,v_1^4v_2^2\\-3a_1v_1^4v_2+(4a_2-4b_3)\,v_1^3v_2^3+6b_1v_1^3v_2^2+(-15a_3-3b_2)\,v_1^2v_2^4\\-6a_1v_1^2v_2^3+(-6a_2+6b_3)\,v_1v_2^5+3b_1v_1v_2^4+(-3a_3+b_2)\,v_2^6-3a_1v_2^5=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-6a_1&=0\\-3a_1&=0\\3b_1&=0\\6b_1&=0\\-6a_2+6b_3&=0\\4a_2-4b_3&=0\\-15a_3-3b_2&=0\\-3a_3+b_2&=0\\-a_3+3b_2&=0\\3a_3+15b_2&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1&=0\\a_2&=b_3\\a_3&=0\\b_1&=0\\b_2&=0\\b_3&=b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi&=x\\\eta&=y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x(-x^2+3y^2)}{y\,(-3x^2+y^2)}\right)(x) \\
&= \frac{-x^4+6x^2y^2-y^4}{3x^2y-y^3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^4+6x^2y^2-y^4}{3x^2y-y^3}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(x^4-6x^2y^2+y^4\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(-x^2+3y^2)}{y\,(-3x^2+y^2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x(x^2 - 3y^2)}{(x^2 + 2xy - y^2)(x^2 - 2xy - y^2)} \\
S_y &= \frac{-3x^2y + y^3}{(x^2 + 2xy - y^2)(x^2 - 2xy - y^2)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

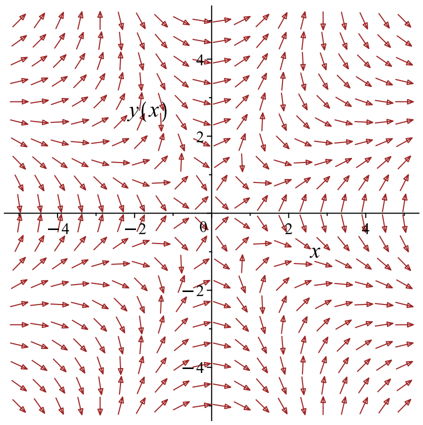
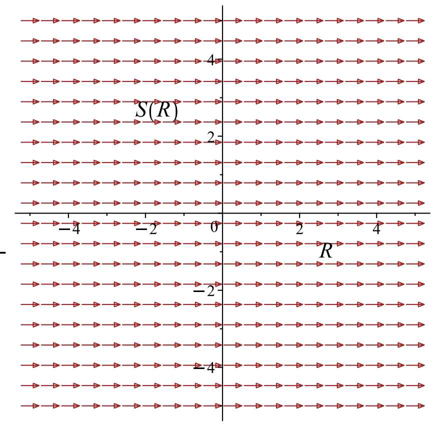
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2 + 2yx - y^2)}{4} + \frac{\ln(x^2 - 2yx - y^2)}{4} = c_1$$

Which simplifies to

$$\frac{\ln(x^2 + 2yx - y^2)}{4} + \frac{\ln(x^2 - 2yx - y^2)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(-x^2+3y^2)}{y(-3x^2+y^2)}$ 	$R = x$ $S = \frac{\ln(x^2+2xy-y^2)}{4} +$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+2yx-y^2)}{4} + \frac{\ln(x^2-2yx-y^2)}{4} = c_1 \tag{1}$$

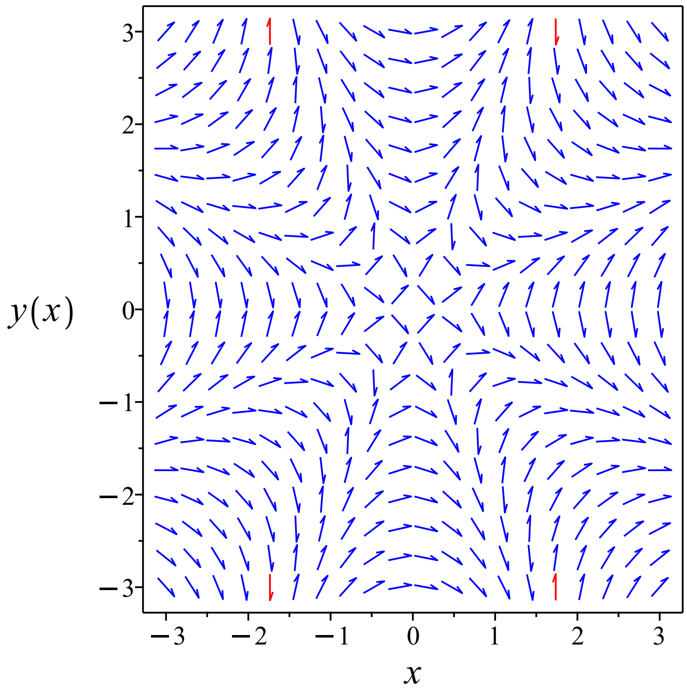


Figure 345: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+2yx-y^2)}{4}+\frac{\ln(x^2-2yx-y^2)}{4}=c_1$$

Verified OK.

12.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-3x^2y + y^3) \, \mathrm{d}y &= (-x^3 + 3x \, y^2) \, \mathrm{d}x \\
(x^3 - 3x \, y^2) \, \mathrm{d}x + (-3x^2y + y^3) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x^3 - 3x \, y^2 \\
N(x, y) &= -3x^2y + y^3
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^3 - 3x \, y^2) \\
&= -6xy
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-3x^2y + y^3) \\
&= -6xy
\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int x^3 - 3x\,y^2 \mathrm{d}x$$
$$\phi = \frac{(x^2-3y^2)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -3\left(x^2-3y^2\right)y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -3x^2y + y^3$. Therefore equation (4) becomes

$$-3x^2y + y^3 = -3\left(x^2-3y^2\right)y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -8y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-8y^3\right) \mathrm{d}y$$
$$f(y) = -2y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2-3y^2)^2}{4} - 2y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 - 3y^2)^2}{4} - 2y^4$$

Summary

The solution(s) found are the following

$$\frac{(x^2 - 3y^2)^2}{4} - 2y^4 = c_1 \tag{1}$$

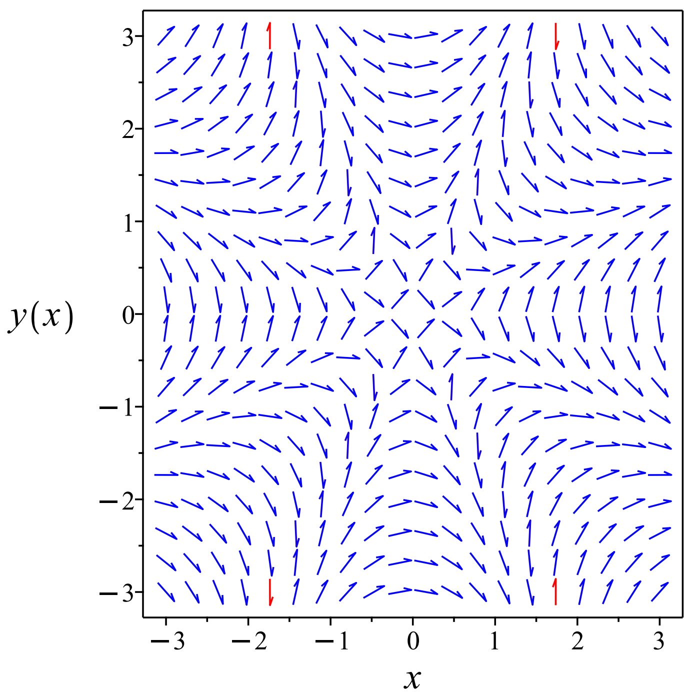


Figure 346: Slope field plot

Verification of solutions

$$\frac{(x^2 - 3y^2)^2}{4} - 2y^4 = c_1$$

Verified OK.

12.4.4 Maple step by step solution

Let's solve

$-3y^2x+(y^3-3yx^2)\,y'=-x^3$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $-6xy=-6xy$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(x^3-3x\,y^2)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{(x^2-3y^2)^2}{4}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-3x^2y+y^3=-3(x^2-3y^2)\,y+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=3(x^2-3y^2)\,y-3x^2y+y^3$

- Solve for $f_1(y)$

  $f_1(y)=-2y^4$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = \frac{(x^2-3y^2)^2}{4} - 2y^4$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{(x^2-3y^2)^2}{4} - 2y^4 = c_1$

- Solve for y

  $\left\{y = \sqrt{3x^2 - 2\sqrt{2x^4+c_1}}, y = \sqrt{3x^2 + 2\sqrt{2x^4+c_1}}, y = -\sqrt{3x^2 - 2\sqrt{2x^4+c_1}}, y = -\sqrt{3x^2 + 2\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 119

```
dsolve((x^3-3*x*y(x)^2)+(y(x)^3-3*x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{3c_1x^2 - \sqrt{8c_1^2x^4+1}}}{\sqrt{c_1}}$$

$$y(x) = \frac{\sqrt{3c_1x^2 + \sqrt{8c_1^2x^4+1}}}{\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{3c_1x^2 - \sqrt{8c_1^2x^4+1}}}{\sqrt{c_1}}$$

$$y(x) = -\frac{\sqrt{3c_1x^2 + \sqrt{8c_1^2x^4+1}}}{\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 7.89 (sec). Leaf size: 245

```
DSolve[(x^3-3*x*y[x]^2)+(y[x]^3-3*x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\sqrt{3x^2 - \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{3x^2 - \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{3x^2 + \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to \sqrt{3x^2 + \sqrt{8x^4 + e^{4c_1}}}$$
$$y(x) \to -\sqrt{3x^2 - 2\sqrt{2}\sqrt{x^4}}$$
$$y(x) \to \sqrt{3x^2 - 2\sqrt{2}\sqrt{x^4}}$$
$$y(x) \to -\sqrt{2\sqrt{2}\sqrt{x^4} + 3x^2}$$
$$y(x) \to \sqrt{2\sqrt{2}\sqrt{x^4} + 3x^2}$$

12.5 problem 279

Internal problem ID [15141]
Internal file name [`OUTPUT/15141_Tuesday_April_23_2024_04_50_46_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 279.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{5yx - 4y^2 + \left(y^2 - 8yx + \frac{5x^2}{2}\right) y' = 6x^2}$$

12.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$5u(x)\,x^2 - 4u(x)^2\,x^2 + \left(u(x)^2\,x^2 - 8u(x)\,x^2 + \frac{5x^2}{2}\right)(u'(x)\,x + u(x)) = 6x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^3 - 24u^2 + 15u - 12}{x\,(2u^2 - 16u + 5)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^3-24u^2+15u-12}{2u^2-16u+5}$. Integrating both sides gives

$$\frac{1}{\frac{2u^3-24u^2+15u-12}{2u^2-16u+5}} du = -\frac{1}{x} dx$$
$$\int \frac{1}{\frac{2u^3-24u^2+15u-12}{2u^2-16u+5}} du = \int -\frac{1}{x} dx$$
$$\frac{\ln(2u^3 - 24u^2 + 15u - 12)}{3} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$(2u^3 - 24u^2 + 15u - 12)^{\frac{1}{3}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$(2u^3 - 24u^2 + 15u - 12)^{\frac{1}{3}} = \frac{c_3}{x}$$

Which simplifies to

$$(2u(x)^3 - 24u(x)^2 + 15u(x) - 12)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$(2u(x)^3 - 24u(x)^2 + 15u(x) - 12)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^3}{x^3} - \frac{24y^2}{x^2} + \frac{15y}{x} - 12\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x}$$
$$\left(\frac{2y^3 - 24y^2x + 15yx^2 - 12x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\left(\frac{2y^3 - 24y^2x + 15yx^2 - 12x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x} \tag{1}$$

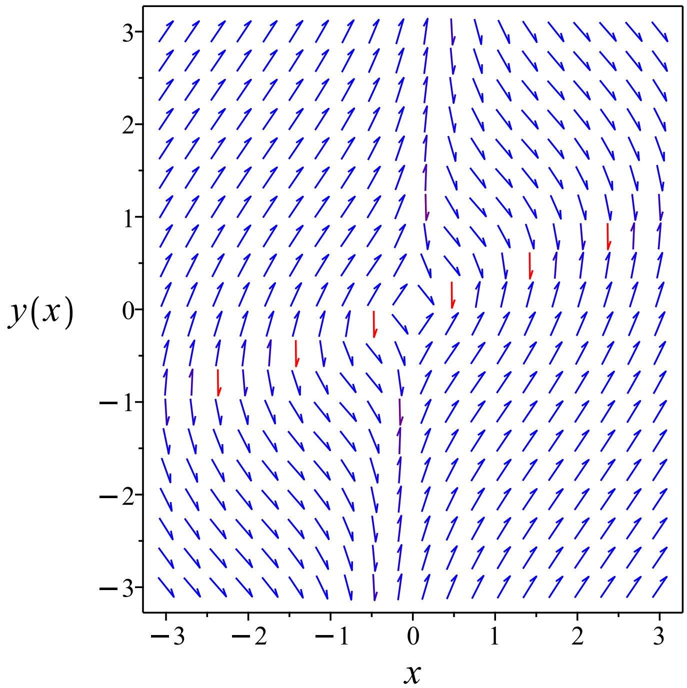


Figure 347: Slope field plot

Verification of solutions

$$\left(\frac{2y^3-24y^2x+15yx^2-12x^3}{x^3}\right)^{\frac{1}{3}}=\frac{c_3\mathrm{e}^{c_2}}{x}$$

Verified OK.

12.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{12x^2-10xy+8y^2}{5x^2-16xy+2y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{2(6x^2 - 5xy + 4y^2)(b_3 - a_2)}{5x^2 - 16xy + 2y^2} - \frac{4(6x^2 - 5xy + 4y^2)^2 a_3}{(5x^2 - 16xy + 2y^2)^2} \\ & - \left(\frac{24x - 10y}{5x^2 - 16xy + 2y^2} - \frac{2(6x^2 - 5xy + 4y^2)(10x - 16y)}{(5x^2 - 16xy + 2y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \left(\frac{-10x + 16y}{5x^2 - 16xy + 2y^2} \right. \\ & \left. - \frac{2(6x^2 - 5xy + 4y^2)(-16x + 4y)}{(5x^2 - 16xy + 2y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{60x^4a_2 + 144x^4a_3 + 117x^4b_2 - 60x^4b_3 - 384x^3ya_2 - 240x^3ya_3 + 192x^3yb_2 + 384x^3yb_3 + 192x^2y^2a_2 + 1}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -60x^4a_2 - 144x^4a_3 - 117x^4b_2 + 60x^4b_3 + 384x^3ya_2 + 240x^3ya_3 - 192x^3yb_2 \\ & - 384x^3yb_3 - 192x^2y^2a_2 - 150x^2y^2a_3 + 384x^2y^2b_2 + 192x^2y^2b_3 + 40x\,y^3a_2 \\ & + 192x\,y^3a_3 - 64x\,y^3b_2 - 40x\,y^3b_3 - 16y^4a_2 - 172y^4a_3 + 4y^4b_2 + 16y^4b_3 \\ & - 142x^3b_1 + 142x^2ya_1 - 32x^2yb_1 + 32x\,y^2a_1 + 108x\,y^2b_1 - 108y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-60a_2v_1^4+384a_2v_1^3v_2-192a_2v_1^2v_2^2+40a_2v_1v_2^3-16a_2v_2^4\\
&-144a_3v_1^4+240a_3v_1^3v_2-150a_3v_1^2v_2^2+192a_3v_1v_2^3-172a_3v_2^4\\
&-117b_2v_1^4-192b_2v_1^3v_2+384b_2v_1^2v_2^2-64b_2v_1v_2^3+4b_2v_2^4+60b_3v_1^4\\
&-384b_3v_1^3v_2+192b_3v_1^2v_2^2-40b_3v_1v_2^3+16b_3v_2^4+142a_1v_1^2v_2\\
&+32a_1v_1v_2^2-108a_1v_2^3-142b_1v_1^3-32b_1v_1^2v_2+108b_1v_1v_2^2=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-60a_2-144a_3-117b_2+60b_3)\,v_1^4+(384a_2+240a_3-192b_2-384b_3)\,v_1^3v_2\\
&-142b_1v_1^3+(-192a_2-150a_3+384b_2+192b_3)\,v_1^2v_2^2\\
&+(142a_1-32b_1)\,v_1^2v_2+(40a_2+192a_3-64b_2-40b_3)\,v_1v_2^3\\
&+(32a_1+108b_1)\,v_1v_2^2+(-16a_2-172a_3+4b_2+16b_3)\,v_2^4-108a_1v_2^3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-108a_1&=0\\
-142b_1&=0\\
32a_1+108b_1&=0\\
142a_1-32b_1&=0\\
-192a_2-150a_3+384b_2+192b_3&=0\\
-60a_2-144a_3-117b_2+60b_3&=0\\
-16a_2-172a_3+4b_2+16b_3&=0\\
40a_2+192a_3-64b_2-40b_3&=0\\
384a_2+240a_3-192b_2-384b_3&=0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1&=0\\
a_2&=b_3\\
a_3&=0\\
b_1&=0\\
b_2&=0\\
b_3&=b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{12x^2 - 10xy + 8y^2}{5x^2 - 16xy + 2y^2}\right)(x) \\
&= \frac{-12x^3 + 15x^2y - 24x\,y^2 + 2y^3}{5x^2 - 16xy + 2y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-12x^3 + 15x^2y - 24x\,y^2 + 2y^3}{5x^2 - 16xy + 2y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(-12x^3 + 15x^2y - 24x\,y^2 + 2y^3\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{12x^2 - 10xy + 8y^2}{5x^2 - 16xy + 2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{12x^2 - 10xy + 8y^2}{12x^3 - 15x^2y + 24x\,y^2 - 2y^3} \\ S_y &= \frac{-5x^2 + 16xy - 2y^2}{12x^3 - 15x^2y + 24x\,y^2 - 2y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

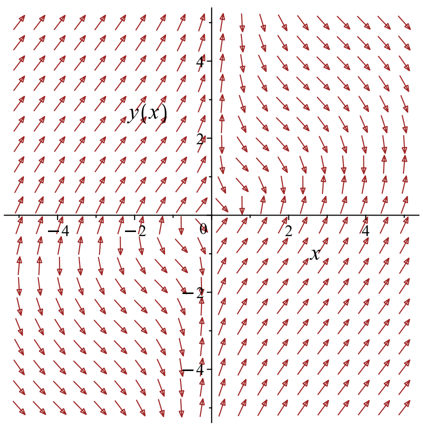
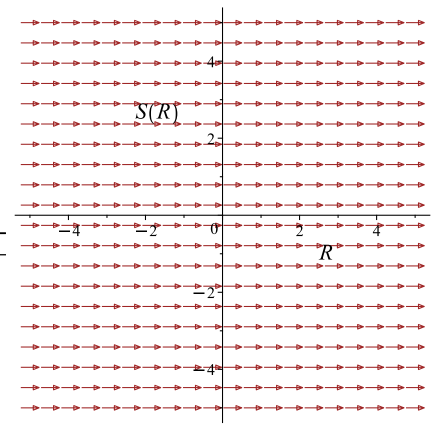
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y^3 - 24y^2x + 15yx^2 - 12x^3)}{3} = c_1$$

Which simplifies to

$$\frac{\ln(2y^3 - 24y^2x + 15yx^2 - 12x^3)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{12x^2-10xy+8y^2}{5x^2-16xy+2y^2}$ 	$R = x$ $S = \frac{\ln(-12x^3+15x^2y-}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2y^3-24y^2x+15yx^2-12x^3)}{3} = c_1 \tag{1}$$

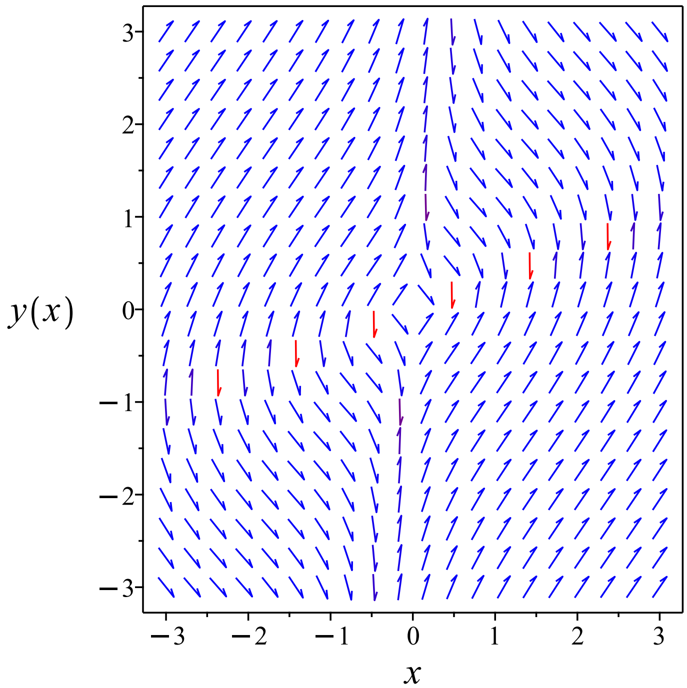


Figure 348: Slope field plot

Verification of solutions

$$\frac{\ln (2y^3 - 24y^2x + 15yx^2 - 12x^3)}{3} = c_1$$

Verified OK.

12.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(y^2 - 8xy + \frac{5}{2}x^2\right)\mathrm{d}y = \left(6x^2 - 5xy + 4y^2\right)\mathrm{d}x$$
$$\left(-6x^2 + 5xy - 4y^2\right)\mathrm{d}x + \left(y^2 - 8xy + \frac{5}{2}x^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -6x^2 + 5xy - 4y^2$$
$$N(x,y) = y^2 - 8xy + \frac{5}{2}x^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-6x^2 + 5xy - 4y^2\right)\\ &= 5x - 8y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y^2 - 8xy + \frac{5}{2}x^2\right)\\ &= 5x - 8y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -6x^2 + 5xy - 4y^2\,\mathrm{d}x$$

$$\phi = -2\left(x^2 - \frac{5}{4}xy + 2y^2\right)x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -2\left(-\frac{5x}{4} + 4y\right)x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2 - 8xy + \frac{5}{2}x^2$. Therefore equation (4) becomes

$$y^2 - 8xy + \frac{5}{2}x^2 = \frac{(5x - 16y)\,x}{2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2)\,\mathrm{d}y$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\left(x^2 - \frac{5}{4}xy + 2y^2\right)x + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\left(x^2 - \frac{5}{4}xy + 2y^2\right)x + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$-2\left(x^2 - \frac{5yx}{4} + 2y^2\right)x + \frac{y^3}{3} = c_1 \tag{1}$$

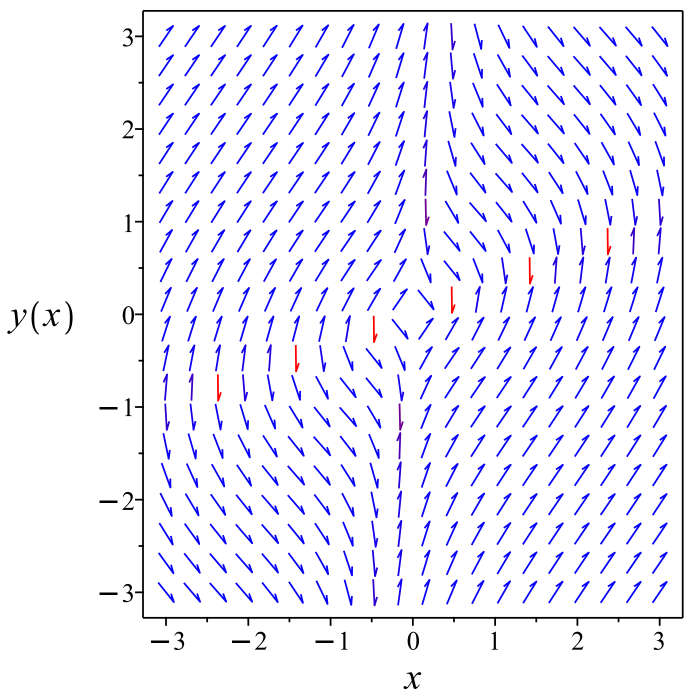


Figure 349: Slope field plot

Verification of solutions

$$-2\left(x^{2}-\frac{5yx}{4}+2y^{2}\right)x+\frac{y^{3}}{3}=c_{1}$$

Verified OK.

12.5.4 Maple step by step solution

Let's solve

$5yx-4y^{2}+\left(y^{2}-8yx+\frac{5x^{2}}{2}\right)y'=6x^{2}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - ∘ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - ∘ Evaluate derivatives

    $5x-8y=5x-8y$
  - ∘ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_{1},M(x,y)=F'(x,y),N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(-6x^{2}+5xy-4y^{2})\,dx+f_{1}(y)$

- Evaluate integral

  $F(x,y)=-2x^{3}+\frac{5x^{2}y}{2}-4x\,y^{2}+f_{1}(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y^{2}-8xy+\frac{5}{2}x^{2}=\frac{5x^{2}}{2}-8xy+\frac{d}{dy}f_{1}(y)$

- Isolate for $\frac{d}{dy}f_{1}(y)$

$\frac{d}{dy}f_1(y) = y^2$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = -2x^3 + \frac{5}{2}x^2y - 4x\,y^2 + \frac{1}{3}y^3$

- Substitute $F(x, y)$ into the solution of the ODE

  $-2x^3 + \frac{5}{2}x^2y - 4x\,y^2 + \frac{1}{3}y^3 = c_1$

- Solve for y

$$\left\{y = \frac{\left(416x^3+12c_1+2\sqrt{3898x^6+2496c_1x^3+36c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{27x^2}{\left(416x^3+12c_1+2\sqrt{3898x^6+2496c_1x^3+36c_1^2}\right)^{\frac{1}{3}}} + 4x, y = -\frac{(416x\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 443

```
dsolve((5*x*y(x)-4*y(x)^2-6*x^2)+(y(x)^2-8*x*y(x)+25/10*x^2)*diff(y(x),x)=0,y(x), singsol=al
```

$$y = \frac{\frac{\left(416x^3c_1^3+2+2\sqrt{3898c_1^6x^6+416x^3c_1^3+1}\right)^{\frac{1}{3}}}{2} + \frac{27x^2c_1^2}{\left(416x^3c_1^3+2+2\sqrt{3898c_1^6x^6+416x^3c_1^3+1}\right)^{\frac{1}{3}}} + 4c_1x}{c_1}$$

$$y = \frac{54i\sqrt{3}\,c_1^2x^2 - i\left(416x^3c_1^3+2+2\sqrt{3898c_1^6x^6+416x^3c_1^3+1}\right)^{\frac{2}{3}}\sqrt{3} - 54x^2c_1^2 + 16c_1x\left(416x^3c_1^3+2+2\sqrt{3\ldots}\right.}{4\left(416x^3c_1^3+2+2\sqrt{3898c_1^6x^6+416x^3\ldots}\right.}$$

$$y = -\frac{54i\sqrt{3}\,c_1^2x^2 - i\left(416x^3c_1^3+2+2\sqrt{3898c_1^6x^6+416x^3c_1^3+1}\right)^{\frac{2}{3}}\sqrt{3} + 54x^2c_1^2 - 16c_1x\left(416x^3c_1^3+2+2\ldots\right.}{4\left(416x^3c_1^3+2+2\sqrt{3898c_1^6x^6+416\ldots}\right.}$$

✓ Solution by Mathematica

Time used: 29.95 (sec). Leaf size: 741

```
DSolve[(5*x*y[x]-4*y[x]^2-6*x^2)+(y[x]^2-8*x*y[x]+25/10*x^2)*y'[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to \frac{\sqrt[3]{208x^3+\sqrt{3898x^6+416e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{2^{2/3}} + \frac{27x^2}{\sqrt[3]{2}\sqrt[3]{208x^3+\sqrt{3898x^6+416e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}} + 4x$$

$$y(x) \to -\frac{\left(1-i\sqrt{3}\right)\sqrt[3]{208x^3+\sqrt{3898x^6+416e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{2\ 2^{2/3}} - \frac{27\left(1+i\sqrt{3}\right)x^2}{2\sqrt[3]{2}\sqrt[3]{208x^3+\sqrt{3898x^6+416e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}} + 4x$$

$$y(x) \to -\frac{\left(1+i\sqrt{3}\right)\sqrt[3]{208x^3+\sqrt{3898x^6+416e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}}{2\ 2^{2/3}} - \frac{27\left(1-i\sqrt{3}\right)x^2}{2\sqrt[3]{2}\sqrt[3]{208x^3+\sqrt{3898x^6+416e^{3c_1}x^3+e^{6c_1}}+e^{3c_1}}} + 4x$$

$$y(x) \to \frac{27\ 2^{2/3}x^2+8\sqrt[3]{\sqrt{3898}\sqrt{x^6}+208x^3}x+\sqrt[3]{2}\left(\sqrt{3898}\sqrt{x^6}+208x^3\right)^{2/3}}{2\sqrt[3]{\sqrt{3898}\sqrt{x^6}+208x^3}}$$

$$y(x) \to \frac{27i2^{2/3}\sqrt{3}x^2-27\ 2^{2/3}x^2+16\sqrt[3]{\sqrt{3898}\sqrt{x^6}+208x^3}x-i\sqrt[3]{2}\sqrt{3}\left(\sqrt{3898}\sqrt{x^6}+208x^3\right)^{2/3}-\sqrt[3]{2}\left(\sqrt{3898}\right.}{4\sqrt[3]{\sqrt{3898}\sqrt{x^6}+208x^3}}$$

$$y(x) \to \frac{\left(\sqrt{3898}\sqrt{x^6}+208x^3\right)^{2/3}\text{Root}\left[\#1^3-16\&,3\right]-54\sqrt[3]{-1}2^{2/3}x^2+16\sqrt[3]{\sqrt{3898}\sqrt{x^6}+208x^3}x}{4\sqrt[3]{\sqrt{3898}\sqrt{x^6}+208x^3}}$$

12.6 problem 280

Internal problem ID [15142]
Internal file name [`OUTPUT/15142_Tuesday_April_23_2024_04_50_57_PM_14616075/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 280.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{3y^2x + (3yx^2 - 6y^2 - 1)\, y' = x^2}$$

12.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3x^2y - 6y^2 - 1\right)\mathrm{d}y &= \left(-3x\,y^2 + x^2\right)\mathrm{d}x\\ \left(3x\,y^2 - x^2\right)\mathrm{d}x + \left(3x^2y - 6y^2 - 1\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3x\,y^2 - x^2\\ N(x, y) &= 3x^2y - 6y^2 - 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x\,y^2 - x^2\right)\\ &= 6xy\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3x^2y - 6y^2 - 1\right)\\ &= 6xy\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3x\, y^2 - x^2 \, \mathrm{d}x$$
$$\phi = \frac{3}{2}x^2y^2 - \frac{1}{3}x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x^2y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^2y - 6y^2 - 1$. Therefore equation (4) becomes

$$3x^2y - 6y^2 - 1 = 3x^2y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -6y^2 - 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-6y^2 - 1\right) \mathrm{d}y$$
$$f(y) = -2y^3 - y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3}{2}x^2y^2 - \frac{1}{3}x^3 - 2y^3 - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3}{2}x^2y^2 - \frac{1}{3}x^3 - 2y^3 - y$$

Summary
The solution(s) found are the following

$$\frac{3y^2x^2}{2} - \frac{x^3}{3} - 2y^3 - y = c_1 \tag{1}$$

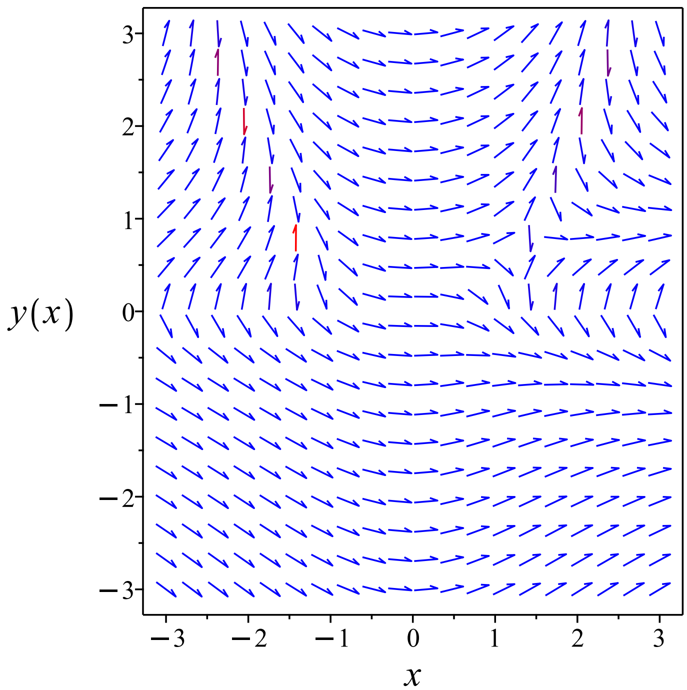


Figure 350: Slope field plot

Verification of solutions

$$\frac{3y^2x^2}{2} - \frac{x^3}{3} - 2y^3 - y = c_1$$

Verified OK.

12.6.2 Maple step by step solution

Let's solve

$3y^2x + (3yx^2 - 6y^2 - 1)\, y' = x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$
- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
- Evaluate derivatives

  $6xy=6xy$
- Condition met, ODE is exact

* Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
* Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (3x\,y^2-x^2)\,dx+f_1(y)$
* Evaluate integral

  $F(x,y)=\frac{3x^2y^2}{2}-\frac{x^3}{3}+f_1(y)$
* Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
* Compute derivative

  $3x^2y-6y^2-1=3x^2y+\frac{d}{dy}f_1(y)$
* Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-6y^2-1$
* Solve for $f_1(y)$

  $f_1(y)=-2y^3-y$
* Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=\frac{3}{2}x^2y^2-\frac{1}{3}x^3-2y^3-y$
* Substitute $F(x,y)$ into the solution of the ODE

  $\frac{3}{2}x^2y^2-\frac{1}{3}x^3-2y^3-y=c_1$
* Solve for y

$$\left\{ y = \frac{\left(-108x^2-144x^3-432c_1+27x^6+12\sqrt{-54x^9-162c_1x^6+144x^6+216x^5+864c_1x^3-27x^4+648c_1x^2+1296c_1^2+96}\right)^{\frac{1}{3}}}{12} - \frac{}{\left(-10\right.} \right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 771

```
dsolve((3*x*y(x)^2-x^2)+(3*x^2*y(x)-6*y(x)^2-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{\left(-108x^2 - 144x^3 + 432c_1 + 27x^6 + 12\sqrt{-54x^9 + 162c_1x^6 + 144x^6 + 216x^5 - 864c_1x^3 - 27x^4 - 648c_1}\right.}{12}$$

$$+ \frac{3x^4 - 8}{4\left(-108x^2 - 144x^3 + 432c_1 + 27x^6 + 12\sqrt{-54x^9 + 162c_1x^6 + 144x^6 + 216x^5 - 864c_1x^3 - 27x^4 - 64}\right.}$$

$$+ \frac{x^2}{4}$$

$$y = \frac{24 + i\left(-24 + 9x^4 - \left(-108x^2 - 144x^3 + 432c_1 + 27x^6 + 12\sqrt{-54x^9 + (162c_1 + 144)x^6 + 216x^5 - 27x}\right.\right.}{}$$

$$y = \frac{24 + i\left(-9x^4 + \left(-108x^2 - 144x^3 + 432c_1 + 27x^6 + 12\sqrt{-54x^9 + (162c_1 + 144)x^6 + 216x^5 - 27x^4 - 8}\right.\right.}{}$$

✓ Solution by Mathematica

Time used: 7.603 (sec). Leaf size: 570

```
DSolve[(3*x*y[x]^2-x^2)+(3*x^2*y[x]-6*y[x]^2-1)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x)\to \frac{x^2}{4} - \frac{\sqrt[3]{-\frac{27x^6}{4}+36x^3+27x^2+\sqrt{4\left(6-\frac{9x^4}{4}\right)^3+\left(-\frac{27x^6}{4}+36x^3+27x^2+108c_1\right)^2}+108c_1}}{6\sqrt[3]{2}} + \frac{6-\frac{9x^4}{4}}{3\ 2^{2/3}\sqrt[3]{-\frac{27x^6}{4}+36x^3+27x^2+\sqrt{4\left(6-\frac{9x^4}{4}\right)^3+\left(-\frac{27x^6}{4}+36x^3+27x^2+108c_1\right)^2}+108c_1}}$$

$$y(x)\to \frac{x^2}{4} + \frac{\left(1-i\sqrt{3}\right)\sqrt[3]{-\frac{27x^6}{4}+36x^3+27x^2+\sqrt{4\left(6-\frac{9x^4}{4}\right)^3+\left(-\frac{27x^6}{4}+36x^3+27x^2+108c_1\right)^2}+108c_1}}{12\sqrt[3]{2}} - \frac{\left(1+i\sqrt{3}\right)\left(6-\frac{9x^4}{4}\right)}{6\ 2^{2/3}\sqrt[3]{-\frac{27x^6}{4}+36x^3+27x^2+\sqrt{4\left(6-\frac{9x^4}{4}\right)^3+\left(-\frac{27x^6}{4}+36x^3+27x^2+108c_1\right)^2}+108c_1}}$$

$$y(x)\to \frac{x^2}{4} + \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-\frac{27x^6}{4}+36x^3+27x^2+\sqrt{4\left(6-\frac{9x^4}{4}\right)^3+\left(-\frac{27x^6}{4}+36x^3+27x^2+108c_1\right)^2}+108c_1}}{12\sqrt[3]{2}} - \frac{\left(1-i\sqrt{3}\right)\left(6-\frac{9x^4}{4}\right)}{6\ 2^{2/3}\sqrt[3]{-\frac{27x^6}{4}+36x^3+27x^2+\sqrt{4\left(6-\frac{9x^4}{4}\right)^3+\left(-\frac{27x^6}{4}+36x^3+27x^2+108c_1\right)^2}+108c_1}}$$

12.7 problem 281

Internal problem ID [15143]
Internal file name [`OUTPUT/15143_Tuesday_April_23_2024_04_50_59_PM_25828603/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 281.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y - x y^2 \ln(x) + x y' = 0}$$

12.7.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(\ln(x)\, xy - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 282: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x\,y^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{xy}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(\ln(x)\,xy - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2 y} \\ S_y &= \frac{1}{x\,y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{xy} = \frac{\ln(x)^2}{2} + c_1$$

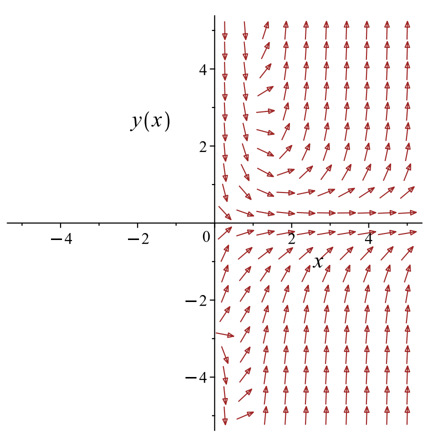
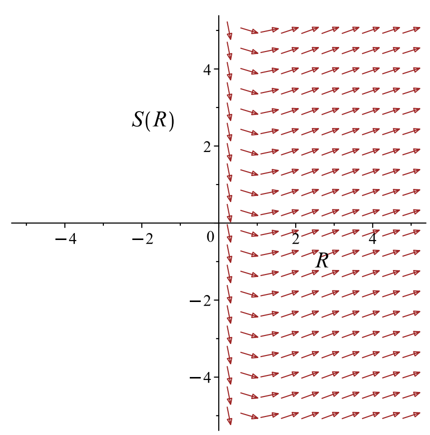
Which simplifies to

$$-\frac{1}{xy} = \frac{\ln(x)^2}{2} + c_1$$

Which gives

$$y = -\frac{2}{x\left(\ln(x)^2 + 2c_1\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(x)xy-1)}{x}$ 	$R = x$ $S = -\frac{1}{xy}$	$\frac{dS}{dR} = \frac{\ln(R)}{R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{2}{x\left(\ln\left(x\right)^{2}+2c_1\right)} \tag{1}$$

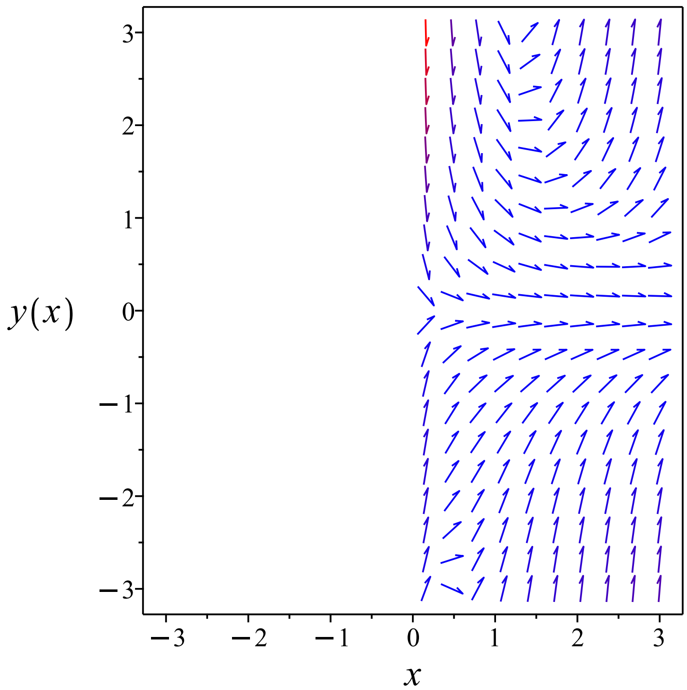


Figure 351: Slope field plot

Verification of solutions

$$y = -\frac{2}{x\left(\ln\left(x\right)^{2}+2c_1\right)}$$

Verified OK.

12.7.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(\ln\left(x\right)xy-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y+\ln\left(x\right)y^{2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \ln(x) \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{xy} + \ln(x) \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + \ln(x) \\ w' &= \frac{w}{x} - \ln(x) \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\ln(x)
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = -\ln(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-\ln(x)) \\
\frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-\ln(x)) \\
d\left(\frac{w}{x}\right) &= \left(-\frac{\ln(x)}{x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -\frac{\ln(x)}{x} dx \\
\frac{w}{x} &= -\frac{\ln(x)^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = -\frac{\ln(x)^2 x}{2} + c_1 x
$$

which simplifies to

$$
w(x) = x\left(-\frac{\ln(x)^2}{2} + c_1\right)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = x\left(-\frac{\ln(x)^2}{2} + c_1\right)
$$

Or

$$y = \frac{1}{x\left(-\frac{\ln(x)^2}{2} + c_1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x\left(-\frac{\ln(x)^2}{2} + c_1\right)} \tag{1}$$

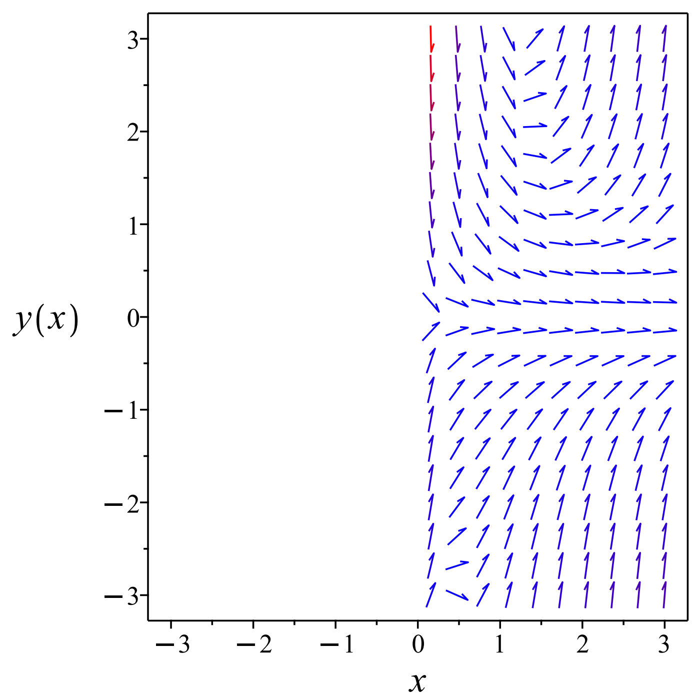


Figure 352: Slope field plot

Verification of solutions

$$y = \frac{1}{x\left(-\frac{\ln(x)^2}{2} + c_1\right)}$$

Verified OK.

12.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(-y+x\,y^2\ln(x)\right)\mathrm{d}x\\ \left(-x\,y^2\ln(x)+y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\,y^2\ln(x)+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\, y^{2} \ln \left(x \right)+y\right)\\ &= -2 \ln \left(x \right) x y +1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}\left(\left(-2 \ln \left(x \right) x y +1\right) - (1)\right)\\ &= -2 y \ln \left(x \right)\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y \left(\ln \left(x \right) x y -1\right)}\left((1) - \left(-2 \ln \left(x \right) x y +1\right)\right)\\ &= -\frac{2 \ln \left(x \right) x}{\ln \left(x \right) x y -1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{x M - y N}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-2\ln(x)\,xy + 1)}{x\,(-x\,y^2 \ln(x) + y) - y\,(x)} \\
&= -\frac{2}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^2 y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2 y^2}\left(-x\,y^2 \ln(x) + y\right) \\
&= \frac{-\ln(x)\,xy + 1}{x^2 y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2 y^2}(x) \\
&= \frac{1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-\ln(x)\,xy + 1}{x^2 y}\right) + \left(\frac{1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-\ln(x)\,xy + 1}{x^2 y}\,\mathrm{d}x$$
$$\phi = -\frac{\ln(x)^2}{2} - \frac{1}{xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x\,y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{1}{x\,y^2} = \frac{1}{x\,y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x)^2}{2} - \frac{1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x)^2}{2} - \frac{1}{xy}$$

The solution becomes

$$y = -\frac{2}{x\left(\ln(x)^2 + 2c_1\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x\left(\ln(x)^2 + 2c_1\right)} \tag{1}$$

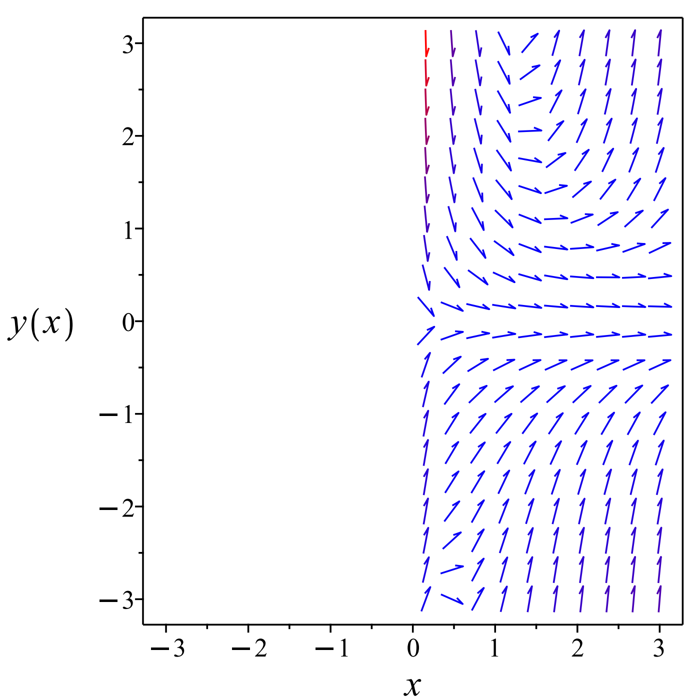


Figure 353: Slope field plot

Verification of solutions

$$y = -\frac{2}{x\left(\ln(x)^2 + 2c_1\right)}$$

Verified OK.

12.7.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(\ln(x)\, xy - 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 \ln(x) - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \ln(x)$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\ln(x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x} \\ f_1 f_2 &= -\frac{\ln(x)}{x} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\ln(x)\, u''(x) - \left(\frac{1}{x} - \frac{\ln(x)}{x}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \ln(x)^2 c_2$$

The above shows that

$$u'(x) = \frac{2\ln(x) c_2}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2}{x\left(c_1 + \ln(x)^2 c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2}{x\left(c_3 + \ln(x)^2\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{x\left(c_3 + \ln(x)^2\right)} \tag{1}$$

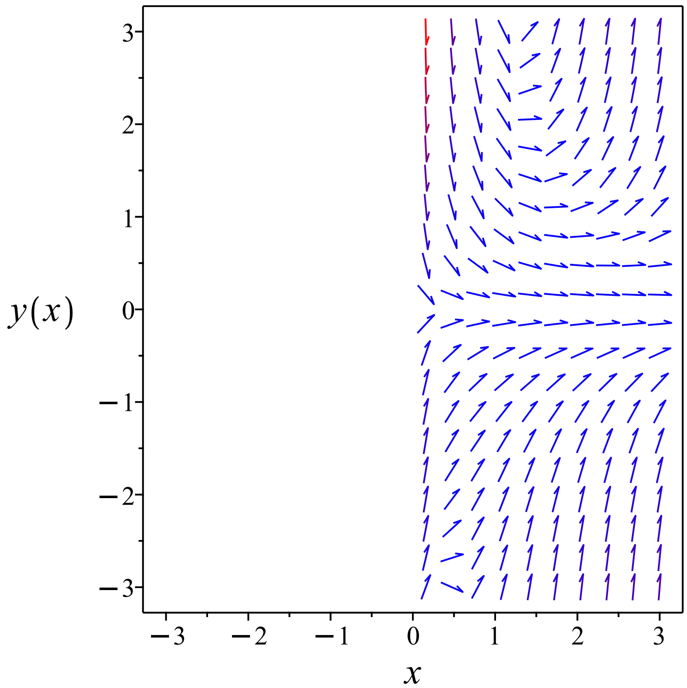


Figure 354: Slope field plot

Verification of solutions

$$y = -\frac{2}{x\left(c_3 + \ln\left(x\right)^2\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((y(x)-x*y(x)^2*ln(x))+(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = -\frac{2}{\left(\ln\left(x\right)^{2}-2c_{1}\right)x}$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 27

```
DSolve[(y[x]-x*y[x]^2*Log[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{-x\log^{2}(x)+2c_{1}x}$$
$$y(x) \to 0$$

12.8 problem 282

Internal problem ID [15144]
Internal file name [`OUTPUT/15144_Tuesday_April_23_2024_04_51_00_PM_58513738/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 282.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2xy\,\mathrm{e}^{x^2}+\mathrm{e}^{x^2}y' = x\sin(x)}$$

12.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= \mathrm{e}^{-x^2} x\sin(x) \end{aligned}$$

Hence the ode is

$$y' + 2yx = \mathrm{e}^{-x^2} x\sin(x)$$

The integrating factor μ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{-x^2} x \sin(x)\right)$$
$$\frac{d}{dx}\left(y e^{x^2}\right) = \left(e^{x^2}\right)\left(e^{-x^2} x \sin(x)\right)$$
$$d\left(y e^{x^2}\right) = (x \sin(x)) dx$$

Integrating gives

$$y e^{x^2} = \int x \sin(x) dx$$
$$y e^{x^2} = \sin(x) - \cos(x) x + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = e^{-x^2}(\sin(x) - \cos(x) x) + c_1 e^{-x^2}$$

which simplifies to

$$y = e^{-x^2}(\sin(x) - \cos(x) x + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}(\sin(x) - \cos(x) x + c_1) \tag{1}$$

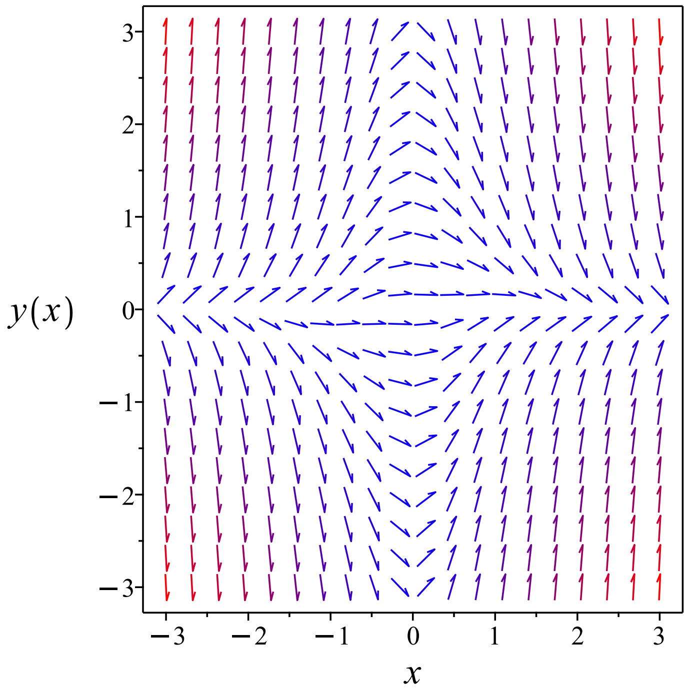


Figure 355: Slope field plot

Verification of solutions

$$y = e^{-x^2}(\sin(x) - \cos(x)\,x + c_1)$$

Verified OK.

12.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -x\left(2y\,e^{x^2} - \sin(x)\right)e^{-x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 284: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x^2}} dy \end{aligned}$$

Which results in

$$S = y\,\mathrm{e}^{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -x\left(2y\,\mathrm{e}^{x^2} - \sin(x)\right)\mathrm{e}^{-x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy\,\mathrm{e}^{x^2} \\ S_y &= \mathrm{e}^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\sin(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) - R\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = \sin(x) - \cos(x)\, x + c_1$$

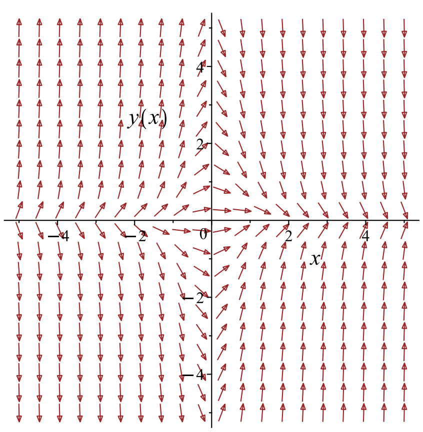
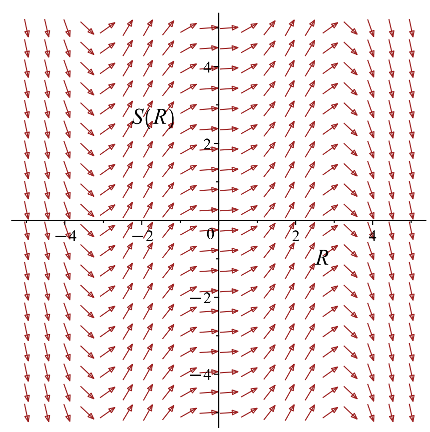
Which simplifies to

$$e^{x^2} y = \sin(x) - \cos(x)\, x + c_1$$

Which gives

$$y = e^{-x^2}\left(\sin(x) - \cos(x)\, x + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x\left(2y\, e^{x^2} - \sin(x)\right) e^{-x^2}$ 	$R = x$ $S = y\, e^{x^2}$	$\frac{dS}{dR} = R\sin(R)$ 

Summary
The solution(s) found are the following

$$y = e^{-x^2}\left(\sin(x) - \cos(x)\, x + c_1\right) \tag{1}$$

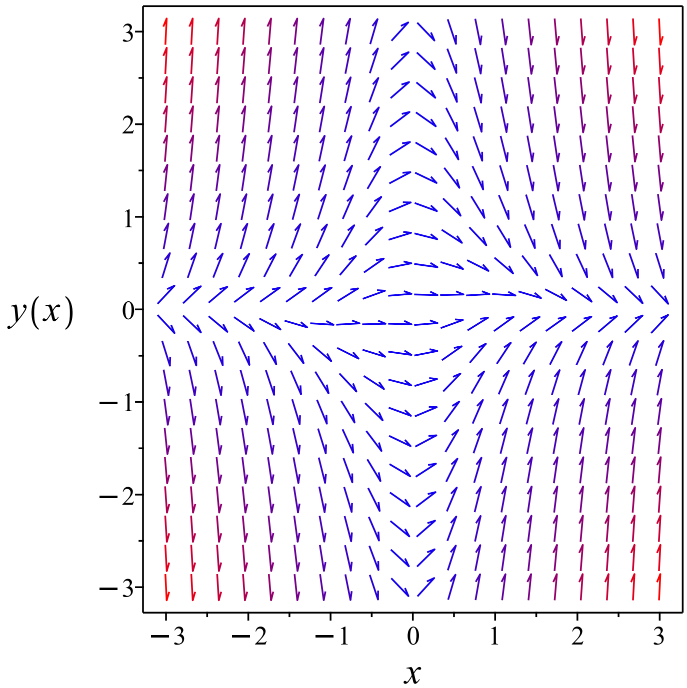


Figure 356: Slope field plot

Verification of solutions

$$y = e^{-x^2}(\sin(x) - \cos(x)\,x + c_1)$$

Verified OK.

12.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{x^2}\right) \mathrm{d}y &= \left(-2xy\,\mathrm{e}^{x^2} + x\sin(x)\right) \mathrm{d}x \\ \left(2xy\,\mathrm{e}^{x^2} - x\sin(x)\right) \mathrm{d}x + \left(\mathrm{e}^{x^2}\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy\,\mathrm{e}^{x^2} - x\sin(x) \\ N(x, y) &= \mathrm{e}^{x^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy\,\mathrm{e}^{x^2} - x\sin(x)\right) \\ &= 2x\,\mathrm{e}^{x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{x^2}\right) \\ &= 2x\,\mathrm{e}^{x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy \, \mathrm{e}^{x^2} - x \sin(x) \, \mathrm{d}x$$
$$\phi = \cos(x) \, x - \sin(x) + y \, \mathrm{e}^{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{x^2} = \mathrm{e}^{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) \, x - \sin(x) + y \, \mathrm{e}^{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) \, x - \sin(x) + y \, \mathrm{e}^{x^2}$$

The solution becomes

$$y = e^{-x^2}(\sin(x) - \cos(x)\, x + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}(\sin(x) - \cos(x)\, x + c_1) \tag{1}$$

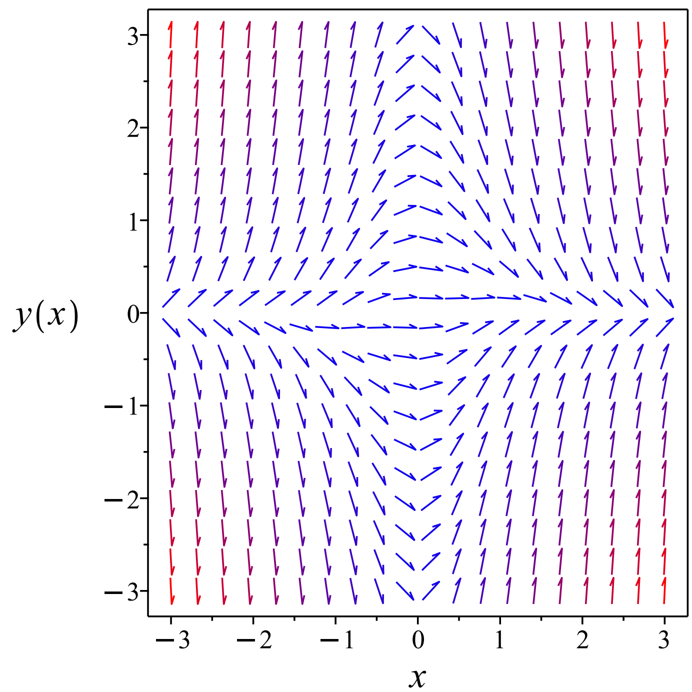


Figure 357: Slope field plot

Verification of solutions

$$y = e^{-x^2}(\sin(x) - \cos(x)\, x + c_1)$$

Verified OK.

12.8.4 Maple step by step solution

Let's solve

$2xy\,\mathrm{e}^{x^2}+\mathrm{e}^{x^2}y'=x\sin\left(x\right)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-2yx+\frac{x\sin(x)}{\mathrm{e}^{x^2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+2yx=\frac{x\sin(x)}{\mathrm{e}^{x^2}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+2yx\right)=\frac{\mu(x)x\sin(x)}{\mathrm{e}^{x^2}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+2yx\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x)=\left(\mathrm{e}^{x^2}\right)^2\mathrm{e}^{-x^2}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)x\sin(x)}{\mathrm{e}^{x^2}}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)x\sin(x)}{\mathrm{e}^{x^2}}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)x\sin(x)}{\mathrm{e}^{x^2}}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\left(\mathrm{e}^{x^2}\right)^2\mathrm{e}^{-x^2}$

  $y=\frac{\int\mathrm{e}^{-x^2}x\sin(x)\mathrm{e}^{x^2}dx+c_1}{\left(\mathrm{e}^{x^2}\right)^2\mathrm{e}^{-x^2}}$

- Evaluate the integrals on the rhs

$$y = \frac{\sin(x) - \cos(x) x + c_1}{\left(e^{x^2}\right)^2 e^{-x^2}}$$

- Simplify

$$y = e^{-x^2} (\sin(x) - \cos(x) x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve((2*x*y(x)*exp(x^2)-x*sin(x))+(exp(x^2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = (\sin(x) - x \cos(x) + c_1) e^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 23

```
DSolve[(2*x*y[x]*Exp[x^2]-x*Sin[x])+Exp[x^2]*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-x^2} (\sin(x) - x \cos(x) + c_1)$$

12.9 problem 283

Internal problem ID [15145]
Internal file name [`OUTPUT/15145_Tuesday_April_23_2024_04_51_01_PM_50072634/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 283.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_exponential_symmetries]]
```

$$\boxed{y' - \frac{1}{2x - y^2} = 0}$$

12.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{1}{y^2 - 2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{b_3 - a_2}{y^2 - 2x} - \frac{a_3}{(y^2 - 2x)^2} + \frac{2xa_2 + 2ya_3 + 2a_1}{(y^2 - 2x)^2} - \frac{2y(xb_2 + yb_3 + b_1)}{(y^2 - 2x)^2} = 0 \quad (5E)$$

Putting the above in normal form gives

$$\frac{y^4 b_2 - 4x\, y^2 b_2 + 4x^2 b_2 - 2xyb_2 + y^2 a_2 - 3y^2 b_3 + 2xb_3 + 2ya_3 - 2yb_1 + 2a_1 - a_3}{(-y^2 + 2x)^2} = 0$$

Setting the numerator to zero gives

$$y^4 b_2 - 4x\, y^2 b_2 + 4x^2 b_2 - 2xyb_2 + y^2 a_2 - 3y^2 b_3 + 2xb_3 + 2ya_3 - 2yb_1 + 2a_1 - a_3 = 0 \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_2^4 - 4b_2 v_1 v_2^2 + a_2 v_2^2 + 4b_2 v_1^2 - 2b_2 v_1 v_2 - 3b_3 v_2^2 + 2a_3 v_2 - 2b_1 v_2 + 2b_3 v_1 + 2a_1 - a_3 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2 v_1^2 - 4b_2 v_1 v_2^2 - 2b_2 v_1 v_2 + 2b_3 v_1 + b_2 v_2^4 + (a_2 - 3b_3)\, v_2^2 + (2a_3 - 2b_1)\, v_2 + 2a_1 - a_3 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-4b_2 &= 0 \\
-2b_2 &= 0 \\
4b_2 &= 0 \\
2b_3 &= 0 \\
2a_1 - a_3 &= 0 \\
a_2 - 3b_3 &= 0 \\
2a_3 - 2b_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 2a_1 \\
b_1 &= 2a_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2y + 1 \\
\eta &= 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 2 - \left(-\frac{1}{y^2 - 2x}\right)(2y + 1) \\
&= \frac{-2y^2 + 4x - 2y - 1}{-y^2 + 2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2y^2+4x-2y-1}{-y^2+2x}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{2} - \frac{\ln(2y^2 - 4x + 2y + 1)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{1}{y^2 - 2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2y^2 - 4x + 2y + 1} \\ S_y &= \frac{-y^2 + 2x}{-2y^2 + 4x - 2y - 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

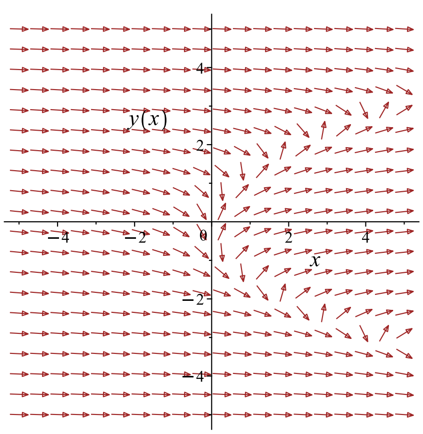
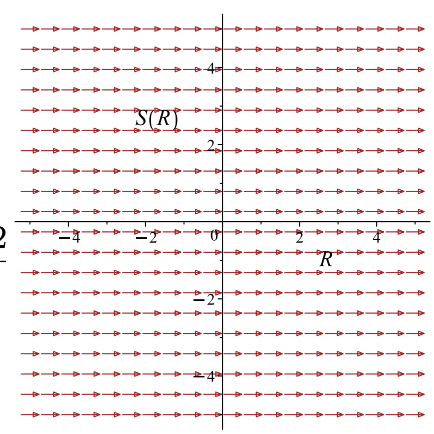
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{2} - \frac{\ln(2y^2 - 4x + 2y + 1)}{4} = c_1$$

Which simplifies to

$$\frac{y}{2} - \frac{\ln(2y^2 - 4x + 2y + 1)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{1}{y^2-2x}$ 	$R = x$ $S = \frac{y}{2} - \frac{\ln(2y^2 - 4x + 2}{4}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{y}{2}-\frac{\ln\left(2y^{2}-4x+2y+1\right)}{4}=c_1 \tag{1}$$

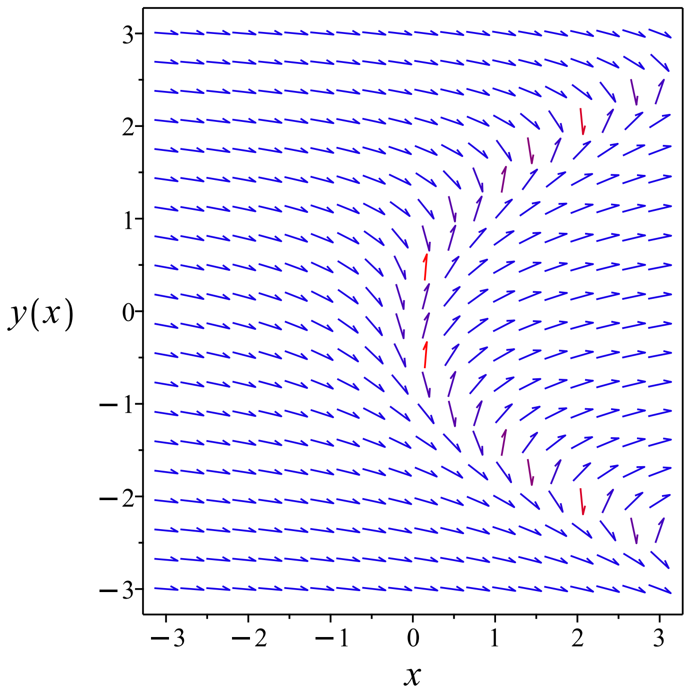


Figure 358: Slope field plot

Verification of solutions

$$\frac{y}{2}-\frac{\ln\left(2y^{2}-4x+2y+1\right)}{4}=c_1$$

Verified OK.

12.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^2-2x)\,\mathrm{d}y&=(-1)\,\mathrm{d}x\\ (1)\,\mathrm{d}x+(y^2-2x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=1\\ N(x,y)&=y^2-2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(1)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2 - 2x)\\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y^2 - 2x}((0) - (-2))\\ &= -\frac{2}{-y^2 + 2x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= 1((-2) - (0))\\ &= -2\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -2 \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2y}\\ &= \mathrm{e}^{-2y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2y}(1)\\ &= \mathrm{e}^{-2y}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-2y}\left(y^2 - 2x\right) \\
&= \left(y^2 - 2x\right)\mathrm{e}^{-2y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{-2y}\right) + \left(\left(y^2 - 2x\right)\mathrm{e}^{-2y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{-2y}\,\mathrm{d}x \\
\phi &= \mathrm{e}^{-2y}x + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -2\,\mathrm{e}^{-2y}x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \left(y^2 - 2x\right)\mathrm{e}^{-2y}$. Therefore equation (4) becomes

$$
\left(y^2 - 2x\right)\mathrm{e}^{-2y} = -2\,\mathrm{e}^{-2y}x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \mathrm{e}^{-2y}y^2
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\mathrm{e}^{-2y}y^2\right)\mathrm{d}y$$

$$f(y) = -\frac{(2y^2+2y+1)\,\mathrm{e}^{-2y}}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-2y}x - \frac{(2y^2+2y+1)\,\mathrm{e}^{-2y}}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-2y}x - \frac{(2y^2+2y+1)\,\mathrm{e}^{-2y}}{4}$$

Summary

The solution(s) found are the following

$$\mathrm{e}^{-2y}x - \frac{(2y^2+2y+1)\,\mathrm{e}^{-2y}}{4} = c_1 \tag{1}$$

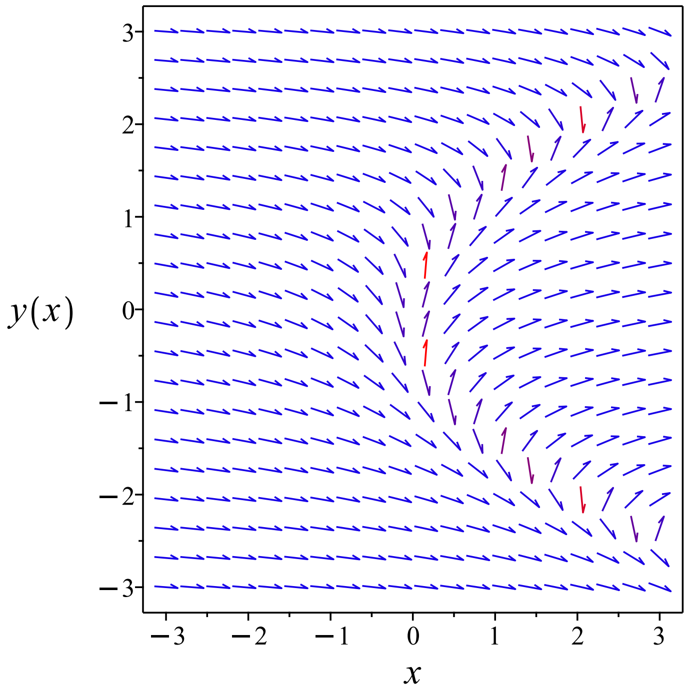


Figure 359: Slope field plot

Verification of solutions

$$
e^{-2y}x - \frac{(2y^2 + 2y + 1)\, e^{-2y}}{4} = c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=1/(2*x-y(x)^2),y(x), singsol=all)
```

$$x-\frac{y^2}{2}-\frac{y}{2}-\frac{1}{4}-\mathrm{e}^{2y}c_1=0$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 31

```
DSolve[y'[x]==1/(2*x-y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x=\frac{1}{4}\left(2y(x)^2+2y(x)+1\right)+c_1e^{2y(x)},y(x)\right]$$

12.10 problem 284

Internal problem ID [15146]
Internal file name [`OUTPUT/15146_Tuesday_April_23_2024_04_51_02_PM_9403647/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 284.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy' - y' = -x^2 + 3x}$$

12.10.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{x(x-3)}{x-1}\, dx \\ &= -\frac{x^2}{2} + 2x + 2\ln(x-1) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2}{2} + 2x + 2\ln(x-1) + c_1 \tag{1}$$

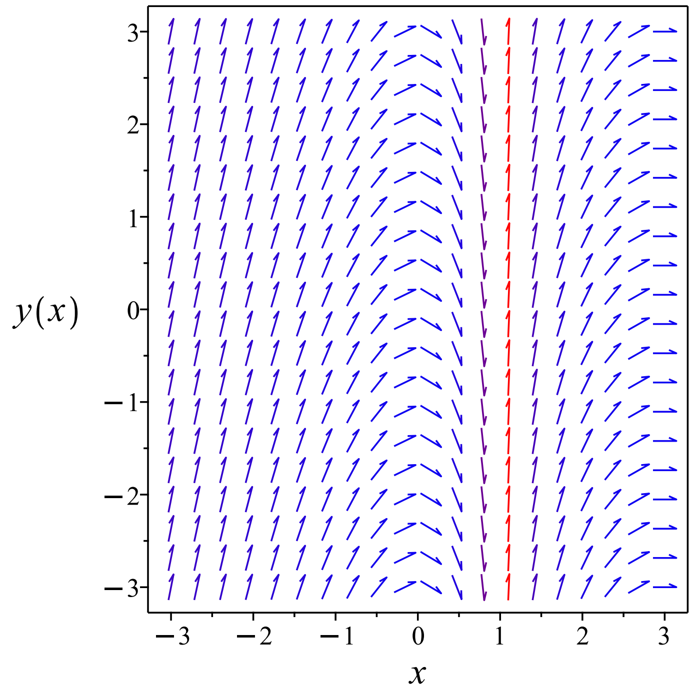


Figure 360: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{2} + 2x + 2\ln(x-1) + c_1$$

Verified OK.

12.10.2 Maple step by step solution

Let's solve

$xy' - y' = -x^2 + 3x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{-x^2+3x}{x-1}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{-x^2+3x}{x-1} dx + c_1$

- Evaluate integral

$$y = -\frac{x^2}{2} + 2x + 2\ln(x-1) + c_1$$

- Solve for y

$$y = -\frac{x^2}{2} + 2x + 2\ln(x-1) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2+x*diff(y(x),x)=3*x+diff(y(x),x),y(x), singsol=all)
```

$$y = -\frac{x^2}{2} + 2x + 2\ln(x-1) + c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 24

```
DSolve[x^2+x*y'[x]==3*x+y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}(x-1)^2 + x + 2\log(x-1) + c_1$$

12.11 problem 285

Internal problem ID [15147]
Internal file name [`OUTPUT/15147_Tuesday_April_23_2024_04_51_02_PM_16174934/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 285.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y'yx - y^2 = x^4}$$

12.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,u(x)\,x^2 - u(x)^2\,x^2 = x^4$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{x}{u} \end{aligned}$$

Where $f(x) = x$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{u}}\,du &= x\,dx \\
\int \frac{1}{\frac{1}{u}}\,du &= \int x\,dx \\
\frac{u^2}{2} &= \frac{x^2}{2} + c_2
\end{aligned}
$$

The solution is

$$
\frac{u(x)^2}{2} - \frac{x^2}{2} - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2}{2x^2} - \frac{x^2}{2} - c_2 = 0 \\
\frac{y^2}{2x^2} - \frac{x^2}{2} - c_2 = 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\frac{y^2}{2x^2} - \frac{x^2}{2} - c_2 = 0 \tag{1}
$$

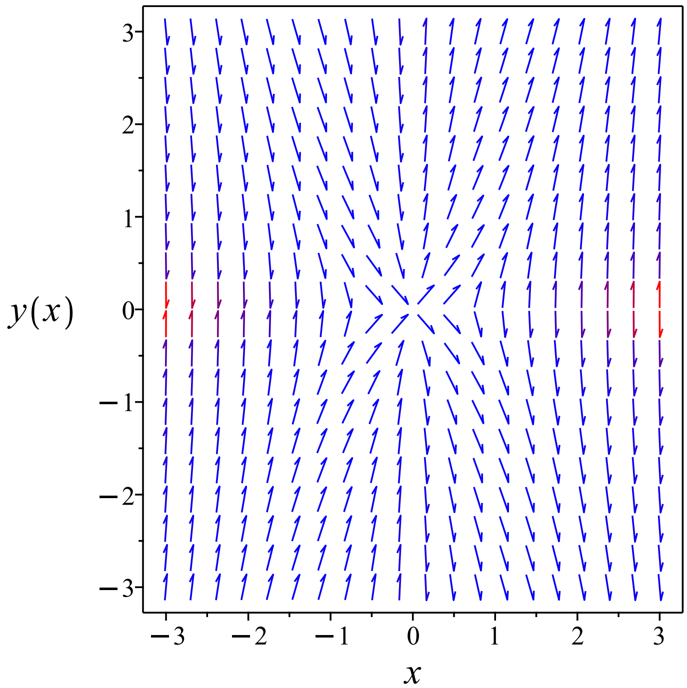


Figure 361: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - \frac{x^2}{2} - c_2 = 0$$

Verified OK.

12.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^4 + y^2}{yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 288: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \frac{x^2}{y} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^4 + y^2}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{x^3} \\ S_y &= \frac{y}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

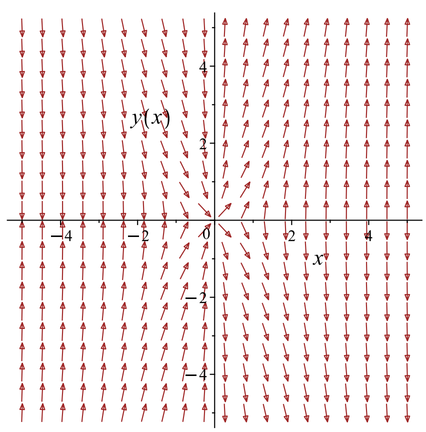
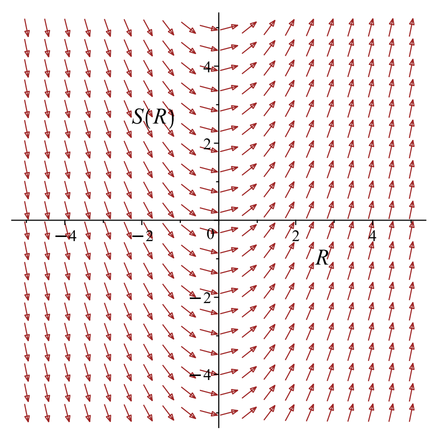
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^2} = \frac{x^2}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^2} = \frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^4+y^2}{yx}$ 	$R = x$ $S = \frac{y^2}{2x^2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x^2} = \frac{x^2}{2} + c_1 \tag{1}$$

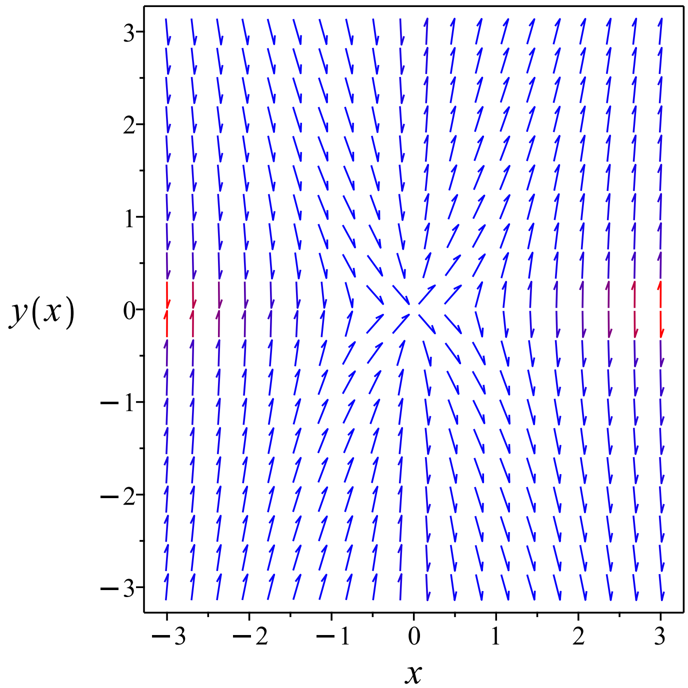


Figure 362: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} = \frac{x^2}{2} + c_1$$

Verified OK.

12.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^4 + y^2}{yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x^3\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= x^3 \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{x} + x^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{x} + x^3 \\ w' &= \frac{2w}{x} + 2x^3 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= 2x^3 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = 2x^3$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(2x^3\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(2x^3\right) \\ \mathrm{d}\left(\frac{w}{x^2}\right) &= (2x)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int 2x\,\mathrm{d}x \\ \frac{w}{x^2} &= x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = x^4 + c_1 x^2$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x^4 + c_1 x^2$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x^2 + c_1}\,x \\ y(x) &= -\sqrt{x^2 + c_1}\,x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + c_1}\,x \tag{1}$$

$$y = -\sqrt{x^2 + c_1}\,x \tag{2}$$

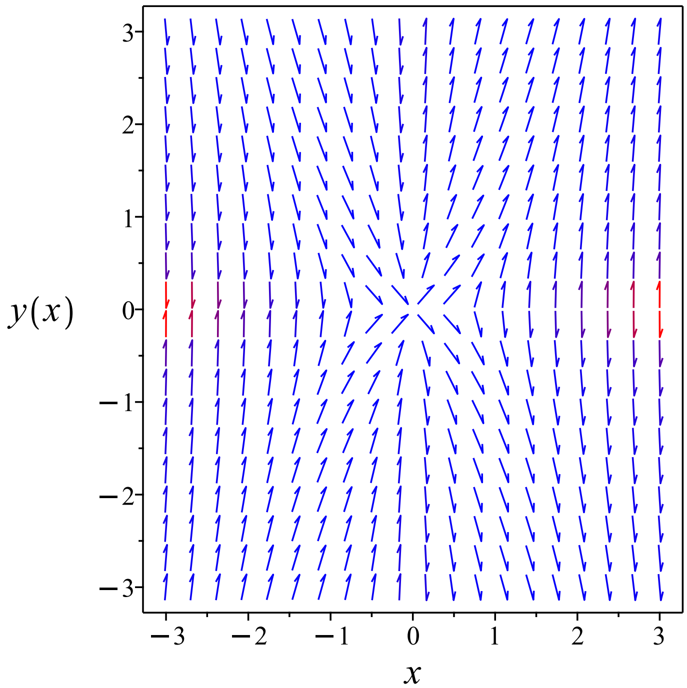


Figure 363: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + c_1}\, x$$

Verified OK.

$$y = -\sqrt{x^2 + c_1}\, x$$

Verified OK.

12.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy) \, \mathrm{d}y &= \left(x^4 + y^2\right) \mathrm{d}x \\ \left(-x^4 - y^2\right) \mathrm{d}x + (xy) \, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^4 - y^2 \\ N(x, y) &= xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^4 - y^2\right) \\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy)\\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{xy}((-2y) - (y))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{3}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)}\\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}(-x^4 - y^2)\\ &= \frac{-x^4 - y^2}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^3}(xy)\\ &= \frac{y}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-x^4 - y^2}{x^3}\right) + \left(\frac{y}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^4 - y^2}{x^3}\,\mathrm{d}x$$
$$\phi = \frac{-x^4 + y^2}{2x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^2}$. Therefore equation (4) becomes

$$\frac{y}{x^2} = \frac{y}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^4 + y^2}{2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^4 + y^2}{2x^2}$$

Summary

The solution(s) found are the following

$$\frac{-x^4 + y^2}{2x^2} = c_1 \tag{1}$$

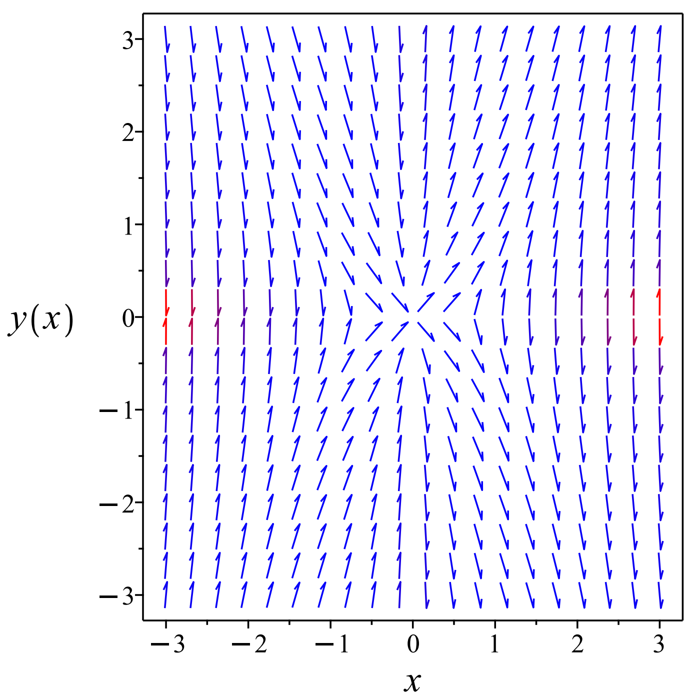


Figure 364: Slope field plot

Verification of solutions

$$\frac{-x^4 + y^2}{2x^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x*y(x)*diff(y(x),x)-y(x)^2=x^4,y(x), singsol=all)
```

$$y = \sqrt{x^2 + c_1}\, x$$
$$y = -\sqrt{x^2 + c_1}\, x$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 34

```
DSolve[x*y[x]*y'[x]-y[x]^2==x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\sqrt{x^2 + c_1}$$
$$y(x) \to x\sqrt{x^2 + c_1}$$

12.12 problem 286

Internal problem ID [15148]
Internal file name [`OUTPUT/15148_Tuesday_April_23_2024_04_51_03_PM_78289460/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 286.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{\frac{1}{x^2 - yx + y^2} - \frac{y'}{2y^2 - yx} = 0}$$

12.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\frac{1}{x^2 - u(x)\,x^2 + u(x)^2\,x^2} - \frac{u'(x)\,x + u(x)}{2u(x)^2\,x^2 - u(x)\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2 - 3u + 2)}{(u^2 - u + 1)\,x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2-3u+2)}{u^2-u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2-3u+2)}{u^2-u+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u(u^2-3u+2)}{u^2-u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u)}{2} - \ln(u-1) + \frac{3\ln(u-2)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln(u)}{2} - \ln(u-1) + \frac{3\ln(u-2)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{\sqrt{u}\,(u-2)^{\frac{3}{2}}}{u-1} = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= \frac{c_3^2\left(\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4c_3^2 - 2\_Z^2c_3^2 - c_3^2\right)^2 + 1\right)^2}{x\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4c_3^2 - 2\_Z^2c_3^2 - c_3^2\right)^6} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3^2\left(\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4c_3^2 - 2\_Z^2c_3^2 - c_3^2\right)^2 + 1\right)^2}{x\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4c_3^2 - 2\_Z^2c_3^2 - c_3^2\right)^6} \tag{1}$$

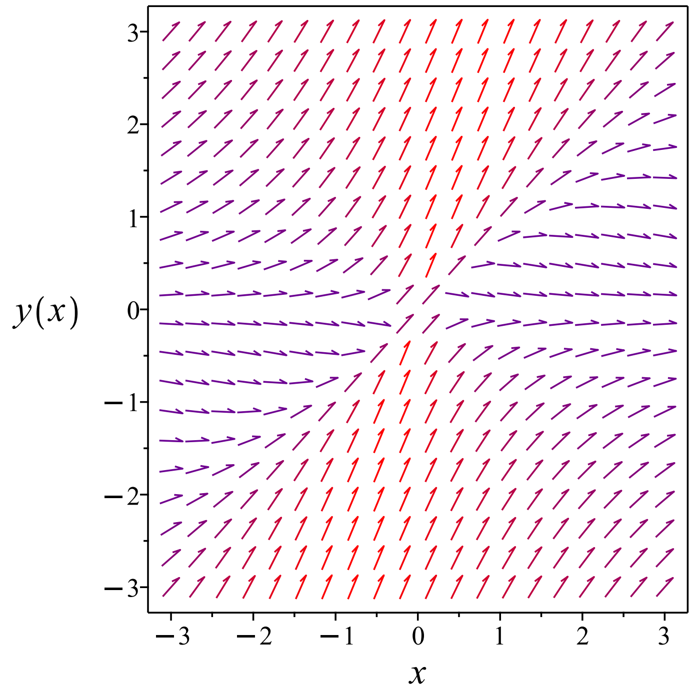


Figure 365: Slope field plot

Verification of solutions

$$y = \frac{c_3^2\left(\operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4c_3^2 - 2\_Z^2c_3^2 - c_3^2\right)^2 + 1\right)^2}{x \operatorname{RootOf}\left(x^2\_Z^8 + 2x^2\_Z^6 - \_Z^4c_3^2 - 2\_Z^2c_3^2 - c_3^2\right)^6}$$

Verified OK.

12.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(-x + 2y)}{x^2 - xy + y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{y(-x+2y)(b_3-a_2)}{x^2-xy+y^2} - \frac{y^2(-x+2y)^2 a_3}{(x^2-xy+y^2)^2} \\
&- \left(-\frac{y}{x^2-xy+y^2} - \frac{y(-x+2y)(2x-y)}{(x^2-xy+y^2)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{-x+2y}{x^2-xy+y^2} + \frac{2y}{x^2-xy+y^2} - \frac{y(-x+2y)^2}{(x^2-xy+y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^4b_2 - 6x^3yb_2 + x^2y^2a_2 - 2x^2y^2a_3 + 4x^2y^2b_2 - x^2y^2b_3 + 2x\,y^3a_2 + 8x\,y^3a_3 - 2x\,y^3b_2 - 2x\,y^3b_3 - 2y^4a_2 -}{(x^2-xy+y^2)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x^4b_2 - 6x^3yb_2 + x^2y^2a_2 - 2x^2y^2a_3 + 4x^2y^2b_2 - x^2y^2b_3 + 2x\,y^3a_2 \\
&+ 8x\,y^3a_3 - 2x\,y^3b_2 - 2x\,y^3b_3 - 2y^4a_2 - 5y^4a_3 + y^4b_2 + 2y^4b_3 \\
&+ x^3b_1 - x^2ya_1 - 4x^2yb_1 + 4x\,y^2a_1 + x\,y^2b_1 - y^3a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&a_2v_1^2v_2^2 + 2a_2v_1v_2^3 - 2a_2v_2^4 - 2a_3v_1^2v_2^2 + 8a_3v_1v_2^3 - 5a_3v_2^4 + 2b_2v_1^4 \\
&- 6b_2v_1^3v_2 + 4b_2v_1^2v_2^2 - 2b_2v_1v_2^3 + b_2v_2^4 - b_3v_1^2v_2^2 - 2b_3v_1v_2^3 + 2b_3v_2^4 \\
&- a_1v_1^2v_2 + 4a_1v_1v_2^2 - a_1v_2^3 + b_1v_1^3 - 4b_1v_1^2v_2 + b_1v_1v_2^2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &2b_2v_1^4 - 6b_2v_1^3v_2 + b_1v_1^3 + (a_2 - 2a_3 + 4b_2 - b_3)\, v_1^2v_2^2 \\ &+ (-a_1 - 4b_1)\, v_1^2v_2 + (2a_2 + 8a_3 - 2b_2 - 2b_3)\, v_1v_2^3 \\ &+ (4a_1 + b_1)\, v_1v_2^2 + (-2a_2 - 5a_3 + b_2 + 2b_3)\, v_2^4 - a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -6b_2 &= 0 \\ 2b_2 &= 0 \\ -a_1 - 4b_1 &= 0 \\ 4a_1 + b_1 &= 0 \\ -2a_2 - 5a_3 + b_2 + 2b_3 &= 0 \\ a_2 - 2a_3 + 4b_2 - b_3 &= 0 \\ 2a_2 + 8a_3 - 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y(-x+2y)}{x^2 - xy + y^2}\right)(x) \\ &= \frac{2x^2 y - 3x\,y^2 + y^3}{x^2 - xy + y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2 y - 3x\,y^2 + y^3}{x^2 - xy + y^2}} dy\end{aligned}$$

Which results in

$$S = -\ln(-x+y) + \frac{\ln(y)}{2} + \frac{3\ln(y-2x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(-x+2y)}{x^2 - xy + y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{-x+y} - \frac{3}{y-2x} \\
S_y &= \frac{1}{x-y} + \frac{1}{2y} - \frac{3}{4x-2y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(-x+y) + \frac{\ln(y)}{2} + \frac{3\ln(y-2x)}{2} = c_1
$$

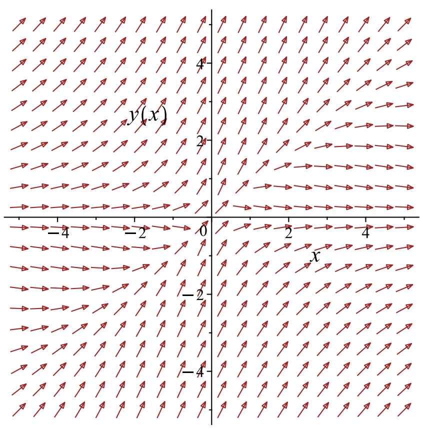
Which simplifies to

$$
-\ln(-x+y) + \frac{\ln(y)}{2} + \frac{3\ln(y-2x)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y(-x+2y)}{x^2-xy+y^2}$



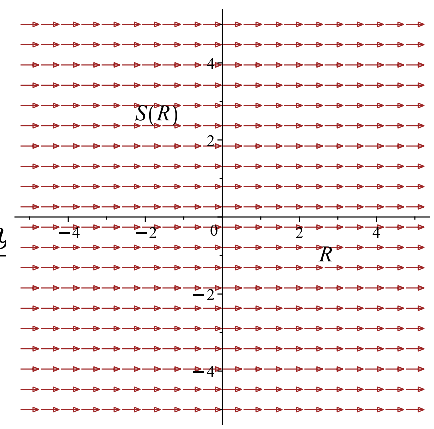
</td><td>

$R = x$

$S = -\ln(-x+y) + \frac{\ln(y}{2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$-\ln(-x+y) + \frac{\ln(y)}{2} + \frac{3\ln(y-2x)}{2} = c_1 \tag{1}$$

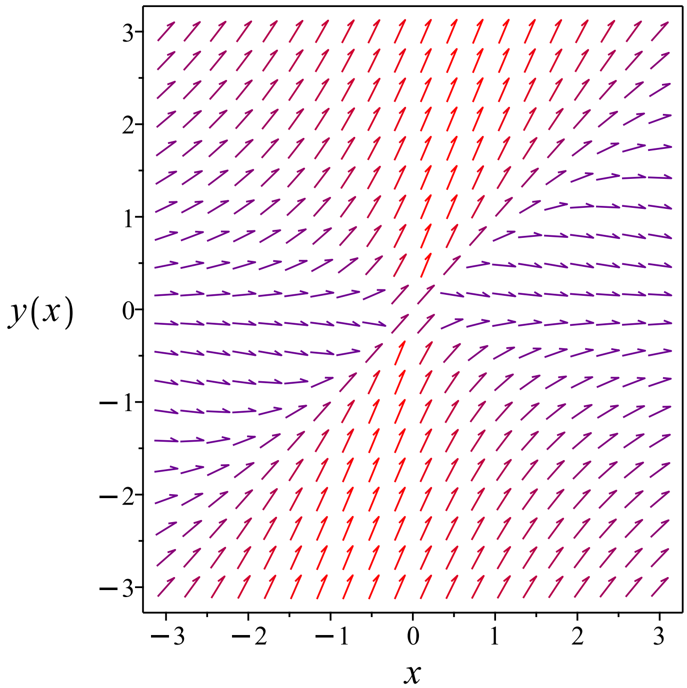


Figure 366: Slope field plot

Verification of solutions

$$-\ln(-x+y)+\frac{\ln(y)}{2}+\frac{3\ln(y-2x)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.938 (sec). Leaf size: 40

```
dsolve(1/(x^2-x*y(x)+y(x)^2)=diff(y(x),x)/(2*y(x)^2-x*y(x)),y(x), singsol=all)
```

$$y = \left(\operatorname{RootOf}\left(\_Z^8 c_1 x^2 + 2\_Z^6 c_1 x^2 - \_Z^4 - 2\_Z^2 - 1\right)^2 + 2\right) x$$

✓ Solution by Mathematica

Time used: 60.201 (sec). Leaf size: 1805

```
DSolve[1/(x^2-x*y[x]+y[x]^2)==y'[x]/(2*y[x]^2-x*y[x]),y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{6}\left(-\sqrt{3}\sqrt{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}+\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+}}\right.$$

$$-\sqrt{3}\sqrt{-\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}-\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6}}}}$$

$$\left.+9x\right)$$

$$y(x) \to \frac{1}{6}\left(-\sqrt{3}\sqrt{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}+\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+}}\right.$$

$$+\sqrt{3}\sqrt{-\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6c_1}}-\frac{e^{4c_1}}{\sqrt[3]{54e^{2c_1}x^4+6\sqrt{3}\sqrt{e^{4c_1}x^4\left(27x^4+e^{4c_1}\right)}+e^{6}}}}$$

$$\left.+9x\right)$$

$$y(x)$$

$$\to \frac{1}{}\left(-\sqrt{}\sqrt{\sqrt[3]{\ldots}\ \frac{e^{4c_1}}{}}\right.$$

12.13 problem 287

Internal problem ID [15149]
Internal file name [`OUTPUT/15149_Tuesday_April_23_2024_04_51_24_PM_26802498/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 287.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(2x-1)\,y'-2y=\frac{1-4x}{x^2}}$$

12.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{2}{2x-1}$$
$$q(x)=\frac{1-4x}{(2x-1)\,x^2}$$

Hence the ode is

$$y'-\frac{2y}{2x-1}=\frac{1-4x}{(2x-1)\,x^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{2x-1}dx} \\
&= \frac{1}{2x-1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1-4x}{(2x-1)\,x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{2x-1}\right) &= \left(\frac{1}{2x-1}\right)\left(\frac{1-4x}{(2x-1)\,x^2}\right) \\
d\left(\frac{y}{2x-1}\right) &= \left(\frac{1-4x}{(2x-1)^2\,x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{2x-1} &= \int \frac{1-4x}{(2x-1)^2\,x^2}\, dx \\
\frac{y}{2x-1} &= -\frac{1}{x}+\frac{2}{2x-1}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{2x-1}$ results in

$$
y = (2x-1)\left(-\frac{1}{x}+\frac{2}{2x-1}\right)+c_1(2x-1)
$$

which simplifies to

$$
y = \frac{2c_1x^2-c_1x+1}{x}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{2c_1x^2-c_1x+1}{x} \tag{1}
$$

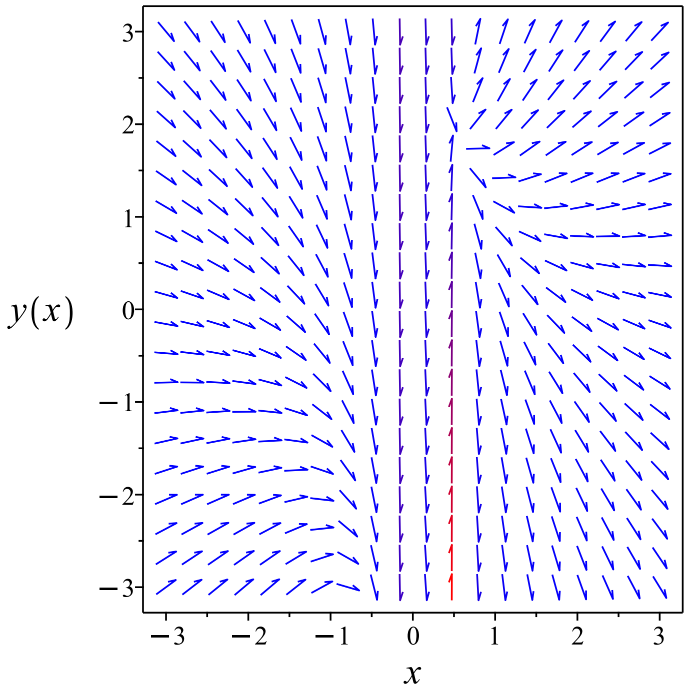


Figure 367: Slope field plot

Verification of solutions

$$y = \frac{2c_1x^2 - c_1x + 1}{x}$$

Verified OK.

12.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x^2y - 4x + 1}{(2x - 1)\,x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 290: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= 2x - 1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{2x-1} dy
\end{aligned}$$

Which results in

$$S = \frac{y}{2x-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x^2 y - 4x + 1}{(2x-1)\, x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2y}{(2x-1)^2} \\
S_y &= \frac{1}{2x-1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1-4x}{(2x-1)^2\, x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1-4R}{(2R-1)^2\, R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + \frac{2}{2R-1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{2x-1} = -\frac{1}{x} + \frac{2}{2x-1} + c_1$$

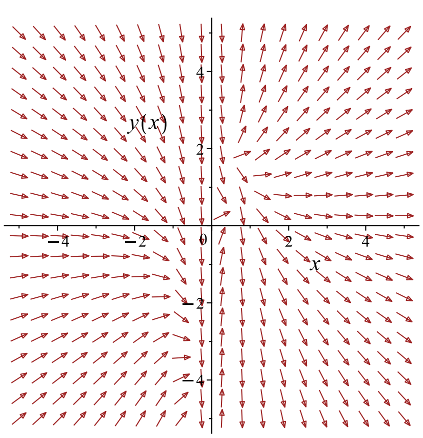
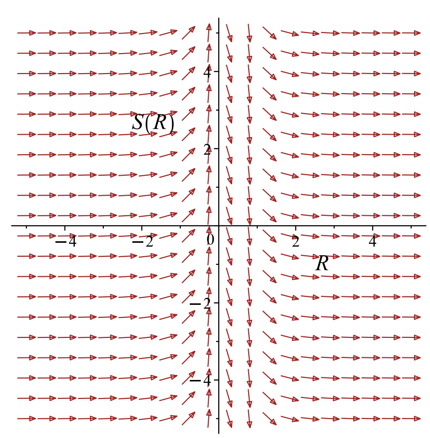
Which simplifies to

$$\frac{y}{2x-1} = -\frac{1}{x} + \frac{2}{2x-1} + c_1$$

Which gives

$$y = \frac{2c_1x^2 - c_1x + 1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^2y-4x+1}{(2x-1)x^2}$ 	$R = x$ $S = \frac{y}{2x-1}$	$\frac{dS}{dR} = \frac{1-4R}{(2R-1)^2R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{2c_1x^2 - c_1x + 1}{x} \tag{1}$$

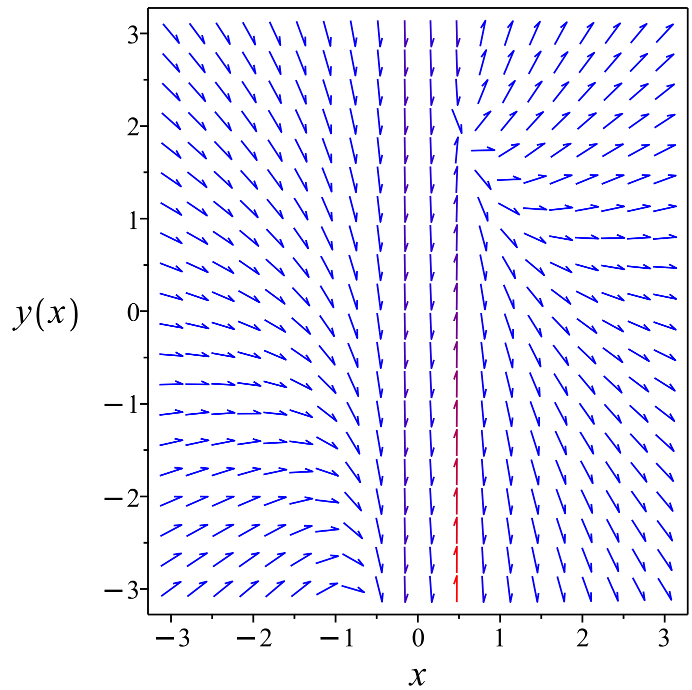


Figure 368: Slope field plot

Verification of solutions

$$y = \frac{2c_1x^2 - c_1x + 1}{x}$$

Verified OK.

12.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x-1)\,\mathrm{d}y &= \left(2y + \frac{1-4x}{x^2}\right)\mathrm{d}x\\ \left(-2y - \frac{1-4x}{x^2}\right)\mathrm{d}x + (2x-1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2y - \frac{1-4x}{x^2}\\ N(x,y) &= 2x-1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - \frac{1-4x}{x^2}\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x-1)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x-1}((-2)-(2))\\ &= -\frac{4}{2x-1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{4}{2x-1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(2x-1)}\\ &= \frac{1}{(2x-1)^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(2x-1)^2}\left(-2y - \frac{1-4x}{x^2}\right)\\ &= \frac{-2x^2y+4x-1}{(2x-1)^2\,x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{(2x-1)^2}(2x-1)\\ &= \frac{1}{2x-1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-2x^2y+4x-1}{\left(2x-1\right)^2x^2}\right)+\left(\frac{1}{2x-1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{-2x^2y+4x-1}{\left(2x-1\right)^2x^2}\,\mathrm{d}x \\
\phi &= \frac{xy-1}{\left(2x-1\right)x}+f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y}=\frac{1}{2x-1}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{1}{2x-1}$. Therefore equation (4) becomes

$$
\frac{1}{2x-1}=\frac{1}{2x-1}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y)=0
$$

Therefore

$$
f(y)=c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{xy-1}{(2x-1)\,x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{xy-1}{(2x-1)\,x}$$

The solution becomes

$$y = \frac{2c_1x^2 - c_1x + 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{2c_1x^2 - c_1x + 1}{x} \tag{1}$$

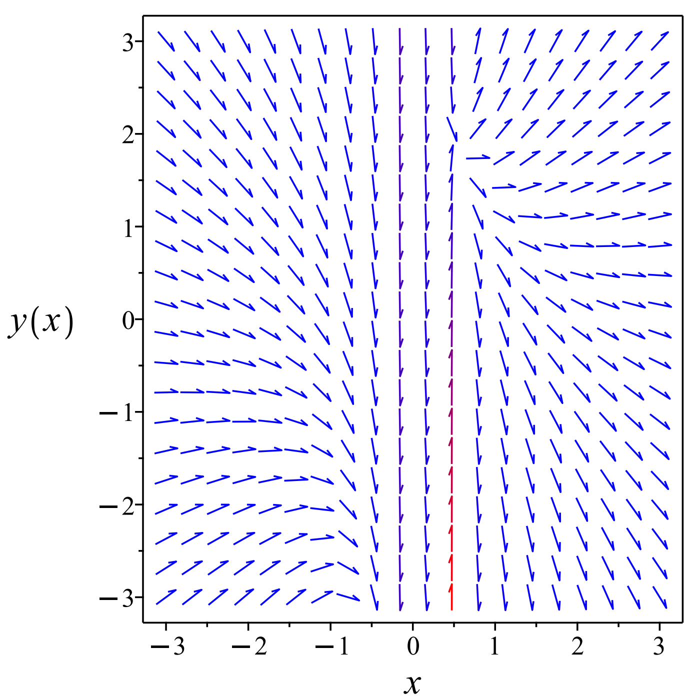


Figure 369: Slope field plot

Verification of solutions

$$y = \frac{2c_1x^2 - c_1x + 1}{x}$$

Verified OK.

12.13.4 Maple step by step solution

Let's solve

$(2x-1)\,y' - 2y = \frac{1-4x}{x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{2x-1} - \frac{-1+4x}{(2x-1)x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{2x-1} = -\frac{-1+4x}{(2x-1)x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{2x-1}\right) = -\frac{\mu(x)(-1+4x)}{(2x-1)x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{2y}{2x-1}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{2x-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{2x-1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{\mu(x)(-1+4x)}{(2x-1)x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)(-1+4x)}{(2x-1)x^2} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)(-1+4x)}{(2x-1)x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{2x-1}$

$$y = (2x-1)\left(\int -\frac{-1+4x}{(2x-1)^2 x^2}dx + c_1\right)$$

- Evaluate the integrals on the rhs

$$y = (2x-1)\left(-\frac{1}{x} + \frac{2}{2x-1} + c_1\right)$$

- Simplify

$$y = \frac{2c_1x^2 - c_1x + 1}{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve((2*x-1)*diff(y(x),x)-2*y(x)=(1-4*x)/x^2,y(x), singsol=all)
```

$$y = (2x-1)c_1 + \frac{1}{x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 18

```
DSolve[(2*x-1)*y'[x]-2*y[x]==(1-4*x)/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x} + 2c_1x - c_1$$

12.14 problem 288

Internal problem ID [15150]
Internal file name [`OUTPUT/15150_Tuesday_April_23_2024_04_51_24_PM_57554736/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 288.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + (3x + y + 1)\, y' = -x - 3}$$

12.14.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-X - x_0 + Y(X) + y_0 - 3}{3X + 3x_0 + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + Y(X)}{3X + Y(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= \frac{-X+Y}{3X+Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = -X+Y$ and $N = 3X+Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-1}{u+3} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-1}{u(X)+3} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{u(X)-1}{u(X)+3} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) Xu(X) + 3\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 2u(X) + 1 = 0
$$

Or

$$
X(u(X)+3)\left(\frac{d}{dX}u(X)\right) + (u(X)+1)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{(u+1)^2}{X\,(u+3)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(u+1)^2}{u+3}$. Integrating both sides gives

$$\frac{1}{\frac{(u+1)^2}{u+3}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{(u+1)^2}{u+3}}\,du = \int -\frac{1}{X}\,dX$$

$$-\frac{2}{u+1} + \ln(u+1) = -\ln(X) + c_2$$

The solution is

$$-\frac{2}{u(X)+1} + \ln(u(X)+1) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$-\frac{2}{\frac{Y(X)}{X}+1} + \ln\left(\frac{Y(X)}{X}+1\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$-\frac{2X}{Y(X)+X} + \ln\left(\frac{Y(X)+X}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - 1$$

Then the solution in y becomes

$$-\frac{2(1+x)}{y-1+x} + \ln\left(\frac{y-1+x}{1+x}\right) + \ln(1+x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{2(1+x)}{y-1+x} + \ln\left(\frac{y-1+x}{1+x}\right) + \ln(1+x) - c_2 = 0 \tag{1}$$

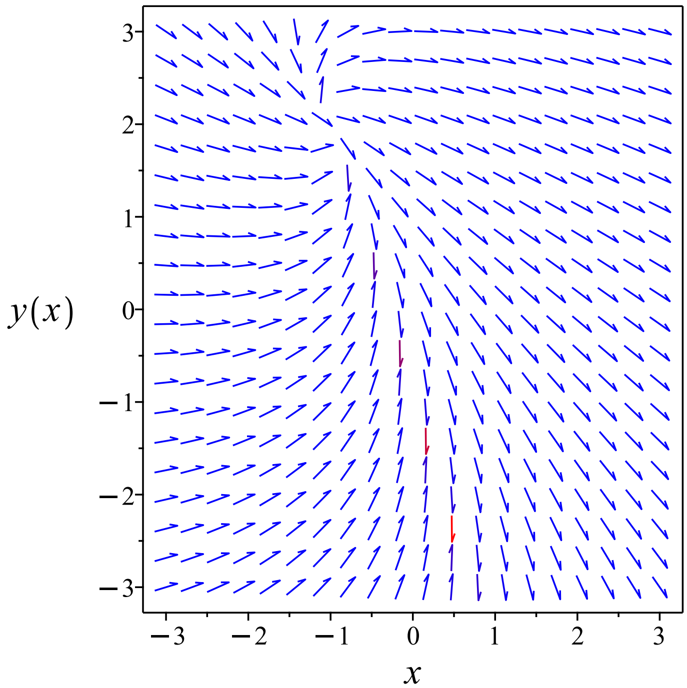


Figure 370: Slope field plot

Verification of solutions

$$-\frac{2(1+x)}{y-1+x}+\ln\left(\frac{y-1+x}{1+x}\right)+\ln(1+x)-c_2=0$$

Verified OK.

12.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{-x+y-3}{3x+y+1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-x+y-3)(b_3-a_2)}{3x+y+1} - \frac{(-x+y-3)^2 a_3}{(3x+y+1)^2} - \left(-\frac{1}{3x+y+1} - \frac{3(-x+y-3)}{(3x+y+1)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{1}{3x+y+1} - \frac{-x+y-3}{(3x+y+1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - x^2a_3 + 5x^2b_2 - 3x^2b_3 + 2xya_2 + 2xya_3 + 6xyb_2 - 2xyb_3 - y^2a_2 + 3y^2a_3 + y^2b_2 + y^2b_3 + 2xa_2 - 6}{(3x+y+} = 0$$

Setting the numerator to zero gives

$$3x^2a_2 - x^2a_3 + 5x^2b_2 - 3x^2b_3 + 2xya_2 + 2xya_3 + 6xyb_2 - 2xyb_3 - y^2a_2 + 3y^2a_3 + y^2b_2 + y^2b_3 + 2xa_2 - 6xa_3 - 4xb_1 + 2xb_2 - 10xb_3 + 4ya_1 + 2ya_2 - 2ya_3 + 2yb_2 - 6yb_3 - 8a_1 + 3a_2 - 9a_3 - 4b_1 + b_2 - 3b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2v_1^2 + 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 + 2a_3v_1v_2 + 3a_3v_2^2 + 5b_2v_1^2 + 6b_2v_1v_2 + b_2v_2^2 - 3b_3v_1^2 - 2b_3v_1v_2 + b_3v_2^2 + 4a_1v_2 + 2a_2v_1 + 2a_2v_2 - 6a_3v_1 - 2a_3v_2 - 4b_1v_1 + 2b_2v_1 + 2b_2v_2 - 10b_3v_1 - 6b_3v_2 - 8a_1 + 3a_2 - 9a_3 - 4b_1 + b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(3a_2 - a_3 + 5b_2 - 3b_3)\, v_1^2 + (2a_2 + 2a_3 + 6b_2 - 2b_3)\, v_1 v_2 \\ &+ (2a_2 - 6a_3 - 4b_1 + 2b_2 - 10b_3)\, v_1 + (-a_2 + 3a_3 + b_2 + b_3)\, v_2^2 \\ &+ (4a_1 + 2a_2 - 2a_3 + 2b_2 - 6b_3)\, v_2 - 8a_1 + 3a_2 - 9a_3 - 4b_1 + b_2 - 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-a_2 + 3a_3 + b_2 + b_3 &= 0\\ 2a_2 + 2a_3 + 6b_2 - 2b_3 &= 0\\ 3a_2 - a_3 + 5b_2 - 3b_3 &= 0\\ 4a_1 + 2a_2 - 2a_3 + 2b_2 - 6b_3 &= 0\\ 2a_2 - 6a_3 - 4b_1 + 2b_2 - 10b_3 &= 0\\ -8a_1 + 3a_2 - 9a_3 - 4b_1 + b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= b_3\\ a_2 &= -2b_2 + b_3\\ a_3 &= -b_2\\ b_1 &= b_2 - 2b_3\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1 + x\\ \eta &= y - 2\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y - 2 - \left(\frac{-x + y - 3}{3x + y + 1}\right)(1 + x)\\ &= \frac{x^2 + 2xy + y^2 - 2x - 2y + 1}{3x + y + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+2xy+y^2-2x-2y+1}{3x+y+1}} dy \end{aligned}$$

Which results in

$$S = \ln(x + y - 1) - \frac{2x + 2}{x + y - 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + y - 3}{3x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x - y + 3}{(x + y - 1)^2} \\ S_y &= \frac{3x + y + 1}{(x + y - 1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(y-1+x)\ln(y-1+x)-2x-2}{y-1+x} = c_1$$

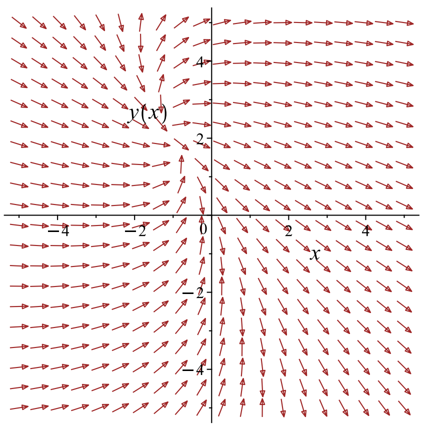
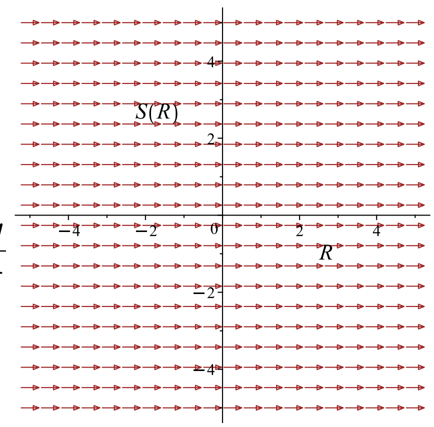
Which simplifies to

$$\frac{(y-1+x)\ln(y-1+x)-2x-2}{y-1+x} = c_1$$

Which gives

$$y = e^{\text{LambertW}\left(2(1+x)e^{-c_1}\right)+c_1} - x + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+y-3}{3x+y+1}$ 	$R = x$ $S = \frac{(x+y-1)\ln(x+y}{x+y-}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = e^{\mathrm{LambertW}\left(2(1+x)e^{-c_1}\right)+c_1} - x + 1 \tag{1}$$

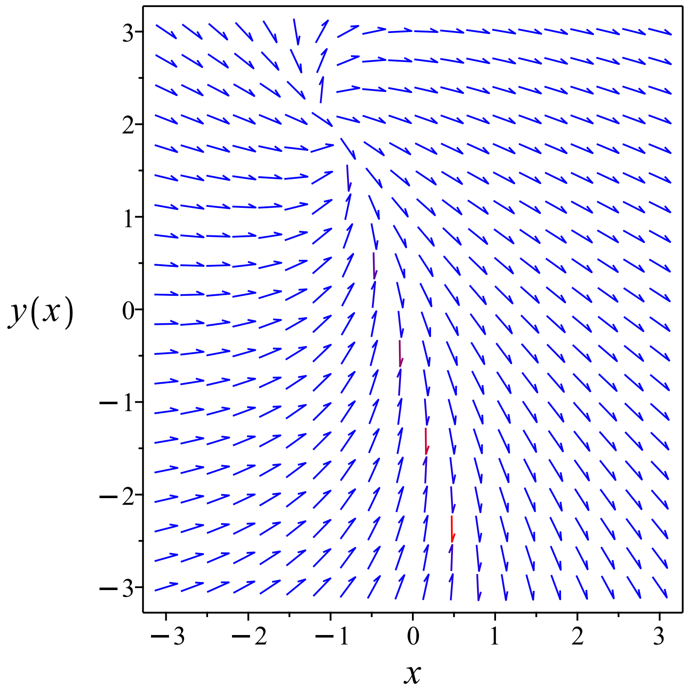


Figure 371: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(2(1+x)e^{-c_1}\right)+c_1} - x + 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 29

```
dsolve((x-y(x)+3)+(3*x+y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = 2 - \frac{(x+1)\left(\operatorname{LambertW}\left(-2c_1(x+1)\right) - 2\right)}{\operatorname{LambertW}\left(-2c_1\left(x+1\right)\right)}$$

✓ Solution by Mathematica

Time used: 0.771 (sec). Leaf size: 163

```
DSolve[(x-y[x]+3)+(3*x+y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{2^{2/3}\left(x\left(-\log\left(\frac{3\ 2^{2/3}(y(x)+x-1)}{y(x)+3x+1}\right)\right) + (x-1)\log\left(\frac{6\ 2^{2/3}(x+1)}{y(x)+3x+1}\right) + \log\left(\frac{3\ 2^{2/3}(y(x)+x-1)}{y(x)+3x+1}\right) + y(x)\left(\log\left(\frac{6}{y(}\right.\right.\right.}{9(y(x)+x-1)}\right.$$

12.15 problem 289

Internal problem ID [15151]
Internal file name [`OUTPUT/15151_Tuesday_April_23_2024_04_51_26_PM_81467620/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 289.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \cos\left(\frac{y}{2} + \frac{x}{2}\right) - \cos\left(-\frac{y}{2} + \frac{x}{2}\right) = 0}$$

12.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 2\sin\left(\frac{y}{2}\right)\sin\left(\frac{x}{2}\right) \end{aligned}$$

Where $f(x) = 2\sin\left(\frac{x}{2}\right)$ and $g(y) = \sin\left(\frac{y}{2}\right)$. Integrating both sides gives

$$\frac{1}{\sin\left(\frac{y}{2}\right)}\, dy = 2\sin\left(\frac{x}{2}\right)\, dx$$

$$\int \frac{1}{\sin\left(\frac{y}{2}\right)}\, dy = \int 2\sin\left(\frac{x}{2}\right)\, dx$$

$$2\ln\left(\csc\left(\frac{y}{2}\right) - \cot\left(\frac{y}{2}\right)\right) = -4\cos\left(\frac{x}{2}\right) + c_1$$

Raising both side to exponential gives

$$\left(\csc\left(\frac{y}{2}\right)-\cot\left(\frac{y}{2}\right)\right)^{2}=\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)+c_1}$$

Which simplifies to

$$\frac{1-\cos\left(\frac{y}{2}\right)}{\cos\left(\frac{y}{2}\right)+1}=c_2\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)}$$

Summary

The solution(s) found are the following

$$y=2\pi-2\arccos\left(\frac{c_2\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)+c_1}-1}{1+c_2\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)+c_1}}\right) \tag{1}$$

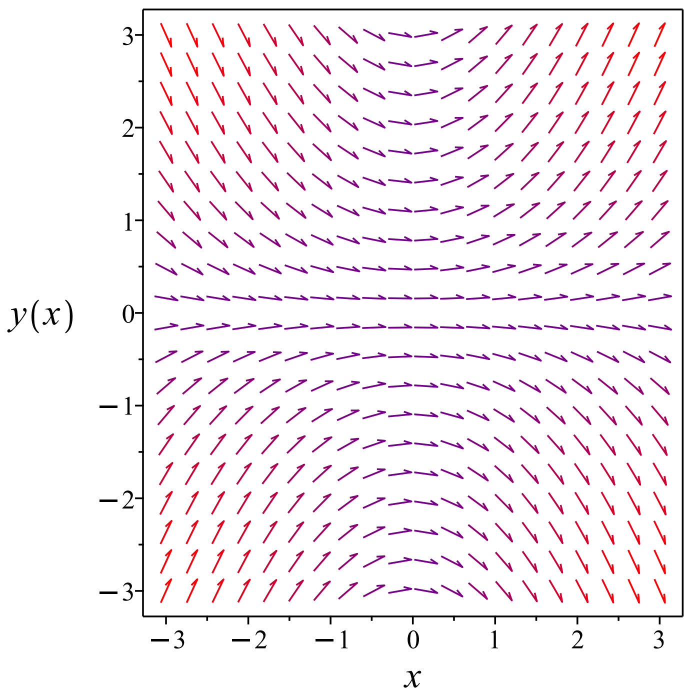


Figure 372: Slope field plot

Verification of solutions

$$y=2\pi-2\arccos\left(\frac{c_2\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)+c_1}-1}{1+c_2\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)+c_1}}\right)$$

Verified OK.

12.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\cos\left(\frac{y}{2}+\frac{x}{2}\right)+\cos\left(-\frac{y}{2}+\frac{x}{2}\right) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 293: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{2\sin\left(\frac{x}{2}\right)}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2\sin\left(\frac{x}{2}\right)}} dx \end{aligned}$$

Which results in

$$S = -4\cos\left(\frac{x}{2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\cos\left(\frac{y}{2} + \frac{x}{2}\right) + \cos\left(-\frac{y}{2} + \frac{x}{2}\right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 2\sin\left(\frac{x}{2}\right) \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \csc\left(\frac{y}{2}\right) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \csc\left(\frac{R}{2}\right)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln\left(\csc\left(\frac{R}{2}\right) + \cot\left(\frac{R}{2}\right)\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-4\cos\left(\frac{x}{2}\right) = -2\ln\left(\csc\left(\frac{y}{2}\right) + \cot\left(\frac{y}{2}\right)\right) + c_1$$

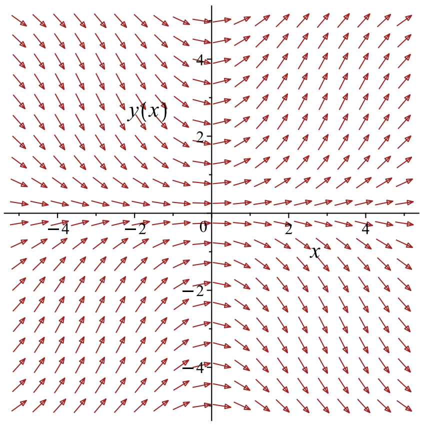
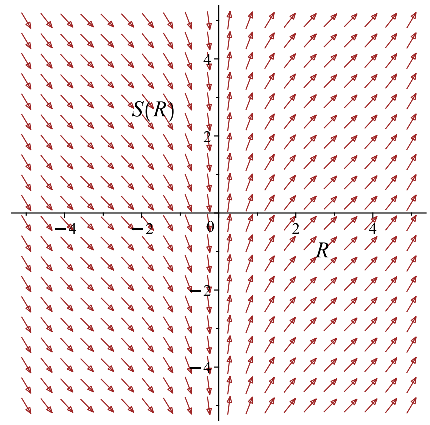
Which simplifies to

$$-4\cos\left(\frac{x}{2}\right) = -2\ln\left(\csc\left(\frac{y}{2}\right) + \cot\left(\frac{y}{2}\right)\right) + c_1$$

Which gives

$$y = 2\arctan\left(\frac{2\,\mathrm{e}^{2\cos\left(\frac{x}{2}\right)+\frac{c_1}{2}}}{\mathrm{e}^{c_1+4\cos\left(\frac{x}{2}\right)} + 1}, \frac{\mathrm{e}^{c_1+4\cos\left(\frac{x}{2}\right)} - 1}{\mathrm{e}^{c_1+4\cos\left(\frac{x}{2}\right)} + 1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cos\left(\frac{y}{2}+\frac{x}{2}\right)+\cos\left(-\frac{y}{2}+\frac{x}{2}\right)$ 	$R = y$ $S = -4\cos\left(\frac{x}{2}\right)$	$\frac{dS}{dR} = \csc\left(\frac{R}{2}\right)$ 

Summary

The solution(s) found are the following

$$y = 2\arctan\left(\frac{2\,e^{2\cos\left(\frac{x}{2}\right)+\frac{c_1}{2}}}{e^{c_1+4\cos\left(\frac{x}{2}\right)}+1}, \frac{e^{c_1+4\cos\left(\frac{x}{2}\right)}-1}{e^{c_1+4\cos\left(\frac{x}{2}\right)}+1}\right) \tag{1}$$

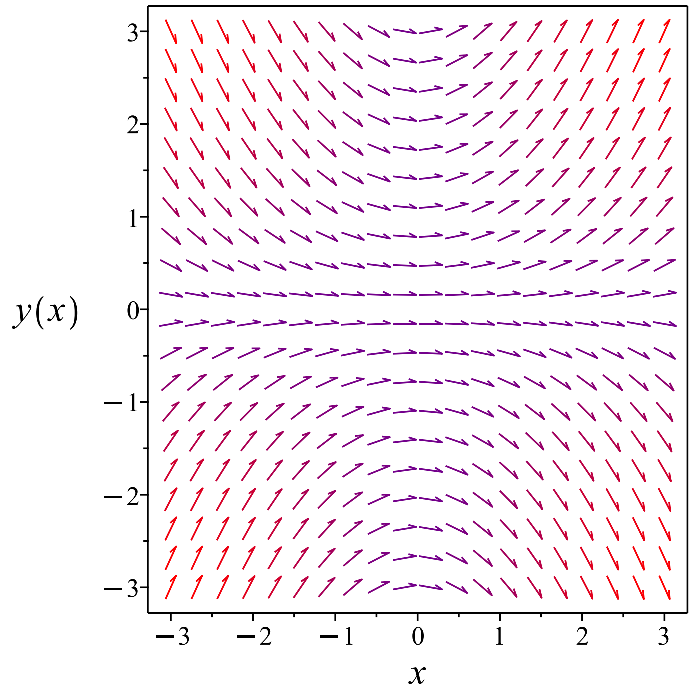


Figure 373: Slope field plot

Verification of solutions

$$y = 2 \arctan\left(\frac{2\,\mathrm{e}^{2\cos\left(\frac{x}{2}\right)+\frac{c_1}{2}}}{\mathrm{e}^{c_1+4\cos\left(\frac{x}{2}\right)}+1}, \frac{\mathrm{e}^{c_1+4\cos\left(\frac{x}{2}\right)}-1}{\mathrm{e}^{c_1+4\cos\left(\frac{x}{2}\right)}+1}\right)$$

Verified OK.

12.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, dx + N(x, y) \, dy = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2 \sin\left(\frac{y}{2}\right)}\right) dy = \left(\sin\left(\frac{x}{2}\right)\right) dx$$
$$\left(-\sin\left(\frac{x}{2}\right)\right) dx + \left(\frac{1}{2 \sin\left(\frac{y}{2}\right)}\right) dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\sin\left(\frac{x}{2}\right)$$
$$N(x, y) = \frac{1}{2 \sin\left(\frac{y}{2}\right)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\sin\left(\frac{x}{2}\right)\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2\sin\left(\frac{y}{2}\right)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\sin\left(\frac{x}{2}\right)\mathrm{d}x\end{aligned}$$

$$\phi = 2\cos\left(\frac{x}{2}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2\sin\left(\frac{y}{2}\right)}$. Therefore equation (4) becomes

$$\frac{1}{2\sin\left(\frac{y}{2}\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= \frac{1}{2\sin\left(\frac{y}{2}\right)}\\ &= \frac{\csc\left(\frac{y}{2}\right)}{2}\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\csc\left(\frac{y}{2}\right)}{2}\right)\mathrm{d}y$$

$$f(y) = -\ln\left(\csc\left(\frac{y}{2}\right) + \cot\left(\frac{y}{2}\right)\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\cos\left(\frac{x}{2}\right) - \ln\left(\csc\left(\frac{y}{2}\right) + \cot\left(\frac{y}{2}\right)\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\cos\left(\frac{x}{2}\right) - \ln\left(\csc\left(\frac{y}{2}\right) + \cot\left(\frac{y}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$2\cos\left(\frac{x}{2}\right) - \ln\left(\csc\left(\frac{y}{2}\right) + \cot\left(\frac{y}{2}\right)\right) = c_1 \tag{1}$$

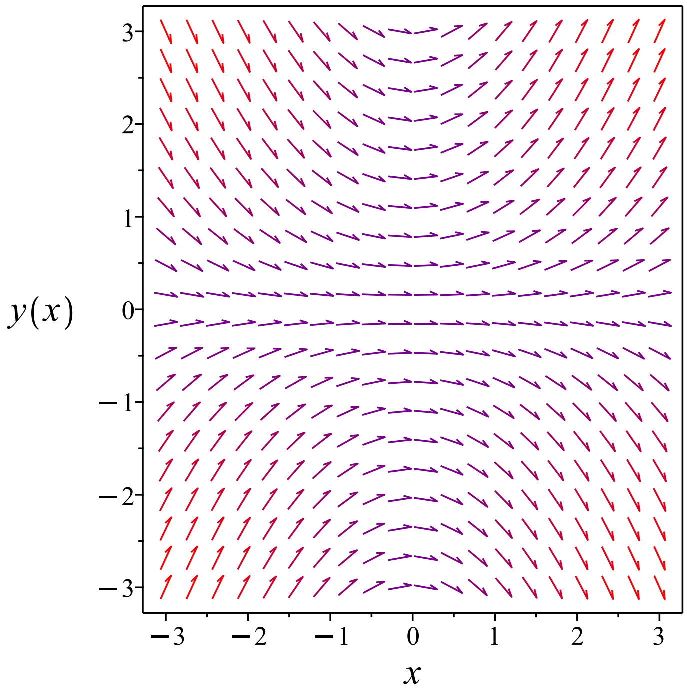


Figure 374: Slope field plot

Verification of solutions

$$2\cos\left(\frac{x}{2}\right)-\ln\left(\csc\left(\frac{y}{2}\right)+\cot\left(\frac{y}{2}\right)\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 62

```
dsolve(diff(y(x),x)+cos((x+y(x))/2)=cos((x-y(x))/2),y(x), singsol=all)
```

$$y = 2 \arctan\left(\frac{2\,\mathrm{e}^{-2\cos\left(\frac{x}{2}\right)} c_1}{\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)} c_1^2 + 1}, \frac{-\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)} c_1^2 + 1}{\mathrm{e}^{-4\cos\left(\frac{x}{2}\right)} c_1^2 + 1}\right)$$

✓ Solution by Mathematica

Time used: 0.486 (sec). Leaf size: 70

```
DSolve[y'[x]+Cos[(x+y[x])/2]==Cos[(x-y[x])/2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2 \arccos\left(\tanh\left(\frac{1}{2}\left(4\cos\left(\frac{x}{2}\right) - c_1\right)\right)\right)$$
$$y(x) \to 2 \arccos\left(\tanh\left(\frac{1}{2}\left(4\cos\left(\frac{x}{2}\right) - c_1\right)\right)\right)$$
$$y(x) \to 0$$
$$y(x) \to -2\pi$$
$$y(x) \to 2\pi$$

12.16 problem 290

Internal problem ID [15152]
Internal file name [`OUTPUT/15152_Tuesday_April_23_2024_04_51_30_PM_48630013/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 290.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'(3x^2-2x)-y(6x-2)=0}$$

12.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2y(3x-1)}{x(3x-2)} \end{aligned}$$

Where $f(x) = \frac{6x-2}{x(3x-2)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{6x-2}{x\,(3x-2)}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{6x-2}{x\,(3x-2)}\,dx \\
\ln\left(y\right) &= \ln\left(x(3x-2)\right) + c_1 \\
y &= \mathrm{e}^{\ln(x(3x-2))+c_1} \\
&= c_1 x(3x-2)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x(3x-2) \tag{1}$$

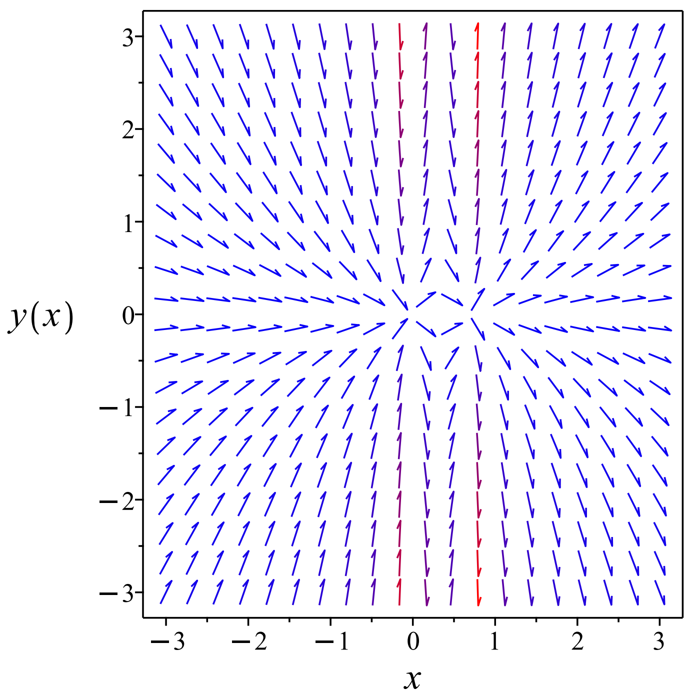


Figure 375: Slope field plot

Verification of solutions

$$y = c_1 x(3x-2)$$

Verified OK.

12.16.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{6x-2}{x(3x-2)} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{(6x-2)\, y}{x(3x-2)} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{6x-2}{x(3x-2)}dx} \\ &= \frac{1}{x(3x-2)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x(3x-2)}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x(3x-2)} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x(3x-2)}$ results in

$$y = c_1(3x^2 - 2x)$$

which simplifies to

$$y = c_1 x(3x-2)$$

Summary

The solution(s) found are the following

$$y = c_1 x(3x-2) \tag{1}$$

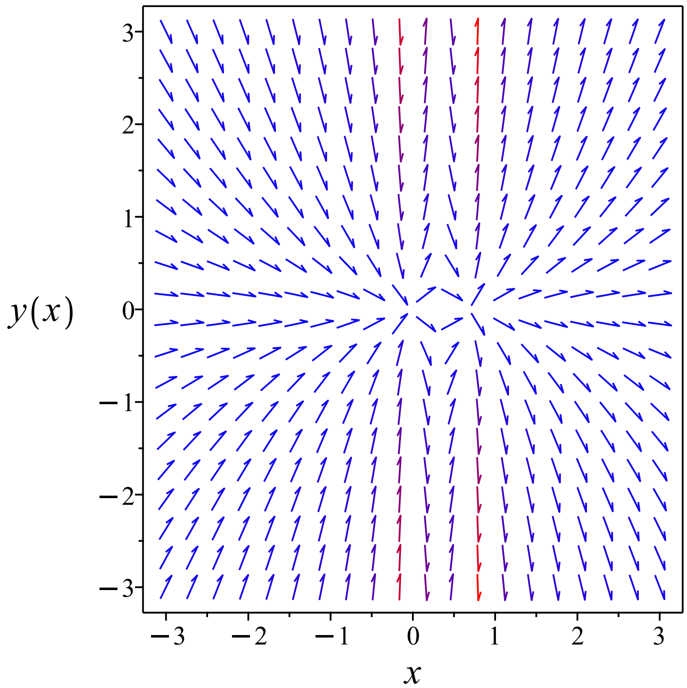


Figure 376: Slope field plot

Verification of solutions

$$y = c_1 x(3x - 2)$$

Verified OK.

12.16.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,(3x^2 - 2x) - u(x)\,x(6x - 2) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{3u}{3x - 2} \end{aligned}$$

Where $f(x) = \frac{3}{3x-2}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{3}{3x-2}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{3}{3x-2}\,dx \\
\ln(u) &= \ln(3x-2) + c_2 \\
u &= \mathrm{e}^{\ln(3x-2)+c_2} \\
&= c_2(3x-2)
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= xc_2(3x-2)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = xc_2(3x-2) \tag{1}$$

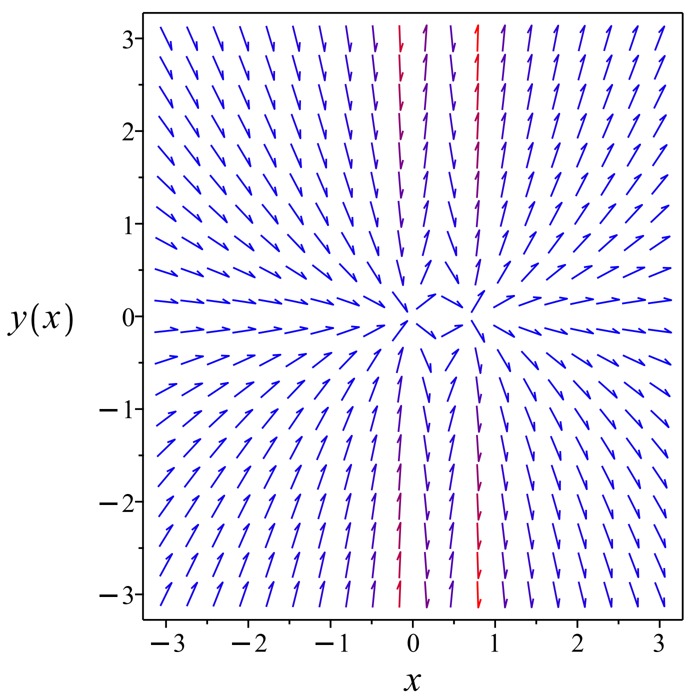


Figure 377: Slope field plot

Verification of solutions

$$y = xc_2(3x - 2)$$

Verified OK.

12.16.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2y(3x-1)}{x\,(3x-2)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 295: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x(3x-2) \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x(3x-2)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x(3x-2)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y(3x-1)}{x(3x-2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y(3x-1)}{x^2(3x-2)^2} \\ S_y &= \frac{1}{x(3x-2)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x(3x-2)} = c_1$$

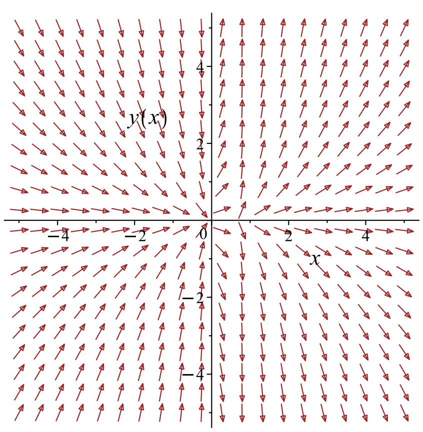
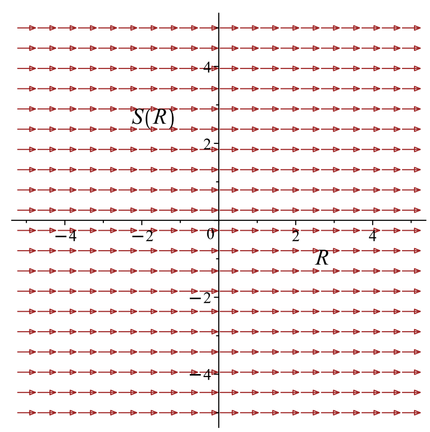
Which simplifies to

$$\frac{y}{x(3x-2)} = c_1$$

Which gives

$$y = c_1 x(3x-2)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y(3x-1)}{x(3x-2)}$ 	$R = x$ $S = \frac{y}{x(3x-2)}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 x(3x-2) \tag{1}$$

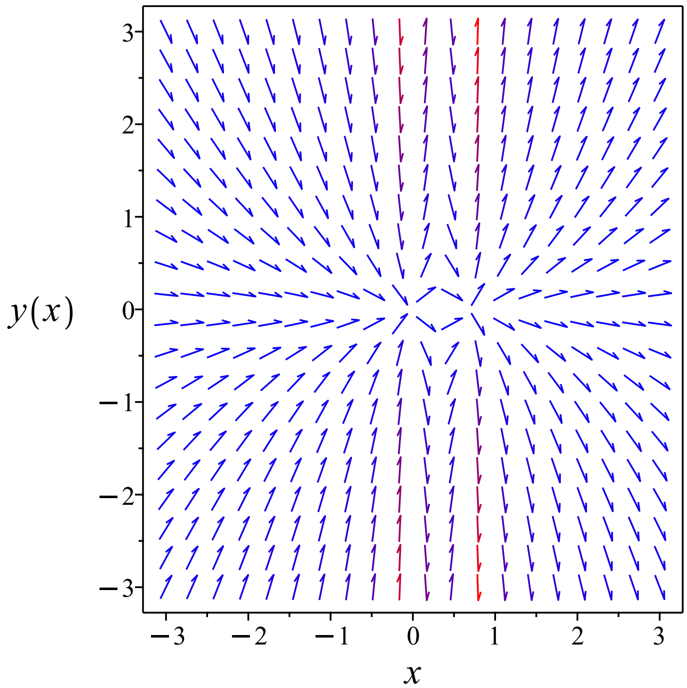


Figure 378: Slope field plot

Verification of solutions

$$y = c_1 x(3x - 2)$$

Verified OK.

12.16.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2y}\right) \mathrm{d}y = \left(\frac{3x - 1}{x (3x - 2)}\right) \mathrm{d}x$$
$$\left(-\frac{3x - 1}{x (3x - 2)}\right) \mathrm{d}x + \left(\frac{1}{2y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{3x - 1}{x (3x - 2)}$$
$$N(x, y) = \frac{1}{2y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3x - 1}{x (3x - 2)}\right) \\ &= 0 \end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{align}
\frac{\partial \phi}{\partial x} &= M \tag{1} \\
\frac{\partial \phi}{\partial y} &= N \tag{2}
\end{align}
$$

Integrating (1) w.r.t. x gives

$$
\begin{align}
\int \frac{\partial \phi}{\partial x} \, dx &= \int M \, dx \nonumber \\
\int \frac{\partial \phi}{\partial x} \, dx &= \int -\frac{3x-1}{x(3x-2)} \, dx \nonumber \\
\phi &= -\frac{\ln(3x^2 - 2x)}{2} + f(y) \tag{3}
\end{align}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$
\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln (y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln (3x^2 - 2x)}{2} + \frac{\ln (y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln (3x^2 - 2x)}{2} + \frac{\ln (y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{2c_1} x(3x - 2)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2c_1} x(3x - 2) \tag{1}$$

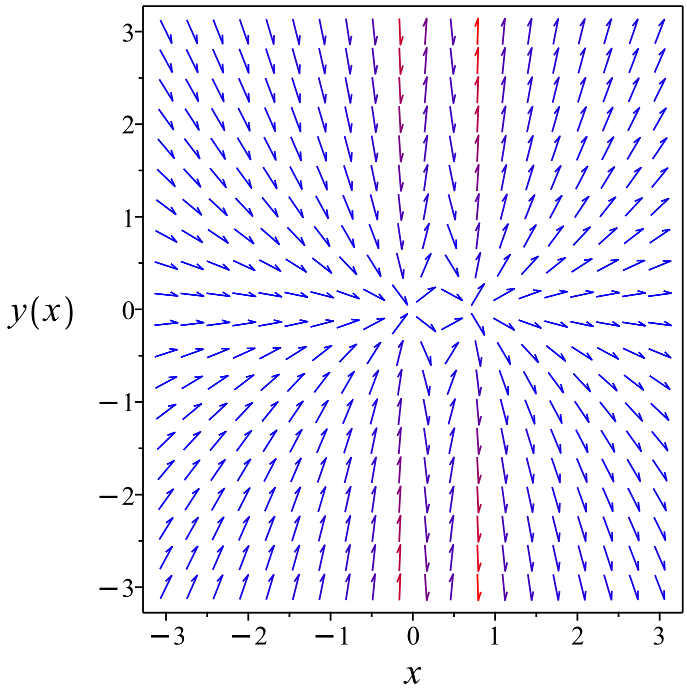


Figure 379: Slope field plot

Verification of solutions

$$y = e^{2c_1}x(3x - 2)$$

Verified OK.

12.16.6 Maple step by step solution

Let's solve

$y'(3x^2 - 2x) - y(6x - 2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{6x-2}{3x^2-2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{6x-2}{3x^2-2x}dx + c_1$

- Evaluate integral

$$\ln(y) = \ln(x(3x-2)) + c_1$$

- Solve for y

$$y = e^{c_1} x(3x-2)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)*(3*x^2-2*x)-y(x)*(6*x-2)=0,y(x), singsol=all)
```

$$y = c_1 x(3x-2)$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 19

```
DSolve[y'[x]*(3*x^2-2*x)-y[x]*(6*x-2)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(2-3x)x$$
$$y(x) \to 0$$

12.17 problem 291

Internal problem ID [15153]
Internal file name [`OUTPUT/15153_Tuesday_April_23_2024_04_51_30_PM_87528537/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 291.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{xy^2y' - y^3 = \frac{x^4}{3}}$$

12.17.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x^3 u(x)^2 \left(u'(x)\, x + u(x)\right) - u(x)^3\, x^3 = \frac{x^4}{3}$$

Integrating both sides gives

$$\int 3u^2 du = x + c_2$$
$$u^3 = x + c_2$$

Solving for u gives these solutions

$$
\begin{aligned}
u_1 &= (x+c_2)^{\frac{1}{3}} \\
u_2 &= -\frac{(x+c_2)^{\frac{1}{3}}}{2} - \frac{i\sqrt{3}\,(x+c_2)^{\frac{1}{3}}}{2} \\
u_3 &= -\frac{(x+c_2)^{\frac{1}{3}}}{2} + \frac{i\sqrt{3}\,(x+c_2)^{\frac{1}{3}}}{2}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x(x+c_2)^{\frac{1}{3}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x(x+c_2)^{\frac{1}{3}} \tag{1}
$$

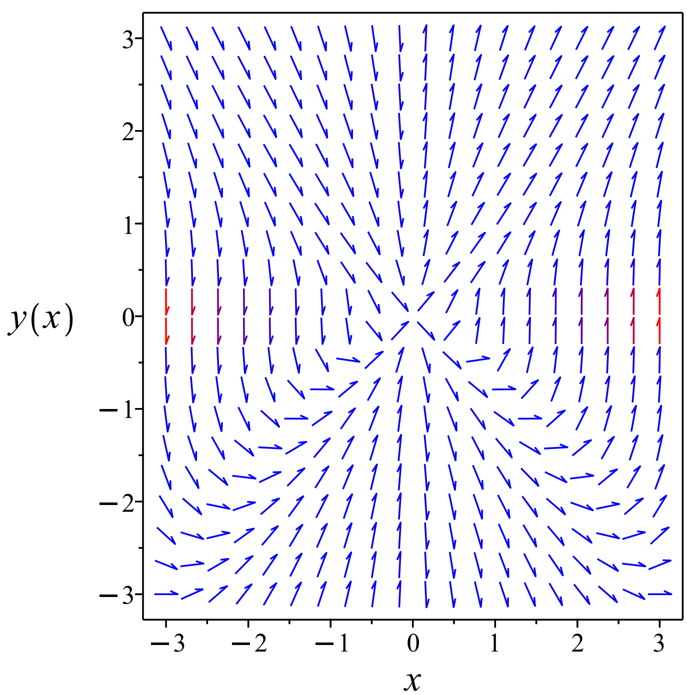


Figure 380: Slope field plot

Verification of solutions

$$
y = x(x+c_2)^{\frac{1}{3}}
$$

Verified OK.

12.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^4 + 3y^3}{3x\,y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 298: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{x^3}{y^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS\tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy\\ &= \int \frac{1}{\frac{x^3}{y^2}}dy\end{aligned}$$

Which results in

$$S=\frac{y^3}{3x^3}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y}\tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{x^4+3y^3}{3x\,y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^3}{x^4} \\
S_y &= \frac{y^2}{x^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{3} + c_1 \tag{4}
$$

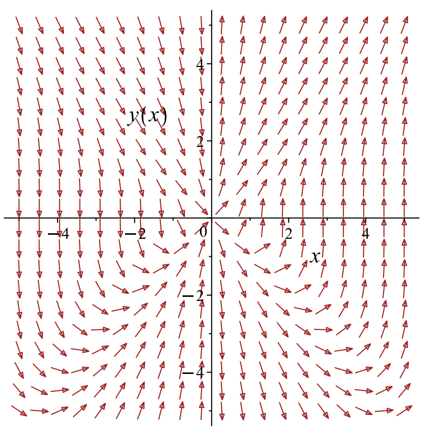
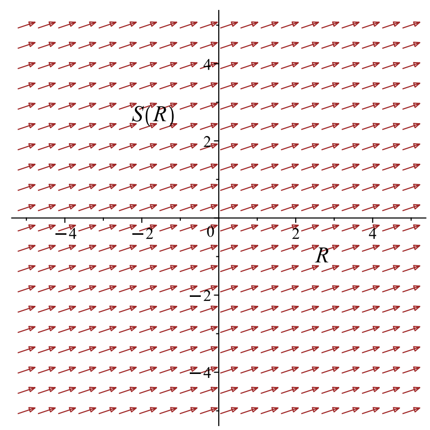
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y^3}{3x^3} = \frac{x}{3} + c_1
$$

Which simplifies to

$$
\frac{y^3}{3x^3} = \frac{x}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^4+3y^3}{3x\,y^2}$ 	$R = x$ $S = \frac{y^3}{3x^3}$	$\frac{dS}{dR} = \frac{1}{3}$ 

Summary

The solution(s) found are the following

$$\frac{y^3}{3x^3} = \frac{x}{3} + c_1 \tag{1}$$

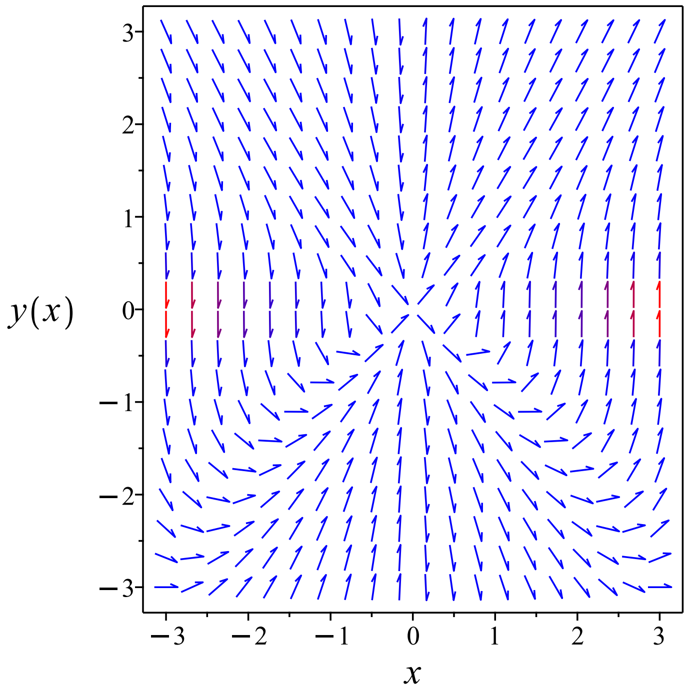


Figure 381: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^3} = \frac{x}{3} + c_1$$

Verified OK.

12.17.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^4 + 3y^3}{3x\,y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + \frac{x^3}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= \frac{x^3}{3} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = \frac{y^3}{x} + \frac{x^3}{3} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= \frac{w(x)}{x} + \frac{x^3}{3} \\
w' &= \frac{3w}{x} + x^3
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= x^3
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = x^3$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(x^3\right) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(x^3\right) \\ d\left(\frac{w}{x^3}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int dx \\ \frac{w}{x^3} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = c_1 x^3 + x^4$$

which simplifies to

$$w(x) = x^3(x + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x^3(x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= (x + c_1)^{\frac{1}{3}}\, x \\ y(x) &= \frac{(x + c_1)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right) x}{2} \\ y(x) &= -\frac{(x + c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right) x}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (x + c_1)^{\frac{1}{3}} x \tag{1}$$

$$y = \frac{(x + c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) x}{2} \tag{2}$$

$$y = -\frac{(x + c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right) x}{2} \tag{3}$$

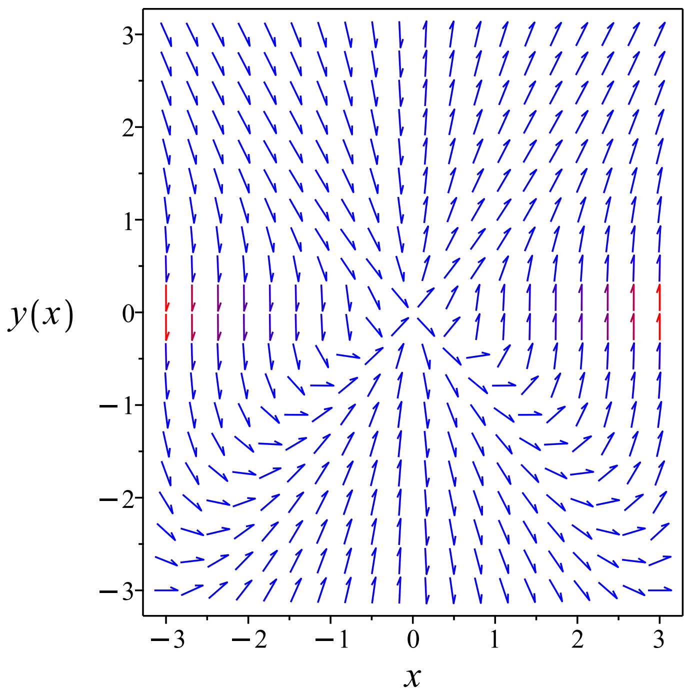


Figure 382: Slope field plot

Verification of solutions

$$y = (x + c_1)^{\frac{1}{3}} x$$

Verified OK.

$$y = \frac{(x + c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) x}{2}$$

Verified OK.

$$y = -\frac{(x + c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right) x}{2}$$

Verified OK.

12.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x\,y^2)\,\mathrm{d}y = \left(y^3 + \frac{x^4}{3}\right)\mathrm{d}x$$

$$\left(-y^3 - \frac{x^4}{3}\right)\mathrm{d}x + (x\,y^2)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y^3 - \frac{x^4}{3}\\ N(x,y) &= x\,y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^3 - \frac{x^4}{3}\right)\\ &= -3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\,y^2)\\ &= y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\,y^2}\left(\left(-3y^2\right) - \left(y^2\right)\right)\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{4}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(-y^3 - \frac{x^4}{3}\right) \\ &= \frac{-x^4 - 3y^3}{3x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^4}(x\,y^2) \\ &= \frac{y^2}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^4 - 3y^3}{3x^4}\right) + \left(\frac{y^2}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^4 - 3y^3}{3x^4} \, \mathrm{d}x$$

$$\phi = -\frac{x}{3} + \frac{y^3}{3x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y^2}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2}{x^3}$. Therefore equation (4) becomes

$$\frac{y^2}{x^3} = \frac{y^2}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x}{3} + \frac{y^3}{3x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x}{3} + \frac{y^3}{3x^3}$$

Summary

The solution(s) found are the following

$$-\frac{x}{3} + \frac{y^3}{3x^3} = c_1 \tag{1}$$

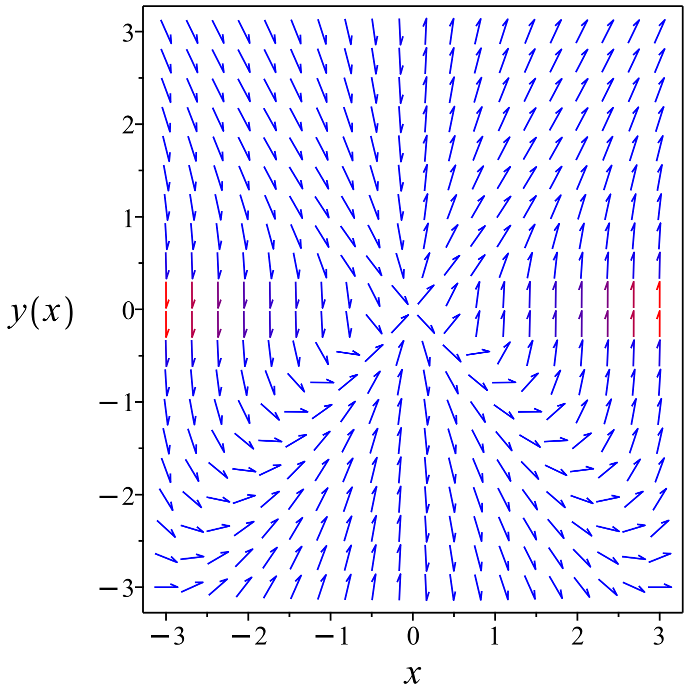


Figure 383: Slope field plot

Verification of solutions

$$-\frac{x}{3}+\frac{y^3}{3x^3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
dsolve(x*y(x)^2*diff(y(x),x)-y(x)^3=1/3*x^4,y(x), singsol=all)
```

$$y = (x+c_1)^{\frac{1}{3}} x$$
$$y = -\frac{(x+c_1)^{\frac{1}{3}} \left(1+i\sqrt{3}\right) x}{2}$$
$$y = \frac{(x+c_1)^{\frac{1}{3}} \left(i\sqrt{3}-1\right) x}{2}$$

✓ Solution by Mathematica

Time used: 0.174 (sec). Leaf size: 54

```
DSolve[x*y[x]^2*y'[x]-y[x]^3==1/3*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\sqrt[3]{x+c_1}$$
$$y(x) \to -\sqrt[3]{-1}x\sqrt[3]{x+c_1}$$
$$y(x) \to (-1)^{2/3}x\sqrt[3]{x+c_1}$$

12.18 problem 292

Internal problem ID [15154]
Internal file name [`OUTPUT/15154_Tuesday_April_23_2024_04_51_32_PM_90493082/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 292.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _dAlembert]
```

$$\boxed{e^{\frac{x}{y}}+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)y'=-1}$$

With initial conditions

$$[y(1)=1]$$

12.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}
y' &= f(x,y)\\
&= -\frac{y\left(e^{\frac{x}{y}}+1\right)e^{-\frac{x}{y}}}{-x+y}
\end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x<1\vee 1<x\}$$

But the point $x_0=1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

12.18.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$e^{\frac{1}{u(x)}}+e^{\frac{1}{u(x)}}\left(1-\frac{1}{u(x)}\right)(u'(x)\,x+u(x))=-1$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u\left(e^{-\frac{1}{u}}+u\right)}{x(u-1)}
\end{aligned}$$

Where $f(x)=-\frac{1}{x}$ and $g(u)=\frac{\left(e^{-\frac{1}{u}}+u\right)u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{\left(e^{-\frac{1}{u}}+u\right)u}{u-1}}\,du=-\frac{1}{x}\,dx$$

$$\int\frac{1}{\frac{\left(e^{-\frac{1}{u}}+u\right)u}{u-1}}\,du=\int-\frac{1}{x}\,dx$$

$$\frac{1}{u}+\ln\left(e^{-\frac{1}{u}}+u\right)=-\ln(x)+c_2$$

The solution is

$$\frac{1}{u(x)}+\ln\left(e^{-\frac{1}{u(x)}}+u(x)\right)+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x}{y}+\ln\left(e^{-\frac{x}{y}}+\frac{y}{x}\right)+\ln(x)-c_2=0$$

$$\frac{\ln\left(\frac{e^{-\frac{x}{y}}x+y}{x}\right)y+(-c_2+\ln(x))\,y+x}{y}=0$$

Substituting initial conditions and solving for c_2 gives $c_2 = 1+\ln\left(e^{-1}+1\right)$. Hence the so-
Summary
The solution(s) found are the following
lution becomes

$$\frac{\ln\left(\frac{e^{-\frac{x}{y}}x+y}{x}\right)y+\left(-1-\ln\left(e^{-1}+1\right)+\ln\left(x\right)\right)y+x}{y}=0 \tag{1}$$

Verification of solutions

$$\frac{\ln\left(\frac{e^{-\frac{x}{y}}x+y}{x}\right)y+\left(-1-\ln\left(e^{-1}+1\right)+\ln\left(x\right)\right)y+x}{y}=0$$

Verified OK.

12.18.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y\left(e^{\frac{x}{y}}+1\right)e^{-\frac{x}{y}}}{-x+y} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}(b_3-a_2)}{-x+y} - \frac{y^2\left(\mathrm{e}^{\frac{x}{y}}+1\right)^2\mathrm{e}^{-\frac{2x}{y}}a_3}{(-x+y)^2} \\
&- \left(-\frac{1}{-x+y} + \frac{\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y} - \frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{(-x+y)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y} + \frac{x}{y(-x+y)} - \frac{\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}x}{y(-x+y)} \right. \\
&\left. + \frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{(-x+y)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{\left(2\,\mathrm{e}^{\frac{2x}{y}}x\,y^2b_2 - \mathrm{e}^{\frac{2x}{y}}y^3a_2 - \mathrm{e}^{\frac{2x}{y}}y^3b_2 + \mathrm{e}^{\frac{2x}{y}}y^3b_3 + \mathrm{e}^{\frac{2x}{y}}xyb_1 - \mathrm{e}^{\frac{2x}{y}}y^2a_1 + \mathrm{e}^{\frac{x}{y}}x^3b_2 - \mathrm{e}^{\frac{x}{y}}x^2ya_2 + \mathrm{e}^{\frac{x}{y}}x^2yb_3 + \mathrm{e}^{\frac{x}{y}}x\,y \cdots\right.}{(x-y)^2\,y}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2\,\mathrm{e}^{\frac{2x}{y}}x\,y^2b_2 + \mathrm{e}^{\frac{2x}{y}}y^3a_2 + \mathrm{e}^{\frac{2x}{y}}y^3b_2 - \mathrm{e}^{\frac{2x}{y}}y^3b_3 - \mathrm{e}^{\frac{2x}{y}}xyb_1 + \mathrm{e}^{\frac{2x}{y}}y^2a_1 \\
&- \mathrm{e}^{\frac{x}{y}}x^3b_2 + \mathrm{e}^{\frac{x}{y}}x^2ya_2 - \mathrm{e}^{\frac{x}{y}}x^2yb_3 - \mathrm{e}^{\frac{x}{y}}x\,y^2a_2 + \mathrm{e}^{\frac{x}{y}}x\,y^2a_3 + \mathrm{e}^{\frac{x}{y}}x\,y^2b_3 \\
&+ \mathrm{e}^{\frac{x}{y}}y^3a_2 - 2\,\mathrm{e}^{\frac{x}{y}}y^3a_3 - \mathrm{e}^{\frac{x}{y}}y^3b_3 - \mathrm{e}^{\frac{x}{y}}x^2b_1 + \mathrm{e}^{\frac{x}{y}}xya_1 - y^3a_3 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-2\,\mathrm{e}^{\frac{2x}{y}}x\,y^2b_2 + \mathrm{e}^{\frac{2x}{y}}y^3a_2 + \mathrm{e}^{\frac{2x}{y}}y^3b_2 - \mathrm{e}^{\frac{2x}{y}}y^3b_3 - \mathrm{e}^{\frac{2x}{y}}xyb_1 + \mathrm{e}^{\frac{2x}{y}}y^2a_1 \\
&- \mathrm{e}^{\frac{x}{y}}x^3b_2 + \mathrm{e}^{\frac{x}{y}}x^2ya_2 - \mathrm{e}^{\frac{x}{y}}x^2yb_3 - \mathrm{e}^{\frac{x}{y}}x\,y^2a_2 + \mathrm{e}^{\frac{x}{y}}x\,y^2a_3 + \mathrm{e}^{\frac{x}{y}}x\,y^2b_3 \\
&+ \mathrm{e}^{\frac{x}{y}}y^3a_2 - 2\,\mathrm{e}^{\frac{x}{y}}y^3a_3 - \mathrm{e}^{\frac{x}{y}}y^3b_3 - \mathrm{e}^{\frac{x}{y}}x^2b_1 + \mathrm{e}^{\frac{x}{y}}xya_1 - y^3a_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \mathrm{e}^{\frac{x}{y}}, \mathrm{e}^{\frac{2x}{y}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, e^{\frac{x}{y}} = v_3, e^{\frac{2x}{y}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &v_3 v_1^2 v_2 a_2 - v_3 v_1 v_2^2 a_2 + v_3 v_2^3 a_2 + v_4 v_2^3 a_2 + v_3 v_1 v_2^2 a_3 - 2 v_3 v_2^3 a_3 \\ &- v_3 v_1^3 b_2 - 2 v_4 v_1 v_2^2 b_2 + v_4 v_2^3 b_2 - v_3 v_1^2 v_2 b_3 + v_3 v_1 v_2^2 b_3 - v_3 v_2^3 b_3 \\ &- v_4 v_2^3 b_3 + v_3 v_1 v_2 a_1 + v_4 v_2^2 a_1 - v_2^3 a_3 - v_3 v_1^2 b_1 - v_4 v_1 v_2 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &-v_3 v_1^3 b_2 + (-b_3 + a_2) v_1^2 v_2 v_3 - v_3 v_1^2 b_1 + (-a_2 + a_3 + b_3) v_1 v_2^2 v_3 \\ &- 2 v_4 v_1 v_2^2 b_2 + v_3 v_1 v_2 a_1 - v_4 v_1 v_2 b_1 + (a_2 - 2a_3 - b_3) v_2^3 v_3 \\ &+ (a_2 + b_2 - b_3) v_2^3 v_4 - v_2^3 a_3 + v_4 v_2^2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2b_2 &= 0 \\ -b_2 &= 0 \\ -b_3 + a_2 &= 0 \\ -a_2 + a_3 + b_3 &= 0 \\ a_2 - 2a_3 - b_3 &= 0 \\ a_2 + b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y}\right)(x) \\
&= \frac{-y^2\mathrm{e}^{\frac{x}{y}} - xy}{\mathrm{e}^{\frac{x}{y}}x - y\,\mathrm{e}^{\frac{x}{y}}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^2 \mathrm{e}^{\frac{x}{y}} - xy}{\mathrm{e}^{\frac{x}{y}} x - y \,\mathrm{e}^{\frac{x}{y}}}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(y\,\mathrm{e}^{\frac{x}{y}} + x\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y\left(\mathrm{e}^{\frac{x}{y}} + 1\right)\mathrm{e}^{-\frac{x}{y}}}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\mathrm{e}^{\frac{x}{y}} + 1}{y\,\mathrm{e}^{\frac{x}{y}} + x} \\
S_y &= -\frac{\mathrm{e}^{\frac{x}{y}}(x - y)}{y\left(y\,\mathrm{e}^{\frac{x}{y}} + x\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\mathrm{e}^{\frac{x}{y}} y + x\right) = c_1$$

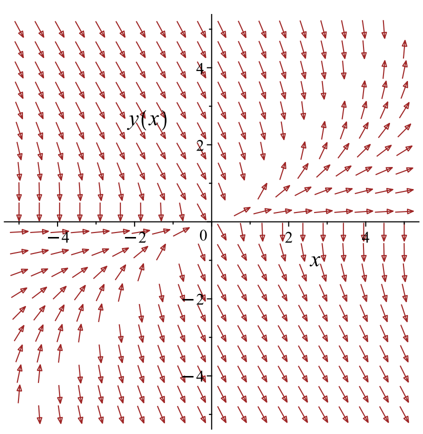
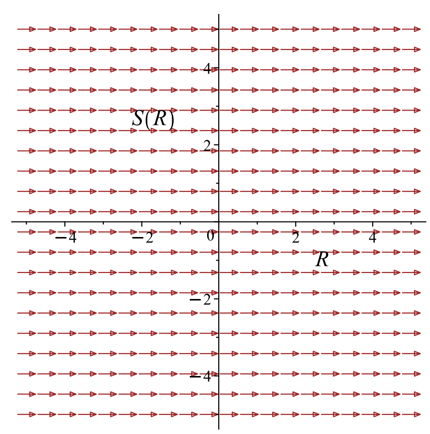
Which simplifies to

$$\ln\left(\mathrm{e}^{\frac{x}{y}} y + x\right) = c_1$$

Which gives

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{\mathrm{e}^{c_1 - x}}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{e}^{-\frac{x}{y}}}{-x+y}$ 	$R = x$ $S = \ln\left(y\,\mathrm{e}^{\frac{x}{y}} + x\right)$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{\mathrm{LambertW}\left(-\frac{1}{\mathrm{e}^{c_1}-1}\right)}$$

$$c_1 = \ln\left(\mathrm{e}+1\right)$$

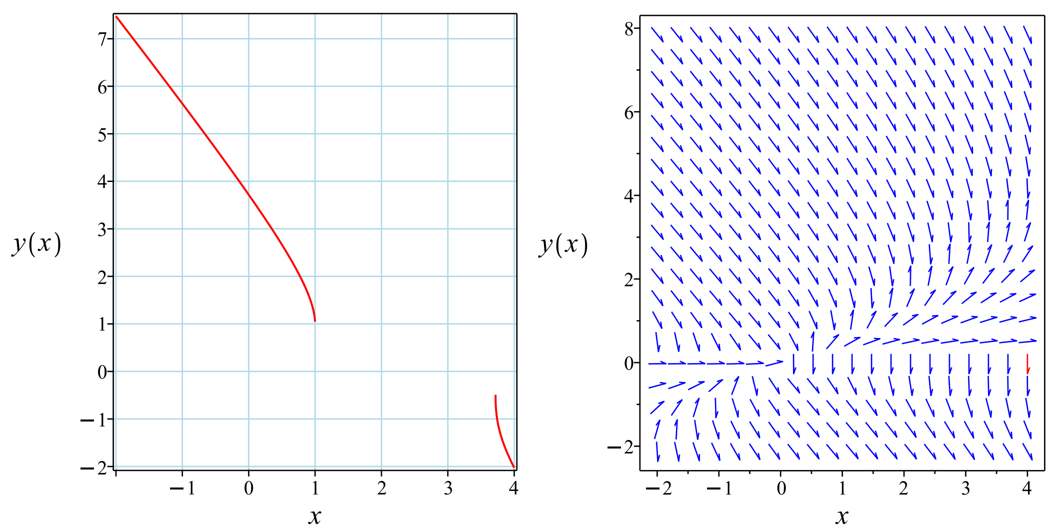
Substituting c_1 found above in the general solution gives

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{-x+1+\mathrm{e}}\right)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{-x+1+\mathrm{e}}\right)} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{x}{\mathrm{LambertW}\left(-\frac{x}{-x+1+\mathrm{e}}\right)}$$

Verified OK.

12.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\right)\mathrm{d}y&=\left(-\mathrm{e}^{\frac{x}{y}}-1\right)\mathrm{d}x\\ \left(\mathrm{e}^{\frac{x}{y}}+1\right)\mathrm{d}x+\left(\mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \mathrm{e}^{\frac{x}{y}} + 1 \\
N(x,y) &= \mathrm{e}^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{\frac{x}{y}} + 1\right) \\
&= -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right) \\
&= -\frac{x\,\mathrm{e}^{\frac{x}{y}}}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{\frac{x}{y}} + 1\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = y\,\mathrm{e}^{\frac{x}{y}} + x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= e^{\frac{x}{y}} - \frac{x\, e^{\frac{x}{y}}}{y} + f'(y) \qquad (4)\\ &= -\frac{e^{\frac{x}{y}} (x-y)}{y} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right)$. Therefore equation (4) becomes

$$e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) = -\frac{e^{\frac{x}{y}} (x-y)}{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\, e^{\frac{x}{y}} + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\, e^{\frac{x}{y}} + x$$

The solution becomes

$$y = -\frac{x}{\text{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{\text{LambertW}\left(-\frac{1}{-1+c_1}\right)}$$

$$c_1 = \mathrm{e} + 1$$

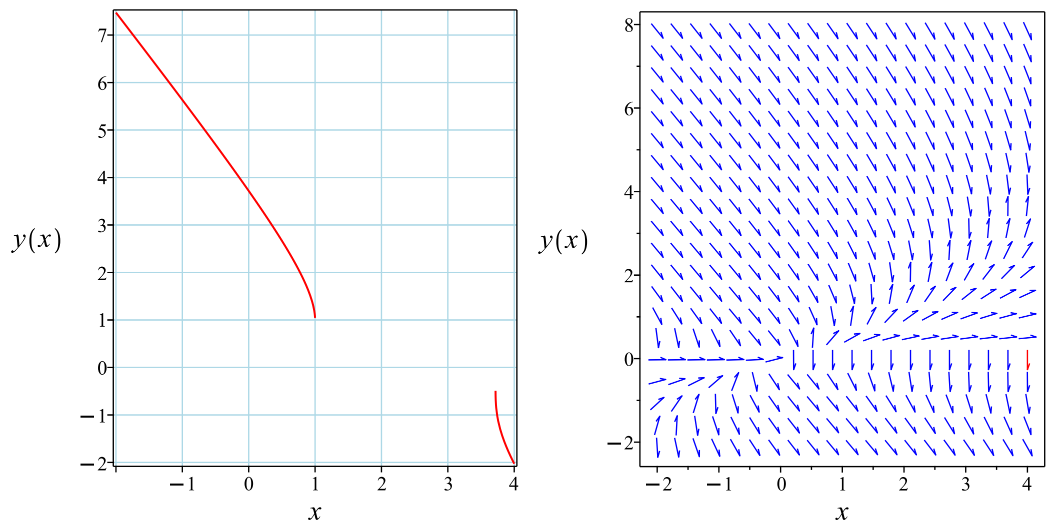
Substituting c_1 found above in the general solution gives

$$y = -\frac{x}{\operatorname{LambertW}\left(-\frac{x}{-x+1+\mathrm{e}}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\operatorname{LambertW}\left(-\frac{x}{-x+1+\mathrm{e}}\right)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{x}{\operatorname{LambertW}\left(-\frac{x}{-x+1+\mathrm{e}}\right)}$$

Verified OK.

12.18.5 Maple step by step solution

Let's solve

$\left[e^{\frac{x}{y}}+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right) y'=-1, y(1)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

  ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

  $-\frac{x\,e^{\frac{x}{y}}}{y^2}=\frac{e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{y}-\frac{e^{\frac{x}{y}}}{y}$

  ◦ Simplify

  $-\frac{x\,e^{\frac{x}{y}}}{y^2}=-\frac{x\,e^{\frac{x}{y}}}{y^2}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(e^{\frac{x}{y}}+1\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=y\,e^{\frac{x}{y}}+x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=e^{\frac{x}{y}}-\frac{x\,e^{\frac{x}{y}}}{y}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \mathrm{e}^{\frac{x}{y}}\left(1-\frac{x}{y}\right) - \mathrm{e}^{\frac{x}{y}} + \frac{x\,\mathrm{e}^{\frac{x}{y}}}{y}$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = y\,\mathrm{e}^{\frac{x}{y}} + x$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y\,\mathrm{e}^{\frac{x}{y}} + x = c_1$$

- Solve for y

$$y = -\frac{x}{\mathit{LambertW}\left(-\frac{x}{-x+c_1}\right)}$$

- Use initial condition $y(1) = 1$

$$1 = -\frac{1}{\mathit{LambertW}\left(-\frac{1}{-1+c_1}\right)}$$

- Solve for c_1

$$c_1 = \mathrm{e} + 1$$

- Substitute $c_1 = \mathrm{e} + 1$ into general solution and simplify

$$y = -\frac{x}{\mathit{LambertW}\left(\frac{x}{x-1-\mathrm{e}}\right)}$$

- Solution to the IVP

$$y = -\frac{x}{\mathit{LambertW}\left(\frac{x}{x-1-\mathrm{e}}\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 21

```
dsolve([(1+exp(x/y(x)))+(exp(x/y(x))*(1-x/y(x)))*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y = -\frac{x}{\operatorname{LambertW}\left(\frac{x}{-1+x-\mathrm{e}}\right)}$$

✓ Solution by Mathematica

Time used: 1.228 (sec). Leaf size: 21

```
DSolve[{(1+Exp[x/y[x]])+(Exp[x/y[x]]*(1-x/y[x]))*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularS
```

$$y(x) \to -\frac{x}{W\left(\frac{x}{x-e-1}\right)}$$

12.19 problem 293

Internal problem ID [15155]
Internal file name [`OUTPUT/15155_Tuesday_April_23_2024_04_51_39_PM_41577422/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 293.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - y'yx = -x^2}$$

12.19.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - (u'(x)\,x + u(x))\,u(x)\,x^2 = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{ux} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{u}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\frac{1}{u}}\,du &= \int \frac{1}{x}\,dx \\
\frac{u^2}{2} &= \ln\left(x\right) + c_2
\end{aligned}
$$

The solution is

$$
\frac{u(x)^2}{2} - \ln\left(x\right) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2}{2x^2} - \ln\left(x\right) - c_2 = 0 \\
\frac{y^2}{2x^2} - \ln\left(x\right) - c_2 = 0
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\frac{y^2}{2x^2} - \ln\left(x\right) - c_2 = 0 \tag{1}
$$

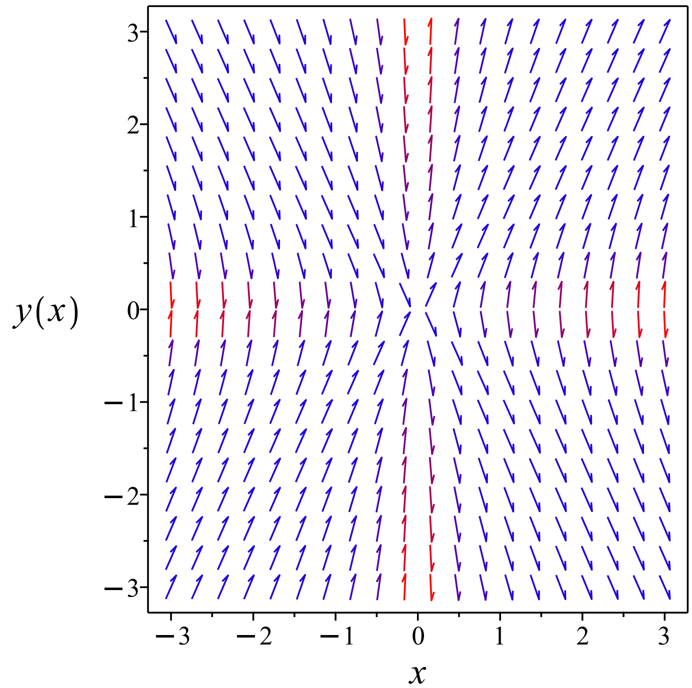


Figure 386: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - \ln(x) - c_2 = 0$$

Verified OK.

12.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 301: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^2}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2}{y}} dy
\end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + y^2}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{x^3} \\
S_y &= \frac{y}{x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

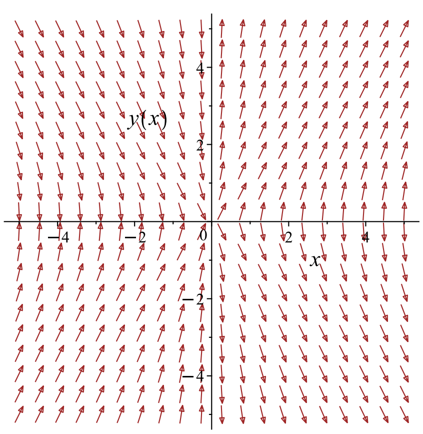
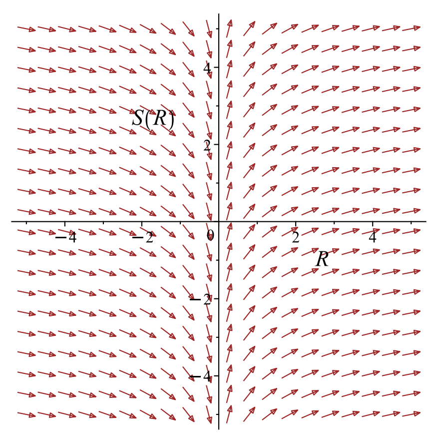
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^2} = \ln(x) + c_1$$

Which simplifies to

$$\frac{y^2}{2x^2} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{yx}$ 	$R = x$ $S = \frac{y^2}{2x^2}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^2} = \ln(x) + c_1 \tag{1}$$

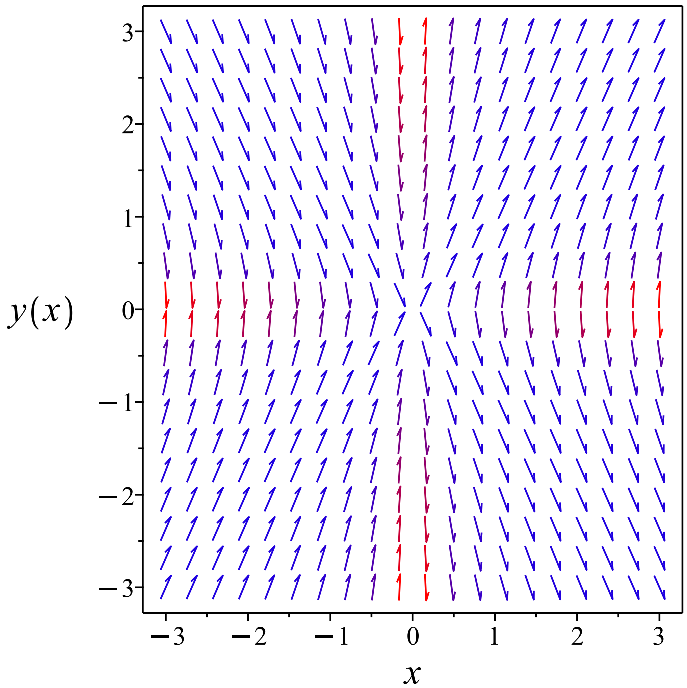


Figure 387: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} = \ln(x) + c_1$$

Verified OK.

12.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{x} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{x} + x \\ w' &= \frac{2w}{x} + 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= 2x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(2x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)(2x) \\
\mathrm{d}\left(\frac{w}{x^2}\right) &= \left(\frac{2}{x}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^2} &= \int \frac{2}{x}\,\mathrm{d}x \\
\frac{w}{x^2} &= 2\ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = 2\ln(x)\,x^2 + c_1 x^2$$

which simplifies to

$$w(x) = x^2(2\ln(x) + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x^2(2\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}
y(x) &= \sqrt{2\ln(x) + c_1}\,x \\
y(x) &= -\sqrt{2\ln(x) + c_1}\,x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2\ln(x) + c_1}\, x \tag{1}$$

$$y = -\sqrt{2\ln(x) + c_1}\, x \tag{2}$$

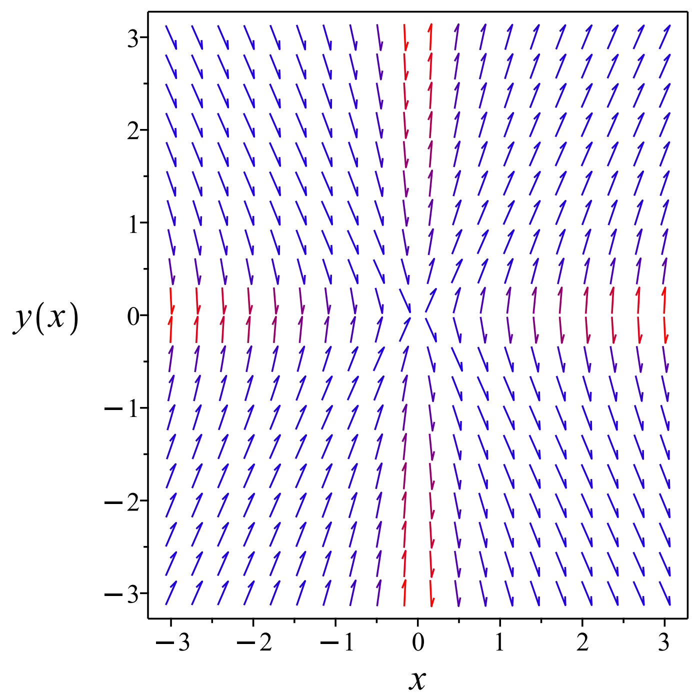


Figure 388: Slope field plot

Verification of solutions

$$y = \sqrt{2\ln(x) + c_1}\, x$$

Verified OK.

$$y = -\sqrt{2\ln(x) + c_1}\, x$$

Verified OK.

12.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x\\ \left(x^2 + y^2\right)\mathrm{d}x + (-xy)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2\\ N(x,y) &= -xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-xy)\\ &= -y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{xy}((2y) - (-y))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)}\\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}(x^2 + y^2)\\ &= \frac{x^2 + y^2}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^3}(-xy)\\ &= -\frac{y}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{x^2+y^2}{x^3}\right) + \left(-\frac{y}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^3}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{y^2}{2x^2} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{y}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{x^2}$. Therefore equation (4) becomes

$$-\frac{y}{x^2} = -\frac{y}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{2x^2} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{2x^2} + \ln(x)$$

Summary

The solution(s) found are the following

$$-\frac{y^2}{2x^2} + \ln(x) = c_1 \tag{1}$$

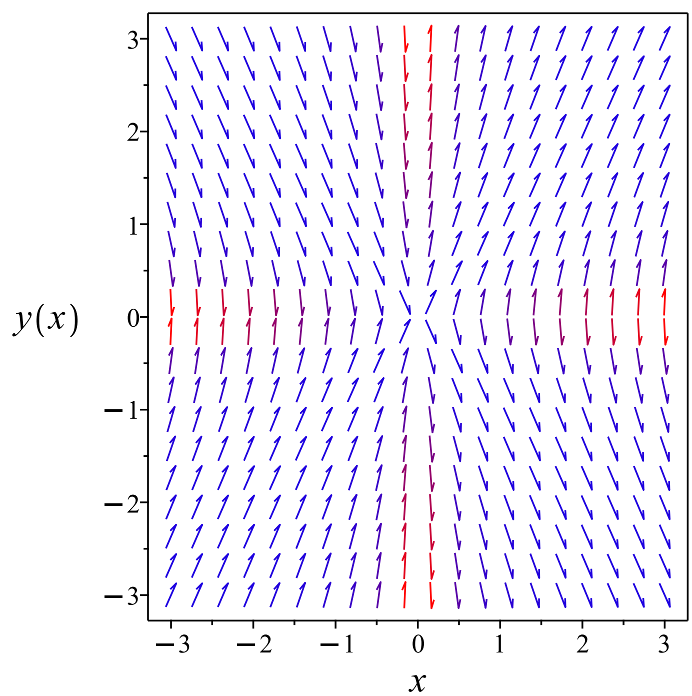


Figure 389: Slope field plot

Verification of solutions

$$-\frac{y^2}{2x^2}+\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve((x^2+y(x)^2)-x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y=\sqrt{2\ln(x)+c_1}\,x$$
$$y=-\sqrt{2\ln(x)+c_1}\,x$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 36

```
DSolve[(x^2+y[x]^2)-x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\sqrt{2\log(x)+c_1}$$
$$y(x)\to x\sqrt{2\log(x)+c_1}$$

12.20 problem 294

Internal problem ID [15156]
Internal file name [`OUTPUT/15156_Tuesday_April_23_2024_04_51_42_PM_42148948/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 294.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y+(x-y+3)\,y'=-x-2}$$

12.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-x+y-2}{-x+y-3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(-x+y-2)(b_3-a_2)}{-x+y-3} - \frac{(-x+y-2)^2 a_3}{(-x+y-3)^2} \\
&- \left(\frac{1}{-x+y-3} - \frac{-x+y-2}{(-x+y-3)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{1}{-x+y-3} + \frac{-x+y-2}{(-x+y-3)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 - 2xyb_2 + 2xyb_3 + y^2a_2 - y^2a_3 + y^2b_2 - y^2b_3 + 6xa_2 - 4xa_3}{(x-y+3)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&x^2a_2 - x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 - 2xyb_2 + 2xyb_3 \\
&+ y^2a_2 - y^2a_3 + y^2b_2 - y^2b_3 + 6xa_2 - 4xa_3 + 5xb_2 - 5xb_3 - 5ya_2 \\
&+ 5ya_3 - 6yb_2 + 4yb_3 + a_1 + 6a_2 - 4a_3 - b_1 + 9b_2 - 6b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&a_2v_1^2 - 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 + 2a_3v_1v_2 - a_3v_2^2 + b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 \\
&- b_3v_1^2 + 2b_3v_1v_2 - b_3v_2^2 + 6a_2v_1 - 5a_2v_2 - 4a_3v_1 + 5a_3v_2 + 5b_2v_1 \\
&- 6b_2v_2 - 5b_3v_1 + 4b_3v_2 + a_1 + 6a_2 - 4a_3 - b_1 + 9b_2 - 6b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(a_2 - a_3 + b_2 - b_3)\, v_1^2 + (-2a_2 + 2a_3 - 2b_2 + 2b_3)\, v_1 v_2 \\ &+ (6a_2 - 4a_3 + 5b_2 - 5b_3)\, v_1 + (a_2 - a_3 + b_2 - b_3)\, v_2^2 \\ &+ (-5a_2 + 5a_3 - 6b_2 + 4b_3)\, v_2 + a_1 + 6a_2 - 4a_3 - b_1 + 9b_2 - 6b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-5a_2 + 5a_3 - 6b_2 + 4b_3 &= 0 \\ -2a_2 + 2a_3 - 2b_2 + 2b_3 &= 0 \\ a_2 - a_3 + b_2 - b_3 &= 0 \\ 6a_2 - 4a_3 + 5b_2 - 5b_3 &= 0 \\ a_1 + 6a_2 - 4a_3 - b_1 + 9b_2 - 6b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -5b_2 + b_1 \\ a_2 &= -b_2 \\ a_3 &= b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= -b_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1 \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(-\frac{-x + y - 2}{-x + y - 3}\right)(1) \\ &= \frac{2x - 2y + 5}{x - y + 3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x-2y+5}{x-y+3}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{2} - \frac{\ln(-2x + 2y - 5)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x + y - 2}{-x + y - 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{4x - 4y + 10} \\ S_y &= \frac{x - y + 3}{2x - 2y + 5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{2} - \frac{\ln(-2x + 2y - 5)}{4} = -\frac{x}{2} + c_1$$

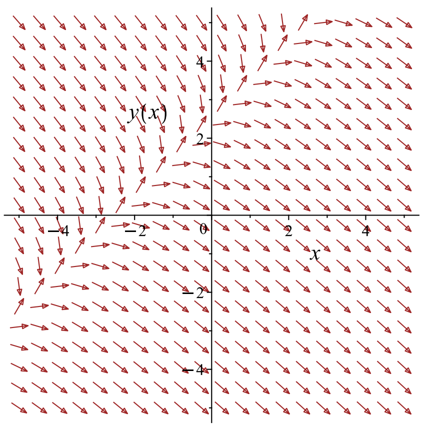
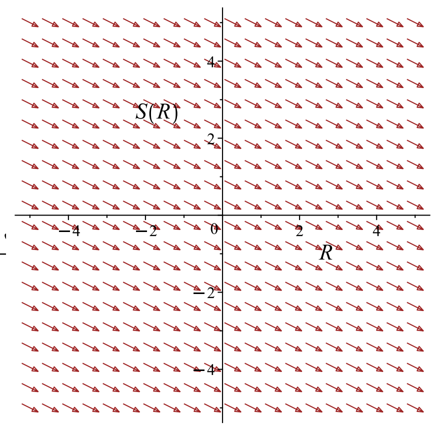
Which simplifies to

$$\frac{y}{2} - \frac{\ln(-2x + 2y - 5)}{4} = -\frac{x}{2} + c_1$$

Which gives

$$y = x - \frac{\text{LambertW}\left(-\mathrm{e}^{4x+5-4c_1}\right)}{2} + \frac{5}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x+y-2}{-x+y-3}$ 	$R = x$ $S = \frac{y}{2} - \frac{\ln(-2x + 2y -}{4}$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$y = x - \frac{\text{LambertW}\left(-e^{4x+5-4c_1}\right)}{2} + \frac{5}{2} \tag{1}$$

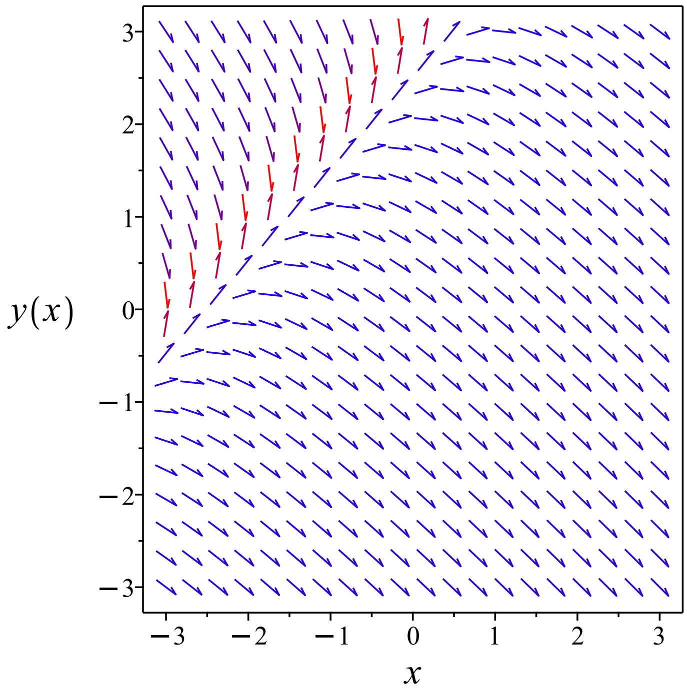


Figure 390: Slope field plot

Verification of solutions

$$y = x - \frac{\text{LambertW}\left(-\mathrm{e}^{4x+5-4c_1}\right)}{2} + \frac{5}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve((x-y(x)+2)+(x-y(x)+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = x - \frac{\mathrm{LambertW}\left(-c_1 \mathrm{e}^{5+4x}\right)}{2} + \frac{5}{2}$$

✓ Solution by Mathematica

Time used: 3.14 (sec). Leaf size: 35

```
DSolve[(x-y[x]+2)+(x-y[x]+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}W\left(-e^{4x-1+c_1}\right) + x + \frac{5}{2}$$
$$y(x) \to x + \frac{5}{2}$$

12.21 problem 295

Internal problem ID [15157]
Internal file name [`OUTPUT/15157_Tuesday_April_23_2024_04_51_43_PM_11273920/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 295.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y^2x + y - xy' = 0}$$

12.21.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^3 + u(x)\,x - x(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= u^2x
\end{aligned}$$

Where $f(x) = x$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\,du = x\,dx$$

$$\int \frac{1}{u^2}\,du = \int x\,dx$$

$$-\frac{1}{u} = \frac{x^2}{2} + c_2$$

The solution is

$$-\frac{1}{u(x)} - \frac{x^2}{2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0$$

$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} - \frac{x^2}{2} - c_2 = 0 \tag{1}$$

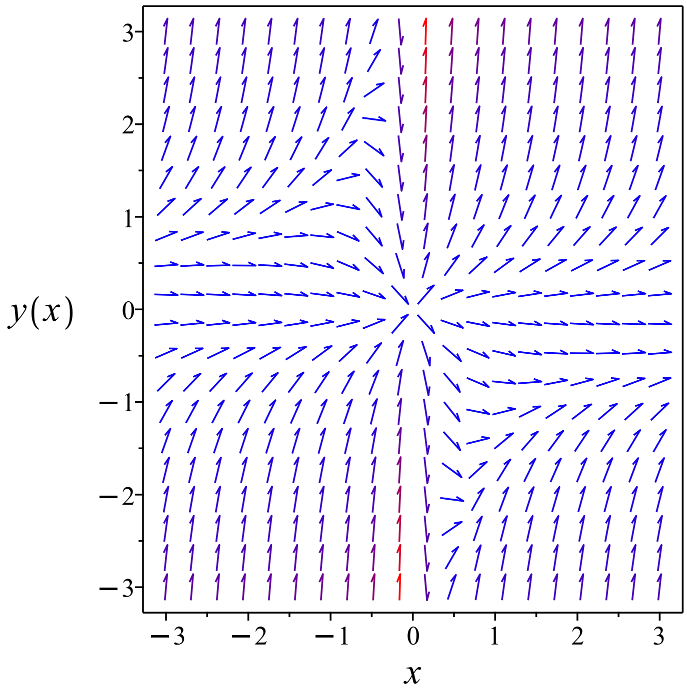


Figure 391: Slope field plot

Verification of solutions

$$-\frac{x}{y}-\frac{x^2}{2}-c_2=0$$

Verified OK.

12.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=\frac{y(xy+1)}{x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 303: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^2}{x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(xy + 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y} \\
S_y &= \frac{x}{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = \frac{x^2}{2} + c_1$$

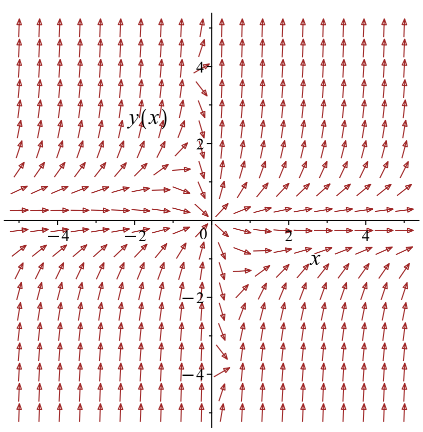
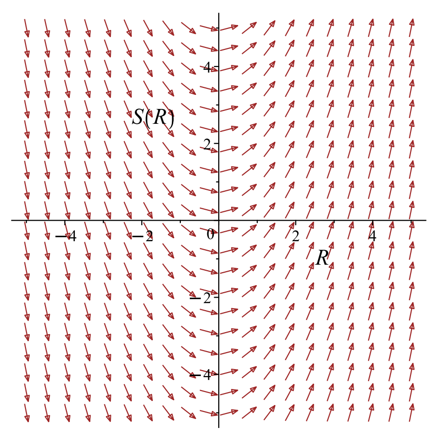
Which simplifies to

$$-\frac{x}{y} = \frac{x^2}{2} + c_1$$

Which gives

$$y = -\frac{2x}{x^2 + 2c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{y(xy+1)}{x}$

</td><td>$R = x$
$S = -\frac{x}{y}$</td><td>$\frac{dS}{dR} = R$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{2x}{x^2 + 2c_1} \tag{1}$$

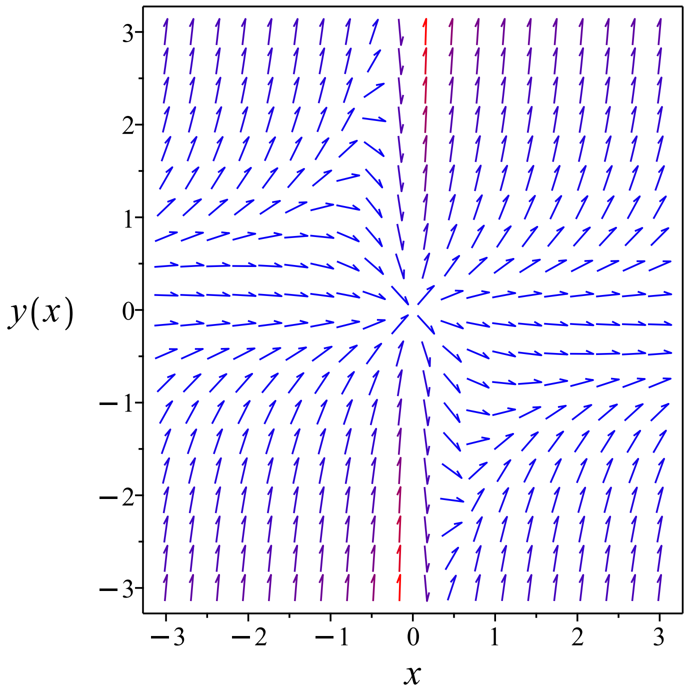


Figure 392: Slope field plot

Verification of solutions

$$y = -\frac{2x}{x^2 + 2c_1}$$

Verified OK.

12.21.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(xy + 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= 1 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{xy} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} + 1 \\ w' &= -\frac{w}{x} - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -1
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{w(x)}{x} = -1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x} dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-1) \\
\frac{\mathrm{d}}{\mathrm{d}x}(xw) &= (x)(-1) \\
\mathrm{d}(xw) &= (-x)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xw &= \int -x \,\mathrm{d}x \\
xw &= -\frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
w(x) = -\frac{x}{2} + \frac{c_1}{x}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = -\frac{x}{2} + \frac{c_1}{x}
$$

Or

$$
y = \frac{1}{-\frac{x}{2} + \frac{c_1}{x}}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{1}{-\frac{x}{2} + \frac{c_1}{x}} \tag{1}
$$

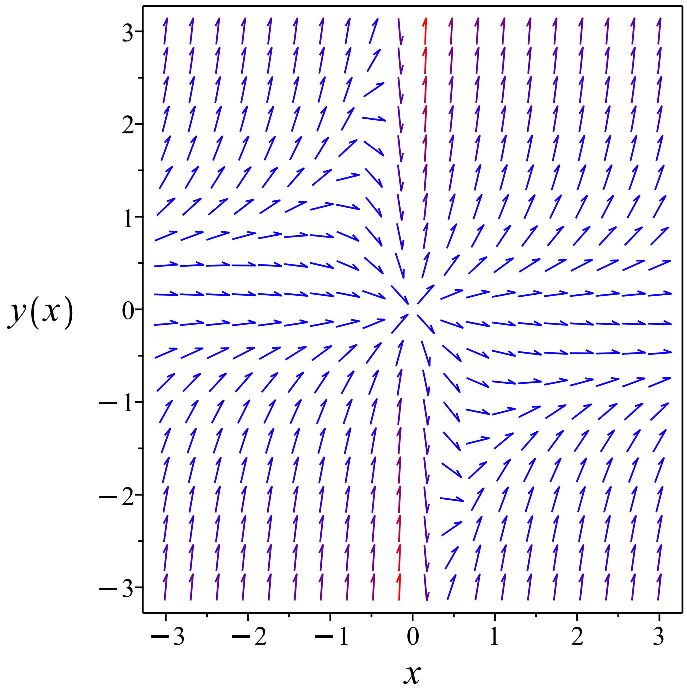


Figure 393: Slope field plot

Verification of solutions

$$y = \frac{1}{-\frac{x}{2} + \frac{c_1}{x}}$$

Verified OK.

12.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x)\,\mathrm{d}y &= \left(-x\,y^2 - y\right)\mathrm{d}x \\
\left(x\,y^2 + y\right)\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x\,y^2 + y \\
N(x, y) &= -x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\,y^2 + y\right) \\
&= 2xy + 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x}((2xy+1)-(-1)) \\ &= \frac{-2xy-2}{x} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{x\,y^2+y}((-1)-(2xy+1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \,\mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(x\,y^2+y) \\ &= \frac{xy+1}{y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2}(-x) \\ &= -\frac{x}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{xy+1}{y}\right) + \left(-\frac{x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\mathrm{d}x &= \int \frac{xy+1}{y}\,\mathrm{d}x \\
\phi &= \frac{x(xy+2)}{2y} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x^2}{2y} - \frac{x(xy+2)}{2y^2} + f'(y) \\
&= -\frac{x}{y^2} + f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$
-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy+2)}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy+2)}{2y}$$

The solution becomes

$$y = \frac{2x}{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{-x^2 + 2c_1} \tag{1}$$

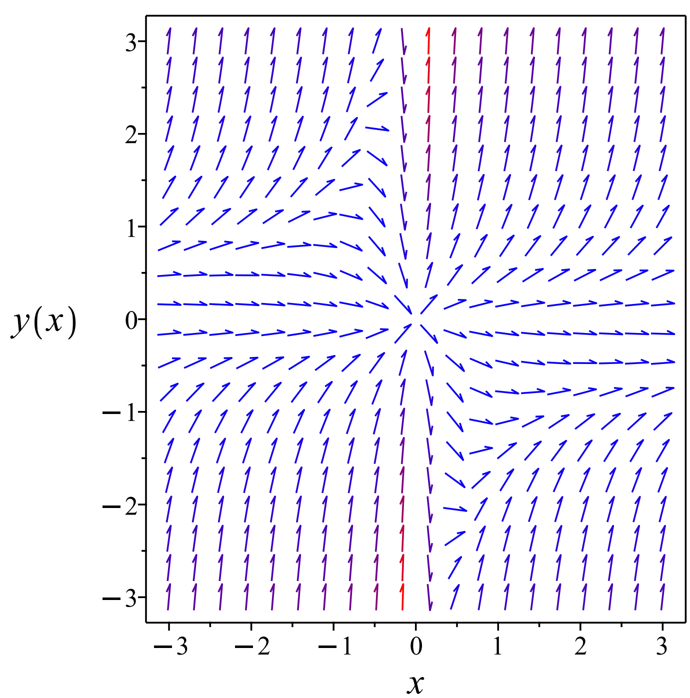


Figure 394: Slope field plot

Verification of solutions

$$y = \frac{2x}{-x^2 + 2c_1}$$

Verified OK.

12.21.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(xy + 1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= \frac{1}{x} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - \frac{u'(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 x^2 + c_1$$

The above shows that

$$u'(x) = 2c_2 x$$

Using the above in (1) gives the solution

$$y = -\frac{2c_2 x}{c_2 x^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{2x}{x^2 + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{2x}{x^2 + c_3} \tag{1}$$

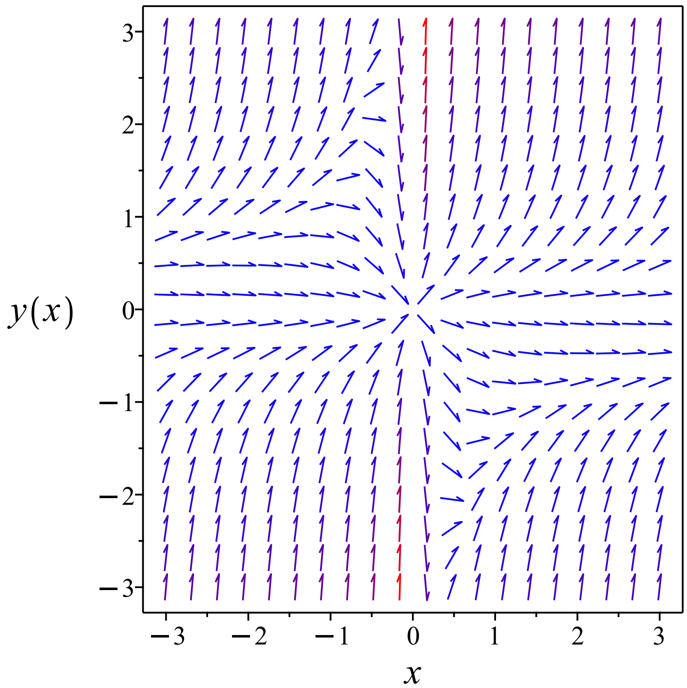


Figure 395: Slope field plot

Verification of solutions

$$y = -\frac{2x}{x^2 + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((x*y(x)^2+y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = -\frac{2x}{x^2 - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 23

```
DSolve[(x*y[x]^2+y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2x}{x^2 - 2c_1}$$
$$y(x) \to 0$$

12.22 problem 296

Internal problem ID [15158]
Internal file name [`OUTPUT/15158_Tuesday_April_23_2024_04_51_44_PM_75256378/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 296.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{y^2 + 2yy' = -x^2 - 2x}$$

12.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + 2u(x)\,x(u'(x)\,x + u(x)) = -x^2 - 2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(x+2)\,(-u^2-1)}{2xu} \end{aligned}$$

Where $f(x) = \frac{x+2}{2x}$ and $g(u) = \frac{-u^2-1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{-u^2-1}{u}}\,du &= \frac{x+2}{2x}\,dx \\
\int \frac{1}{\frac{-u^2-1}{u}}\,du &= \int \frac{x+2}{2x}\,dx \\
-\frac{\ln\left(u^2+1\right)}{2} &= \frac{x}{2} + \ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{\sqrt{u^2+1}} = \mathrm{e}^{\frac{x}{2}+\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{1}{\sqrt{u^2+1}} = c_3 \mathrm{e}^{\frac{x}{2}+\ln(x)}
$$

The solution is

$$
\frac{1}{\sqrt{u\left(x\right)^2+1}} = c_3 \mathrm{e}^{\frac{x}{2}+\ln(x)+c_2}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{1}{\sqrt{\frac{y^2}{x^2}+1}} &= c_3 \mathrm{e}^{\frac{x}{2}+\ln(x)+c_2} \\
\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} &= c_3 \mathrm{e}^{\frac{x}{2}+c_2} x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} = c_3 \mathrm{e}^{\frac{x}{2}+c_2} x \tag{1}
$$

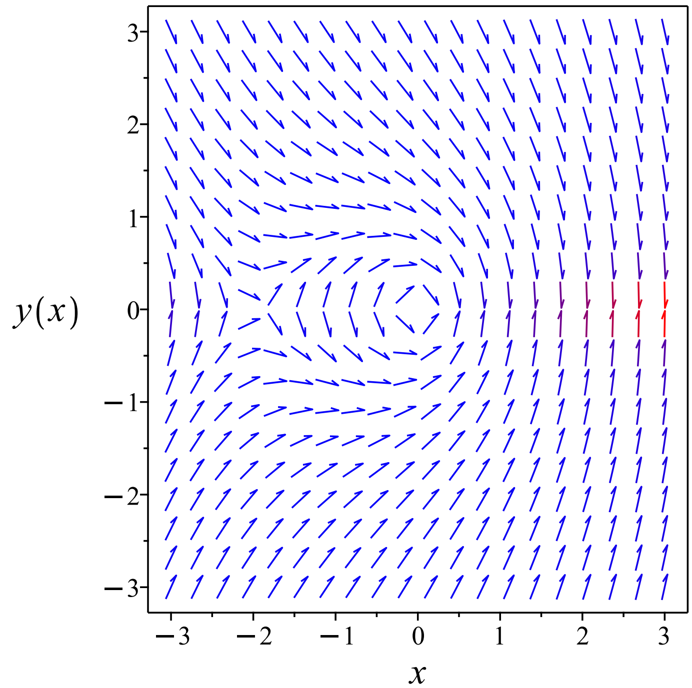


Figure 396: Slope field plot

Verification of solutions

$$\frac{1}{\sqrt{\frac{x^2+y^2}{x^2}}} = c_3 e^{\frac{x}{2}+c_2} x$$

Verified OK.

12.22.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2+y^2+2x}{2y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 305: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{e^{-x}}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-x}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{x} y^{2}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2 + 2x}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{x} y^{2}}{2} \\ S_y &= y\, e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{e^{x} x(x+2)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{e^{R} R(R+2)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{e^{R} R^{2}}{2} + c_1 \tag{4}$$

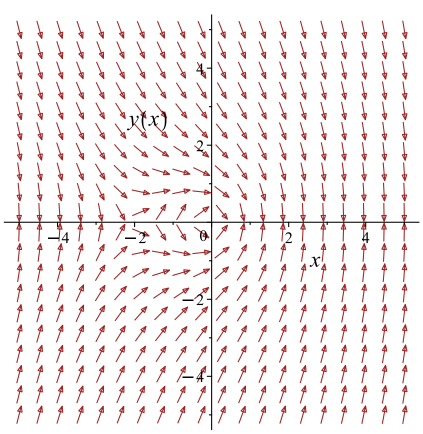
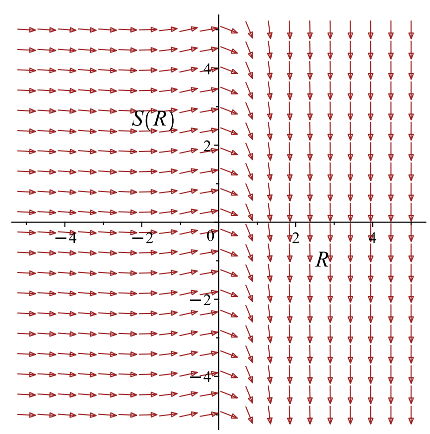
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{x} y^{2}}{2} = -\frac{e^{x} x^{2}}{2} + c_1$$

Which simplifies to

$$\frac{e^{x} y^{2}}{2} = -\frac{e^{x} x^{2}}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2+2x}{2y}$ 	$R = x$ $S = \frac{e^{x} y^{2}}{2}$	$\frac{dS}{dR} = -\frac{e^{R} R(R+2)}{2}$ 

Summary

The solution(s) found are the following

$$\frac{e^{x} y^{2}}{2} = -\frac{e^{x} x^{2}}{2} + c_1 \tag{1}$$

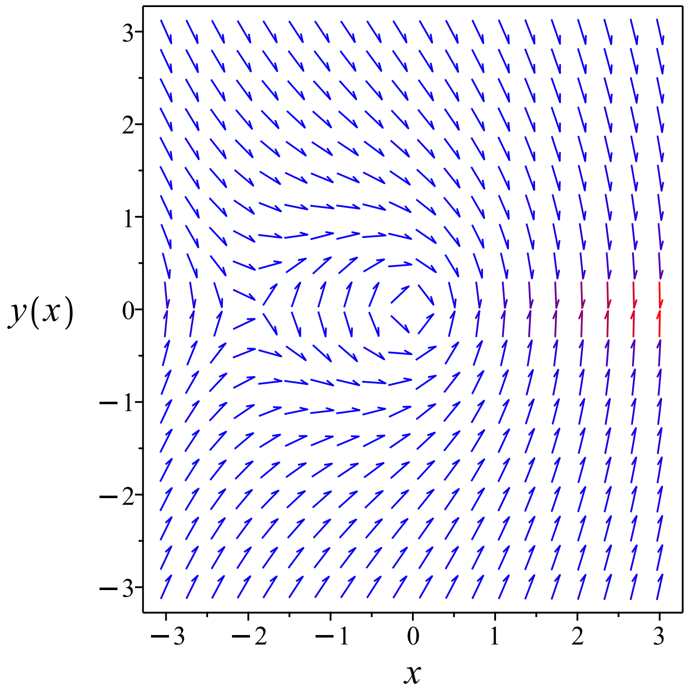


Figure 397: Slope field plot

Verification of solutions

$$\frac{e^x y^2}{2} = -\frac{e^x x^2}{2} + c_1$$

Verified OK.

12.22.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x^2 + y^2 + 2x}{2y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2}y - \frac{1}{2}x^2 - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2} \\ f_1(x) &= -\frac{1}{2}x^2 - x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2} - \frac{1}{2}x^2 - x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{2} - \frac{x^2}{2} - x \\ w' &= -x^2 - w - 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -x^2 - 2x \end{aligned}$$

Hence the ode is

$$w'(x) + w(x) = -x^2 - 2x$$

The integrating factor μ is

$$\mu = e^{\int 1dx}$$
$$= e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(-x^2 - 2x\right)$$
$$\frac{d}{dx}(e^{x} w) = (e^{x})\left(-x^2 - 2x\right)$$
$$d(e^{x} w) = \left(-e^{x} x(x+2)\right)\, dx$$

Integrating gives

$$e^{x} w = \int -e^{x} x(x+2)\, dx$$
$$e^{x} w = -e^{x} x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$w(x) = -e^{-x} e^{x} x^2 + c_1 e^{-x}$$

which simplifies to

$$w(x) = -x^2 + c_1 e^{-x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -x^2 + c_1 e^{-x}$$

Solving for y gives

$$y(x) = \sqrt{-x^2 + c_1 e^{-x}}$$
$$y(x) = -\sqrt{-x^2 + c_1 e^{-x}}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + c_1 e^{-x}} \tag{1}$$
$$y = -\sqrt{-x^2 + c_1 e^{-x}} \tag{2}$$

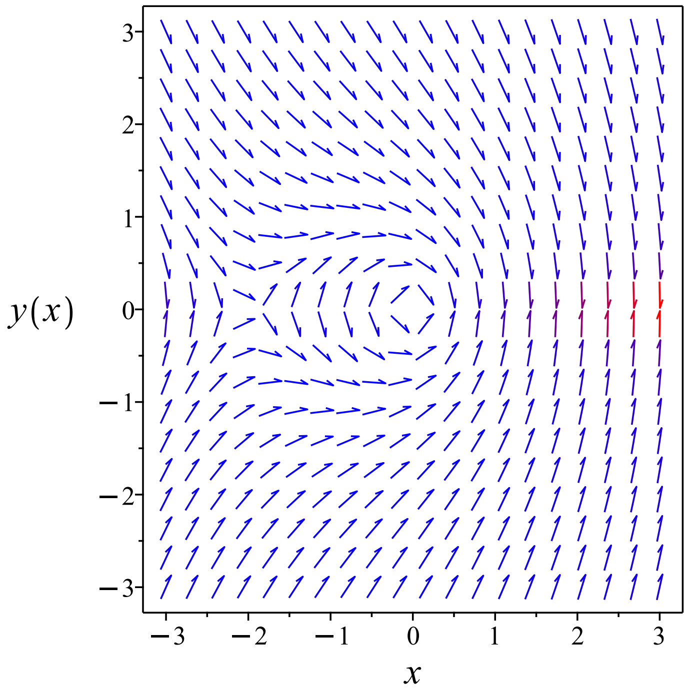


Figure 398: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + c_1 e^{-x}}$$

Verified OK.

$$y = -\sqrt{-x^2 + c_1 e^{-x}}$$

Verified OK.

12.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y)\,\mathrm{d}y&=\left(-x^2-y^2-2x\right)\mathrm{d}x\\ \left(x^2+y^2+2x\right)\mathrm{d}x+(2y)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+y^2+2x\\ N(x,y)&=2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x^2+y^2+2x\right)\\ &=2y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2y}((2y) - (0)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}\left(x^2 + y^2 + 2x\right) \\
&= \mathrm{e}^{x}\left(x^2 + y^2 + 2x\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(2y) \\
&= 2y\,\mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{x}\left(x^2 + y^2 + 2x\right)\right) + \left(2y\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{x} \left(x^2 + y^2 + 2x\right) \mathrm{d}x$$

$$\phi = \left(x^2 + y^2\right) \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y \, \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y \, \mathrm{e}^{x}$. Therefore equation (4) becomes

$$2y \, \mathrm{e}^{x} = 2y \, \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \left(x^2 + y^2\right) \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \left(x^2 + y^2\right) \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$\left(x^2+y^2\right) e^x = c_1 \tag{1}$$

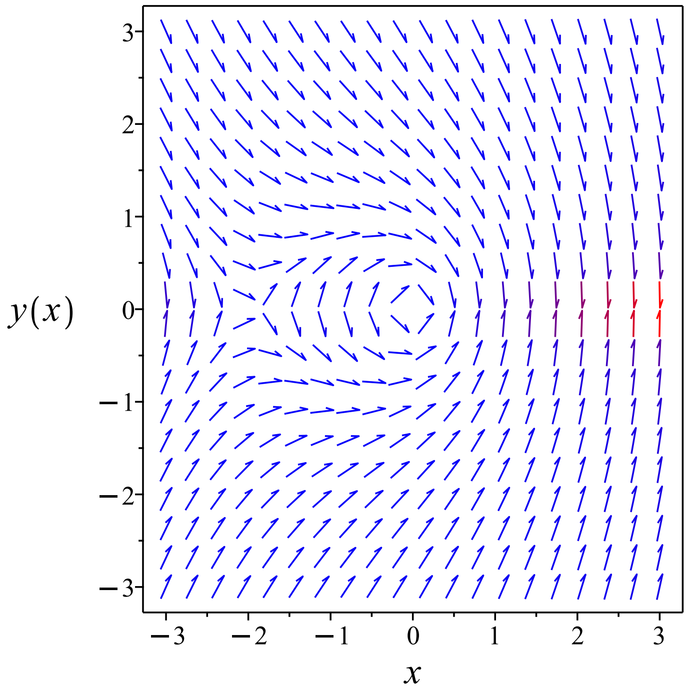


Figure 399: Slope field plot

Verification of solutions

$$\left(x^2+y^2\right) e^x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve((x^2+y(x)^2+2*x)+(2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \sqrt{c_1 e^{-x} - x^2}$$
$$y = -\sqrt{c_1 e^{-x} - x^2}$$

✓ Solution by Mathematica

Time used: 5.559 (sec). Leaf size: 47

```
DSolve[(x^2+y[x]^2+2*x)+(2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 + c_1 e^{-x}}$$
$$y(x) \to \sqrt{-x^2 + c_1 e^{-x}}$$

12.23 problem 297

Internal problem ID [15159]
Internal file name [`OUTPUT/15159_Tuesday_April_23_2024_04_51_45_PM_62653458/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 297.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$
\boxed{(x-1)\left(y^{2}-y+1\right)-(y-1)\left(x^{2}+x+1\right)y'=0}
$$

12.23.1 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{(x-1)\left(y^{2}-y+1\right)}{(y-1)\left(x^{2}+x+1\right)}
\end{aligned}
$$

Where $f(x)=\frac{x-1}{x^{2}+x+1}$ and $g(y)=\frac{y^{2}-y+1}{y-1}$. Integrating both sides gives

$$
\frac{1}{\frac{y^{2}-y+1}{y-1}}\,dy = \frac{x-1}{x^{2}+x+1}\,dx
$$

$$
\int \frac{1}{\frac{y^{2}-y+1}{y-1}}\,dy = \int \frac{x-1}{x^{2}+x+1}\,dx
$$

$$\frac{\ln \left(y^{2}-y+1\right)}{2}-\frac{\sqrt{3}\, \arctan \left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3}=\frac{\ln \left(x^{2}+x+1\right)}{2}-\sqrt{3}\, \arctan \left(\frac{(2x+1)\sqrt{3}}{3}\right)+c_{1}$$

Which results in

Expression too large to display

Summary

The solution(s) found are the following

Expression too large to display (1)

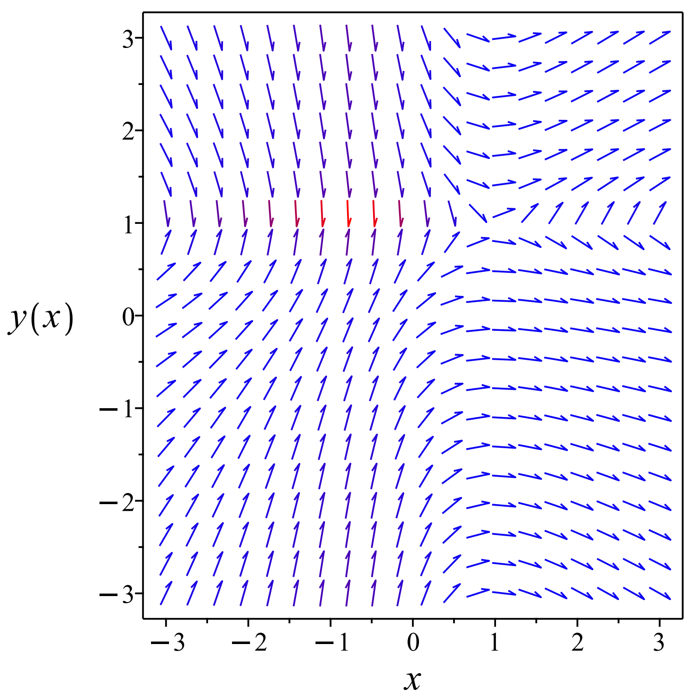


Figure 400: Slope field plot

Verification of solutions

Expression too large to display

Warning, solution could not be verified

12.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x\,y^2 - xy - y^2 + x + y - 1}{x^2 y - x^2 + xy - x + y - 1} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 307: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2+x+1}{x-1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2+x+1}{x-1}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x\,y^2 - xy - y^2 + x + y - 1}{x^2y - x^2 + xy - x + y - 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x-1}{x^2+x+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y-1}{y^2-y+1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R-1}{R^2-R+1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R^2-R+1)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2R-1)\sqrt{3}}{3}\right)}{3} + c_1 \tag{4}
$$

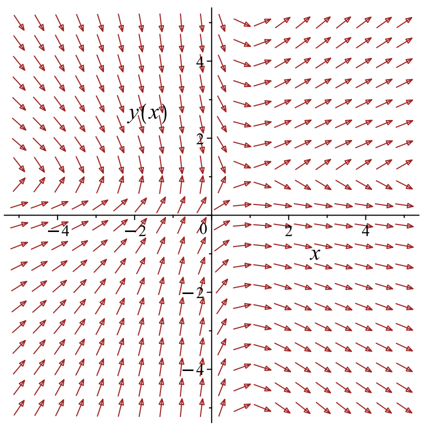
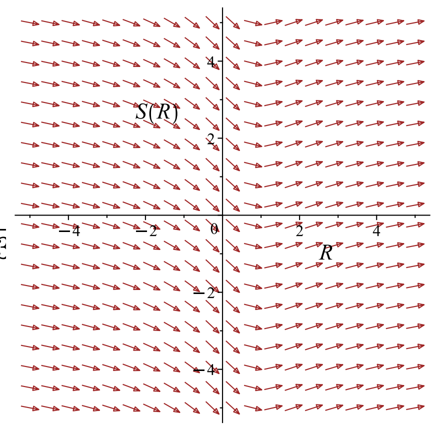
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(x^2+x+1)}{2} - \sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) = \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} + c_1
$$

Which simplifies to

$$
\frac{\ln(x^2+x+1)}{2} - \sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) = \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,y^2-xy-y^2+x+y-1}{x^2y-x^2+xy-x+y-1}$ 	$R = y$ $S = \frac{\ln(x^2+x+1)}{2} - \sqrt{3}$	$\frac{dS}{dR} = \frac{R-1}{R^2-R+1}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+x+1)}{2} - \sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) = \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} + c_1 \tag{1}$$

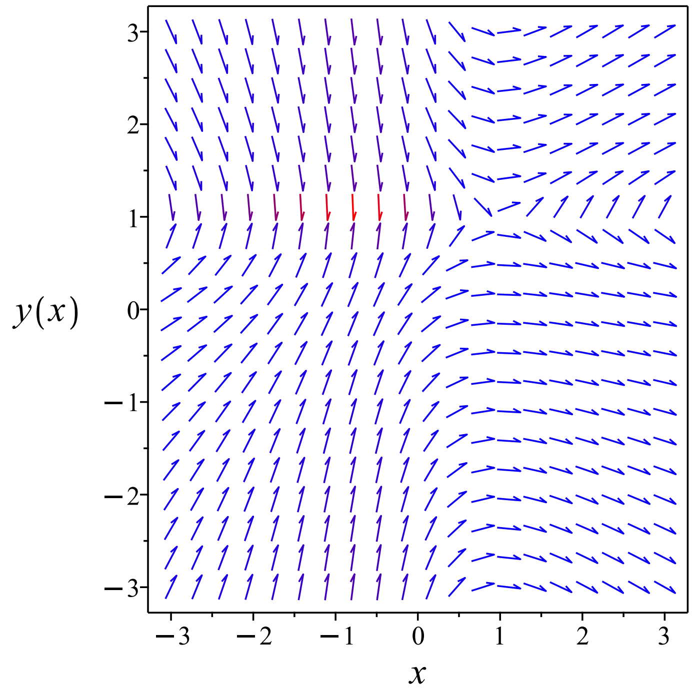


Figure 401: Slope field plot

Verification of solutions

$$\frac{\ln\left(x^2+x+1\right)}{2}-\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)=\frac{\ln\left(y^2-y+1\right)}{2}-\frac{\sqrt{3}\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3}+c_1$$

Verified OK.

12.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{y-1}{y^2-y+1}\right) \mathrm{d} y &= \left(\frac{x-1}{x^2+x+1}\right) \mathrm{d} x \\
\left(-\frac{x-1}{x^2+x+1}\right) \mathrm{d} x+\left(\frac{y-1}{y^2-y+1}\right) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{x-1}{x^2+x+1} \\
N(x, y) &= \frac{y-1}{y^2-y+1}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x-1}{x^2+x+1}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y-1}{y^2-y+1}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x-1}{x^2+x+1}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\ln(x^2+x+1)}{2} + \sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y-1}{y^2-y+1}$. Therefore equation (4) becomes

$$\frac{y-1}{y^2-y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y-1}{y^2-y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y-1}{y^2-y+1}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+x+1)}{2} + \sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) + \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+x+1)}{2} + \sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) + \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+x+1)}{2} + \sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) + \frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} = c_1 \tag{1}$$

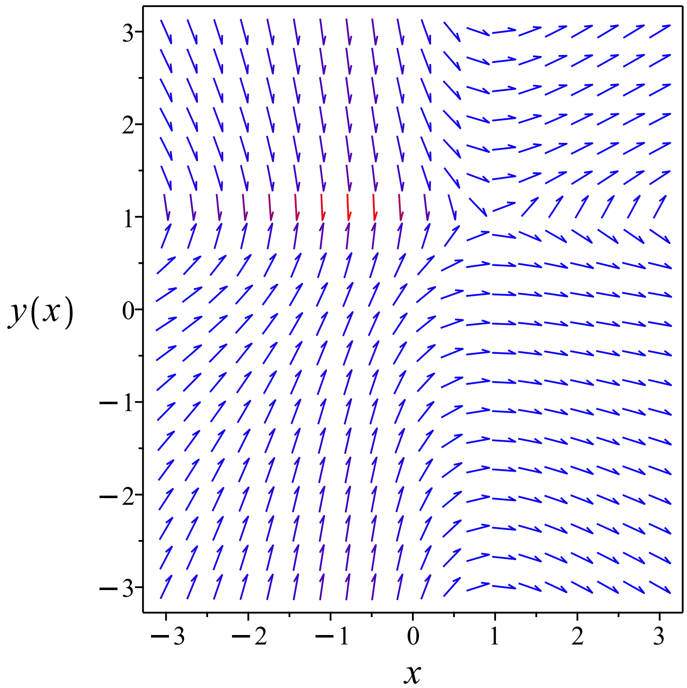


Figure 402: Slope field plot

Verification of solutions

$$-\frac{\ln\left(x^2+x+1\right)}{2}+\sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)+\frac{\ln\left(y^2-y+1\right)}{2}-\frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3}=c_1$$

Verified OK.

12.23.4 Maple step by step solution

Let's solve

$(x-1)\left(y^2-y+1\right)-(y-1)\left(x^2+x+1\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-1)}{y^2-y+1}=\frac{x-1}{x^2+x+1}$

- Integrate both sides with respect to x

$$\int \frac{y'(y-1)}{y^2-y+1}dx = \int \frac{x-1}{x^2+x+1}dx + c_1$$

- Evaluate integral

$$\frac{\ln(y^2-y+1)}{2} - \frac{\sqrt{3}\,\arctan\left(\frac{(2y-1)\sqrt{3}}{3}\right)}{3} = \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\,\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 2397

```
dsolve(((x-1)*(y(x)^2-y(x)+1))=((y(x)-1)*(x^2+x+1))*diff(y(x),x),y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 0.578 (sec). Leaf size: 96

```
DSolve[((x-1)*(y[x]^2-y[x]+1))==((y[x]-1)*(x^2+x+1))*y'[x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{2}\log\left(\#1^2-\#1+1\right) - \frac{\arctan\left(\frac{2\#1-1}{\sqrt{3}}\right)}{\sqrt{3}}\&\right]\left[-\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{2}\log\left(x^2+x+1\right) + c_1\right]$$
$$y(x) \to \sqrt[3]{-1}$$
$$y(x) \to -(-1)^{2/3}$$

12.24 problem 298

Internal problem ID [15160]
Internal file name [`OUTPUT/15160_Tuesday_April_23_2024_04_51_46_PM_38551458/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 298.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{(x - 2yx - y^2)\, y' + y^2 = 0}$$

12.24.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{y^2}{2xy + y^2 - x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + xy a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x^2 b_4 + xy b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &2xb_4 + yb_5 + b_2 + \frac{y^2(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{2xy + y^2 - x} \\ &- \frac{y^4(xa_5 + 2ya_6 + a_3)}{(2xy + y^2 - x)^2} + \frac{y^2(2y-1)\left(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1\right)}{(2xy + y^2 - x)^2} \\ &- \left(\frac{2y}{2xy + y^2 - x} - \frac{y^2(2x+2y)}{(2xy + y^2 - x)^2}\right)\left(x^2b_4\right. \\ &\left. + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^3y^2b_4 - 2x^2y^3a_4 + 8x^2y^3b_4 + 4x^2y^3b_5 - 2x\,y^4a_4 - x\,y^4a_5 + 2x\,y^4b_4 + 5x\,y^4b_5 + 2x\,y^4b_6 - y^5a_5 + y^5b_5 +}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &6x^3y^2b_4 - 2x^2y^3a_4 + 8x^2y^3b_4 + 4x^2y^3b_5 - 2x\,y^4a_4 - x\,y^4a_5 + 2x\,y^4b_4 \\ &+ 5x\,y^4b_5 + 2x\,y^4b_6 - y^5a_5 + y^5b_5 + 2y^5b_6 - 6x^3yb_4 + x^2y^2a_4 \\ &+ 2x^2y^2b_2 - 4x^2y^2b_4 - 3x^2y^2b_5 + 4x\,y^3b_2 - 2x\,y^3b_5 - y^4a_2 \\ &+ y^4a_3 - y^4a_6 + y^4b_2 + y^4b_3 + 2x^3b_4 - 2x^2yb_2 + x^2yb_5 - 2x\,y^2b_1 \\ &- 2x\,y^2b_2 + x\,y^2b_3 + 2y^3a_1 - y^3a_3 + x^2b_2 + 2xyb_1 - y^2a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2a_4v_1^2v_2^3 - 2a_4v_1v_2^4 - a_5v_1v_2^4 - a_5v_2^5 + 6b_4v_1^3v_2^2 + 8b_4v_1^2v_2^3 + 2b_4v_1v_2^4 \\
&+ 4b_5v_1^2v_2^3 + 5b_5v_1v_2^4 + b_5v_2^5 + 2b_6v_1v_2^4 + 2b_6v_2^5 - a_2v_2^4 + a_3v_2^4 \\
&+ a_4v_1^2v_2^2 - a_6v_2^4 + 2b_2v_1^2v_2^2 + 4b_2v_1v_2^3 + b_2v_2^4 + b_3v_2^4 - 6b_4v_1^3v_2 \\
&- 4b_4v_1^2v_2^2 - 3b_5v_1^2v_2^2 - 2b_5v_1v_2^3 + 2a_1v_2^3 - a_3v_2^3 - 2b_1v_1v_2^2 - 2b_2v_1^2v_2 \\
&- 2b_2v_1v_2^2 + b_3v_1v_2^2 + 2b_4v_1^3 + b_5v_1^2v_2 - a_1v_2^2 + 2b_1v_1v_2 + b_2v_1^2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&6b_4v_1^3v_2^2 - 6b_4v_1^3v_2 + 2b_4v_1^3 + (-2a_4 + 8b_4 + 4b_5)\, v_1^2v_2^3 \\
&+ (a_4 + 2b_2 - 4b_4 - 3b_5)\, v_1^2v_2^2 + (-2b_2 + b_5)\, v_1^2v_2 + b_2v_1^2 \\
&+ (-2a_4 - a_5 + 2b_4 + 5b_5 + 2b_6)\, v_1v_2^4 + (4b_2 - 2b_5)\, v_1v_2^3 \\
&+ (-2b_1 - 2b_2 + b_3)\, v_1v_2^2 + 2b_1v_1v_2 + (-a_5 + b_5 + 2b_6)\, v_2^5 \\
&+ (-a_2 + a_3 - a_6 + b_2 + b_3)\, v_2^4 + (2a_1 - a_3)\, v_2^3 - a_1v_2^2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-a_1 &= 0 \\
2b_1 &= 0 \\
-6b_4 &= 0 \\
2b_4 &= 0 \\
6b_4 &= 0 \\
2a_1 - a_3 &= 0 \\
-2b_2 + b_5 &= 0 \\
4b_2 - 2b_5 &= 0 \\
-2a_4 + 8b_4 + 4b_5 &= 0 \\
-a_5 + b_5 + 2b_6 &= 0 \\
-2b_1 - 2b_2 + b_3 &= 0 \\
a_4 + 2b_2 - 4b_4 - 3b_5 &= 0 \\
-a_2 + a_3 - a_6 + b_2 + b_3 &= 0 \\
-2a_4 - a_5 + 2b_4 + 5b_5 + 2b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -a_6 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 2b_6 \\
a_6 &= a_6 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2xy \\
\eta &= y^2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y^2 - \left(\frac{y^2}{2xy + y^2 - x}\right)(2xy) \\
&= \frac{y^4 - x\,y^2}{2xy + y^2 - x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^4 - x\,y^2}{2xy + y^2 - x}} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{y} - 2\ln(y) + \ln(y^2 - x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{2xy + y^2 - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{-y^2 + x} \\ S_y &= \frac{-2xy - y^2 + x}{y^2(-y^2 + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

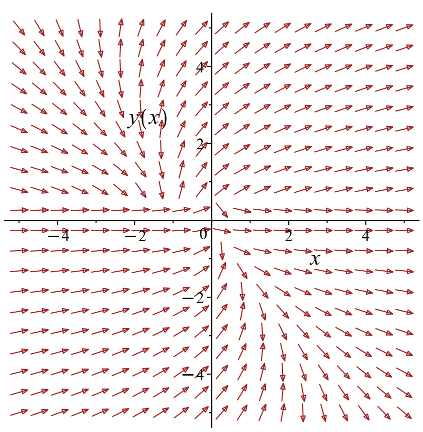
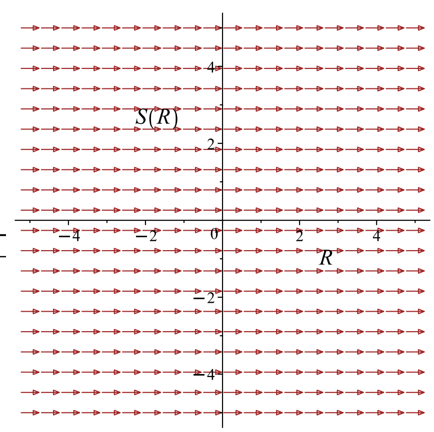
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-2\ln (y)\, y + \ln (y^2 - x)\, y - 1}{y} = c_1$$

Which simplifies to

$$\frac{-2\ln (y)\, y + \ln (y^2 - x)\, y - 1}{y} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{2xy+y^2-x}$ 	$R = x$ $S = \frac{-2\ln (y)\, y + \ln (y^2 -}{y}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{-2\ln (y)\, y + \ln (y^2 - x)\, y - 1}{y} = c_1 \tag{1}$$

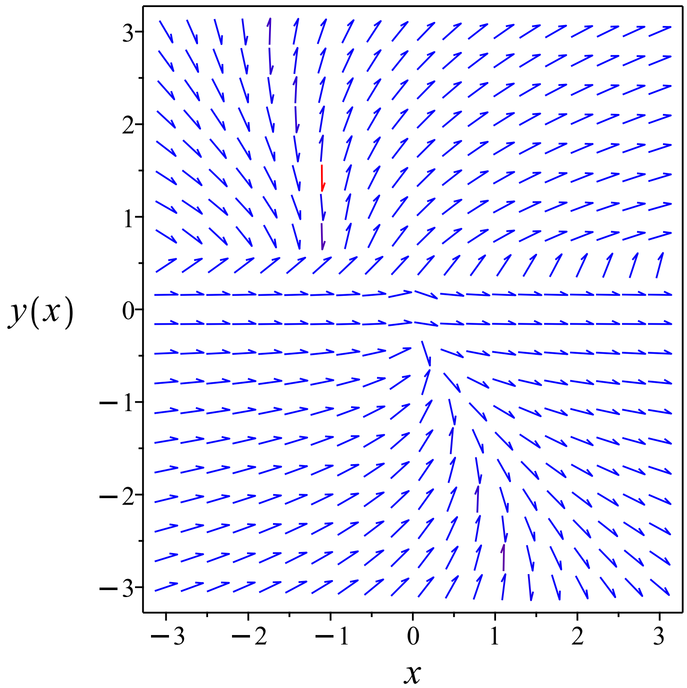


Figure 403: Slope field plot

Verification of solutions

$$\frac{-2\ln(y)\,y+\ln(y^2-x)\,y-1}{y}=c_1$$

Verified OK.

12.24.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(-2xy - y^2 + x)\, \mathrm{d}y &= (-y^2)\, \mathrm{d}x \\
(y^2)\, \mathrm{d}x + (-2xy - y^2 + x)\, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y^2 \\
N(x, y) &= -2xy - y^2 + x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2) \\
&= 2y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy - y^2 + x) \\
&= -2y + 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{-2xy - y^2 + x}((2y) - (-2y+1))\\ &= \frac{-4y+1}{2xy + y^2 - x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{y^2}((-2y+1) - (2y))\\ &= \frac{-4y+1}{y^2}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int \frac{-4y+1}{y^2} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{1}{y} - 4\ln(y)}\\ &= \frac{\mathrm{e}^{-\frac{1}{y}}}{y^4}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\mathrm{e}^{-\frac{1}{y}}}{y^4}(y^2)\\ &= \frac{\mathrm{e}^{-\frac{1}{y}}}{y^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{-\frac{1}{y}}}{y^4}\left(-2xy - y^2 + x\right) \\
&= \frac{\left(-2xy - y^2 + x\right)\mathrm{e}^{-\frac{1}{y}}}{y^4}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\mathrm{e}^{-\frac{1}{y}}}{y^2}\right) + \left(\frac{\left(-2xy - y^2 + x\right)\mathrm{e}^{-\frac{1}{y}}}{y^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{\mathrm{e}^{-\frac{1}{y}}}{y^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{\mathrm{e}^{-\frac{1}{y}}x}{y^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2\,\mathrm{e}^{-\frac{1}{y}}x}{y^3} + \frac{\mathrm{e}^{-\frac{1}{y}}x}{y^4} + f'(y) \tag{4}
$$

$$
= \frac{x(-2y + 1)\,\mathrm{e}^{-\frac{1}{y}}}{y^4} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(-2xy-y^2+x)\,\mathrm{e}^{-\frac{1}{y}}}{y^4}$. Therefore equation (4) becomes

$$\frac{(-2xy - y^2 + x)\,\mathrm{e}^{-\frac{1}{y}}}{y^4} = \frac{x(-2y+1)\,\mathrm{e}^{-\frac{1}{y}}}{y^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\mathrm{e}^{-\frac{1}{y}}}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{\mathrm{e}^{-\frac{1}{y}}}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\mathrm{e}^{-\frac{1}{y}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{e}^{-\frac{1}{y}} x}{y^2} - \mathrm{e}^{-\frac{1}{y}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{e}^{-\frac{1}{y}} x}{y^2} - \mathrm{e}^{-\frac{1}{y}}$$

Summary

The solution(s) found are the following

$$\frac{\mathrm{e}^{-\frac{1}{y}} x}{y^2} - \mathrm{e}^{-\frac{1}{y}} = c_1 \tag{1}$$

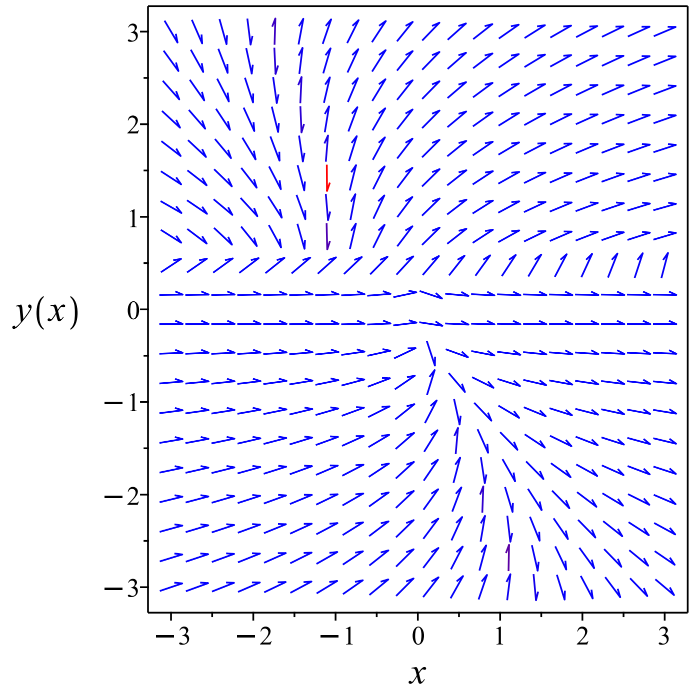


Figure 404: Slope field plot

Verification of solutions

$$\frac{e^{-\frac{1}{y}} x}{y^2} - e^{-\frac{1}{y}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x-2*x*y(x)-y(x)^2)*diff(y(x),x)+y(x)^2=0,y(x), singsol=all)
```

$$y = \frac{1}{\text{RootOf}\left(-\_Z^2 x + e^{\_Z} c_1 + 1\right)}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 23

```
DSolve[(x-2*x*y[x]-y[x]^2)*y'[x]+y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = y(x)^2 + c_1 e^{\frac{1}{y(x)}} y(x)^2, y(x)\right]$$

12.25 problem 299

Internal problem ID [15161]
Internal file name [`OUTPUT/15161_Tuesday_April_23_2024_04_51_48_PM_96396761/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 299.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`], [_Abel, `2nd type
   `, `class A`]]
```

$$\boxed{\cos(x)\,y + (2y - \sin(x))\,y' = 0}$$

12.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y - \sin(x))\,\mathrm{d}y &= (-y\cos(x))\,\mathrm{d}x\\ (y\cos(x))\,\mathrm{d}x + (2y - \sin(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y\cos(x)\\ N(x,y) &= 2y - \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cos(x))\\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y - \sin(x))\\ &= -\cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2y - \sin(x)}((\cos(x)) - (-\cos(x))) \\ &= -\frac{2\cos(x)}{-2y + \sin(x)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\sec(x)}{y}((-\cos(x)) - (\cos(x))) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(y\cos(x)) \\ &= \frac{\cos(x)}{y} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(2y - \sin(x)) \\
&= \frac{2y - \sin(x)}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\cos(x)}{y}\right) + \left(\frac{2y - \sin(x)}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{\cos(x)}{y} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{\sin(x)}{y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{\sin(x)}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y - \sin(x)}{y^2}$. Therefore equation (4) becomes

$$
\frac{2y - \sin(x)}{y^2} = -\frac{\sin(x)}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\sin(x)}{y} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\sin(x)}{y} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{\sin(x)\mathrm{e}^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{\sin(x)\mathrm{e}^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}} \tag{1}$$

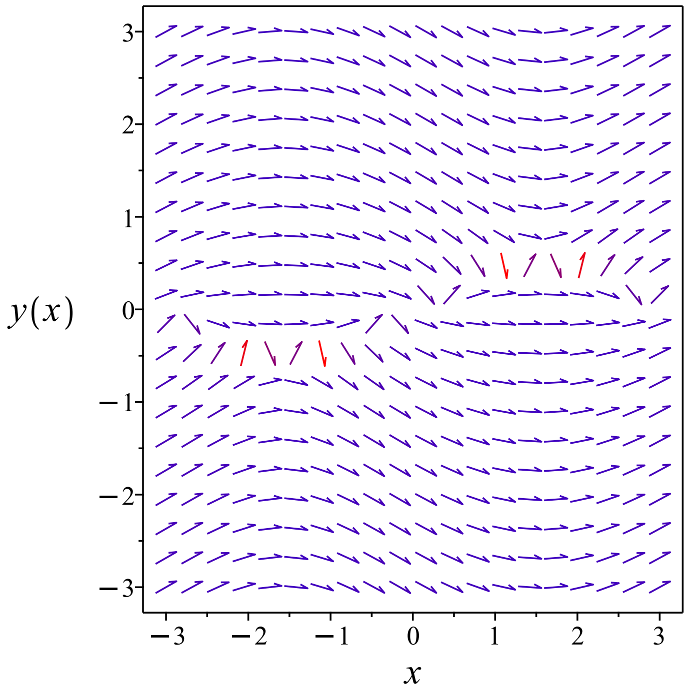


Figure 405: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(-\frac{\sin(x)\,e^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(y(x)*cos(x)+(2*y(x)-sin(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = -\frac{\sin(x)}{2\,\text{LambertW}\left(-\frac{\sin(x)e^{\frac{c_1}{2}}}{2}\right)}$$

✓ Solution by Mathematica

Time used: 10.969 (sec). Leaf size: 349

```
DSolve[y[x]*Cos[x]+(2*y[x]-Sin[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{\sqrt[3]{-2}\left(\frac{2^{2/3}\cos(x)(4y(x)+\sin(x))}{\sqrt[3]{-\cos^3(x)}(\sin(x)-2y(x))}+(-2)^{2/3}\right)\left(\frac{\left(-\cos^3(x)\right)^{2/3}\sec^2(x)(4y(x)+\sin(x))}{\sqrt[3]{2}(\sin(x)-2y(x))}+(-2)^{2/3}\right)\left(-\log\left(\frac{}{\sqrt[3]{}}\right.\right.}{}\right.$$

12.26 problem 300

Internal problem ID [15162]
Internal file name [`OUTPUT/15162_Tuesday_April_23_2024_04_51_51_PM_91527092/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 300.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - e^{x+2y} = 1}$$

12.26.1 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = 1 + e^{x+2y} \tag{1}$$

And using the substitution $u = e^{-2y}$ then

$$u' = -2y' e^{-2y}$$

The above shows that

$$\begin{aligned} y' &= -\frac{u'(x) e^{2y}}{2} \\ &= -\frac{u'(x)}{2u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{2u} = \frac{e^x}{u} + 1$$

The above simplifies to

$$
\begin{aligned}
-\frac{u'(x)}{2} &= e^{x}+u(x) \\
u'(x)+2u(x) &= -2\,e^{x} \qquad (2)
\end{aligned}
$$

Now ode (2) is solved for $u(x)$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
u'(x)+p(x)u(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= -2\,e^{x}
\end{aligned}
$$

Hence the ode is

$$
u'(x)+2u(x) = -2\,e^{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2dx} \\
&= e^{2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu u) &= (\mu)\left(-2\,e^{x}\right) \\
\frac{d}{dx}\left(e^{2x}u\right) &= \left(e^{2x}\right)\left(-2\,e^{x}\right) \\
d\left(e^{2x}u\right) &= \left(-2\,e^{3x}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2x}u &= \int -2\,e^{3x}\,dx \\
e^{2x}u &= -\frac{2\,e^{3x}}{3}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$
u(x) = -\frac{2\,e^{-2x}e^{3x}}{3}+c_1e^{-2x}
$$

which simplifies to

$$u(x) = -\frac{(2\,\mathrm{e}^{3x} - 3c_1)\,\mathrm{e}^{-2x}}{3}$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-2y}$ gives

$$\begin{aligned} y &= -\frac{\ln(u(x))}{2} \\ &= -\frac{\ln\left(-\frac{(2\,\mathrm{e}^{3x}-3c_1)\,\mathrm{e}^{-2x}}{3}\right)}{2} \\ &= \frac{\ln(3)}{2} - \frac{\ln\left((-2\,\mathrm{e}^{3x} + 3c_1)\,\mathrm{e}^{-2x}\right)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(3)}{2} - \frac{\ln\left((-2\,\mathrm{e}^{3x} + 3c_1)\,\mathrm{e}^{-2x}\right)}{2} \tag{1}$$

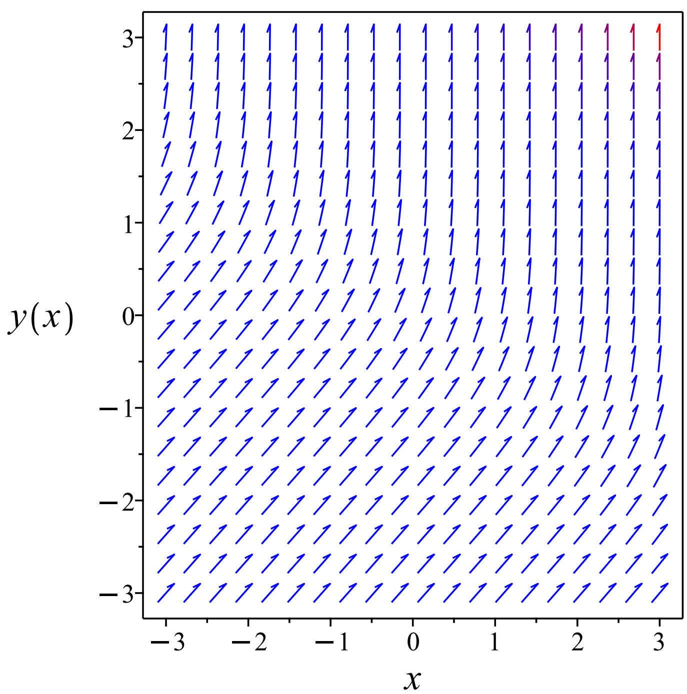


Figure 406: Slope field plot

Verification of solutions

$$y = \frac{\ln(3)}{2} - \frac{\ln\left((-2\,\mathrm{e}^{3x} + 3c_1)\,\mathrm{e}^{-2x}\right)}{2}$$

Verified OK.

12.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 1 + e^{x+2y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `first order special form ID 1`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 310: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \mathrm{e}^{-3x} \\
\eta(x,y) &= 1+\mathrm{e}^{-3x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{1+\mathrm{e}^{-3x}}{\mathrm{e}^{-3x}} \\
&= 1+\mathrm{e}^{3x}
\end{aligned}
$$

This is easily solved to give

$$
y = x + \frac{\mathrm{e}^{3x}}{3} + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = -x - \frac{\mathrm{e}^{3x}}{3} + y
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{\mathrm{e}^{-3x}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \frac{\mathrm{e}^{3x}}{3}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 1 + \mathrm{e}^{x+2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -1 - \mathrm{e}^{3x} \\ R_y &= 1 \\ S_x &= \mathrm{e}^{3x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\mathrm{e}^{3x}}{-\mathrm{e}^{3x} + \mathrm{e}^{x+2y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{-1 + \mathrm{e}^{2R+2S(R)}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\mathrm{LambertW}\left(-\mathrm{e}^{2c_1}\mathrm{e}^{-2R}\right)}{2} - c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\mathrm{e}^{3x}}{3} = \frac{\mathrm{LambertW}\left(-\mathrm{e}^{2c_1}\mathrm{e}^{2x+\frac{2\,\mathrm{e}^{3x}}{3}-2y}\right)}{2} - c_1$$

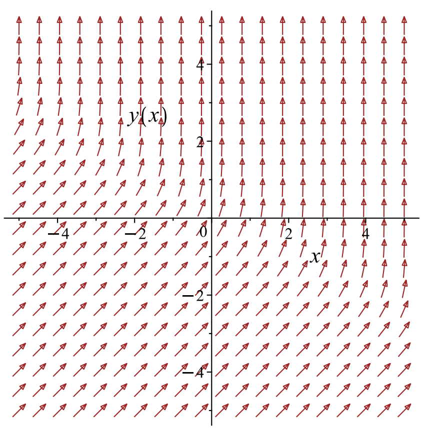
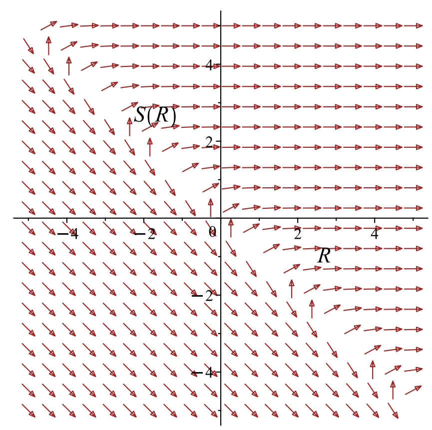
Which simplifies to

$$\frac{\mathrm{e}^{3x}}{3}-\frac{\operatorname{LambertW}\left(-\mathrm{e}^{2c_1+2x+\frac{2\,\mathrm{e}^{3x}}{3}-2y}\right)}{2}+c_1=0$$

Which gives

$$y=c_1+x+\frac{\mathrm{e}^{3x}}{3}-\frac{\ln\left(-\frac{2\,\mathrm{e}^{\frac{2\,\mathrm{e}^{3x}}{3}+2c_1+3x}}{3}-2\,\mathrm{e}^{\frac{2\,\mathrm{e}^{3x}}{3}+2c_1}c_1\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x,y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R,S)
$\frac{dy}{dx}=1+\mathrm{e}^{x+2y}$ 	$R=-x-\frac{\mathrm{e}^{3x}}{3}+y$ $S=\frac{\mathrm{e}^{3x}}{3}$	$\frac{dS}{dR}=\frac{1}{-1+\mathrm{e}^{2R+2S(R)}}$ 

Summary
The solution(s) found are the following

$$y=c_1+x+\frac{\mathrm{e}^{3x}}{3}-\frac{\ln\left(-\frac{2\,\mathrm{e}^{\frac{2\,\mathrm{e}^{3x}}{3}+2c_1+3x}}{3}-2\,\mathrm{e}^{\frac{2\,\mathrm{e}^{3x}}{3}+2c_1}c_1\right)}{2}\tag{1}$$

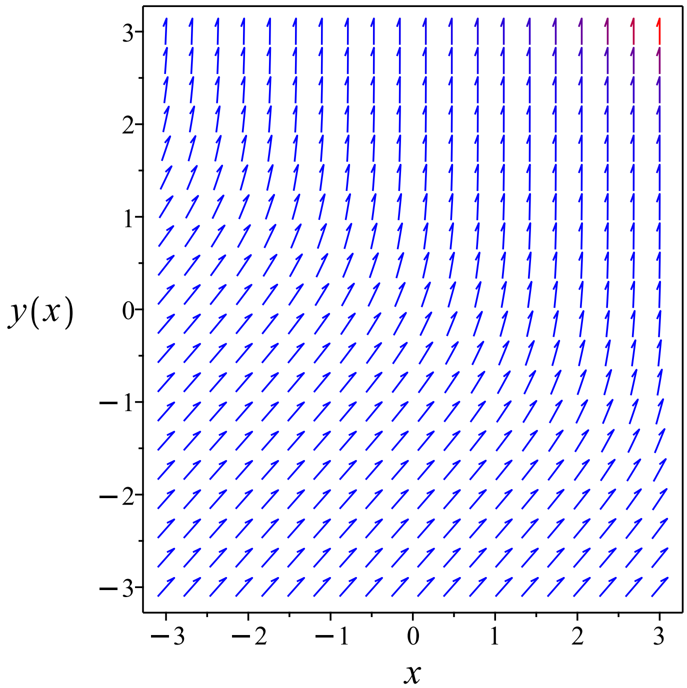


Figure 407: Slope field plot

Verification of solutions

$$y = c_1 + x + \frac{e^{3x}}{3} - \frac{\ln\left(-\frac{2\,e^{\frac{2\,e^{3x}}{3}+2c_1+3x}}{3} - 2\,e^{\frac{2\,e^{3x}}{3}+2c_1} c_1\right)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)-1=exp(x+2*y(x)),y(x), singsol=all)
```

$$y = -\frac{x}{2} + \frac{\ln (3)}{2} + \frac{\ln \left(\frac{e^{3x}}{-2\,e^{3x}+c_1}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.931 (sec). Leaf size: 26

```
DSolve[y'[x]-1==Exp[x+2*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \frac{1}{2} \log \left(-\frac{2}{3}\left(e^{3x} + 3c_1\right)\right)$$

12.27 problem 301

Internal problem ID [15163]
Internal file name [`OUTPUT/15163_Tuesday_April_23_2024_04_51_52_PM_36157548/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 301.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{4yx^3 - 2y^2x + \left(y^2 + 2yx^2 - x^4\right) y' = -2x^5}$$

12.27.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2x(-x^4 - 2x^2y + y^2)}{-x^4 + 2x^2y + y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \frac{2x(-x^4 - 2x^2y + y^2)(b_3 - a_2)}{-x^4 + 2x^2y + y^2} - \frac{4x^2(-x^4 - 2x^2y + y^2)^2 a_3}{(-x^4 + 2x^2y + y^2)^2} \\ &- \left(\frac{-2x^4 - 4x^2y + 2y^2}{-x^4 + 2x^2y + y^2} + \frac{2x(-4x^3 - 4xy)}{-x^4 + 2x^2y + y^2} \right. \\ &\left. - \frac{2x(-x^4 - 2x^2y + y^2)(-4x^3 + 4xy)}{(-x^4 + 2x^2y + y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{2x(-2x^2 + 2y)}{-x^4 + 2x^2y + y^2} \right. \\ &\left. - \frac{2x(-x^4 - 2x^2y + y^2)(2x^2 + 2y)}{(-x^4 + 2x^2y + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4x^{10}a_3 + 4x^9a_2 - 2x^9b_3 + 18x^8ya_3 + 2x^8a_1 + 7x^8b_2 - 16x^7ya_2 + 8x^7yb_3 - 8x^6y^2a_3 + 8x^7b_1 - 16x^6ya_1 +}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}&-4x^{10}a_3 - 4x^9a_2 + 2x^9b_3 - 18x^8ya_3 - 2x^8a_1 - 7x^8b_2 + 16x^7ya_2 - 8x^7yb_3 \\ &+ 8x^6y^2a_3 - 8x^7b_1 + 16x^6ya_1 - 4x^6yb_2 + 24x^5y^2a_2 - 12x^5y^2b_3 + 28x^4y^3a_3 \\ &+ 12x^4y^2a_1 - 6x^4y^2b_2 + 16x^3y^3a_2 - 8x^3y^3b_3 + 12x^2y^4a_3 - 8x^3y^2b_1 \\ &+ 16x^2y^3a_1 + 4x^2y^3b_2 - 4x\,y^4a_2 + 2x\,y^4b_3 - 2y^5a_3 - 2y^4a_1 + y^4b_2 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4a_3v_1^{10} - 4a_2v_1^9 - 18a_3v_1^8v_2 + 2b_3v_1^9 - 2a_1v_1^8 + 16a_2v_1^7v_2 + 8a_3v_1^6v_2^2 - 7b_2v_1^8 \\
&- 8b_3v_1^7v_2 + 16a_1v_1^6v_2 + 24a_2v_1^5v_2^2 + 28a_3v_1^4v_2^3 - 8b_1v_1^7 - 4b_2v_1^6v_2 - 12b_3v_1^5v_2^2 \\
&+ 12a_1v_1^4v_2^2 + 16a_2v_1^3v_2^3 + 12a_3v_1^2v_2^4 - 6b_2v_1^4v_2^2 - 8b_3v_1^3v_2^3 + 16a_1v_1^2v_2^3 \\
&- 4a_2v_1v_2^4 - 2a_3v_2^5 - 8b_1v_1^3v_2^2 + 4b_2v_1^2v_2^3 + 2b_3v_1v_2^4 - 2a_1v_2^4 + b_2v_2^4 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-4a_3v_1^{10} + (-4a_2 + 2b_3)\, v_1^9 - 18a_3v_1^8v_2 + (-2a_1 - 7b_2)\, v_1^8 + (16a_2 - 8b_3)\, v_1^7v_2 \\
&- 8b_1v_1^7 + 8a_3v_1^6v_2^2 + (16a_1 - 4b_2)\, v_1^6v_2 + (24a_2 - 12b_3)\, v_1^5v_2^2 + 28a_3v_1^4v_2^3 \\
&+ (12a_1 - 6b_2)\, v_1^4v_2^2 + (16a_2 - 8b_3)\, v_1^3v_2^3 - 8b_1v_1^3v_2^2 + 12a_3v_1^2v_2^4 \\
&+ (16a_1 + 4b_2)\, v_1^2v_2^3 + (-4a_2 + 2b_3)\, v_1v_2^4 - 2a_3v_2^5 + (-2a_1 + b_2)\, v_2^4 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-18a_3 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
8a_3 &= 0 \\
12a_3 &= 0 \\
28a_3 &= 0 \\
-8b_1 &= 0 \\
-2a_1 - 7b_2 &= 0 \\
-2a_1 + b_2 &= 0 \\
12a_1 - 6b_2 &= 0 \\
16a_1 - 4b_2 &= 0 \\
16a_1 + 4b_2 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
16a_2 - 8b_3 &= 0 \\
24a_2 - 12b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 2y - \left(\frac{2x(-x^4 - 2x^2y + y^2)}{-x^4 + 2x^2y + y^2}\right)(x) \\
&= \frac{-2x^6 - 2x^4y - 2x^2y^2 - 2y^3}{x^4 - 2x^2y - y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^6 - 2x^4y - 2x^2y^2 - 2y^3}{x^4 - 2x^2y - y^2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(x^4+y^2)}{2} - \frac{\ln(x^2+y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x(-x^4-2x^2y+y^2)}{-x^4+2x^2y+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x^3}{x^4+y^2} - \frac{x}{x^2+y} \\ S_y &= \frac{y}{x^4+y^2} - \frac{1}{2x^2+2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

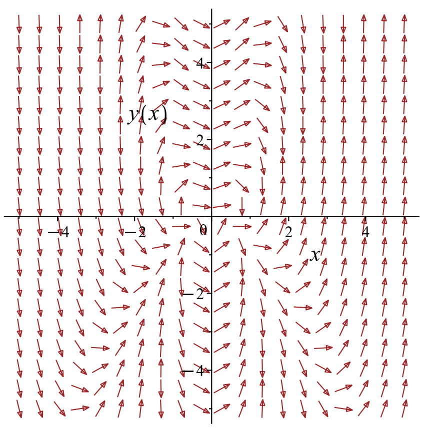
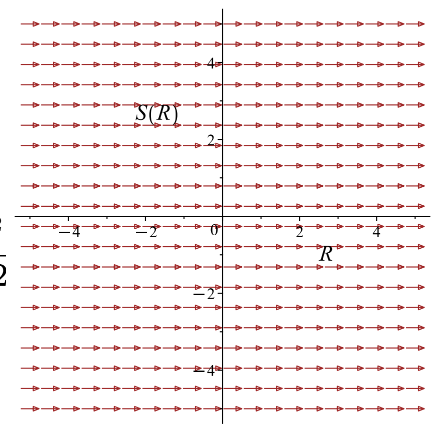
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2+x^4)}{2} - \frac{\ln(x^2+y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y^2+x^4)}{2} - \frac{\ln(x^2+y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x(-x^4-2x^2y+y^2)}{-x^4+2x^2y+y^2}$ 	$R = x$ $S = \frac{\ln(x^4+y^2)}{2} - \frac{\ln(x^2}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+x^4)}{2} - \frac{\ln(x^2+y)}{2} = c_1 \tag{1}$$

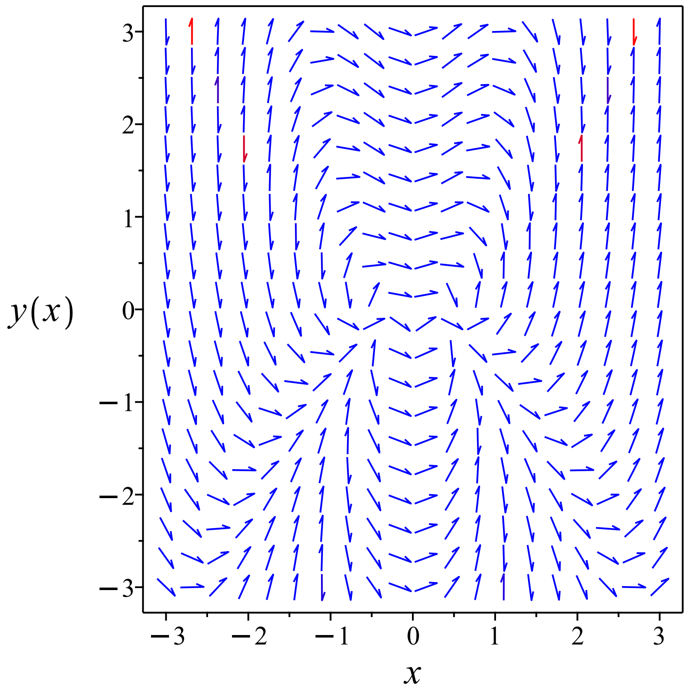


Figure 408: Slope field plot

Verification of solutions

$$\frac{\ln(y^2+x^4)}{2}-\frac{\ln(x^2+y)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 53

```
dsolve(2*(x^5+2*x^3*y(x)-y(x)^2*x)+(y(x)^2+2*x^2*y(x)-x^4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{c_1}{2} - \frac{\sqrt{-4x^4 + 4c_1x^2 + c_1^2}}{2}$$
$$y = \frac{c_1}{2} + \frac{\sqrt{-4x^4 + 4c_1x^2 + c_1^2}}{2}$$

✓ Solution by Mathematica

Time used: 15.349 (sec). Leaf size: 87

```
DSolve[2*(x^5+2*x^3*y[x]-y[x]^2*x)+(y[x]^2+2*x^2*y[x]-x^4)*y'[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \frac{1}{2}\left(e^{2c_1} - \sqrt{-4x^4 + 4e^{2c_1}x^2 + e^{4c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{-4x^4 + 4e^{2c_1}x^2 + e^{4c_1}} + e^{2c_1}\right)$$

12.28 problem 302

Internal problem ID [15164]
Internal file name [`OUTPUT/15164_Tuesday_April_23_2024_04_51_53_PM_4906061/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 302.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{x^2y^ny' - 2xy' + y = 0}$$

12.28.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y}{x\,(y^nx-2)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(b_3 - a_2)}{x\,(y^n x - 2)} - \frac{y^2 a_3}{x^2\,(y^n x - 2)^2} \\ &- \left(\frac{y}{x^2\,(y^n x - 2)} + \frac{y\,y^n}{x\,(y^n x - 2)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{x\,(y^n x - 2)} + \frac{y^n n}{(y^n x - 2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{y^{2n} x^4 b_2 - y^n n\,x^3 b_2 - y^n n\,x^2 y b_3 - y^n n\,x^2 b_1 - 3y^n x^3 b_2 - y^n x^2 y a_2 - 2y^n x\,y^2 a_3 + y^n x^2 b_1 - 2y^n x y a_1 + 2b_2 x}{x^2\,(y^n x - 2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &y^{2n} x^4 b_2 - y^n n\,x^3 b_2 - y^n n\,x^2 y b_3 - y^n n\,x^2 b_1 - 3y^n x^3 b_2 - y^n x^2 y a_2 \\ &- 2y^n x\,y^2 a_3 + y^n x^2 b_1 - 2y^n x y a_1 + 2b_2 x^2 + y^2 a_3 - 2x b_1 + 2y a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, y^n\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, y^n = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &v_3^2 v_1^4 b_2 - v_3 n v_1^3 b_2 - v_3 n v_1^2 v_2 b_3 - v_3 n v_1^2 b_1 - v_3 v_1^2 v_2 a_2 - 2v_3 v_1 v_2^2 a_3 \\ &- 3v_3 v_1^3 b_2 - 2v_3 v_1 v_2 a_1 + v_3 v_1^2 b_1 + v_2^2 a_3 + 2b_2 v_1^2 + 2v_2 a_1 - 2v_1 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} & v_3^2 v_1^4 b_2 + (-n b_2 - 3 b_2) v_1^3 v_3 + (-n b_3 - a_2) v_1^2 v_2 v_3 + (-n b_1 + b_1) v_1^2 v_3 \\ & + 2 b_2 v_1^2 - 2 v_3 v_1 v_2^2 a_3 - 2 v_3 v_1 v_2 a_1 - 2 v_1 b_1 + v_2^2 a_3 + 2 v_2 a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_3 &= 0 \\ b_2 &= 0 \\ -2a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_2 &= 0 \\ -nb_1 + b_1 &= 0 \\ -nb_2 - 3b_2 &= 0 \\ -nb_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -nb_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -nx \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y}{x\,(y^n x - 2)}\right)(-nx) \\ &= \frac{y\,y^n x - yn - 2y}{y^n x - 2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y\,y^n x - yn - 2y}{y^n x - 2}} dy\end{aligned}$$

Which results in

$$S = \frac{2\ln(y)}{n+2} + \frac{\ln\left(-\mathrm{e}^{n\ln(y)} x + n + 2\right)}{n+2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y}{x\,(y^n x - 2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^n}{(n+2)\,(n-y^n x+2)} \\
S_y &= \frac{-y^n x+2}{y\,(n-y^n x+2)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{(n+2)\,x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{(n+2)\,R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R)}{n+2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\ln(y)+\ln(n-y^n x+2)}{n+2} = \frac{\ln(x)}{n+2} + c_1
$$

Which simplifies to

$$
\frac{2\ln(y)+\ln(n-y^n x+2)}{n+2} = \frac{\ln(x)}{n+2} + c_1
$$

Summary

The solution(s) found are the following

$$
\frac{2\ln(y)+\ln(n-y^n x+2)}{n+2} = \frac{\ln(x)}{n+2} + c_1 \tag{1}
$$

Verification of solutions

$$
\frac{2\ln(y)+\ln(n-y^n x+2)}{n+2} = \frac{\ln(x)}{n+2} + c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 32

```
dsolve(x^2*y(x)^n*diff(y(x),x)=2*x*diff(y(x),x)-y(x),y(x), singsol=all)
```

$$y^{2n}(y^n x - n - 2)^n x^{-n} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.18 (sec). Leaf size: 41

```
DSolve[x^2*y[x]^n*y'[x]==2*x*y'[x]-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{n(\log(x) - \log(-xy(x)^n + n + 2))}{n+2} - \frac{2n\log(y(x))}{n+2} = c_1, y(x)\right]$$

12.29 problem 303

Internal problem ID [15165]
Internal file name [`OUTPUT/15165_Tuesday_April_23_2024_04_51_55_PM_12544078/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 303.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(3y+3x+a^2)\,y'-4y=b^2+4x}$$

12.29.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{b^2+4x+4y}{a^2+3x+3y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(b^2+4x+4y)(b_3-a_2)}{a^2+3x+3y} - \frac{(b^2+4x+4y)^2 a_3}{(a^2+3x+3y)^2} - \left(\frac{4}{a^2+3x+3y} - \frac{3(b^2+4x+4y)}{(a^2+3x+3y)^2}\right)(xa_2+ya_3+a_1) - \left(\frac{4}{a^2+3x+3y} - \frac{3(b^2+4x+4y)}{(a^2+3x+3y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{a^4b_2 - a^2b^2a_2 + a^2b^2b_3 - b^4a_3 - 8a^2xa_2 + 2a^2xb_2 + 4a^2xb_3 - 4a^2ya_2 - 4a^2ya_3 + 6a^2yb_2 - 8b^2xa_3 + 3b^2xb}{}$$

$$= 0$$

Setting the numerator to zero gives

$$a^4b_2 - a^2b^2a_2 + a^2b^2b_3 - b^4a_3 - 8a^2xa_2 + 2a^2xb_2 + 4a^2xb_3 - 4a^2ya_2 - 4a^2ya_3 + 6a^2yb_2 - 8b^2xa_3 + 3b^2xb_2 + 3b^2xb_3 - 3b^2ya_2 - 5b^2ya_3 + 6b^2yb_3 - 4a^2a_1 - 4a^2b_1 + 3b^2a_1 + 3b^2b_1 - 12x^2a_2 - 16x^2a_3 + 9x^2b_2 + 12x^2b_3 - 24xya_2 - 32xya_3 + 18xyb_2 + 24xyb_3 - 12y^2a_2 - 16y^2a_3 + 9y^2b_2 + 12y^2b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a^4b_2 - a^2b^2a_2 + a^2b^2b_3 - b^4a_3 - 8a^2a_2v_1 - 4a^2a_2v_2 - 4a^2a_3v_2 + 2a^2b_2v_1 + 6a^2b_2v_2 + 4a^2b_3v_1 - 3b^2a_2v_2 - 8b^2a_3v_1 - 5b^2a_3v_2 + 3b^2b_2v_1 + 3b^2b_3v_1 + 6b^2b_3v_2 - 4a^2a_1 - 4a^2b_1 + 3b^2a_1 + 3b^2b_1 - 12a_2v_1^2 - 24a_2v_1v_2 - 12a_2v_2^2 - 16a_3v_1^2 - 32a_3v_1v_2 - 16a_3v_2^2 + 9b_2v_1^2 + 18b_2v_1v_2 + 9b_2v_2^2 + 12b_3v_1^2 + 24b_3v_1v_2 + 12b_3v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-12a_2 - 16a_3 + 9b_2 + 12b_3) v_1^2 + (-24a_2 - 32a_3 + 18b_2 + 24b_3) v_1 v_2 \\
&+ \left(-8a^2a_2 + 2a^2b_2 + 4a^2b_3 - 8b^2a_3 + 3b^2b_2 + 3b^2b_3\right) v_1 \\
&+ (-12a_2 - 16a_3 + 9b_2 + 12b_3) v_2^2 \\
&+ \left(-4a^2a_2 - 4a^2a_3 + 6a^2b_2 - 3b^2a_2 - 5b^2a_3 + 6b^2b_3\right) v_2 + a^4b_2 \\
&- a^2b^2a_2 + a^2b^2b_3 - b^4a_3 - 4a^2a_1 - 4a^2b_1 + 3b^2a_1 + 3b^2b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-24a_2 - 32a_3 + 18b_2 + 24b_3 &= 0 \\
-12a_2 - 16a_3 + 9b_2 + 12b_3 &= 0 \\
-8a^2a_2 + 2a^2b_2 + 4a^2b_3 - 8b^2a_3 + 3b^2b_2 + 3b^2b_3 &= 0 \\
-4a^2a_2 - 4a^2a_3 + 6a^2b_2 - 3b^2a_2 - 5b^2a_3 + 6b^2b_3 &= 0 \\
a^4b_2 - a^2b^2a_2 + a^2b^2b_3 - b^4a_3 - 4a^2a_1 - 4a^2b_1 + 3b^2a_1 + 3b^2b_1 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= \frac{1}{3}a^2a_3 + \frac{1}{3}b^2a_3 - b_1 \\
a_2 &= a_3 \\
a_3 &= a_3 \\
b_1 &= b_1 \\
b_2 &= \frac{4a_3}{3} \\
b_3 &= \frac{4a_3}{3}
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(\frac{b^2+4x+4y}{a^2+3x+3y}\right)(-1) \\
&= \frac{a^2+b^2+7x+7y}{a^2+3x+3y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{a^2+b^2+7x+7y}{a^2+3x+3y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3y}{7} + \left(\frac{4a^2}{49} - \frac{3b^2}{49}\right) \ln\left(a^2+b^2+7x+7y\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{b^2+4x+4y}{a^2+3x+3y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4a^2 - 3b^2}{7a^2 + 7b^2 + 49x + 49y} \\
S_y &= \frac{a^2 + 3x + 3y}{a^2 + b^2 + 7x + 7y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{4}{7} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{4}{7}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{4R}{7} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{3y}{7} + \frac{(4a^2 - 3b^2)\ln(a^2 + b^2 + 7x + 7y)}{49} = \frac{4x}{7} + c_1
$$

Which simplifies to

$$
\frac{3y}{7} + \frac{(4a^2 - 3b^2)\ln(a^2 + b^2 + 7x + 7y)}{49} = \frac{4x}{7} + c_1
$$

Which gives

$$
y = \frac{4\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2+49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right) a^2}{21} - \frac{\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2+49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right) b^2}{7} - \frac{a^2}{7} - \frac{b^2}{7} - x
$$

Summary

The solution(s) found are the following

$$
y = \frac{4\,\text{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2+49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right)a^2}{21} - \frac{\text{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2+49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right)b^2}{7} - \frac{a^2}{7} - \frac{b^2}{7} - x \tag{1}
$$

Verification of solutions

$$
y = \frac{4\,\text{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2+49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right)a^2}{21} - \frac{\text{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2+49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right)b^2}{7} - \frac{a^2}{7} - \frac{b^2}{7} - x
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 79

```
dsolve((3*(x+y(x))+a^2)*diff(y(x),x)=4*(x+y(x))+b^2,y(x), singsol=all)
```

$$y = \frac{(4a^2-3b^2)\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{\frac{3a^2+3b^2-49c_1+49x}{4a^2-3b^2}}}{4a^2-3b^2}\right)}{21} - \frac{a^2}{7} - \frac{b^2}{7} - x$$

✓ Solution by Mathematica

Time used: 60.042 (sec). Leaf size: 97

```
DSolve[(3*(x+y[x])+a^2)*y'[x]==4*(x+y[x])+b^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{21}\left(-3\left(a^2+b^2+7x\right)+\left(4a^2-3b^2\right)W\left(-4\left(2^{\frac{3b^2}{2a^2}-2}e^{\frac{49x-3b^2(-1+c_1)}{4a^2}-1+c_1}\right)^{\frac{4a^2}{4a^2-3b^2}}\right)\right)$$

12.30 problem 304

Internal problem ID [15166]
Internal file name [`OUTPUT/15166_Tuesday_April_23_2024_04_51_56_PM_34556715/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 304.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{-y^2 + 2y'yx = -x}$$

12.30.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 - x}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 312: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{x}{y} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 - x}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

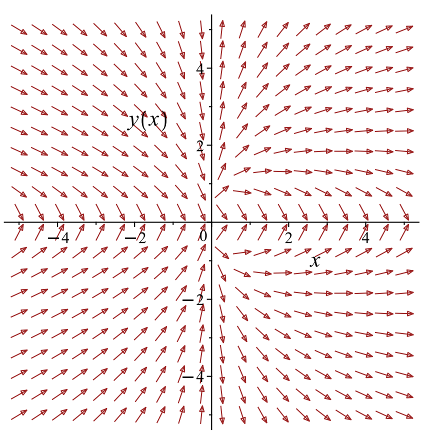
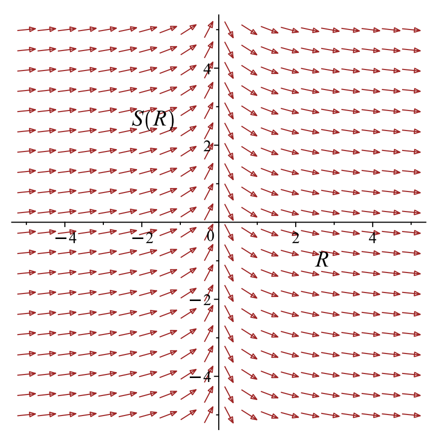
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2-x}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

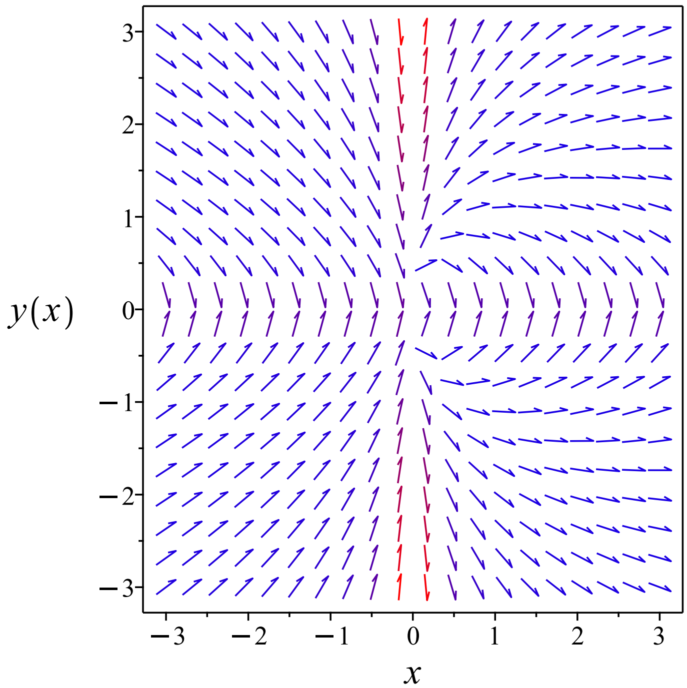


Figure 409: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

12.30.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 - x}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - \frac{1}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= -\frac{1}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} - \frac{1}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{w(x)}{2x} - \frac{1}{2} \\ w' &= \frac{w}{x} - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -1 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-1) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-1) \\ d\left(\frac{w}{x}\right) &= \left(-\frac{1}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int -\frac{1}{x} dx \\ \frac{w}{x} &= -\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = -\ln(x)\,x + c_1 x$$

which simplifies to

$$w(x) = x(-\ln(x) + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(-\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{-x(\ln(x) - c_1)} \\ y(x) &= -\sqrt{x(-\ln(x) + c_1)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x(\ln(x) - c_1)} \tag{1}$$

$$y = -\sqrt{x(-\ln(x) + c_1)} \tag{2}$$

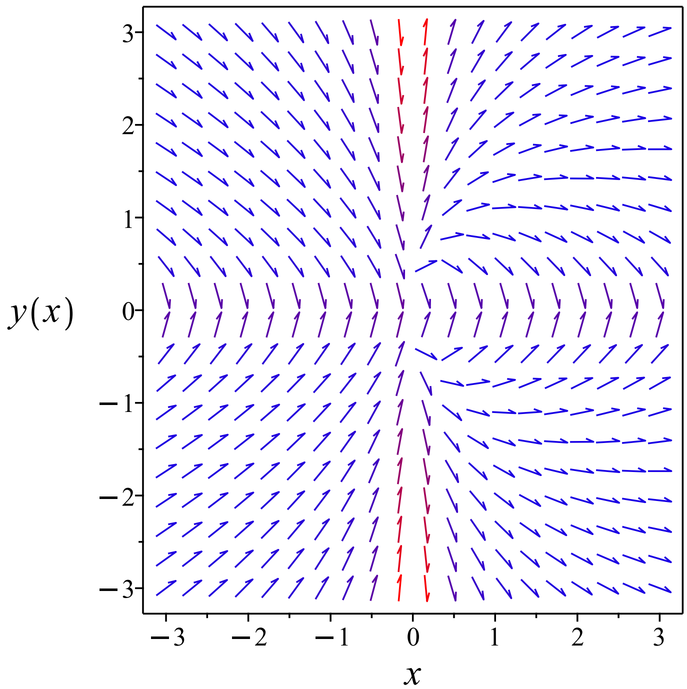


Figure 410: Slope field plot

Verification of solutions

$$y = \sqrt{-x(\ln(x) - c_1)}$$

Verified OK.

$$y = -\sqrt{x(-\ln(x) + c_1)}$$

Verified OK.

12.30.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(y^2-x\right)\mathrm{d}x\\ \left(-y^2+x\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^2+x\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^2 + x)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2xy}((-2y) - (2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(-y^2 + x)\\ &= \frac{-y^2 + x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(2xy)\\ &= \frac{2y}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-y^2 + x}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{-y^2 + x}{x^2}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{y^2}{x} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$\frac{2y}{x} = \frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^2}{x} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^2}{x} + \ln(x)$$

Summary

The solution(s) found are the following

$$\frac{y^2}{x} + \ln(x) = c_1 \tag{1}$$

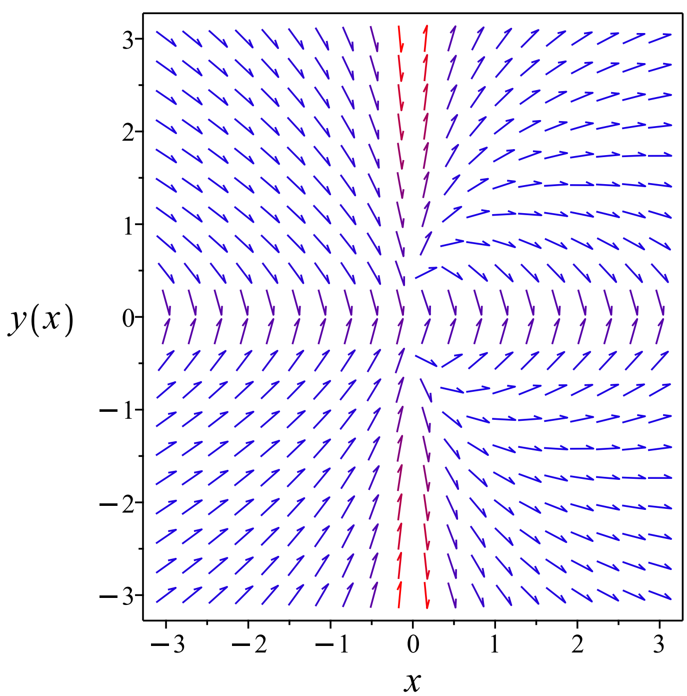


Figure 411: Slope field plot

Verification of solutions

$$\frac{y^2}{x}+\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve((x-y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y=\sqrt{-x(\ln(x)-c_1)}$$
$$y=-\sqrt{(-\ln(x)+c_1)\,x}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 44

```
DSolve[(x-y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\sqrt{x}\sqrt{-\log(x)+c_1}$$
$$y(x)\to \sqrt{x}\sqrt{-\log(x)+c_1}$$

12.31 problem 305

Internal problem ID [15167]
Internal file name [`OUTPUT/15167_Tuesday_April_23_2024_04_51_58_PM_88565658/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 305.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{xy' + y - y^2 \ln(x) = 0}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}\right]$$

12.31.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(\ln(x) y - 1)}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{1}{2}$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(\ln(x)\, y - 1)}{x}\right) \\
&= \frac{\ln(x)\, y - 1}{x} + \frac{\ln(x)\, y}{x}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

12.31.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{y(\ln(x)\, y - 1)}{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 314: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x\, y^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x\,y^2} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{xy}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(\ln(x)\,y - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x^2 y} \\
S_y &= \frac{1}{x\,y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(x)}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{R} - \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{xy} = -\frac{\ln(x)}{x} - \frac{1}{x} + c_1$$

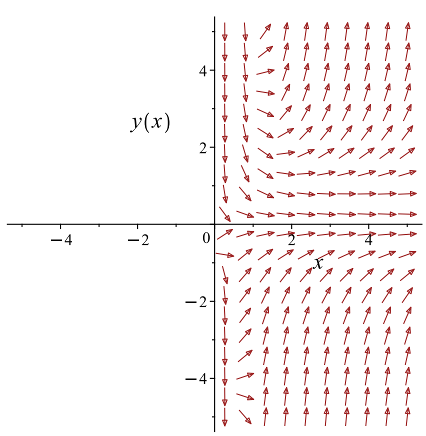
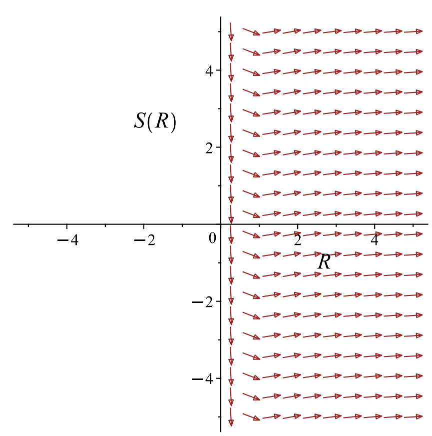
Which simplifies to

$$\frac{-c_1 xy + \ln(x)\, y + y - 1}{xy} = 0$$

Which gives

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(\ln(x)y-1)}{x}$ 	$R = x$ $S = -\frac{1}{xy}$	$\frac{dS}{dR} = \frac{\ln(R)}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{-1 + c_1}$$

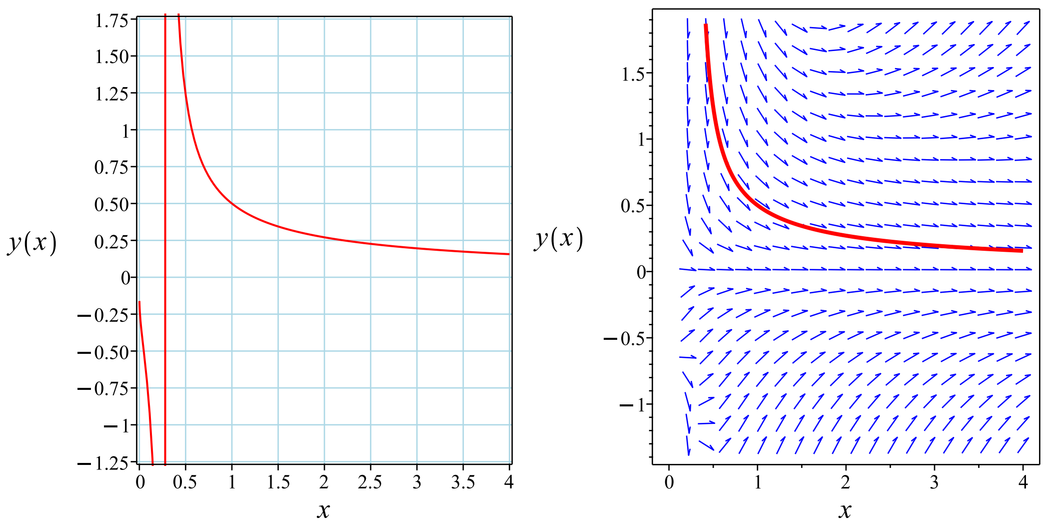
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x + \ln(x) + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x + \ln(x) + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x + \ln(x) + 1}$$

Verified OK.

12.31.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(\ln(x)\, y-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{\ln(x)}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{\ln(x)}{x} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{xy} + \frac{\ln(x)}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= -\frac{w(x)}{x} + \frac{\ln(x)}{x} \\
w' &= \frac{w}{x} - \frac{\ln(x)}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{\ln(x)}{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = -\frac{\ln(x)}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{\ln(x)}{x}\right) \\
\frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(-\frac{\ln(x)}{x}\right) \\
d\left(\frac{w}{x}\right) &= \left(-\frac{\ln(x)}{x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -\frac{\ln(x)}{x^2} dx \\
\frac{w}{x} &= \frac{\ln(x)}{x} + \frac{1}{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x\left(\frac{\ln(x)}{x} + \frac{1}{x}\right) + c_1 x$$

which simplifies to

$$w(x) = c_1 x + \ln(x) + 1$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x + \ln(x) + 1$$

Or

$$y = \frac{1}{c_1 x + \ln(x) + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{c_1 + 1}$$

$$c_1 = 1$$

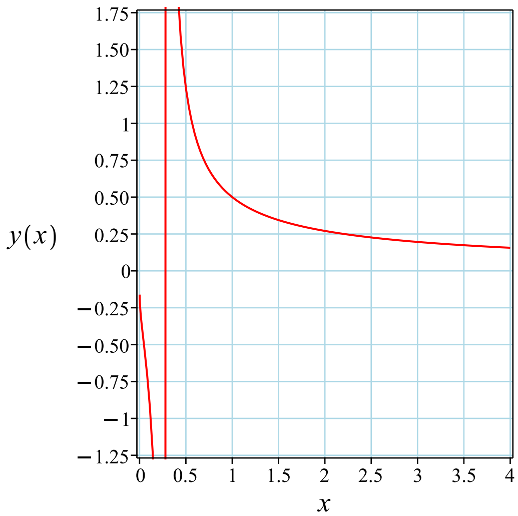
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x + \ln(x) + 1}$$

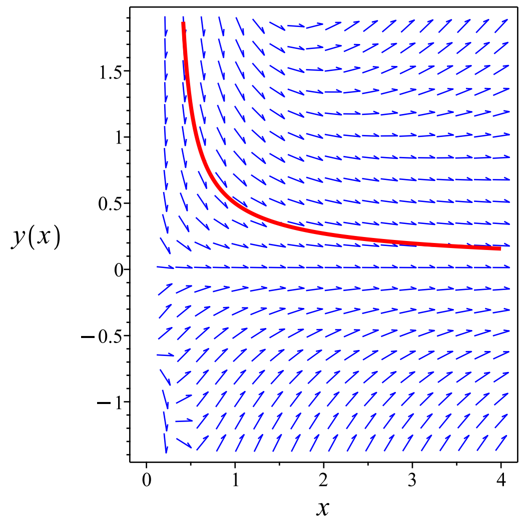
Summary

The solution(s) found are the following

$$y = \frac{1}{x + \ln(x) + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x + \ln(x) + 1}$$

Verified OK.

12.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-y + y^2\ln(x)\right)\mathrm{d}x\\ \left(-y^2\ln(x) + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y^2\ln(x) + y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^2\ln(x) + y\right)\\ &= -2\ln(x)\,y + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}\left(\left(-2\ln(x)\,y + 1\right) - (1)\right)\\ &= -\frac{2\ln(x)\,y}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{y(\ln(x)\,y-1)}((1)-(-2\ln(x)\,y+1)) \\ &= -\frac{2\ln(x)}{\ln(x)\,y-1} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(1)-(-2\ln(x)\,y+1)}{x(-y^2\ln(x)+y)-y(x)} \\ &= -\frac{2}{xy} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R\,\mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right)\mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^2y^2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2y^2}\left(-y^2\ln\left(x\right)+y\right) \\
&= \frac{-\ln\left(x\right)y+1}{x^2y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2y^2}(x) \\
&= \frac{1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-\ln\left(x\right)y+1}{x^2y}\right)+\left(\frac{1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{-\ln\left(x\right)y+1}{x^2y}\,\mathrm{d}x \\
\phi &= \frac{\ln\left(x\right)y+y-1}{xy}+f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\ln(x)+1}{xy} - \frac{\ln(x)\, y + y - 1}{x\, y^2} + f'(y) \tag{4}$$
$$= \frac{1}{x\, y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x\, y^2}$. Therefore equation (4) becomes

$$\frac{1}{x\, y^2} = \frac{1}{x\, y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x)\, y + y - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x)\, y + y - 1}{xy}$$

The solution becomes

$$y = \frac{1}{-c_1 x + \ln(x) + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{-1 + c_1}$$

$$c_1 = -1$$

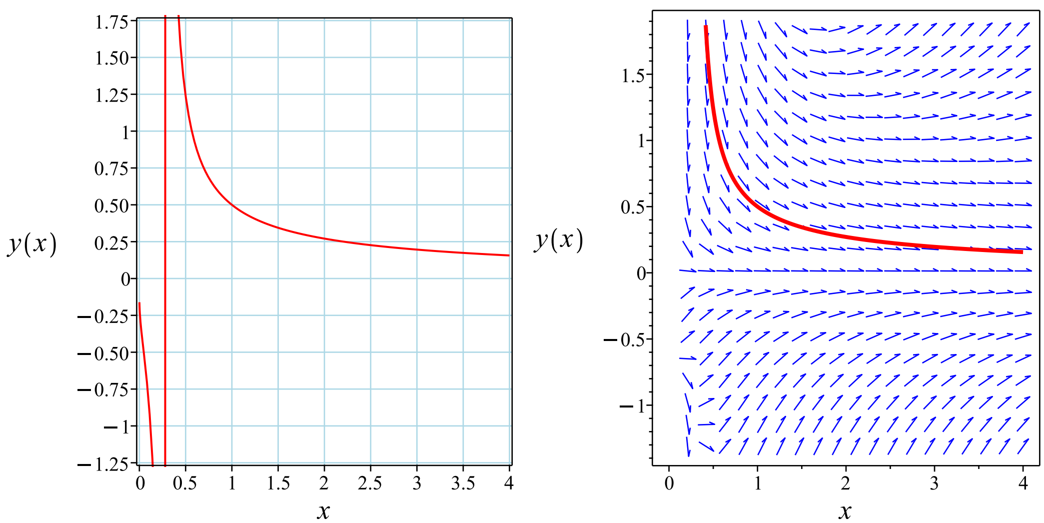
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x + \ln(x) + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x + \ln(x) + 1}$$

Verified OK.

12.31.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{y(\ln(x)\,y-1)}{x}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2\ln(x)}{x} - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{\ln(x)}{x}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\frac{\ln(x)u}{x}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned}f_2' &= -\frac{\ln(x)}{x^2} + \frac{1}{x^2}\\ f_1f_2 &= -\frac{\ln(x)}{x^2}\\ f_2^2f_0 &= 0\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{\ln(x)\,u''(x)}{x} - \left(-\frac{2\ln(x)}{x^2} + \frac{1}{x^2}\right)u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{-c_2\ln(x) + c_1x - c_2}{x}$$

The above shows that

$$u'(x) = \frac{c_2 \ln(x)}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{-c_2 \ln(x) + c_1 x - c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{-c_3 x + \ln(x) + 1}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{c_3 - 1}$$

$$c_3 = -1$$

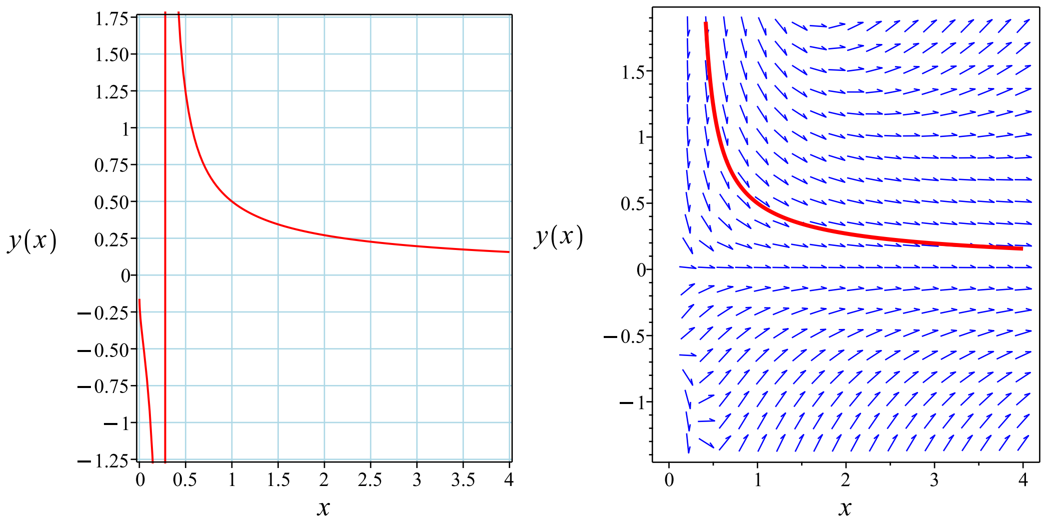
Substituting c_3 found above in the general solution gives

$$y = \frac{1}{x + \ln(x) + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x + \ln(x) + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x + \ln(x) + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([x*diff(y(x),x)+y(x)=y(x)^2*ln(x),y(1) = 1/2],y(x), singsol=all)
```

$$y = \frac{1}{1 + x + \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 12

```
DSolve[{x*y'[x]+y[x]==y[x]^2*Log[x],{y[1]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x + \log(x) + 1}$$

12.32 problem 306

Internal problem ID [15168]
Internal file name [`OUTPUT/15168_Tuesday_April_23_2024_04_51_59_PM_39469266/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 306.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$
\boxed{-\cos\left(\ln\left(y\right)\right) y' = -\sin\left(\ln\left(x\right)\right)}
$$

12.32.1 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{\sin\left(\ln\left(x\right)\right)}{\cos\left(\ln\left(y\right)\right)}
\end{aligned}
$$

Where $f(x) = \sin\left(\ln\left(x\right)\right)$ and $g(y) = \frac{1}{\cos(\ln(y))}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{\cos(\ln(y))}}\,dy &= \sin\left(\ln\left(x\right)\right)\,dx \\
\int \frac{1}{\frac{1}{\cos(\ln(y))}}\,dy &= \int \sin\left(\ln\left(x\right)\right)\,dx
\end{aligned}
$$

$$\frac{\cos(\ln(y))\, y}{2}+\frac{\sin(\ln(y))\, y}{2}=-\frac{\cos(\ln(x))\, x}{2}+\frac{\sin(\ln(x))\, x}{2}+c_1$$

Which results in

$$y = \mathrm{e}^{\mathrm{RootOf}\left(-2\sin(\ln(x))\sin(\_Z)\mathrm{e}^{\_Z}x+2\sin(\_Z)^2\mathrm{e}^{2\_Z}+2\sin(\_Z)\cos(\ln(x))\mathrm{e}^{\_Z}x+4c_1x\sin(\ln(x))-4\sin(\_Z)\mathrm{e}^{\_Z}c_1-4\cos(\ln(x))c_1x-x^2\sin(2\right.}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{RootOf}\left(-2\sin(\ln(x))\sin(\_Z)\mathrm{e}^{\_Z}x+2\sin(\_Z)^2\mathrm{e}^{2\_Z}+2\sin(\_Z)\cos(\ln(x))\mathrm{e}^{\_Z}x+4c_1x\sin(\ln(x))-4\sin(\_Z)\mathrm{e}^{\_Z}c_1-4\cos(\ln(x))c_1x-x^2\sin(2\right.} \quad (1)$$

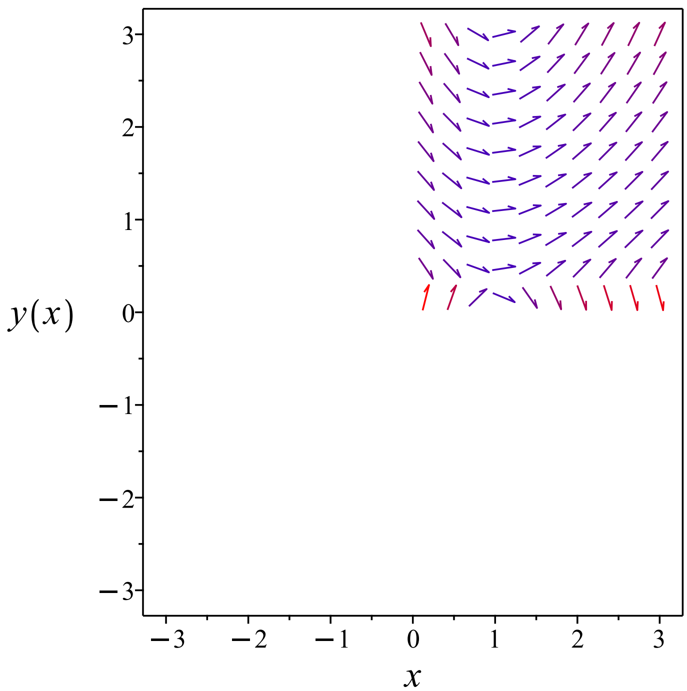


Figure 416: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{RootOf}\left(-2\sin(\ln(x))\sin(\_Z)\mathrm{e}^{\_Z}x+2\sin(\_Z)^2\mathrm{e}^{2\_Z}+2\sin(\_Z)\cos(\ln(x))\mathrm{e}^{\_Z}x+4c_1x\sin(\ln(x))-4\sin(\_Z)\mathrm{e}^{\_Z}c_1-4\cos(\ln(x))c_1x-x^2\sin(2\right.}$$

Warning, solution could not be verified

12.32.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(\ln(x))}{\cos(\ln(y))} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 316: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{1}{\sin\left(\ln\left(x\right)\right)} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{\sin(\ln(x))}} dx\end{aligned}$$

Which results in

$$S = -\frac{\cos\left(\ln\left(x\right)\right) x}{2} + \frac{\sin\left(\ln\left(x\right)\right) x}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sin\left(\ln\left(x\right)\right)}{\cos\left(\ln\left(y\right)\right)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \sin(\ln(x)) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(\ln(y)) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(\ln(R))
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 + \frac{R(\cos(\ln(R)) + \sin(\ln(R)))}{2} \tag{4}
$$

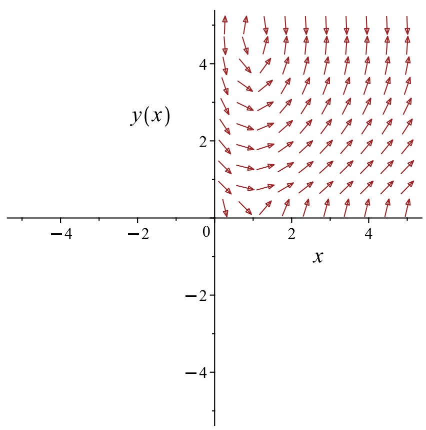
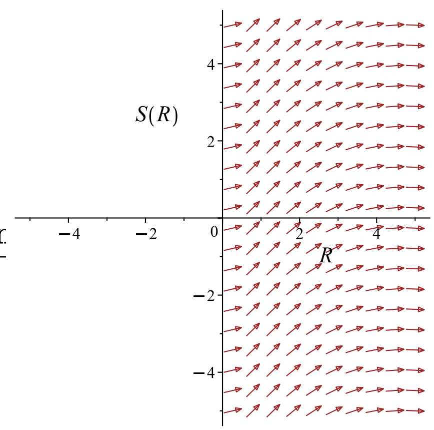
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} = c_1 + \frac{y(\cos(\ln(y)) + \sin(\ln(y)))}{2}
$$

Which simplifies to

$$
\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} = c_1 + \frac{y(\cos(\ln(y)) + \sin(\ln(y)))}{2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(\ln(x))}{\cos(\ln(y))}$ 	$R = y$ $S = \frac{x(-\cos(\ln(x)) + \text{sir}}{2}$	$\frac{dS}{dR} = \cos(\ln(R))$ 

Summary
The solution(s) found are the following

$$\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} = c_1 + \frac{y(\cos(\ln(y)) + \sin(\ln(y)))}{2} \tag{1}$$

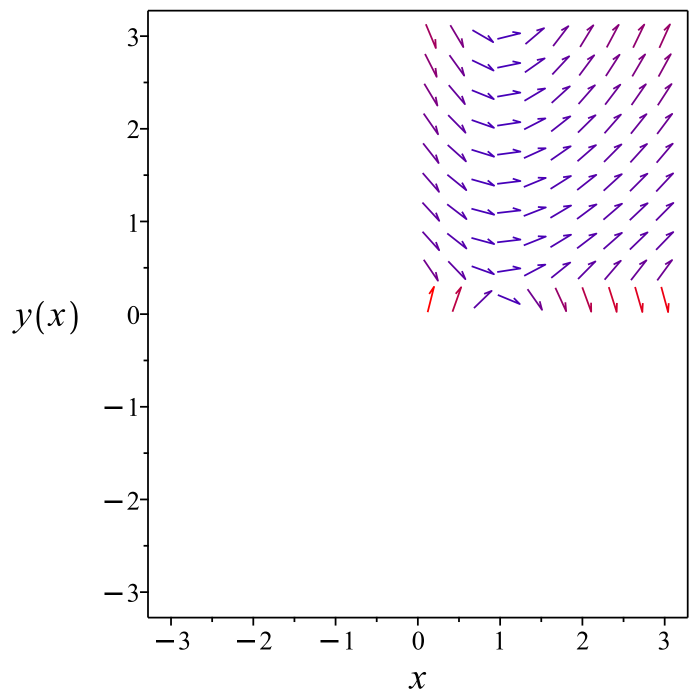


Figure 417: Slope field plot

Verification of solutions

$$\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} = c_1 + \frac{y(\cos(\ln(y)) + \sin(\ln(y)))}{2}$$

Verified OK.

12.32.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\cos(\ln(y)))\,\mathrm{d}y = (\sin(\ln(x)))\,\mathrm{d}x$$
$$(-\sin(\ln(x)))\,\mathrm{d}x + (\cos(\ln(y)))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\sin(\ln(x))$$
$$N(x, y) = \cos(\ln(y))$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(\ln(x))) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(\ln(y))) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -\sin(\ln(x))\,\mathrm{d}x$$

$$\phi=-\frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\cos(\ln(y))$. Therefore equation (4) becomes

$$\cos(\ln(y))=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\cos(\ln(y))$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (\cos(\ln(y)))\,\mathrm{d}y$$

$$f(y)=\frac{\cos(\ln(y))\,y}{2}+\frac{\sin(\ln(y))\,y}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} + \frac{\cos(\ln(y))\, y}{2} + \frac{\sin(\ln(y))\, y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} + \frac{\cos(\ln(y))\, y}{2} + \frac{\sin(\ln(y))\, y}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} + \frac{\cos(\ln(y))\, y}{2} + \frac{\sin(\ln(y))\, y}{2} = c_1 \tag{1}$$

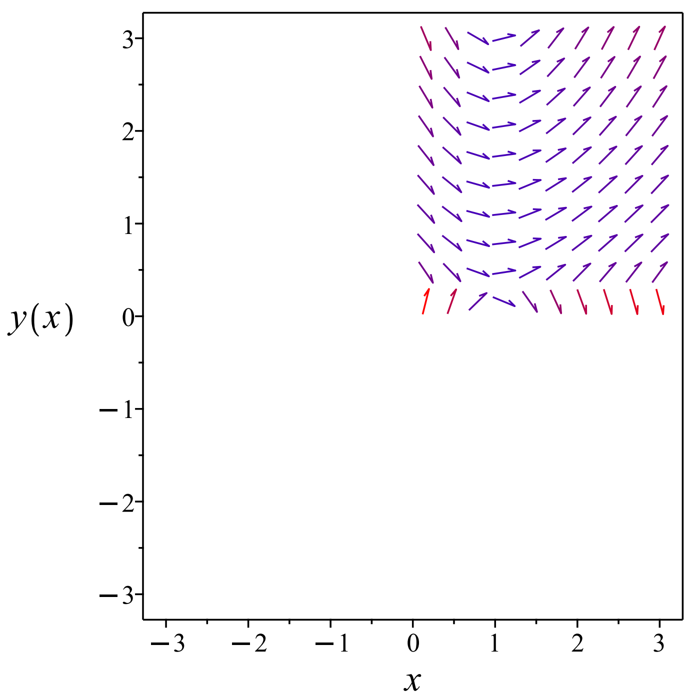


Figure 418: Slope field plot

Verification of solutions

$$-\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{2} + \frac{\cos(\ln(y))\, y}{2} + \frac{\sin(\ln(y))\, y}{2} = c_1$$

Verified OK.

12.32.4 Maple step by step solution

Let's solve

$-\cos(\ln(y))\, y' = -\sin(\ln(x))$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int -\cos(\ln(y))\, y' dx = \int -\sin(\ln(x))\, dx + c_1$

- Evaluate integral

  $-\frac{\cos(\ln(y))y}{2} - \frac{\sin(\ln(y))y}{2} = \frac{\cos(\ln(x))x}{2} - \frac{\sin(\ln(x))x}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 81

```
dsolve(sin(ln(x))-cos(ln(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = e^{\operatorname{RootOf}\left(-2\cos(\ln(x))x^2\sin(\ln(x))-2\sin(\ln(x))\sin(\_Z)e^{\_Z}x+2\sin(\_Z)\cos(\ln(x))e^{\_Z}x-2e^{2\_Z}\cos(\_Z)^2+4c_1x\sin(\ln(x))-4\cos(\ln(x))c_1x-\right.}$$

✓ Solution by Mathematica

Time used: 0.386 (sec). Leaf size: 47

```
DSolve[Sin[Log[x]]-Cos[Log[y[x]]]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{2}\#1\sin(\log(\#1)) + \frac{1}{2}\#1\cos(\log(\#1))\&\right]\left[\frac{1}{2}x\sin(\log(x)) - \frac{1}{2}x\cos(\log(x)) + c_1\right]$$

12.33 problem 307

Internal problem ID [15169]
Internal file name [OUTPUT/15169_Tuesday_April_23_2024_04_52_38_PM_32068494/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 307.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{y' - \sqrt{\frac{9y^2 - 6y + 2}{x^2 - 2x + 5}} = 0}$$

12.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\right)\mathrm{d}x$$

$$\left(-\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\right)\\&=\frac{-9y+3}{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,(x^2-2x+5)}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{18y-6}{2\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,(x^2-2x+5)}\right) - (0)\right) \\ &= \frac{-9y+3}{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,(x^2-2x+5)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}}\left((0) - \left(-\frac{18y-6}{2\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,(x^2-2x+5)}\right)\right) \\ &= \frac{-9y+3}{9y^2-6y+2} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{-9y+3}{9y^2-6y+2} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{\ln(9y^2-6y+2)}{2}} \\ &= \frac{1}{\sqrt{9y^2-6y+2}} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{\sqrt{9y^2-6y+2}}\left(-\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\right) \\ &= -\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}}{\sqrt{9y^2-6y+2}} \end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{9y^2-6y+2}}(1) \\
&= \frac{1}{\sqrt{9y^2-6y+2}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{gathered}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}=0 \\
\left(-\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}}{\sqrt{9y^2-6y+2}}\right)+\left(\frac{1}{\sqrt{9y^2-6y+2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0
\end{gathered}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int-\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}}{\sqrt{9y^2-6y+2}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi=-\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)}{\sqrt{9y^2-6y+2}}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial\phi}{\partial y} &= \frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)(18y-6)}{2\left(9y^2-6y+2\right)^{\frac{3}{2}}} \\
&\quad-\frac{\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)(18y-6)}{2\sqrt{9y^2-6y+2}\,\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}}+f'(y) \\
&= 0+f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{9y^2-6y+2}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{9y^2-6y+2}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{9y^2-6y+2}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{9y^2-6y+2}}\right) \mathrm{d}y$$
$$f(y) = \frac{\operatorname{arcsinh}(3y-1)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)}{\sqrt{9y^2-6y+2}} + \frac{\operatorname{arcsinh}(3y-1)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)}{\sqrt{9y^2-6y+2}} + \frac{\operatorname{arcsinh}(3y-1)}{3}$$

Summary

The solution(s) found are the following

$$-\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)}{\sqrt{9y^2-6y+2}} + \frac{\operatorname{arcsinh}(3y-1)}{3} = c_1 \tag{1}$$

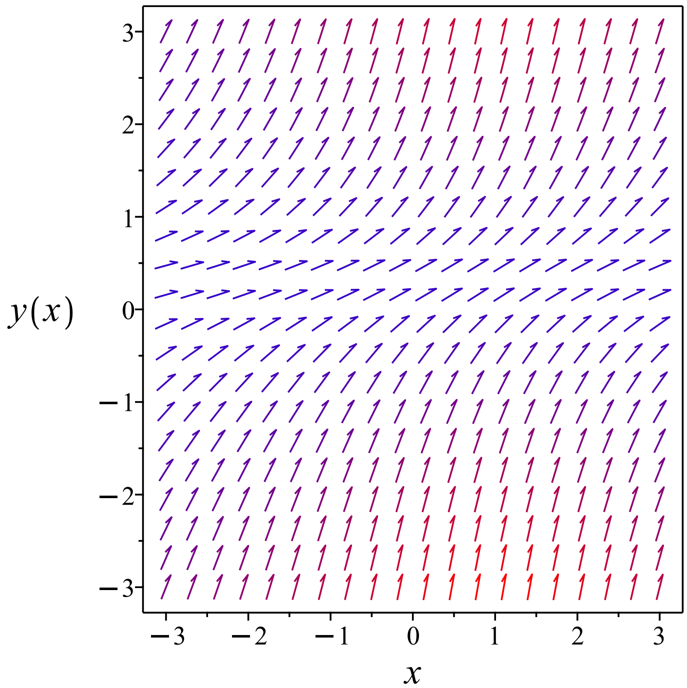


Figure 419: Slope field plot

Verification of solutions

$$-\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)}{\sqrt{9y^2-6y+2}}+\frac{\operatorname{arcsinh}\left(3y-1\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 71

```
dsolve(diff(y(x),x)=sqrt(  (9*y(x)^2-6*y(x)+2)/ (x^2-2*x+5)   ),y(x), singsol=all)
```

$$-\frac{\sqrt{\frac{9y^2-6y+2}{x^2-2x+5}}\,\sqrt{x^2-2x+5}\,\operatorname{arcsinh}\left(\frac{x}{2}-\frac{1}{2}\right)}{\sqrt{9y^2-6y+2}}+\frac{\operatorname{arcsinh}\left(3y-1\right)}{3}+c_1=0$$

✓ Solution by Mathematica

Time used: 5.598 (sec). Leaf size: 160

```
DSolve[y'[x]==Sqrt[  (9*y[x]^2-6*y[x]+2)/ (x^2-2*x+5)   ],y[x],x,IncludeSingularSolutions ->
```

$$\begin{aligned}y(x)\to\frac{1}{96}\Big(e^{3c_1}\Big(x^3+\left(\sqrt{x^2-2x+5}-3\right)x^2-2\left(\sqrt{x^2-2x+5}-3\right)x\\+2\left(\sqrt{x^2-2x+5}-2\right)\Big)-64e^{-3c_1}\Big(-x^3+\left(\sqrt{x^2-2x+5}+3\right)x^2\\-2\left(\sqrt{x^2-2x+5}+3\right)x+2\left(\sqrt{x^2-2x+5}+2\right)\Big)+32\Big)\end{aligned}$$

$$y(x)\to\frac{1}{3}-\frac{i}{3}$$

$$y(x)\to\frac{1}{3}+\frac{i}{3}$$

12.34 problem 308

Internal problem ID [15170]
Internal file name [`OUTPUT/15170_Tuesday_April_23_2024_04_52_40_PM_62807825/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 308.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(5x - 7y + 1)\, y' + y = 1 - x}$$

12.34.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{X + x_0 + Y(X) + y_0 - 1}{-5X - 5x_0 + 7Y\,(X) + 7y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = \frac{1}{2}$$
$$y_0 = \frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X + Y(X)}{-5X + 7Y\,(X)}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= \frac{X+Y}{-5X+7Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = -X - Y$ and $N = 5X - 7Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u+1}{7u-5} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)+1}{7u(X)-5} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{u(X)+1}{7u(X)-5} - u(X)}{X} = 0
$$

Or

$$
7\left(\frac{d}{dX}u(X)\right) Xu(X) - 5\left(\frac{d}{dX}u(X)\right) X + 7u(X)^2 - 6u(X) - 1 = 0
$$

Or

$$
-1 + X(7u(X) - 5)\left(\frac{d}{dX}u(X)\right) + 7u(X)^2 - 6u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{7u^2 - 6u - 1}{X(7u-5)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{7u^2-6u-1}{7u-5}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{7u^2-6u-1}{7u-5}}\,du &= -\frac{1}{X}\,dX \\
\int \frac{1}{\frac{7u^2-6u-1}{7u-5}}\,du &= \int -\frac{1}{X}\,dX \\
\frac{\ln(u-1)}{4} + \frac{3\ln(7u+1)}{4} &= -\ln(X) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\frac{\ln(u-1) + 3\ln(7u+1)}{4} &= -\ln(X) + c_2 \\
\ln(u-1) + 3\ln(7u+1) &= (4)\left(-\ln(X) + c_2\right) \\
&= -4\ln(X) + 4c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u-1)+3\ln(7u+1)} = e^{-4\ln(X)+4c_2}
$$

Which simplifies to

$$
\begin{aligned}
(u-1)(7u+1)^3 &= \frac{4c_2}{X^4} \\
&= \frac{c_3}{X^4}
\end{aligned}
$$

Which simplifies to

$$
u(X) = \text{RootOf}\left(343\_Z^4 - 196\_Z^3 - \frac{c_3 e^{4c_2}}{X^4} - 126\_Z^2 - 20\_Z - 1\right)
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
Y(X) = X\,\text{RootOf}\left(343\_Z^4X^4 - 196\_Z^3X^4 - 126\_Z^2X^4 - c_3e^{4c_2} - 20\_ZX^4 - X^4\right)
$$

Using the solution for $Y(X)$

$$
Y(X) = X\,\text{RootOf}\left(343\_Z^4X^4 - 196\_Z^3X^4 - 126\_Z^2X^4 - c_3e^{4c_2} - 20\_ZX^4 - X^4\right)
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y + \frac{1}{2} \\
X &= x + \frac{1}{2}
\end{aligned}
$$

Then the solution in y becomes

$$
y - \frac{1}{2} = \left(x - \frac{1}{2}\right) \operatorname{RootOf}\left(\left(5488x^4 - 10976x^3 + 8232x^2 - 2744x + 343\right)\_Z^4 + \left(-3136x^4 + 6272x^3 - 47\right.
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y - \frac{1}{2} = \left(x - \frac{1}{2}\right) \operatorname{RootOf}\Big(&\left(5488x^4 - 10976x^3 + 8232x^2 - 2744x + 343\right)\_Z^4 \\
&+ \left(-3136x^4 + 6272x^3 - 4704x^2 + 1568x - 196\right)\_Z^3 \\
&+ \left(-2016x^4 + 4032x^3 - 3024x^2 + 1008x - 126\right)\_Z^2 \\
&+ \left(-320x^4 + 640x^3 - 480x^2 + 160x - 20\right)\_Z - 16c_3 e^{4c_2} - 16x^4 + 32x^3 \\
&- 24x^2 + 8x - 1\Big)
\end{aligned} \tag{1}
$$

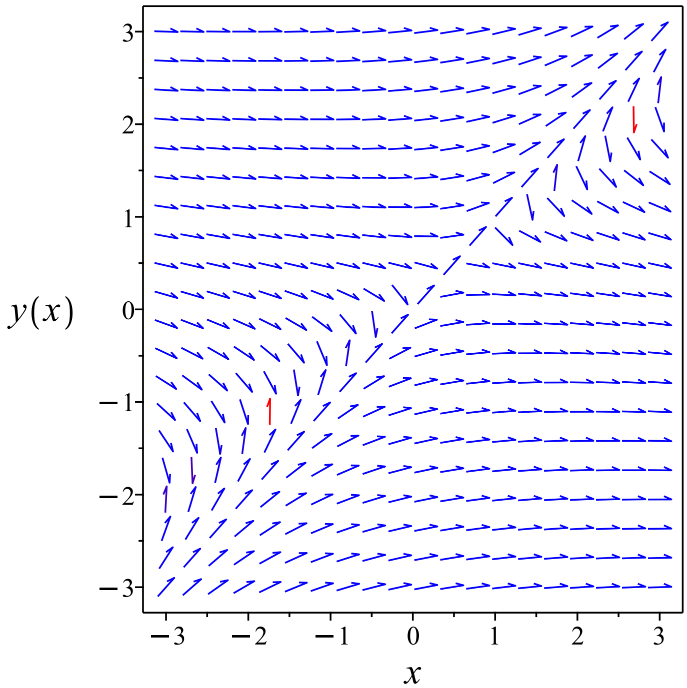


Figure 420: Slope field plot

Verification of solutions

$$y-\frac{1}{2}=\left(x-\frac{1}{2}\right)\operatorname{RootOf}\left(\left(5488x^4-10976x^3+8232x^2-2744x+343\right)\_Z^4+\left(-3136x^4+6272x^3-4704x^2+1568x-196\right)\_Z^3+\left(-2016x^4+4032x^3-3024x^2+1008x-126\right)\_Z^2+\left(-320x^4+640x^3-480x^2+160x-20\right)\_Z-16c_3e^{4c_2}-16x^4+32x^3-24x^2+8x-1\right)$$

Verified OK.

12.34.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x+y-1}{-5x+7y-1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(x+y-1)(b_3-a_2)}{-5x+7y-1} - \frac{(x+y-1)^2 a_3}{(-5x+7y-1)^2} \\
&- \left(\frac{1}{-5x+7y-1} + \frac{5x+5y-5}{(-5x+7y-1)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{1}{-5x+7y-1} - \frac{7(x+y-1)}{(-5x+7y-1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{5x^2a_2 - x^2a_3 + 37x^2b_2 - 5x^2b_3 - 14xya_2 - 2xya_3 - 70xyb_2 + 14xyb_3 - 7y^2a_2 - 13y^2a_3 + 49y^2b_2 + 7y^2b_3}{(5x}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&5x^2a_2 - x^2a_3 + 37x^2b_2 - 5x^2b_3 - 14xya_2 - 2xya_3 - 70xyb_2 + 14xyb_3 \\
&- 7y^2a_2 - 13y^2a_3 + 49y^2b_2 + 7y^2b_3 + 2xa_2 + 2xa_3 + 12xb_1 + 4xb_2 + 4xb_3 \\
&- 12ya_1 + 8ya_2 + 8ya_3 - 14yb_2 - 14yb_3 + 6a_1 - a_2 - a_3 - 6b_1 + b_2 + b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&5a_2v_1^2 - 14a_2v_1v_2 - 7a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 13a_3v_2^2 + 37b_2v_1^2 - 70b_2v_1v_2 \\
&+ 49b_2v_2^2 - 5b_3v_1^2 + 14b_3v_1v_2 + 7b_3v_2^2 - 12a_1v_2 + 2a_2v_1 + 8a_2v_2 + 2a_3v_1 + 8a_3v_2 \\
&+ 12b_1v_1 + 4b_2v_1 - 14b_2v_2 + 4b_3v_1 - 14b_3v_2 + 6a_1 - a_2 - a_3 - 6b_1 + b_2 + b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(5a_2 - a_3 + 37b_2 - 5b_3)\, v_1^2 + (-14a_2 - 2a_3 - 70b_2 + 14b_3)\, v_1v_2 \\
&+ (2a_2 + 2a_3 + 12b_1 + 4b_2 + 4b_3)\, v_1 + (-7a_2 - 13a_3 + 49b_2 + 7b_3)\, v_2^2 \\
&+ (-12a_1 + 8a_2 + 8a_3 - 14b_2 - 14b_3)\, v_2 + 6a_1 - a_2 - a_3 - 6b_1 + b_2 + b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-14a_2 - 2a_3 - 70b_2 + 14b_3 &= 0 \\
-7a_2 - 13a_3 + 49b_2 + 7b_3 &= 0 \\
5a_2 - a_3 + 37b_2 - 5b_3 &= 0 \\
-12a_1 + 8a_2 + 8a_3 - 14b_2 - 14b_3 &= 0 \\
2a_2 + 2a_3 + 12b_1 + 4b_2 + 4b_3 &= 0 \\
6a_1 - a_2 - a_3 - 6b_1 + b_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_1 \\
a_2 &= 12b_1 + 7b_3 \\
a_3 &= -14b_1 - 7b_3 \\
b_1 &= b_1 \\
b_2 &= -2b_1 - b_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 7x - 7y \\
\eta &= -x + y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -x + y - \left(\frac{x+y-1}{-5x+7y-1}\right)(7x-7y) \\
&= \frac{2x^2+12xy-14y^2-8x+8y}{5x-7y+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^2+12xy-14y^2-8x+8y}{5x-7y+1}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3\ln(x+7y-4)}{8} + \frac{\ln(-x+y)}{8}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x+y-1}{-5x+7y-1}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x+y-1}{2\left(x+7y-4\right)\left(x-y\right)} \\
S_y &= \frac{21}{8x+56y-32} - \frac{1}{8x-8y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

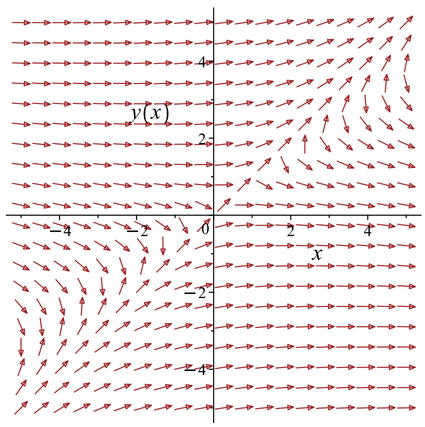
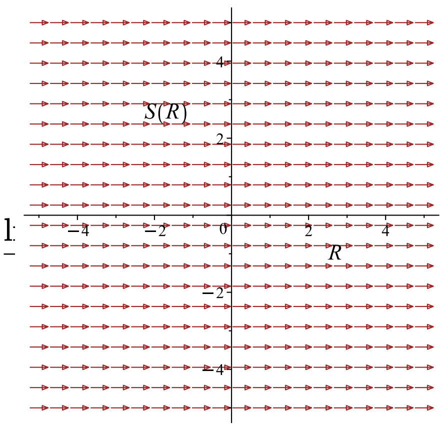
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln\left(x+7y-4\right)}{8} + \frac{\ln\left(-x+y\right)}{8} = c_1$$

Which simplifies to

$$\frac{3\ln\left(x+7y-4\right)}{8} + \frac{\ln\left(-x+y\right)}{8} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y-1}{-5x+7y-1}$ 	$R = x$ $S = \frac{3\ln(x+7y-4)}{8} + \text{l}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(x+7y-4)}{8} + \frac{\ln(-x+y)}{8} = c_1 \tag{1}$$

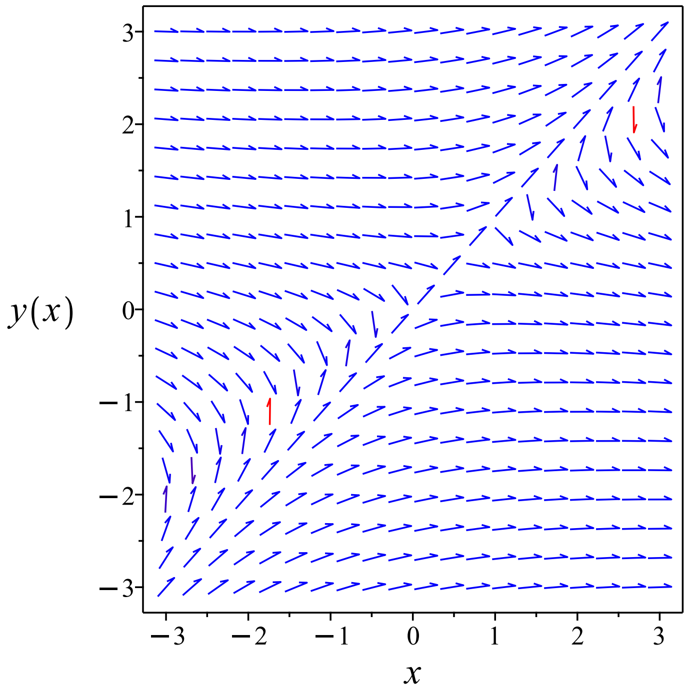


Figure 421: Slope field plot

Verification of solutions

$$\frac{3\ln\left(x+7y-4\right)}{8}+\frac{\ln\left(-x+y\right)}{8}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.734 (sec). Leaf size: 92

```
dsolve((5*x-7*y(x)+1)*diff(y(x),x)+(x+y(x)-1)=0,y(x), singsol=all)
```

$$y = \frac{-\operatorname{RootOf}\left(7\_Z^{16} + \left(-128c_1x^4 + 256c_1x^3 - 192c_1x^2 + 64c_1x - 8c_1\right)\_Z^4 - 16c_1x^4 + 32c_1x^3 - 24c_1x^2 +}{2c_1\left(2x-1\right)^3}$$

✓ Solution by Mathematica

Time used: 60.319 (sec). Leaf size: 8165

```
DSolve[(5*x-7*y[x]+1)*y'[x]+(x+y[x]-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

12.35 problem 309

Internal problem ID [15171]
Internal file name [OUTPUT/15171_Tuesday_April_23_2024_04_52_43_PM_5007905/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 309.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + 2y - 1)\, y' = -1 - x}$$

With initial conditions

$$[y(1) = 2]$$

12.35.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x + y + 1}{2x + 2y - 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\left\{x < -\frac{3}{2} \vee -\frac{3}{2} < x\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < -\frac{1}{2} \vee -\frac{1}{2} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x + y + 1}{2x + 2y - 1}\right) \\ &= -\frac{1}{2x + 2y - 1} + \frac{2x + 2y + 2}{(2x + 2y - 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\left\{x < -\frac{3}{2} \vee -\frac{3}{2} < x\right\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < -\frac{1}{2} \vee -\frac{1}{2} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

12.35.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{x + y + 1}{2x + 2y - 1} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(x+y+1)(b_3-a_2)}{2x+2y-1} - \frac{(x+y+1)^2 a_3}{(2x+2y-1)^2} \\ - \left(-\frac{1}{2x+2y-1} + \frac{2x+2y+2}{(2x+2y-1)^2}\right)(xa_2+ya_3+a_1) \\ - \left(-\frac{1}{2x+2y-1} + \frac{2x+2y+2}{(2x+2y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xa_2}{(2x+2y-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} 2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 \\ + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 - 2xa_2 - 2xa_3 - 7xb_2 - xb_3 \\ + ya_2 - 5ya_3 - 4yb_2 - 4yb_3 - 3a_1 - a_2 - a_3 - 3b_1 + b_2 + b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} 2a_2v_1^2 + 4a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 \\ + 4b_2v_2^2 - 2b_3v_1^2 - 4b_3v_1v_2 - 2b_3v_2^2 - 2a_2v_1 + a_2v_2 - 2a_3v_1 - 5a_3v_2 \\ - 7b_2v_1 - 4b_2v_2 - b_3v_1 - 4b_3v_2 - 3a_1 - a_2 - a_3 - 3b_1 + b_2 + b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_2 - a_3 + 4b_2 - 2b_3) v_1^2 + (4a_2 - 2a_3 + 8b_2 - 4b_3) v_1 v_2 \\
&+ (-2a_2 - 2a_3 - 7b_2 - b_3) v_1 + (2a_2 - a_3 + 4b_2 - 2b_3) v_2^2 \\
&+ (a_2 - 5a_3 - 4b_2 - 4b_3) v_2 - 3a_1 - a_2 - a_3 - 3b_1 + b_2 + b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_2 - 2a_3 - 7b_2 - b_3 &= 0 \\
a_2 - 5a_3 - 4b_2 - 4b_3 &= 0 \\
2a_2 - a_3 + 4b_2 - 2b_3 &= 0 \\
4a_2 - 2a_3 + 8b_2 - 4b_3 &= 0 \\
-3a_1 - a_2 - a_3 - 3b_1 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 2b_2 - b_1 \\
a_2 &= -2b_2 \\
a_3 &= -2b_2 \\
b_1 &= b_1 \\
b_2 &= b_2 \\
b_3 &= b_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(-\frac{x + y + 1}{2x + 2y - 1}\right)(-1) \\
&= \frac{x + y - 2}{2x + 2y - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+y-2}{2x+2y-1}} dy \end{aligned}$$

Which results in

$$S = 2y + 3\ln(x + y - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y + 1}{2x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3}{x + y - 2} \\ S_y &= 2 + \frac{3}{x + y - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2y + 3\ln(x + y - 2) = -x + c_1$$

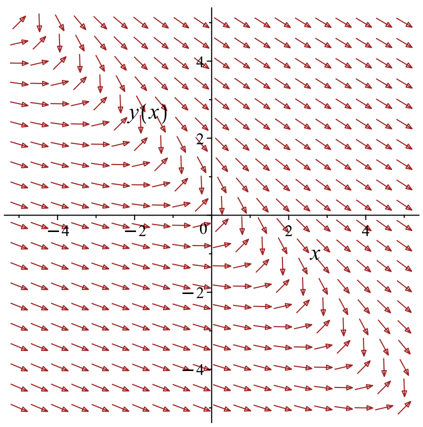
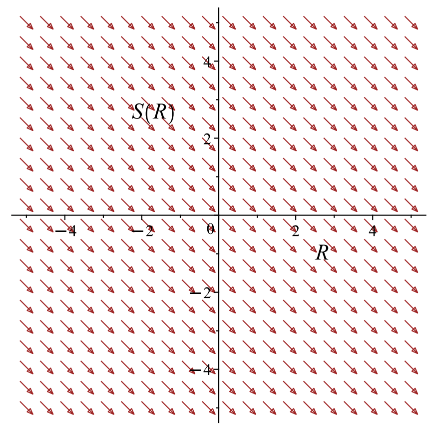
Which simplifies to

$$2y + 3\ln(x + y - 2) = -x + c_1$$

Which gives

$$y = \frac{3\,\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{c_1}{3}-\frac{4}{3}}}{3}\right)}{2} - x + 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y+1}{2x+2y-1}$ 	$R = x$ $S = 2y + 3\ln(x + y - 2)$	$\frac{dS}{dR} = -1$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{3\,\text{LambertW}\left(\frac{2\,e^{-1+\frac{c_1}{3}}}{3}\right)}{2} + 1$$

$$c_1 = 5$$

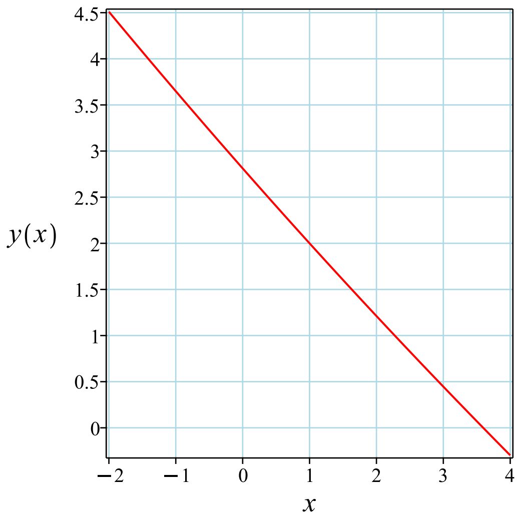
Substituting c_1 found above in the general solution gives

$$y = \frac{3\,\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{1}{3}}}{3}\right)}{2} - x + 2$$

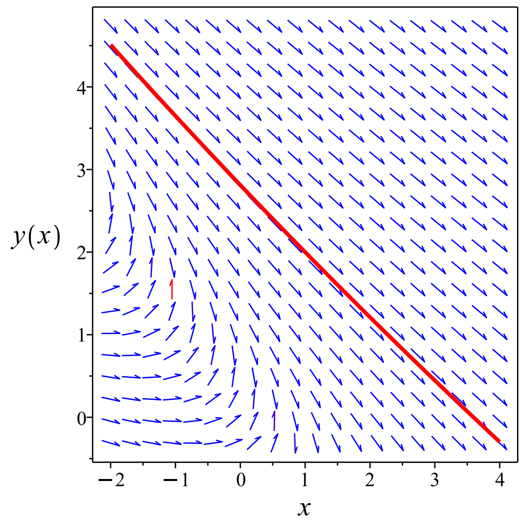
Summary

The solution(s) found are the following

$$y = \frac{3\,\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{1}{3}}}{3}\right)}{2} - x + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{3}\right)}{2} - x + 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 20

```
dsolve([(x+y(x)+1)+(2*x+2*y(x)-1)*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y = -x + \frac{3\,\mathrm{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{1}{3}+\frac{x}{3}}}{3}\right)}{2} + 2$$

✓ Solution by Mathematica

Time used: 3.539 (sec). Leaf size: 28

```
DSolve[{(x+y[x]+1)+(2*x+2*y[x]-1)*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{3}{2}W\left(\frac{2}{3}e^{\frac{x+1}{3}}\right) - x + 2$$

12.36 problem 310

Internal problem ID [15172]
Internal file name [`OUTPUT/15172_Tuesday_April_23_2024_04_52_45_PM_17898937/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 310.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y^3 + 2(x^2 - y^2x)\, y' = 0}$$

12.36.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^3}{2x\,(y^2 - x)} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y^3(b_3 - a_2)}{2x\,(y^2 - x)} - \frac{y^6 a_3}{4x^2\,(y^2 - x)^2} \\ & - \left(-\frac{y^3}{2x^2\,(y^2 - x)} + \frac{y^3}{2x\,(y^2 - x)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(\frac{3y^2}{2x\,(y^2 - x)} - \frac{y^4}{x\,(y^2 - x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2y^4b_2 + y^6a_3 - 2x^3y^2b_2 - 2x^2y^3a_2 + 4x^2y^3b_3 - 4x\,y^4a_3 - 2x\,y^4b_1 + 2y^5a_1 + 4x^4b_2 + 6x^2y^2b_1 - 4x\,y^3a_1}{4x^2\,(-y^2 + x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^2y^4b_2 + y^6a_3 - 2x^3y^2b_2 - 2x^2y^3a_2 + 4x^2y^3b_3 - 4x\,y^4a_3 \\ & - 2x\,y^4b_1 + 2y^5a_1 + 4x^4b_2 + 6x^2y^2b_1 - 4x\,y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & a_3v_2^6 + 2b_2v_1^2v_2^4 + 2a_1v_2^5 - 2a_2v_1^2v_2^3 - 4a_3v_1v_2^4 - 2b_1v_1v_2^4 \\ & - 2b_2v_1^3v_2^2 + 4b_3v_1^2v_2^3 - 4a_1v_1v_2^3 + 6b_1v_1^2v_2^2 + 4b_2v_1^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^4 - 2b_2v_1^3v_2^2 + 2b_2v_1^2v_2^4 + (-2a_2 + 4b_3)\, v_1^2v_2^3 + 6b_1v_1^2v_2^2 + (-4a_3 - 2b_1)\, v_1v_2^4 - 4a_1v_1v_2^3 + a_3v_2^6 + 2a_1v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_3 &= 0 \\
-4a_1 &= 0 \\
2a_1 &= 0 \\
6b_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
4b_2 &= 0 \\
-2a_2 + 4b_3 &= 0 \\
-4a_3 - 2b_1 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^3}{2x\,(y^2-x)}\right)(2x) \\
&= \frac{xy}{-y^2+x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{xy}{-y^2+x}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{y^2}{2x} + \ln(y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y^3}{2x\,(y^2-x)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y^2}{2x^2} \\
S_y &= \frac{-y^2 + x}{xy}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2 \ln (y)\, x - y^2}{2x} = c_1
$$

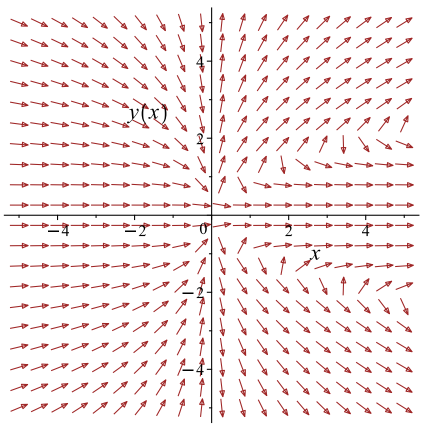
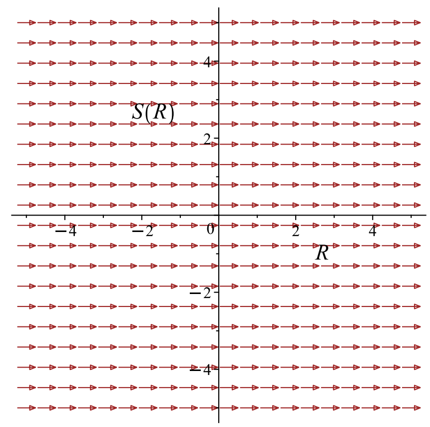
Which simplifies to

$$
\frac{2 \ln (y)\, x - y^2}{2x} = c_1
$$

Which gives

$$
y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{2c_1}}{x}\right)}{2} + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^3}{2x(y^2-x)}$ 	$R = x$ $S = \frac{2\ln(y)\,x - y^2}{2x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{2c_1}}{x}\right)}{2}+c_1} \tag{1}$$

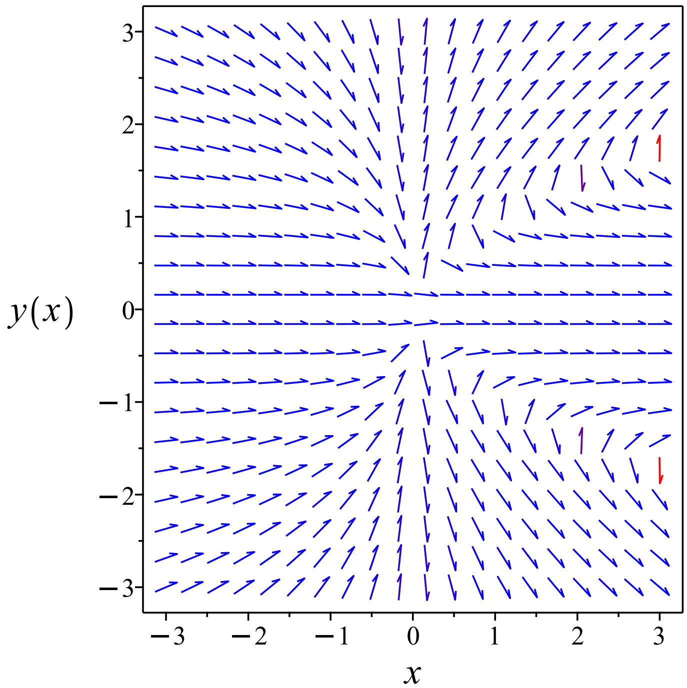


Figure 423: Slope field plot

Verification of solutions

$$y = e^{-\frac{\text{LambertW}\left(-\frac{e^{2c_1}}{x}\right)}{2}+c_1}$$

Verified OK.

12.36.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-2x\, y^2 + 2x^2\right) \mathrm{d}y &= \left(-y^3\right) \mathrm{d}x \\
\left(y^3\right) \mathrm{d}x + \left(-2x\, y^2 + 2x^2\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y^3 \\
N(x, y) &= -2x\, y^2 + 2x^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^3\right) \\
&= 3y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-2x\, y^2 + 2x^2\right) \\
&= -2y^2 + 4x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = y^3$ and $N = 2x^2 - 2y^2 x$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{y^2}{x^2} \\
N &= \frac{2x^2 - 2y^2 x}{x^2 y}
\end{aligned}
$$

To solve an ode of the form

$$
M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx} \phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{-2x\,y^{2}+2x^{2}}{x^{2}y}\right)\mathrm{d}y = \left(-\frac{y^{2}}{x^{2}}\right)\mathrm{d}x$$

$$\left(\frac{y^{2}}{x^{2}}\right)\mathrm{d}x + \left(\frac{-2x\,y^{2}+2x^{2}}{x^{2}y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \frac{y^{2}}{x^{2}}\\
N(x,y) &= \frac{-2x\,y^{2}+2x^{2}}{x^{2}y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^{2}}{x^{2}}\right)\\
&= \frac{2y}{x^{2}}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-2x\,y^{2}+2x^{2}}{x^{2}y}\right)\\
&= \frac{2y}{x^{2}}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{y^{2}}{x^{2}}\,\mathrm{d}x$$

$$\phi = -\frac{y^{2}}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2x\,y^2+2x^2}{x^2 y}$. Therefore equation (4) becomes

$$\frac{-2x\,y^2+2x^2}{x^2 y} = -\frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{x} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{x} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)}{2}+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)}{2}+\frac{c_1}{2}} \tag{1}$$

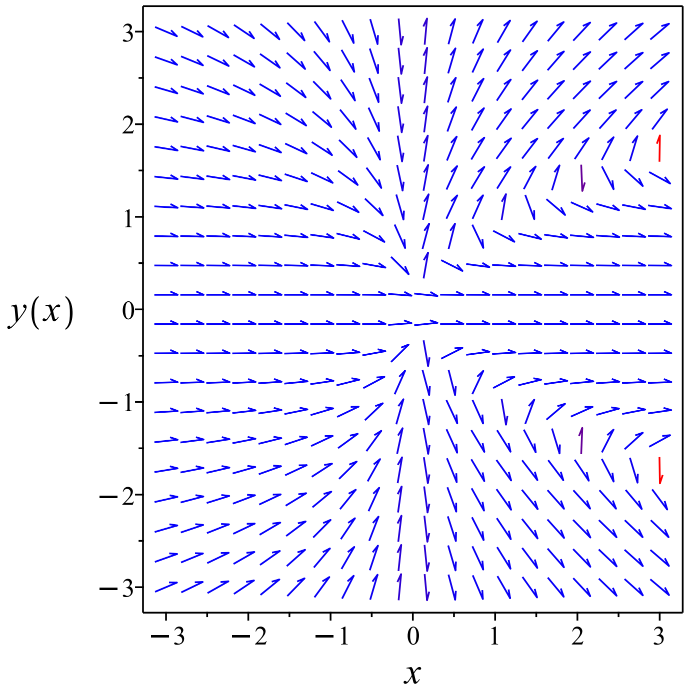


Figure 424: Slope field plot

Verification of solutions

$$y = e^{-\frac{\mathrm{LambertW}\left(-\frac{e^{c_1}}{x}\right)}{2}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(y(x)^3+2*(x^2-x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y = \frac{e^{\frac{c_1}{2}}}{\sqrt{-\frac{e^{c_1}}{x \operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)}}}$$

✓ Solution by Mathematica

Time used: 2.795 (sec). Leaf size: 60

```
DSolve[y[x]^3+2*(x^2-x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -i\sqrt{x}\sqrt{W\left(-\frac{e^{c_1}}{x}\right)}$$

$$y(x) \to i\sqrt{x}\sqrt{W\left(-\frac{e^{c_1}}{x}\right)}$$

$$y(x) \to 0$$

12.37 problem 311

Internal problem ID [15173]
Internal file name [`OUTPUT/15173_Tuesday_April_23_2024_04_52_47_PM_36121211/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 311.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational]
```

$$\boxed{y' - \frac{2(y+2)^2}{(x+y-1)^2} = 0}$$

12.37.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2(Y(X)+y_0+2)^2}{(X+x_0+Y(X)+y_0-1)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 3$$
$$y_0 = -2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)^2}{X^2+2XY(X)+Y(X)^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{2Y^2}{X^2+2XY+Y^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 2Y^2$ and $N = X^2+2XY+Y^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u^2}{(u+1)^2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)^2}{(u(X)+1)^2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)^2}{(u(X)+1)^2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)u(X)^2 X + 2\left(\frac{d}{dX}u(X)\right)u(X)\,X + u(X)^3 + \left(\frac{d}{dX}u(X)\right)X + u(X) = 0$$

Or

$$X(u(X)+1)^2\left(\frac{d}{dX}u(X)\right) + u(X)^3 + u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u(u^2+1)}{X\,(u+1)^2} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u(u^2+1)}{(u+1)^2}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{(u+1)^2}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{u(u^2+1)}{(u+1)^2}} du = \int -\frac{1}{X} dX$$

$$2\arctan(u) + \ln(u) = -\ln(X) + c_2$$

The solution is

$$2\arctan(u(X)) + \ln(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$2\arctan\left(\frac{Y(X)}{X}\right) + \ln\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$2\arctan\left(\frac{Y(X)}{X}\right) + \ln\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x + 3$$

Then the solution in y becomes

$$2\arctan\left(\frac{y+2}{x-3}\right) + \ln\left(\frac{y+2}{x-3}\right) + \ln(x-3) - c_2 = 0$$

Summary

The solution(s) found are the following

$$2\arctan\left(\frac{y+2}{x-3}\right) + \ln\left(\frac{y+2}{x-3}\right) + \ln(x-3) - c_2 = 0 \tag{1}$$

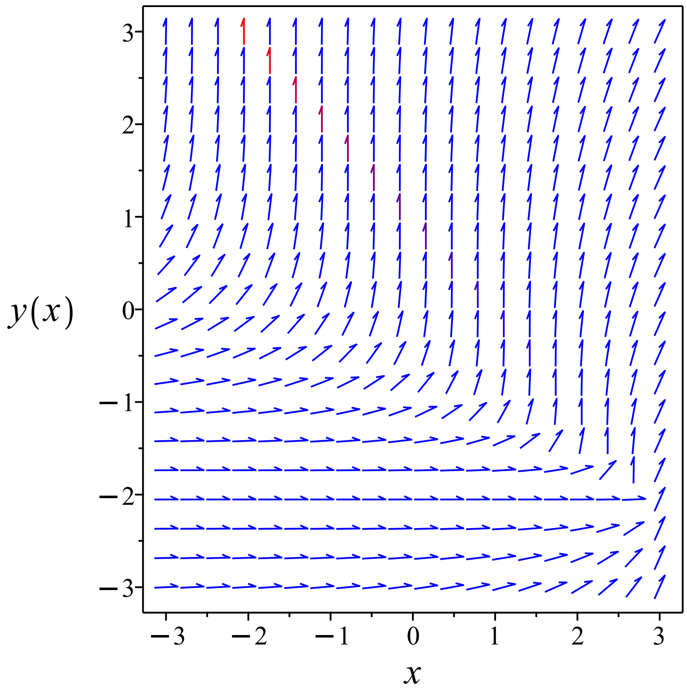


Figure 425: Slope field plot

Verification of solutions

$$2\arctan\left(\frac{y+2}{x-3}\right)+\ln\left(\frac{y+2}{x-3}\right)+\ln(x-3)-c_2=0$$

Verified OK.

12.37.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{2(y+2)^2}{(x+y-1)^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{2(y+2)^2 (b_3 - a_2)}{(x+y-1)^2} - \frac{4(y+2)^4 a_3}{(x+y-1)^4} + \frac{4(y+2)^2 (xa_2 + ya_3 + a_1)}{(x+y-1)^3} - \left(\frac{4y+8}{(x+y-1)^2} - \frac{4(y+2)^2}{(x+y-1)^3}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4b_2 + 2x^2y^2a_2 + 2x^2y^2b_2 - 2x^2y^2b_3 + 4x\,y^3a_3 + 4x\,y^3b_2 - 2y^4a_2 + y^4b_2 + 2y^4b_3 - 12x^3b_2 + 8x^2ya_2 - 4x^2}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&x^4b_2 + 2x^2y^2a_2 + 2x^2y^2b_2 - 2x^2y^2b_3 + 4x\,y^3a_3 + 4x\,y^3b_2 - 2y^4a_2 \\
&+ y^4b_2 + 2y^4b_3 - 12x^3b_2 + 8x^2ya_2 - 4x^2yb_1 - 4x^2yb_2 + 4x\,y^2a_1 \\
&+ 16x\,y^2a_3 - 4x\,y^2b_1 + 20x\,y^2b_3 + 4y^3a_1 - 4y^3a_2 - 20y^3a_3 - 4y^3b_2 \\
&+ 16y^3b_3 + 8x^2a_2 - 8x^2b_1 + 38x^2b_2 + 8x^2b_3 + 16xya_1 + 16xya_3 \\
&+ 8xyb_1 + 24xyb_2 + 32xyb_3 + 12y^2a_1 + 6y^2a_2 - 96y^2a_3 + 12y^2b_1 \\
&+ 6y^2b_2 + 6y^2b_3 + 16xa_1 + 32xb_1 - 28xb_2 - 16xb_3 + 8ya_2 - 144ya_3 \\
&+ 12yb_1 - 4yb_2 - 32yb_3 - 16a_1 - 8a_2 - 64a_3 - 24b_1 + b_2 + 8b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2a_2v_1^2v_2^2 - 2a_2v_2^4 + 4a_3v_1v_2^3 + b_2v_1^4 + 2b_2v_1^2v_2^2 + 4b_2v_1v_2^3 + b_2v_2^4 - 2b_3v_1^2v_2^2 \\
&+ 2b_3v_2^4 + 4a_1v_1v_2^2 + 4a_1v_2^3 + 8a_2v_1^2v_2 - 4a_2v_2^3 + 16a_3v_1v_2^2 - 20a_3v_2^3 \\
&- 4b_1v_1^2v_2 - 4b_1v_1v_2^2 - 12b_2v_1^3 - 4b_2v_1^2v_2 - 4b_2v_2^3 + 20b_3v_1v_2^2 + 16b_3v_2^3 \\
&+ 16a_1v_1v_2 + 12a_1v_2^2 + 8a_2v_1^2 + 6a_2v_2^2 + 16a_3v_1v_2 - 96a_3v_2^2 - 8b_1v_1^2 \\
&+ 8b_1v_1v_2 + 12b_1v_2^2 + 38b_2v_1^2 + 24b_2v_1v_2 + 6b_2v_2^2 + 8b_3v_1^2 + 32b_3v_1v_2 \\
&+ 6b_3v_2^2 + 16a_1v_1 + 8a_2v_2 - 144a_3v_2 + 32b_1v_1 + 12b_1v_2 - 28b_2v_1 \\
&- 4b_2v_2 - 16b_3v_1 - 32b_3v_2 - 16a_1 - 8a_2 - 64a_3 - 24b_1 + b_2 + 8b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&b_2v_1^4 - 12b_2v_1^3 + (2a_2 + 2b_2 - 2b_3)\,v_1^2v_2^2 \\
&+ (8a_2 - 4b_1 - 4b_2)\,v_1^2v_2 + (8a_2 - 8b_1 + 38b_2 + 8b_3)\,v_1^2 \\
&+ (4a_3 + 4b_2)\,v_1v_2^3 + (4a_1 + 16a_3 - 4b_1 + 20b_3)\,v_1v_2^2 \\
&+ (16a_1 + 16a_3 + 8b_1 + 24b_2 + 32b_3)\,v_1v_2 + (16a_1 + 32b_1 - 28b_2 - 16b_3)\,v_1 \\
&+ (-2a_2 + b_2 + 2b_3)\,v_2^4 + (4a_1 - 4a_2 - 20a_3 - 4b_2 + 16b_3)\,v_2^3 \\
&+ (12a_1 + 6a_2 - 96a_3 + 12b_1 + 6b_2 + 6b_3)\,v_2^2 \\
&+ (8a_2 - 144a_3 + 12b_1 - 4b_2 - 32b_3)\,v_2 \\
&- 16a_1 - 8a_2 - 64a_3 - 24b_1 + b_2 + 8b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-12b_2 &= 0 \\
4a_3 + 4b_2 &= 0 \\
-2a_2 + b_2 + 2b_3 &= 0 \\
2a_2 + 2b_2 - 2b_3 &= 0 \\
8a_2 - 4b_1 - 4b_2 &= 0 \\
4a_1 + 16a_3 - 4b_1 + 20b_3 &= 0 \\
16a_1 + 32b_1 - 28b_2 - 16b_3 &= 0 \\
8a_2 - 8b_1 + 38b_2 + 8b_3 &= 0 \\
4a_1 - 4a_2 - 20a_3 - 4b_2 + 16b_3 &= 0 \\
16a_1 + 16a_3 + 8b_1 + 24b_2 + 32b_3 &= 0 \\
8a_2 - 144a_3 + 12b_1 - 4b_2 - 32b_3 &= 0 \\
-16a_1 - 8a_2 - 64a_3 - 24b_1 + b_2 + 8b_3 &= 0 \\
12a_1 + 6a_2 - 96a_3 + 12b_1 + 6b_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -3b_3 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 2b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x - 3 \\
\eta &= y + 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 2 - \left(\frac{2(y+2)^2}{(x+y-1)^2}\right)(x-3) \\
&= \frac{x^2y + y^3 + 2x^2 - 6xy + 6y^2 - 12x + 21y + 26}{x^2 + 2xy + y^2 - 2x - 2y + 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2y+y^3+2x^2-6xy+6y^2-12x+21y+26}{x^2+2xy+y^2-2x-2y+1}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(y+2) + 2\arctan\left(\frac{2y+4}{2x-6}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{2(y+2)^2}{(x+y-1)^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-2y - 4}{x^2 + y^2 - 6x + 4y + 13} \\
S_y &= \frac{(x + y - 1)^2}{(y + 2)(x^2 + y^2 - 6x + 4y + 13)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

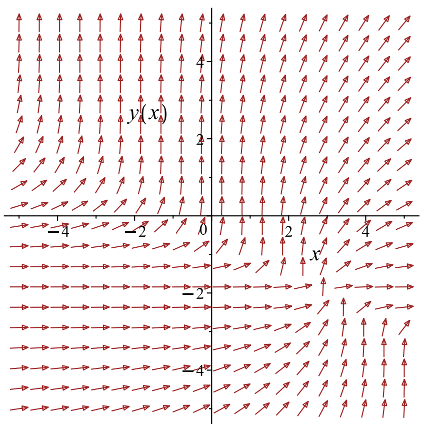
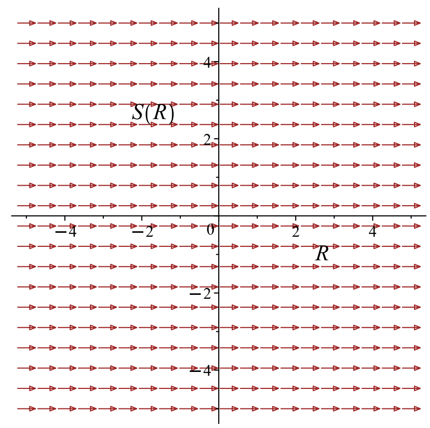
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(y + 2) + 2 \arctan\left(\frac{y + 2}{x - 3}\right) = c_1
$$

Which simplifies to

$$
\ln(y + 2) + 2 \arctan\left(\frac{y + 2}{x - 3}\right) = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2(y+2)^2}{(x+y-1)^2}$ 	$R = x$ $S = \ln(y+2) + 2\,\mathrm{arctan}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\ln(y+2) + 2\arctan\left(\frac{y+2}{x-3}\right) = c_1 \tag{1}$$

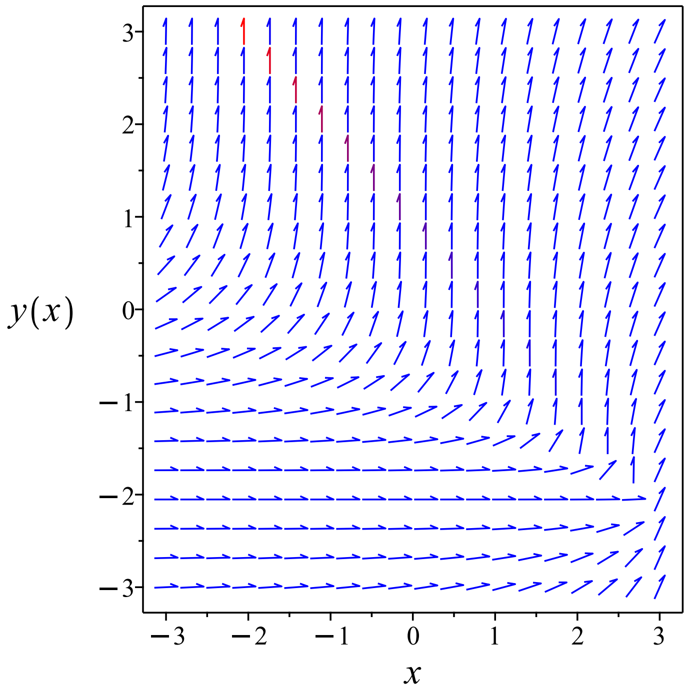


Figure 426: Slope field plot

Verification of solutions

$$\ln(y+2) + 2\arctan\left(\frac{y+2}{x-3}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x)=2*(  (y(x)+2)/(x+y(x)-1) )^2,y(x), singsol=all)
```

$$y = -2 + (-x + 3) \tan \left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\tan\left(\_Z\right)\right) + \ln\left(x - 3\right) + c_1\right)\right)$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 27

```
DSolve[y'[x]==2*(  (y[x]+2)/(x+y[x]-1) )^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2\arctan\left(\frac{3 - x}{y(x) + 2}\right) + \log(y(x) + 2) = c_1, y(x)\right]$$

12.38 problem 312

Internal problem ID [15174]
Internal file name [`OUTPUT/15174_Tuesday_April_23_2024_04_52_49_PM_90769389/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 312.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{4x^2y'^2 - y^2 - y^3x = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{yx+1}\, y}{2x} \tag{1}$$

$$y' = -\frac{\sqrt{yx+1}\, y}{2x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\sqrt{xy+1}\, y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\sqrt{xy+1}\, y(b_3 - a_2)}{2x} - \frac{(xy+1)\, y^2 a_3}{4x^2} \\ &- \left(\frac{y^2}{4\sqrt{xy+1}\, x} - \frac{\sqrt{xy+1}\, y}{2x^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{y}{4\sqrt{xy+1}} + \frac{\sqrt{xy+1}}{2x}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{-\sqrt{xy+1}\, x\, y^3 a_3 - 3x^3 y b_2 - x^2 y^2 a_2 - x^2 y^2 b_3 + x\, y^3 a_3 + 4 b_2 x^2 \sqrt{xy+1} - y^2 a_3 \sqrt{xy+1} - 3x^2 y b_1 + x\, y^2 a_{}}{4x^2 \sqrt{xy+1}} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\sqrt{xy+1}\, x\, y^3 a_3 - 3x^3 y b_2 - x^2 y^2 a_2 - x^2 y^2 b_3 + x\, y^3 a_3 + 4 b_2 x^2 \sqrt{xy+1} \\ &- y^2 a_3 \sqrt{xy+1} - 3x^2 y b_1 + x\, y^2 a_1 - 2x^2 b_2 + 2y^2 a_3 - 2x b_1 + 2y a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-\sqrt{xy+1}\, x\, y^3 a_3 - 2(xy+1)\, x^2 b_2 + 2(xy+1)\, y^2 a_3 - x^3 y b_2 \\ &- x^2 y^2 a_2 - x^2 y^2 b_3 - x\, y^3 a_3 - 2(xy+1)\, x b_1 + 2(xy+1)\, y a_1 \\ &+ 4 b_2 x^2 \sqrt{xy+1} - y^2 a_3 \sqrt{xy+1} - x^2 y b_1 - x\, y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-\sqrt{xy+1}\,x\,y^{3}a_{3}-3x^{3}yb_{2}-x^{2}y^{2}a_{2}-x^{2}y^{2}b_{3}+x\,y^{3}a_{3}+4b_{2}x^{2}\sqrt{xy+1}-y^{2}a_{3}\sqrt{xy+1}-3x^{2}yb_{1}+x\,y^{2}a_{1}-2x^{2}b_{2}+2y^{2}a_{3}-2xb_{1}+2ya_{1}=0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{xy+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_{1}, y=v_{2}, \sqrt{xy+1}=v_{3}\right\}$$

The above PDE (6E) now becomes

$$-v_{3}v_{1}v_{2}^{3}a_{3}-v_{1}^{2}v_{2}^{2}a_{2}+v_{1}v_{2}^{3}a_{3}-3v_{1}^{3}v_{2}b_{2}-v_{1}^{2}v_{2}^{2}b_{3}+v_{1}v_{2}^{2}a_{1}-v_{2}^{2}a_{3}v_{3}-3v_{1}^{2}v_{2}b_{1}+4b_{2}v_{1}^{2}v_{3}+2v_{2}^{2}a_{3}-2v_{1}^{2}b_{2}+2v_{2}a_{1}-2v_{1}b_{1}=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_{1}, v_{2}, v_{3}\}$$

Equation (7E) now becomes

$$-3v_{1}^{3}v_{2}b_{2}+(-a_{2}-b_{3})\,v_{1}^{2}v_{2}^{2}-3v_{1}^{2}v_{2}b_{1}+4b_{2}v_{1}^{2}v_{3}-2v_{1}^{2}b_{2}-v_{3}v_{1}v_{2}^{3}a_{3}+v_{1}v_{2}^{3}a_{3}+v_{1}v_{2}^{2}a_{1}-2v_{1}b_{1}-v_{2}^{2}a_{3}v_{3}+2v_{2}^{2}a_{3}+2v_{2}a_{1}=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
2a_1 &= 0 \\
-a_3 &= 0 \\
2a_3 &= 0 \\
-3b_1 &= 0 \\
-2b_1 &= 0 \\
-3b_2 &= 0 \\
-2b_2 &= 0 \\
4b_2 &= 0 \\
-a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{\sqrt{xy+1}\, y}{2x}\right)(-x) \\
&= y + \frac{y\sqrt{xy+1}}{2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y + \frac{y\sqrt{xy+1}}{2}} dy \end{aligned}$$

Which results in

$$S = -\frac{8 \ln\left(\sqrt{xy+1}+2\right)}{3} + 2 \ln\left(\sqrt{xy+1}+1\right) + \frac{2 \ln\left(\sqrt{xy+1}-1\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\sqrt{xy+1}\, y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{\left(\sqrt{xy+1}+2\right) x} \\ S_y &= \frac{2}{y\left(\sqrt{xy+1}+2\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{8\ln\left(\sqrt{yx+1}+2\right)}{3} + 2\ln\left(\sqrt{yx+1}+1\right) + \frac{2\ln\left(\sqrt{yx+1}-1\right)}{3} = \ln(x) + c_1$$

Which simplifies to

$$-\frac{8\ln\left(\sqrt{yx+1}+2\right)}{3} + 2\ln\left(\sqrt{yx+1}+1\right) + \frac{2\ln\left(\sqrt{yx+1}-1\right)}{3} = \ln(x) + c_1$$

Summary

The solution(s) found are the following

$$-\frac{8\ln\left(\sqrt{yx+1}+2\right)}{3} + 2\ln\left(\sqrt{yx+1}+1\right) + \frac{2\ln\left(\sqrt{yx+1}-1\right)}{3} = \ln(x) + c_1 \tag{1}$$

Verification of solutions

$$-\frac{8\ln\left(\sqrt{yx+1}+2\right)}{3} + 2\ln\left(\sqrt{yx+1}+1\right) + \frac{2\ln\left(\sqrt{yx+1}-1\right)}{3} = \ln(x) + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned} y' &= -\frac{\sqrt{xy+1}\,y}{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\sqrt{xy+1}\, y(b_3 - a_2)}{2x} - \frac{(xy+1)\, y^2 a_3}{4x^2} \\ &- \left(-\frac{y^2}{4\sqrt{xy+1}\, x} + \frac{\sqrt{xy+1}\, y}{2x^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{y}{4\sqrt{xy+1}} - \frac{\sqrt{xy+1}}{2x}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{-y^3 a_3 \sqrt{xy+1}\, x + 3x^3 y b_2 + x^2 y^2 a_2 + x^2 y^2 b_3 - x\, y^3 a_3 + 4 b_2 x^2 \sqrt{xy+1} - y^2 a_3 \sqrt{xy+1} + 3x^2 y b_1 - x\, y^2 a_1}{4x^2 \sqrt{xy+1}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-y^3 a_3 \sqrt{xy+1}\, x + 3x^3 y b_2 + x^2 y^2 a_2 + x^2 y^2 b_3 - x\, y^3 a_3 + 4 b_2 x^2 \sqrt{xy+1} \\ &- y^2 a_3 \sqrt{xy+1} + 3x^2 y b_1 - x\, y^2 a_1 + 2x^2 b_2 - 2y^2 a_3 + 2x b_1 - 2y a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-y^3 a_3 \sqrt{xy+1}\, x + 2(xy+1)\, x^2 b_2 - 2(xy+1)\, y^2 a_3 + x^3 y b_2 \\ &+ x^2 y^2 a_2 + x^2 y^2 b_3 + x\, y^3 a_3 + 2(xy+1)\, x b_1 - 2(xy+1)\, y a_1 \\ &+ 4 b_2 x^2 \sqrt{xy+1} - y^2 a_3 \sqrt{xy+1} + x^2 y b_1 + x\, y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$-y^3 a_3 \sqrt{xy+1}\, x + 3x^3 y b_2 + x^2 y^2 a_2 + x^2 y^2 b_3 - x\, y^3 a_3 + 4 b_2 x^2 \sqrt{xy+1} - y^2 a_3 \sqrt{xy+1} + 3x^2 y b_1 - x\, y^2 a_1 + 2x^2 b_2 - 2y^2 a_3 + 2x b_1 - 2y a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{xy+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{xy+1} = v_3\right\}$$

The above PDE (6E) now becomes

$$-v_2^3 a_3 v_3 v_1 + v_1^2 v_2^2 a_2 - v_1 v_2^3 a_3 + 3v_1^3 v_2 b_2 + v_1^2 v_2^2 b_3 - v_1 v_2^2 a_1 - v_2^2 a_3 v_3 + 3v_1^2 v_2 b_1 + 4 b_2 v_1^2 v_3 - 2v_2^2 a_3 + 2v_1^2 b_2 - 2v_2 a_1 + 2v_1 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$3v_1^3 v_2 b_2 + (a_2 + b_3)\, v_1^2 v_2^2 + 3v_1^2 v_2 b_1 + 4 b_2 v_1^2 v_3 + 2v_1^2 b_2 - v_2^3 a_3 v_3 v_1 - v_1 v_2^3 a_3 - v_1 v_2^2 a_1 + 2v_1 b_1 - v_2^2 a_3 v_3 - 2v_2^2 a_3 - 2v_2 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0\\
-a_1 &= 0\\
-2a_3 &= 0\\
-a_3 &= 0\\
2b_1 &= 0\\
3b_1 &= 0\\
2b_2 &= 0\\
3b_2 &= 0\\
4b_2 &= 0\\
a_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{\sqrt{xy+1}\,y}{2x}\right)(-x) \\
&= y - \frac{y\sqrt{xy+1}}{2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y - \frac{y\sqrt{xy+1}}{2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{2\ln\left(\sqrt{xy+1}+1\right)}{3} + 2\ln\left(\sqrt{xy+1}-1\right) - \frac{8\ln\left(\sqrt{xy+1}-2\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{xy+1}\,y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{\left(\sqrt{xy+1}-2\right)x} \\ S_y &= -\frac{2}{y\left(\sqrt{xy+1}-2\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln\left(\sqrt{yx+1}+1\right)}{3} + 2\ln\left(\sqrt{yx+1}-1\right) - \frac{8\ln\left(\sqrt{yx+1}-2\right)}{3} = \ln(x) + c_1$$

Which simplifies to

$$\frac{2\ln\left(\sqrt{yx+1}+1\right)}{3}+2\ln\left(\sqrt{yx+1}-1\right)-\frac{8\ln\left(\sqrt{yx+1}-2\right)}{3}=\ln\left(x\right)+c_1$$

Summary

The solution(s) found are the following

$$\frac{2\ln\left(\sqrt{yx+1}+1\right)}{3}+2\ln\left(\sqrt{yx+1}-1\right)-\frac{8\ln\left(\sqrt{yx+1}-2\right)}{3}=\ln\left(x\right)+c_1 \quad (1)$$

Verification of solutions

$$\frac{2\ln\left(\sqrt{yx+1}+1\right)}{3}+2\ln\left(\sqrt{yx+1}-1\right)-\frac{8\ln\left(\sqrt{yx+1}-2\right)}{3}=\ln\left(x\right)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   <- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.187 (sec). Leaf size: 1759

```
dsolve(4*x^2*diff(y(x),x)^2-y(x)^2=x*y(x)^3,y(x), singsol=all)
```

$$y = 0$$

Expression too large to display

Expression too large to display

✓ Solution by Mathematica

Time used: 105.55 (sec). Leaf size: 1401

```
DSolve[4*x^2*y'[x]^2-y[x]^2==x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,1\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,2\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,3\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,4\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,5\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,6\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,7\right]$$

$$y(x) \to \text{Root}\left[\#1^8\left(x^9-2e^{3c_1}x^6+e^{6c_1}x^3\right)+\#1^7\left(-24x^8-120e^{3c_1}x^5\right)+\#1^6\left(252x^7-444e^{3c_1}x^4\right)+\#1^5\left(-1512x^6+56e^{3c_1}x^3\right)+\#1^4\left(5670x^5-66e^{3c_1}x^2\right)+\#1^3\left(-13608x^4+48e^{3c_1}x\right)+\#1^2\left(20412x^3-16e^{3c_1}\right)-17496\#1x^2+6561x\&,8\right]$$

12.39 problem 313

Internal problem ID [15175]
Internal file name [OUTPUT/15175_Tuesday_April_23_2024_04_53_32_PM_38247935/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Section 12. Miscellaneous problems. Exercises page 93
Problem number: 313.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[_rational, _dAlembert]
```

$$\boxed{y' + xy'^2 - y = 0}$$

12.39.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$x\,p^2 + p - y = 0$$

Solving for y from the above results in

$$y = x\,p^2 + p \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2$$
$$g = p$$

Hence (2) becomes

$$-p^2 + p = (2xp + 1)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving for p from the above gives

$$p = 0$$
$$p = 1$$

Substituting these in (1A) gives

$$y = 0$$
$$y = 1 + x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\, x + 1} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{2x(p)\, p + 1}{-p^2 + p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{p - 1}$$
$$q(p) = -\frac{1}{p\,(p - 1)}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p-1} = -\frac{1}{p\,(p-1)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{2}{p-1}dp}\\ &= (p-1)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)\left(-\frac{1}{p\,(p-1)}\right)\\ \frac{\mathrm{d}}{\mathrm{d}p}\left((p-1)^2\,x\right) &= \left((p-1)^2\right)\left(-\frac{1}{p\,(p-1)}\right)\\ \mathrm{d}\left((p-1)^2\,x\right) &= \left(\frac{-p+1}{p}\right)\,\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}(p-1)^2\,x &= \int \frac{-p+1}{p}\,\mathrm{d}p\\ (p-1)^2\,x &= -p + \ln(p) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (p-1)^2$ results in

$$x(p) = \frac{-p+\ln(p)}{(p-1)^2} + \frac{c_1}{(p-1)^2}$$

which simplifies to

$$x(p) = \frac{-p+\ln(p)+c_1}{(p-1)^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{-1+\sqrt{1+4yx}}{2x}\\ p &= -\frac{1+\sqrt{1+4yx}}{2x}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$x = -\frac{2\left(2\ln \left(2\right) x -2 \ln \left(\frac{-1+\sqrt{1+4yx}}{x}\right) x -2 c_1 x +\sqrt{1+4yx}-1\right) x}{\left(-1+\sqrt{1+4yx}-2x \right)^{2}}$$

$$x = \frac{4 x^{2} \ln \left(\frac{-1-\sqrt{1+4yx}}{x}\right)+2 \left(2 c_1 x -2 \ln \left(2\right) x +\sqrt{1+4yx}+1\right) x}{\left(1+\sqrt{1+4yx}+2x \right)^{2}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = 1+x \tag{2}$$

$$x = -\frac{2\left(2\ln \left(2\right) x -2 \ln \left(\frac{-1+\sqrt{1+4yx}}{x}\right) x -2 c_1 x +\sqrt{1+4yx}-1\right) x}{\left(-1+\sqrt{1+4yx}-2x \right)^{2}} \tag{3}$$

$$x = \frac{4 x^{2} \ln \left(\frac{-1-\sqrt{1+4yx}}{x}\right)+2 \left(2 c_1 x -2 \ln \left(2\right) x +\sqrt{1+4yx}+1\right) x}{\left(1+\sqrt{1+4yx}+2x \right)^{2}} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = 1+x$$

Verified OK.

$$x = -\frac{2\left(2\ln \left(2\right) x -2 \ln \left(\frac{-1+\sqrt{1+4yx}}{x}\right) x -2 c_1 x +\sqrt{1+4yx}-1\right) x}{\left(-1+\sqrt{1+4yx}-2x \right)^{2}}$$

Verified OK.

$$x = \frac{4 x^{2} \ln \left(\frac{-1-\sqrt{1+4yx}}{x}\right)+2 \left(2 c_1 x -2 \ln \left(2\right) x +\sqrt{1+4yx}+1\right) x}{\left(1+\sqrt{1+4yx}+2x \right)^{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 59

```
dsolve(diff(y(x),x)+x*diff(y(x),x)^2-y(x)=0,y(x), singsol=all)
```

$$y = 2\,e^{\operatorname{RootOf}\left(-x\,e^{2\_Z}+2\,e^{\_Z}x+\_Z+c_1-x-e^{\_Z}\right)}x + \operatorname{RootOf}\left(-x\,e^{2\_Z}+2\,e^{\_Z}x+\_Z+c_1-x-e^{\_Z}\right)+c_1-x$$

✓ Solution by Mathematica

Time used: 0.881 (sec). Leaf size: 46

```
DSolve[y'[x]+x*y'[x]^2-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\left\{x=\frac{\log(K[1])-K[1]}{(K[1]-1)^2}+\frac{c_1}{(K[1]-1)^2},y(x)=xK[1]^2+K[1]\right\},\{y(x),K[1]\}\right]$$

13 Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98

13.1 problem 318

Internal problem ID [15176]
Internal file name [`OUTPUT/15176_Tuesday_April_23_2024_04_53_33_PM_62491544/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 318.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 2\cos(x) + 2\sin(x)}$$

13.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\cos(x) + 2\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x) + 2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\sin(x), \cos(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \sin(x) + A_2 \cos(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) - 2A_2 \sin(x) = 2\cos(x) + 2\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\sin(x) - \cos(x)\, x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (x \sin(x) - \cos(x) x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x \sin(x) - \cos(x) x \tag{1}$$

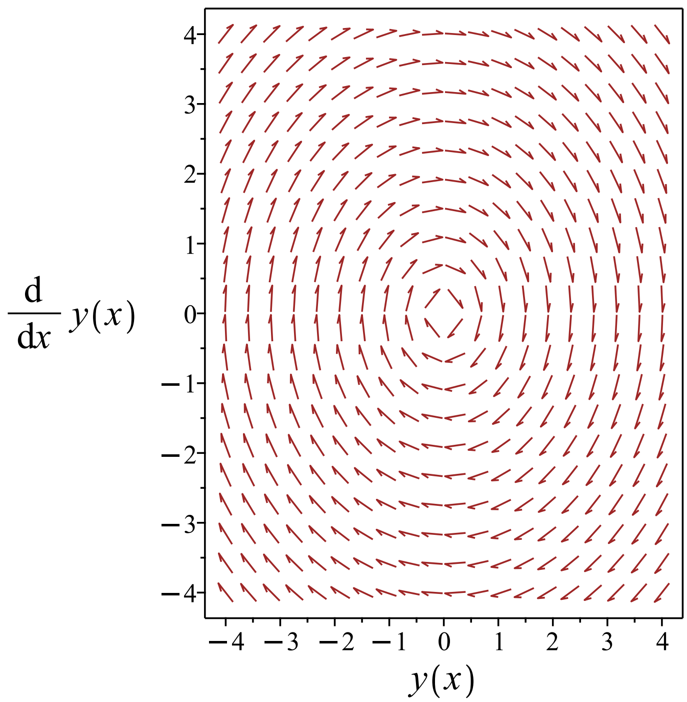


Figure 427: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x \sin(x) - \cos(x) x$$

Verified OK.

13.1.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' + y &= 0 \qquad (1)\\
Ay'' + By' + Cy &= 0 \qquad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 0 \qquad (3)\\
C &= 1
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \qquad (4)
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \qquad (6)
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \qquad (7)
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 319: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x) + 2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\sin(x), \cos(x)x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\sin(x) + A_2\cos(x)x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\cos(x) - 2A_2\sin(x) = 2\cos(x) + 2\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\sin(x) - \cos(x)x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (x\sin(x) - \cos(x)x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x \sin(x) - \cos(x) x \tag{1}$$

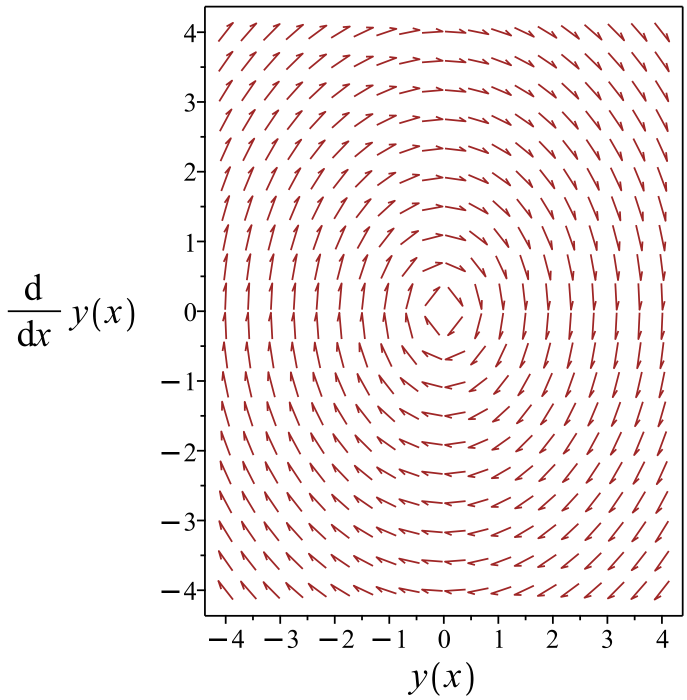


Figure 428: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x \sin(x) - \cos(x) x$$

Verified OK.

13.1.3 Maple step by step solution

Let's solve

$y'' + y = 2\cos(x) + 2\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\cos(x) + 2\sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int (1 - \cos(2x) + \sin(2x)) dx\right) + \sin(x) \left(\int (1 + \sin(2x) + \cos(2x)) dx\right)$

  ◦ Compute integrals

    $y_p(x) = \frac{(-2x+1)\cos(x)}{2} + \frac{(2x+1)\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{(-2x+1)\cos(x)}{2} + \frac{(2x+1)\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+y(x)=2*(cos(x)+sin(x)),y(x), singsol=all)
```

$$y = (c_1 - x + 1) \cos(x) + \sin(x) (x + c_2)$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==2*(Cos[x]+Sin[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + 1 + c_1) \cos(x) + (x + c_2) \sin(x)$$

13.2 problem 319

Internal problem ID [15177]
Internal file name [`OUTPUT/15177_Tuesday_April_23_2024_04_53_34_PM_30366996/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 319.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _quadrature]]
```

$$\boxed{xy''' = 2}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$xv''(x) = 0$$

Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

But since $y' = v(x)$ then we now need to solve the ode $y' = c_1 x + c_2$. Integrating both sides gives

$$\begin{aligned} y &= \int c_1 x + c_2 \, \mathrm{d}x \\ &= \frac{1}{2} c_1 x^2 + c_2 x + c_3 \end{aligned}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$xy''' = 0$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}$$
$$|W| = 2$$

The determinant simplifies to

$$|W| = 2$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$$
$$= x^2$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} 1 & x^2 \\ 0 & 2x \end{bmatrix} \\
&= 2x
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\
&= 1
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)} \, dx \\
&= (-1)^2 \int \frac{(2)\left(x^2\right)}{(x)\,(2)} \, dx \\
&= \int \frac{2x^2}{2x} \, dx \\
&= \int (x) \; dx \\
&= \frac{x^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)} \, dx \\
&= (-1)^1 \int \frac{(2)\,(2x)}{(x)\,(2)} \, dx \\
&= -\int \frac{4x}{2x} \, dx \\
&= -\int (2) \; dx \\
&= -2x
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(2)\,(1)}{(x)\,(2)}\,dx \\
&= \int \frac{2}{2x}\,dx \\
&= \int \left(\frac{1}{x}\right) dx \\
&= \ln(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x^2}{2}\right) \\
&+ (-2x)\,(x) \\
&+ (\ln(x))\,(x^2)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = x^2 \left(-\frac{3}{2} + \ln(x)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \Big(y \\
&= \frac{1}{2} c_1 x^2 + c_2 x + c_3\Big) + \left(x^2\left(-\frac{3}{2} + \ln(x)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^2}{2} + c_2 x + c_3 + x^2\left(-\frac{3}{2} + \ln(x)\right) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^2}{2} + c_2 x + c_3 + x^2\left(-\frac{3}{2} + \ln(x)\right)
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$3)=2,y(x), singsol=all)
```

$$y = x^2 \ln(x) + \frac{(c_1 - 3)\, x^2}{2} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 28

```
DSolve[x*y'''[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 \log(x) + \left(-\frac{3}{2} + c_3\right) x^2 + c_2 x + c_1$$

13.3 problem 320

Internal problem ID [15178]
Internal file name [`OUTPUT/15178_Tuesday_April_23_2024_04_53_35_PM_80715557/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 320.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' - y'^2 = 0}$$

13.3.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2} dp = x + c_1$$
$$-\frac{1}{p} = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\frac{1}{x + c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{x + c_1}$$

Integrating both sides gives

$$y = \int -\frac{1}{x + c_1} \, \mathrm{d}x$$
$$= -\ln(x + c_1) + c_2$$

Summary

The solution(s) found are the following

$$y = -\ln(x + c_1) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln(x + c_1) + c_2$$

Verified OK.

13.3.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$y'' = \frac{dp}{dx}$$
$$= \frac{dy}{dx} \frac{dp}{dy}$$
$$= p \frac{dp}{dy}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right)-p(y)^{2}=0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned}\int \frac{1}{p}dp &= y+c_1\\ \ln(p) &= y+c_1\\ p &= e^{y+c_1}\\ p &= c_1 e^{y}\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_1 e^{y}$$

Integrating both sides gives

$$\begin{aligned}\int \frac{e^{-y}}{c_1}dy &= x+c_2\\ -\frac{e^{-y}}{c_1} &= x+c_2\end{aligned}$$

Solving for y gives these solutions

$$y_1=\ln\left(-\frac{1}{(x+c_2)\,c_1}\right)$$

Summary

The solution(s) found are the following

$$y=\ln\left(-\frac{1}{(x+c_2)\,c_1}\right)\tag{1}$$

Verification of solutions

$$y=\ln\left(-\frac{1}{(x+c_2)\,c_1}\right)$$

Verified OK.

13.3.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{x+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{x+c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{x+c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{1}{x+c_1} dx + c_2$

- Compute integrals

  $y = -\ln(x + c_1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=diff(y(x),x)^2,y(x), singsol=all)
```

$$y = -\ln(-c_1 x - c_2)$$

✓ Solution by Mathematica

Time used: 0.165 (sec). Leaf size: 15

```
DSolve[y''[x]==y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log(x + c_1)$$

13.4 problem 321

Internal problem ID [15179]
Internal file name [`OUTPUT/15179_Tuesday_April_23_2024_04_53_35_PM_63287211/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 321.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{(x-1)\,y''=1}$$

Simplyfing the ode gives

$$y''=\frac{1}{x-1}$$

13.4.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \ln(x-1) + c_1$$

Integrating again gives

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2 \tag{1}$$

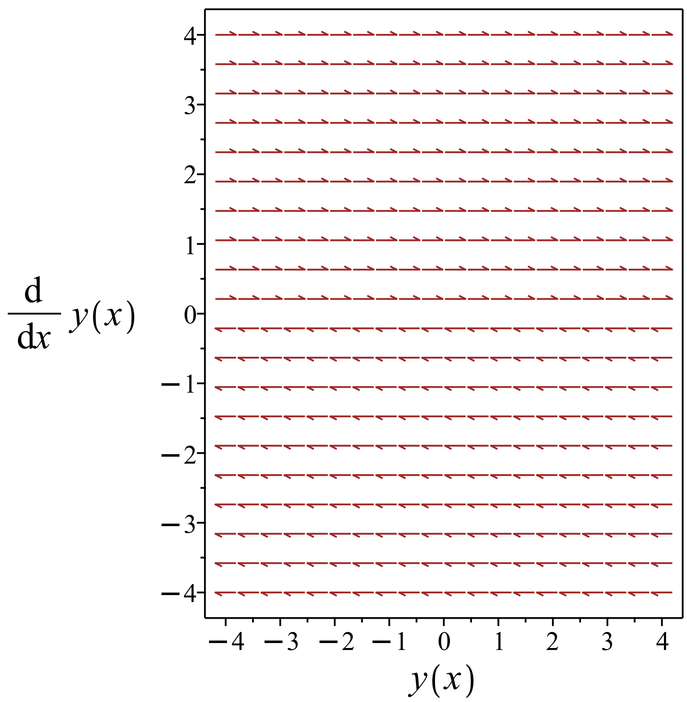


Figure 429: Slope field plot

Verification of solutions

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2$$

Verified OK.

13.4.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = \frac{1}{x-1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\
&= 0
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x}{x-1}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x-1} dx$$

Hence

$$u_1 = -x - \ln(x-1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x-1}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x-1} dx$$

Hence

$$u_2 = \ln(x-1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x - \ln(x-1) + \ln(x-1)\, x$$

Which simplifies to

$$y_p(x) = \ln(x-1)\,(x-1) - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (\ln(x-1)\,(x-1) - x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + \ln(x-1)(x-1) - x \tag{1}$$

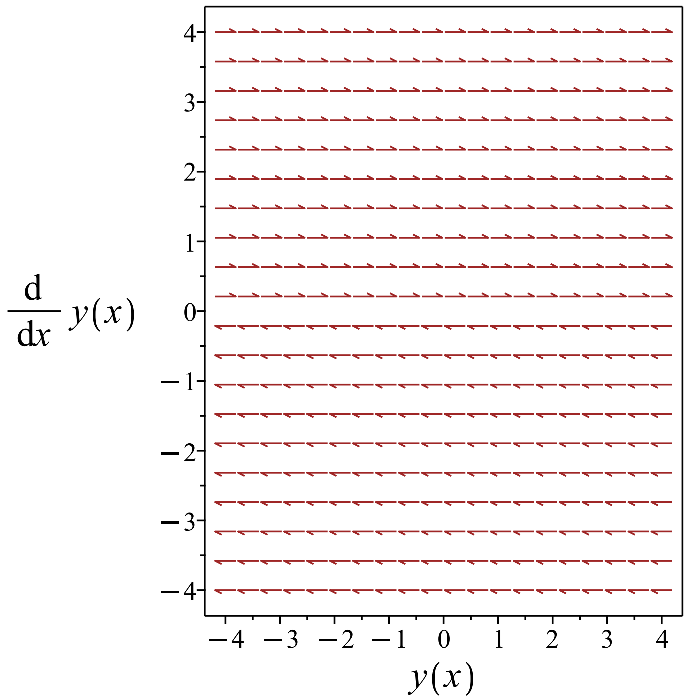


Figure 430: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + \ln(x-1)(x-1) - x$$

Verified OK.

13.4.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int \frac{1}{x-1} dx$$
$$y' = \ln(x-1) + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \ln(x-1) + c_1 \, \mathrm{d}x \\ &= \ln(x-1)(x-1) - x + 1 + c_1 x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2 \tag{1}$$

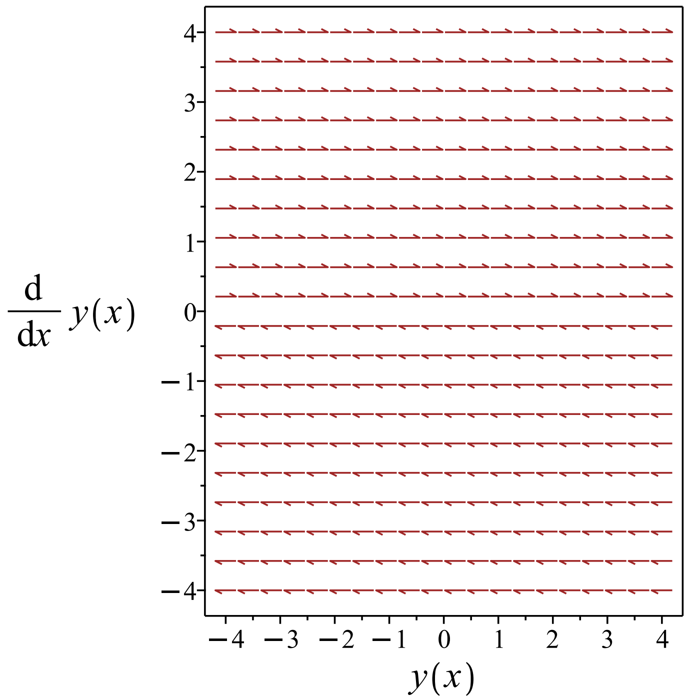


Figure 431: Slope field plot

Verification of solutions

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2$$

Verified OK.

13.4.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \frac{1}{x-1} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int \frac{1}{x-1}\, dx \\ &= \ln(x-1) + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \ln(x-1) + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \ln(x-1) + c_1\, dx \\ &= \ln(x-1)(x-1) - x + 1 + c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2 \tag{1}$$

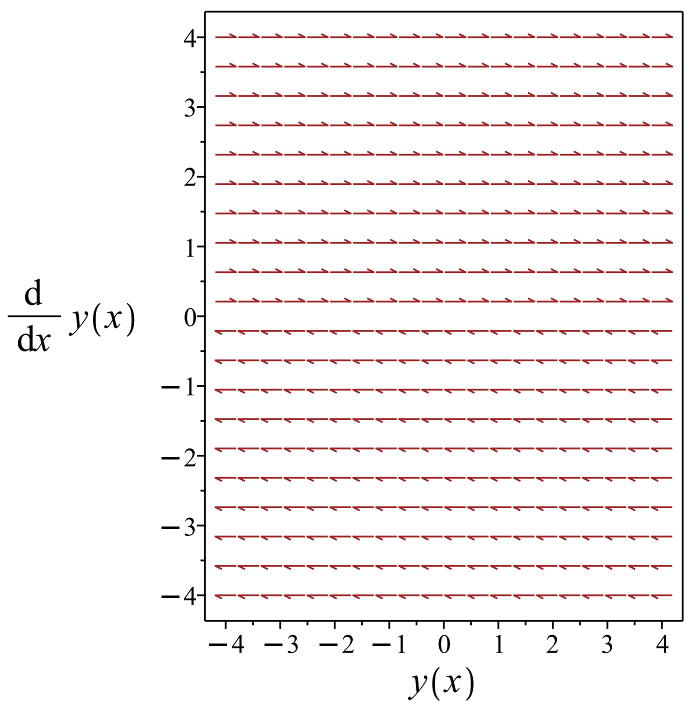


Figure 432: Slope field plot

Verification of solutions

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2$$

Verified OK.

13.4.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 322: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= 1 \int \frac{1}{1}\, dx \\
&= 1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x + c_1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x}{x-1}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x-1} dx$$

Hence

$$u_1 = -x - \ln(x-1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x-1}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x-1} dx$$

Hence

$$u_2 = \ln(x-1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x - \ln(x-1) + \ln(x-1)\, x$$

Which simplifies to

$$y_p(x) = \ln(x-1)(x-1) - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (\ln(x-1)(x-1) - x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2x + c_1 + \ln(x-1)(x-1) - x \tag{1}$$

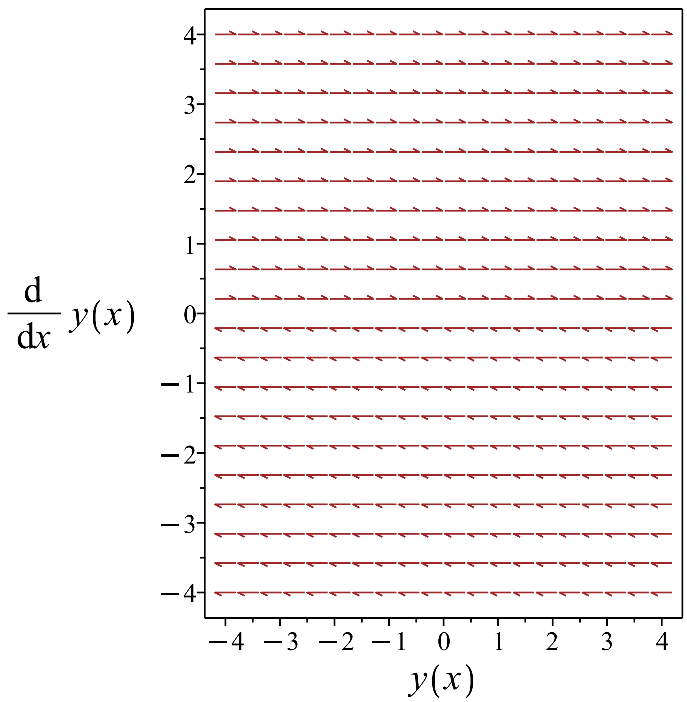


Figure 433: Slope field plot

Verification of solutions

$$y = c_2x + c_1 + \ln(x-1)(x-1) - x$$

Verified OK.

13.4.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 0 \\
r(x) &= 0 \\
s(x) &= \frac{1}{x-1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y' + (q(x)-p'(x))\,y)' = s(x)
$$

Integrating gives

$$
p(x)\,y' + (q(x)-p'(x))\,y = \int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
y' = \int \frac{1}{x-1}\,dx
$$

We now have a first order ode to solve which is

$$
y' = \ln(x-1) + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \ln(x-1) + c_1 \,\mathrm{d}x \\
&= \ln(x-1)\,(x-1) - x + 1 + c_1 x + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \ln(x-1)\,(x-1) - x + 1 + c_1 x + c_2 \tag{1}
$$

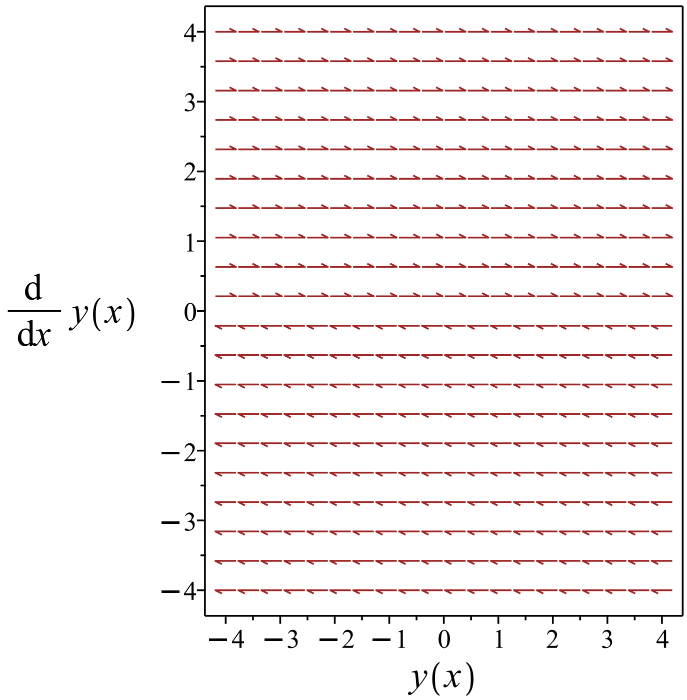


Figure 434: Slope field plot

Verification of solutions

$$y = \ln(x-1)(x-1) - x + 1 + c_1 x + c_2$$

Verified OK.

13.4.7 Maple step by step solution

Let's solve

$y'' = \frac{1}{x-1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

$r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \frac{1}{x-1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\left(\int \frac{x}{x-1}dx\right) + x\left(\int \frac{1}{x-1}dx\right)$

  - Compute integrals

    $y_p(x) = \ln(x-1)(x-1) - x$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 + \ln(x-1)(x-1) - x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((x-1)*diff(y(x),x$2)=1,y(x), singsol=all)
```

$$y = \ln (x-1) (x-1) + (c_1 - 1) x + c_2 + 1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 22

```
DSolve[(x-1)*y''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x-1) \log(x-1) + (-1 + c_2)x + c_1$$

13.5 problem 322

Internal problem ID [15180]
Internal file name [`OUTPUT/15180_Tuesday_April_23_2024_04_53_37_PM_51133802/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 322.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^4 = 1}$$

Solving the given ode for y' results in 4 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 1 \tag{1}$$
$$y' = -1 \tag{2}$$
$$y' = i \tag{3}$$
$$y' = -i \tag{4}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int 1 \, \mathrm{d}x \\ &= x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x + c_1 \tag{1}$$

Verification of solutions

$$y = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -1 \,\mathrm{d}x \\ &= -x + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -x + c_2 \tag{1}$$

Verification of solutions

$$y = -x + c_2$$

Verified OK.

Solving equation (3)

Integrating both sides gives

$$\begin{aligned} y &= \int i \,\mathrm{d}x \\ &= ix + c_3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = ix + c_3 \tag{1}$$

Verification of solutions

$$y = ix + c_3$$

Verified OK.

Solving equation (4)

Integrating both sides gives

$$y = \int -i \, \mathrm{d}x$$
$$= -ix + c_4$$

Summary
The solution(s) found are the following

$$y = -ix + c_4 \tag{1}$$

Verification of solutions

$$y = -ix + c_4$$

Verified OK.

13.5.1 Maple step by step solution

Let's solve

$y'^4 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^4 dx = \int 1 dx + c_1$

- Cannot compute integral

  $\int y'^4 dx = x + c_1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)*diff(y(x),x)^3=1,y(x), singsol=all)
```

$$y = -ix + c_1$$
$$y = ix + c_1$$
$$y = x + c_1$$
$$y = -x + c_1$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 43

```
DSolve[y'[x]*y'[x]^3==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + c_1$$
$$y(x) \to c_1 - ix$$
$$y(x) \to ix + c_1$$
$$y(x) \to x + c_1$$

13.6 problem 323

Internal problem ID [15181]
Internal file name [`OUTPUT/15181_Tuesday_April_23_2024_04_53_37_PM_20816000/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 323.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

13.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

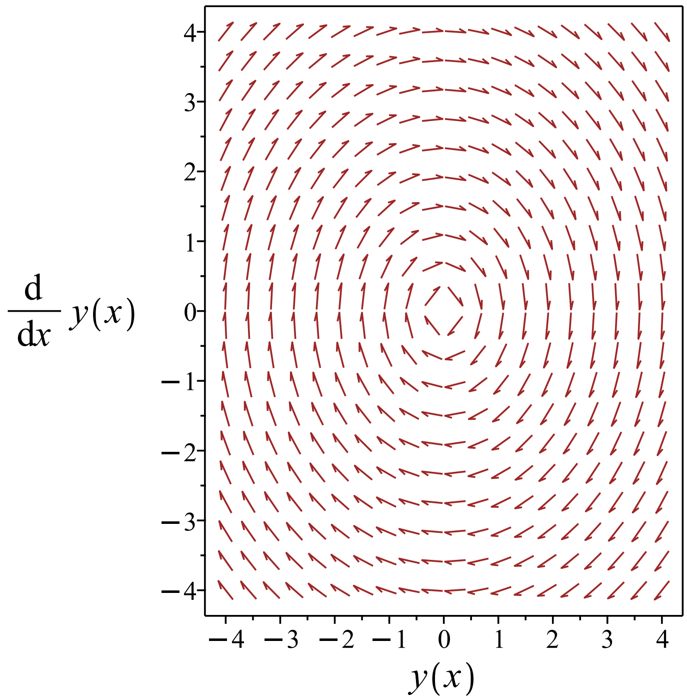


Figure 435: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Verified OK.

13.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\, dx = 0$$

$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$
$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x+c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3+x$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x+c_2 \tag{1}$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3+x \tag{2}$$

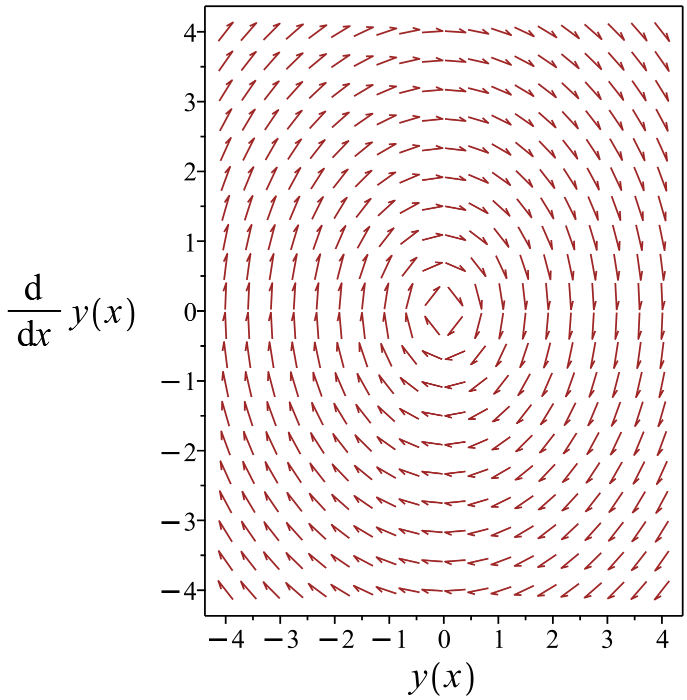


Figure 436: Slope field plot

<u>Verification of solutions</u>

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2$$

Verified OK.

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x$$

Verified OK.

13.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

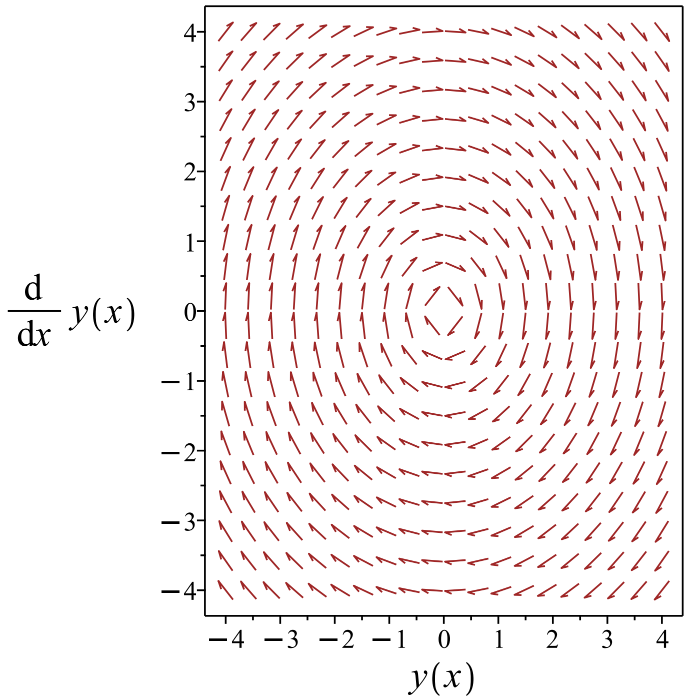


Figure 437: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Verified OK.

13.6.4 Maple step by step solution

Let's solve

$y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y = c_1 \sin(x) + c_2 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x)$$

13.7 problem 324

Internal problem ID [15182]
Internal file name [`OUTPUT/15182_Tuesday_April_23_2024_04_53_38_PM_24997785/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 324.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 2}$$

13.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda \, \mathrm{e}^{\lambda x} + 2 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^x\right) + (1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x + 1 \tag{1}$$

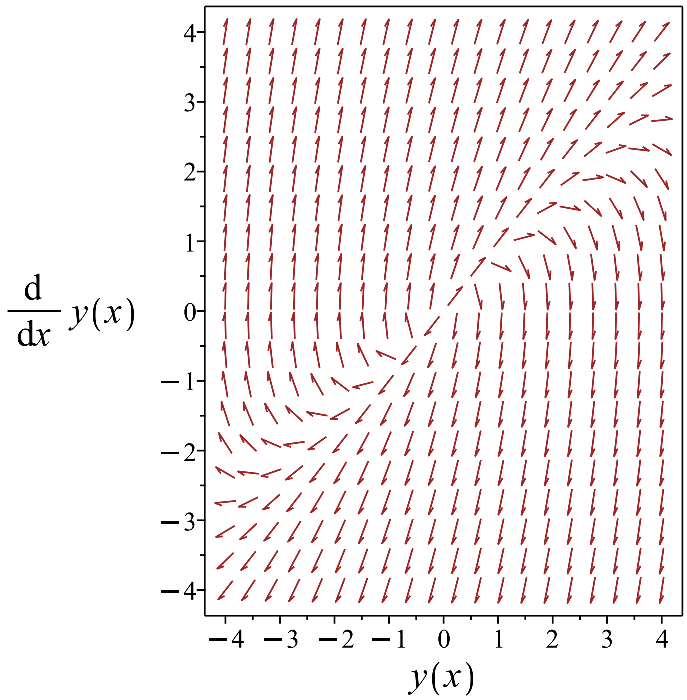


Figure 438: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x} + 1$$

Verified OK.

13.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^x, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 = 2
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 e^{2x}\right) + (1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{2x} + 1 \tag{1}$$

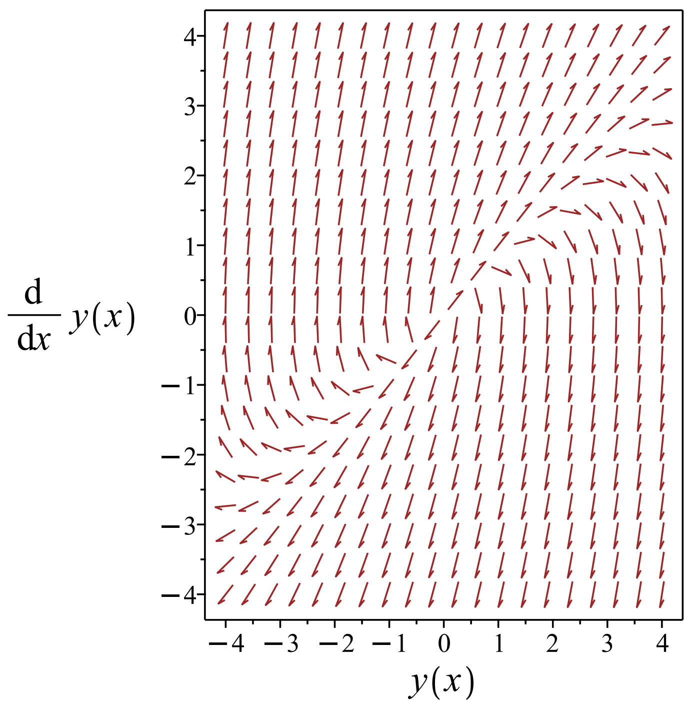


Figure 439: Slope field plot

Verification of solutions

$$y = e^x c_1 + c_2 e^{2x} + 1$$

Verified OK.

13.7.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,e^x\left(\int e^{-x} dx\right) + 2\,e^{2x}\left(\int e^{-2x} dx\right)$

- Compute integrals

  $y_p(x) = 1$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 e^{2x} + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=2,y(x), singsol=all)
```

$$y = e^{2x} c_1 + c_2 e^x + 1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 21

```
DSolve[y''[x]-3*y'[x]+2*y[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_2 e^{2x} + 1$$

13.8 problem 325

Internal problem ID [15183]
Internal file name [`OUTPUT/15183_Tuesday_April_23_2024_04_53_39_PM_25005712/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 325.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \left(1 + {y'}^2\right)^{\frac{3}{2}} = 0}$$

13.8.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \left(1 + p(x)^2\right)^{\frac{3}{2}} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{(p^2+1)^{\frac{3}{2}}} dp = \int dx$$
$$\frac{p(x)}{\sqrt{1+p(x)^2}} = x + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'}{\sqrt{1+y'^2}} = x + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\,\mathrm{d}x \\ &= \sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\,(c_1+x+1)\,(c_1+x-1)+c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\,(c_1+x+1)\,(c_1+x-1)+c_2 \tag{1}$$

Verification of solutions

$$y = \sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\,(c_1+x+1)\,(c_1+x-1)+c_2$$

Verified OK.

13.8.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) = \left(1+p(y)^2\right)^{\frac{3}{2}}
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{p}{(p^2+1)^{\frac{3}{2}}}dp &= \int dy \\
-\frac{1}{\sqrt{1+p\left(y\right)^2}} &= y+c_1
\end{aligned}
$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$
-\frac{1}{\sqrt{1+y'^2}} = y+c_1
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
y' = \frac{\sqrt{-y^2-2yc_1-c_1^2+1}}{y+c_1} \tag{1}
$$

$$
y' = -\frac{\sqrt{-y^2-2yc_1-c_1^2+1}}{y+c_1} \tag{2}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
\int \frac{y+c_1}{\sqrt{-c_1^2-2c_1y-y^2+1}}dy &= \int dx \\
\frac{(y+c_1+1)\,(y+c_1-1)}{\sqrt{-y^2-2yc_1-c_1^2+1}} &= x+c_2
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y+c_1}{\sqrt{-c_1^2-2c_1y-y^2+1}}dy = \int dx$$
$$-\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2yc_1-c_1^2+1}} = c_3+x$$

Summary

The solution(s) found are the following

$$\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2yc_1-c_1^2+1}} = x+c_2 \tag{1}$$
$$-\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2yc_1-c_1^2+1}} = c_3+x \tag{2}$$

Verification of solutions

$$\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2yc_1-c_1^2+1}} = x+c_2$$

Verified OK.

$$-\frac{(y+c_1+1)(y+c_1-1)}{\sqrt{-y^2-2yc_1-c_1^2+1}} = c_3+x$$

Verified OK.

13.8.3 Maple step by step solution

Let's solve

$y'' = \left(1+y'^2\right)^{\frac{3}{2}}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = \left(1+u(x)^2\right)^{\frac{3}{2}}$

- Separate variables

$$\frac{u'(x)}{\left(1+u(x)^2\right)^{\frac{3}{2}}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{\left(1+u(x)^2\right)^{\frac{3}{2}}} dx = \int 1dx + c_1$$

- Evaluate integral

$$\frac{u(x)}{\sqrt{1+u(x)^2}} = x + c_1$$

- Solve for $u(x)$

$$u(x) = c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}$$

- Solve 1st ODE for $u(x)$

$$u(x) = c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}$$

- Make substitution $u = y'$

$$y' = c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}$$

- Integrate both sides to solve for y

$$\int y'dx = \int \left(c_1\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}} + x\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\right) dx + c_2$$

- Compute integrals

$$y = \sqrt{-\frac{1}{c_1^2+2c_1x+x^2-1}}\,(c_1+x+1)\,(c_1+x-1) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)^2+1)^(3/2), _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [y, -_b^2-1]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$2)=(1+diff(y(x),x)^2)^(3/2),y(x), singsol=all)
```

$$y = -ix + c_1$$
$$y = ix + c_1$$
$$y = (c_1 + x + 1)(c_1 + x - 1)\sqrt{-\frac{1}{(c_1 + x + 1)(c_1 + x - 1)}} + c_2$$

✓ Solution by Mathematica

Time used: 0.253 (sec). Leaf size: 59

```
DSolve[y''[x]==(1+y'[x]^2)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - i\sqrt{x^2 + 2c_1 x - 1 + c_1^2}$$
$$y(x) \to i\sqrt{x^2 + 2c_1 x - 1 + c_1^2} + c_2$$

13.9 problem 326

Internal problem ID [15184]
Internal file name [`OUTPUT/15184_Tuesday_April_23_2024_04_53_40_PM_53989808/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 13. Basic concepts and definitions. Exercises page 98
Problem number: 326.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{{y'}^2+yy''=1}$$

13.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left({y'}^2+yy''\right) dx = \int 1 dx$$
$$yy' = x + c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x+c_1}{y} \end{aligned}$$

Where $f(x) = x + c_1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{y}}\, dy &= x + c_1\, dx \\ \int \frac{1}{\frac{1}{y}}\, dy &= \int x + c_1\, dx \\ \frac{y^2}{2} &= \frac{1}{2}x^2 + c_1 x + c_2 \end{aligned}$$

The solution is

$$\frac{y^2}{2} - \frac{x^2}{2} - c_1 x - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - \frac{x^2}{2} - c_1 x - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - \frac{x^2}{2} - c_1 x - c_2 = 0$$

Verified OK.

13.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)^2 + yp(y)\left(\frac{d}{dy}p(y)\right) = 1
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p^2-1}{yp}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = \frac{p^2-1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{p^2-1}{p}}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{\frac{p^2-1}{p}}\,dp &= \int -\frac{1}{y}\,dy \\
\frac{\ln(p-1)}{2} + \frac{\ln(p+1)}{2} &= -\ln(y) + c_1
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{2}\right)(\ln(p-1) + \ln(p+1)) &= -\ln(y) + 2c_1 \\
\ln(p-1) + \ln(p+1) &= (2)(-\ln(y) + 2c_1) \\
&= -2\ln(y) + 4c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(p-1)+\ln(p+1)} = e^{-2\ln(y)+2c_1}
$$

Which simplifies to

$$
\begin{aligned}
p^2 - 1 &= \frac{2c_1}{y^2} \\
&= \frac{c_2}{y^2}
\end{aligned}
$$

The solution is

$$
p(y)^2 - 1 = \frac{c_2}{y^2}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y'^2 - 1 = \frac{c_2}{y^2}
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
y' = \frac{\sqrt{y^2 + c_2}}{y} \tag{1}
$$

$$
y' = -\frac{\sqrt{y^2 + c_2}}{y} \tag{2}
$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$
\begin{aligned}
\int \frac{y}{\sqrt{y^2 + c_2}} dy &= \int dx \\
\sqrt{y^2 + c_2} &= c_3 + x
\end{aligned}
$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{y}{\sqrt{y^2 + c_2}} dy &= \int dx \\
-\sqrt{y^2 + c_2} &= x + c_4
\end{aligned}
$$

Summary
The solution(s) found are the following

$$\sqrt{y^2+c_2} = c_3 + x \tag{1}$$
$$-\sqrt{y^2+c_2} = x + c_4 \tag{2}$$

Verification of solutions

$$\sqrt{y^2+c_2} = c_3 + x$$

Verified OK.

$$-\sqrt{y^2+c_2} = x + c_4$$

Verified OK.

13.9.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'^2 + yy'' = 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'^2 + yy''\right) dx = \int 1dx$$
$$yy' = x + c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x+c_1}{y} \end{aligned}$$

Where $f(x) = x + c_1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = x + c_1\, dx$$
$$\int \frac{1}{\frac{1}{y}}\, dy = \int x + c_1\, dx$$
$$\frac{y^2}{2} = \frac{1}{2}x^2 + c_1 x + c_2$$

The solution is

$$\frac{y^2}{2} - \frac{x^2}{2} - c_1 x - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - \frac{x^2}{2} - c_1 x - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - \frac{x^2}{2} - c_1 x - c_2 = 0$$

Verified OK.

13.9.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= y\\ a_1 &= y'\\ a_0 &= -1\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1\\ \int y\, dy' + \int y'\, dy + \int -1\, dx &= c_1\end{aligned}$$

Which results in

$$2yy' - x = c_1$$

Which is now solved In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\frac{x}{2}+\frac{c_1}{2}}{y} \end{aligned}$$

Where $f(x)=\frac{x}{2}+\frac{c_1}{2}$ and $g(y)=\frac{1}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{y}}\,dy &= \frac{x}{2}+\frac{c_1}{2}\,dx \\ \int \frac{1}{\frac{1}{y}}\,dy &= \int \frac{x}{2}+\frac{c_1}{2}\,dx \\ \frac{y^2}{2} &= \frac{1}{4}x^2+\frac{1}{2}c_1x+c_2 \end{aligned}$$

The solution is

$$\frac{y^2}{2}-\frac{x^2}{4}-\frac{c_1x}{2}-c_2=0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2}-\frac{x^2}{4}-\frac{c_1x}{2}-c_2=0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2}-\frac{x^2}{4}-\frac{c_1x}{2}-c_2=0$$

Warning, solution could not be verified

13.9.5 Maple step by step solution

Let's solve

$y'^2 + yy'' = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)^2 + yu(y)\left(\frac{d}{dy}u(y)\right) = 1$

- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2+1} = \frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2+1}dy = \int \frac{1}{y}dy + c_1$

- Evaluate integral

  $-\frac{\ln(u(y)-1)}{2} - \frac{\ln(u(y)+1)}{2} = \ln(y) + c_1$

- Solve for $u(y)$

  $\left\{u(y) = \frac{y\,e^{c_1}\left(y\,e^{c_1}-\sqrt{(e^{c_1})^2y^2+1}\right)+1}{y\,e^{c_1}\left(y\,e^{c_1}-\sqrt{(e^{c_1})^2y^2+1}\right)}, u(y) = \frac{y\,e^{c_1}\left(y\,e^{c_1}+\sqrt{(e^{c_1})^2y^2+1}\right)+1}{y\,e^{c_1}\left(y\,e^{c_1}+\sqrt{(e^{c_1})^2y^2+1}\right)}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{y\,e^{c_1}\left(y\,e^{c_1}-\sqrt{(e^{c_1})^2y^2+1}\right)+1}{y\,e^{c_1}\left(y\,e^{c_1}-\sqrt{(e^{c_1})^2y^2+1}\right)}$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}-\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}-\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}$$

- Separate variables

$$\frac{y' y\left(y\,\mathrm{e}^{c_1}-\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}-\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1} = \frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y\left(y\,\mathrm{e}^{c_1}-\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}-\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1}dx = \int \frac{1}{\mathrm{e}^{c_1}}dx + c_2$$

- Evaluate integral

$$-\frac{\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}}{(\mathrm{e}^{c_1})^2} = \frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = \frac{\sqrt{-1+(\mathrm{e}^{c_1})^4 c_2^2+2(\mathrm{e}^{c_1})^3 c_2 x+(\mathrm{e}^{c_1})^2 x^2}}{\mathrm{e}^{c_1}}, y = -\frac{\sqrt{-1+(\mathrm{e}^{c_1})^4 c_2^2+2(\mathrm{e}^{c_1})^3 c_2 x+(\mathrm{e}^{c_1})^2 x^2}}{\mathrm{e}^{c_1}}\right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = \frac{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}$$

- Separate variables

$$\frac{y' y\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1} = \frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)}{y\,\mathrm{e}^{c_1}\left(y\,\mathrm{e}^{c_1}+\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}\right)+1}dx = \int \frac{1}{\mathrm{e}^{c_1}}dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{(\mathrm{e}^{c_1})^2 y^2+1}}{(\mathrm{e}^{c_1})^2} = \frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y=\frac{\sqrt{-1+(\mathrm{e}^{c_1})^4c_2^2+2(\mathrm{e}^{c_1})^3c_2x+(\mathrm{e}^{c_1})^2x^2}}{\mathrm{e}^{c_1}}, y=-\frac{\sqrt{-1+(\mathrm{e}^{c_1})^4c_2^2+2(\mathrm{e}^{c_1})^3c_2x+(\mathrm{e}^{c_1})^2x^2}}{\mathrm{e}^{c_1}}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve(diff(y(x),x)^2+y(x)*diff(y(x),x$2)=1,y(x), singsol=all)
```

$$y=\sqrt{-2c_1x+x^2+2c_2}$$
$$y=-\sqrt{-2c_1x+x^2+2c_2}$$

✓ Solution by Mathematica

Time used: 0.593 (sec). Leaf size: 79

```
DSolve[y'[x]^2+y[x]*y''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\sqrt{(x+c_2)^2-e^{2c_1}}$$
$$y(x)\to \sqrt{(x+c_2)^2-e^{2c_1}}$$
$$y(x)\to -\sqrt{(x+c_2)^2}$$
$$y(x)\to \sqrt{(x+c_2)^2}$$

14 Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107

14.1 problem 327

Internal problem ID [15185]
Internal file name [`OUTPUT/15185_Tuesday_April_23_2024_04_53_41_PM_43313790/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 327.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _quadrature]]
```

$$\boxed{y'''' = x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' = 0$$

The characteristic equation is

$$\lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_4x^3 + c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = x^3$$

Now the particular solution to the given ODE is found

$$y'''' = x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, x^3\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3, x^4\}]$$

Since x^3 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^4, x^5\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^5 + A_1x^4$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$120xA_2 + 24A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{120}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^5}{120}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_4x^3 + c_3x^2 + c_2x + c_1\right) + \left(\frac{x^5}{120}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_4x^3 + c_3x^2 + c_2x + c_1 + \frac{1}{120}x^5 \tag{1}$$

Verification of solutions

$$y = c_4x^3 + c_3x^2 + c_2x + c_1 + \frac{1}{120}x^5$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$4)=x,y(x), singsol=all)
```

$$y = \frac{x^5}{120} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + \frac{(3c_1^2 + 2c_3)\, x}{2} + c_4$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 31

```
DSolve[y''''[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^5}{120} + c_4 x^3 + c_3 x^2 + c_2 x + c_1$$

14.2 problem 328

Internal problem ID [15186]
Internal file name [`OUTPUT/15186_Tuesday_April_23_2024_04_53_42_PM_41726877/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 328.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _quadrature]]
```

$$\boxed{y''' = x + \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' = 0$$

The characteristic equation is

$$\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$

Now the particular solution to the given ODE is found

$$y''' = x + \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(x), \sin(x)\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3\}, \{\cos(x), \sin(x)\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3, x^4\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^4 + A_1 x^3 + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_2 x + 6A_1 + A_3 \sin(x) - A_4 \cos(x) = x + \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{24}, A_3 = 0, A_4 = -1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4}{24} - \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_3 x^2 + c_2 x + c_1) + \left(\frac{x^4}{24} - \sin(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3 x^2 + c_2 x + c_1 + \frac{x^4}{24} - \sin(x) \tag{1}$$

Verification of solutions

$$y = c_3 x^2 + c_2 x + c_1 + \frac{x^4}{24} - \sin(x)$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$3)=x+cos(x),y(x), singsol=all)
```

$$y = \frac{x^4}{24} + \frac{c_1 x^2}{2} - \sin(x) + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 29

```
DSolve[y'''[x]==x+Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{24} + c_3 x^2 - \sin(x) + c_2 x + c_1$$

14.3 problem 329

Internal problem ID [15187]
Internal file name [`OUTPUT/15187_Tuesday_April_23_2024_04_53_43_PM_24351852/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 329.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y''(x+2)^5 = 1}$$

With initial conditions

$$\left[y(-1) = \frac{1}{12}, y'(-1) = -\frac{1}{4}\right]$$

Simplyfing the ode gives

$$y'' = \frac{1}{(x+2)^5}$$

14.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ F &= \frac{1}{(x+2)^5} \end{aligned}$$

Hence the ode is

$$y'' = \frac{1}{(x+2)^5}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $F = \frac{1}{(x+2)^5}$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

14.3.2 Solving as second order ode quadrature ode

Integrating once gives

$$y' = -\frac{1}{4(x+2)^4} + c_1$$

Integrating again gives

$$y = \frac{1}{12(x+2)^3} + c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{12(x+2)^3} + c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{12}$ and $x = -1$ in the above gives

$$\frac{1}{12} = -c_1 + c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{1}{4 (x+2)^4} + c_1$$

substituting $y' = -\frac{1}{4}$ and $x = -1$ in the above gives

$$-\frac{1}{4} = c_1 - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

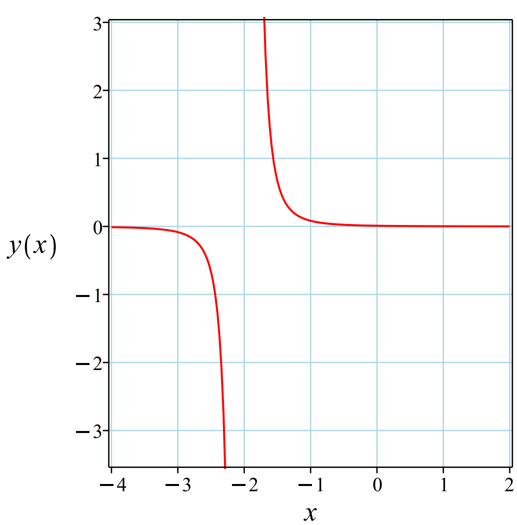
Substituting these values back in above solution results in

$$y = \frac{1}{12 (x+2)^3}$$

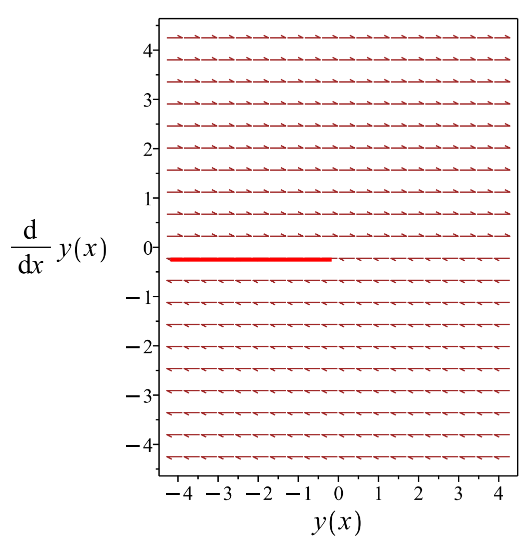
Summary

The solution(s) found are the following

$$y = \frac{1}{12 (x+2)^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{12 (x+2)^3}$$

Verified OK.

14.3.3 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = \frac{1}{(x+2)^5}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x}{(x+2)^5}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x+2)^5} dx$$

Hence

$$u_1 = \frac{1}{3(x+2)^3} - \frac{1}{2(x+2)^4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{(x+2)^5}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x+2)^5} dx$$

Hence

$$u_2 = -\frac{1}{4(x+2)^4}$$

Which simplifies to

$$u_1 = \frac{2x+1}{6(x+2)^4}$$

$$u_2 = -\frac{1}{4(x+2)^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x+1}{6\left(x+2\right)^4} - \frac{x}{4\left(x+2\right)^4}$$

Which simplifies to

$$y_p(x) = \frac{1}{12\left(x+2\right)^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{1}{12\left(x+2\right)^3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + \frac{1}{12\left(x+2\right)^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{12}$ and $x = -1$ in the above gives

$$\frac{1}{12} = c_1 - c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 - \frac{1}{4\left(x+2\right)^4}$$

substituting $y' = -\frac{1}{4}$ and $x = -1$ in the above gives

$$-\frac{1}{4} = c_2 - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

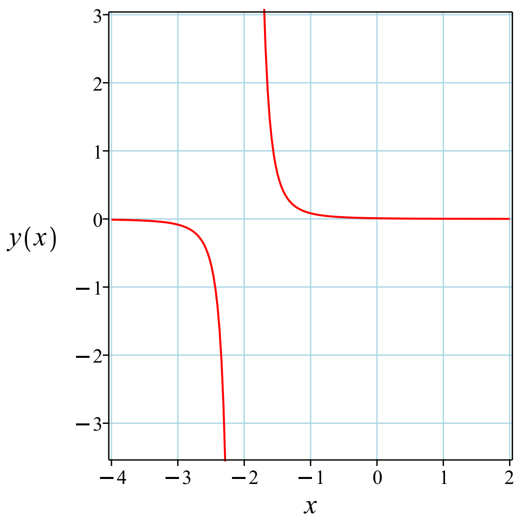
Substituting these values back in above solution results in

$$y = \frac{1}{12 (x+2)^3}$$

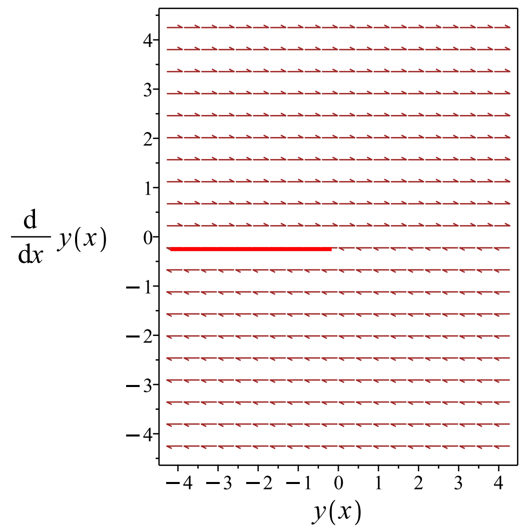
Summary

The solution(s) found are the following

$$y = \frac{1}{12 (x+2)^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{12 (x+2)^3}$$

Verified OK.

14.3.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int \frac{1}{(x+2)^5} dx$$

$$y' = -\frac{1}{4 (x+2)^4} + c_1$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{4c_1x^4 + 32c_1x^3 + 96c_1x^2 + 128c_1x + 64c_1 - 1}{4(x+2)^4} dx \\
&= \frac{1}{12(x+2)^3} + c_1x + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{1}{12(x+2)^3} + c_1x + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{12}$ and $x = -1$ in the above gives

$$
\frac{1}{12} = -c_1 + c_2 + \frac{1}{12} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{1}{4(x+2)^4} + c_1
$$

substituting $y' = -\frac{1}{4}$ and $x = -1$ in the above gives

$$
-\frac{1}{4} = c_1 - \frac{1}{4} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

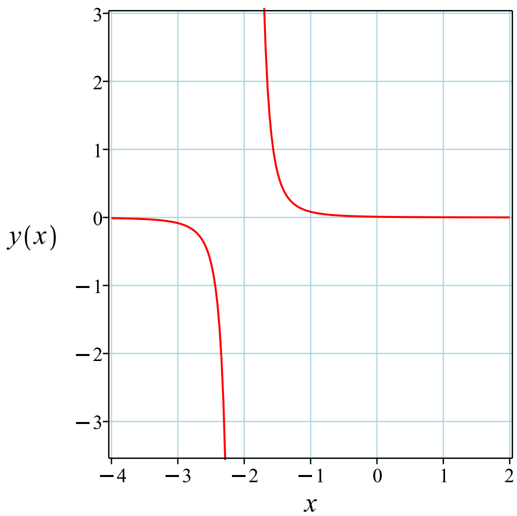
Substituting these values back in above solution results in

$$
y = \frac{1}{12(x+2)^3}
$$

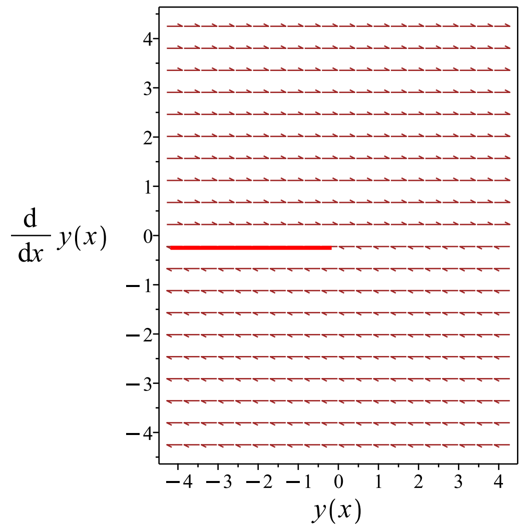
Summary

The solution(s) found are the following

$$
y = \frac{1}{12(x+2)^3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{12 (x+2)^3}$$

Verified OK.

14.3.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \frac{1}{(x+2)^5} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int \frac{1}{(x+2)^5} \, dx \\ &= -\frac{1}{4 (x+2)^4} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $p = -\frac{1}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{4} = c_1 - \frac{1}{4}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(x) = -\frac{1}{4(x+2)^4}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{4(x+2)^4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{4(x+2)^4} \,\mathrm{d}x \\ &= \frac{1}{12(x+2)^3} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = \frac{1}{12}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{12} = c_2 + \frac{1}{12}$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

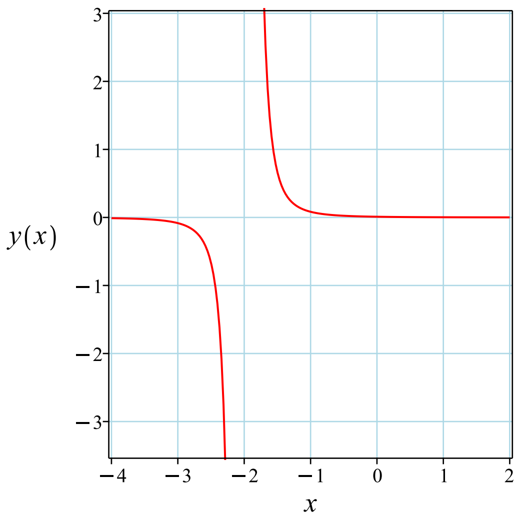
$$y = \frac{1}{12(x+2)^3}$$

Initial conditions are used to solve for the constants of integration.

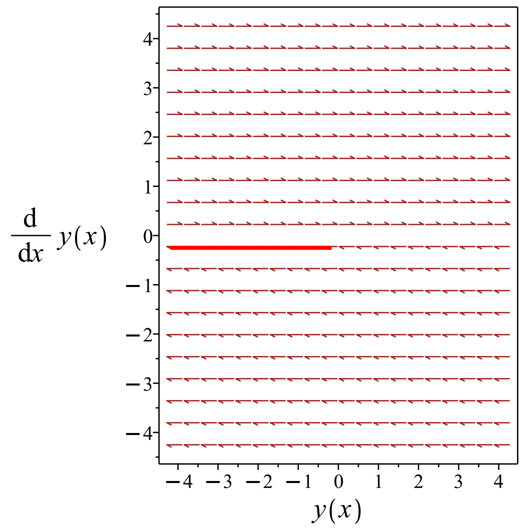
Summary

The solution(s) found are the following

$$y = \frac{1}{12(x+2)^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{12\left(x+2\right)^{3}}$$

Verified OK.

14.3.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= 1 \int \frac{1}{1}\, dx \\
&= 1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x + c_1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x}{(x+2)^5}}{1} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x+2)^5} dx$$

Hence

$$u_1 = \frac{1}{3(x+2)^3} - \frac{1}{2(x+2)^4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{(x+2)^5}}{1} \, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x+2)^5} dx$$

Hence

$$u_2 = -\frac{1}{4(x+2)^4}$$

Which simplifies to

$$u_1 = \frac{2x+1}{6(x+2)^4}$$

$$u_2 = -\frac{1}{4(x+2)^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x+1}{6(x+2)^4} - \frac{x}{4(x+2)^4}$$

Which simplifies to

$$y_p(x) = \frac{1}{12(x+2)^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{1}{12(x+2)^3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + \frac{1}{12(x+2)^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{12}$ and $x = -1$ in the above gives

$$\frac{1}{12} = c_1 - c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 - \frac{1}{4\left(x+2\right)^4}$$

substituting $y' = -\frac{1}{4}$ and $x = -1$ in the above gives

$$-\frac{1}{4} = c_2 - \frac{1}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

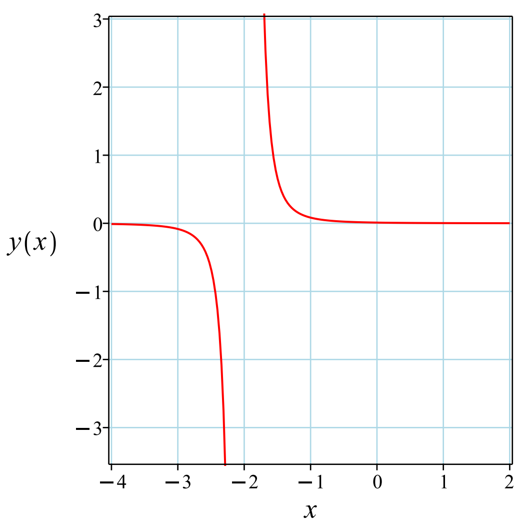
$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

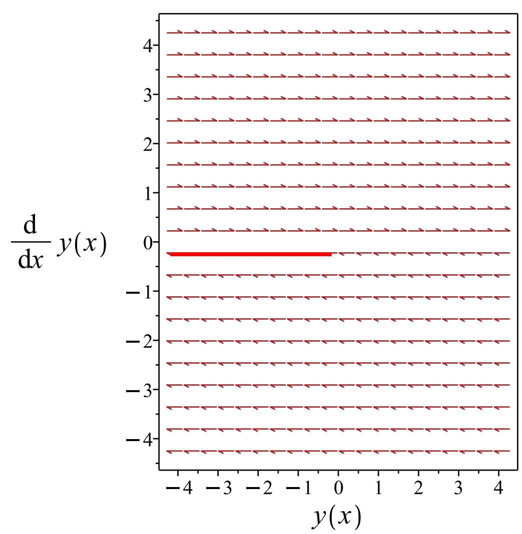
$$y = \frac{1}{12\left(x+2\right)^3}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{12\left(x+2\right)^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{12\left(x+2\right)^3}$$

Verified OK.

14.3.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= 0\\ r(x) &= 0\\ s(x) &= \frac{1}{(x+2)^5}\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= 0\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int \frac{1}{(x+2)^5}\,dx$$

We now have a first order ode to solve which is

$$y' = -\frac{1}{4\,(x+2)^4} + c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{4c_1x^4 + 32c_1x^3 + 96c_1x^2 + 128c_1x + 64c_1 - 1}{4\left(x+2\right)^4} \mathrm{d}x \\
&= \frac{1}{12\left(x+2\right)^3} + c_1x + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{1}{12\left(x+2\right)^3} + c_1x + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{12}$ and $x = -1$ in the above gives

$$
\frac{1}{12} = -c_1 + c_2 + \frac{1}{12} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{1}{4\left(x+2\right)^4} + c_1
$$

substituting $y' = -\frac{1}{4}$ and $x = -1$ in the above gives

$$
-\frac{1}{4} = c_1 - \frac{1}{4} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

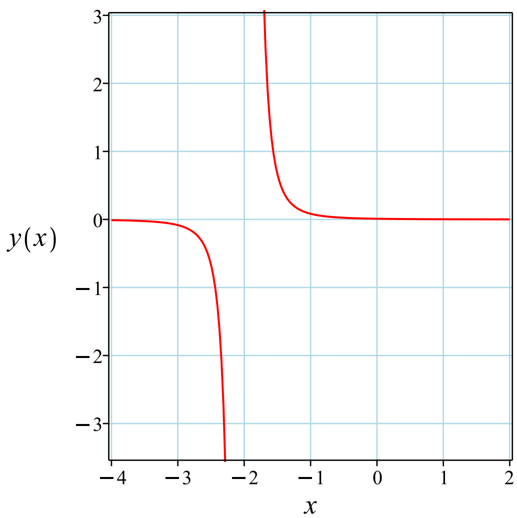
Substituting these values back in above solution results in

$$
y = \frac{1}{12\left(x+2\right)^3}
$$

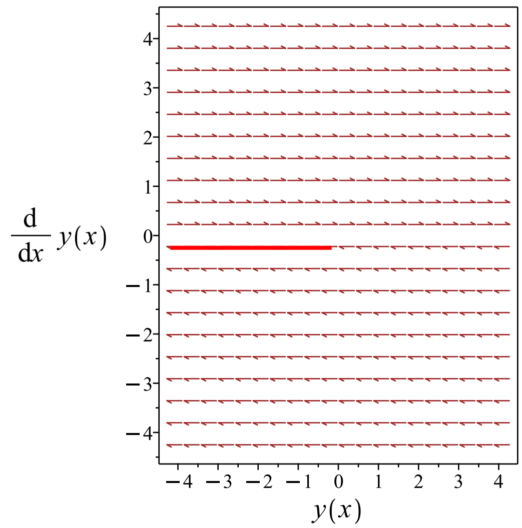
<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{1}{12\left(x+2\right)^3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{12 (x + 2)^3}$$

Verified OK.

14.3.8 Maple step by step solution

Let's solve

$$\left[y'' = \frac{1}{(x+2)^5}, y(-1) = \frac{1}{12}, y'\Big|_{\{x=-1\}} = -\frac{1}{4}\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{(x+2)^5}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\left(\int \frac{x}{(x+2)^5} dx\right) + x\left(\int \frac{1}{(x+2)^5} dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{1}{12(x+2)^3}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 + \frac{1}{12(x+2)^3}$

☐ Check validity of solution $y = c_2 x + c_1 + \frac{1}{12(x+2)^3}$

  ◦ Use initial condition $y(-1) = \frac{1}{12}$

  $\frac{1}{12} = c_1 - c_2 + \frac{1}{12}$

  ◦ Compute derivative of the solution

  $y' = c_2 - \frac{1}{4(x+2)^4}$

- Use the initial condition $y'\Big|_{\{x=-1\}} = -\frac{1}{4}$

  $-\frac{1}{4} = c_2 - \frac{1}{4}$
- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = \frac{1}{12(x+2)^3}$

- Solution to the IVP

  $y = \frac{1}{12(x+2)^3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)*(x+2)^5=1,y(-1) = 1/12, D(y)(-1) = -1/4],y(x), singsol=all)
```

$$y = \frac{1}{12\,(x+2)^3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 14

```
DSolve[{y''[x]*(x+2)^5==1,{y[-1]==1/12,y'[-1]==-1/4}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{12(x+2)^3}$$

14.4 problem 330

Internal problem ID [15188]
Internal file name [`OUTPUT/15188_Tuesday_April_23_2024_04_53_44_PM_56595732/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 330.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = e^{x} x}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

14.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 0 \\ F &= e^x x \end{aligned}$$

Hence the ode is

$$y'' = e^x x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = e^x x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.4.2 Solving as second order ode quadrature ode

Integrating once gives

$$y' = (x - 1) e^x + c_1$$

Integrating again gives

$$y = (x - 2) e^x + c_1 x + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (x - 2) e^x + c_1 x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -2 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x} + (x-2)\,e^{x} + c_1$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = e^{x}x - 2\,e^{x} + 2 + x$$

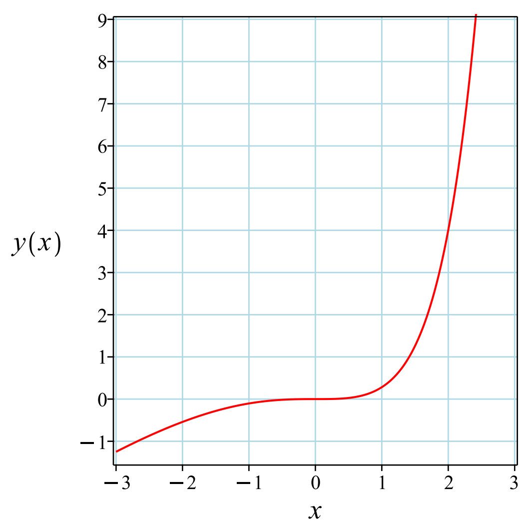
Which simplifies to

$$y = (x-2)\,e^{x} + x + 2$$

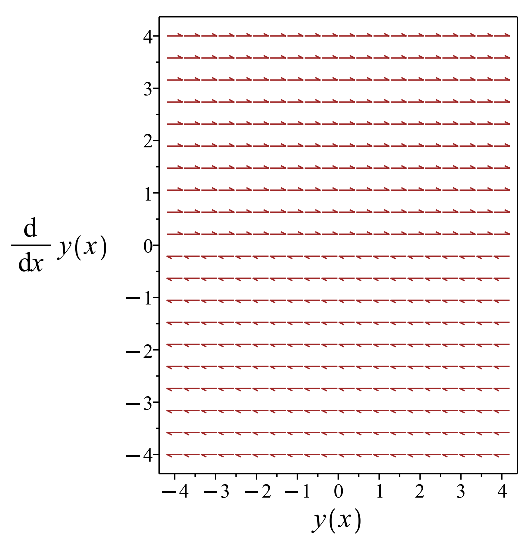
Summary

The solution(s) found are the following

$$y = (x-2)\,e^{x} + x + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x-2)\,e^{x} + x + 2$$

Verified OK.

14.4.3 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = e^x x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^x x + 2A_1 e^x + A_2 e^x = e^x x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x x - 2\, e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (e^x x - 2\, e^x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + e^x x - 2\, e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + e^x x - e^x$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = e^x x - 2e^x + 2 + x$$

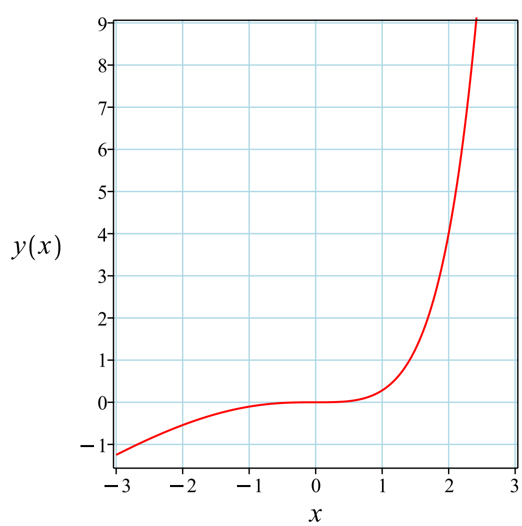
Which simplifies to

$$y = (x - 2)e^x + x + 2$$

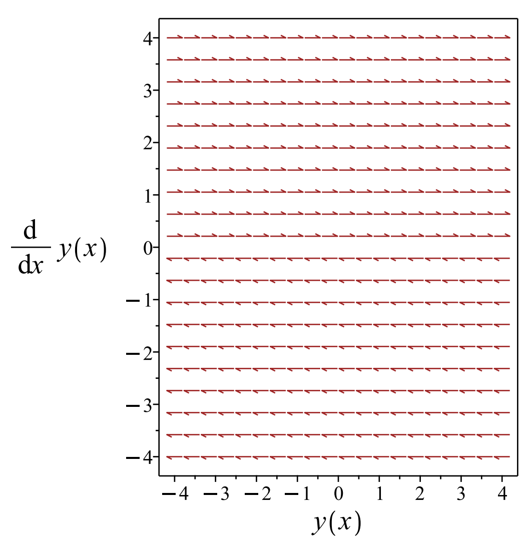
Summary

The solution(s) found are the following

$$y = (x - 2)e^x + x + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x - 2)\,e^{x} + x + 2$$

Verified OK.

14.4.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int e^{x} x dx$$
$$y' = (x - 1)\,e^{x} + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int e^{x} x - e^{x} + c_1 \, \mathrm{d}x \\ &= c_1 x + e^{x} x - 2\,e^{x} + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x + e^{x} x - 2\,e^{x} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -2 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x} x - e^{x} + c_1$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = e^x x - 2e^x + 2 + x$$

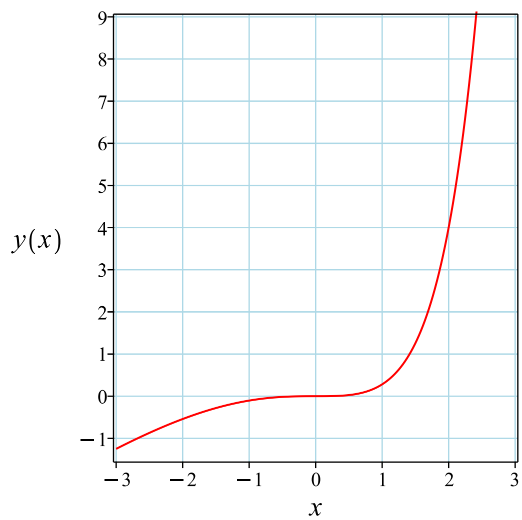
Which simplifies to

$$y = (x - 2)e^x + x + 2$$

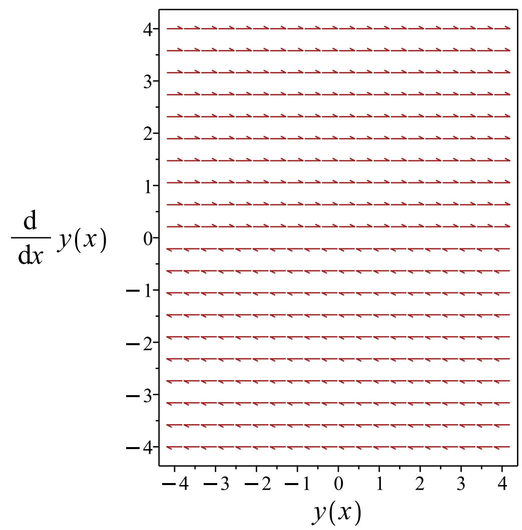
Summary

The solution(s) found are the following

$$y = (x - 2)e^x + x + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x - 2)e^x + x + 2$$

Verified OK.

14.4.5 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - e^x x = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int e^x x \, dx \\ &= (x-1) e^x + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -1 + c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$p(x) = e^x x - e^x + 1$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^x x - e^x + 1$$

Integrating both sides gives

$$\begin{aligned} y &= \int e^x x - e^x + 1 \, dx \\ &= e^x x - 2 e^x + c_2 + x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -2 + c_2$$

$$c_2 = 2$$

Substituting c_2 found above in the general solution gives

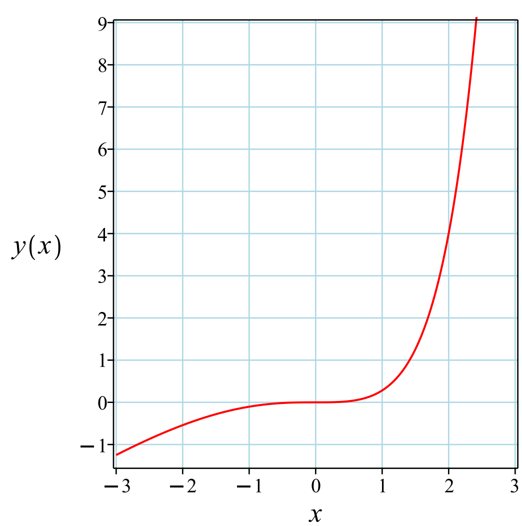
$$y = e^x x - 2e^x + 2 + x$$

Initial conditions are used to solve for the constants of integration.

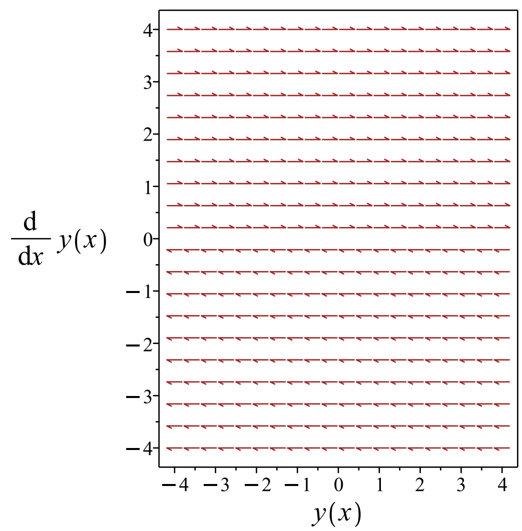
Summary

The solution(s) found are the following

$$y = e^x x - 2e^x + 2 + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^x x - 2e^x + 2 + x$$

Verified OK.

14.4.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= 1 \int \frac{1}{1}\,dx \\
&= 1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}x, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} x + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{x} x + 2A_1 e^{x} + A_2 e^{x} = e^{x} x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{x} x - 2\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + (e^{x} x - 2\,e^{x}) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x + c_1 + e^{x} x - 2\,e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 + e^x x - e^x$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = e^x x - 2e^x + 2 + x$$

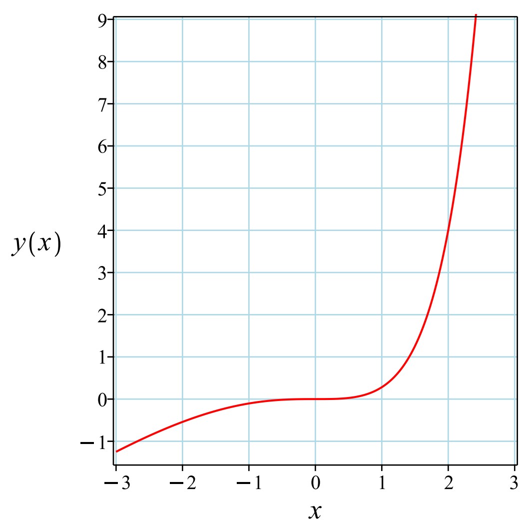
Which simplifies to

$$y = (x - 2)e^x + x + 2$$

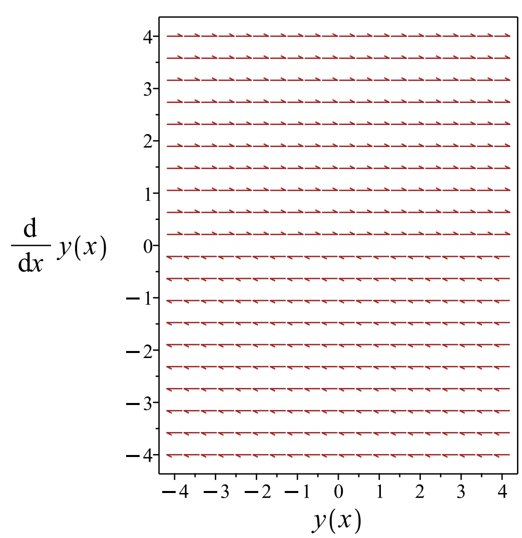
Summary

The solution(s) found are the following

$$y = (x - 2)e^x + x + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x - 2)e^x + x + 2$$

Verified OK.

14.4.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= \mathrm{e}^{x} x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + \left(q(x) - p'(x)\right) y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + \left(q(x) - p'(x)\right) y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int \mathrm{e}^{x} x \, dx$$

We now have a first order ode to solve which is

$$y' = (x - 1)\, \mathrm{e}^{x} + c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int e^x x - e^x + c_1 \, dx \\
&= c_1 x + e^x x - 2 e^x + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x + e^x x - 2 e^x + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = -2 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = e^x x - e^x + c_1
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = -1 + c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = e^x x - 2 e^x + 2 + x
$$

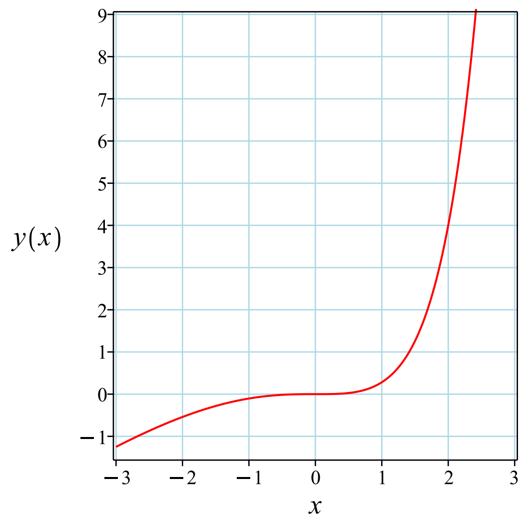
Which simplifies to

$$
y = (x - 2) e^x + x + 2
$$

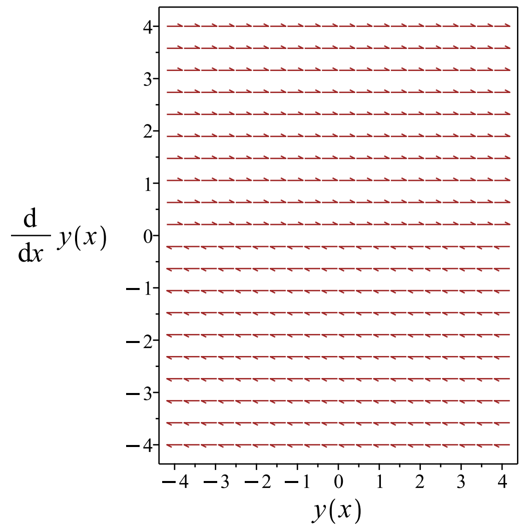
Summary

The solution(s) found are the following

$$
y = (x - 2) e^x + x + 2 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x-2)\,\mathrm{e}^{x} + x + 2$$

Verified OK.

14.4.8 Maple step by step solution

Let's solve

$$\left[y'' = \mathrm{e}^{x} x, y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

$y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^x x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\left(\int e^x x^2 dx\right) + x\left(\int e^x x dx\right)$

  ◦ Compute integrals

  $y_p(x) = (x-2)\, e^x$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 x + (x-2)\, e^x$

□ Check validity of solution $y = c_1 + c_2 x + (x-2)\, e^x$

  ◦ Use initial condition $y(0) = 0$

  $0 = -2 + c_1$

  ◦ Compute derivative of the solution

  $y' = c_2 + e^x + (x-2)\, e^x$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = (x - 2)\,\mathrm{e}^{x} + x + 2$

- Solution to the IVP

  $y = (x - 2)\,\mathrm{e}^{x} + x + 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)=x*exp(x),y(0) = 0, D(y)(0) = 0],y(x), singsol=all)
```

$$y = (x - 2)\,\mathrm{e}^{x} + x + 2$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 15

```
DSolve[{y''[x]==x*Exp[x],{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(x - 2) + x + 2$$

14.5 problem 331

Internal problem ID [15189]
Internal file name [`OUTPUT/15189_Tuesday_April_23_2024_04_53_45_PM_48279975/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 331.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second__order__integrable__as__is", "second__order__ode__missing__y", "second__order__ode__quadrature", "second__order__linear__constant__coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{y'' = 2\ln(x)\, x}$$

14.5.1 Solving as second order ode quadrature ode

Integrating once gives

$$y' = \ln(x)\, x^2 - \frac{x^2}{2} + c_1$$

Integrating again gives

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln (x)}{3} + c_1 x + c_2 \tag{1}$$

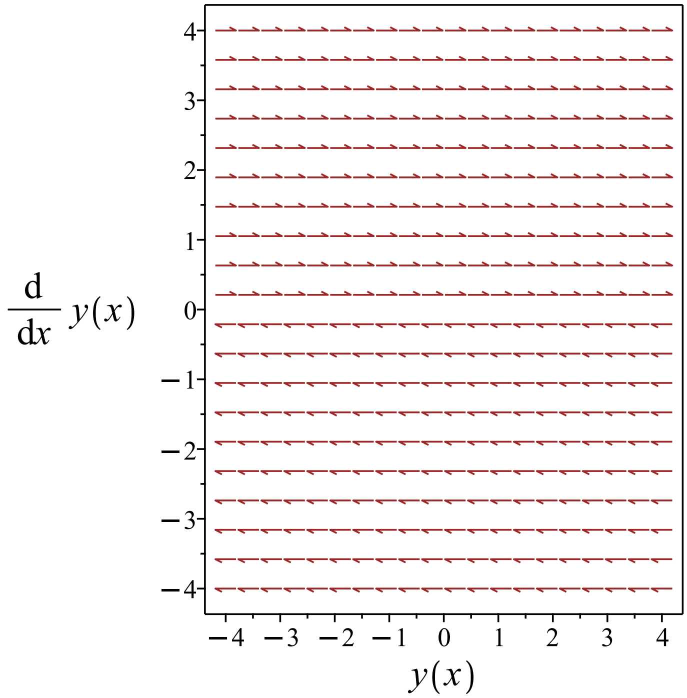


Figure 452: Slope field plot

Verification of solutions

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln (x)}{3} + c_1 x + c_2$$

Verified OK.

14.5.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = 2\ln (x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=0, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) thro927ughout by $e^{\lambda x}$ gives

$$\lambda^2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=0, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{(0)^2-(4)(1)(0)}\\&=0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2}=0$. Therefore the solution is

$$y=c_1 1+c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h=c_2 x+c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1 y_1+u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\ln(x)\, x^2}{1}\, dx$$

Which simplifies to

$$u_1 = -\int 2 \ln(x)\, x^2 dx$$

Hence

$$u_1 = -\frac{2x^3 \ln(x)}{3} + \frac{2x^3}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2 \ln(x)\, x}{1}\, dx$$

Which simplifies to

$$u_2 = \int 2 \ln(x)\, x dx$$

Hence

$$u_2 = \ln(x)\, x^2 - \frac{x^2}{2}$$

Which simplifies to

$$u_1 = -\frac{2x^3(-1 + 3\ln(x))}{9}$$
$$u_2 = x^2\left(-\frac{1}{2} + \ln(x)\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2x^3(-1 + 3\ln(x))}{9} + x^3\left(-\frac{1}{2} + \ln(x)\right)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-5 + 6\ln(x))}{18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(\frac{x^3(-5 + 6\ln(x))}{18}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + \frac{x^3(-5 + 6 \ln(x))}{18} \tag{1}$$

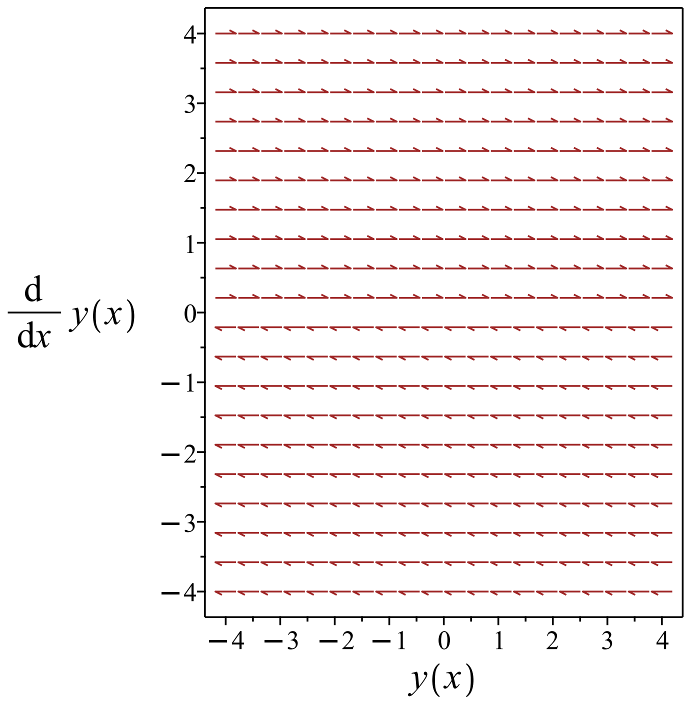


Figure 453: Slope field plot

Verification of solutions

$$y = c_2 x + c_1 + \frac{x^3(-5 + 6 \ln(x))}{18}$$

Verified OK.

14.5.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int 2 \ln(x) x dx$$

$$y' = \ln(x) x^2 - \frac{x^2}{2} + c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int -\frac{x^2}{2} + \ln(x)\, x^2 + c_1 \,\mathrm{d}x$$
$$= -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2 \tag{1}$$

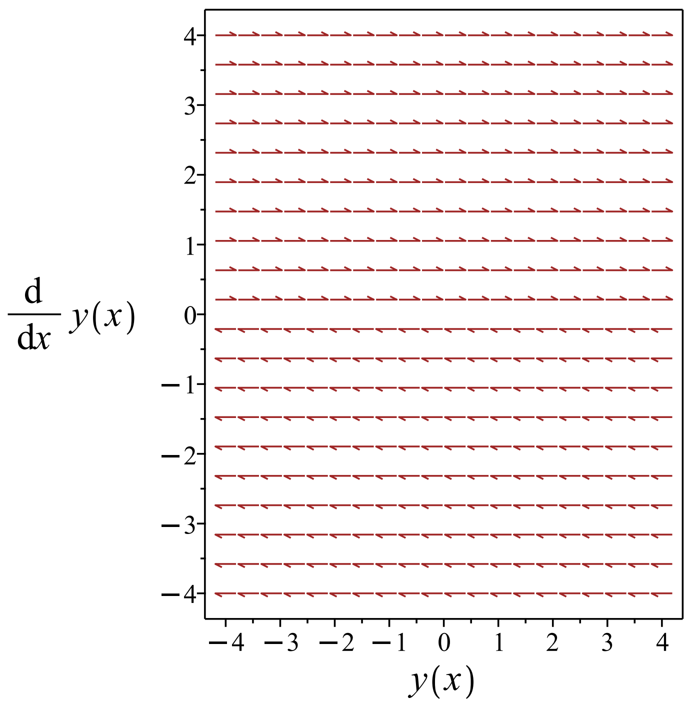


Figure 454: Slope field plot

Verification of solutions

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2$$

Verified OK.

14.5.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 2\ln(x)\, x = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned} p(x) &= \int 2\ln(x)\, x \, \mathrm{d}x \\ &= -\frac{x^2}{2} + \ln(x)\, x^2 + c_1 \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{x^2}{2} + \ln(x)\, x^2 + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{x^2}{2} + \ln(x)\, x^2 + c_1 \, \mathrm{d}x \\ &= -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2 \tag{1}$$

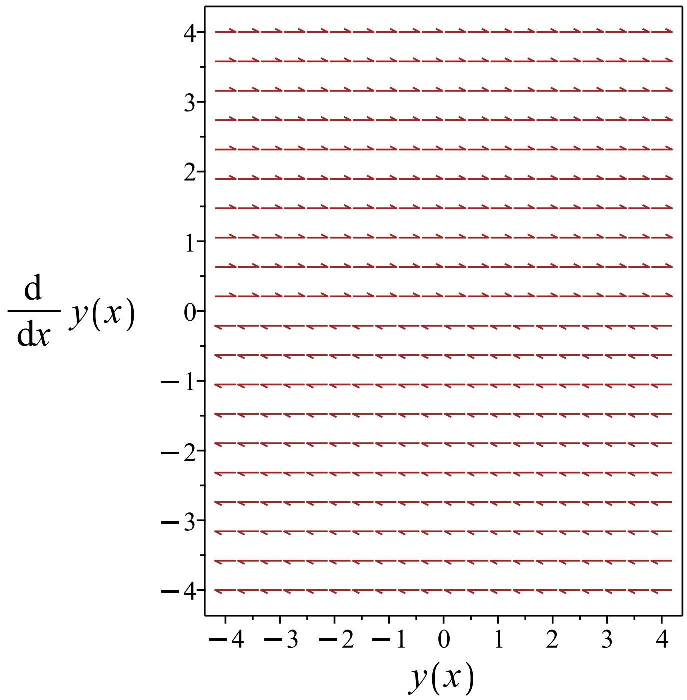


Figure 455: Slope field plot

Verification of solutions

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2$$

Verified OK.

14.5.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(1) + c_2(1(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2x + c_1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x \\ \frac{d}{dx}(1) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (x)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\ln(x)\,x^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 2\ln(x)\,x^2 dx$$

Hence

$$u_1 = -\frac{2x^3\ln(x)}{3} + \frac{2x^3}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\ln(x)\,x}{1}\,dx$$

Which simplifies to

$$u_2 = \int 2\ln(x)\,x dx$$

Hence

$$u_2 = \ln(x)\, x^2 - \frac{x^2}{2}$$

Which simplifies to

$$\begin{aligned} u_1 &= -\frac{2x^3(-1+3\ln(x))}{9} \\ u_2 &= x^2\left(-\frac{1}{2}+\ln(x)\right) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2x^3(-1+3\ln(x))}{9} + x^3\left(-\frac{1}{2}+\ln(x)\right)$$

Which simplifies to

$$y_p(x) = \frac{x^3(-5+6\ln(x))}{18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1) + \left(\frac{x^3(-5+6\ln(x))}{18}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_2x + c_1 + \frac{x^3(-5+6\ln(x))}{18} \tag{1}$$

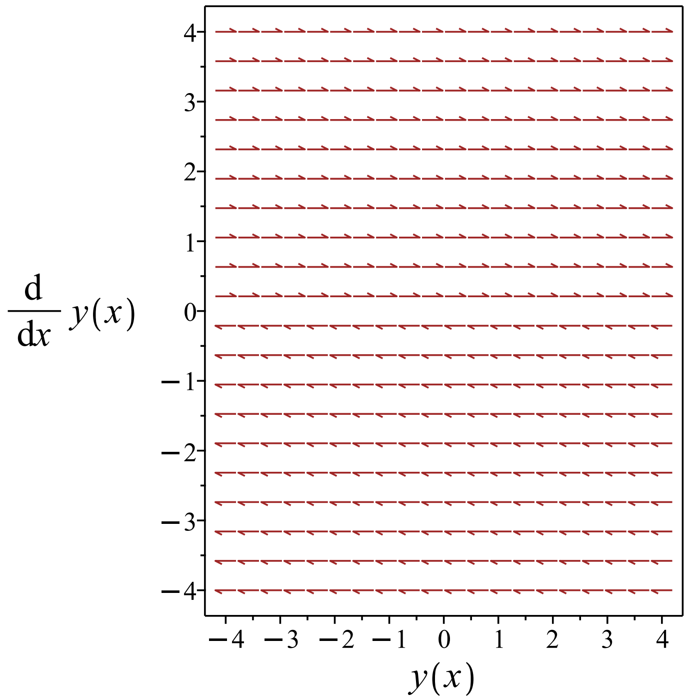


Figure 456: Slope field plot

Verification of solutions

$$y = c_2x + c_1 + \frac{x^3(-5 + 6\ln(x))}{18}$$

Verified OK.

14.5.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= 0\\
r(x) &= 0\\
s(x) &= 2\ln(x)\,x
\end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y' + (q(x)-p'(x))\,y)' = s(x)
$$

Integrating gives

$$
p(x)\,y' + (q(x)-p'(x))\,y = \int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
y' = \int 2\ln(x)\,x\,dx
$$

We now have a first order ode to solve which is

$$
y' = -\frac{x^2}{2} + \ln(x)\,x^2 + c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int -\frac{x^2}{2} + \ln(x)\,x^2 + c_1\;\mathrm{d}x \\
&= -\frac{5x^3}{18} + \frac{x^3\ln(x)}{3} + c_1x + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{5x^3}{18} + \frac{x^3\ln(x)}{3} + c_1x + c_2 \tag{1}
$$

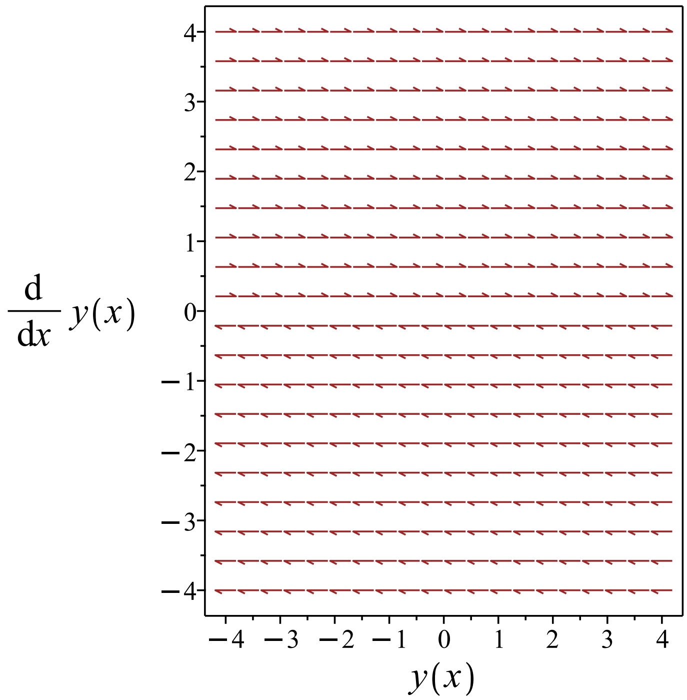


Figure 457: Slope field plot

Verification of solutions

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2$$

Verified OK.

14.5.7 Maple step by step solution

Let's solve

$y'' = 2\ln(x)\, x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial

$r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\ln(x)\, x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\left(\int \ln(x)\, x^2 dx\right) + 2x\left(\int \ln(x)\, x dx\right)$

  - Compute integrals

    $y_p(x) = \frac{x^3(-5+6\ln(x))}{18}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x + c_1 + \frac{x^3(-5+6\ln(x))}{18}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)=2*x*ln(x),y(x), singsol=all)
```

$$y = -\frac{5x^3}{18} + \frac{x^3 \ln(x)}{3} + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 28

```
DSolve[y''[x]==2*x*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5x^3}{18} + \frac{1}{3}x^3 \log(x) + c_2 x + c_1$$

14.6 problem 332

Internal problem ID [15190]
Internal file name [`OUTPUT/15190_Tuesday_April_23_2024_04_53_47_PM_83686283/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 332.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = 0}$$

14.6.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\,dx = 0$$

$$xy' - 2y = c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{2y + c_1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = 2y + c_1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{2y + c_1}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{2y + c_1}\,dy &= \int \frac{1}{x}\,dx \\ \frac{\ln(2y + c_1)}{2} &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{2y + c_1} = \mathrm{e}^{\ln(x) + c_2}$$

Which simplifies to

$$\sqrt{2y + c_1} = c_3 x$$

Summary

The solution(s) found are the following

$$y = \frac{c_3^2 \mathrm{e}^{2c_2} x^2}{2} - \frac{c_1}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_3^2 \mathrm{e}^{2c_2} x^2}{2} - \frac{c_1}{2}$$

Verified OK.

14.6.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x - p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{p}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\, dp &= \frac{1}{x}\, dx \\ \int \frac{1}{p}\, dp &= \int \frac{1}{x}\, dx \\ \ln(p) &= \ln(x) + c_1 \\ p &= \mathrm{e}^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 x$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 x \, \mathrm{d}x \\ &= \frac{c_1 x^2}{2} + c_2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 x^2}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} + c_2$$

Verified OK.

14.6.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)(0) + (-1)(0) + (0)(-1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-xv'' + (1)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{1}{x}\, dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= \mathrm{e}^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 x \,\mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-1)\left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{c_1 x^2}{2} - c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1 x^2}{2} - c_2 \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1 x^2}{2} - c_2
$$

Verified OK.

14.6.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
xy'' - y' = 0
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (xy'' - y')\,dx &= 0 \\
xy' - 2y &= c_1
\end{aligned}
$$

Which is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{2y + c_1}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(y) = 2y + c_1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{2y + c_1} \, dy &= \frac{1}{x} \, dx \\
\int \frac{1}{2y + c_1} \, dy &= \int \frac{1}{x} \, dx \\
\frac{\ln (2y + c_1)}{2} &= \ln (x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{2y + c_1} = e^{\ln(x) + c_2}
$$

Which simplifies to

$$
\sqrt{2y + c_1} = c_3 x
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_3^2 e^{2c_2} x^2}{2} - \frac{c_1}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_3^2 e^{2c_2} x^2}{2} - \frac{c_1}{2}
$$

Verified OK.

14.6.5 Solving using Kovacic algorithm

Writing the ode as

$$
xy'' - y' = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x \\
B &= -1 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 + \frac{c_2 x^2}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 + \frac{c_2 x^2}{2}
$$

Verified OK.

14.6.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$xy' - 2y = c_1$$

We now have a first order ode to solve which is

$$xy' - 2y = c_1$$

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{2y+c_1}{x}
\end{aligned}
$$

Where $f(x)=\frac{1}{x}$ and $g(y)=2y+c_1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{2y+c_1}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{2y+c_1}\,dy &= \int \frac{1}{x}\,dx \\
\frac{\ln(2y+c_1)}{2} &= \ln(x)+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{2y+c_1} = \mathrm{e}^{\ln(x)+c_2}
$$

Which simplifies to

$$
\sqrt{2y+c_1} = c_3 x
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_3^2\mathrm{e}^{2c_2}x^2}{2} - \frac{c_1}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_3^2\mathrm{e}^{2c_2}x^2}{2} - \frac{c_1}{2}
$$

Verified OK.

14.6.7 Maple step by step solution

Let's solve

$y''x - y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

  $y''x - y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x - \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

  $\frac{\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t)}{x} = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = 2\frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$

- 1st solution of the ODE

  $y_1(t) = 1$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 + c_2 e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^2 + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x$2)=diff(y(x),x),y(x), singsol=all)
```

$$y = c_2 x^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 17

```
DSolve[x*y''[x]==y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 x^2}{2} + c_2$$

14.7 problem 333

Internal problem ID [15191]
Internal file name [`OUTPUT/15191_Tuesday_April_23_2024_04_53_48_PM_25907837/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 333.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_y]]`

$$\boxed{xy'' + y' = 0}$$

14.7.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$xy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

14.7.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{p}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\, dp &= -\frac{1}{x}\, dx \\
\int \frac{1}{p}\, dp &= \int -\frac{1}{x}\, dx \\
\ln(p) &= -\ln(x) + c_1 \\
p &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1}{x}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{c_1}{x}\, \mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

14.7.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x \\
B &= 1 \\
C &= 0 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)\,(0) + (1)\,(0) + (0)\,(1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
xv'' + (1)\,v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x}\,\mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (1)\,(c_1 \ln(x) + c_2) \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

14.7.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$xy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \frac{c_1}{x}\, \mathrm{d}x$$
$$= c_1 \ln(x) + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

14.7.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = x$$
$$B = 1 \tag{3}$$
$$C = 0$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{x}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \ln(x)$$

Verified OK.

14.7.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x \\
q(x) &= 1 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
xy' = c_1
$$

We now have a first order ode to solve which is

$$
xy' = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{c_1}{x}\, \mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = c_1 \ln(x) + c_2
$$

Verified OK.

14.7.7 Maple step by step solution

Let's solve

$y''x + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

  $y''x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x + \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

  $\frac{\frac{d^2}{dt^2}y(t)}{x} = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(t) = 1$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

  $y(t) = c_2t + c_1$

- Change variables back using $t = \ln(x)$

  $y = c_1 + c_2 \ln(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y = c_2 \ln (x) + c_1$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 13

```
DSolve[x*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x) + c_2$$

14.8 problem 334

Internal problem ID [15192]
Internal file name [`OUTPUT/15192_Tuesday_April_23_2024_04_53_49_PM_72080280/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 334.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - (2x^2 + 1)\, y' = 0}$$

14.8.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x + (-2x^2 - 1)\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{p(2x^2+1)}{x} \end{aligned}$$

Where $f(x) = \frac{2x^2+1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\, dp &= \frac{2x^2+1}{x}\, dx \\ \int \frac{1}{p}\, dp &= \int \frac{2x^2+1}{x}\, dx \\ \ln(p) &= x^2 + \ln(x) + c_1 \\ p &= e^{x^2+\ln(x)+c_1} \\ &= c_1 e^{x^2+\ln(x)} \end{aligned}$$

Which simplifies to

$$p(x) = c_1 x\, e^{x^2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 x\, e^{x^2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 x\, e^{x^2}\, dx \\ &= \frac{c_1 e^{x^2}}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{x^2}}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 e^{x^2}}{2} + c_2$$

Verified OK.

14.8.2 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-2x^2 - 1)\, y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x^2 - 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 341: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$
L = [1, 2]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = x^2 + \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{8x^3} - \frac{9}{128x^7} + \frac{27}{1024x^{11}} - \frac{405}{32768x^{15}} + \frac{1701}{262144x^{19}} - \frac{15309}{4194304x^{23}} + \frac{72171}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (x^2) + \left(\frac{3}{4x^2}\right) \\
&= x^2 + \frac{3}{4x^2}
\end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{1} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{1} - 1\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^4+3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(x) \\ &= -\frac{1}{2x} - x \\ &= -\frac{1}{2x} - x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - x\right)(0) + \left(\left(\frac{1}{2x^2} - 1\right) + \left(-\frac{1}{2x} - x\right)^2 - \left(\frac{4x^4+3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= e^{\int \left(-\frac{1}{2x} - x\right)dx} \\ &= \frac{e^{-\frac{x^2}{2}}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x^2-1}{x}\,dx} \\
&= z_1 e^{\frac{x^2}{2}+\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\,\mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{x^2}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{x^2}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \frac{c_2 \mathrm{e}^{x^2}}{2} \tag{1}
$$

Verification of solutions

$$y = c_1 + \frac{c_2 \mathrm{e}^{x^2}}{2}$$

Verified OK.

14.8.3 Maple step by step solution

Let's solve

$y''x + (-2x^2 - 1) y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x^2+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2x^2+1)y'}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2x^2+1}{x}, P_3(x) = 0\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (-2x^2 - 1) y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k->k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + a_1(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+r-1) - 2a_{k-1}(k+r-\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term must be 0

  $a_1(1+r)\,(-1+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_{k+1}(k+1+r) - 2a_{k-1}) = 0$
- Shift index using $k->k+1$

  $(k+r)\,(a_{k+2}(k+2+r) - 2a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{2a_k}{k+2+r}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{2a_k}{k+2}$
- Solution for $r = 0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{2a_k}{k+2}, -a_1=0\right]$$

- Recursion relation for $r=2$

$a_{k+2}=\frac{2a_k}{k+4}$

- Solution for $r=2$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{2a_k}{k+4}, 3a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2}=\frac{2a_k}{k+2}, -a_1=0, b_{k+2}=\frac{2b_k}{k+4}, 3b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x$2)=(1+2*x^2)*diff(y(x),x),y(x), singsol=all)
```

$$y=c_1+e^{x^2}c_2$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[x*y''[x]==(1+2*x^2)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{c_1 e^{x^2}}{2}+c_2$$

14.9 problem 335

Internal problem ID [15193]
Internal file name [`OUTPUT/15193_Tuesday_April_23_2024_04_53_50_PM_17322247/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 335.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = x^2}$$

14.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\, dx = \int x^2 dx$$
$$xy' - 2y = \frac{x^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x^3 + 3c_1}{3x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^3 + 3c_1}{3x}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right)$$
$$d\left(\frac{y}{x^2}\right) = \left(\frac{x^3 + 3c_1}{3x^3}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x^3 + 3c_1}{3x^3} dx$$
$$\frac{y}{x^2} = \frac{x}{3} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Verified OK.

14.9.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x - p(x) - x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = x$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ d\left(\frac{p}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int dx \\ \frac{p}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x + x^2$$

which simplifies to

$$p(x) = x(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x(x + c_1)$$

Integrating both sides gives

$$\begin{aligned}y &= \int x(x + c_1)\, dx \\ &= \frac{1}{3}x^3 + \frac{1}{2}c_1 x^2 + c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3}x^3 + \frac{1}{2}c_1x^2 + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 + \frac{1}{2}c_1x^2 + c_2$$

Verified OK.

14.9.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x \\
B &= -1 \\
C &= 0 \\
F &= x^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)\,(0) + (-1)\,(0) + (0)\,(-1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-xv'' + (1)\,v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}v(x) &= \int c_1 x \, \mathrm{d}x\\ &= \frac{c_1 x^2}{2} + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv\\ &= (-1)\left(\frac{c_1 x^2}{2} + c_2\right)\\ &= -\frac{c_1 x^2}{2} - c_2\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= -1\\ y_2 &= x^2\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4}{-2x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^2}{2} dx$$

Hence

$$u_1 = \frac{x^3}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-x^2}{-2x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2} dx$$

Hence

$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{3}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{c_1 x^2}{2} - c_2\right) + \left(\frac{x^3}{3}\right) \\ &= -\frac{1}{2}c_1 x^2 - c_2 + \frac{1}{3}x^3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{2}c_1 x^2 - c_2 + \frac{1}{3}x^3 \tag{1}$$

Verification of solutions

$$y = -\frac{1}{2}c_1 x^2 - c_2 + \frac{1}{3}x^3$$

Verified OK.

14.9.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' - y' = x^2$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int (xy'' - y')\, dx &= \int x^2 dx \\ xy' - 2y &= \frac{x^3}{3} + c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x^3 + 3c_1}{3x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^3 + 3c_1}{3x}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right)$$
$$d\left(\frac{y}{x^2}\right) = \left(\frac{x^3 + 3c_1}{3x^3}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x^3 + 3c_1}{3x^3} dx$$
$$\frac{y}{x^2} = \frac{x}{3} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Verified OK.

14.9.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 343: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 x^2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{x^2}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^4}{2}}{x^2} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2}{2} dx$$

Hence

$$u_1 = -\frac{x^3}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^2} \, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\frac{x^3}{3}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + \frac{1}{2} c_2 x^2 + \frac{1}{3} x^3 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 + \frac{1}{2} c_2 x^2 + \frac{1}{3} x^3$$

Verified OK.

14.9.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= x^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$xy' - 2y = \int x^2\,dx$$

We now have a first order ode to solve which is

$$xy' - 2y = \frac{x^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x^3 + 3c_1}{3x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x^3 + 3c_1}{3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x^3 + 3c_1}{3x}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{x^3 + 3c_1}{3x^3}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x^3 + 3c_1}{3x^3}\,\mathrm{d}x$$
$$\frac{y}{x^2} = \frac{x}{3} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{x}{3} - \frac{c_1}{2x^2}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{3}x^3 - \frac{1}{2}c_1 + c_2 x^2$$

Verified OK.

14.9.7 Maple step by step solution

Let's solve

$y''x - y' = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\,x - u(x) = x^2$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} + x$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{u(x)}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu(x)\,x$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)-\frac{u(x)}{x}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\mu(x)\,xdx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\mu(x)\,xdx+c_1$
- Solve for $u(x)$

  $u(x)=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x}$

  $u(x)=x\left(\int 1dx+c_1\right)$
- Evaluate the integrals on the rhs

  $u(x)=x(x+c_1)$
- Solve 1st ODE for $u(x)$

  $u(x)=x(x+c_1)$
- Make substitution $u=y'$

  $y'=x(x+c_1)$
- Integrate both sides to solve for y

  $\int y'dx=\int x(x+c_1)\,dx+c_2$
- Compute integrals

  $y=\frac{1}{3}x^3+\frac{1}{2}c_1x^2+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_a^2+_b(_a))/_a, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)=diff(y(x),x)+x^2,y(x), singsol=all)
```

$$y = \frac{1}{3}x^3 + \frac{1}{2}c_1x^2 + c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[x*y''[x]==y'[x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} + \frac{c_1x^2}{2} + c_2$$

14.10 problem 336

Internal problem ID [15194]
Internal file name [`OUTPUT/15194_Tuesday_April_23_2024_04_53_52_PM_3577603/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 336.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x \ln(x) y'' - y' = 0}$$

14.10.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x \ln(x) p'(x) - p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(x,p) \\
&= f(x)g(p) \\
&= \frac{p}{x\ln(x)}
\end{aligned}
$$

Where $f(x)=\frac{1}{x\ln(x)}$ and $g(p)=p$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p}\,dp &= \frac{1}{x\ln(x)}\,dx \\
\int \frac{1}{p}\,dp &= \int \frac{1}{x\ln(x)}\,dx \\
\ln(p) &= \ln(\ln(x)) + c_1 \\
p &= e^{\ln(\ln(x))+c_1} \\
&= c_1\ln(x)
\end{aligned}
$$

Since $p=y'$ then the new first order ode to solve is

$$
y' = c_1\ln(x)
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1\ln(x)\,\mathrm{d}x \\
&= c_1(\ln(x)\,x - x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(\ln(x)\,x - x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = c_1(\ln(x)\,x - x) + c_2
$$

Verified OK.

14.10.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= \ln(x)\, x \\ B &= -1 \\ C &= 0 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (\ln(x)\, x)\,(0) + (-1)\,(0) + (0)\,(-1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-\ln(x)\,xv'' + (1)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$-\ln(x)\,xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{x\ln(x)} \end{aligned}$$

Where $f(x) = \frac{1}{x\ln(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{1}{x\ln(x)}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{1}{x\ln(x)}\,dx \\ \ln(u) &= \ln(\ln(x)) + c_1 \\ u &= e^{\ln(\ln(x))+c_1} \\ &= c_1\ln(x) \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1\ln(x) \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1\ln(x)\,\mathrm{d}x \\ &= c_1(\ln(x)\,x - x) + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-1)\,(c_1(\ln(x)\,x - x) + c_2) \\ &= -xc_1\ln(x) + c_1x - c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -x c_1 \ln(x) + c_1 x - c_2 \tag{1}$$

Verification of solutions

$$y = -x c_1 \ln(x) + c_1 x - c_2$$

Verified OK.

14.10.3 Maple step by step solution

Let's solve

$x \ln(x) y'' - y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $x \ln(x) u'(x) - u(x) = 0$

- Separate variables

  $\frac{u'(x)}{u(x)} = \frac{1}{x \ln(x)}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)} dx = \int \frac{1}{x \ln(x)} dx + c_1$

- Evaluate integral

  $\ln(u(x)) = \ln(\ln(x)) + c_1$

- Solve for $u(x)$

  $u(x) = e^{c_1} \ln(x)$

- Solve 1st ODE for $u(x)$

  $u(x) = e^{c_1} \ln(x)$

- Make substitution $u = y'$

  $y' = e^{c_1} \ln(x)$

- Integrate both sides to solve for y

  $\int y' dx = \int e^{c_1} \ln(x) dx + c_2$

- Compute integrals

$$y = e^{c_1}(\ln(x)\, x - x) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*ln(x)*diff(y(x),x$2)=diff(y(x),x),y(x), singsol=all)
```

$$y = \ln(x)\, c_2 x - c_2 x + c_1$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 19

```
DSolve[x*Log[x]*y''[x]==y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(-x) + c_1 x \log(x) + c_2$$

14.11 problem 337

Internal problem ID [15195]
Internal file name [`OUTPUT/15195_Tuesday_April_23_2024_04_53_53_PM_67792014/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 337.
ODE order: 1.
ODE degree: 0.

The type(s) of ODE detected by this program : **"separable"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yx - y' \ln\left(\frac{y'}{x}\right) = 0}$$

14.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{yx}{\operatorname{LambertW}(y)} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{y}{\operatorname{LambertW}(y)}$. Integrating both sides gives

$$\frac{1}{\frac{y}{\operatorname{LambertW}(y)}}\, dy = x\, dx$$

$$\int \frac{1}{\frac{y}{\operatorname{LambertW}(y)}}\, dy = \int x\, dx$$

$$\frac{\operatorname{LambertW}(y)^2}{2} + \operatorname{LambertW}(y) = \frac{x^2}{2} + c_1$$

Which results in

$$y = \left(-1+\sqrt{x^2+2c_1+1}\right) \mathrm{e}^{-1+\sqrt{x^2+2c_1+1}}$$

$$y = \left(-1-\sqrt{x^2+2c_1+1}\right) \mathrm{e}^{-1-\sqrt{x^2+2c_1+1}}$$

Summary

The solution(s) found are the following

$$y = \left(-1+\sqrt{x^2+2c_1+1}\right) \mathrm{e}^{-1+\sqrt{x^2+2c_1+1}} \tag{1}$$

$$y = \left(-1-\sqrt{x^2+2c_1+1}\right) \mathrm{e}^{-1-\sqrt{x^2+2c_1+1}} \tag{2}$$

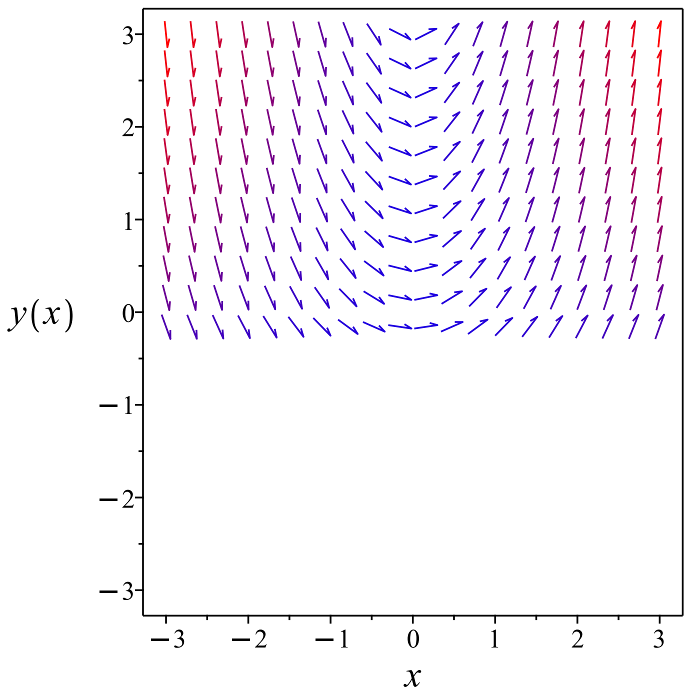


Figure 458: Slope field plot

Verification of solutions

$$y = \left(-1+\sqrt{x^2+2c_1+1}\right) \mathrm{e}^{-1+\sqrt{x^2+2c_1+1}}$$

Verified OK.

$$y = \left(-1-\sqrt{x^2+2c_1+1}\right) \mathrm{e}^{-1-\sqrt{x^2+2c_1+1}}$$

Verified OK.

14.11.2 Maple step by step solution

Let's solve

$yx - y' \ln\left(\frac{y'}{x}\right) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \mathit{LambertW}(y)}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y' \mathit{LambertW}(y)}{y} dx = \int x dx + c_1$

- Evaluate integral

  $\frac{\mathit{LambertW}(y)^2}{2} + \mathit{LambertW}(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \left(-1 - \sqrt{x^2 + 2c_1 + 1}\right) \mathrm{e}^{-1-\sqrt{x^2+2c_1+1}}, y = \left(-1 + \sqrt{x^2 + 2c_1 + 1}\right) \mathrm{e}^{-1+\sqrt{x^2+2c_1+1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 63

```
dsolve(x*y(x)=diff(y(x),x)*ln(diff(y(x),x)/x),y(x), singsol=all)
```

$$y = \left(-1-\sqrt{x^2-2c_1+1}\right) \mathrm{e}^{-1-\sqrt{x^2-2c_1+1}}$$
$$y = \left(-1+\sqrt{x^2-2c_1+1}\right) \mathrm{e}^{-1+\sqrt{x^2-2c_1+1}}$$

✓ Solution by Mathematica

Time used: 4.223 (sec). Leaf size: 83

```
DSolve[x*y[x]==y'[x]*Log[y'[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-1-\sqrt{x^2+1+2c_1}}\left(1+\sqrt{x^2+1+2c_1}\right)$$
$$y(x) \to e^{-1+\sqrt{x^2+1+2c_1}}\left(-1+\sqrt{x^2+1+2c_1}\right)$$
$$y(x) \to 0$$

14.12 problem 338

Internal problem ID [15196]
Internal file name [`OUTPUT/15196_Tuesday_April_23_2024_04_53_54_PM_68807839/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 338.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_poly_yn
    ]]
```

$$\boxed{2y'' - \frac{y'}{x} - \frac{x^2}{y'} = 0}$$

With initial conditions

$$\left[y(1) = \frac{\sqrt{2}}{5}, y'(1) = \frac{\sqrt{2}}{2}\right]$$

14.12.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2p'(x)\,xp(x) - x^3 - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Writing the ode as

$$p'(x) = \frac{x^3 + p^2}{2xp}$$
$$p'(x) = \omega(x, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2 \xi_p - \omega_x \xi - \omega_p \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 347: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,p) &= 0\\ \eta(x,p) &= \frac{x}{p}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{x}{p}} dy\end{aligned}$$

Which results in

$$S = \frac{p^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p)S_p}{R_x + \omega(x,p)R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = \frac{x^3 + p^2}{2xp}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_p &= 0 \\
S_x &= -\frac{p^2}{2x^2} \\
S_p &= \frac{p}{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{x}{2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R}{2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^2}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$
\frac{p(x)^2}{2x} = \frac{x^2}{4} + c_1
$$

Which simplifies to

$$
\frac{p(x)^2}{2x} = \frac{x^2}{4} + c_1
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = \frac{\sqrt{2}}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{1}{4} = \frac{1}{4} + c_1
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2x} = \frac{x^2}{4}$$

The above simplifies to

$$-x^3 + 2p^2 = 0$$

Solving for $p(x)$ from the above gives

$$p(x) = \frac{\sqrt{2}\, x^{\frac{3}{2}}}{2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\sqrt{2}\, x^{\frac{3}{2}}}{2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sqrt{2}\, x^{\frac{3}{2}}}{2}\, \mathrm{d}x \\ &= \frac{x^{\frac{5}{2}}\sqrt{2}}{5} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = \frac{\sqrt{2}}{5}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{2}}{5} = \frac{\sqrt{2}}{5} + c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$y = \frac{x^{\frac{5}{2}}\sqrt{2}}{5}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{x^{\frac{5}{2}}\sqrt{2}}{5} \tag{1}$$

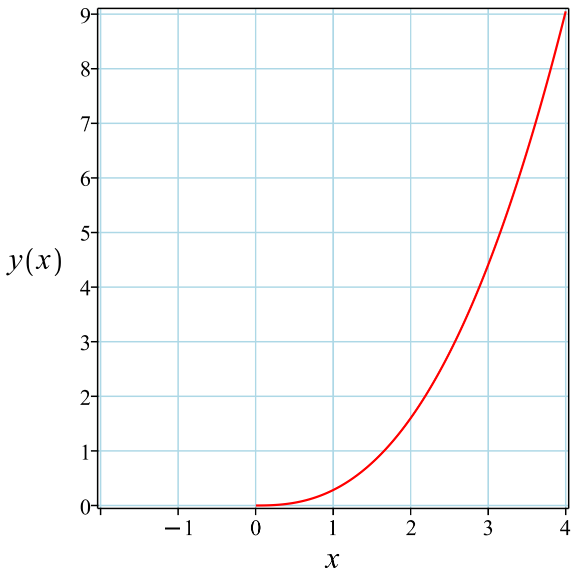


Figure 459: Solution plot

Verification of solutions

$$y = \frac{x^{\frac{5}{2}}\sqrt{2}}{5}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/2)*(_a^3+_b(_a)^2)/(_a*_b(_a)), _b(_
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 3/2*_b]
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 12

```
dsolve([2*diff(y(x),x$2)=diff(y(x),x)/x+x^2/diff(y(x),x),y(1) = 1/5*sqrt(2), D(y)(1) = 1/2*s
```

$$y = \frac{\sqrt{2}\, x^{\frac{5}{2}}}{5}$$

✓ Solution by Mathematica

Time used: 0.12 (sec). Leaf size: 26

```
DSolve[{2*y''[x]==y'[x]/x+x^2/y'[x],{y[1]==Sqrt[2]/5,y'[1]==Sqrt[2]/2}},y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{5}\sqrt{2}x^{3/2}\sqrt{x^2}$$

14.13 problem 339

Internal problem ID [15197]
Internal file name [`OUTPUT/15197_Tuesday_April_23_2024_04_53_56_PM_69806491/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 339.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x], [_3rd_order, _missing_y], [
   _3rd_order, _with_linear_symmetries], [_3rd_order, _reducible
   , _mu_y2]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

✗ Solution by Maple

```
dsolve(diff(y(x),x$3)=sqrt(1-diff(y(x),x$2)^2),y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica
Time used: 0.287 (sec). Leaf size: 34

```
DSolve[y'''[x]==Sqrt[1-y''[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3x - \cos(x + c_1) + c_2$$
$$y(x) \to \text{Interval}[\{-1, 1\}] + c_3x + c_2$$

14.14 problem 340

Internal problem ID [15198]
Internal file name [`OUTPUT/15198_Tuesday_April_23_2024_04_53_56_PM_24783477/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 340.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{xy''' - y'' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$xv''(x) - v'(x) = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xv''(x) - v'(x))\,dx = 0$$
$$v'(x)\,x - 2v(x) = c_1$$

Which is now solved for $v(x)$. In canonical form the ODE is

$$\begin{aligned} v' &= F(x,v) \\ &= f(x)g(v) \\ &= \frac{2v + c_1}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(v) = 2v + c_1$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{2v + c_1}\, dv &= \frac{1}{x}\, dx \\
\int \frac{1}{2v + c_1}\, dv &= \int \frac{1}{x}\, dx \\
\frac{\ln(2v + c_1)}{2} &= \ln(x) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{2v + c_1} = \mathrm{e}^{\ln(x) + c_2}$$

Which simplifies to

$$\sqrt{2v + c_1} = c_3 x$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \frac{c_3^2 \mathrm{e}^{2c_2} x^2}{2} - \frac{c_1}{2}$. Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{c_3^2 \mathrm{e}^{2c_2} x^2}{2} - \frac{c_1}{2}\, \mathrm{d}x \\
&= \frac{c_3^2 \mathrm{e}^{2c_2} x^3}{6} - \frac{c_1 x}{2} + c_4
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3^2 \mathrm{e}^{2c_2} x^3}{6} - \frac{c_1 x}{2} + c_4 \tag{1}$$

Verification of solutions

$$y = \frac{c_3^2 \mathrm{e}^{2c_2} x^3}{6} - \frac{c_1 x}{2} + c_4$$

Verified OK.

14.14.1 Maple step by step solution

Let's solve

$xy''' - y'' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{y''}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} = 0$

- Multiply by denominators of the ODE

  $xy''' - y'' = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

$$x\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - \frac{\frac{d^2}{dt^2}y(t)}{x^2} + \frac{\frac{d}{dt}y(t)}{x^2} = 0$$

- Simplify

$$\frac{\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t)}{x^2} = 0$$

- Isolate 3rd derivative

$$\frac{d^3}{dt^3}y(t) = 4\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) = 0$$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y(t)$

  - Define new variable $y_2(t)$

    $y_2(t) = \frac{d}{dt}y(t)$

  - Define new variable $y_3(t)$

    $y_3(t) = \frac{d^2}{dt^2}y(t)$

  - Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

    $\frac{d}{dt}y_3(t) = 4y_3(t) - 3y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) - 3y_2(t)\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 4 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{9}\\ \frac{1}{3}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{3t} \cdot \left[\begin{array}{c}\frac{1}{9}\\ \frac{1}{3}\\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_2 e^t \cdot \left[\begin{array}{c}1\\ 1\\ 1\end{array}\right] + c_3 e^{3t} \cdot \left[\begin{array}{c}\frac{1}{9}\\ \frac{1}{3}\\ 1\end{array}\right] + \left[\begin{array}{c}c_1\\ 0\\ 0\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = c_2 e^t + \frac{c_3 e^{3t}}{9} + c_1$$

- Change variables back using $t = \ln(x)$

$$y = c_2 x + \frac{1}{9} c_3 x^3 + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*diff(y(x),x$3)-diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y = c_3x^3 + c_2x + c_1$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 21

```
DSolve[x*y'''[x]-y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1x^3}{6} + c_3x + c_2$$

14.15 problem 341

Internal problem ID [15199]
Internal file name [`OUTPUT/15199_Tuesday_April_23_2024_04_53_57_PM_68681805/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 341.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \sqrt{1 + y'^2} = 0}$$

14.15.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \sqrt{1 + p(x)^2} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{\sqrt{p^2+1}}dp = x + c_1$$
$$\operatorname{arcsinh}(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \sinh(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \sinh(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \sinh(x + c_1)\,\mathrm{d}x \\ &= \cosh(x + c_1) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cosh(x + c_1) + c_2 \tag{1}$$

Verification of solutions

$$y = \cosh(x + c_1) + c_2$$

Verified OK.

14.15.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) = \sqrt{1+p(y)^2}$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{\sqrt{p^2+1}}dp = \int dy$$
$$\sqrt{1+p(y)^2} = y + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\sqrt{1+y'^2} = y + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2+2yc_1+c_1^2-1} \tag{1}$$
$$y' = -\sqrt{y^2+2yc_1+c_1^2-1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_1^2+2c_1y+y^2-1}}dy = \int dx$$
$$\ln\left(y+c_1+\sqrt{c_1^2+2c_1y+y^2-1}\right) = x + c_2$$

Raising both side to exponential gives

$$y+c_1+\sqrt{c_1^2+2c_1y+y^2-1} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$y+c_1+\sqrt{(c_1+y+1)(c_1+y-1)} = \mathrm{e}^{x}c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_1^2+2c_1y+y^2-1}}dy = \int dx$$

$$-\ln\left(y+c_1+\sqrt{c_1^2+2c_1y+y^2-1}\right) = x+c_4$$

Raising both side to exponential gives

$$\frac{1}{y+c_1+\sqrt{c_1^2+2c_1y+y^2-1}} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+c_1+\sqrt{(c_1+y+1)(c_1+y-1)}} = c_5\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{(\mathrm{e}^{2x}c_3^2-2\,\mathrm{e}^{x}c_1c_3+1)\,\mathrm{e}^{-x}}{2c_3} \tag{1}$$

$$y = -\frac{(-c_5^2\mathrm{e}^{2x}+2c_1c_5\mathrm{e}^{x}-1)\,\mathrm{e}^{-x}}{2c_5} \tag{2}$$

Verification of solutions

$$y = \frac{(\mathrm{e}^{2x}c_3^2-2\,\mathrm{e}^{x}c_1c_3+1)\,\mathrm{e}^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(-c_5^2\mathrm{e}^{2x}+2c_1c_5\mathrm{e}^{x}-1)\,\mathrm{e}^{-x}}{2c_5}$$

Verified OK.

14.15.3 Maple step by step solution

Let's solve

$y'' = \sqrt{1+y'^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

$$u'(x) = \sqrt{1 + u(x)^2}$$

- Separate variables

$$\frac{u'(x)}{\sqrt{1+u(x)^2}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{\sqrt{1+u(x)^2}}dx = \int 1dx + c_1$$

- Evaluate integral

$$\operatorname{arcsinh}(u(x)) = x + c_1$$

- Solve for $u(x)$

$$u(x) = \sinh(x + c_1)$$

- Solve 1st ODE for $u(x)$

$$u(x) = \sinh(x + c_1)$$

- Make substitution $u = y'$

$$y' = \sinh(x + c_1)$$

- Integrate both sides to solve for y

$$\int y'dx = \int \sinh(x + c_1)\, dx + c_2$$

- Compute integrals

$$y = \cosh(x + c_1) + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)-(diff(y(x), x)), y(x)`   *
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)^2+1)^(1/2), _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)=sqrt(1+diff(y(x),x)^2),y(x), singsol=all)
```

$$y = -ix + c_1$$
$$y = ix + c_1$$
$$y = \cosh(x + c_1) + c_2$$

✓ Solution by Mathematica

Time used: 0.231 (sec). Leaf size: 29

```
DSolve[y''[x]==Sqrt[1+y'[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(e^{-x-c_1} + e^{x+c_1}\right) + c_2$$

14.16 problem 342

Internal problem ID [15200]
Internal file name [`OUTPUT/15200_Tuesday_April_23_2024_04_53_58_PM_61534917/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 342.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' - {y'}^2 = 0}$$

14.16.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2} dp = x + c_1$$
$$-\frac{1}{p} = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\frac{1}{x + c_1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{x + c_1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{1}{x + c_1} \, \mathrm{d}x \\ &= -\ln(x + c_1) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\ln(x + c_1) + c_2 \tag{1}$$

Verification of solutions

$$y = -\ln(x + c_1) + c_2$$

Verified OK.

14.16.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right)-p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned}\int \frac{1}{p}dp &= y+c_1\\ \ln(p) &= y+c_1\\ p &= e^{y+c_1}\\ p &= c_1 e^{y}\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_1 e^{y}$$

Integrating both sides gives

$$\begin{aligned}\int \frac{e^{-y}}{c_1}dy &= x+c_2\\ -\frac{e^{-y}}{c_1} &= x+c_2\end{aligned}$$

Solving for y gives these solutions

$$y_1=\ln\left(-\frac{1}{(x+c_2)\,c_1}\right)$$

Summary

The solution(s) found are the following

$$y=\ln\left(-\frac{1}{(x+c_2)\,c_1}\right) \tag{1}$$

Verification of solutions

$$y=\ln\left(-\frac{1}{(x+c_2)\,c_1}\right)$$

Verified OK.

14.16.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 0$

- Separate variables

  $\frac{u'(x)}{u(x)^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{u(x)} = x + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{1}{x+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{1}{x+c_1}$

- Make substitution $u = y'$

  $y' = -\frac{1}{x+c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{1}{x+c_1} dx + c_2$

- Compute integrals

  $y = -\ln(x + c_1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=diff(y(x),x)^2,y(x), singsol=all)
```

$$y = -\ln\left(-c_1 x - c_2\right)$$

✓ Solution by Mathematica

Time used: 1.667 (sec). Leaf size: 16

```
DSolve[y''[x]==1+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log(\cos(x + c_1))$$

14.17 problem 343

Internal problem ID [15201]
Internal file name [`OUTPUT/15201_Tuesday_April_23_2024_04_53_59_PM_61062009/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 343.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \sqrt{-y'^2 + 1} = 0}$$

14.17.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \sqrt{-p(x)^2 + 1} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{\sqrt{-p^2+1}}dp = x + c_1$$
$$\arcsin(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \sin(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \sin(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \sin(x + c_1)\,\mathrm{d}x \\ &= -\cos(x + c_1) + c_2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\cos(x + c_1) + c_2 \tag{1}$$

<u>Verification of solutions</u>

$$y = -\cos(x + c_1) + c_2$$

Verified OK.

14.17.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) = \sqrt{-p(y)^2+1}$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{\sqrt{-p^2+1}}dp = \int dy$$

$$\frac{(p(y)-1)(p(y)+1)}{\sqrt{-p(y)^2+1}} = y + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{(y'-1)(y'+1)}{\sqrt{-y'^2+1}} = y + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 - 2yc_1 - c_1^2 + 1} \tag{1}$$

$$y' = -\sqrt{-y^2 - 2yc_1 - c_1^2 + 1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-c_1^2 - 2c_1y - y^2 + 1}}dy = x + c_2$$

$$\arcsin(y + c_1) = x + c_2$$

Solving for y gives these solutions

$$y_1 = -c_1 + \sin(x + c_2)$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-c_1^2 - 2c_1y - y^2 + 1}}dy = c_3 + x$$

$$-\arcsin(y + c_1) = c_3 + x$$

Solving for y gives these solutions

$$y_1 = -c_1 - \sin(c_3 + x)$$

Summary

The solution(s) found are the following

$$y = -c_1 + \sin(x + c_2) \tag{1}$$

$$y = -c_1 - \sin(c_3 + x) \tag{2}$$

Verification of solutions

$$y = -c_1 + \sin(x + c_2)$$

Verified OK.

$$y = -c_1 - \sin(c_3 + x)$$

Verified OK.

14.17.3 Maple step by step solution

Let's solve

$y'' = \sqrt{-y'^2 + 1}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = \sqrt{-u(x)^2 + 1}$

- Separate variables

  $\frac{u'(x)}{\sqrt{-u(x)^2+1}} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\sqrt{-u(x)^2+1}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\arcsin(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = \sin(x + c_1)$

- Solve 1st ODE for $u(x)$

$u(x) = \sin(x + c_1)$

- Make substitution $u = y'$

  $y' = \sin(x + c_1)$

- Integrate both sides to solve for y

  $\int y' dx = \int \sin(x + c_1) \, dx + c_2$

- Compute integrals

  $y = -\cos(x + c_1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)+diff(y(x), x), y(x)`   ***
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (-_b(_a)^2+1)^(1/2), _b(_a), HINT = [[1
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 2.297 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)=sqrt(1-diff(y(x),x)^2),y(x), singsol=all)
```

$$y = -x + c_1$$
$$y = x + c_1$$
$$y = -\cos(x + c_1) + c_2$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 24

```
DSolve[y''[x]==Sqrt[1-y'[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x + c_1) + c_2$$
$$y(x) \to \text{Interval}[\{-1, 1\}] + c_2$$

14.18 problem 344

Internal problem ID [15202]
Internal file name [`OUTPUT/15202_Tuesday_April_23_2024_04_54_00_PM_28031435/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 344.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' - y'^2 = 1}$$

14.18.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 1 - p(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p^2+1} dp = x + c_1$$
$$\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = \tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x + c_1) \,\mathrm{d}x \\ &= \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2$$

Verified OK.

14.18.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{p^2+1}dp = \int dy$$
$$\frac{\ln(p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\sqrt{p^2+1} = e^{y+c_1}$$

Which simplifies to

$$\sqrt{p^2+1} = c_2 e^{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \operatorname{RootOf}\left(\_Z^2 - c_2^2 e^{2y} + 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\operatorname{RootOf}\left(\_Z^2 - c_2^2 e^{2y} + 1\right)}dy = \int dx$$
$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = c_3 + x$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = c_3 + x \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\operatorname{RootOf}\left(\_Z^2 - c_2^2 e^{2\_a} + 1\right)}d\_a = c_3 + x$$

Verified OK.

14.18.3 Maple step by step solution

Let's solve

$y'' - y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)^2 = 1$

- Separate variables

  $\frac{u'(x)}{1+u(x)^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{1+u(x)^2}dx = \int 1dx + c_1$

- Evaluate integral

  $\arctan(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = \tan(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \tan(x + c_1)$

- Make substitution $u = y'$

  $y' = \tan(x + c_1)$

- Integrate both sides to solve for y

  $\int y'dx = \int \tan(x + c_1)\, dx + c_2$

- Compute integrals

  $y = \frac{\ln\left(1+\tan(x+c_1)^2\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)=1+diff(y(x),x)^2,y(x), singsol=all)
```

$$y = -\ln\left(-c_2 \cos\left(x\right) + c_1 \sin\left(x\right)\right)$$

✓ Solution by Mathematica

Time used: 1.595 (sec). Leaf size: 16

```
DSolve[y''[x]==1+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log(\cos(x + c_1))$$

14.19 problem 345

Internal problem ID [15203]
Internal file name [`OUTPUT/15203_Tuesday_April_23_2024_04_54_01_PM_24959990/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 345.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \sqrt{y' + 1} = 0}$$

14.19.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - \sqrt{p(x) + 1} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{\sqrt{p+1}} dp = \int dx$$

$$2\sqrt{p(x) + 1} = x + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$2\sqrt{y' + 1} = x + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{4}c_1^2 + \frac{1}{2}c_1 x + \frac{1}{4}x^2 - 1 \, \mathrm{d}x \\ &= \frac{x^3}{12} + \frac{c_1 x^2}{4} + \frac{(c_1 + 2)(c_1 - 2)x}{4} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3}{12} + \frac{c_1 x^2}{4} + \frac{(c_1 + 2)(c_1 - 2)x}{4} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{x^3}{12} + \frac{c_1 x^2}{4} + \frac{(c_1 + 2)(c_1 - 2)x}{4} + c_2$$

Verified OK.

14.19.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) = \sqrt{p(y) + 1}$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int \frac{p}{\sqrt{p+1}}dp = \int dy$$

$$\frac{2\sqrt{p(y)+1}\,(p(y)-2)}{3} = y + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{2\sqrt{y'+1}\,(y'-2)}{3} = y + c_1$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \left(\frac{\left(6y+6c_1+2\sqrt{-16+9y^2+18yc_1+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{2}{\left(6y+6c_1+2\sqrt{-16+9y^2+18yc_1+9c_1^2}\right)^{\frac{1}{3}}}\right)^2 - 1 \tag{1}$$

$$y' = \left(-\frac{\left(6y+6c_1+2\sqrt{-16+9y^2+18yc_1+9c_1^2}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(6y+6c_1+2\sqrt{-16+9y^2+18yc_1+9c_1^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{}}{} \tag{2}$$

$$y' = \left(-\frac{\left(6y+6c_1+2\sqrt{-16+9y^2+18yc_1+9c_1^2}\right)^{\frac{1}{3}}}{4} - \frac{1}{\left(6y+6c_1+2\sqrt{-16+9y^2+18yc_1+9c_1^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{}}{} \tag{3}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{4\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}}{\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}+4\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}+16}dy$$
$$=\int dx$$

$$4\left(\int^{y}\frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}+4\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}\right.$$
$$=x+c_2$$

Solving equation (2)

Integrating both sides gives

$$\int$$

$$-\frac{8\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}}{i\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}-16i\sqrt{3}-}$$
$$=\int dx$$

$$-8\left(\int^{y}\frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+}\right.}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-\right.}\right.$$
$$=c_3+x$$

Solving equation (3)

Integrating both sides gives

$$\int\frac{8\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{2}{3}}}{i\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6y+6c_1+2\sqrt{9c_1^2+18c_1y+9y^2-16}\right)^{\frac{4}{3}}-16i\sqrt{3}-}$$
$$=\int dx$$

$$8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}}\right.$$

$$= x+c_4$$

Summary

The solution(s) found are the following

$$4\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}+4\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}\right. \quad (1)$$

$$= x+c_2$$

$$-8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}}\right. \quad (2)$$

$$= c_3+x$$

$$8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}}\right. \quad (3)$$

$$= x+c_4$$

Verification of solutions

$$4\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}}{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}+4\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{2}{3}}\right.$$
$$= x + c_2$$

Verified OK.

$$-8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+\right.}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}+\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-\right.}\right.$$
$$= c_3 + x$$

Verified OK.

$$8\left(\int^{y} \frac{\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9\right.}{i\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-16}\right)^{\frac{4}{3}}\sqrt{3}-\left(6\_a+6c_1+2\sqrt{9\_a^2+18\_ac_1+9c_1^2-1\right.}\right.$$
$$= x + c_4$$

Verified OK.

14.19.3 Maple step by step solution

Let's solve

$y'' = \sqrt{y'+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = \sqrt{u(x)+1}$

- Separate variables

  $\frac{u'(x)}{\sqrt{u(x)+1}} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\sqrt{u(x)+1}}dx = \int 1dx + c_1$

- Evaluate integral

$2\sqrt{u(x)+1} = x + c_1$

- Solve for $u(x)$

  $u(x) = \frac{1}{4}c_1^2 + \frac{1}{2}c_1 x + \frac{1}{4}x^2 - 1$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{1}{4}c_1^2 + \frac{1}{2}c_1 x + \frac{1}{4}x^2 - 1$

- Make substitution $u = y'$

  $y' = \frac{1}{4}c_1^2 + \frac{1}{2}c_1 x + \frac{1}{4}x^2 - 1$

- Integrate both sides to solve for y

  $\int y' dx = \int \left(\frac{1}{4}c_1^2 + \frac{1}{2}c_1 x + \frac{1}{4}x^2 - 1\right) dx + c_2$

- Compute integrals

  $y = \frac{x^3}{12} + \frac{c_1 x^2}{4} + \frac{(c_1+2)(c_1-2)x}{4} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)-1/2, y(x)`   *** Sublevel
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)+1)^(1/2), _b(_a), HINT = [[1, 0
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [_a, 2+2*_b]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)=sqrt(1+diff(y(x),x)),y(x), singsol=all)
```

$$y = -x + c_1$$
$$y = \frac{1}{12}x^3 + \frac{1}{4}c_1x^2 + \frac{1}{4}c_1^2x - x + c_2$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 30

```
DSolve[y''[x]==Sqrt[1+y'[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}x\left(x^2 + 3c_1x + 3\left(-4 + c_1{}^2\right)\right) + c_2$$

14.20 problem 346

Internal problem ID [15204]
Internal file name [`OUTPUT/15204_Tuesday_April_23_2024_04_54_05_PM_26597427/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 346.
ODE order: 2.
ODE degree: 0.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' \ln(y') = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

14.20.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x) \ln(p(x)) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p \ln(p)} dp = \int dx$$
$$\ln(\ln(p)) = x + c_1$$

Raising both side to exponential gives

$$\ln(p) = e^{x+c_1}$$

Which simplifies to

$$\ln(p) = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{c_2}$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$p(x) = 1$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 1$$

Integrating both sides gives

$$y = \int 1 \, dx$$
$$= c_3 + x$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3$$

$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

$$y = x$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = x \tag{1}$$

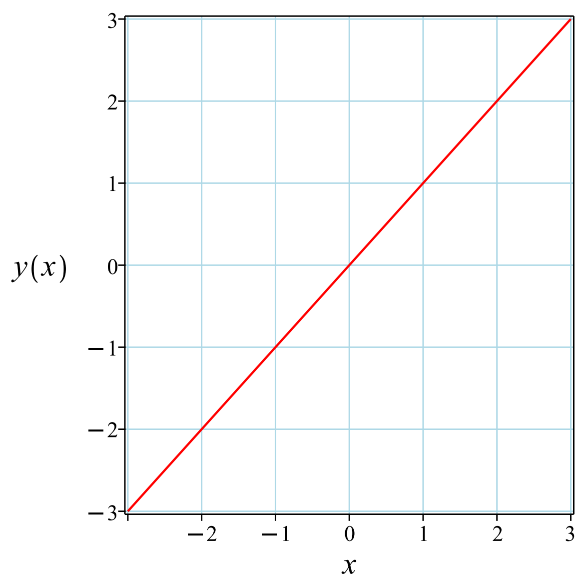


Figure 460: Solution plot

Verification of solutions

$$y = x$$

Verified OK.

14.20.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) - p(y)\ln\left(p(y)\right) = 0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\ln\left(p\right)}dp &= \int dy \\
-\operatorname{expIntegral}_1\left(-\ln\left(p\right)\right) &= y + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{-\operatorname{expIntegral}_1(-\ln(p))} = e^{y+c_1}
$$

Which simplifies to

$$
e^{-\operatorname{expIntegral}_1(-\ln(p))} = c_2 e^{y}
$$

Unable to solve for constant of integration due to RootOf in solution.

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = e^{\operatorname{RootOf}(\operatorname{expIntegral}_1(-\_Z)+\ln(c_2)+y)}
$$

Integrating both sides gives

$$
\begin{aligned}
\int e^{-\operatorname{RootOf}(\operatorname{expIntegral}_1(-\_Z)+\ln(c_2)+y)}dy &= \int dx \\
\int^{y} e^{-\operatorname{RootOf}(\operatorname{expIntegral}_1(-\_Z)+\ln(c_2)+\_a)}d\_a &= c_3 + x
\end{aligned}
$$

Unable to solve for constant of integration due to RootOf in solution.

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\int^{y} \mathrm{e}^{-\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(-\_Z)+\ln(c_2)+\_a\right)}d\_a = c_3 + x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$\int^{0} \mathrm{e}^{-\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(-\_Z)+\ln(c_2)+\_a\right)}d\_a = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(-\_Z)+\ln(c_2)+\operatorname{RootOf}\left(-\left(\int^{\_Z} \mathrm{e}^{-\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(-\_Z)+\ln(c_2)+\_a\right)}d\_a\right)+c_3+x\right)\right)}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \mathrm{e}^{\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(-\_Z)+\ln(c_2)+\operatorname{RootOf}\left(-\left(\int^{\_Z} \mathrm{e}^{\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(\_Z)+\ln(c_2)+\_a\right)}d\_a\right)+c_3+x\right)\Big|_{\left\{\int^{\_Z} \mathrm{e}^{\operatorname{RootOf}\left(\operatorname{expIntegral}_{1}(\_Z)+\right.}\right.}\right.} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_2, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

14.20.3 Maple step by step solution

Let's solve

$$\left[y'' - y' \ln(y') = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - u(x)\ln(u(x)) = 0$

- Separate variables

  $\frac{u'(x)}{u(x)\ln(u(x))} = 1$

- Integrate both sides with respect to x

$\int \frac{u'(x)}{u(x)\ln(u(x))}dx = \int 1dx + c_1$

- Evaluate integral

  $\ln(\ln(u(x))) = x + c_1$

- Solve for $u(x)$

  $u(x) = e^{e^{x+c_1}}$

- Solve 1st ODE for $u(x)$

  $u(x) = e^{e^{x+c_1}}$

- Make substitution $u = y'$

  $y' = e^{e^{x+c_1}}$

- Integrate both sides to solve for y

  $\int y'dx = \int e^{e^{x+c_1}}dx + c_2$

- Compute integrals

  $y = -\mathrm{Ei}_1(-e^{x+c_1}) + c_2$

☐ Check validity of solution $y = -\mathrm{Ei}_1(-e^{x+c_1}) + c_2$

  - Use initial condition $y(0) = 0$

    $0 = -\mathrm{Ei}_1(-e^{c_1}) + c_2$

  - Compute derivative of the solution

    $y' = e^{e^{x+c_1}}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = e^{e^{c_1}}$

  - Solve for c_1 and c_2
  - The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)*ln(_b(_a)), _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)=diff(y(x),x)*ln(diff(y(x),x)),y(0) = 0, D(y)(0) = 1],y(x), singsol=al
```

$$y = -\operatorname{expIntegral}_1\left(-2i\pi\_Z5\,\mathrm{e}^{x}\right) + \operatorname{expIntegral}_1\left(-2i\pi\_Z5\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]==y'[x]*Log[y'[x]],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

{}

14.21 problem 347

Internal problem ID [15205]
Internal file name [`OUTPUT/15205_Tuesday_April_23_2024_04_54_07_PM_36436185/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 347.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' = -2}$$

With initial conditions

$$[y(0) = 0, y'(0) = -2]$$

14.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ F &= -2 \end{aligned}$$

Hence the ode is

$$y'' + y' = -2$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.21.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = -2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = -2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-x}) + (-2x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 e^{-x} - 2x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_2 e^{-x} - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

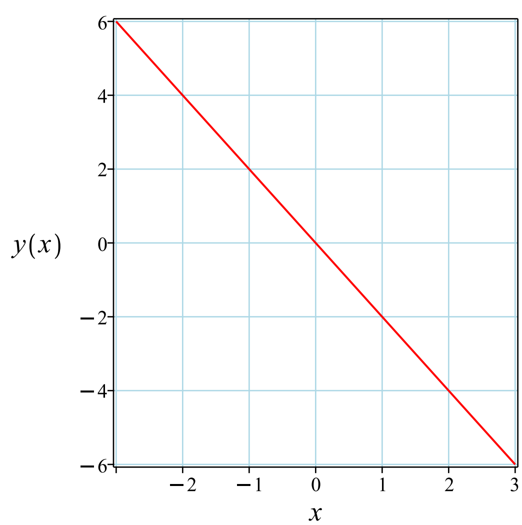
Substituting these values back in above solution results in

$$y = -2x$$

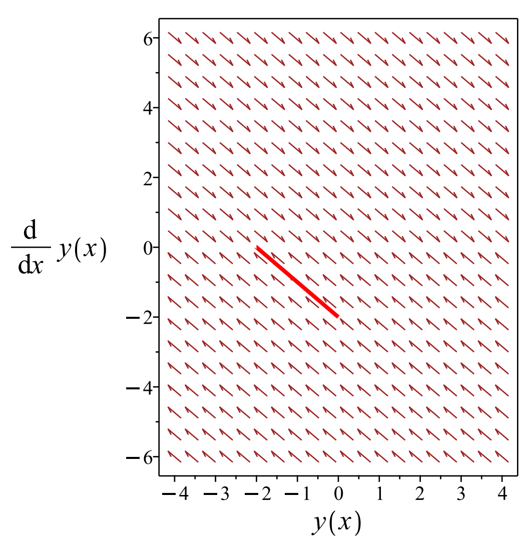
Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

14.21.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int (y'' + y')\, dx &= \int (-2)\, dx \\ y' + y &= -2x + c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 1 \\ q(x) &= -2x + c_1\end{aligned}$$

Hence the ode is

$$y' + y = -2x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-2x + c_1) \\ \frac{d}{dx}(y\, e^{x}) &= (e^{x})(-2x + c_1) \\ d(y\, e^{x}) &= ((-2x + c_1)\, e^{x})\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\, e^{x} &= \int (-2x + c_1)\, e^{x}\, dx \\ y\, e^{x} &= -e^{x}(2x - c_1 - 2) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = -e^{-x} e^x (2x - c_1 - 2) + c_2 e^{-x}$$

which simplifies to

$$y = -2x + c_1 + 2 + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -2x + c_1 + 2 + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 2 + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_2 e^{-x} - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 0$$

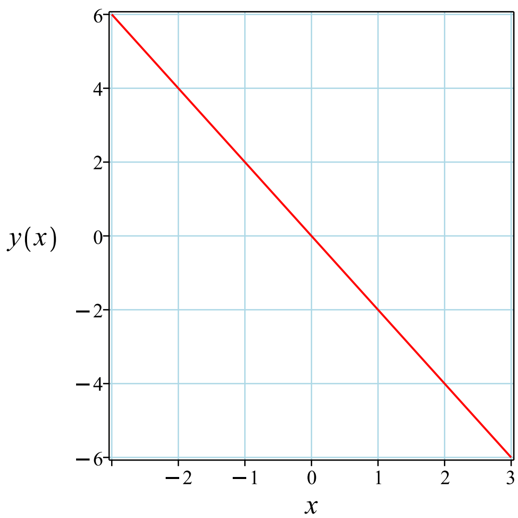
Substituting these values back in above solution results in

$$y = -2x$$

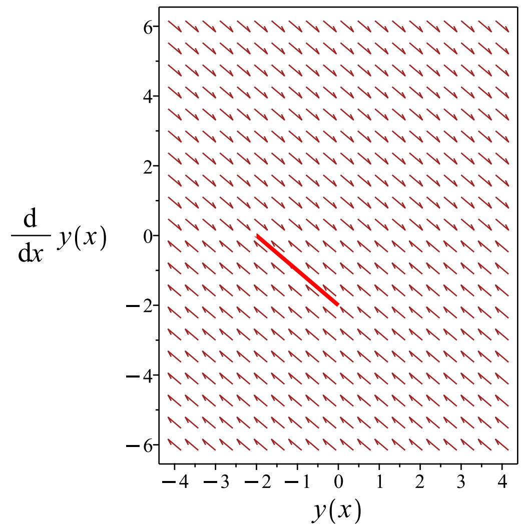
Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

14.21.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) + 2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p-2} dp = \int dx$$
$$-\ln(-p-2) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{-p-2} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{-p-2} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $p = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{-1-2c_2}{c_2}$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $p = -\frac{e^{-x}}{c_2} - 2 = p = -2$ and this result satisfies the given initial condition. Since $p = y'$ then the new first order ode to solve is

$$y' = -2$$

Integrating both sides gives

$$\begin{aligned}y &= \int -2 \, dx \\ &= -2x + c_3\end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3$$

$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

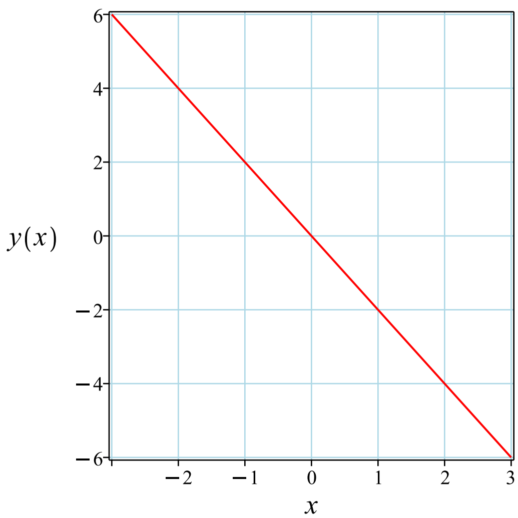
$$y = -2x$$

Initial conditions are used to solve for the constants of integration.

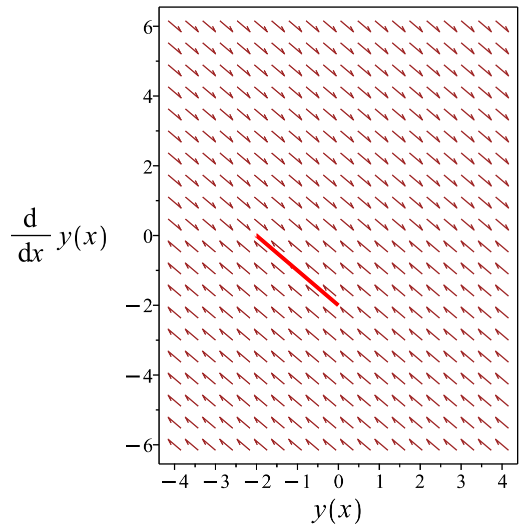
Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

14.21.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = -2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (-2)\, dx$$

$$y' + y = -2x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = -2x + c_1$$

Hence the ode is

$$y' + y = -2x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(-2x + c_1) \\ \frac{d}{dx}(y\, e^x) &= (e^x)(-2x + c_1) \\ d(y\, e^x) &= ((-2x + c_1)\, e^x)\; dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\, e^x &= \int (-2x + c_1)\, e^x\, dx \\ y\, e^x &= -e^x(2x - c_1 - 2) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = -e^{-x} e^x(2x - c_1 - 2) + c_2 e^{-x}$$

which simplifies to

$$y = -2x + c_1 + 2 + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -2x + c_1 + 2 + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 2 + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_2 e^{-x} - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

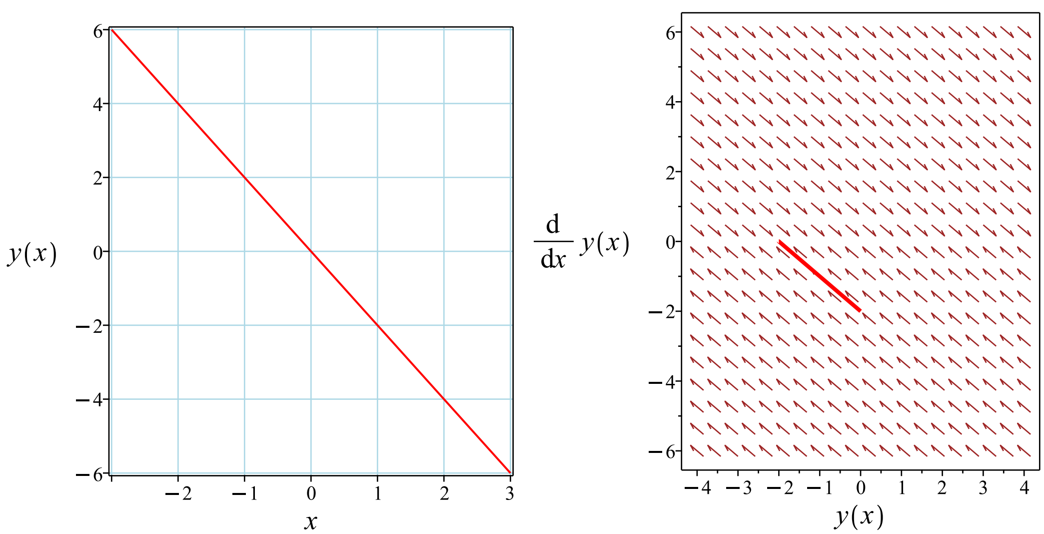
$$c_1 = -2$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -2x$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

14.21.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 356: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = -2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2\right) + (-2x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 - 2x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -c_1 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

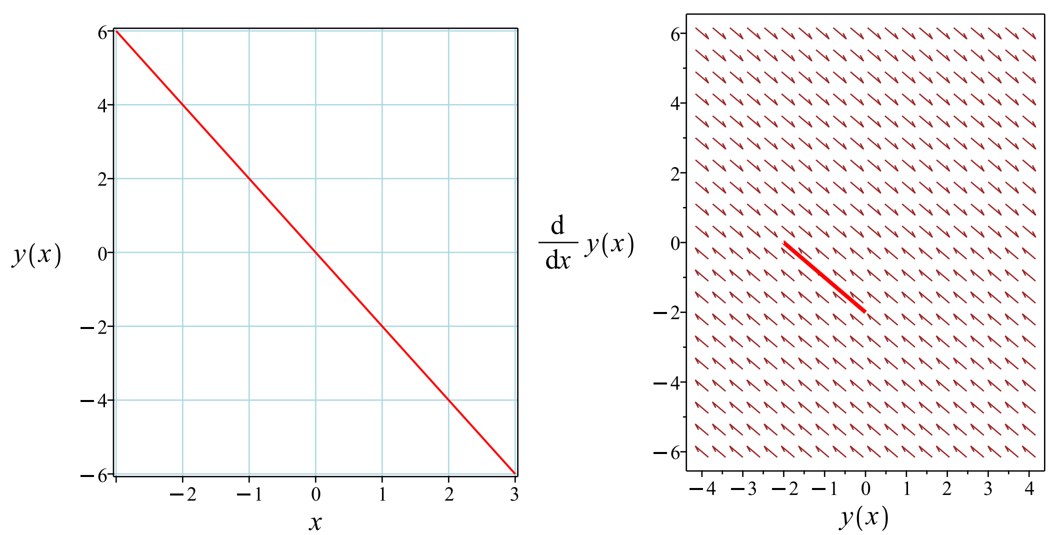
Substituting these values back in above solution results in

$$y = -2x$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

14.21.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= -2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int -2\, dx$$

We now have a first order ode to solve which is

$$y' + y = -2x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -2x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = -2x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(-2x + c_1) \\ \frac{d}{dx}(y\,e^{x}) &= (e^{x})(-2x + c_1) \\ d(y\,e^{x}) &= ((-2x + c_1)\,e^{x})\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,e^{x} &= \int (-2x + c_1)\,e^{x}\,dx \\ y\,e^{x} &= -e^{x}(2x - c_1 - 2) + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = -e^{-x}e^{x}(2x - c_1 - 2) + c_2e^{-x}$$

which simplifies to

$$y = -2x + c_1 + 2 + c_2e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -2x + c_1 + 2 + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 2 + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_2 e^{-x} - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 0$$

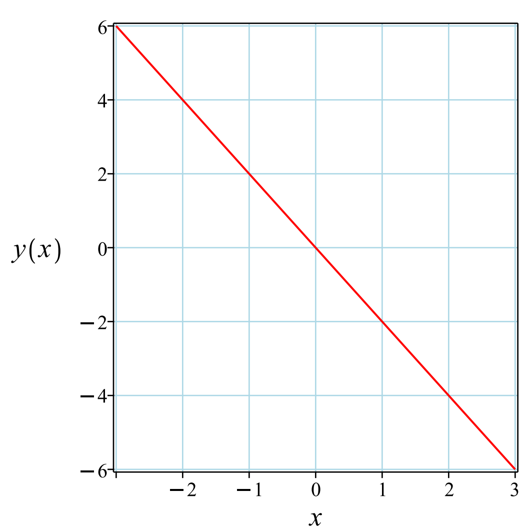
Substituting these values back in above solution results in

$$y = -2x$$

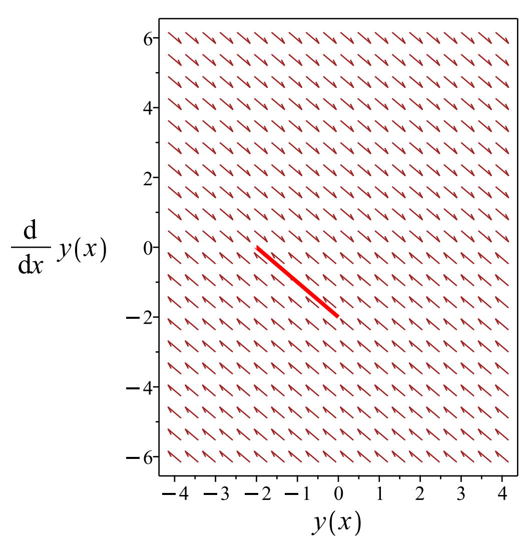
Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

14.21.8 Maple step by step solution

Let's solve

$\left[y'' + y' = -2, y(0) = 0, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\,e^{-x}\left(\int e^{x} dx\right) - 2\left(\int 1 dx\right)$
- Compute integrals

  $y_p(x) = -2x + 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 - 2x + 2$

☐ Check validity of solution $y = c_1 e^{-x} + c_2 - 2x + 2$

- Use initial condition $y(0) = 0$

  $0 = 2 + c_1 + c_2$
- Compute derivative of the solution

  $y' = -c_1 e^{-x} - 2$
- Use the initial condition $y'\Big|_{\{x=0\}} = -2$

  $-2 = -c_1 - 2$
- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = -2\}$
- Substitute constant values into general solution and simplify

  $y = -2x$

- Solution to the IVP

  $y = -2x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)-2, _b(_a)`   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 7

```
dsolve([diff(y(x),x$2)+diff(y(x),x)+2=0,y(0) = 0, D(y)(0) = -2],y(x), singsol=all)
```

$$y = -2x$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 8

```
DSolve[{y''[x]+y'[x]+2==0,{y[0]==0,y'[0]==-2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x$$

14.22 problem 348

Internal problem ID [15206]
Internal file name [`OUTPUT/15206_Tuesday_April_23_2024_04_54_08_PM_4933972/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 348.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_xy]]
```

$$\boxed{y'' - y'(y' + 1) = 0}$$

14.22.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + (-p(x) - 1)\, p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{p\,(p+1)} dp = \int dx$$
$$-\ln\left(p+1\right)+\ln\left(p\right) = x+c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\ln(p+1)+\ln(p)} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{p}{p+1} = c_2\mathrm{e}^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{c_2\mathrm{e}^{x}}{-1+c_2\mathrm{e}^{x}}$$

Integrating both sides gives

$$y = \int -\frac{c_2\mathrm{e}^{x}}{-1+c_2\mathrm{e}^{x}}\,\mathrm{d}x$$
$$= -\ln\left(-1+c_2\mathrm{e}^{x}\right)+c_3$$

Summary
The solution(s) found are the following

$$y = -\ln\left(-1+c_2\mathrm{e}^{x}\right)+c_3 \tag{1}$$

Verification of solutions

$$y = -\ln\left(-1+c_2\mathrm{e}^{x}\right)+c_3$$

Verified OK.

14.22.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right)+(-p(y)-1)\,p(y)=0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{p+1}dp &= \int dy \\
\ln(p+1) &= y+c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
p+1 = \mathrm{e}^{y+c_1}
$$

Which simplifies to

$$
p+1 = c_2\mathrm{e}^{y}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = c_2\mathrm{e}^{y}-1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{c_2\mathrm{e}^{y}-1}dy &= \int dx \\
\ln(c_2\mathrm{e}^{y}-1)-\ln(\mathrm{e}^{y}) &= c_3+x
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(c_2\mathrm{e}^{y}-1)-\ln(\mathrm{e}^{y})} = \mathrm{e}^{c_3+x}
$$

Which simplifies to

$$
c_2-\mathrm{e}^{-y} = c_4\mathrm{e}^{x}
$$

Summary
The solution(s) found are the following

$$y = -\ln\left(-c_4 e^x + c_2\right) \tag{1}$$

Verification of solutions

$$y = -\ln\left(-c_4 e^x + c_2\right)$$

Verified OK.

14.22.3 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$f(x) = -1$$
$$g(y) = -1$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln\left(y'\right) \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln\left(y'\right) + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln\left(y'\right) + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy} e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = -1$ and $f = -1$, then

$$\begin{aligned}\int -g dy &= \int 1 dy \\ &= y \\ \int -f dx &= \int 1 dx \\ &= x\end{aligned}$$

Substituting the above into Eq(6A) gives

$$y' = c_2 e^y e^x$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x) g(y) \\ &= c_2 e^y e^x\end{aligned}$$

Where $f(x) = c_2 e^x$ and $g(y) = e^y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{e^y} dy &= c_2 e^x dx \\ \int \frac{1}{e^y} dy &= \int c_2 e^x dx \\ -e^{-y} &= c_2 e^x + c_3\end{aligned}$$

The solution is

$$-e^{-y} - c_2 e^x - c_3 = 0$$

Summary

The solution(s) found are the following

$$-e^{-y} - c_2 e^x - c_3 = 0 \tag{1}$$

Verification of solutions

$$-e^{-y} - c_2 e^x - c_3 = 0$$

Verified OK.

14.22.4 Maple step by step solution

Let's solve

$y'' + (-y' - 1)\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + (-u(x) - 1)\, u(x) = 0$

- Separate variables

  $\frac{u'(x)}{(-u(x)-1)u(x)} = -1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{(-u(x)-1)u(x)} dx = \int (-1)\, dx + c_1$

- Evaluate integral

  $\ln (u(x) + 1) - \ln (u(x)) = -x + c_1$

- Solve for $u(x)$

  $u(x) = \frac{1}{\mathrm{e}^{-x+c_1}-1}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{1}{\mathrm{e}^{-x+c_1}-1}$

- Make substitution $u = y'$

  $y' = \frac{1}{\mathrm{e}^{-x+c_1}-1}$

- Integrate both sides to solve for y

  $\int y' dx = \int \frac{1}{\mathrm{e}^{-x+c_1}-1} dx + c_2$

- Compute integrals

  $y = \ln (\mathrm{e}^{-x+c_1}) - \ln (\mathrm{e}^{-x+c_1} - 1) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)=diff(y(x),x)*(1+diff(y(x),x)),y(x), singsol=all)
```

$$y = -\ln\left(-c_1 \mathrm{e}^{x} - c_2\right)$$

✓ Solution by Mathematica

Time used: 1.619 (sec). Leaf size: 31

```
DSolve[y''[x]==y'[x]*(1+y'[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \log\left(-1 + e^{x+c_1}\right)$$
$$y(x) \to c_2 - i\pi$$

14.23 problem 349

Internal problem ID [15207]
Internal file name [`OUTPUT/15207_Tuesday_April_23_2024_04_54_09_PM_87500085/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 349.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{3y'' - \left(1+{y'}^2\right)^{\frac{3}{2}} = 0}$$

14.23.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$3p'(x) - \left(1+p(x)^2\right)^{\frac{3}{2}} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{3}{(p^2+1)^{\frac{3}{2}}} dp = \int dx$$

$$\frac{3p(x)}{\sqrt{1+p(x)^2}} = x + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{3y'}{\sqrt{1+y'^2}} = x + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \sqrt{-\frac{1}{c_1^2+2c_1x+x^2-9}}\,(x+c_1)\,\mathrm{d}x \\ &= (c_1+x+3)(c_1+x-3)\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-9}} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = (c_1+x+3)(c_1+x-3)\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-9}} + c_2 \tag{1}$$

Verification of solutions

$$y = (c_1+x+3)(c_1+x-3)\sqrt{-\frac{1}{c_1^2+2c_1x+x^2-9}} + c_2$$

Verified OK.

14.23.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
3p(y)\left(\frac{d}{dy}p(y)\right) = \left(1+p(y)^2\right)^{\frac{3}{2}}
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{3p}{(p^2+1)^{\frac{3}{2}}}dp &= \int dy \\
-\frac{3}{\sqrt{1+p(y)^2}} &= y+c_1
\end{aligned}
$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$
-\frac{3}{\sqrt{1+y'^2}} = y+c_1
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
y' = \frac{\sqrt{-y^2-2yc_1-c_1^2+9}}{y+c_1} \tag{1}
$$

$$
y' = -\frac{\sqrt{-y^2-2yc_1-c_1^2+9}}{y+c_1} \tag{2}
$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$
\begin{aligned}
\int \frac{y+c_1}{\sqrt{-c_1^2-2c_1y-y^2+9}}dy &= \int dx \\
\frac{(y+c_1+3)(y+c_1-3)}{\sqrt{-y^2-2yc_1-c_1^2+9}} &= x+c_2
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y+c_1}{\sqrt{-c_1^2-2c_1y-y^2+9}}dy = \int dx$$
$$-\frac{(y+c_1+3)\,(y+c_1-3)}{\sqrt{-y^2-2yc_1-c_1^2+9}} = c_3+x$$

Summary

The solution(s) found are the following

$$\frac{(y+c_1+3)\,(y+c_1-3)}{\sqrt{-y^2-2yc_1-c_1^2+9}} = x+c_2 \tag{1}$$
$$-\frac{(y+c_1+3)\,(y+c_1-3)}{\sqrt{-y^2-2yc_1-c_1^2+9}} = c_3+x \tag{2}$$

Verification of solutions

$$\frac{(y+c_1+3)\,(y+c_1-3)}{\sqrt{-y^2-2yc_1-c_1^2+9}} = x+c_2$$

Verified OK.

$$-\frac{(y+c_1+3)\,(y+c_1-3)}{\sqrt{-y^2-2yc_1-c_1^2+9}} = c_3+x$$

Verified OK.

14.23.3 Maple step by step solution

Let's solve

$3y'' = \left(1+y'^2\right)^{\frac{3}{2}}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $3u'(x) = \left(1+u(x)^2\right)^{\frac{3}{2}}$

- Separate variables

$$\frac{u'(x)}{\left(1+u(x)^2\right)^{\frac{3}{2}}} = \frac{1}{3}$$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{\left(1+u(x)^2\right)^{\frac{3}{2}}} dx = \int \frac{1}{3} dx + c_1$$

- Evaluate integral

$$\frac{u(x)}{\sqrt{1+u(x)^2}} = \frac{x}{3} + c_1$$

- Solve for $u(x)$

$$u(x) = \sqrt{-\frac{1}{9c_1^2+6c_1x+x^2-9}}\,(x+3c_1)$$

- Solve 1st ODE for $u(x)$

$$u(x) = \sqrt{-\frac{1}{9c_1^2+6c_1x+x^2-9}}\,(x+3c_1)$$

- Make substitution $u = y'$

$$y' = \sqrt{-\frac{1}{9c_1^2+6c_1x+x^2-9}}\,(x+3c_1)$$

- Integrate both sides to solve for y

$$\int y' dx = \int \sqrt{-\frac{1}{9c_1^2+6c_1x+x^2-9}}\,(x+3c_1)\,dx + c_2$$

- Compute integrals

$$y = (3c_1+x+3)\,(3c_1+x-3)\sqrt{-\frac{1}{9c_1^2+6c_1x+x^2-9}} + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/3)*(_b(_a)^2+1)^(3/2), _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [y, -_b^2-1]
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 49

```
dsolve(3*diff(y(x),x$2)=(1+diff(y(x),x)^2)^(3/2),y(x), singsol=all)
```

$$y = -ix + c_1$$
$$y = ix + c_1$$
$$y = (c_1 + x + 3)(c_1 + x - 3)\sqrt{-\frac{1}{(c_1 + x + 3)(c_1 + x - 3)}} + c_2$$

✓ Solution by Mathematica

Time used: 0.26 (sec). Leaf size: 63

```
DSolve[3*y''[x]==(1+y'[x]^2)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - i\sqrt{x^2 + 6c_1 x - 9 + 9{c_1}^2}$$
$$y(x) \to i\sqrt{x^2 + 6c_1 x - 9 + 9{c_1}^2} + c_2$$

14.24 problem 350

Internal problem ID [15208]
Internal file name [`OUTPUT/15208_Tuesday_April_23_2024_04_54_10_PM_64380128/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 350.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x], [_3rd_order, _missing_y], [
   _3rd_order, _with_linear_symmetries], [_3rd_order, _reducible
   , _mu_y2], [_3rd_order, _reducible, _mu_poly_yn]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^2, _b(_a), HINT = [[1, 0], [_a,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [_a, -_b]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)^2=0,y(x), singsol=all)
```

$$y = \ln(x + c_1)(x + c_1) + (c_2 - 1)x - c_1 + c_3$$

✓ Solution by Mathematica

Time used: 0.315 (sec). Leaf size: 28

```
DSolve[y'''[x]+y''[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-1 + c_3)x + (x - c_1)\log(x - c_1) + c_2$$

14.25 problem 351

Internal problem ID [15209]
Internal file name [`OUTPUT/15209_Tuesday_April_23_2024_04_54_11_PM_3706120/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 351.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' - y'^2 = 0}$$

14.25.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)-p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= \frac{p}{y}\end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p}\,dp &= \frac{1}{y}\,dy\\ \int \frac{1}{p}\,dp &= \int \frac{1}{y}\,dy\\ \ln(p) &= \ln(y)+c_1\\ p &= e^{\ln(y)+c_1}\\ &= c_1 y\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y' = yc_1$$

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{c_1 y}dy &= \int dx\\ \frac{\ln(y)}{c_1} &= x+c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\frac{\ln(y)}{c_1}} = e^{x+c_2}$$

Which simplifies to

$$y^{\frac{1}{c_1}} = e^x c_3$$

Summary

The solution(s) found are the following

$$y = (e^x c_3)^{c_1} \tag{1}$$

Verification of solutions

$$y = (e^x c_3)^{c_1}$$

Verified OK.

14.25.2 Maple step by step solution

Let's solve

$yy'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y) \left(\frac{d}{dy}u(y)\right) - u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int \frac{1}{y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = \ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = y\,\mathrm{e}^{c_1}$

- Solve 1st ODE for $u(y)$

  $u(y) = y\,\mathrm{e}^{c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = y\,\mathrm{e}^{c_1}$

- Separate variables

  $\frac{y'}{y} = e^{c_1}$
- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int e^{c_1} dx + c_2$
- Evaluate integral

  $\ln(y) = e^{c_1} x + c_2$
- Solve for y

  $y = e^{e^{c_1} x + c_2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(y(x)*diff(y(x),x$2)=diff(y(x),x)^2,y(x), singsol=all)
```

$$y = 0$$
$$y = e^{c_1 x} c_2$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 14

```
DSolve[y[x]*y''[x]==y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^{c_1 x}$$

14.26 problem 352

Internal problem ID [15210]
Internal file name [`OUTPUT/15210_Tuesday_April_23_2024_04_54_12_PM_96369871/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 352.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{y'' - 2yy' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

14.26.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2yy')\,dx = 0$$
$$-y^2 + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y^2 + c_1} dy = x + c_2$$
$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \sec\left(c_2\sqrt{c_1}\right)^2 c_1 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

14.26.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) - 2yp(y) = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int 2y \,\mathrm{d}y \\ &= y^2 + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $y = 1$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + 1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(y) = y^2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = y^2$$

Integrating both sides gives

$$\int \frac{1}{y^2} dy = x + c_2$$

$$-\frac{1}{y} = x + c_2$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{x + c_2}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_2}$$

$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$y = -\frac{1}{x - 1}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = -\frac{1}{x - 1} \tag{1}$$

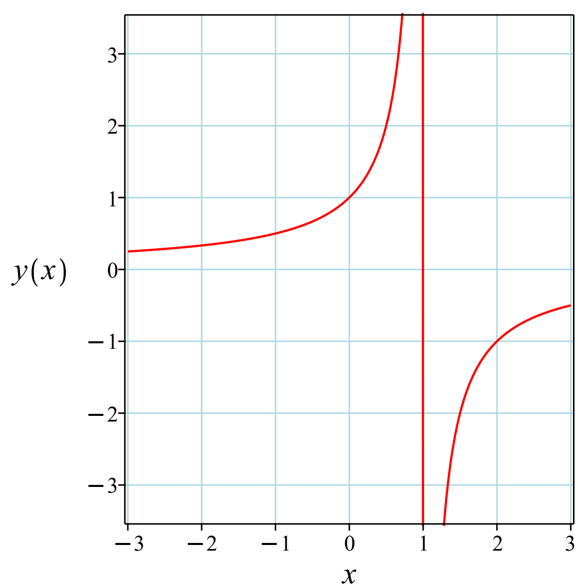


Figure 467: Solution plot

Verification of solutions

$$y = -\frac{1}{x-1}$$

Verified OK.

14.26.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 2yy' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2yy')\, dx = 0$$
$$-y^2 + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{y^2 + c_1} dy = x + c_2$$
$$\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \sec\left(c_2\sqrt{c_1}\right)^2 c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

14.26.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= 1\\ a_1 &= -2y\\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1\\ \int 1\, dy' + \int -2y\, dy + \int 0\, dx &= c_1\end{aligned}$$

Which results in

$$-y^2 + y' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y^2 + c_1} dy &= x + c_2\\ \frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} &= x + c_2\end{aligned}$$

Solving for y gives these solutions

$$y_1 = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \sec\left(c_2\sqrt{c_1}\right)^2 c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

14.26.5 Maple step by step solution

Let's solve

$\left[y'' - 2yy' = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) - 2yu(y) = 0$
- Separate variables

  $\frac{d}{dy}u(y) = 2y$
- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}u(y)\right) dy = \int 2y dy + c_1$
- Evaluate integral

  $u(y) = y^2 + c_1$
- Solve for $u(y)$

  $u(y) = y^2 + c_1$
- Solve 1st ODE for $u(y)$

  $u(y) = y^2 + c_1$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = y^2 + c_1$
- Separate variables

  $\frac{y'}{y^2+c_1} = 1$
- Integrate both sides with respect to x

  $\int \frac{y'}{y^2+c_1} dx = \int 1 dx + c_2$
- Evaluate integral

  $\frac{\arctan\left(\frac{y}{\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$
- Solve for y

  $y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$

☐ Check validity of solution $y = \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)\sqrt{c_1}$

- Use initial condition $y(0) = 1$

  $$1 = \tan\left(c_2\sqrt{c_1}\right)\sqrt{c_1}$$

- Compute derivative of the solution

  $$y' = c_1\left(1 + \tan\left(c_2\sqrt{c_1} + x\sqrt{c_1}\right)^2\right)$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $$1 = c_1\left(1 + \tan\left(c_2\sqrt{c_1}\right)^2\right)$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-2*_a*_b(_a) = 0, _b(_a), HINT =
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)=2*y(x)*diff(y(x),x),y(0) = 1, D(y)(0) = 1],y(x), singsol=all)
```

$$y = -\frac{1}{x - 1}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]==2*y[x]*y'[x],{y[0]==1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$\{\}$$

14.27 problem 353

Internal problem ID [15211]
Internal file name [`OUTPUT/15211_Tuesday_April_23_2024_04_54_12_PM_97607109/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 353.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{3y'y'' - 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

14.27.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$3p(y)^2 \left(\frac{d}{dy}p(y)\right) - 2y = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y,p) \\ &= f(y)g(p) \\ &= \frac{2y}{3p^2} \end{aligned}$$

Where $f(y) = \frac{2y}{3}$ and $g(p) = \frac{1}{p^2}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{p^2}}\,dp &= \frac{2y}{3}\,dy \\ \int \frac{1}{\frac{1}{p^2}}\,dp &= \int \frac{2y}{3}\,dy \\ \frac{p^3}{3} &= \frac{y^2}{3} + c_1 \end{aligned}$$

The solution is

$$\frac{p(y)^3}{3} - \frac{y^2}{3} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $y = 1$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{p^3}{3} - \frac{y^2}{3} = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^3}{3} - \frac{y^2}{3} = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = y^{\frac{2}{3}} \tag{1}$$

$$y' = -\frac{y^{\frac{2}{3}}}{2} + \frac{i\sqrt{3}\, y^{\frac{2}{3}}}{2} \tag{2}$$

$$y' = -\frac{y^{\frac{2}{3}}}{2} - \frac{i\sqrt{3}\, y^{\frac{2}{3}}}{2} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{y^{\frac{2}{3}}} dy = \int dx$$

$$3y^{\frac{1}{3}} = x + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_2$$

$$c_2 = 3$$

Substituting c_2 found above in the general solution gives

$$3y^{\frac{1}{3}} = x + 3$$

Solving for y from the above gives

$$y = \frac{(x + 3)^3}{27}$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{-\frac{y^{\frac{2}{3}}}{2} + \frac{i\sqrt{3}\, y^{\frac{2}{3}}}{2}} dy = \int dx$$

$$\frac{6y^{\frac{1}{3}}}{i\sqrt{3} - 1} = c_3 + x$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{6}{i\sqrt{3}-1} = c_3$$

$$c_3 = \frac{6}{i\sqrt{3}-1}$$

Substituting c_3 found above in the general solution gives

$$\frac{6y^{\frac{1}{3}}}{i\sqrt{3}-1} = \frac{i\sqrt{3}\,x - x + 6}{i\sqrt{3}-1}$$

The above simplifies to

$$-i\sqrt{3}\,x + 6y^{\frac{1}{3}} + x - 6 = 0$$

Solving for y from the above gives

$$y = -\frac{i(x-3)\,x\sqrt{3}}{6} + \frac{x^3}{27} - \frac{x^2}{6} - \frac{x}{2} + 1$$

Solving equation (3)

Integrating both sides gives

$$\int \frac{1}{-\frac{y^{\frac{2}{3}}}{2} - \frac{i\sqrt{3}\,y^{\frac{2}{3}}}{2}}dy = \int dx$$

$$-\frac{6y^{\frac{1}{3}}}{1+i\sqrt{3}} = x + c_4$$

Initial conditions are used to solve for c_4. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{6}{1+i\sqrt{3}} = c_4$$

$$c_4 = -\frac{6}{1+i\sqrt{3}}$$

Substituting c_4 found above in the general solution gives

$$-\frac{6y^{\frac{1}{3}}}{1+i\sqrt{3}} = \frac{i\sqrt{3}\,x + x - 6}{1+i\sqrt{3}}$$

The above simplifies to

$$-i\sqrt{3}\,x - 6y^{\frac{1}{3}} - x + 6 = 0$$

Solving for y from the above gives

$$y = \frac{i(x-3)\,x\sqrt{3}}{6} + \frac{x^3}{27} - \frac{x^2}{6} - \frac{x}{2} + 1$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{(x+3)^3}{27} \tag{1}$$

$$y = -\frac{i(x-3)\,x\sqrt{3}}{6} + \frac{x^3}{27} - \frac{x^2}{6} - \frac{x}{2} + 1 \tag{2}$$

$$y = \frac{i(x-3)\,x\sqrt{3}}{6} + \frac{x^3}{27} - \frac{x^2}{6} - \frac{x}{2} + 1 \tag{3}$$

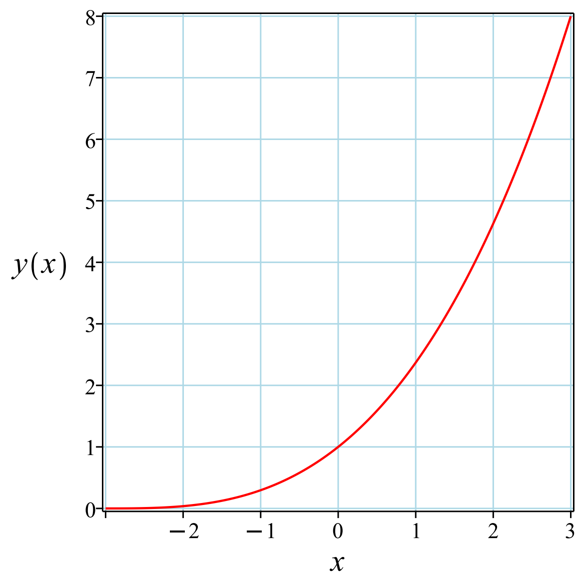


Figure 468: Solution plot

Verification of solutions

$$y = \frac{(x+3)^3}{27}$$

Verified OK.

$$y = -\frac{i(x-3)\,x\sqrt{3}}{6} + \frac{x^3}{27} - \frac{x^2}{6} - \frac{x}{2} + 1$$

Warning, solution could not be verified

$$y = \frac{i(x-3)\,x\sqrt{3}}{6} + \frac{x^3}{27} - \frac{x^2}{6} - \frac{x}{2} + 1$$

Warning, solution could not be verified

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying differential order: 2; missing variables
   `, `-> Computing symmetries using: way = 3
   -> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(2/3)*_a/_b(_a) = 0, _b(_a),
      symmetry methods on request
   `, `1st order, trying reduction of order with given symmetries:`[_a, 2/3*_b]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([3*diff(y(x),x)*diff(y(x),x$2)=2*y(x),y(0) = 1, D(y)(0) = 1],y(x), singsol=all)
```

$$y = \frac{(x+3)^3}{27}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{3*y'[x]*y''[x]==2*y[x],{y[0]==1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

{}

14.28 problem 354

Internal problem ID [15212]
Internal file name [`OUTPUT/15212_Tuesday_April_23_2024_04_54_15_PM_17262081/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 354.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{2y'' - 3y^2 = 0}$$

With initial conditions

$$[y(-2) = 1, y'(-2) = -1]$$

14.28.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$2y'y'' - 3y'y^2 = 0$$

Integrating the above w.r.t x gives

$$\int (2y'y'' - 3y'y^2)\, dx = 0$$
$$y'^2 - y^3 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^3 + c_1} \tag{1}$$

$$y' = -\sqrt{y^3 + c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^3 + c_1}} dy = \int dx$$
$$\int^y \frac{1}{\sqrt{\_a^3 + c_1}} d\_a = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^3 + c_1}} dy = \int dx$$
$$\int^y -\frac{1}{\sqrt{\_a^3 + c_1}} d\_a = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\int^y \frac{1}{\sqrt{\_a^3 + c_1}} d\_a = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = -2$ in the above gives

$$\int^1 \frac{1}{\sqrt{\_a^3 + c_1}} d\_a = -2 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \sqrt{\operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\sqrt{\_a^3 + c_1}} d\_a\right) + x + c_2\right)^3 + c_1}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = \sqrt{\operatorname{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\sqrt{\_a^3 + c_1}} d\_a\right) - 2 + c_2\right)^3 + c_1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$\int^{y} -\frac{1}{\sqrt{\_a^3 + c_1}} d\_a = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = -2$ in the above gives

$$-\left(\int^{1} \frac{1}{\sqrt{\_a^3 + c_1}} d\_a\right) = c_3 - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sqrt{\operatorname{RootOf}\left(-\left(\int^{\_Z} -\frac{1}{\sqrt{\_a^3 + c_1}} d\_a\right) + c_3 + x\right)^3 + c_1}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = -\sqrt{\operatorname{RootOf}\left(\int^{\_Z} \frac{1}{\sqrt{\_a^3 + c_1}} d\_a + c_3 - 2\right)^3 + c_1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

14.28.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
2p(y)\left(\frac{d}{dy}p(y)\right) - 3y^2 = 0
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= \frac{3y^2}{2p}
\end{aligned}
$$

Where $f(y) = \frac{3y^2}{2}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{p}}\,dp &= \frac{3y^2}{2}\,dy \\
\int \frac{1}{\frac{1}{p}}\,dp &= \int \frac{3y^2}{2}\,dy \\
\frac{p^2}{2} &= \frac{y^3}{2} + c_1
\end{aligned}
$$

The solution is

$$
\frac{p(y)^2}{2} - \frac{y^3}{2} - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $y = 1$ and $p = -1$ in the above solution gives an equation to solve for the constant of integration.

$$
-c_1 = 0
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{y^3}{2} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = -y^{\frac{3}{2}}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -y^{\frac{3}{2}}$$

Integrating both sides gives

$$\int -\frac{1}{y^{\frac{3}{2}}} dy = \int dx$$

$$\frac{2}{\sqrt{y}} = x + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = -2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -2 + c_2$$

$$c_2 = 4$$

Substituting c_2 found above in the general solution gives

$$\frac{2}{\sqrt{y}} = x + 4$$

The above simplifies to

$$-x\sqrt{y} - 4\sqrt{y} + 2 = 0$$

Solving for y from the above gives

$$y = \frac{4}{(x + 4)^2}$$

Initial conditions are used to solve for the constants of integration.

Summary
The solution(s) found are the following

$$y = \frac{4}{(x+4)^2} \tag{1}$$

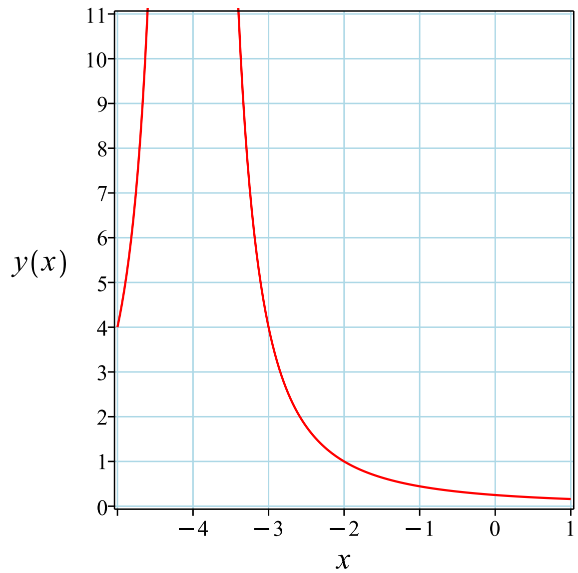


Figure 469: Solution plot

Verification of solutions

$$y = \frac{4}{(x+4)^2}$$

Verified OK.

14.28.3 Maple step by step solution

Let's solve

$$\left[2y'' - 3y^2 = 0, y(-2) = 1, y'\Big|_{\{x=-2\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Define new dependent variable u

  $u(x) = y'$
- Compute y''

$$u'(x) = y''$$

- Use chain rule on the lhs

$$y'\left(\frac{d}{dy}u(y)\right) = y''$$

- Substitute in the definition of u

$$u(y)\left(\frac{d}{dy}u(y)\right) = y''$$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$2u(y)\left(\frac{d}{dy}u(y)\right) - 3y^2 = 0$$

- Integrate both sides with respect to y

$$\int \left(2u(y)\left(\frac{d}{dy}u(y)\right) - 3y^2\right) dy = \int 0dy + c_1$$

- Evaluate integral

$$-y^3 + u(y)^2 = c_1$$

- Solve for $u(y)$

$$\left\{u(y) = \sqrt{y^3 + c_1}, u(y) = -\sqrt{y^3 + c_1}\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \sqrt{y^3 + c_1}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \sqrt{y^3 + c_1}$$

- Separate variables

$$\frac{y'}{\sqrt{y^3+c_1}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{y^3+c_1}}dx = \int 1dx + c_2$$

- Evaluate integral

$$\frac{-\frac{2\mathrm{I}}{3}\sqrt{3}\,(-c_1)^{\frac{1}{3}}\sqrt{\frac{\mathrm{I}\left(y+\frac{(-c_1)^{\frac{1}{3}}}{2}-\frac{\mathrm{I}\sqrt{3}\,(-c_1)^{\frac{1}{3}}}{2}\right)\sqrt{3}}{(-c_1)^{\frac{1}{3}}}}\sqrt{\frac{y-(-c_1)^{\frac{1}{3}}}{-\frac{3(-c_1)^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\sqrt{3}\,(-c_1)^{\frac{1}{3}}}{2}}}\sqrt{\frac{-\mathrm{I}\left(y+\frac{(-c_1)^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\sqrt{3}\,(-c_1)^{\frac{1}{3}}}{2}\right)\sqrt{3}}{(-c_1)^{\frac{1}{3}}}}\,\mathit{EllipticF}\left(\frac{\sqrt{3}\sqrt{\frac{\mathrm{I}\left(y-\right.}{}}}{}\right.}{\sqrt{y^3+c_1}}$$

- Solve for y

$$\left\{-\frac{(-c_1)^{\frac{1}{3}}\left(2\,\mathrm{I}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(\mathrm{I}\sqrt{3}-3\right)}\,(x+c_2)}{4(-c_1)^{\frac{2}{3}}},\frac{\sqrt{2}\,\sqrt{1-\mathrm{I}\sqrt{3}}}{2}\right)^2-\mathrm{I}\sqrt{3}+1\right)}{2}\right\}$$

- Solve 2nd ODE for $u(y)$

  $u(y)=-\sqrt{y^3+c_1}$

- Revert to original variables with substitution $u(y)=y', y=y$

  $y'=-\sqrt{y^3+c_1}$

- Separate variables

  $\frac{y'}{\sqrt{y^3+c_1}}=-1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y^3+c_1}}dx=\int (-1)\,dx+c_2$

- Evaluate integral

$$-\frac{\frac{2\mathrm{I}}{3}\sqrt{3}\,(-c_1)^{\frac{1}{3}}\sqrt{\frac{\mathrm{I}\left(y+\frac{(-c_1)^{\frac{1}{3}}}{2}-\frac{\mathrm{I}\sqrt{3}\,(-c_1)^{\frac{1}{3}}}{2}\right)\sqrt{3}}{(-c_1)^{\frac{1}{3}}}}\sqrt{\frac{y-(-c_1)^{\frac{1}{3}}}{-\frac{3(-c_1)^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\sqrt{3}\,(-c_1)^{\frac{1}{3}}}{2}}}\sqrt{\frac{-\mathrm{I}\left(y+\frac{(-c_1)^{\frac{1}{3}}}{2}+\frac{\mathrm{I}\sqrt{3}\,(-c_1)^{\frac{1}{3}}}{2}\right)\sqrt{3}}{(-c_1)^{\frac{1}{3}}}}\,\mathit{EllipticF}\left(\frac{\sqrt{3}\sqrt{\frac{\mathrm{I}\left(y-\right.}{}}}{}\right.}{\sqrt{y^3+c_1}}$$

- Solve for y

$$\left\{-\frac{(-c_1)^{\frac{1}{3}}\left(2\,\mathrm{I}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(\mathrm{I}\sqrt{3}-3\right)}\,(-x+c_2)}{4(-c_1)^{\frac{2}{3}}},\frac{\sqrt{2}\,\sqrt{1-\mathrm{I}\sqrt{3}}}{2}\right)^2-\mathrm{I}\sqrt{3}+1\right)}{2}\right\}$$

- ☐ Check validity of solution $-\frac{(-c_1)^{\frac{1}{3}}\left(2\,\mathrm{I}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(\mathrm{I}\sqrt{3}-3\right)}\,(-x+c_2)}{4(-c_1)^{\frac{2}{3}}},\frac{\sqrt{2}\,\sqrt{1-\mathrm{I}\sqrt{3}}}{2}\right)^2-\mathrm{I}\sqrt{3}+1\right)}{2}$
  - Use initial condition $y(-2)=1$

    $-\frac{(-c_1)^{\frac{1}{3}}\left(2\,\mathrm{I}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(\mathrm{I}\sqrt{3}-3\right)}\,(2+c_2)}{4(-c_1)^{\frac{2}{3}}},\frac{\sqrt{2}\,\sqrt{1-\mathrm{I}\sqrt{3}}}{2}\right)^2-\mathrm{I}\sqrt{3}+1\right)}{2}$
  - Compute derivative of the solution

$$\frac{\frac{I}{2}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(-x+c_2)}{4(-c_1)^{\frac{2}{3}}}, \frac{\sqrt{2}\,\sqrt{1-I\sqrt{3}}}{2}\right)\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,\mathrm{JacobiCN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(-x+c_2)}{4(-c_1)^{\frac{2}{3}}}, \cdots\right.}{(-c_1)^{\frac{1}{3}}}$$

- Use the initial condition $y'\Big|_{\{x=-2\}} = -1$

$$\frac{\frac{I}{2}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(2+c_2)}{4(-c_1)^{\frac{2}{3}}}, \frac{\sqrt{2}\,\sqrt{1-I\sqrt{3}}}{2}\right)\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,\mathrm{JacobiCN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(2+c_2)}{4(-c_1)^{\frac{2}{3}}}, \cdots\right.}{(-c_1)^{\frac{1}{3}}}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

☐ Check validity of solution $-\frac{(-c_1)^{\frac{1}{3}}\left(2I\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(x+c_2)}{4(-c_1)^{\frac{2}{3}}}, \frac{\sqrt{2}\,\sqrt{1-I\sqrt{3}}}{2}\right)^2 - I\sqrt{3}+1\right)}{2}$

- Use initial condition $y(-2) = 1$

$$-\frac{(-c_1)^{\frac{1}{3}}\left(2I\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(-2+c_2)}{4(-c_1)^{\frac{2}{3}}}, \frac{\sqrt{2}\,\sqrt{1-I\sqrt{3}}}{2}\right)^2 - I\sqrt{3}+1\right)}{2}$$

- Compute derivative of the solution

$$\frac{-\frac{I}{2}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(x+c_2)}{4(-c_1)^{\frac{2}{3}}}, \frac{\sqrt{2}\,\sqrt{1-I\sqrt{3}}}{2}\right)\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,\mathrm{JacobiCN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(x+c_2)}{4(-c_1)^{\frac{2}{3}}}, \cdots\right.}{(-c_1)^{\frac{1}{3}}}$$

- Use the initial condition $y'\Big|_{\{x=-2\}} = -1$

$$\frac{-\frac{I}{2}\sqrt{3}\,\mathrm{JacobiSN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(-2+c_2)}{4(-c_1)^{\frac{2}{3}}}, \frac{\sqrt{2}\,\sqrt{1-I\sqrt{3}}}{2}\right)\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,\mathrm{JacobiCN}\left(\frac{\sqrt{2}\,\sqrt{(-c_1)^{\frac{2}{3}}c_1\left(I\sqrt{3}-3\right)}\,(-2+c_2)}{4(-c_1)^{\frac{2}{3}}}, \cdots\right.}{(-c_1)^{\frac{1}{3}}}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
<- 2nd_order WeierstrassP successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve([2*diff(y(x),x$2)=3*y(x)^2,y(-2) = 1, D(y)(-2) = -1],y(x), singsol=all)
```

$$y = \frac{4}{(x+4)^2}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 12

```
DSolve[{2*y''[x]==3*y[x]^2,{y[-2]==1,y'[-2]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{(x+4)^2}$$

14.29 problem 355

Internal problem ID [15213]
Internal file name [`OUTPUT/15213_Tuesday_April_23_2024_04_54_16_PM_81762950/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 355.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{yy'' + {y'}^2 = 0}$$

14.29.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + {y'}^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary
The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.29.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$y'' = \frac{dp}{dx}$$
$$= \frac{dy}{dx}\frac{dp}{dy}$$
$$= p\frac{dp}{dy}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p}{y}
\end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{y}\,dy \\
\ln(p) &= -\ln(y) + c_1 \\
p &= e^{-\ln(y)+c_1} \\
&= \frac{c_1}{y}
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{c_1}{y}$$

Integrating both sides gives

$$\begin{aligned}
\int \frac{y}{c_1} dy &= x + c_2 \\
\frac{y^2}{2c_1} &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= \sqrt{2c_1c_2 + 2c_1x} \\
y_2 &= -\sqrt{2c_1c_2 + 2c_1x}
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.29.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.29.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}
a_2 &= y \\
a_1 &= y' \\
a_0 &= 0
\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}
\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\
\int y\, dy' + \int y'\, dy + \int 0\, dx &= c_1
\end{aligned}$$

Which results in

$$2yy' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}
\int \frac{2y}{c_1} dy &= x + c_2 \\
\frac{y^2}{c_1} &= x + c_2
\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}
y_1 &= \sqrt{c_1 c_2 + c_1 x} \\
y_2 &= -\sqrt{c_1 c_2 + c_1 x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 c_2 + c_1 x} \tag{1}$$
$$y = -\sqrt{c_1 c_2 + c_1 x} \tag{2}$$

Verification of solutions

$$y = \sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

$$y = -\sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

14.29.5 Maple step by step solution

Let's solve

$y y'' + y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy} u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy} u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy} u(y)\right)$ to reduce order of ODE

  $y u(y) \left(\frac{d}{dy} u(y)\right) + u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy} u(y)}{u(y)} = -\frac{1}{y}$

- Integrate both sides with respect to y

$\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{e^{c_1}}{y}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{e^{c_1}}{y}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{e^{c_1}}{y}$

- Separate variables

  $yy' = e^{c_1}$

- Integrate both sides with respect to x

  $\int yy'dx = \int e^{c_1}dx + c_2$

- Evaluate integral

  $\frac{y^2}{2} = e^{c_1}x + c_2$

- Solve for y

  $\{y = \sqrt{2\,e^{c_1}x + 2c_2}, y = -\sqrt{2\,e^{c_1}x + 2c_2}\}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y = 0$$
$$y = \sqrt{2c_1x + 2c_2}$$
$$y = -\sqrt{2c_1x + 2c_2}$$

✓ Solution by Mathematica

Time used: 0.154 (sec). Leaf size: 20

```
DSolve[y[x]*y''[x]+y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2\sqrt{2x - c_1}$$

14.30 problem 356

Internal problem ID [15214]
Internal file name [`OUTPUT/15214_Tuesday_April_23_2024_04_54_17_PM_94662908/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 356.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' - y' - {y'}^2 = 0}$$

14.30.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)+(-p(y)-1)\,p(y)=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= \frac{p+1}{y}\end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(p)=p+1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p+1}\,dp &= \frac{1}{y}\,dy\\ \int\frac{1}{p+1}\,dp &= \int\frac{1}{y}\,dy\\ \ln(p+1) &= \ln(y)+c_1\end{aligned}$$

Raising both side to exponential gives

$$p+1=\mathrm{e}^{\ln(y)+c_1}$$

Which simplifies to

$$p+1=c_2y$$

Which simplifies to

$$p(y)=c_2y\,\mathrm{e}^{c_1}-1$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_2y\,\mathrm{e}^{c_1}-1$$

Integrating both sides gives

$$\begin{aligned}\int\frac{1}{c_2y\,\mathrm{e}^{c_1}-1}dy &= \int dx\\ \frac{\ln(c_2y\,\mathrm{e}^{c_1}-1)\,\mathrm{e}^{-c_1}}{c_2} &= c_3+x\end{aligned}$$

Raising both side to exponential gives

$$e^{\frac{\ln(c_2 y\,e^{c_1}-1)e^{-c_1}}{c_2}} = e^{c_3+x}$$

Which simplifies to

$$(c_2 y\,e^{c_1}-1)^{\frac{e^{-c_1}}{c_2}} = c_4 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left((c_4 e^{x})^{c_2 e^{c_1}}+1\right)e^{-c_1}}{c_2} \tag{1}$$

Verification of solutions

$$y = \frac{\left((c_4 e^{x})^{c_2 e^{c_1}}+1\right)e^{-c_1}}{c_2}$$

Verified OK.

14.30.2 Maple step by step solution

Let's solve

$yy''+(-y'-1)\,y'=0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x)=y'$

- Compute y''

  $u'(x)=y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right)=y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$

- Make substitutions $y'=u(y)\,,y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$yu(y)\left(\frac{d}{dy}u(y)\right)+(-u(y)-1)\,u(y)=0$$

- Separate variables

$$\frac{\frac{d}{dy}u(y)}{-u(y)-1}=-\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int\frac{\frac{d}{dy}u(y)}{-u(y)-1}dy=\int-\frac{1}{y}dy+c_1$$

- Evaluate integral

$$-\ln\left(-u(y)-1\right)=-\ln\left(y\right)+c_1$$

- Solve for $u(y)$

$$u(y)=-\frac{\mathrm{e}^{c_1}+y}{\mathrm{e}^{c_1}}$$

- Solve 1st ODE for $u(y)$

$$u(y)=-\frac{\mathrm{e}^{c_1}+y}{\mathrm{e}^{c_1}}$$

- Revert to original variables with substitution $u(y)=y', y=y$

$$y'=-\frac{\mathrm{e}^{c_1}+y}{\mathrm{e}^{c_1}}$$

- Separate variables

$$\frac{y'}{\mathrm{e}^{c_1}+y}=-\frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int\frac{y'}{\mathrm{e}^{c_1}+y}dx=\int-\frac{1}{\mathrm{e}^{c_1}}dx+c_2$$

- Evaluate integral

$$\ln\left(\mathrm{e}^{c_1}+y\right)=-\frac{x}{\mathrm{e}^{c_1}}+c_2$$

- Solve for y

$$y=\mathrm{e}^{\frac{c_2\mathrm{e}^{c_1}-x}{\mathrm{e}^{c_1}}}-\mathrm{e}^{c_1}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)*(_b(_a)+1)/_a = 0, _b(_a)
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(y(x)*diff(y(x),x$2)=diff(y(x),x)+diff(y(x),x)^2,y(x), singsol=all)
```

$$y = 0$$
$$y = \frac{e^{c_1(x+c_2)} + 1}{c_1}$$

✓ Solution by Mathematica

Time used: 1.452 (sec). Leaf size: 26

```
DSolve[y[x]*y''[x]==y'[x]+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1 + e^{c_1(x+c_2)}}{c_1}$$
$$y(x) \to \text{Indeterminate}$$

14.31 problem 357

Internal problem ID [15215]
Internal file name [`OUTPUT/15215_Tuesday_April_23_2024_04_54_18_PM_31305956/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 357.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{yy'' - y'^2 = 1}$$

14.31.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= \frac{p^2+1}{yp}\end{aligned}$$

Where $f(y)=\frac{1}{y}$ and $g(p)=\frac{p^2+1}{p}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{p^2+1}{p}}\,dp &= \frac{1}{y}\,dy\\ \int \frac{1}{\frac{p^2+1}{p}}\,dp &= \int \frac{1}{y}\,dy\\ \frac{\ln(p^2+1)}{2} &= \ln(y)+c_1\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{p^2+1} = \mathrm{e}^{\ln(y)+c_1}$$

Which simplifies to

$$\sqrt{p^2+1} = c_2 y$$

Which simplifies to

$$\sqrt{p(y)^2+1} = c_2 y\,\mathrm{e}^{c_1}$$

The solution is

$$\sqrt{p(y)^2+1} = c_2 y\,\mathrm{e}^{c_1}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$\sqrt{1+y'^2} = c_2 y\,\mathrm{e}^{c_1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-1+c_2^2y^2\mathrm{e}^{2c_1}} \tag{1}$$

$$y' = -\sqrt{-1+c_2^2y^2\mathrm{e}^{2c_1}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}} dy = \int dx$$

$$\frac{\ln\left(\frac{c_2^2 \mathrm{e}^{2c_1} y}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + \sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}\right)}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} = c_3 + x$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(\frac{c_2^2 \mathrm{e}^{2c_1} y}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + \sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}\right)}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}}} = \mathrm{e}^{c_3 + x}$$

Which simplifies to

$$\left(\frac{c_2^2 \mathrm{e}^{2c_1} y + \sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}\, \sqrt{c_2^2 \mathrm{e}^{2c_1}}}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}}\right)^{\frac{1}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}}} = c_4 \mathrm{e}^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}} dy = \int dx$$

$$-\frac{\ln\left(\frac{c_2^2 \mathrm{e}^{2c_1} y}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + \sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}\right)}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} = x + c_5$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(\frac{c_2^2 \mathrm{e}^{2c_1} y}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + \sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}\right)}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}}} = \mathrm{e}^{x + c_5}$$

Which simplifies to

$$\left(\frac{c_2^2 \mathrm{e}^{2c_1} y + \sqrt{-1+c_2^2 y^2 \mathrm{e}^{2c_1}}\, \sqrt{c_2^2 \mathrm{e}^{2c_1}}}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}}\right)^{-\frac{1}{\sqrt{c_2^2 \mathrm{e}^{2c_1}}}} = c_6 \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{c_2^2 \mathrm{e}^{2c_1}}\, \mathrm{e}^{-2c_1} \left((c_4 \mathrm{e}^x)^{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + (c_4 \mathrm{e}^x)^{-\sqrt{c_2^2 \mathrm{e}^{2c_1}}} \right)}{2c_2^2} \tag{1}$$

$$y = \frac{\sqrt{c_2^2 \mathrm{e}^{2c_1}}\, \mathrm{e}^{-2c_1} \left((c_6 \mathrm{e}^x)^{-\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + (c_6 \mathrm{e}^x)^{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} \right)}{2c_2^2} \tag{2}$$

Verification of solutions

$$y = \frac{\sqrt{c_2^2 \mathrm{e}^{2c_1}}\, \mathrm{e}^{-2c_1} \left((c_4 \mathrm{e}^x)^{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + (c_4 \mathrm{e}^x)^{-\sqrt{c_2^2 \mathrm{e}^{2c_1}}} \right)}{2c_2^2}$$

Verified OK.

$$y = \frac{\sqrt{c_2^2 \mathrm{e}^{2c_1}}\, \mathrm{e}^{-2c_1} \left((c_6 \mathrm{e}^x)^{-\sqrt{c_2^2 \mathrm{e}^{2c_1}}} + (c_6 \mathrm{e}^x)^{\sqrt{c_2^2 \mathrm{e}^{2c_1}}} \right)}{2c_2^2}$$

Verified OK.

14.31.2 Maple step by step solution

Let's solve

$yy'' - y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy} u(y) \right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy} u(y) \right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$y u(y) \left(\frac{d}{dy}u(y)\right) - u(y)^2 = 1$$

- Separate variables

$$\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2+1} = \frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2+1}dy = \int \frac{1}{y}dy + c_1$$

- Evaluate integral

$$\frac{\ln\left(u(y)^2+1\right)}{2} = \ln(y) + c_1$$

- Solve for $u(y)$

$$\left\{u(y) = \sqrt{\left(e^{c_1}\right)^2 y^2 - 1}, u(y) = -\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \sqrt{\left(e^{c_1}\right)^2 y^2 - 1}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \sqrt{\left(e^{c_1}\right)^2 y^2 - 1}$$

- Separate variables

$$\frac{y'}{\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}}dx = \int 1dx + c_2$$

- Evaluate integral

$$\frac{\ln\left(\frac{\left(e^{c_1}\right)^2 y}{\sqrt{\left(e^{c_1}\right)^2}} + \sqrt{\left(e^{c_1}\right)^2 y^2 - 1}\right)}{\sqrt{\left(e^{c_1}\right)^2}} = x + c_2$$

- Solve for y

$$y = \frac{\sqrt{\left(e^{c_1}\right)^2}\left(\left(e^{c_2\sqrt{\left(e^{c_1}\right)^2}+x\sqrt{\left(e^{c_1}\right)^2}}\right)^2 + 1\right)}{2\,e^{c_2\sqrt{\left(e^{c_1}\right)^2}+x\sqrt{\left(e^{c_1}\right)^2}}\left(e^{c_1}\right)^2}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}$$

- Separate variables

$$\frac{y'}{\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{\left(e^{c_1}\right)^2 y^2 - 1}} dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$\frac{\ln\left(\frac{\left(e^{c_1}\right)^2 y}{\sqrt{\left(e^{c_1}\right)^2}} + \sqrt{\left(e^{c_1}\right)^2 y^2 - 1}\right)}{\sqrt{\left(e^{c_1}\right)^2}} = -x + c_2$$

- Solve for y

$$y = \frac{\sqrt{\left(e^{c_1}\right)^2}\left(\left(e^{c_2 \sqrt{\left(e^{c_1}\right)^2} - x\sqrt{\left(e^{c_1}\right)^2}}\right)^2 + 1\right)}{2\, e^{c_2 \sqrt{\left(e^{c_1}\right)^2} - x\sqrt{\left(e^{c_1}\right)^2}} \left(e^{c_1}\right)^2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-(_b(_a)^2+1)/_a = 0, _b(_a), HIN
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 55

```
dsolve(y(x)*diff(y(x),x$2)=1+diff(y(x),x)^2,y(x), singsol=all)
```

$$y = \frac{c_1\left(\mathrm{e}^{\frac{x+c_2}{c_1}} + \mathrm{e}^{\frac{-x-c_2}{c_1}}\right)}{2}$$
$$y = \frac{c_1\left(\mathrm{e}^{\frac{x+c_2}{c_1}} + \mathrm{e}^{\frac{-x-c_2}{c_1}}\right)}{2}$$

✓ Solution by Mathematica

Time used: 60.132 (sec). Leaf size: 80

```
DSolve[y[x]*y''[x]==1+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{-c_1} \tanh\left(e^{c_1}(x + c_2)\right)}{\sqrt{-\mathrm{sech}^2\left(e^{c_1}(x + c_2)\right)}}$$
$$y(x) \to \frac{e^{-c_1} \tanh\left(e^{c_1}(x + c_2)\right)}{\sqrt{-\mathrm{sech}^2\left(e^{c_1}(x + c_2)\right)}}$$

14.32 problem 358

Internal problem ID [15216]
Internal file name [`OUTPUT/15216_Tuesday_April_23_2024_04_54_22_PM_80498237/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 358.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{2yy'' - {y'}^2 = 1}$$

14.32.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{p^2+1}{2yp}
\end{aligned}$$

Where $f(y) = \frac{1}{2y}$ and $g(p) = \frac{p^2+1}{p}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{p^2+1}{p}}\, dp &= \frac{1}{2y}\, dy \\
\int \frac{1}{\frac{p^2+1}{p}}\, dp &= \int \frac{1}{2y}\, dy \\
\frac{\ln(p^2+1)}{2} &= \frac{\ln(y)}{2} + c_1
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{p^2+1} = e^{\frac{\ln(y)}{2}+c_1}$$

Which simplifies to

$$\sqrt{p^2+1} = c_2\sqrt{y}$$

Which simplifies to

$$\sqrt{p(y)^2+1} = c_2\sqrt{y}\, e^{c_1}$$

The solution is

$$\sqrt{p(y)^2+1} = c_2\sqrt{y}\, e^{c_1}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\sqrt{1+y'^2} = c_2\sqrt{y}\, e^{c_1}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-1 + c_2^2 e^{2c_1} y} \tag{1}$$

$$y' = -\sqrt{-1 + c_2^2 e^{2c_1} y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-1+c_2^2 \mathrm{e}^{2c_1} y}} dy = \int dx$$

$$\frac{2\sqrt{-1+c_2^2 \mathrm{e}^{2c_1} y}\, \mathrm{e}^{-2c_1}}{c_2^2} = c_3 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-1+c_2^2 \mathrm{e}^{2c_1} y}} dy = \int dx$$

$$-\frac{2\sqrt{-1+c_2^2 \mathrm{e}^{2c_1} y}\, \mathrm{e}^{-2c_1}}{c_2^2} = x + c_4$$

Summary

The solution(s) found are the following

$$y = \frac{(c_3^2 \mathrm{e}^{4c_1} c_2^4 + 2c_3 \mathrm{e}^{4c_1} c_2^4 x + x^2 \mathrm{e}^{4c_1} c_2^4 + 4)\, \mathrm{e}^{-2c_1}}{4c_2^2} \tag{1}$$

$$y = \frac{(c_4^2 \mathrm{e}^{4c_1} c_2^4 + 2c_4 \mathrm{e}^{4c_1} c_2^4 x + x^2 \mathrm{e}^{4c_1} c_2^4 + 4)\, \mathrm{e}^{-2c_1}}{4c_2^2} \tag{2}$$

Verification of solutions

$$y = \frac{(c_3^2 \mathrm{e}^{4c_1} c_2^4 + 2c_3 \mathrm{e}^{4c_1} c_2^4 x + x^2 \mathrm{e}^{4c_1} c_2^4 + 4)\, \mathrm{e}^{-2c_1}}{4c_2^2}$$

Verified OK.

$$y = \frac{(c_4^2 \mathrm{e}^{4c_1} c_2^4 + 2c_4 \mathrm{e}^{4c_1} c_2^4 x + x^2 \mathrm{e}^{4c_1} c_2^4 + 4)\, \mathrm{e}^{-2c_1}}{4c_2^2}$$

Verified OK.

14.32.2 Maple step by step solution

Let's solve

$2yy'' - y'^2 = 1$

- Highest derivative means the order of the ODE is 2

  y''
- Define new dependent variable u

  $u(x) = y'$
- Compute y''

  $u'(x) = y''$
- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $2yu(y)\left(\frac{d}{dy}u(y)\right) - u(y)^2 = 1$
- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2+1} = \frac{1}{2y}$
- Integrate both sides with respect to y

  $\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)}{u(y)^2+1}dy = \int \frac{1}{2y}dy + c_1$
- Evaluate integral

  $\frac{\ln\left(u(y)^2+1\right)}{2} = \frac{\ln(y)}{2} + c_1$
- Solve for $u(y)$

  $\left\{u(y) = \frac{\sqrt{-e^{-2c_1}\left(e^{-2c_1}-y\right)}}{e^{-2c_1}}, u(y) = -\frac{\sqrt{-e^{-2c_1}\left(e^{-2c_1}-y\right)}}{e^{-2c_1}}\right\}$
- Solve 1st ODE for $u(y)$

  $u(y) = \frac{\sqrt{-e^{-2c_1}\left(e^{-2c_1}-y\right)}}{e^{-2c_1}}$
- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}}{\mathrm{e}^{-2c_1}}$$

- Separate variables

$$\frac{y'}{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}} = \frac{1}{\mathrm{e}^{-2c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}}dx = \int \frac{1}{\mathrm{e}^{-2c_1}}dx + c_2$$

- Evaluate integral

$$\frac{2\sqrt{\mathrm{e}^{-2c_1}y-\left(\mathrm{e}^{-2c_1}\right)^2}}{\mathrm{e}^{-2c_1}} = \frac{x}{\mathrm{e}^{-2c_1}} + c_2$$

- Solve for y

$$y = \frac{c_2^2\left(\mathrm{e}^{-2c_1}\right)^2+2c_2\mathrm{e}^{-2c_1}x+4\left(\mathrm{e}^{-2c_1}\right)^2+x^2}{4\,\mathrm{e}^{-2c_1}}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}}{\mathrm{e}^{-2c_1}}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}}{\mathrm{e}^{-2c_1}}$$

- Separate variables

$$\frac{y'}{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}} = -\frac{1}{\mathrm{e}^{-2c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{-\mathrm{e}^{-2c_1}\left(\mathrm{e}^{-2c_1}-y\right)}}dx = \int -\frac{1}{\mathrm{e}^{-2c_1}}dx + c_2$$

- Evaluate integral

$$\frac{2\sqrt{\mathrm{e}^{-2c_1}y-\left(\mathrm{e}^{-2c_1}\right)^2}}{\mathrm{e}^{-2c_1}} = -\frac{x}{\mathrm{e}^{-2c_1}} + c_2$$

- Solve for y

$$y = \frac{c_2^2\left(\mathrm{e}^{-2c_1}\right)^2-2c_2\mathrm{e}^{-2c_1}x+4\left(\mathrm{e}^{-2c_1}\right)^2+x^2}{4\,\mathrm{e}^{-2c_1}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x), y(x)`   *** Sublevel 2 **
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   <- quadrature successful
<- 2nd order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(2*y(x)*diff(y(x),x$2)=1+diff(y(x),x)^2,y(x), singsol=all)
```

$$y = \frac{(c_1^2 + 1)\, x^2}{4c_2} + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 34

```
DSolve[2*y[x]*y''[x]==1+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(1 + c_1{}^2)\, x^2}{4c_2} + c_1 x + c_2$$
$$y(x) \to \text{Indeterminate}$$

14.33 problem 359

Internal problem ID [15217]
Internal file name [`OUTPUT/15217_Tuesday_April_23_2024_04_54_24_PM_12687429/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 359.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y^3 y'' = -1}$$

With initial conditions

$$[y(1) = 1, y'(1) = 0]$$

14.33.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^3 p(y) \left(\frac{d}{dy} p(y)\right) = -1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y, p) \\ &= f(y) g(p) \\ &= -\frac{1}{y^3 p} \end{aligned}$$

Where $f(y) = -\frac{1}{y^3}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$\frac{1}{\frac{1}{p}}\, dp = -\frac{1}{y^3}\, dy$$

$$\int \frac{1}{\frac{1}{p}}\, dp = \int -\frac{1}{y^3}\, dy$$

$$\frac{p^2}{2} = \frac{1}{2y^2} + c_1$$

The solution is

$$\frac{p(y)^2}{2} - \frac{1}{2y^2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $y = 1$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 - \frac{1}{2} = 0$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2 y^2 + y^2 - 1}{2y^2} = 0$$

The above simplifies to

$$p^2 y^2 + y^2 - 1 = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = \frac{\sqrt{-y^2+1}}{y}$$
$$p(y) = -\frac{\sqrt{-y^2+1}}{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{-y^2+1}}{y}$$

Integrating both sides gives

$$\int \frac{y}{\sqrt{-y^2+1}} dy = \int dx$$
$$\frac{(y-1)(y+1)}{\sqrt{-y^2+1}} = x + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 1 + c_2$$

$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$\frac{(y-1)(y+1)}{\sqrt{-y^2+1}} = x - 1$$

The above simplifies to

$$-x\sqrt{-y^2+1} + y^2 + \sqrt{-y^2+1} - 1 = 0$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{-y^2+1}}{y}$$

Integrating both sides gives

$$\int -\frac{y}{\sqrt{-y^2+1}}dy = \int dx$$
$$-\frac{(y-1)(y+1)}{\sqrt{-y^2+1}} = c_3 + x$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3 + 1$$

$$c_3 = -1$$

Substituting c_3 found above in the general solution gives

$$-\frac{(y-1)(y+1)}{\sqrt{-y^2+1}} = x - 1$$

The above simplifies to

$$-x\sqrt{-y^2+1} - y^2 + \sqrt{-y^2+1} + 1 = 0$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$-x\sqrt{-y^2+1} + y^2 + \sqrt{-y^2+1} - 1 = 0 \tag{1}$$
$$-x\sqrt{-y^2+1} - y^2 + \sqrt{-y^2+1} + 1 = 0 \tag{2}$$

Verification of solutions

$$-x\sqrt{-y^2+1} + y^2 + \sqrt{-y^2+1} - 1 = 0$$

Verified OK.

$$-x\sqrt{-y^2+1} - y^2 + \sqrt{-y^2+1} + 1 = 0$$

Verified OK.

14.33.2 Maple step by step solution

Let's solve

$$\left[y^3 y'' = -1, y(1) = 1, y'\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Define new dependent variable u

$$u(x) = y'$$

- Compute y''

$$u'(x) = y''$$

- Use chain rule on the lhs

$$y'\left(\frac{d}{dy}u(y)\right) = y''$$

- Substitute in the definition of u

$$u(y)\left(\frac{d}{dy}u(y)\right) = y''$$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$$y^3 u(y)\left(\frac{d}{dy}u(y)\right) = -1$$

- Separate variables

$$u(y)\left(\frac{d}{dy}u(y)\right) = -\frac{1}{y^3}$$

- Integrate both sides with respect to y

$$\int u(y)\left(\frac{d}{dy}u(y)\right)dy = \int -\frac{1}{y^3}dy + c_1$$

- Evaluate integral

$$\frac{u(y)^2}{2} = \frac{1}{2y^2} + c_1$$

- Solve for $u(y)$

$$\left\{u(y) = \frac{\sqrt{2c_1 y^2+1}}{y}, u(y) = -\frac{\sqrt{2c_1 y^2+1}}{y}\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \frac{\sqrt{2c_1 y^2+1}}{y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{\sqrt{2y^2c_1+1}}{y}$$

- Separate variables

$$\frac{y'y}{\sqrt{2y^2c_1+1}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{\sqrt{2y^2c_1+1}}dx = \int 1dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2y^2c_1+1}}{2c_1} = x + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{2}\,\sqrt{c_1\left(4c_1^2c_2^2+8c_1^2c_2x+4c_1^2x^2-1\right)}}{2c_1}, y = \frac{\sqrt{2}\,\sqrt{c_1\left(4c_1^2c_2^2+8c_1^2c_2x+4c_1^2x^2-1\right)}}{2c_1}\right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\sqrt{2c_1y^2+1}}{y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{2y^2c_1+1}}{y}$$

- Separate variables

$$\frac{y'y}{\sqrt{2y^2c_1+1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{\sqrt{2y^2c_1+1}}dx = \int (-1)\,dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2y^2c_1+1}}{2c_1} = -x + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{2}\,\sqrt{c_1\left(4c_1^2c_2^2-8c_1^2c_2x+4c_1^2x^2-1\right)}}{2c_1}, y = \frac{\sqrt{2}\,\sqrt{c_1\left(4c_1^2c_2^2-8c_1^2c_2x+4c_1^2x^2-1\right)}}{2c_1}\right\}$$

□ Check validity of solution $y = -\frac{\sqrt{2}\,\sqrt{c_1\left(4c_1^2c_2^2-8c_1^2c_2x+4c_1^2x^2-1\right)}}{2c_1}$

  - Use initial condition $y(1) = 1$

$$1 = -\frac{\sqrt{2}\,\sqrt{c_1\left(4c_1^2c_2^2-8c_1^2c_2+4c_1^2-1\right)}}{2c_1}$$

  - Compute derivative of the solution

$$y' = -\frac{\sqrt{2}\,(-8c_1^2c_2+8c_1^2x)}{4\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2x+4c_1^2x^2-1)}}$$

- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

$$0 = -\frac{\sqrt{2}\,(-8c_1^2c_2+8c_1^2)}{4\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2+4c_1^2-1)}}$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{1}{2}, c_2 = 1\right\}$$

- Substitute constant values into general solution and simplify

$$y = \sqrt{-x\,(x-2)}$$

☐ Check validity of solution $y = \frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2x+4c_1^2x^2-1)}}{2c_1}$

- Use initial condition $y(1) = 1$

$$1 = \frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2+4c_1^2-1)}}{2c_1}$$

- Compute derivative of the solution

$$y' = \frac{\sqrt{2}\,(-8c_1^2c_2+8c_1^2x)}{4\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2x+4c_1^2x^2-1)}}$$

- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

$$0 = \frac{\sqrt{2}\,(-8c_1^2c_2+8c_1^2)}{4\sqrt{c_1\,(4c_1^2c_2^2-8c_1^2c_2+4c_1^2-1)}}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

☐ Check validity of solution $y = -\frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2x+4c_1^2x^2-1)}}{2c_1}$

- Use initial condition $y(1) = 1$

$$1 = -\frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2+4c_1^2-1)}}{2c_1}$$

- Compute derivative of the solution

$$y' = -\frac{\sqrt{2}\,(8c_1^2c_2+8c_1^2x)}{4\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2x+4c_1^2x^2-1)}}$$

- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

$$0 = -\frac{\sqrt{2}\,(8c_1^2c_2+8c_1^2)}{4\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2+4c_1^2-1)}}$$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = -1\right\}$

- Substitute constant values into general solution and simplify

  $y = \sqrt{-x\,(x-2)}$

☐ Check validity of solution $y = \frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2x+4c_1^2x^2-1)}}{2c_1}$

- Use initial condition $y(1) = 1$

  $1 = \frac{\sqrt{2}\,\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2+4c_1^2-1)}}{2c_1}$

- Compute derivative of the solution

  $y' = \frac{\sqrt{2}\,(8c_1^2c_2+8c_1^2x)}{4\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2x+4c_1^2x^2-1)}}$

- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

  $0 = \frac{\sqrt{2}\,(8c_1^2c_2+8c_1^2)}{4\sqrt{c_1\,(4c_1^2c_2^2+8c_1^2c_2+4c_1^2-1)}}$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

• Solution to the IVP

  $y = \sqrt{-x\,(x-2)}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+1/_a^3 = 0, _b(_a), HINT = [[_a,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, -_b]
```

✗ Solution by Maple

```
dsolve([y(x)^3*diff(y(x),x$2)=-1,y(1) = 1, D(y)(1) = 0],y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 15

```
DSolve[{y[x]^3*y''[x]==-1,{y[1]==1,y'[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{-((x-2)x)}$$

14.34 problem 360

Internal problem ID [15218]
Internal file name [`OUTPUT/15218_Tuesday_April_23_2024_04_54_25_PM_19635730/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 360.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order,
   _with_potential_symmetries], [_2nd_order, _reducible, _mu_xy
   ]]
```

$$\boxed{yy'' - y'^2 - y'y^2 = 0}$$

14.34.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y p(y) \left(\frac{d}{dy}p(y)\right) + \left(-p(y) - y^2\right) p(y) = 0$$

Which is now solved as first order ode for $p(y)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dy}p(y) + p(y)p(y) = q(y)$$

Where here

$$p(y) = -\frac{1}{y}$$
$$q(y) = y$$

Hence the ode is

$$\frac{d}{dy}p(y) - \frac{p(y)}{y} = y$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{y}dy}$$
$$= \frac{1}{y}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}y}(\mu p) = (\mu)(y)$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{p}{y}\right) = \left(\frac{1}{y}\right)(y)$$
$$\mathrm{d}\left(\frac{p}{y}\right) = \mathrm{d}y$$

Integrating gives

$$\frac{p}{y} = \int \mathrm{d}y$$
$$\frac{p}{y} = y + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{y}$ results in

$$p(y) = c_1 y + y^2$$

which simplifies to

$$p(y) = y(y + c_1)$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = y(y + c_1)$$

Integrating both sides gives

$$\int \frac{1}{y\left(y + c_1\right)} dy = \int dx$$
$$-\frac{\ln\left(y + c_1\right)}{c_1} + \frac{\ln\left(y\right)}{c_1} = x + c_2$$

The above can be written as

$$\left(-\frac{1}{c_1}\right)\left(\ln\left(y + c_1\right) - \ln\left(y\right)\right) = x + c_2$$
$$\ln\left(y + c_1\right) - \ln\left(y\right) = \left(-c_1\right)\left(x + c_2\right)$$
$$= -\left(x + c_2\right) c_1$$

Raising both side to exponential gives

$$e^{\ln(y+c_1)-\ln(y)} = -c_1 c_2 e^{-c_1 x}$$

Which simplifies to

$$\frac{y + c_1}{y} = c_3 e^{-c_1 x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{-1 + c_3 e^{-c_1 x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{-1 + c_3 e^{-c_1 x}}$$

Verified OK.

14.34.2 Maple step by step solution

Let's solve

$yy'' + (-y' - y^2)\, y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + (-u(y) - y^2)\, u(y) = 0$

- Isolate the derivative

  $\frac{d}{dy}u(y) = \frac{u(y)}{y} + y$

- Group terms with $u(y)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dy}u(y) - \frac{u(y)}{y} = y$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \mu(y)\, y$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\, u(y))$

  $\mu(y)\left(\frac{d}{dy}u(y) - \frac{u(y)}{y}\right) = \left(\frac{d}{dy}\mu(y)\right) u(y) + \mu(y)\left(\frac{d}{dy}u(y)\right)$

- Isolate $\frac{d}{dy}\mu(y)$

  $\frac{d}{dy}\mu(y) = -\frac{\mu(y)}{y}$

- Solve to find the integrating factor

  $\mu(y) = \frac{1}{y}$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}(\mu(y)\, u(y))\right) dy = \int \mu(y)\, y dy + c_1$

- Evaluate the integral on the lhs

  $\mu(y)\, u(y) = \int \mu(y)\, y dy + c_1$

- Solve for $u(y)$

  $u(y) = \frac{\int \mu(y) y dy + c_1}{\mu(y)}$

- Substitute $\mu(y) = \frac{1}{y}$

  $u(y) = y\left(\int 1 dy + c_1\right)$

- Evaluate the integrals on the rhs

  $u(y) = y(y + c_1)$

- Solve 1st ODE for $u(y)$

  $u(y) = y(y + c_1)$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = y(y + c_1)$

- Separate variables

  $\frac{y'}{y(y+c_1)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y+c_1)} dx = \int 1 dx + c_2$

- Evaluate integral

  $-\frac{\ln(y+c_1)}{c_1} + \frac{\ln(y)}{c_1} = x + c_2$

- Solve for y

  $y = -\frac{c_1 \mathrm{e}^{c_2 c_1 + c_1 x}}{-1 + \mathrm{e}^{c_2 c_1 + c_1 x}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)*(_a^2+_b(_a))/_a = 0, _b(
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)-diff(y(x),x)^2=y(x)^2*diff(y(x),x),y(x), singsol=all)
```

$$y = 0$$
$$y = -\frac{c_1 e^{c_1(x+c_2)}}{-1 + e^{c_1(x+c_2)}}$$

✓ Solution by Mathematica

Time used: 1.39 (sec). Leaf size: 43

```
DSolve[y[x]*y''[x]-y'[x]^2==y[x]^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1 e^{c_1(x+c_2)}}{-1 + e^{c_1(x+c_2)}}$$
$$y(x) \to -\frac{1}{x + c_2}$$

14.35 problem 361

Internal problem ID [15219]
Internal file name [`OUTPUT/15219_Tuesday_April_23_2024_04_54_25_PM_8292284/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 361.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{y'' - e^{2y} = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

14.35.1 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - y'e^{2y} = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y'e^{2y})\, dx = 0$$
$$\frac{y'^2}{2} - \frac{e^{2y}}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{e^{2y} + 2c_1} \tag{1}$$

$$y' = -\sqrt{e^{2y} + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{e^{2y} + 2c_1}} dy = \int dx$$

$$-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^{2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{e^{2y} + 2c_1}} dy = \int dx$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^{2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$-\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{\sqrt{e^{2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$-\frac{\operatorname{arctanh}\left(\frac{\sqrt{1+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)\sqrt{2}}{2\sqrt{c_1}} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\tanh\left(\sqrt{c_1}\,(x + c_2)\,\sqrt{2}\right) c_1^{\frac{3}{2}} \sqrt{2}\left(1 - \tanh\left(\sqrt{c_1}\,(x + c_2)\,\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x + c_2)\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-\mathrm{e}^{2c_2\sqrt{c_1}\,\sqrt{2}} + 1\right)\sqrt{c_1}\,\sqrt{2}}{\mathrm{e}^{2c_2\sqrt{c_1}\,\sqrt{2}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{\mathrm{e}^{2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$\frac{\operatorname{arctanh}\left(\frac{\sqrt{1+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)\sqrt{2}}{2\sqrt{c_1}} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\tanh\left(\sqrt{c_1}\,(x + c_3)\,\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1 - \tanh\left(\sqrt{c_1}\,(x + c_3)\,\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,\,(x + c_3)\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(-\mathrm{e}^{2c_3\sqrt{c_1}\,\sqrt{2}} + 1\right)\sqrt{c_1}\,\sqrt{2}}{\mathrm{e}^{2c_3\sqrt{c_1}\,\sqrt{2}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

14.35.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y) \left(\frac{d}{dy}p(y)\right) = e^{2y}
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y, p) \\
&= f(y)g(p) \\
&= \frac{e^{2y}}{p}
\end{aligned}
$$

Where $f(y) = e^{2y}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$
\frac{1}{\frac{1}{p}}\, dp = e^{2y}\, dy
$$

$$
\int \frac{1}{\frac{1}{p}}\, dp = \int e^{2y}\, dy
$$

$$
\frac{p^2}{2} = \frac{e^{2y}}{2} + c_1
$$

The solution is

$$
\frac{p(y)^2}{2} - \frac{e^{2y}}{2} - c_1 = 0
$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
-c_1 = 0
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$\frac{p^2}{2} - \frac{e^{2y}}{2} = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = e^y$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = e^y$$

Integrating both sides gives

$$\int e^{-y} dy = x + c_2$$
$$-e^{-y} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \ln\left(-\frac{1}{x + c_2}\right)$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(-\frac{1}{c_2}\right)$$

$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

$$y = \ln\left(-\frac{1}{x - 1}\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \ln\left(-\frac{1}{x - 1}\right) \tag{1}$$

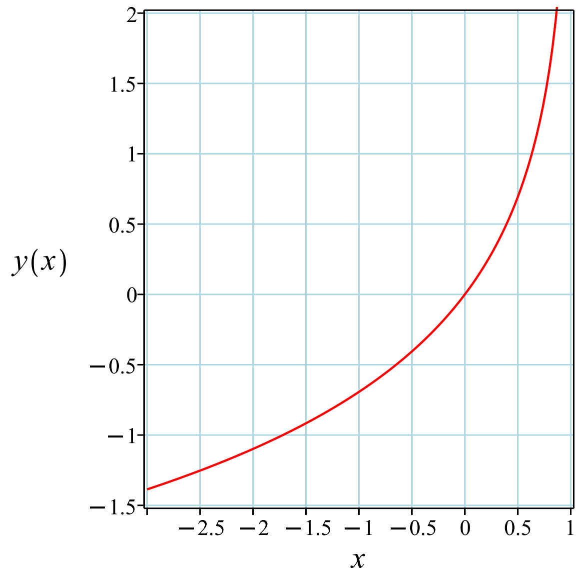


Figure 470: Solution plot

Verification of solutions

$$y = \ln\left(-\frac{1}{x-1}\right)$$

Verified OK.

14.35.3 Maple step by step solution

Let's solve

$$\left[y'' = e^{2y}, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

$u(y)\left(\frac{d}{dy}u(y)\right)=y''$

- Make substitutions $y'=u(y)\,,y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

$u(y)\left(\frac{d}{dy}u(y)\right)=\mathrm{e}^{2y}$

- Integrate both sides with respect to y

$\int u(y)\left(\frac{d}{dy}u(y)\right)dy=\int \mathrm{e}^{2y}dy+c_1$

- Evaluate integral

$\frac{u(y)^2}{2}=\frac{\mathrm{e}^{2y}}{2}+c_1$

- Solve for $u(y)$

$\left\{u(y)=\sqrt{\mathrm{e}^{2y}+2c_1},u(y)=-\sqrt{\mathrm{e}^{2y}+2c_1}\right\}$

- Solve 1st ODE for $u(y)$

$u(y)=\sqrt{\mathrm{e}^{2y}+2c_1}$

- Revert to original variables with substitution $u(y)=y',y=y$

$y'=\sqrt{\mathrm{e}^{2y}+2c_1}$

- Separate variables

$\frac{y'}{\sqrt{\mathrm{e}^{2y}+2c_1}}=1$

- Integrate both sides with respect to x

$\int \frac{y'}{\sqrt{\mathrm{e}^{2y}+2c_1}}dx=\int 1dx+c_2$

- Evaluate integral

$-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{\mathrm{e}^{2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}}=x+c_2$

- Solve for y

$y=\frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2c_1-2c_1\right)}{2}$

- Solve 2nd ODE for $u(y)$

$u(y)=-\sqrt{\mathrm{e}^{2y}+2c_1}$

- Revert to original variables with substitution $u(y)=y',y=y$

$y'=-\sqrt{\mathrm{e}^{2y}+2c_1}$

- Separate variables

$$\frac{y'}{\sqrt{e^{2y}+2c_1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{e^{2y}+2c_1}}dx = \int (-1)\, dx + c_2$$

- Evaluate integral

$$-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{e^{2y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{2\sqrt{c_1}} = -x + c_2$$

- Solve for y

$$y = \frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$$

□ Check validity of solution $y = \frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$

- ○ Use initial condition $y(0) = 0$

$$0 = \frac{\ln\left(2\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$$

- ○ Compute derivative of the solution

$$y' = -\frac{2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(-x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1}$$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = -\frac{2\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)^2\right)}{2\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)^2 c_1 - 2c_1}$$

- ○ Solve for c_1 and c_2
- ○ The solution does not satisfy the initial conditions

□ Check validity of solution $y = \frac{\ln\left(2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$

- ○ Use initial condition $y(0) = 0$

$$0 = \frac{\ln\left(2\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)^2 c_1 - 2c_1\right)}{2}$$

- ○ Compute derivative of the solution

$$y' = \frac{2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\left(1-\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2\right)}{2\tanh\left(\sqrt{c_1}\,(x+c_2)\sqrt{2}\right)^2 c_1 - 2c_1}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = \frac{2\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)c_1^{\frac{3}{2}}\sqrt{2}\,\left(1-\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)^2\right)}{2\tanh\left(c_2\sqrt{c_1}\,\sqrt{2}\right)^2 c_1-2c_1}$$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-exp(2*_a) = 0, _b(_a), HINT = [[
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, _b]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)=exp(2*y(x)),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y = -\frac{\ln\left((x-1)^2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.148 (sec). Leaf size: 13

```
DSolve[{y''[x]==Exp[2*y[x]],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(1-x)$$

14.36 problem 362

Internal problem ID [15220]
Internal file name [`OUTPUT/15220_Tuesday_April_23_2024_05_23_35_PM_67136864/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 362.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{2yy'' - 3{y'}^2 - 4y^2 = 0}$$

14.36.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2yp(y)\left(\frac{d}{dy}p(y)\right) - 3p(y)^2 - 4y^2 = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\,y$ on the above ode results in new ode in $u(y)$

$$2y^2u(y)\left(\left(\frac{d}{dy}u(y)\right)y+u(y)\right)-3u(y)^2\,y^2=4y^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(y,u)\\ &= f(y)g(u)\\ &= \frac{u^2+4}{2uy}\end{aligned}$$

Where $f(y)=\frac{1}{2y}$ and $g(u)=\frac{u^2+4}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4}{u}}\,du=\frac{1}{2y}\,dy$$

$$\int\frac{1}{\frac{u^2+4}{u}}\,du=\int\frac{1}{2y}\,dy$$

$$\frac{\ln(u^2+4)}{2}=\frac{\ln(y)}{2}+c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+4}=\mathrm{e}^{\frac{\ln(y)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+4}=c_3\sqrt{y}$$

Which simplifies to

$$\sqrt{u(y)^2+4}=c_3\sqrt{y}\,\mathrm{e}^{c_2}$$

The solution is

$$\sqrt{u(y)^2+4}=c_3\sqrt{y}\,\mathrm{e}^{c_2}$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\sqrt{\frac{p(y)^2}{y^2}+4}=c_3\sqrt{y}\,\mathrm{e}^{c_2}$$

$$\sqrt{\frac{p(y)^2+4y^2}{y^2}}=c_3\sqrt{y}\,\mathrm{e}^{c_2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\sqrt{\frac{y'^2 + 4y^2}{y^2}} = c_3 \sqrt{y}\, \mathrm{e}^{c_2}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-4 + c_3^2 \mathrm{e}^{2c_2} y}\, y \tag{1}$$

$$y' = -\sqrt{-4 + c_3^2 \mathrm{e}^{2c_2} y}\, y \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-4 + c_3^2 \mathrm{e}^{2c_2} y}\, y} dy = \int dx$$

$$\arctan\left(\frac{\sqrt{-4 + c_3^2 \mathrm{e}^{2c_2} y}}{2}\right) = x + c_4$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-4 + c_3^2 \mathrm{e}^{2c_2} y}\, y} dy = \int dx$$

$$-\arctan\left(\frac{\sqrt{-4 + c_3^2 \mathrm{e}^{2c_2} y}}{2}\right) = x + c_5$$

Summary

The solution(s) found are the following

$$y = \frac{4\left(\tan\left(x + c_4\right)^2 + 1\right) \mathrm{e}^{-2c_2}}{c_3^2} \tag{1}$$

$$y = \frac{4\left(\tan\left(x + c_5\right)^2 + 1\right) \mathrm{e}^{-2c_2}}{c_3^2} \tag{2}$$

Verification of solutions

$$y = \frac{4\left(\tan \left(x+c_4\right)^2+1\right) \mathrm{e}^{-2c_2}}{c_3^2}$$

Verified OK.

$$y = \frac{4\left(\tan \left(x+c_5\right)^2+1\right) \mathrm{e}^{-2c_2}}{c_3^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(2*y(x)*diff(y(x),x$2)-3*diff(y(x),x)^2=4*y(x)^2,y(x), singsol=all)
```

$$y = 0$$

$$y = \frac{4}{\left(c_2 \cos \left(x\right)-c_1 \sin \left(x\right)\right)^2}$$

✓ Solution by Mathematica

Time used: 0.637 (sec). Leaf size: 17

```
DSolve[2*y[x]*y''[x]-3*y'[x]^2==4*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sec^2(x+2c_1)$$

14.37 problem 363

Internal problem ID [15221]
Internal file name [OUTPUT/15221_Tuesday_April_23_2024_05_23_37_PM_64086032/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 14. Differential equations admitting of depression of their order. Exercises page 107
Problem number: 363.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x], [_3rd_order, _exact, _nonlinear], [
   _3rd_order, _with_linear_symmetries], [_3rd_order, _reducible
   , _mu_y2]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(diff(_b(_a), _a), _a))*_b(_a)^2+(diff(_b(_a), _a))^2
   symmetry methods on request
`, `2nd order, trying reduction of order with given symmetries:`[_a, 3/2*_b]
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$3)=3*y(x)*diff(y(x),x),y(0) = 1, D(y)(0) = 1, (D@@2)(y)(0) = 3/2],y(x),
```

$$y = \frac{4}{(x-2)^2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y'''[x]==3*y[x]*y'[x],{y[0]==1,y'[0]==1,y''[0]==3/2}},y[x],x,IncludeSingularSolution
```

{}

15 Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121

15.1 problem 432

Internal problem ID [15222]
Internal file name [`OUTPUT/15222_Tuesday_April_23_2024_05_23_38_PM_28640414/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 432.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

15.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \tag{1}$$

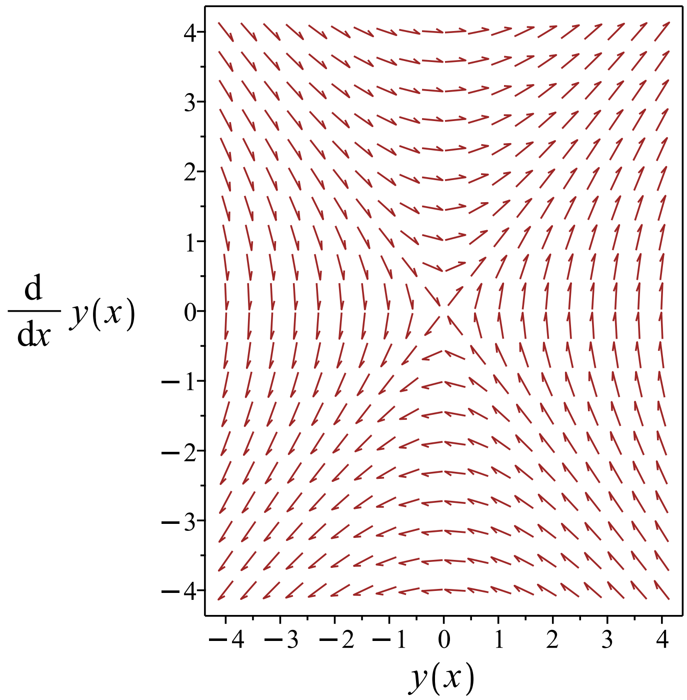


Figure 471: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x}$$

Verified OK.

15.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\,dx = 0$$

$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y+\sqrt{y^2+2c_1} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$y+\sqrt{y^2+2c_1} = c_3\mathrm{e}^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = c_5\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{2x}c_3^2-2c_1\right)\mathrm{e}^{-x}}{2c_3} \tag{1}$$
$$y = -\frac{\left(2c_1c_5^2\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_5} \tag{2}$$

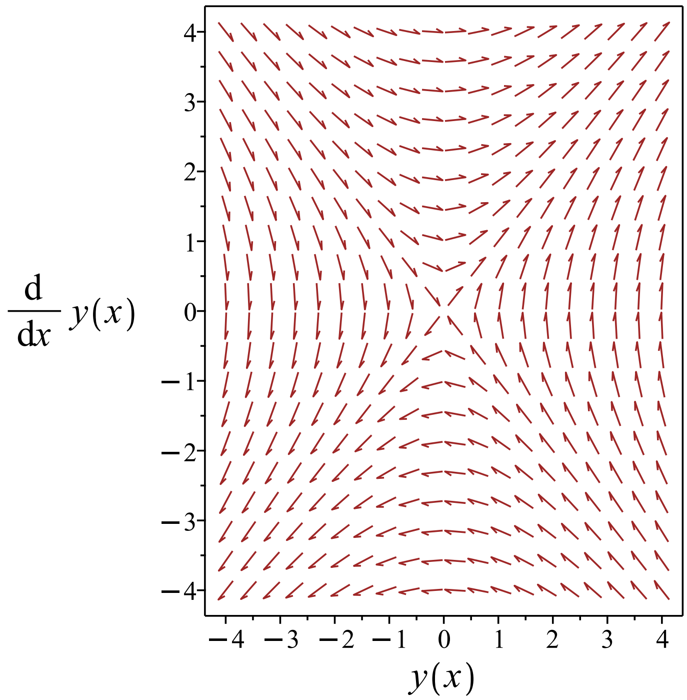


Figure 472: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2c_1)\,e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)\,e^{-x}}{2c_5}$$

Verified OK.

15.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 370: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

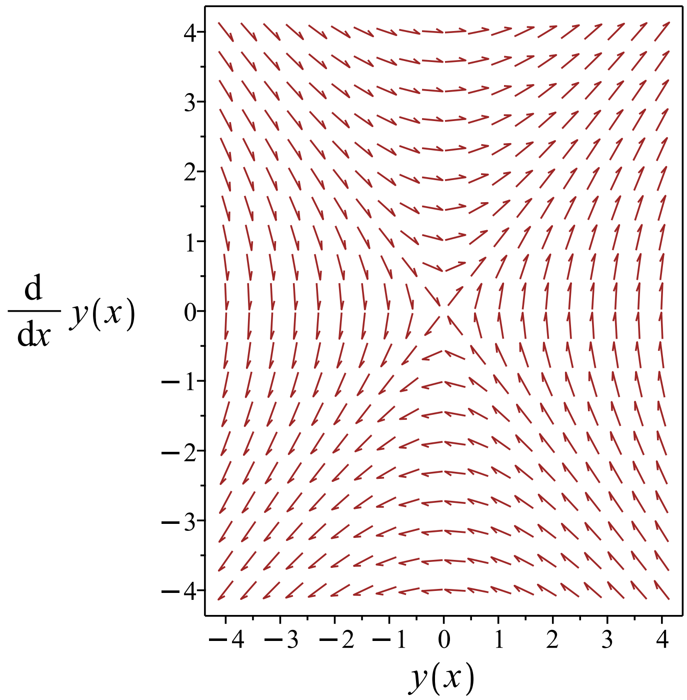


Figure 473: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

Verified OK.

15.1.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y = c_1 e^{-x} + c_2 e^{x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

15.2 problem 433

Internal problem ID [15223]
Internal file name [`OUTPUT/15223_Wednesday_May_08_2024_03_21_26_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 433.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{3y'' - 2y' - 8y = 0}$$

15.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 3, B = -2, C = -8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$3\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - 8 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$3\lambda^2 - 2\lambda - 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = -2, C = -8$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)\,(3)} \pm \frac{1}{(2)\,(3)}\sqrt{-2^2 - (4)\,(3)\,(-8)} \\
&= \frac{1}{3} \pm \frac{5}{3}
\end{aligned}
$$

Hence

$$
\lambda_1 = \frac{1}{3} + \frac{5}{3}
$$
$$
\lambda_2 = \frac{1}{3} - \frac{5}{3}
$$

Which simplifies to

$$
\lambda_1 = 2
$$
$$
\lambda_2 = -\frac{4}{3}
$$

Since roots are real and distinct, then the solution is

$$
y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}
$$
$$
y = c_1 e^{(2)x} + c_2 e^{\left(-\frac{4}{3}\right)x}
$$

Or

$$
y = c_1 e^{2x} + c_2 e^{-\frac{4x}{3}}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{2x} + c_2 e^{-\frac{4x}{3}} \tag{1}
$$

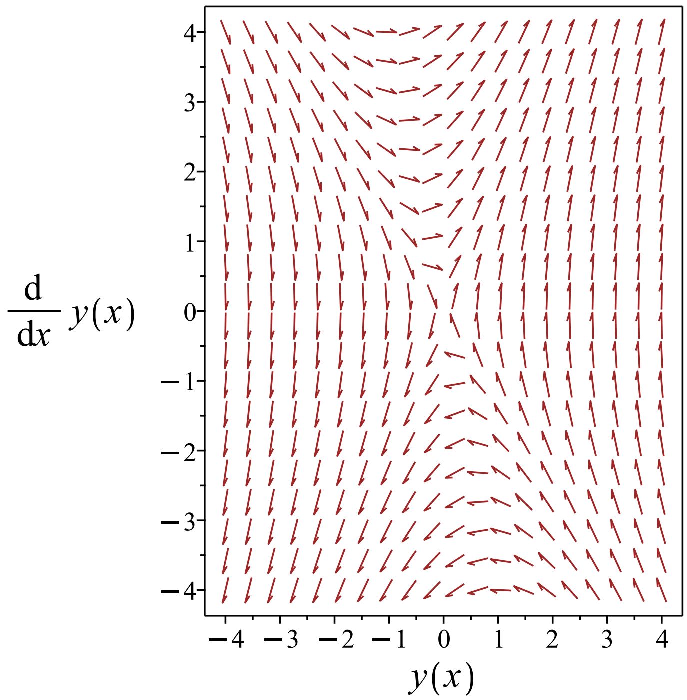


Figure 474: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-\frac{4x}{3}}$$

Verified OK.

15.2.2 Solving using Kovacic algorithm

Writing the ode as

$$3y'' - 2y' - 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= -2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{9} \tag{6}$$

Comparing the above to (5) shows that

$$s = 25$$
$$t = 9$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{9} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 372: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{9}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{3}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{3}\,dx} \\ &= z_1 e^{\frac{x}{3}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{4x}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{3}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{2x}{3}}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{3\,\mathrm{e}^{\frac{10x}{3}}}{10}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{4x}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{4x}{3}}\left(\frac{3\,\mathrm{e}^{\frac{10x}{3}}}{10}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{4x}{3}} + \frac{3 c_2 \mathrm{e}^{2x}}{10} \tag{1}
$$

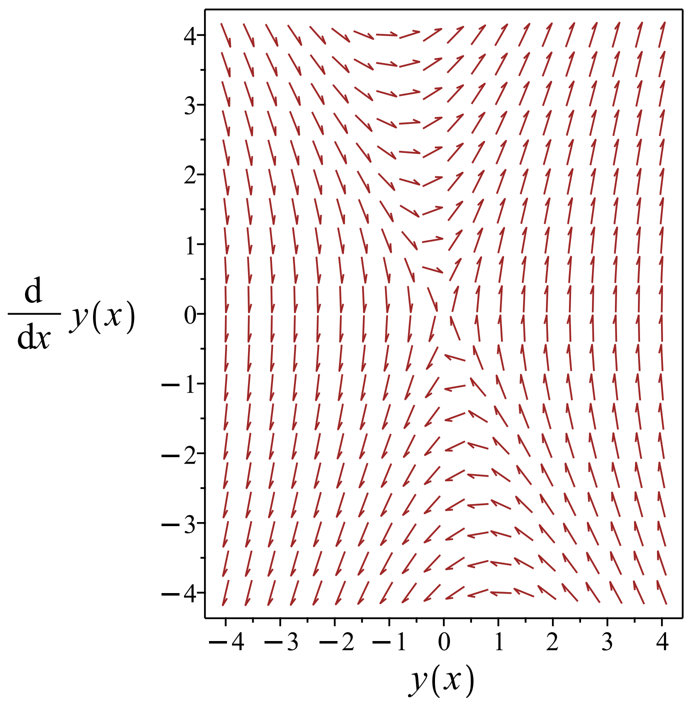


Figure 475: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{4x}{3}} + \frac{3 c_2 \mathrm{e}^{2x}}{10}
$$

Verified OK.

15.2.3 Maple step by step solution

Let's solve

$3y'' - 2y' - 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{3} + \frac{8y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{3} - \frac{8y}{3} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{2}{3}r - \frac{8}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+4)(r-2)}{3} = 0$

- Roots of the characteristic polynomial

  $r = \left(2, -\frac{4}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{4x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{2x} + c_2 e^{-\frac{4x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(3*diff(y(x),x$2)-2*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{\frac{10x}{3}} + c_1\right) \mathrm{e}^{-\frac{4x}{3}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 24

```
DSolve[3*y''[x]-2*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-4x/3} + c_2 e^{2x}$$

15.3 problem 434

Internal problem ID [15224]
Internal file name [`OUTPUT/15224_Wednesday_May_08_2024_03_53_53_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 434.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' + 3y' - y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2, y''(0) = 3]$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + x\,e^x c_2 + x^2 e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^x \\
y_2 &= x e^x \\
y_3 &= x^2 e^x
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^x c_1 + x e^x c_2 + x^2 e^x c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = e^x c_1 + e^x c_2 + x e^x c_2 + 2x e^x c_3 + x^2 e^x c_3
$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + c_2 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = e^x c_1 + 2 e^x c_2 + x e^x c_2 + 2 e^x c_3 + 4x e^x c_3 + x^2 e^x c_3
$$

substituting $y'' = 3$ and $x = 0$ in the above gives

$$
3 = c_1 + 2c_2 + 2c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 1 \\
c_3 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = x e^x + e^x
$$

Which simplifies to

$$y = e^x(x+1)$$

Summary

The solution(s) found are the following

$$y = e^x(x+1) \tag{1}$$

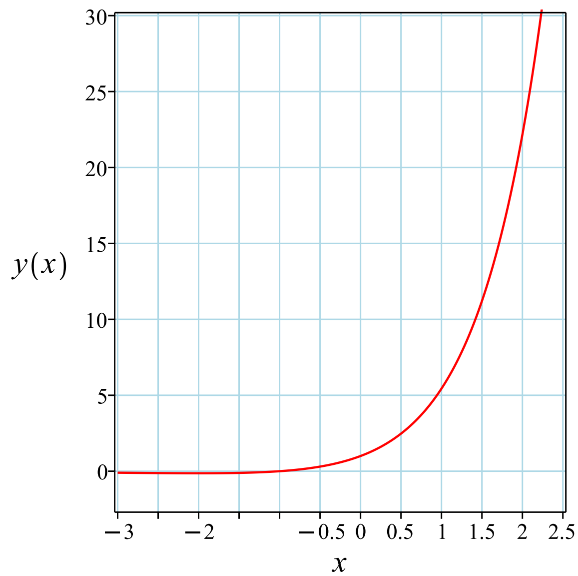


Figure 476: Solution plot

Verification of solutions

$$y = e^x(x+1)$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=0,y(0) = 1, D(y)(0) = 2, (D@@2)(
```

$$y(x) = e^x(1 + x)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 12

```
DSolve[{y'''[x]-3*y''[x]+3*y'[x]-y[x]==0,{y[0]==1,y'[0]==2,y''[0]==3}},y[x],x,IncludeSingula
```

$$y(x) \to e^x(x + 1)$$

15.4 problem 435

Internal problem ID [15225]
Internal file name [`OUTPUT/15225_Wednesday_May_08_2024_03_53_53_PM_81436689/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 435.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

15.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 x e^{-x} \tag{1}$$

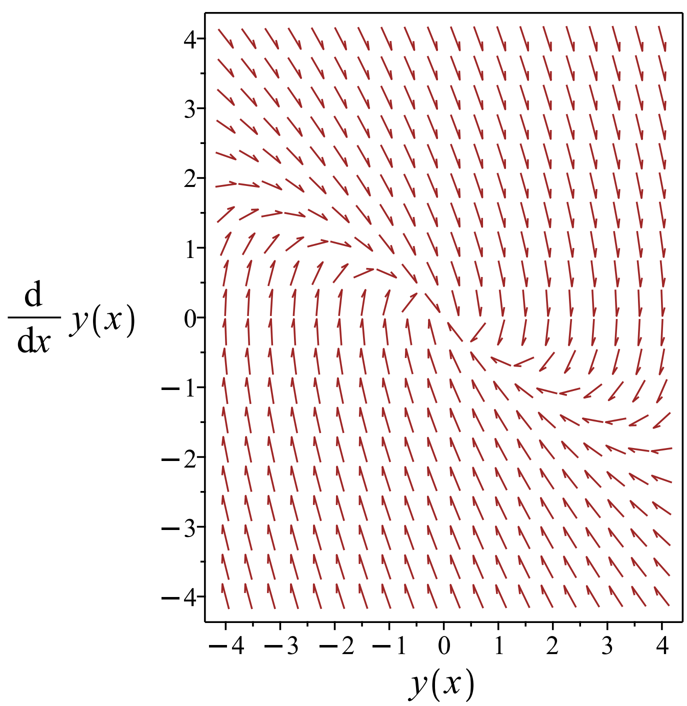


Figure 477: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

Verified OK.

15.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\, dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ (\mathrm{e}^{x} y)'' &= 0 \end{aligned}$$

Integrating once gives

$$(\mathrm{e}^{x} y)' = c_1$$

Integrating again gives

$$(\mathrm{e}^{x} y) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{-x} + \mathrm{e}^{-x} c_2$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{-x} + \mathrm{e}^{-x} c_2 \tag{1}$$

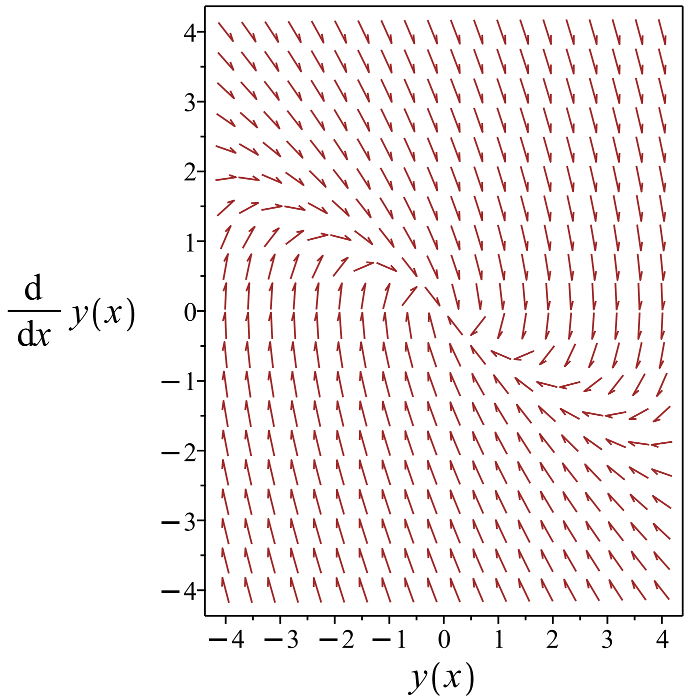


Figure 478: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} c_2$$

Verified OK.

15.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} \tag{1}$$

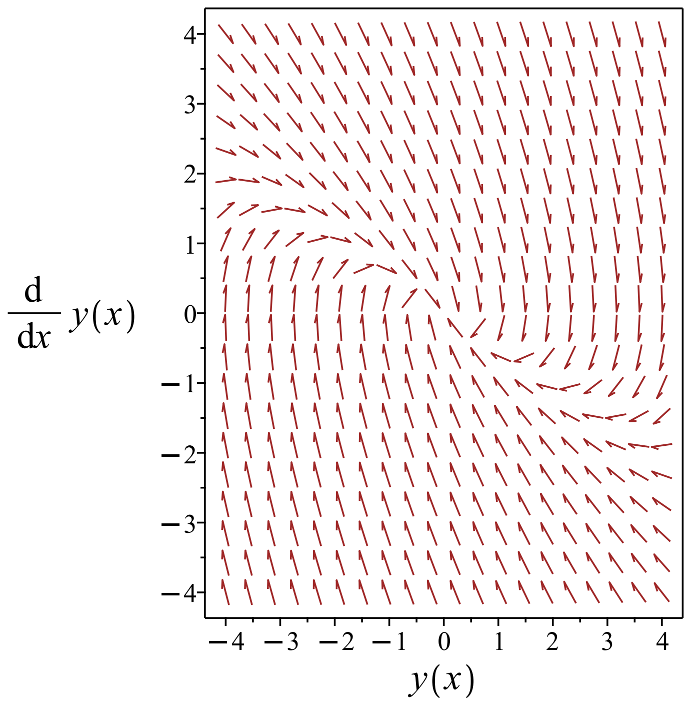


Figure 479: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$$

Verified OK.

15.4.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[y''[x]+2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2x + c_1)$$

15.5 problem 436

Internal problem ID [15226]
Internal file name [`OUTPUT/15226_Wednesday_May_08_2024_03_53_54_PM_95537058/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 436.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 4y' + 3y = 0}$$

With initial conditions

$$[y(0) = 6, y'(0) = 10]$$

15.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 3y = 0$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda \, e^{\lambda x} + 3 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(3)} \\ &= 2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 1 \\ \lambda_2 &= 2 - 1\end{aligned}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{3x} + e^x c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + e^x c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} + e^x c_2$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$10 = 3c_1 + c_2 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 4$$

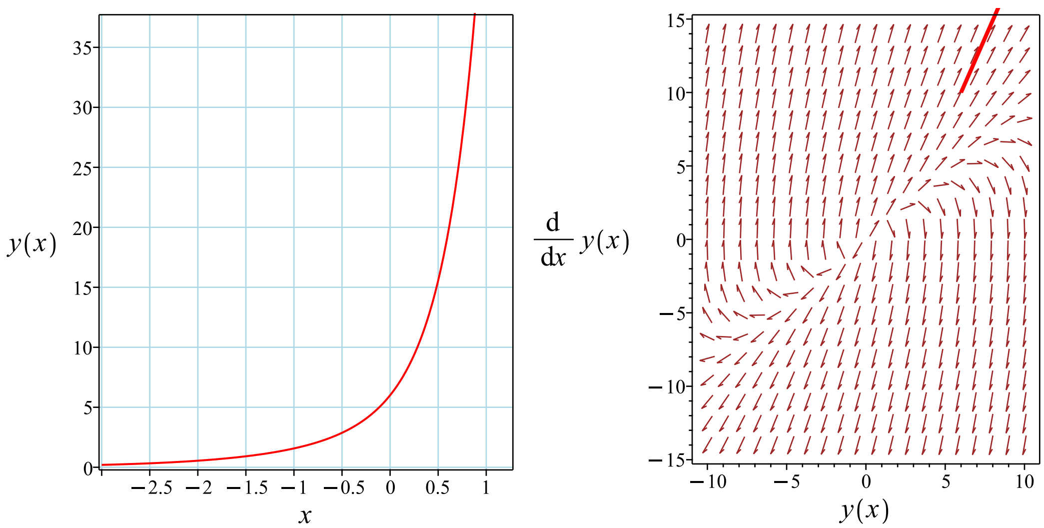
Substituting these values back in above solution results in

$$y = 2 e^{3x} + 4 e^x$$

Summary

The solution(s) found are the following

$$y = 2\,e^{3x} + 4\,e^{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,e^{3x} + 4\,e^{x}$$

Verified OK.

15.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^x c_1 + \frac{c_2 \mathrm{e}^{3x}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$
6 = c_1 + \frac{c_2}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \mathrm{e}^x c_1 + \frac{3c_2 \mathrm{e}^{3x}}{2}
$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$
10 = c_1 + \frac{3c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 4 \\
c_2 &= 4
\end{aligned}
$$

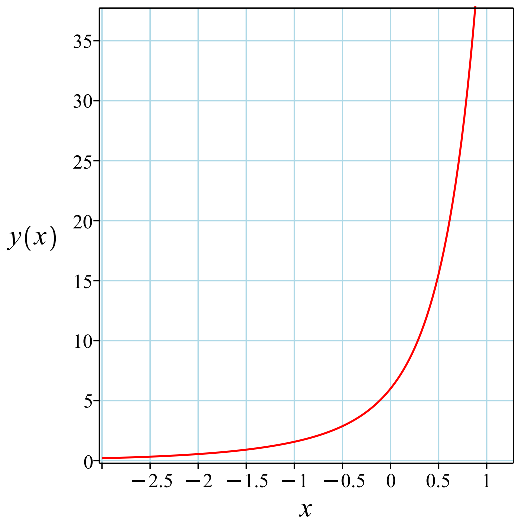
Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{3x} + 4\,\mathrm{e}^x
$$

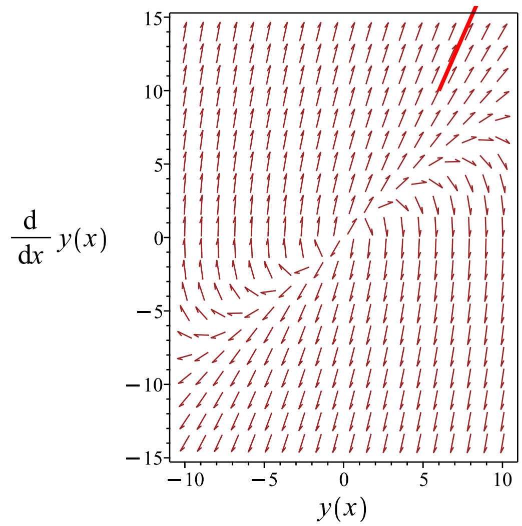
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{3x} + 4\,\mathrm{e}^x \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{3x} + 4\,e^{x}$$

Verified OK.

15.5.4 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 3y = 0, y(0) = 6, y'\Big|_{\{x=0\}} = 10\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the ODE

$y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = e^x c_1 + c_2 e^{3x}$

- □ Check validity of solution $y = e^x c_1 + c_2 e^{3x}$
  - Use initial condition $y(0) = 6$

    $6 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = e^x c_1 + 3c_2 e^{3x}$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 10$

    $10 = c_1 + 3c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 4, c_2 = 2\}$
  - Substitute constant values into general solution and simplify

    $y = 2\,e^{3x} + 4\,e^x$

- Solution to the IVP

  $y = 2\,e^{3x} + 4\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=0,y(0) = 6, D(y)(0) = 10],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{3x} + 4\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 17

```
DSolve[{y''[x]-4*y'[x]+3*y[x]==0,{y[0]==6,y'[0]==10}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to 2e^{x}\left(e^{2x} + 2\right)$$

15.6 problem 437

Internal problem ID [15227]
Internal file name [`OUTPUT/15227_Wednesday_May_08_2024_03_53_55_PM_72649872/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 437.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 6y'' + 11y' + 6y = 0}$$

The characteristic equation is

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-2x} c_2 + \mathrm{e}^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-2x}$$
$$y_3 = \mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{-2x} c_2 + e^{-3x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{-2x} c_2 + e^{-3x} c_3$$

Verified OK.

15.6.1 Maple step by step solution

Let's solve

$y''' + 6y'' + 11y' + 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -6y_3(x) - 11y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -6y_3(x) - 11y_2(x) - 6y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + e^{-2x} c_2 \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{c_1 e^{-3x}}{9} + \frac{e^{-2x} c_2}{4} + c_3 e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+6*diff(y(x),x$2)+11*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 27

```
DSolve[y'''[x]+6*y''[x]+11*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(e^x(c_3 e^x + c_2) + c_1)$$

15.7 problem 438

Internal problem ID [15228]
Internal file name [`OUTPUT/15228_Wednesday_May_08_2024_03_53_55_PM_68571445/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 438.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 2y = 0}$$

15.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -2$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-2)} \\
&= 1 \pm \sqrt{3}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= 1 + \sqrt{3} \\
\lambda_2 &= 1 - \sqrt{3}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 + \sqrt{3} \\
\lambda_2 &= 1 - \sqrt{3}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{\left(1+\sqrt{3}\right)x} + c_2 e^{\left(1-\sqrt{3}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{3}\right)x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{3}\right)x} \tag{1}
$$

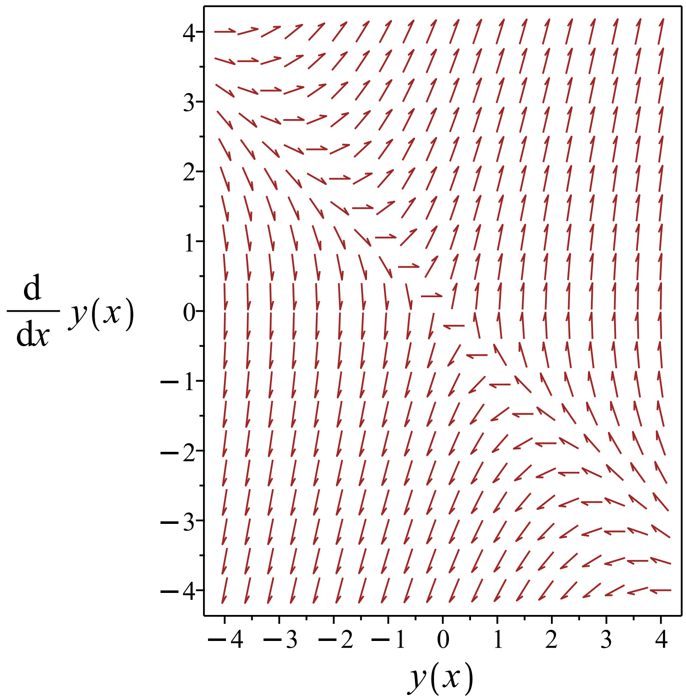


Figure 482: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\left(1+\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{3}\right)x}$$

Verified OK.

15.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x\sqrt{3}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\left(\sqrt{3}-1\right)x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\sqrt{3}\,\mathrm{e}^{2x\sqrt{3}}}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-\left(\sqrt{3}-1\right)x}\right) + c_2\left(e^{-\left(\sqrt{3}-1\right)x}\left(\frac{\sqrt{3}\, e^{2x\sqrt{3}}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-\left(\sqrt{3}-1\right)x} + \frac{c_2\sqrt{3}\, e^{\left(1+\sqrt{3}\right)x}}{6} \tag{1}
$$

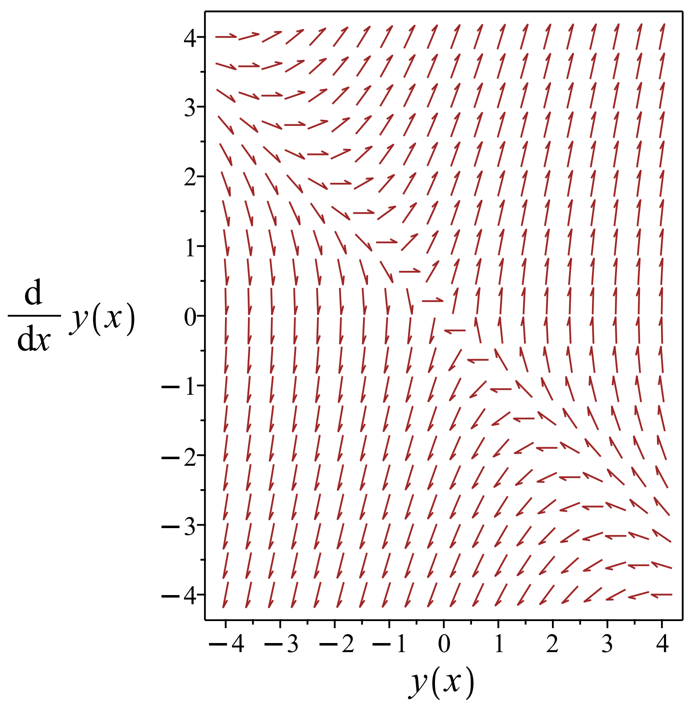


Figure 483: Slope field plot

Verification of solutions

$$
y = c_1 e^{-\left(\sqrt{3}-1\right)x} + \frac{c_2\sqrt{3}\, e^{\left(1+\sqrt{3}\right)x}}{6}
$$

Verified OK.

15.7.3 Maple step by step solution

Let's solve

$y'' - 2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm \left(\sqrt{12}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \sqrt{3}, 1 + \sqrt{3}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(1-\sqrt{3}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(1+\sqrt{3}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(1-\sqrt{3}\right)x} + c_2 e^{\left(1+\sqrt{3}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\left(1+\sqrt{3}\right)x} + c_2 e^{-\left(\sqrt{3}-1\right)x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 34

```
DSolve[y''[x]-2*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{x-\sqrt{3}x}\left(c_2 e^{2\sqrt{3}x} + c_1\right)$$

15.8 problem 439

Internal problem ID [15229]
Internal file name [OUTPUT/15229_Wednesday_May_08_2024_03_53_56_PM_7763517/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 439.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(6)}+2y^{(5)}+y'''' = 0}$$

The characteristic equation is

$$\lambda^6+2\lambda^5+\lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$
$$\lambda_5 = -1$$
$$\lambda_6 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1\mathrm{e}^{-x}+c_2x\,\mathrm{e}^{-x}+c_3+c_4x+c_5x^2+c_6x^3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\,e^{-x}$$
$$y_3 = 1$$
$$y_4 = x$$
$$y_5 = x^2$$
$$y_6 = x^3$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 x\,e^{-x} + c_3 + c_4 x + c_5 x^2 + c_6 x^3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 x\,e^{-x} + c_3 + c_4 x + c_5 x^2 + c_6 x^3$$

Verified OK.

15.8.1 Maple step by step solution

Let's solve

$y^{(6)} + 2y^{(5)} + y'''' = 0$

- Highest derivative means the order of the ODE is 6

  $y^{(6)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Define new variable $y_5(x)$

$y_5(x) = y''''$

- Define new variable $y_6(x)$

  $y_6(x) = y^{(5)}$

- Isolate for $y_6'(x)$ using original ODE

  $y_6'(x) = -2y_6(x) - y_5(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_6(x) = y_5'(x), y_6'(x) = -2y_6(x) -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\vec{y}_1(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I)\cdot \overrightarrow{p}+\overrightarrow{v}=A\cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A-\lambda \cdot I)\cdot \overrightarrow{p}=\overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{array}\right]-(-1)\cdot \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]\right)\cdot \overrightarrow{p}=\left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p}=\left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = e^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3 + c_4 \vec{y}_4 + c_5 \vec{y}_5 + c_6 \vec{y}_6$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \mathrm{e}^{-x} c_2 \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((-x-1) c_2 - c_1) \mathrm{e}^{-x} + c_3$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$6)+2*diff(y(x),x$5)+diff(y(x),x$4)=0,y(x), singsol=all)
```

$$y(x) = (c_6 x + c_5) \mathrm{e}^{-x} + c_4 x^3 + c_3 x^2 + c_2 x + c_1$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 37

```
DSolve[y''''''[x]+2*y'''''[x]+y''''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2(x+4)+c_1)+x(x(c_6x+c_5)+c_4)+c_3$$

15.9 problem 440

Internal problem ID [15230]
Internal file name [`OUTPUT/15230_Wednesday_May_08_2024_03_53_56_PM_48988675/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 440.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 8y' + 5y = 0}$$

15.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = -8, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} - 8\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 8\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -8, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{8}{(2)(4)} \pm \frac{1}{(2)(4)} \sqrt{-8^2 - (4)(4)(5)} \\ &= 1 \pm \frac{i}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + \frac{i}{2} \\ \lambda_2 &= 1 - \frac{i}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1 + \frac{i}{2}$$

$$\lambda_2 = 1 - \frac{i}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^x \left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = e^x \left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right) \tag{1}$$

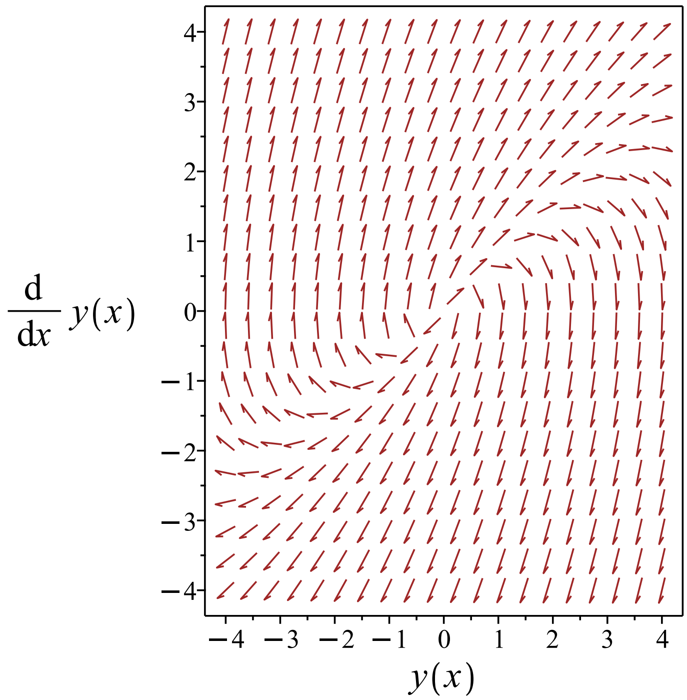


Figure 484: Slope field plot

Verification of solutions

$$y = e^{x}\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Verified OK.

15.9.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 8y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -8 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 382: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8}{4}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos\left(\frac{x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8}{4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1\left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^x \cos\left(\frac{x}{2}\right)\right) + c_2 \left(e^x \cos\left(\frac{x}{2}\right) \left(2 \tan\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x \cos\left(\frac{x}{2}\right) + 2 c_2 e^x \sin\left(\frac{x}{2}\right) \tag{1}$$

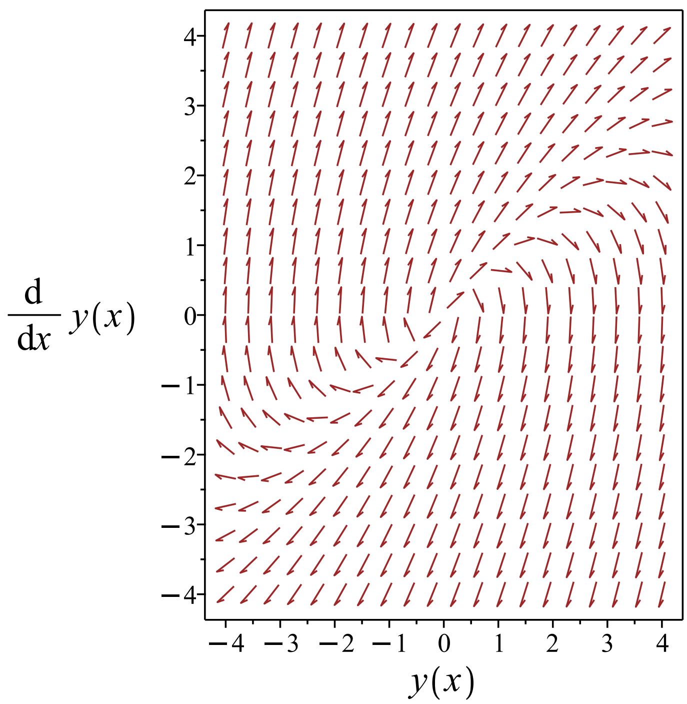


Figure 485: Slope field plot

Verification of solutions

$$y = c_1 e^x \cos\left(\frac{x}{2}\right) + 2 c_2 e^x \sin\left(\frac{x}{2}\right)$$

Verified OK.

15.9.3 Maple step by step solution

Let's solve

$4y'' - 8y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 2y' - \frac{5y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y' + \frac{5y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \frac{\mathrm{I}}{2}, 1 + \frac{\mathrm{I}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos\left(\frac{x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin\left(\frac{x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos\left(\frac{x}{2}\right) + c_2 \mathrm{e}^x \sin\left(\frac{x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(4*diff(y(x),x$2)-8*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x}\left(c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 28

```
DSolve[4*y''[x]-8*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}\left(c_2 \cos\left(\frac{x}{2}\right) + c_1 \sin\left(\frac{x}{2}\right)\right)$$

15.10 problem 441

Internal problem ID [15231]
Internal file name [`OUTPUT/15231_Wednesday_May_08_2024_03_53_57_PM_18391819/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 441.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -1 - i\sqrt{3}$$
$$\lambda_3 = -1 + i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{2x} c_1 + e^{\left(-1-i\sqrt{3}\right)x} c_2 + e^{\left(-1+i\sqrt{3}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{\left(-1-i\sqrt{3}\right)x}$$
$$y_3 = e^{\left(-1+i\sqrt{3}\right)x}$$

Summary

The solution(s) found are the following

$$y = e^{2x}c_1 + e^{(-1-i\sqrt{3})x}c_2 + e^{(-1+i\sqrt{3})x}c_3 \tag{1}$$

Verification of solutions

$$y = e^{2x}c_1 + e^{(-1-i\sqrt{3})x}c_2 + e^{(-1+i\sqrt{3})x}c_3$$

Verified OK.

15.10.1 Maple step by step solution

Let's solve

$y''' - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(-1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{-1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-1-I\sqrt{3}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(-1-I\sqrt{3}\right)^{2}}\\ \frac{1}{-1-I\sqrt{3}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x}\cdot\left(\cos\left(x\sqrt{3}\right)-I\sin\left(x\sqrt{3}\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(-1-I\sqrt{3}\right)^{2}}\\ \frac{1}{-1-I\sqrt{3}}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-x}\cdot\left[\begin{array}{c}\frac{\cos\left(x\sqrt{3}\right)-I\sin\left(x\sqrt{3}\right)}{\left(-1-I\sqrt{3}\right)^{2}}\\ \frac{\cos\left(x\sqrt{3}\right)-I\sin\left(x\sqrt{3}\right)}{-1-I\sqrt{3}}\\ \cos\left(x\sqrt{3}\right)-I\sin\left(x\sqrt{3}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(x\sqrt{3}\right)}{8}-\frac{\sin\left(x\sqrt{3}\right)\sqrt{3}}{8}\\ -\frac{\cos\left(x\sqrt{3}\right)}{4}+\frac{\sin\left(x\sqrt{3}\right)\sqrt{3}}{4}\\ \cos\left(x\sqrt{3}\right)\end{array}\right],\vec{y}_{3}(x)=e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(x\sqrt{3}\right)\sqrt{3}}{8}+\frac{\sin\left(x\sqrt{3}\right)}{8}\\ \frac{\cos\left(x\sqrt{3}\right)\sqrt{3}}{4}+\frac{\sin\left(x\sqrt{3}\right)}{4}\\ -\sin\left(x\sqrt{3}\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=e^{2x}c_{1}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+e^{-x}c_{2}\cdot\left[\begin{array}{c}-\frac{\cos\left(x\sqrt{3}\right)}{8}-\frac{\sin\left(x\sqrt{3}\right)\sqrt{3}}{8}\\ -\frac{\cos\left(x\sqrt{3}\right)}{4}+\frac{\sin\left(x\sqrt{3}\right)\sqrt{3}}{4}\\ \cos\left(x\sqrt{3}\right)\end{array}\right]+c_{3}e^{-x}\cdot\left[\begin{array}{c}-\frac{\cos\left(x\sqrt{3}\right)\sqrt{3}}{8}+\frac{\sin\left(x\sqrt{3}\right)}{8}\\ \frac{\cos\left(x\sqrt{3}\right)\sqrt{3}}{4}+\frac{\sin\left(x\sqrt{3}\right)}{4}\\ -\sin\left(x\sqrt{3}\right)\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=-\frac{e^{-x}\left(\sqrt{3}\,c_{3}+c_{2}\right)\cos\left(x\sqrt{3}\right)}{8}-\frac{e^{-x}\left(c_{2}\sqrt{3}-c_{3}\right)\sin\left(x\sqrt{3}\right)}{8}+\frac{e^{2x}c_{1}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$3)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x} \sin\left(x\sqrt{3}\right) + c_3 \mathrm{e}^{-x} \cos\left(x\sqrt{3}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 42

```
DSolve[y'''[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_1 e^{3x} + c_2 \cos\left(\sqrt{3}x\right) + c_3 \sin\left(\sqrt{3}x\right)\right)$$

15.11 problem 442

Internal problem ID [15232]
Internal file name [`OUTPUT/15232_Wednesday_May_08_2024_03_53_57_PM_58150107/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 442.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y''' + 10y'' + 12y' + 5y = 0}$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + 10\lambda^2 + 12\lambda + 5 = 0$$

The roots of the above equation are

$$\lambda_1 = -1 - 2i$$
$$\lambda_2 = -1 + 2i$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{(-1-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= \mathrm{e}^{(-1+2i)x} \\
y_4 &= \mathrm{e}^{(-1-2i)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{(-1-2i)x} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{(-1-2i)x} c_4
$$

Verified OK.

15.11.1 Maple step by step solution

Let's solve

$y'''' + 4y''' + 10y'' + 12y' + 5y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -4y_4(x) - 10y_3(x) - 12y_2(x) - 5y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_4(x) - 10y_3(x) - 12y_2(x) - 5y_1(x)]$$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -12 & -10 & -4 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -12 & -10 & -4 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[-1-2\,\mathrm{I}, \left[\begin{array}{c} \frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[-1+2\,\mathrm{I}, \left[\begin{array}{c} \frac{11}{125} + \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} - \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\begin{array}{cc} -1, & \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right] \end{array}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_{1}(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_{2}(x) = \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_{2}(x)$ into the homogeneous system

$$\lambda \,\mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x \overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda \,\mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x} \overrightarrow{v} = \mathrm{e}^{\lambda x} \left(\lambda x \overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_{2}(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -12 & -10 & -4 \end{array}\right] - (-1) \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = e^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - 2\,\mathrm{I}, \begin{bmatrix} \frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{(-1-2\,\mathrm{I})x} \cdot \begin{bmatrix} \frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x} \cdot (\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} \frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x}\cdot\begin{bmatrix}\left(\frac{11}{125}-\frac{2\mathrm{I}}{125}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\ \left(-\frac{3}{25}-\frac{4\mathrm{I}}{25}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\ \left(-\frac{1}{5}+\frac{2\mathrm{I}}{5}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\ \cos(2x)-\mathrm{I}\sin(2x)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x)=e^{-x}\cdot\begin{bmatrix}\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\\ -\frac{3\cos(2x)}{25}-\frac{4\sin(2x)}{25}\\ -\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\\ \cos(2x)\end{bmatrix},\vec{y}_4(x)=e^{-x}\cdot\begin{bmatrix}-\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125}\\ \frac{3\sin(2x)}{25}-\frac{4\cos(2x)}{25}\\ \frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5}\\ -\sin(2x)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1e^{-x}\cdot\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}+e^{-x}c_2\cdot\left(x\cdot\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}+\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}\right)+c_3e^{-x}\cdot\begin{bmatrix}\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125}\\ -\frac{3\cos(2x)}{25}-\frac{4\sin(2x)}{25}\\ -\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5}\\ \cos(2x)\end{bmatrix}+c_4e$$

- First component of the vector is the solution to the ODE

$$y=-\left(\frac{(-11c_3+2c_4)\cos(2x)}{125}+\frac{(2c_3+11c_4)\sin(2x)}{125}+(x+1)c_2+c_1\right)e^{-x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$3)+10*diff(y(x),x$2)+12*diff(y(x),x)+5*y(x)=0,y(x), sing
```

$$y(x) = e^{-x}(c_1 + c_2x + c_3 \sin(2x) + c_4 \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 32

```
DSolve[y''''[x]+4*y'''[x]+10*y''[x]+12*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-x}(c_4x + c_2 \cos(2x) + c_1 \sin(2x) + c_3)$$

15.12 problem 443

Internal problem ID [15233]
Internal file name [`OUTPUT/15233_Wednesday_May_08_2024_03_53_58_PM_39063778/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 443.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 2y' + 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

15.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)}\\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

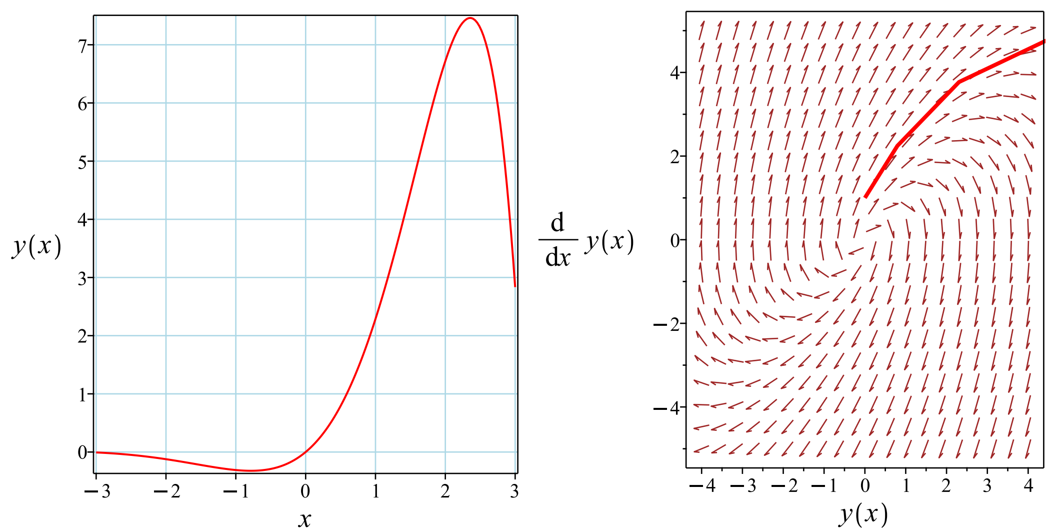
Substituting these values back in above solution results in

$$y = e^{x} \sin(x)$$

Summary

The solution(s) found are the following

$$y = e^x \sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^x \sin(x)$$

Verified OK.

15.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 386: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x \cos(x)) + c_2(\mathrm{e}^x \cos(x)(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^x \cos(x) + \mathrm{e}^x c_2 \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^x \cos(x) - c_1 \mathrm{e}^x \sin(x) + \mathrm{e}^x c_2 \sin(x) + \mathrm{e}^x c_2 \cos(x)
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 1
\end{aligned}
$$

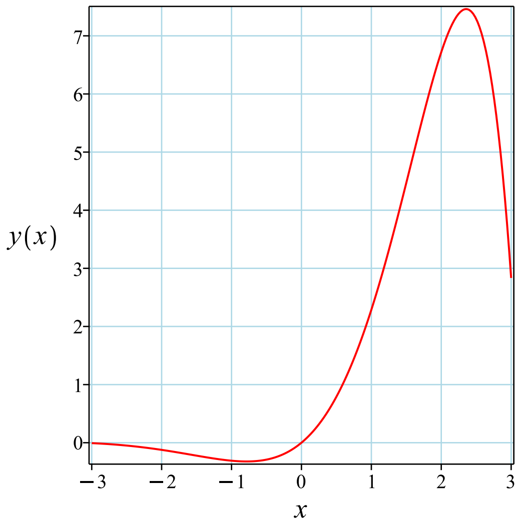
Substituting these values back in above solution results in

$$
y = \mathrm{e}^x \sin(x)
$$

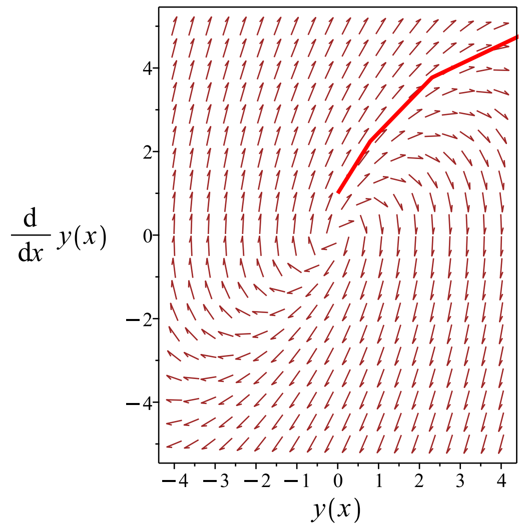
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x \sin(x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^x \sin(x)$$

Verified OK.

15.12.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 2y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$
- 1st solution of the ODE

$y_1(x) = e^x \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = e^x \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x \cos(x) + e^x c_2 \sin(x)$

□ Check validity of solution $y = c_1 e^x \cos(x) + e^x c_2 \sin(x)$

  ◦ Use initial condition $y(0) = 0$

    $0 = c_1$

  ◦ Compute derivative of the solution

    $y' = c_1 e^x \cos(x) - c_1 e^x \sin(x) + e^x c_2 \sin(x) + e^x c_2 \cos(x)$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = c_1 + c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

  ◦ Substitute constant values into general solution and simplify

    $y = e^x \sin(x)$

- Solution to the IVP

  $y = e^x \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = e^{x} \sin(x)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 11

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{x} \sin(x)$$

15.13 problem 444

Internal problem ID [15234]
Internal file name [`OUTPUT/15234_Wednesday_May_08_2024_03_53_58_PM_33212483/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 444.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 2y' + 3y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3]$$

15.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 3y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.13.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(3)}\\ &= 1 \pm i\sqrt{2}\end{aligned}$$

Hence

$$\lambda_1 = 1 + i\sqrt{2}$$
$$\lambda_2 = 1 - i\sqrt{2}$$

Which simplifies to

$$\lambda_1 = 1 + i\sqrt{2}$$
$$\lambda_2 = 1 - i\sqrt{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{x}\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + \mathrm{e}^{x}\left(-c_1\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + c_2\sqrt{2}\, \cos\left(\sqrt{2}\, x\right)\right)$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = c_1 + \sqrt{2}\, c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \sqrt{2}$$

Substituting these values back in above solution results in

$$y = \sqrt{2}\, e^{x} \sin\left(\sqrt{2}\, x\right) + e^{x} \cos\left(\sqrt{2}\, x\right)$$

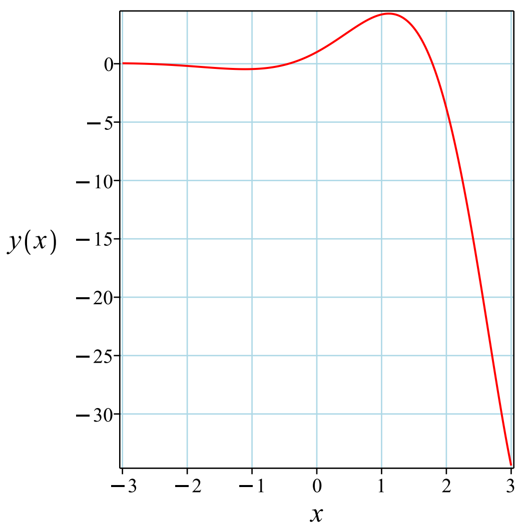
Which simplifies to

$$y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) e^{x}$$

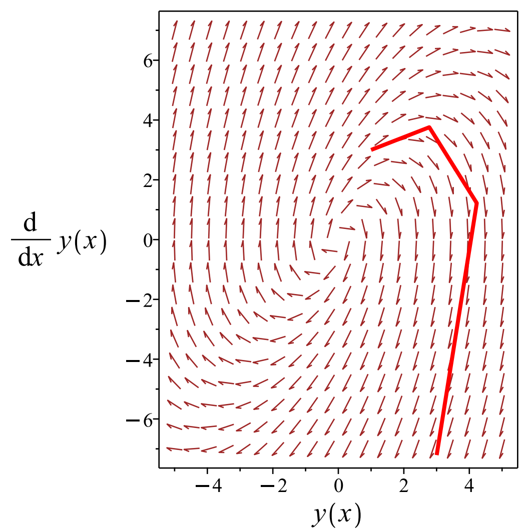
Summary

The solution(s) found are the following

$$y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) e^{x}$$

Verified OK.

15.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 388: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\sqrt{2}\, x\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x} \cos\left(\sqrt{2}\, x\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x} \cos\left(\sqrt{2}\, x\right)\right) + c_2\left(\mathrm{e}^{x} \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\,\tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{x} \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\,\mathrm{e}^{x} \sin\left(\sqrt{2}\, x\right)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = c_1 e^x \cos\left(\sqrt{2}\, x\right) - c_1 e^x \sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, e^x \sin\left(\sqrt{2}\, x\right)}{2} + c_2 e^x \cos\left(\sqrt{2}\, x\right)$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = \sqrt{2}\, e^x \sin\left(\sqrt{2}\, x\right) + e^x \cos\left(\sqrt{2}\, x\right)$$

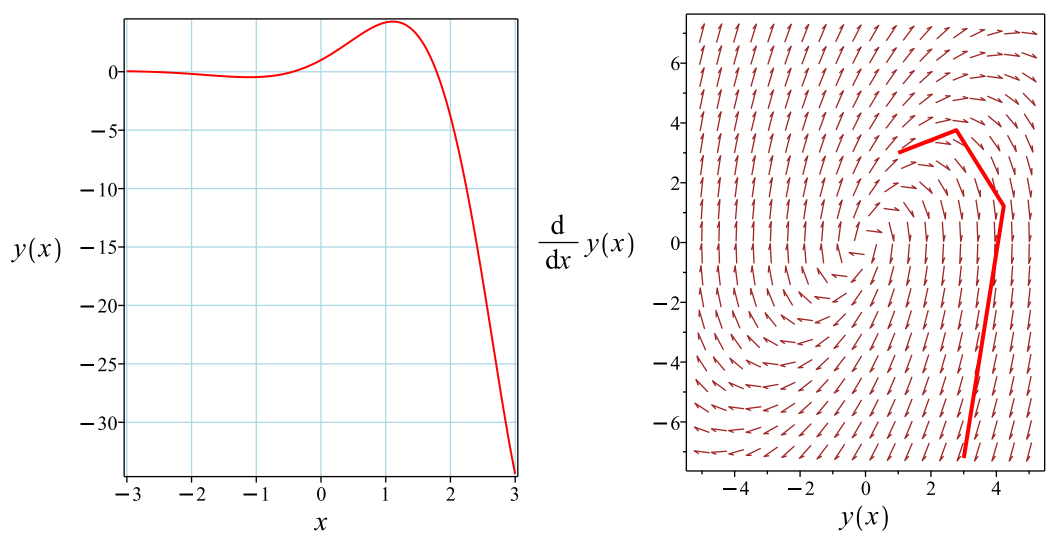
Which simplifies to

$$y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) e^x$$

Summary

The solution(s) found are the following

$$y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) e^x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) e^{x}$$

Verified OK.

15.13.4 Maple step by step solution

Let's solve

$\left[y'' - 2y' + 3y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 3 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \mathrm{I}\sqrt{2}, 1 + \mathrm{I}\sqrt{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{x} \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{x} \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{x} \cos\left(\sqrt{2}\, x\right) + e^{x} \sin\left(\sqrt{2}\, x\right) c_2$

□ Check validity of solution $y = c_1 e^{x} \cos\left(\sqrt{2}\, x\right) + e^{x} \sin\left(\sqrt{2}\, x\right) c_2$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1$

- ○ Compute derivative of the solution

  $y' = c_1 e^{x} \cos\left(\sqrt{2}\, x\right) - c_1 e^{x} \sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + e^{x} \sin\left(\sqrt{2}\, x\right) c_2 + e^{x} \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) c_2$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = c_1 + \sqrt{2}\, c_2$
- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = \sqrt{2}\}$
- Substitute constant values into general solution and simplify

  $y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) \mathrm{e}^{x}$

• Solution to the IVP

  $y = \left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right) \mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+3*y(x)=0,y(0) = 1, D(y)(0) = 3],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x}\left(\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) + \cos\left(\sqrt{2}\, x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 32

```
DSolve[{y''[x]-2*y'[x]+3*y[x]==0,{y[0]==1,y'[0]==3}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{x}\left(\sqrt{2}\sin\left(\sqrt{2}x\right) + \cos\left(\sqrt{2}x\right)\right)$$

15.14 problem 445

Internal problem ID [15235]
Internal file name [`OUTPUT/15235_Wednesday_May_08_2024_03_53_59_PM_56013244/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 445.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' + 4y'' - 2y' - 5y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + 4\lambda^2 - 2\lambda - 5 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\\ \lambda_3 &= -1 - 2i\\ \lambda_4 &= -1 + 2i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{(-1-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{(-1+2i)x}$$
$$y_4 = e^{(-1-2i)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{x} c_2 + e^{(-1+2i)x} c_3 + e^{(-1-2i)x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{x} c_2 + e^{(-1+2i)x} c_3 + e^{(-1-2i)x} c_4$$

Verified OK.

15.14.1 Maple step by step solution

Let's solve

$y'''' + 2y''' + 4y'' - 2y' - 5y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -2y_4(x) - 4y_3(x) + 2y_2(x) + 5y_1(x)$

    Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2y_4(x) - 4y_3(x) + 2y_2(x) + 5y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 2 & -4 & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 2 & -4 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-1-2\,\mathrm{I}, \begin{bmatrix} \frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[-1+2\,\mathrm{I}, \begin{bmatrix} \frac{11}{125} + \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} - \frac{2\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\,\mathrm{I}, \left[\begin{array}{c} \frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-1-2\,\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x} \cdot \left(\cos(2x) - \mathrm{I}\sin(2x)\right) \cdot \left[\begin{array}{c} \frac{11}{125}-\frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5}+\frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-x} \cdot \left[\begin{array}{c} \left(\frac{11}{125}-\frac{2\,\mathrm{I}}{125}\right)\left(\cos(2x) - \mathrm{I}\sin(2x)\right) \\ \left(-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\right)\left(\cos(2x) - \mathrm{I}\sin(2x)\right) \\ \left(-\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\right)\left(\cos(2x) - \mathrm{I}\sin(2x)\right) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-x} \cdot \left[\begin{array}{c} \frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}-\frac{4\sin(2x)}{25} \\ -\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{array}\right], \vec{y}_4(x) = e^{-x} \cdot \left[\begin{array}{c} -\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25}-\frac{4\cos(2x)}{25} \\ \frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right] + e^{x} c_2 \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + c_3 e^{-x} \cdot \left[\begin{array}{c} \frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}-\frac{4\sin(2x)}{25} \\ -\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{array}\right] + c_4 e^{-x} \cdot \left[\begin{array}{c} -\frac{11\sin(2x)}{125}-2\mathrm{c}\ldots \\ \frac{3\sin(2x)}{25}-4\cos\ldots \\ \frac{\sin(2x)}{5}+2\cos(\ldots \\ -\sin(2x) \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{((11c_3-2c_4)\cos(2x)+(-2c_3-11c_4)\sin(2x)-125c_1)\,\mathrm{e}^{-x}}{125} + \mathrm{e}^x c_2$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+4*diff(y(x),x$2)-2*diff(y(x),x)-5*y(x)=0,y(x), singso
```

$$y(x) = (c_3 \sin(2x) + c_4 \cos(2x) + c_1)\,\mathrm{e}^{-x} + c_2\mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 36

```
DSolve[y''''[x]+2*y'''[x]+4*y''[x]-2*y'[x]-5*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-x}\left(c_4 e^{2x} + c_2 \cos(2x) + c_1 \sin(2x) + c_3\right)$$

15.15 problem 446

Internal problem ID [15236]
Internal file name [`OUTPUT/15236_Wednesday_May_08_2024_03_53_59_PM_23615901/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 446.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)}+4y''''+5y'''-6y'-4y=0}$$

The characteristic equation is

$$\lambda^5+4\lambda^4+5\lambda^3-6\lambda-4=0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\\ \lambda_3 &= -1\\ \lambda_4 &= -1-i\\ \lambda_5 &= -1+i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x)=c_1\mathrm{e}^{-x}+\mathrm{e}^{-2x}c_2+\mathrm{e}^{x}c_3+\mathrm{e}^{(-1+i)x}c_4+\mathrm{e}^{(-1-i)x}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{-2x} \\
y_3 &= \mathrm{e}^{x} \\
y_4 &= \mathrm{e}^{(-1+i)x} \\
y_5 &= \mathrm{e}^{(-1-i)x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-2x} c_2 + \mathrm{e}^{x} c_3 + \mathrm{e}^{(-1+i)x} c_4 + \mathrm{e}^{(-1-i)x} c_5 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-2x} c_2 + \mathrm{e}^{x} c_3 + \mathrm{e}^{(-1+i)x} c_4 + \mathrm{e}^{(-1-i)x} c_5
$$

Verified OK.

15.15.1 Maple step by step solution

Let's solve

$y^{(5)} + 4y'''' + 5y''' - 6y' - 4y = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = -4y_5(x) - 5y_4(x) + 6y_2(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = -4y_5(x) - 5y_4(x) + 6y_2(x)$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4 & 6 & 0 & -5 & -4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4 & 6 & 0 & -5 & -4 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c}\frac{1}{16}\\ -\frac{1}{8}\\ \frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]\right], \left[-1, \left[\begin{array}{c}1\\ -1\\ 1\\ -1\\ 1\end{array}\right]\right], \left[1, \left[\begin{array}{c}1\\ 1\\ 1\\ 1\\ 1\end{array}\right]\right], \left[-1-\mathrm{I}, \left[\begin{array}{c}-\frac{1}{4}\\ \frac{1}{4}+\frac{\mathrm{I}}{4}\\ -\frac{\mathrm{I}}{2}\\ -\frac{1}{2}+\frac{\mathrm{I}}{2}\\ 1\end{array}\right]\right], \left[-1+\mathrm{I}, \left[\begin{array}{c}-\frac{1}{4}\\ \frac{1}{4}-\frac{\mathrm{I}}{4}\\ \frac{\mathrm{I}}{2}\\ -\frac{1}{2}-\frac{\mathrm{I}}{2}\\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c}\frac{1}{16}\\ -\frac{1}{8}\\ \frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c}\frac{1}{16}\\ -\frac{1}{8}\\ \frac{1}{4}\\ -\frac{1}{2}\\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1\\ -1\\ 1\\ -1\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - I, \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} + \frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-1-I)x} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4}+\frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4}+\frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x} \cdot \begin{bmatrix} -\frac{\cos(x)}{4}+\frac{I\sin(x)}{4} \\ \left(\frac{1}{4}+\frac{I}{4}\right)(\cos(x) - I\sin(x)) \\ -\frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(-\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = e^{-x} \cdot \begin{bmatrix} -\frac{\cos(x)}{4} \\ \frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_5(x) = e^{-x} \cdot \begin{bmatrix} \frac{\sin(x)}{4} \\ \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4(x) + c_5\vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + e^{-x} c_2 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + e^{x} c_3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 e^{-x} \cdot \begin{bmatrix} -\frac{\cos(x)}{4} \\ \frac{\cos(x)}{4} + \frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix} + c_5 e^{-x} \cdot \Bigg[$$

- First component of the vector is the solution to the ODE

$$y = \left(e^{3x} c_3 + \left(-\frac{\cos(x) c_4}{4} + \frac{\sin(x) c_5}{4} + c_2\right) e^{x} + \frac{c_1}{16}\right) e^{-2x}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$5)+4*diff(y(x),x$4)+5*diff(y(x),x$3)-6*diff(y(x),x)-4*y(x)=0,y(x), singso
```

$$y(x) = e^{-2x}\left(c_2 e^{3x} + (\sin(x) c_4 + \cos(x) c_5 + c_1) e^{x} + c_3\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 44

```
DSolve[y'''''[x]+4*y''''[x]+5*y'''[x]-6*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-2x}\left(c_4 e^{x} + c_5 e^{3x} + c_2 e^{x} \cos(x) + c_1 e^{x} \sin(x) + c_3\right)$$

15.16 problem 447

Internal problem ID [15237]
Internal file name [OUTPUT/15237_Wednesday_May_08_2024_03_54_00_PM_57195971/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 447.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 2y'' - y' - 2y = 0}$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{-2x} c_2 + e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^x$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{-2x} c_2 + e^x c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{-2x} c_2 + e^x c_3$$

Verified OK.

15.16.1 Maple step by step solution

Let's solve

$y''' + 2y'' - y' - 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs

  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -2y_3(x) + y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -2y_3(x) + y_2(x) + 2y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + e^{-x} c_2 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{x} c_3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4 e^{3x} c_3 + 4 e^{x} c_2 + c_1) e^{-2x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_2 e^{3x} + c_1 e^x + c_3) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 28

```
DSolve[y'''[x]+2*y''[x]-y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(c_2 e^x + c_3 e^{3x} + c_1)$$

15.17 problem 448

Internal problem ID [15238]
Internal file name [OUTPUT/15238_Wednesday_May_08_2024_03_54_00_PM_65893225/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 448.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y'' + 2y' = 0}$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = 1 - i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{(1-i)x}c_2 + e^{(1+i)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{(1-i)x}$$
$$y_3 = e^{(1+i)x}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{(1-i)x}c_2 + e^{(1+i)x}c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + e^{(1-i)x}c_2 + e^{(1+i)x}c_3$$

Verified OK.

15.17.1 Maple step by step solution

Let's solve

$y''' - 2y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) - 2y_2(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1 - \mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - \mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \begin{bmatrix} \frac{I}{2} \\ \frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{I}{2} \\ \frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{x} \cdot \begin{bmatrix} \frac{\sin(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} \frac{\cos(x)}{2} \\ -\frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^{x}c_2 \cdot \begin{bmatrix} \frac{\sin(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ \cos(x) \end{bmatrix} + e^{x}c_3 \cdot \begin{bmatrix} \frac{\cos(x)}{2} \\ -\frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(c_2\sin(x) + \cos(x)c_3)e^{x}}{2} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)+2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^x \sin(x) + c_3 e^x \cos(x)$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 34

```
DSolve[y'''[x]-2*y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^x((c_2 - c_1)\cos(x) + (c_1 + c_2)\sin(x)) + c_3$$

15.18 problem 449

Internal problem ID [15239]
Internal file name [`OUTPUT/15239_Wednesday_May_08_2024_03_54_01_PM_22240505/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 449.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - y = 0}$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = \mathrm{e}^{ix}$$
$$y_4 = \mathrm{e}^{-ix}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4$$

Verified OK.

15.18.1 Maple step by step solution

Let's solve

$y'''' - y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \mathrm{e}^{x} c_2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_3\sin(x) - \cos(x)\,c_4 \\ -\cos(x)\,c_3 + \sin(x)\,c_4 \\ c_3\sin(x) + \cos(x)\,c_4 \\ \cos(x)\,c_3 - \sin(x)\,c_4 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1\mathrm{e}^{-x} + \mathrm{e}^{x}c_2 - \cos(x)\,c_4 - c_3\sin(x)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$4)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \sin(x) + c_4 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 30

```
DSolve[y''''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_3 e^{-x} + c_2 \cos(x) + c_4 \sin(x)$$

15.19 problem 450

Internal problem ID [15240]
Internal file name [OUTPUT/15240_Wednesday_May_08_2024_03_54_01_PM_47998857/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 450.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _quadrature]]
```

$$\boxed{y^{(5)} = 0}$$

The characteristic equation is

$$\lambda^5 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$
$$\lambda_5 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_5x^4 + c_4x^3 + c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = x^3$$
$$y_5 = x^4$$

Summary

The solution(s) found are the following

$$y = c_5 x^4 + c_4 x^3 + c_3 x^2 + c_2 x + c_1 \tag{1}$$

Verification of solutions

$$y = c_5 x^4 + c_4 x^3 + c_3 x^2 + c_2 x + c_1$$

Verified OK.

15.19.1 Maple step by step solution

Let's solve

$y^{(5)} = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 0$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 0]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + c_5 \vec{y}_5$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$5)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{24}c_1x^4 + \frac{1}{6}c_2x^3 + \frac{1}{2}c_3x^2 + c_4x + c_5$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 27

```
DSolve[y'''''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(x(c_5x + c_4) + c_3) + c_2) + c_1$$

15.20 problem 451

Internal problem ID [15241]
Internal file name [OUTPUT/15241_Wednesday_May_08_2024_03_54_01_PM_19903849/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 451.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y' - 2y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda - 2 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= -1 \\ \lambda_3 &= -1 \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + \mathrm{e}^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\,\mathrm{e}^{-x} \\ y_3 &= \mathrm{e}^{2x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 x \, e^{-x} + e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 x \, e^{-x} + e^{2x} c_3$$

Verified OK.

15.20.1 Maple step by step solution

Let's solve

$y''' - 3y' - 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_2(x) + 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_2(x) + 2y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$

- Simplify equation

  $\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

  $$\left(\left[\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & 0\end{array}\right]-(-1)\cdot\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\cdot\overrightarrow{p}=\left[\begin{array}{c}1\\ -1\\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\left[\begin{array}{c}1\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue -1

  $$\overrightarrow{y}_2(x)=\mathrm{e}^{-x}\cdot\left(x\cdot\left[\begin{array}{c}1\\ -1\\ 1\end{array}\right]+\left[\begin{array}{c}1\\ 0\\ 0\end{array}\right]\right)$$

- Consider eigenpair

  $$\left[2,\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_3=\mathrm{e}^{2x}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3$
- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{-x} c_2 \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + e^{2x} c_3 \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$
- First component of the vector is the solution to the ODE

  $y = ((x+1) c_2 + c_1) e^{-x} + \frac{e^{2x} c_3}{4}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 x + c_2) e^{-x} + c_1 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 26

```
DSolve[y'''[x]-3*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_2 x + c_3 e^{3x} + c_1 \right)$$

15.21 problem 452

Internal problem ID [15242]
Internal file name [`OUTPUT/15242_Wednesday_May_08_2024_03_54_02_PM_83478964/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 452.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{2y''' - 3y'' + y' = 0}$$

The characteristic equation is

$$2\lambda^3 - 3\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^x c_2 + e^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^x c_2 + e^{\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + e^x c_2 + e^{\frac{x}{2}} c_3$$

Verified OK.

15.21.1 Maple step by step solution

Let's solve

$2y''' - 3y'' + y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{3y''}{2} - \frac{y'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{2} + \frac{y'}{2} = 0$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \frac{3y_3(x)}{2} - \frac{y_2(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3y_3(x)}{2} - \frac{y_2(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_2 \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \mathrm{e}^{x} c_3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = 4 c_2 \mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{x} c_3 + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(2*diff(y(x),x$3)-3*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{\frac{x}{2}} + c_3 e^{x}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 25

```
DSolve[2*y'''[x]-3*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 2c_1 e^{x/2} + c_2 e^{x} + c_3$$

15.22 problem 453

Internal problem ID [15243]
Internal file name [`OUTPUT/15243_Wednesday_May_08_2024_03_54_02_PM_94213759/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.2 Homogeneous differential equations with constant coefficients. Exercises page 121
Problem number: 453.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0, y''(0) = 1]$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = e^{-x}c_1 + c_2 + c_3 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= x
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^{-x} c_1 + c_2 + c_3 x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\mathrm{e}^{-x} c_1 + c_3
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = -c_1 + c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = \mathrm{e}^{-x} c_1
$$

substituting $y'' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 0 \\
c_3 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{-x} + x
$$

Summary

The solution(s) found are the following

$$y = e^{-x} + x \tag{1}$$

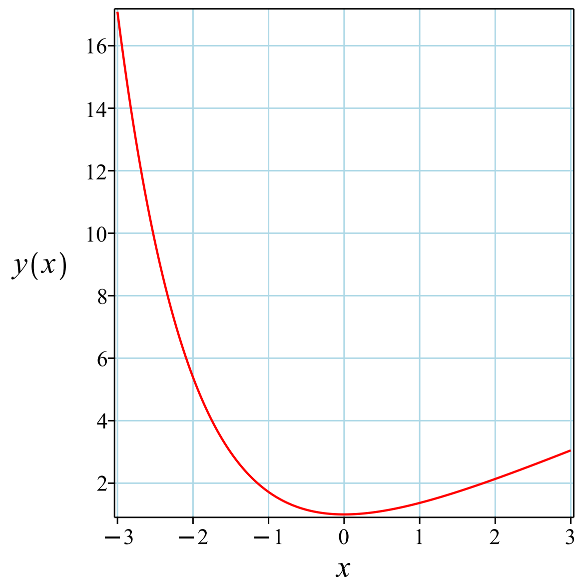


Figure 490: Solution plot

Verification of solutions

$$y = e^{-x} + x$$

Verified OK.

15.22.1 Maple step by step solution

Let's solve

$\left[y''' + y'' = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = e^{-x} c_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = e^{-x} c_1 + c_2$

- Use the initial condition $y(0) = 1$

  $1 = c_1 + c_2$

- Calculate the 1st derivative of the solution

  $y' = -e^{-x} c_1$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -c_1$

- Calculate the 2nd derivative of the solution

  $y'' = e^{-x} c_1$

- Use the initial condition $y''\Big|_{\{x=0\}} = 1$

  $1 = c_1$

- Solve for the unknown coefficients
- The solution does not satisfy the initial conditions

<u>Maple trace</u>

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$3)+diff(y(x),x$2)=0,y(0) = 1, D(y)(0) = 0, (D@@2)(y)(0) = 1],y(x), sings
```

$$y = e^{-x} + x$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 12

```
DSolve[{y'''[x]+y''[x]==0,{y[0]==1,y'[0]==0,y''[0]==1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to x + e^{-x}$$

16 Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132

16.1 problem 474

Internal problem ID [15244]
Internal file name [`OUTPUT/15244_Wednesday_May_08_2024_03_54_02_PM_10666093/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 474.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' = 3}$$

16.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 0, f(x) = 3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+3y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=3, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+3\lambda\, e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+3\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=3, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-3}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{3^2-(4)(1)(0)}\\&=-\frac{3}{2}\pm\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{3}{2}+\frac{3}{2}\\\lambda_2&=-\frac{3}{2}-\frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-3x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 = 3$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 e^{-3x}\right) + (x)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 + c_2 e^{-3x} + x \tag{1}
$$

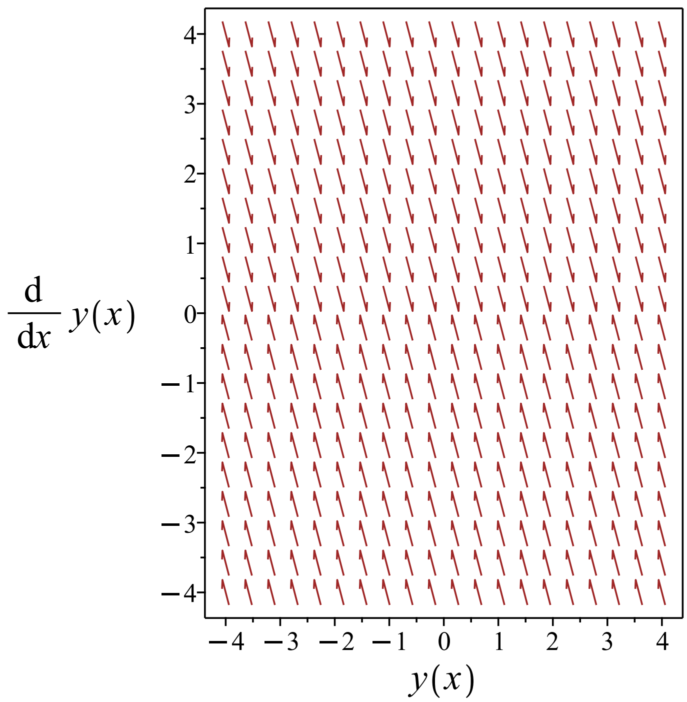


Figure 491: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 e^{-3x} + x
$$

Verified OK.

16.1.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y''+3y')\,dx = \int 3dx$$
$$3y+y' = 3x+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = 3$$
$$q(x) = 3x+c_1$$

Hence the ode is

$$3y+y' = 3x+c_1$$

The integrating factor μ is

$$\mu = e^{\int 3dx}$$
$$= e^{3x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(3x+c_1)$$
$$\frac{d}{dx}(e^{3x}y) = (e^{3x})(3x+c_1)$$
$$d(e^{3x}y) = ((3x+c_1)e^{3x})\,dx$$

Integrating gives

$$e^{3x}y = \int (3x+c_1)e^{3x}\,dx$$
$$e^{3x}y = \frac{(3x+c_1-1)e^{3x}}{3}+c_2$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = \frac{e^{-3x}(3x+c_1-1)e^{3x}}{3}+c_2e^{-3x}$$

which simplifies to

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 e^{-3x}$$

Summary

The solution(s) found are the following

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 e^{-3x} \tag{1}$$

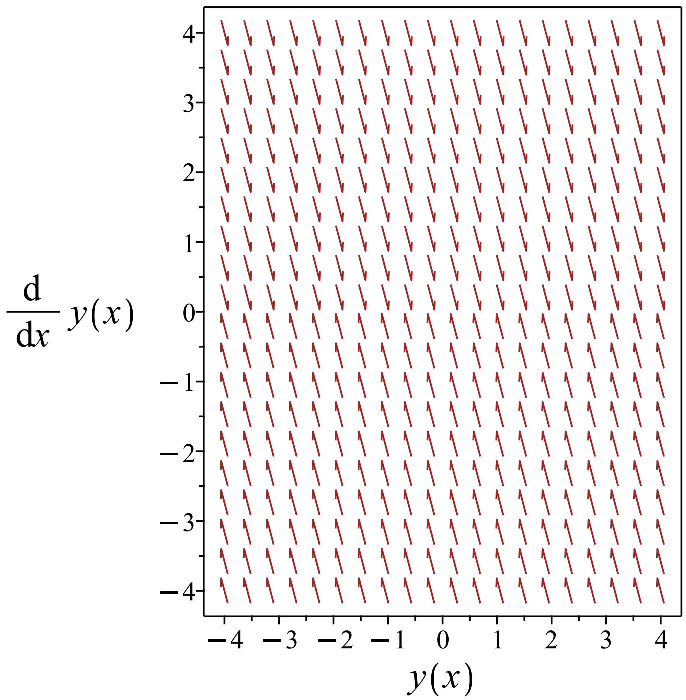


Figure 492: Slope field plot

Verification of solutions

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 e^{-3x}$$

Verified OK.

16.1.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 3p(x) - 3 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-3p+3} dp = \int dx$$
$$-\frac{\ln(-p+1)}{3} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{(-p+1)^{\frac{1}{3}}} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{(-p+1)^{\frac{1}{3}}} = \mathrm{e}^{x} c_2$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{\mathrm{e}^{-3x}}{c_2^3} + 1$$

Integrating both sides gives

$$y = \int \frac{(\mathrm{e}^{3x} c_2^3 - 1)\,\mathrm{e}^{-3x}}{c_2^3} \mathrm{d}x$$
$$= \frac{\mathrm{e}^{-3x}}{3c_2^3} + \ln(\mathrm{e}^{x}) + c_3$$

Summary
The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-3x}}{3c_2^3} + \ln(\mathrm{e}^{x}) + c_3 \tag{1}$$

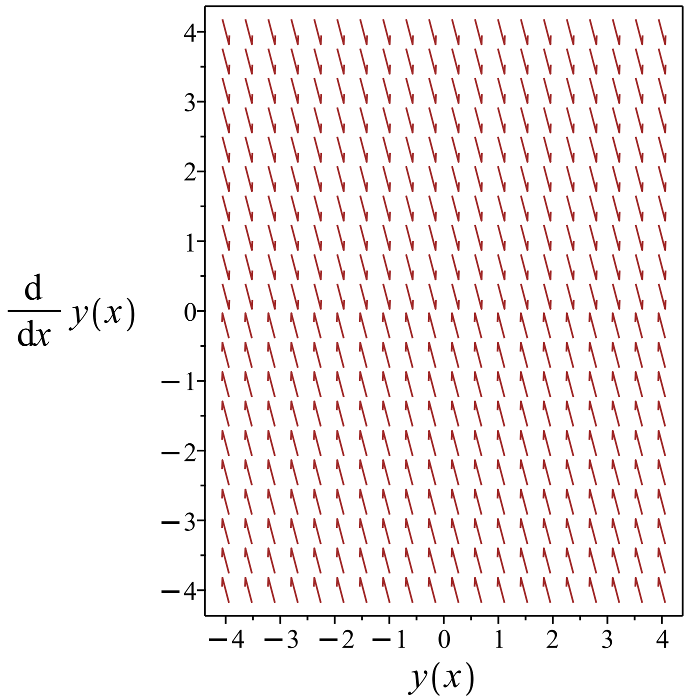


Figure 493: Slope field plot

Verification of solutions

$$y = \frac{e^{-3x}}{3c_2^3} + \ln(e^x) + c_3$$

Verified OK.

16.1.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 3y' = 3$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 3y')\, dx = \int 3dx$$

$$3y + y' = 3x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 3\\ q(x) &= 3x + c_1\end{aligned}$$

Hence the ode is

$$3y + y' = 3x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 3dx}\\ &= e^{3x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(3x + c_1)\\ \frac{d}{dx}\left(e^{3x}y\right) &= \left(e^{3x}\right)(3x + c_1)\\ d\left(e^{3x}y\right) &= \left((3x + c_1)e^{3x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{3x}y &= \int (3x + c_1)e^{3x}\, dx\\ e^{3x}y &= \frac{(3x + c_1 - 1)e^{3x}}{3} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = \frac{e^{-3x}(3x + c_1 - 1)e^{3x}}{3} + c_2e^{-3x}$$

which simplifies to

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2e^{-3x}$$

Summary

The solution(s) found are the following

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2e^{-3x} \tag{1}$$

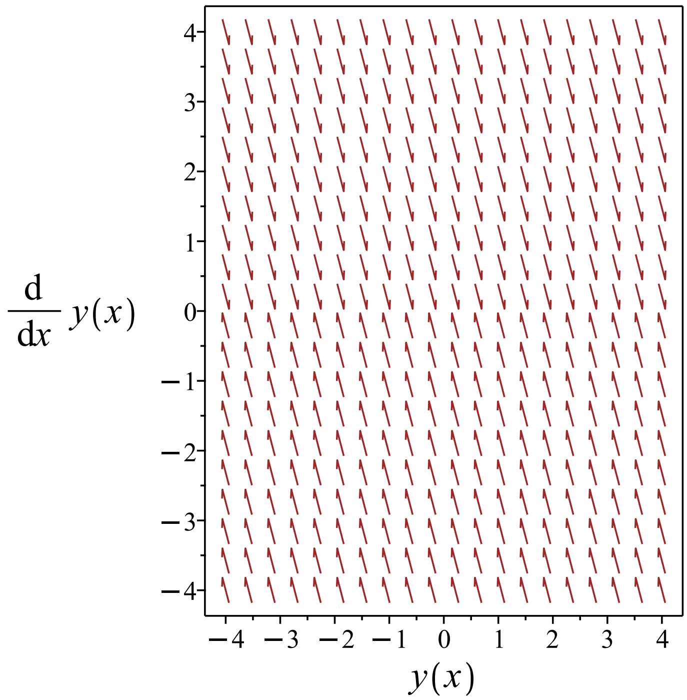


Figure 494: Slope field plot

Verification of solutions

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 e^{-3x}$$

Verified OK.

16.1.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2}{3}
$$

Now we need to find the particular solution y_p to

$$
y'' + 3y' = 3
$$

Since the RHS of the ode $f(x)$ is a constant, because it does not depend on x, then let the particular solution be

$$
y_p = kx
$$

where k is a constant to be determined. Substituting $y = kx$ in the ODE gives

$$
3k = 3
$$

Therefore

$$
k = x
$$

Hence $y_p = x^2$. Therefore the complete solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 \mathrm{e}^{-3x} + \frac{c_2}{3} + (x)
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-3x} + \frac{c_2}{3}\right) + (x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-3x} + \frac{c_2}{3} + x \tag{1}
$$

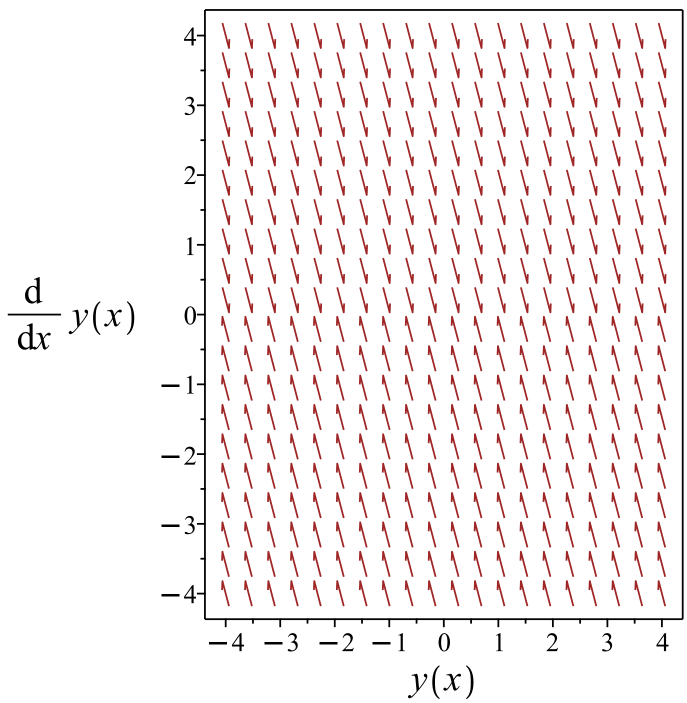


Figure 495: Slope field plot

Verification of solutions

$$
y = c_1 e^{-3x} + \frac{c_2}{3} + x
$$

Verified OK.

16.1.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= 3\\
r(x) &= 0\\
s(x) &= 3
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$3y + y' = \int 3\,dx$$

We now have a first order ode to solve which is

$$3y + y' = 3x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 3 \\
q(x) &= 3x + c_1
\end{aligned}$$

Hence the ode is

$$3y + y' = 3x + c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int 3dx} \\
&= e^{3x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(3x + c_1) \\
\frac{d}{dx}\left(e^{3x}y\right) &= \left(e^{3x}\right)(3x + c_1) \\
d\left(e^{3x}y\right) &= \left((3x + c_1)e^{3x}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{3x}y &= \int (3x + c_1)e^{3x}\, dx \\
e^{3x}y &= \frac{(3x + c_1 - 1)e^{3x}}{3} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = \frac{e^{-3x}(3x + c_1 - 1)e^{3x}}{3} + c_2 e^{-3x}$$

which simplifies to

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 e^{-3x}$$

Summary

The solution(s) found are the following

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 e^{-3x} \tag{1}$$

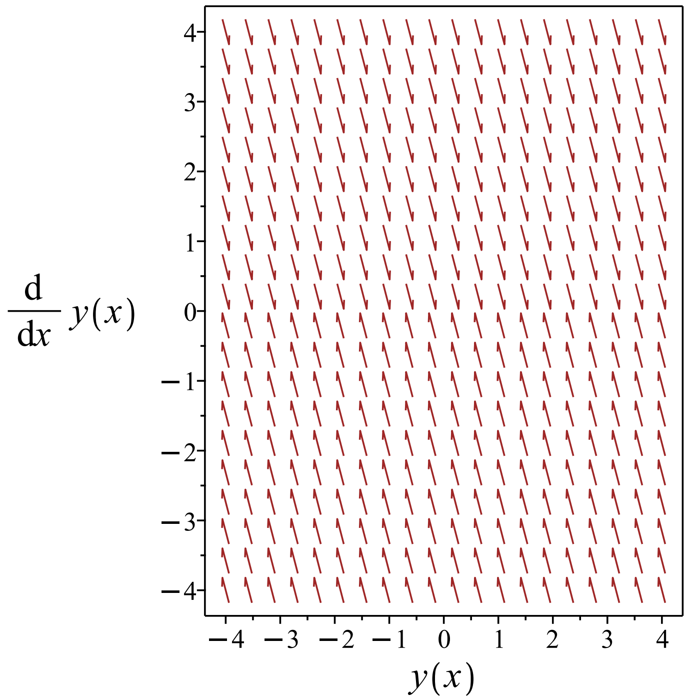


Figure 496: Slope field plot

Verification of solutions

$$y = x + \frac{c_1}{3} - \frac{1}{3} + c_2 \mathrm{e}^{-3x}$$

Verified OK.

16.1.7 Maple step by step solution

Let's solve

$y'' + 3y' = 3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r = 0$

- Factor the characteristic polynomial

  $r(r + 3) = 0$

- Roots of the characteristic polynomial

$r = (-3, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & 1 \\ -3\,\mathrm{e}^{-3x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-3x} \left(\int \mathrm{e}^{3x} dx\right) + \int 1 dx$

  - Compute integrals

    $y_p(x) = -\frac{1}{3} + x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 - \frac{1}{3} + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -3*_b(_a)+3, _b(_a)`   *** Sublevel 2 *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)=3,y(x), singsol=all)
```

$$y(x) = -\frac{c_1 e^{-3x}}{3} + x + c_2$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 20

```
DSolve[y''[x]+3*y'[x]==3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \frac{1}{3} c_1 e^{-3x} + c_2$$

16.2 problem 475

Internal problem ID [15245]
Internal file name [`OUTPUT/15245_Wednesday_May_08_2024_03_54_03_PM_68847940/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 475.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 7y' = (x-1)^2}$$

16.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -7, C = 0, f(x) = (x-1)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 7y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -7, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 7\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 7\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -7, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-7^2 - (4)\,(1)\,(0)}\\ &= \frac{7}{2} \pm \frac{7}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{7}{2} + \frac{7}{2}$$
$$\lambda_2 = \frac{7}{2} - \frac{7}{2}$$

Which simplifies to

$$\lambda_1 = 7$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(7)x} + c_2 e^{(0)x}$$

Or

$$y = c_1 e^{7x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{7x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{7x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-21x^2 A_3 - 14x A_2 + 6x A_3 - 7A_1 + 2A_2 = (x-1)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{37}{343}, A_2 = \frac{6}{49}, A_3 = -\frac{1}{21}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{21}x^3 + \frac{6}{49}x^2 - \frac{37}{343}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{7x} + c_2) + \left(-\frac{1}{21}x^3 + \frac{6}{49}x^2 - \frac{37}{343}x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{7x} + c_2 - \frac{x^3}{21} + \frac{6x^2}{49} - \frac{37x}{343} \tag{1}$$

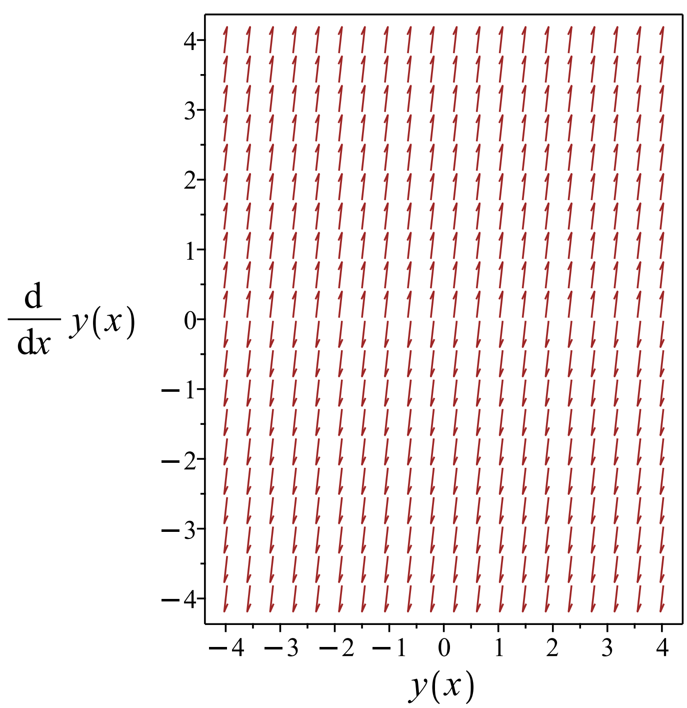


Figure 497: Slope field plot

Verification of solutions

$$y = c_1 e^{7x} + c_2 - \frac{x^3}{21} + \frac{6x^2}{49} - \frac{37x}{343}$$

Verified OK.

16.2.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y''-7y')\,dx = \int (x-1)^2\,dx$$
$$-7y+y' = \frac{(x-1)^3}{3}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -7$$
$$q(x) = \frac{1}{3}x^3-x^2+x-\frac{1}{3}+c_1$$

Hence the ode is

$$-7y+y' = \frac{1}{3}x^3-x^2+x-\frac{1}{3}+c_1$$

The integrating factor μ is

$$\mu = e^{\int (-7)dx}$$
$$= e^{-7x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{3}x^3-x^2+x-\frac{1}{3}+c_1\right)$$
$$\frac{d}{dx}\left(e^{-7x}y\right) = \left(e^{-7x}\right)\left(\frac{1}{3}x^3-x^2+x-\frac{1}{3}+c_1\right)$$
$$d\left(e^{-7x}y\right) = \left(\frac{(x^3-3x^2+3c_1+3x-1)\,e^{-7x}}{3}\right)\,dx$$

Integrating gives

$$e^{-7x}y = \int \frac{(x^3-3x^2+3c_1+3x-1)\,e^{-7x}}{3}\,dx$$
$$e^{-7x}y = -\frac{(343x^3-882x^2+1029c_1+777x-232)\,e^{-7x}}{7203}+c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-7x}$ results in

$$y = -\frac{\mathrm{e}^{7x}(343x^3 - 882x^2 + 1029c_1 + 777x - 232)\,\mathrm{e}^{-7x}}{7203} + c_2\mathrm{e}^{7x}$$

which simplifies to

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2\mathrm{e}^{7x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2\mathrm{e}^{7x} \tag{1}$$

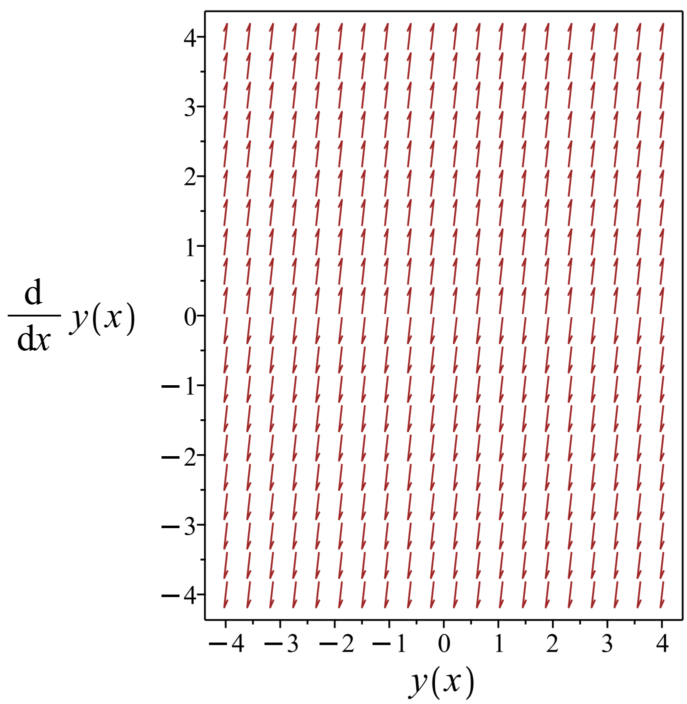


Figure 498: Slope field plot

Verification of solutions

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2\mathrm{e}^{7x}$$

Verified OK.

16.2.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 7p(x) - (x-1)^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -7 \\ q(x) &= (x-1)^2 \end{aligned}$$

Hence the ode is

$$p'(x) - 7p(x) = (x-1)^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-7)dx} \\ &= e^{-7x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left((x-1)^2\right) \\ \frac{d}{dx}\left(e^{-7x}p\right) &= \left(e^{-7x}\right)\left((x-1)^2\right) \\ d\left(e^{-7x}p\right) &= \left((x-1)^2 e^{-7x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-7x}p &= \int (x-1)^2 e^{-7x} dx \\ e^{-7x}p &= -\frac{(49x^2 - 84x + 37) e^{-7x}}{343} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-7x}$ results in

$$p(x) = -\frac{e^{7x}(49x^2 - 84x + 37)\,e^{-7x}}{343} + c_1 e^{7x}$$

which simplifies to

$$p(x) = -\frac{x^2}{7} + \frac{12x}{49} - \frac{37}{343} + c_1 e^{7x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{x^2}{7} + \frac{12x}{49} - \frac{37}{343} + c_1 e^{7x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{x^2}{7} + \frac{12x}{49} - \frac{37}{343} + c_1 e^{7x}\, dx \\ &= \frac{6x^2}{49} - \frac{37x}{343} - \frac{x^3}{21} + \frac{c_1 e^{7x}}{7} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{6x^2}{49} - \frac{37x}{343} - \frac{x^3}{21} + \frac{c_1 e^{7x}}{7} + c_2 \tag{1}$$

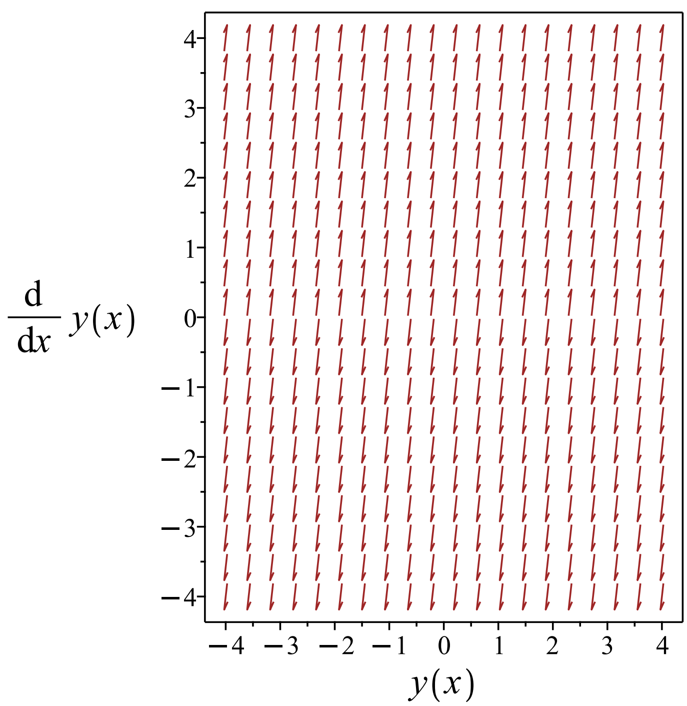


Figure 499: Slope field plot

Verification of solutions

$$y = \frac{6x^2}{49} - \frac{37x}{343} - \frac{x^3}{21} + \frac{c_1 e^{7x}}{7} + c_2$$

Verified OK.

16.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 7y' = (x-1)^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 7y')\, dx = \int (x-1)^2\, dx$$

$$-7y + y' = \frac{(x-1)^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -7$$
$$q(x) = \frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1$$

Hence the ode is

$$-7y + y' = \frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-7)dx}$$
$$= e^{-7x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1\right)$$
$$\frac{d}{dx}\left(e^{-7x} y\right) = \left(e^{-7x}\right)\left(\frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1\right)$$
$$d\left(e^{-7x} y\right) = \left(\frac{(x^3 - 3x^2 + 3c_1 + 3x - 1)\, e^{-7x}}{3}\right) dx$$

Integrating gives

$$e^{-7x}y = \int \frac{(x^3 - 3x^2 + 3c_1 + 3x - 1)\, e^{-7x}}{3}\, dx$$

$$e^{-7x}y = -\frac{(343x^3 - 882x^2 + 1029c_1 + 777x - 232)\, e^{-7x}}{7203} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-7x}$ results in

$$y = -\frac{e^{7x}(343x^3 - 882x^2 + 1029c_1 + 777x - 232)\, e^{-7x}}{7203} + c_2 e^{7x}$$

which simplifies to

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2 e^{7x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2 e^{7x} \tag{1}$$

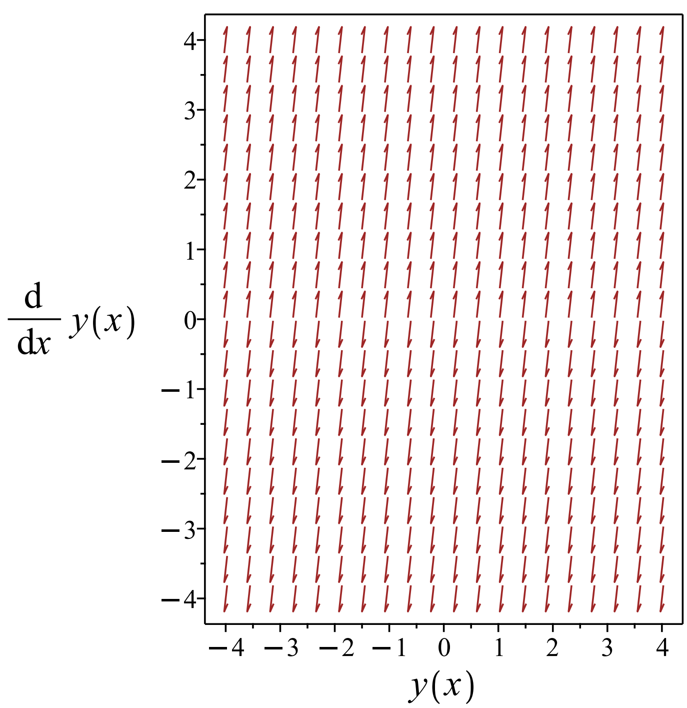


Figure 500: Slope field plot

Verification of solutions

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2 e^{7x}$$

Verified OK.

16.2.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 7y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -7 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-7}{1}\,dx} \\ &= z_1 e^{\frac{7x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{7x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{7x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{7x}}{7}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-7y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1+\frac{c_2 \mathrm{e}^{7x}}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2+x+1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1,x,x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1,\frac{\mathrm{e}^{7x}}{7}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x,x^2,x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p=A_3x^3+A_2x^2+A_1x$$

The unknowns $\{A_1,A_2,A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-21x^2A_3-14xA_2+6xA_3-7A_1+2A_2=(x-1)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{37}{343},A_2=\frac{6}{49},A_3=-\frac{1}{21}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{21}x^3 + \frac{6}{49}x^2 - \frac{37}{343}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{7x}}{7}\right) + \left(-\frac{1}{21}x^3 + \frac{6}{49}x^2 - \frac{37}{343}x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{7x}}{7} - \frac{x^3}{21} + \frac{6x^2}{49} - \frac{37x}{343} \tag{1}$$

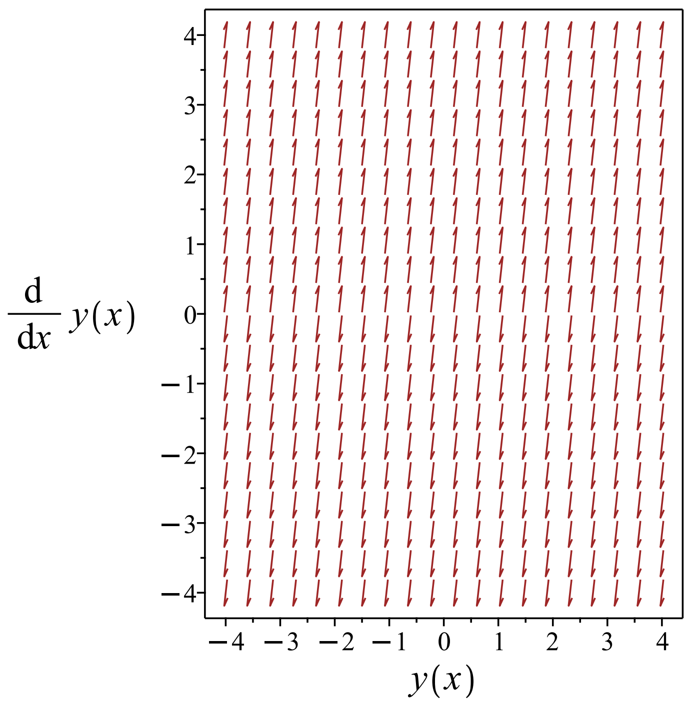


Figure 501: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{7x}}{7} - \frac{x^3}{21} + \frac{6x^2}{49} - \frac{37x}{343}$$

Verified OK.

16.2.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=1\\ q(x)&=-7\\ r(x)&=0\\ s(x)&=(x-1)^2\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=0\\ q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+(q(x)-p'(x))\,y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$-7y+y'=\int (x-1)^2\;dx$$

We now have a first order ode to solve which is

$$-7y+y'=\frac{(x-1)^3}{3}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -7 \\ q(x) &= \frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1 \end{aligned}$$

Hence the ode is

$$-7y + y' = \frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-7)dx} \\ &= e^{-7x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1\right) \\ \frac{d}{dx}\left(e^{-7x}y\right) &= \left(e^{-7x}\right)\left(\frac{1}{3}x^3 - x^2 + x - \frac{1}{3} + c_1\right) \\ d\left(e^{-7x}y\right) &= \left(\frac{(x^3 - 3x^2 + 3c_1 + 3x - 1)\, e^{-7x}}{3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-7x}y &= \int \frac{(x^3 - 3x^2 + 3c_1 + 3x - 1)\, e^{-7x}}{3}\, dx \\ e^{-7x}y &= -\frac{(343x^3 - 882x^2 + 1029c_1 + 777x - 232)\, e^{-7x}}{7203} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-7x}$ results in

$$y = -\frac{e^{7x}(343x^3 - 882x^2 + 1029c_1 + 777x - 232)\, e^{-7x}}{7203} + c_2 e^{7x}$$

which simplifies to

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2 e^{7x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2 e^{7x} \tag{1}$$

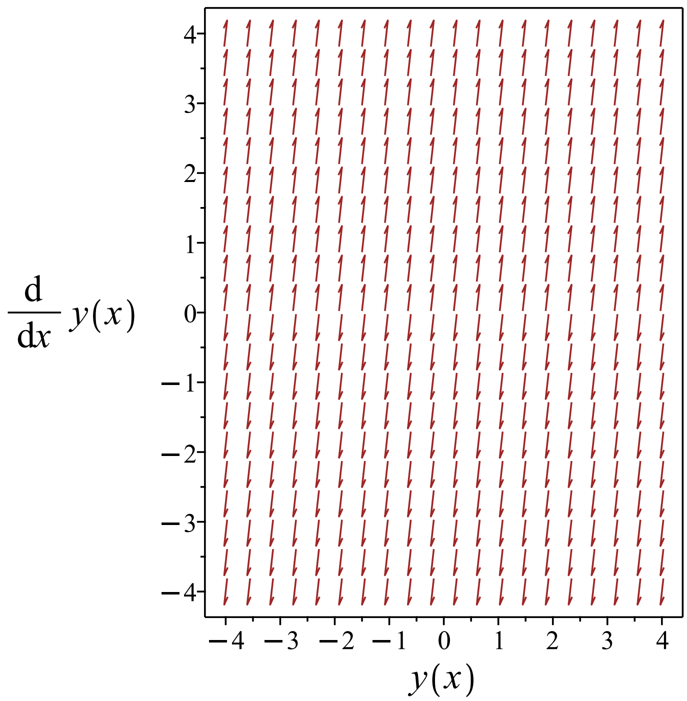


Figure 502: Slope field plot

Verification of solutions

$$y = -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{c_1}{7} - \frac{37x}{343} + \frac{232}{7203} + c_2 e^{7x}$$

Verified OK.

16.2.7 Maple step by step solution

Let's solve

$y'' - 7y' = (x-1)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 7r = 0$

- Factor the characteristic polynomial

  $r(r-7) = 0$

- Roots of the characteristic polynomial

  $r = (0, 7)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{7x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{7x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = (x-1)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{7x} \\ 0 & 7\,e^{7x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 7\,e^{7x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\left(\int (x-1)^2 dx\right)}{7} + \frac{e^{7x}\left(\int (x-1)^2 e^{-7x} dx\right)}{7}$

  - Compute integrals

    $y_p(x) = -\frac{1}{21}x^3 + \frac{6}{49}x^2 - \frac{37}{343}x + \frac{232}{7203}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{7x} - \frac{x^3}{21} + \frac{6x^2}{49} - \frac{37x}{343} + \frac{232}{7203}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2+7*_b(_a)-2*_a+1, _b(_a)`   *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-7*diff(y(x),x)=(x-1)^2,y(x), singsol=all)
```

$$y(x) = \frac{6x^2}{49} - \frac{x^3}{21} + \frac{e^{7x}c_1}{7} - \frac{37x}{343} + c_2$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 38

```
DSolve[y''[x]-7*y'[x]==(x-1)^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{21} + \frac{6x^2}{49} - \frac{37x}{343} + \frac{1}{7}c_1e^{7x} + c_2$$

16.3 problem 476

Internal problem ID [15246]
Internal file name [`OUTPUT/15246_Wednesday_May_08_2024_03_54_05_PM_64304397/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 476.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 3y' = e^x}$$

16.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 0, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+3y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=3, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+3\lambda \,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+3\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=3, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-3}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{3^2-(4)\,(1)\,(0)}\\&=-\frac{3}{2}\pm\frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{3}{2}+\frac{3}{2}\\\lambda_2&=-\frac{3}{2}-\frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 e^{-3x}\right) + \left(\frac{e^x}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 e^{-3x} + \frac{e^x}{4} \tag{1}
$$

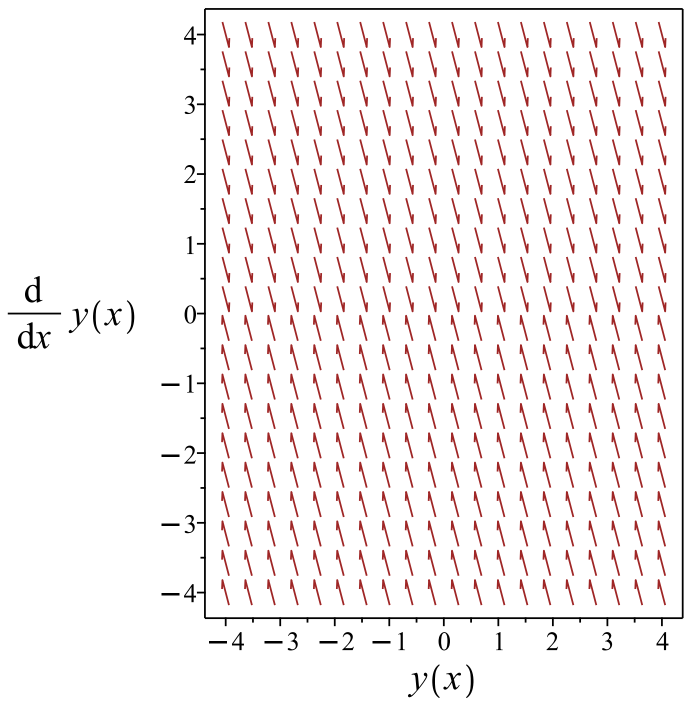


Figure 503: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 e^{-3x} + \frac{e^x}{4}
$$

Verified OK.

16.3.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' + 3y')\, dx &= \int \mathrm{e}^{x} dx \\
3y + y' &= \mathrm{e}^{x} + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 3 \\
q(x) &= \mathrm{e}^{x} + c_1
\end{aligned}
$$

Hence the ode is

$$
3y + y' = \mathrm{e}^{x} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 3dx} \\
&= \mathrm{e}^{3x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{x} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{3x} y\right) &= \left(\mathrm{e}^{3x}\right)\left(\mathrm{e}^{x} + c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{3x} y\right) &= \left(\left(\mathrm{e}^{x} + c_1\right) \mathrm{e}^{3x}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{3x} y &= \int \left(\mathrm{e}^{x} + c_1\right) \mathrm{e}^{3x}\, \mathrm{d}x \\
\mathrm{e}^{3x} y &= \frac{\mathrm{e}^{4x}}{4} + \frac{c_1 \mathrm{e}^{3x}}{3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{3x}$ results in

$$
y = \mathrm{e}^{-3x}\left(\frac{\mathrm{e}^{4x}}{4} + \frac{c_1 \mathrm{e}^{3x}}{3}\right) + c_2 \mathrm{e}^{-3x}
$$

which simplifies to

$$y = \frac{(3\,e^{4x} + 4c_1 e^{3x} + 12c_2)\,e^{-3x}}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{(3\,e^{4x} + 4c_1 e^{3x} + 12c_2)\,e^{-3x}}{12} \tag{1}$$

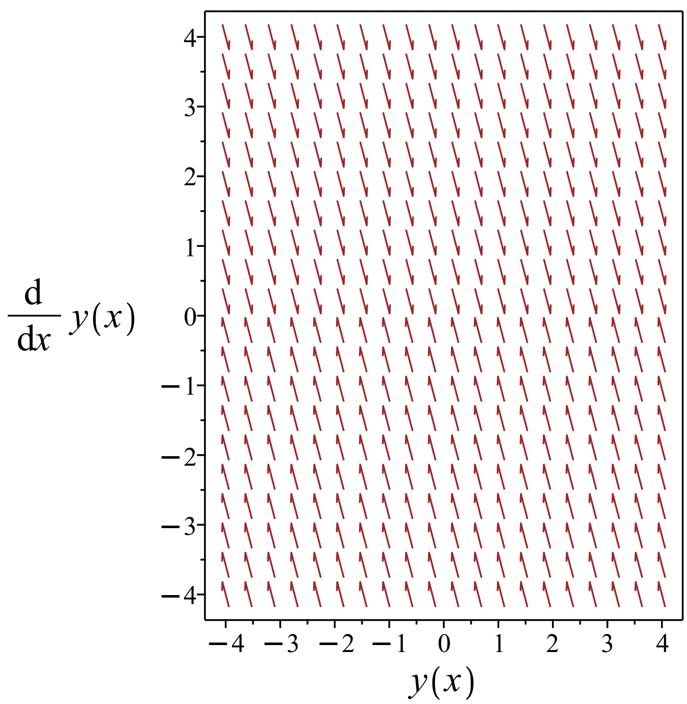


Figure 504: Slope field plot

Verification of solutions

$$y = \frac{(3\,e^{4x} + 4c_1 e^{3x} + 12c_2)\,e^{-3x}}{12}$$

Verified OK.

16.3.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 3p(x) - e^x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= e^x \end{aligned}$$

Hence the ode is

$$p'(x) + 3p(x) = e^x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 3dx} \\ &= e^{3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(e^x\right) \\ \frac{d}{dx}\left(e^{3x}p\right) &= \left(e^{3x}\right)\left(e^x\right) \\ d\left(e^{3x}p\right) &= e^{4x}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{3x}p &= \int e^{4x}\, dx \\ e^{3x}p &= \frac{e^{4x}}{4} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$p(x) = \frac{e^{4x}e^{-3x}}{4} + c_1 e^{-3x}$$

which simplifies to

$$p(x) = \frac{(e^{4x} + 4c_1)\, e^{-3x}}{4}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{(e^{4x} + 4c_1)\, e^{-3x}}{4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{(e^{4x} + 4c_1)\, e^{-3x}}{4}\, dx \\ &= -\frac{c_1 e^{-3x}}{3} + \frac{e^x}{4} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1 e^{-3x}}{3} + \frac{e^x}{4} + c_2 \tag{1}$$

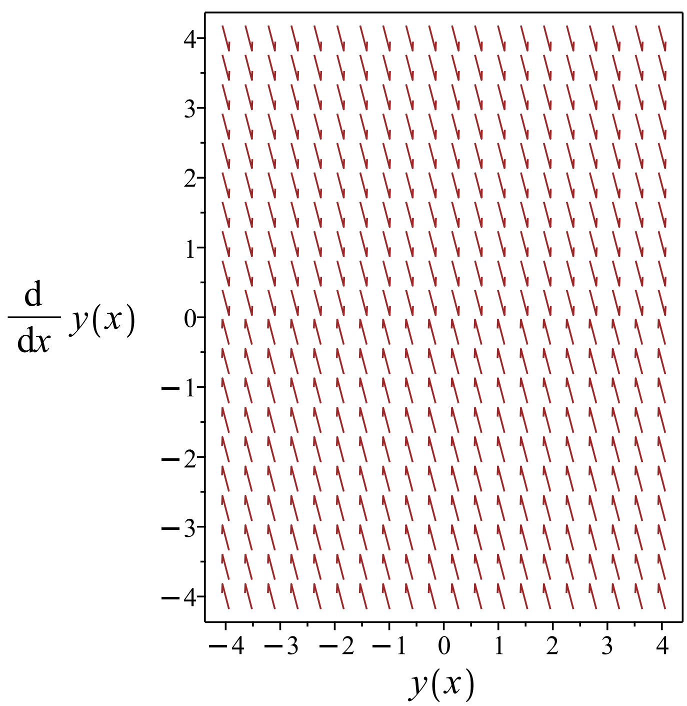


Figure 505: Slope field plot

Verification of solutions

$$y = -\frac{c_1 e^{-3x}}{3} + \frac{e^x}{4} + c_2$$

Verified OK.

16.3.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 3y' = e^x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 3y')\, dx = \int e^x dx$$
$$3y + y' = e^x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 3$$
$$q(x) = e^x + c_1$$

Hence the ode is

$$3y + y' = e^x + c_1$$

The integrating factor μ is

$$\mu = e^{\int 3dx}$$
$$= e^{3x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(e^x + c_1)$$
$$\frac{d}{dx}(e^{3x} y) = (e^{3x})(e^x + c_1)$$
$$d(e^{3x} y) = ((e^x + c_1) e^{3x})\, dx$$

Integrating gives

$$e^{3x}y = \int (e^{x} + c_1)\, e^{3x}\, dx$$
$$e^{3x}y = \frac{e^{4x}}{4} + \frac{c_1 e^{3x}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = e^{-3x}\left(\frac{e^{4x}}{4} + \frac{c_1 e^{3x}}{3}\right) + c_2 e^{-3x}$$

which simplifies to

$$y = \frac{(3\, e^{4x} + 4c_1 e^{3x} + 12c_2)\, e^{-3x}}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{(3\, e^{4x} + 4c_1 e^{3x} + 12c_2)\, e^{-3x}}{12} \tag{1}$$

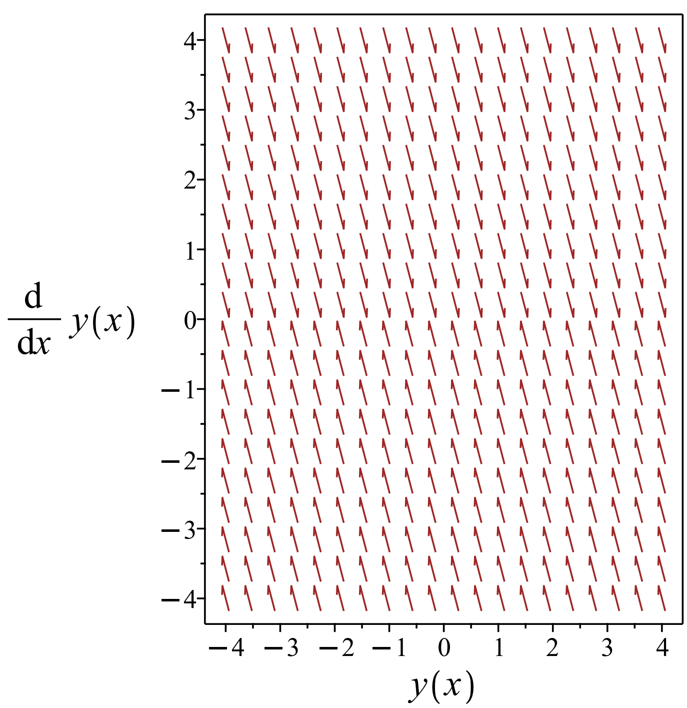


Figure 506: Slope field plot

Verification of solutions

$$y = \frac{(3\,e^{4x} + 4c_1 e^{3x} + 12c_2)\,e^{-3x}}{12}$$

Verified OK.

16.3.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{3}, \mathrm{e}^{-3x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^x = \mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2}{3}\right) + \left(\frac{e^x}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2}{3} + \frac{e^x}{4} \tag{1}$$

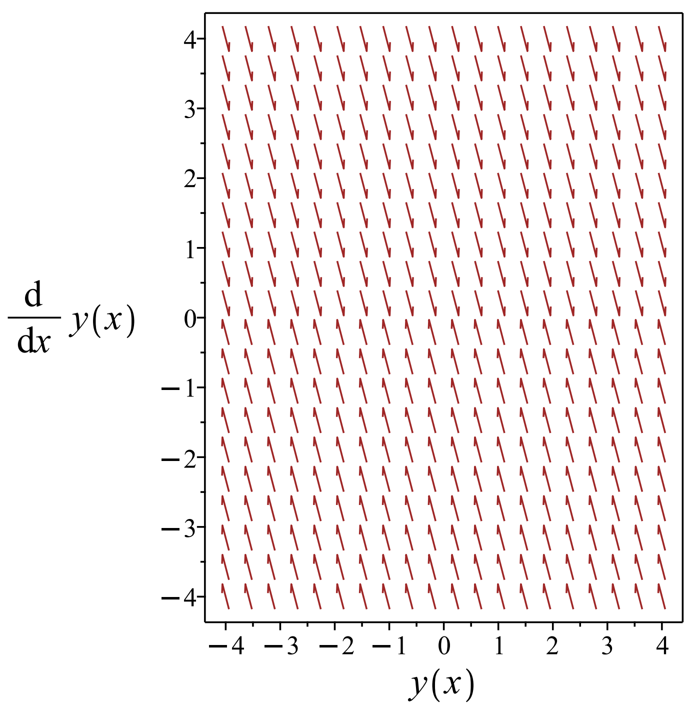


Figure 507: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2}{3} + \frac{e^x}{4}$$

Verified OK.

16.3.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= 3\\
r(x) &= 0\\
s(x) &= \mathrm{e}^{x}
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$3y+y'=\int \mathrm{e}^{x}\,dx$$

We now have a first order ode to solve which is

$$3y+y'=\mathrm{e}^{x}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 3 \\
q(x) &= e^x + c_1
\end{aligned}$$

Hence the ode is

$$3y + y' = e^x + c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int 3dx} \\
&= e^{3x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^x + c_1\right) \\
\frac{d}{dx}\left(e^{3x}y\right) &= \left(e^{3x}\right)\left(e^x + c_1\right) \\
d\left(e^{3x}y\right) &= \left(\left(e^x + c_1\right)e^{3x}\right)\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{3x}y &= \int \left(e^x + c_1\right)e^{3x}\, dx \\
e^{3x}y &= \frac{e^{4x}}{4} + \frac{c_1 e^{3x}}{3} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = e^{-3x}\left(\frac{e^{4x}}{4} + \frac{c_1 e^{3x}}{3}\right) + c_2 e^{-3x}$$

which simplifies to

$$y = \frac{\left(3\,e^{4x} + 4c_1 e^{3x} + 12c_2\right)e^{-3x}}{12}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(3\,e^{4x} + 4c_1 e^{3x} + 12c_2\right)e^{-3x}}{12} \tag{1}$$

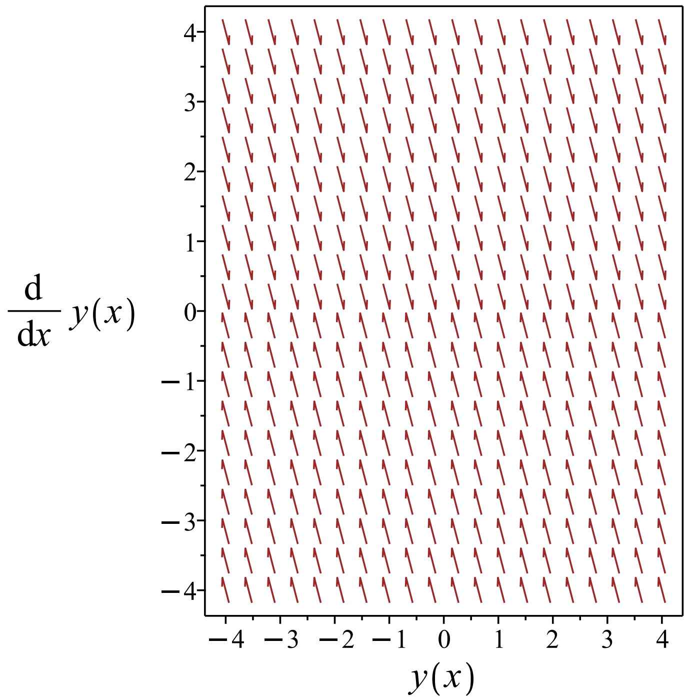


Figure 508: Slope field plot

Verification of solutions

$$y = \frac{(3\,\mathrm{e}^{4x} + 4c_1\mathrm{e}^{3x} + 12c_2)\,\mathrm{e}^{-3x}}{12}$$

Verified OK.

16.3.7 Maple step by step solution

Let's solve

$y'' + 3y' = \mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r = 0$

- Factor the characteristic polynomial

  $r(r+3) = 0$

- Roots of the characteristic polynomial

$r = (-3, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & 1 \\ -3\,\mathrm{e}^{-3x} & 0 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-3x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-3x}\left(\int \mathrm{e}^{4x} dx\right)}{3} + \frac{\left(\int \mathrm{e}^{x} dx\right)}{3}$

  ◦ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 + \frac{\mathrm{e}^{x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -3*_b(_a)+exp(_a), _b(_a)`   *** Sublev
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)=exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-3x}\left(-3c_2 e^{3x} + c_1 - \frac{3\,e^{4x}}{4}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 26

```
DSolve[y''[x]+3*y'[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{4} - \frac{1}{3}c_1 e^{-3x} + c_2$$

16.4 problem 477

Internal problem ID [15247]
Internal file name [`OUTPUT/15247_Wednesday_May_08_2024_03_54_06_PM_32131591/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 477.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 7y' = e^{-7x}}$$

16.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 7, C = 0, f(x) = e^{-7x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+7y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=7, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+7\lambda\, e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+7\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=7, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-7}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{7^2-(4)\,(1)\,(0)}\\&=-\frac{7}{2}\pm\frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{7}{2}+\frac{7}{2}\\\lambda_2&=-\frac{7}{2}-\frac{7}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-7\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-7)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-7x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-7x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-7x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-7x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-7x}\}$$

Since e^{-7x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{-7x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{-7x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 e^{-7x} = e^{-7x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\, e^{-7x}}{7}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 \mathrm{e}^{-7x}\right) + \left(-\frac{x\,\mathrm{e}^{-7x}}{7}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2 \mathrm{e}^{-7x} - \frac{x\,\mathrm{e}^{-7x}}{7} \tag{1}
$$

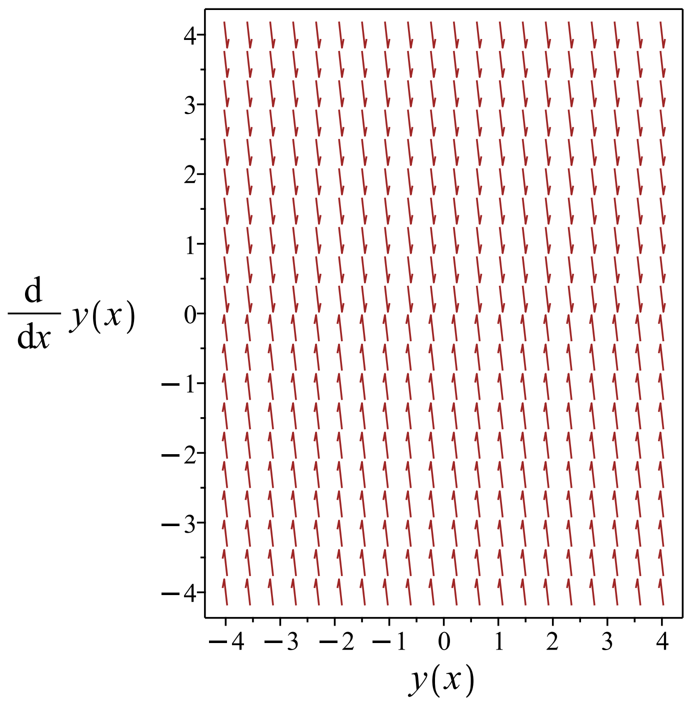


Figure 509: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 \mathrm{e}^{-7x} - \frac{x\,\mathrm{e}^{-7x}}{7}
$$

Verified OK.

16.4.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' + 7y')\, dx &= \int \mathrm{e}^{-7x} dx \\
7y + y' &= -\frac{\mathrm{e}^{-7x}}{7} + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 7 \\
q(x) &= -\frac{\mathrm{e}^{-7x}}{7} + c_1
\end{aligned}
$$

Hence the ode is

$$
7y + y' = -\frac{\mathrm{e}^{-7x}}{7} + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 7dx} \\
&= \mathrm{e}^{7x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{\mathrm{e}^{-7x}}{7} + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{7x} y\right) &= \left(\mathrm{e}^{7x}\right)\left(-\frac{\mathrm{e}^{-7x}}{7} + c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{7x} y\right) &= \left(c_1 \mathrm{e}^{7x} - \frac{1}{7}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{7x} y &= \int c_1 \mathrm{e}^{7x} - \frac{1}{7} \,\mathrm{d}x \\
\mathrm{e}^{7x} y &= -\frac{x}{7} + \frac{c_1 \mathrm{e}^{7x}}{7} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{7x}$ results in

$$y = e^{-7x}\left(-\frac{x}{7} + \frac{c_1 e^{7x}}{7}\right) + c_2 e^{-7x}$$

which simplifies to

$$y = \frac{(-x + 7c_2) e^{-7x}}{7} + \frac{c_1}{7}$$

Summary

The solution(s) found are the following

$$y = \frac{(-x + 7c_2) e^{-7x}}{7} + \frac{c_1}{7} \tag{1}$$

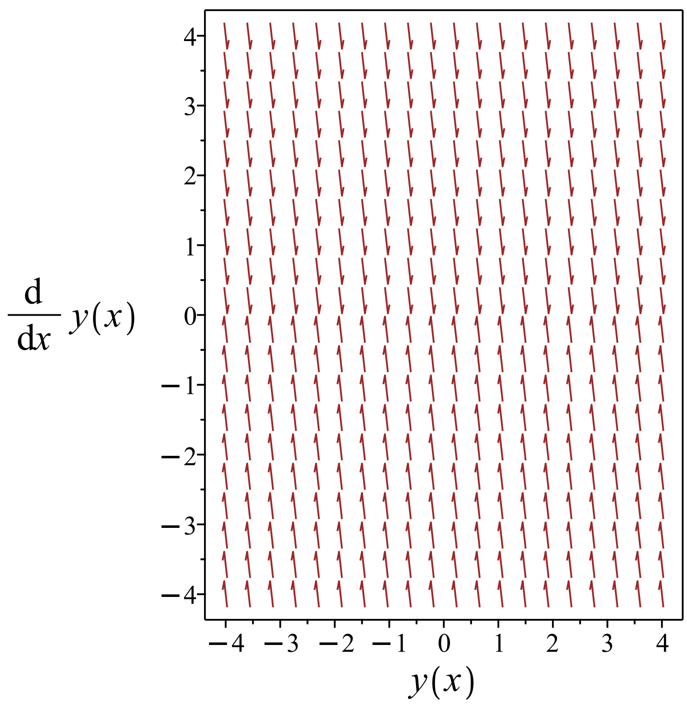


Figure 510: Slope field plot

Verification of solutions

$$y = \frac{(-x + 7c_2) e^{-7x}}{7} + \frac{c_1}{7}$$

Verified OK.

16.4.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 7p(x) - e^{-7x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 7 \\ q(x) &= e^{-7x} \end{aligned}$$

Hence the ode is

$$p'(x) + 7p(x) = e^{-7x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 7dx} \\ &= e^{7x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(e^{-7x}\right) \\ \frac{d}{dx}\left(e^{7x}p\right) &= \left(e^{7x}\right)\left(e^{-7x}\right) \\ d\left(e^{7x}p\right) &= dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{7x}p &= \int dx \\ e^{7x}p &= x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{7x}$ results in

$$p(x) = x\,e^{-7x} + c_1 e^{-7x}$$

which simplifies to

$$p(x) = e^{-7x}(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{-7x}(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int e^{-7x}(x + c_1)\,dx \\ &= -\frac{(7x + 7c_1 + 1)\,e^{-7x}}{49} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{(7x + 7c_1 + 1)\,e^{-7x}}{49} + c_2 \tag{1}$$

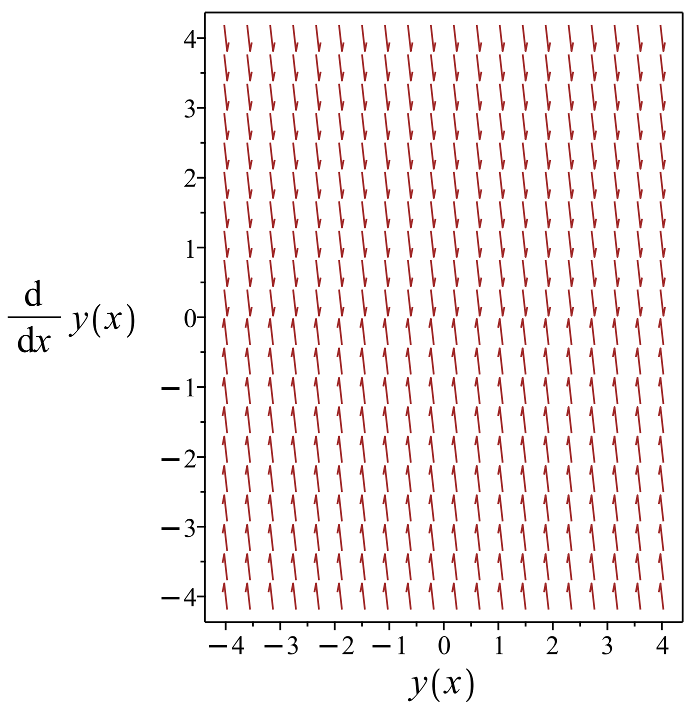


Figure 511: Slope field plot

Verification of solutions

$$y = -\frac{(7x+7c_1+1)\,\mathrm{e}^{-7x}}{49}+c_2$$

Verified OK.

16.4.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y''+7y' = \mathrm{e}^{-7x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y''+7y')\,dx = \int \mathrm{e}^{-7x}dx$$
$$7y+y' = -\frac{\mathrm{e}^{-7x}}{7}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = 7$$
$$q(x) = -\frac{\mathrm{e}^{-7x}}{7}+c_1$$

Hence the ode is

$$7y+y' = -\frac{\mathrm{e}^{-7x}}{7}+c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 7dx}$$
$$= \mathrm{e}^{7x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{\mathrm{e}^{-7x}}{7}+c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{7x}y) = (\mathrm{e}^{7x})\left(-\frac{\mathrm{e}^{-7x}}{7}+c_1\right)$$
$$\mathrm{d}(\mathrm{e}^{7x}y) = \left(c_1\mathrm{e}^{7x}-\frac{1}{7}\right)\,\mathrm{d}x$$

Integrating gives

$$e^{7x}y = \int c_1 e^{7x} - \frac{1}{7}\,dx$$
$$e^{7x}y = -\frac{x}{7} + \frac{c_1 e^{7x}}{7} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{7x}$ results in

$$y = e^{-7x}\left(-\frac{x}{7} + \frac{c_1 e^{7x}}{7}\right) + c_2 e^{-7x}$$

which simplifies to

$$y = \frac{(-x + 7c_2)\,e^{-7x}}{7} + \frac{c_1}{7}$$

Summary

The solution(s) found are the following

$$y = \frac{(-x + 7c_2)\,e^{-7x}}{7} + \frac{c_1}{7} \tag{1}$$

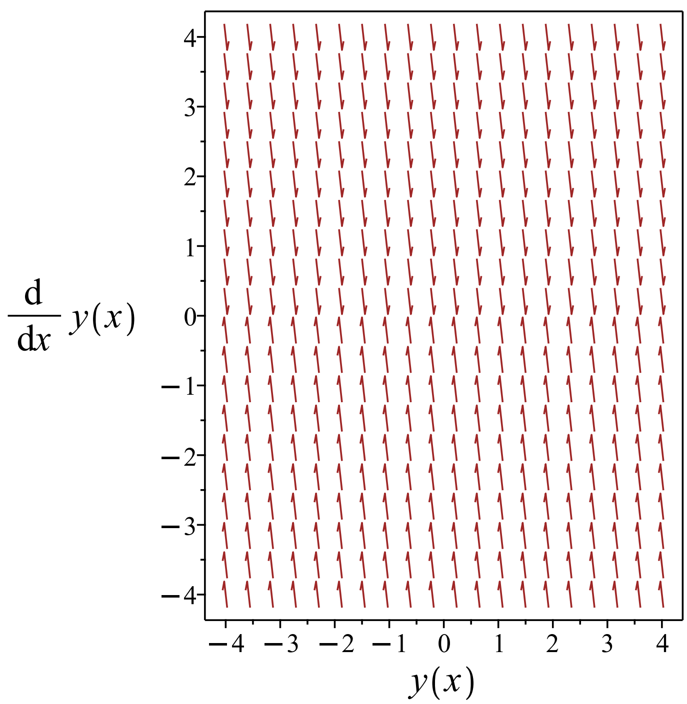


Figure 512: Slope field plot

Verification of solutions

$$y = \frac{(-x + 7c_2)\,\mathrm{e}^{-7x}}{7} + \frac{c_1}{7}$$

Verified OK.

16.4.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dx} \\
&= z_1 e^{-\frac{7x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{7x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-7x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-7x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{7x}}{7}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-7x}\right) + c_2 \left(\mathrm{e}^{-7x} \left(\frac{\mathrm{e}^{7x}}{7}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 7y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-7x} + \frac{c_2}{7}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-7x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-7x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{7}, \mathrm{e}^{-7x}\right\}$$

Since e^{-7x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-7x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-7x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 \mathrm{e}^{-7x} = \mathrm{e}^{-7x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,\mathrm{e}^{-7x}}{7}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-7x} + \frac{c_2}{7}\right) + \left(-\frac{x\,\mathrm{e}^{-7x}}{7}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-7x} + \frac{c_2}{7} - \frac{x\,\mathrm{e}^{-7x}}{7} \tag{1}$$

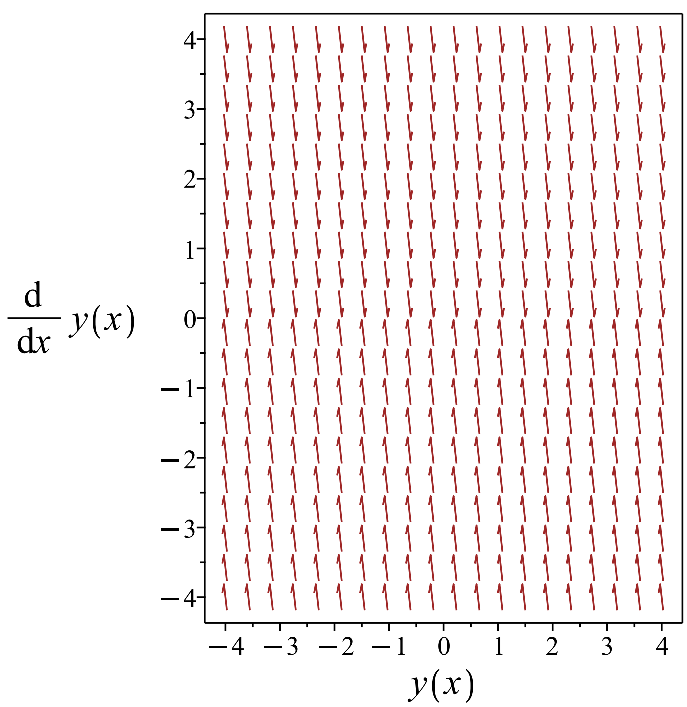


Figure 513: Slope field plot

Verification of solutions

$$y = c_1 e^{-7x} + \frac{c_2}{7} - \frac{x\, e^{-7x}}{7}$$

Verified OK.

16.4.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 7 \\ r(x) &= 0 \\ s(x) &= e^{-7x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$7y + y' = \int e^{-7x}\, dx$$

We now have a first order ode to solve which is

$$7y + y' = -\frac{e^{-7x}}{7} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 7 \\ q(x) &= -\frac{e^{-7x}}{7} + c_1 \end{aligned}$$

Hence the ode is

$$7y + y' = -\frac{e^{-7x}}{7} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 7dx} \\ &= e^{7x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(-\frac{e^{-7x}}{7} + c_1\right) \\ \frac{d}{dx}\left(e^{7x} y\right) &= \left(e^{7x}\right)\left(-\frac{e^{-7x}}{7} + c_1\right) \\ d\left(e^{7x} y\right) &= \left(c_1 e^{7x} - \frac{1}{7}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{7x} y &= \int c_1 e^{7x} - \frac{1}{7} \, dx \\ e^{7x} y &= -\frac{x}{7} + \frac{c_1 e^{7x}}{7} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{7x}$ results in

$$y = e^{-7x}\left(-\frac{x}{7} + \frac{c_1 e^{7x}}{7}\right) + c_2 e^{-7x}$$

which simplifies to

$$y = \frac{(-x + 7c_2)\,\mathrm{e}^{-7x}}{7} + \frac{c_1}{7}$$

Summary

The solution(s) found are the following

$$y = \frac{(-x + 7c_2)\,\mathrm{e}^{-7x}}{7} + \frac{c_1}{7} \tag{1}$$

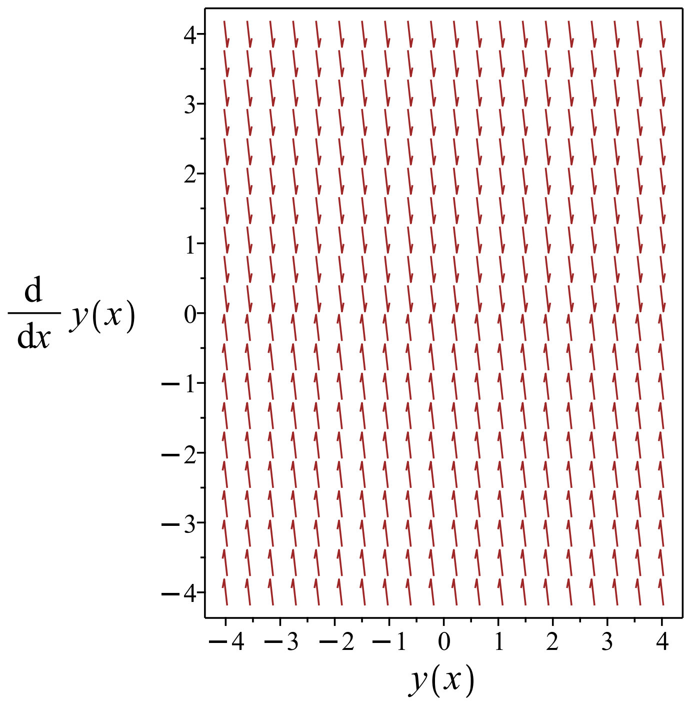


Figure 514: Slope field plot

Verification of solutions

$$y = \frac{(-x + 7c_2)\,\mathrm{e}^{-7x}}{7} + \frac{c_1}{7}$$

Verified OK.

16.4.7 Maple step by step solution

Let's solve

$y'' + 7y' = e^{-7x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 7r = 0$

- Factor the characteristic polynomial

  $r(r + 7) = 0$

- Roots of the characteristic polynomial

  $r = (-7, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-7x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-7x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{-7x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-7x} & 1 \\ -7\,e^{-7x} & 0 \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 7\,e^{-7x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-7x}\left(\int 1 dx\right)}{7} + \frac{\left(\int e^{-7x} dx\right)}{7}$$

- Compute integrals

$$y_p(x) = -\frac{e^{-7x}(7x+1)}{49}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-7x} + c_2 - \frac{e^{-7x}(7x+1)}{49}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -7*_b(_a)+exp(-7*_a), _b(_a)`   *** Sub
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+7*diff(y(x),x)=exp(-7*x),y(x), singsol=all)
```

$$y(x) = \frac{(-7x - 7c_1 - 1) e^{-7x}}{49} + c_2$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 26

```
DSolve[y''[x]+7*y'[x]==Exp[-7*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{49} e^{-7x}(7x + 1 + 7c_1)$$

16.5 problem 478

Internal problem ID [15248]
Internal file name [`OUTPUT/15248_Wednesday_May_08_2024_03_54_07_PM_96590617/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 478.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 8y' + 16y = (1 - x)\,\mathrm{e}^{4x}}$$

16.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -8, C = 16, f(x) = (1 - x)\,\mathrm{e}^{4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 8y' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -8, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 8\lambda\, e^{\lambda x} + 16\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 8\lambda + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -8, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{8}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-8)^2 - (4)\,(1)\,(16)} \\ &= 4\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -4$. Therefore the solution is

$$y = c_1 e^{4x} + c_2 e^{4x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{4x} + c_2 x\, e^{4x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(1 - x)\, e^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{4x}x, e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{4x}x, e^{4x}\}$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{4x}, e^{4x} x\}]$$

Since $e^{4x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{4x}, x^3 e^{4x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{4x} + A_2 x^3 e^{4x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{4x} + 6A_2 x \, e^{4x} = (1 - x) \, e^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{4x}}{2} - \frac{x^3 e^{4x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 x \, e^{4x}\right) + \left(\frac{x^2 e^{4x}}{2} - \frac{x^3 e^{4x}}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^{4x}(c_2 x + c_1) + \frac{x^2 e^{4x}}{2} - \frac{x^3 e^{4x}}{6}$$

Summary
The solution(s) found are the following

$$y = e^{4x}(c_2 x + c_1) + \frac{x^2 e^{4x}}{2} - \frac{x^3 e^{4x}}{6} \tag{1}$$

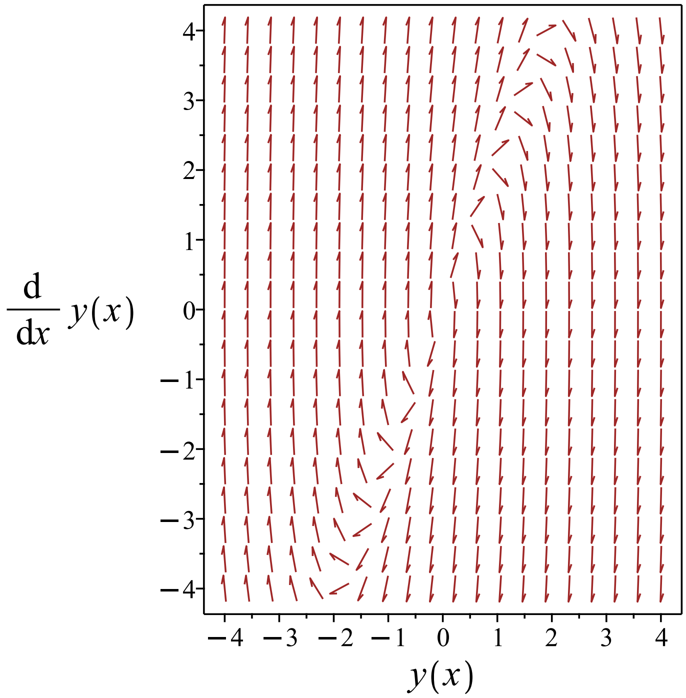


Figure 515: Slope field plot

Verification of solutions

$$y = e^{4x}(c_2 x + c_1) + \frac{x^2 e^{4x}}{2} - \frac{x^3 e^{4x}}{6}$$

Verified OK.

16.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -8$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -8\,dx} \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-4x}(1-x)\,\mathrm{e}^{4x} \\
\left(\mathrm{e}^{-4x}y\right)'' &= \mathrm{e}^{-4x}(1-x)\,\mathrm{e}^{4x}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-4x}y\right)' = -\frac{x(x-2)}{2} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-4x}y\right) = -\frac{x(x^2-6c_1-3x)}{6} + c_2
$$

Hence the solution is

$$
y = \frac{-\frac{x(x^2-6c_1-3x)}{6} + c_2}{\mathrm{e}^{-4x}}
$$

Or

$$
y = -\frac{x^3\mathrm{e}^{4x}}{6} + c_1 x\,\mathrm{e}^{4x} + \frac{x^2\mathrm{e}^{4x}}{2} + c_2\mathrm{e}^{4x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x^3\mathrm{e}^{4x}}{6} + c_1 x\,\mathrm{e}^{4x} + \frac{x^2\mathrm{e}^{4x}}{2} + c_2\mathrm{e}^{4x} \tag{1}
$$

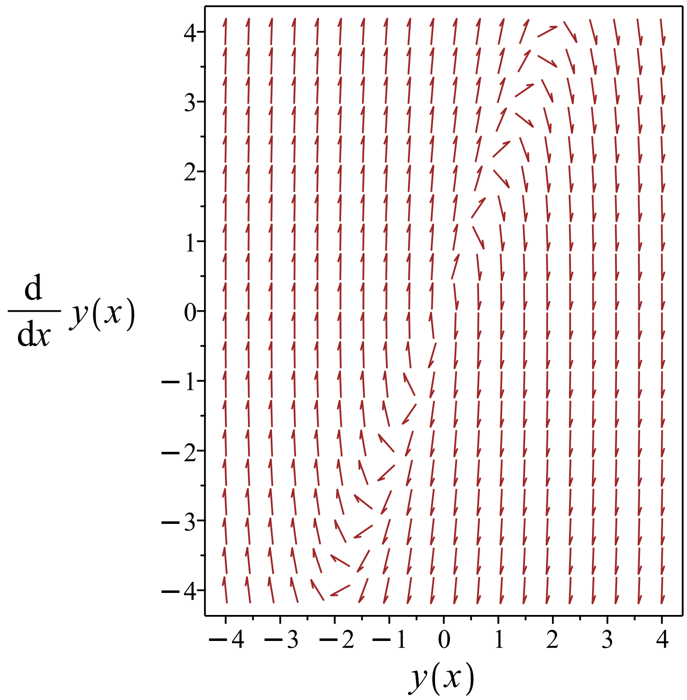


Figure 516: Slope field plot

Verification of solutions

$$y = -\frac{x^3 e^{4x}}{6} + c_1 x\, e^{4x} + \frac{x^2 e^{4x}}{2} + c_2 e^{4x}$$

Verified OK.

16.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 8y' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -8 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8}{1}\,dx} \\ &= z_1 e^{4x} \\ &= z_1\left(\mathrm{e}^{4x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{8x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{4x}\right) + c_2 \left(\mathrm{e}^{4x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 8y' + 16y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{4x} + c_2 x\,\mathrm{e}^{4x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
(1 - x)\,\mathrm{e}^{4x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{4x} x, \mathrm{e}^{4x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{4x} x, \mathrm{e}^{4x}\}
$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2 \mathrm{e}^{4x}, \mathrm{e}^{4x} x\}]
$$

Since $\mathrm{e}^{4x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2 \mathrm{e}^{4x}, x^3 \mathrm{e}^{4x}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{4x} + A_2 x^3 \mathrm{e}^{4x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{4x} + 6A_2 x\,\mathrm{e}^{4x} = (1 - x)\,\mathrm{e}^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 \mathrm{e}^{4x}}{2} - \frac{x^3 \mathrm{e}^{4x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{4x} + c_2 x\,\mathrm{e}^{4x}\right) + \left(\frac{x^2 \mathrm{e}^{4x}}{2} - \frac{x^3 \mathrm{e}^{4x}}{6}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{4x}(c_2 x + c_1) + \frac{x^2 \mathrm{e}^{4x}}{2} - \frac{x^3 \mathrm{e}^{4x}}{6}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{4x}(c_2 x + c_1) + \frac{x^2 \mathrm{e}^{4x}}{2} - \frac{x^3 \mathrm{e}^{4x}}{6} \tag{1}$$

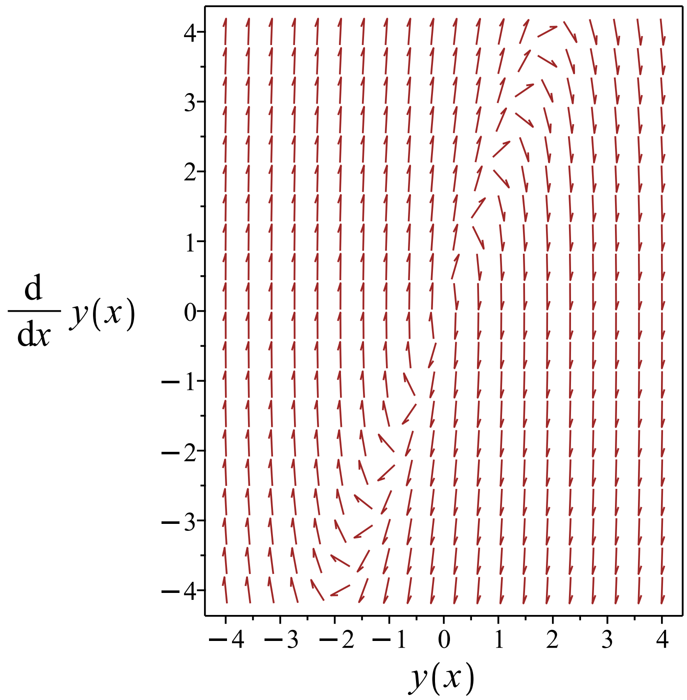


Figure 517: Slope field plot

Verification of solutions

$$y = e^{4x}(c_2 x + c_1) + \frac{x^2 e^{4x}}{2} - \frac{x^3 e^{4x}}{6}$$

Verified OK.

16.5.4 Maple step by step solution

Let's solve

$y'' - 8y' + 16y = (1 - x) e^{4x}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -e^{4x} x + 8y' - 16y + e^{4x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 8y' + 16y = -(x - 1) e^{4x}$

- Characteristic polynomial of homogeneous ODE

$r^2 - 8r + 16 = 0$

- Factor the characteristic polynomial

$(r-4)^2 = 0$

- Root of the characteristic polynomial

$r = 4$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{4x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$y_2(x) = e^{4x}x$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 e^{4x} + c_2 x\, e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -(x-1)\, e^{4x}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{4x} & e^{4x}x \\ 4\, e^{4x} & 4\, e^{4x}x + e^{4x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{8x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{4x}\left(\int x(x-1)\, dx - \left(\int (x-1)\, dx\right) x\right)$

  ○ Compute integrals

  $y_p(x) = -\frac{e^{4x} x^2 (x-3)}{6}$

- Substitute particular solution into general solution to ODE

$y = c_1 e^{4x} + c_2 x\, e^{4x} - \frac{e^{4x} x^2 (x-3)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-8*diff(y(x),x)+16*y(x)=(1-x)*exp(4*x),y(x), singsol=all)
```

$$y(x) = -\frac{(x^3 - 3x^2 + (-6c_1 + 2)x - 6c_2)e^{4x}}{6}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 34

```
DSolve[y''[x]-8*y'[x]+16*y[x]==(1-x)*Exp[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{4x}(-x^3 + 3x^2 + 6c_2x + 6c_1)$$

16.6 problem 479

Internal problem ID [15249]
Internal file name [`OUTPUT/15249_Wednesday_May_08_2024_03_54_08_PM_40851537/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 479.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 10y' + 25y = e^{5x}}$$

16.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -10, C = 25, f(x) = e^{5x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 10y' + 25y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -10, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 10\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 10\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -10, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{10}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-10)^2 - (4)(1)(25)}\\ &= 5\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -5$. Therefore the solution is

$$y = c_1 e^{5x} + c_2 x\, e^{5x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{5x} + c_2 x\, e^{5x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{5x}, e^{5x}\}$$

Since e^{5x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{5x}\}]$$

Since $x\,e^{5x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{5x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{5x} = e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2e^{5x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{5x} + c_2x\,e^{5x}\right) + \left(\frac{x^2e^{5x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{5x}(c_2x + c_1) + \frac{x^2e^{5x}}{2}$$

Summary

The solution(s) found are the following

$$y = e^{5x}(c_2 x + c_1) + \frac{x^2 e^{5x}}{2} \tag{1}$$

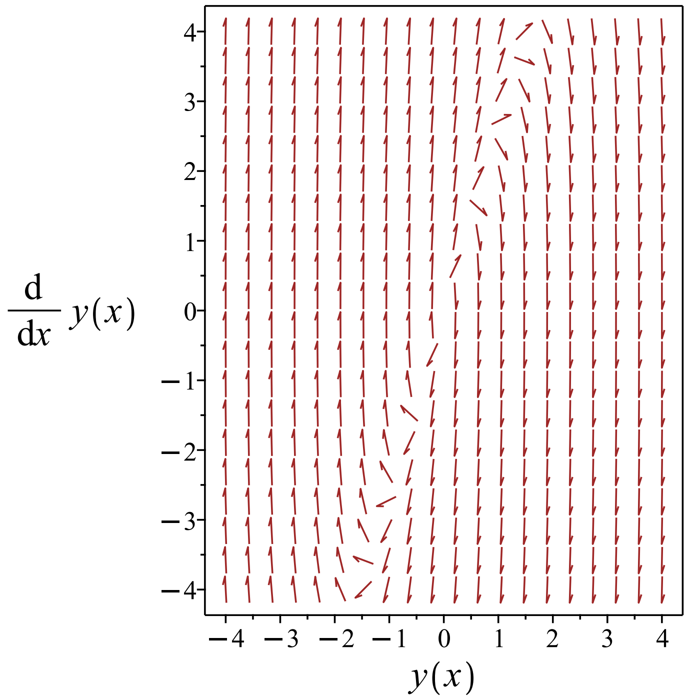


Figure 518: Slope field plot

Verification of solutions

$$y = e^{5x}(c_2 x + c_1) + \frac{x^2 e^{5x}}{2}$$

Verified OK.

16.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -10$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -10\,dx} \\
&= \mathrm{e}^{-5x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-5x}\mathrm{e}^{5x} \\
\left(\mathrm{e}^{-5x}y\right)'' &= \mathrm{e}^{-5x}\mathrm{e}^{5x}
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-5x}y\right)' = x + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-5x}y\right) = \frac{x(x+2c_1)}{2} + c_2
$$

Hence the solution is

$$
y = \frac{\frac{x(x+2c_1)}{2} + c_2}{\mathrm{e}^{-5x}}
$$

Or

$$
y = c_1 x\,\mathrm{e}^{5x} + \frac{x^2\mathrm{e}^{5x}}{2} + c_2\mathrm{e}^{5x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{5x} + \frac{x^2\mathrm{e}^{5x}}{2} + c_2\mathrm{e}^{5x} \tag{1}
$$

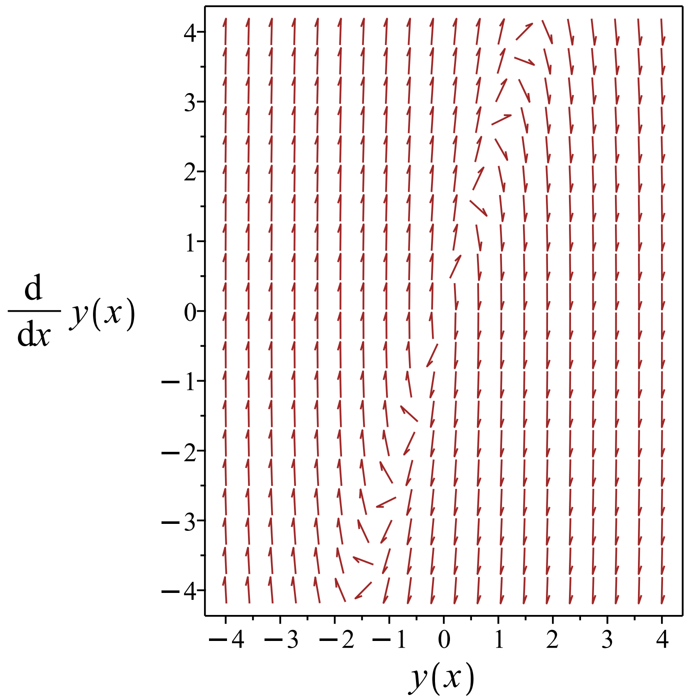


Figure 519: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{5x} + \frac{x^2 e^{5x}}{2} + c_2 e^{5x}$$

Verified OK.

16.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 10y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -10 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-10}{1}\,dx} \\ &= z_1 e^{5x} \\ &= z_1\left(\mathrm{e}^{5x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-10}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{10x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{5x}\right) + c_2 \left(e^{5x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 10y' + 25y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{5x} + c_2 x\, e^{5x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^{5x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{5x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\, e^{5x}, e^{5x}\}
$$

Since e^{5x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\, e^{5x}\}]
$$

Since $x\, e^{5x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2 e^{5x}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{5x} = e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{5x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{5x} + c_2 x\, e^{5x}\right) + \left(\frac{x^2 e^{5x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{5x}(c_2 x + c_1) + \frac{x^2 e^{5x}}{2}$$

Summary

The solution(s) found are the following

$$y = e^{5x}(c_2 x + c_1) + \frac{x^2 e^{5x}}{2} \tag{1}$$

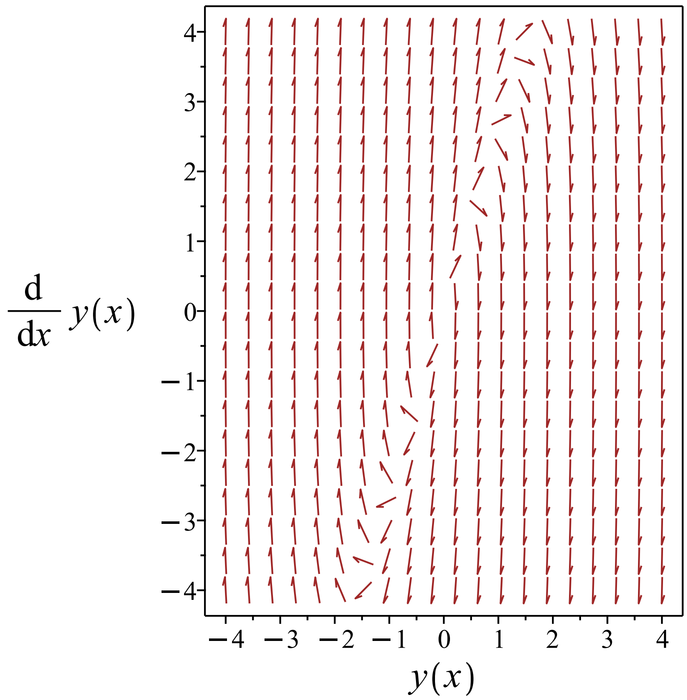


Figure 520: Slope field plot

Verification of solutions

$$y = e^{5x}(c_2 x + c_1) + \frac{x^2 e^{5x}}{2}$$

Verified OK.

16.6.4 Maple step by step solution

Let's solve

$y'' - 10y' + 25y = e^{5x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 10r + 25 = 0$

- Factor the characteristic polynomial

  $(r-5)^2 = 0$

- Root of the characteristic polynomial

$r = 5$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{5x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{5x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{5x} + c_2 x\,e^{5x} + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = e^{5x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} e^{5x} & x\,e^{5x} \\ 5\,e^{5x} & e^{5x} + 5x\,e^{5x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{10x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{5x}\left(-\left(\int x dx\right) + \left(\int 1 dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{x^2 e^{5x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^{5x} + c_1 e^{5x} + \frac{x^2 e^{5x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-10*diff(y(x),x)+25*y(x)=exp(5*x),y(x), singsol=all)
```

$$y(x) = e^{5x}\left(c_2 + c_1 x + \frac{1}{2}x^2\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 27

```
DSolve[y''[x]-10*y'[x]+25*y[x]==Exp[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}e^{5x}\left(x^2 + 2c_2 x + 2c_1\right)$$

16.7 problem 480

Internal problem ID [15250]
Internal file name [`OUTPUT/15250_Wednesday_May_08_2024_03_54_09_PM_19449092/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 480.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{4y'' - 3y' = x\,\mathrm{e}^{\frac{3x}{4}}}$$

16.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4, B = -3, C = 0, f(x) = x\,\mathrm{e}^{\frac{3x}{4}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$4y''-3y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=4, B=-3, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x}-3\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2-3\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=4, B=-3, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{3}{(2)\,(4)}\pm\frac{1}{(2)\,(4)}\sqrt{-3^2-(4)\,(4)\,(0)}\\&=\frac{3}{8}\pm\frac{3}{8}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=\frac{3}{8}+\frac{3}{8}\\\lambda_2&=\frac{3}{8}-\frac{3}{8}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=\frac{3}{4}\\\lambda_2&=0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{\left(\frac{3}{4}\right)x}+c_2e^{(0)x}\end{aligned}$$

Or

$$y = c_1 e^{\frac{3x}{4}} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\frac{3x}{4}} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{\frac{3x}{4}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{x\,e^{\frac{3x}{4}}, e^{\frac{3x}{4}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{\frac{3x}{4}}\right\}$$

Since $e^{\frac{3x}{4}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x\,e^{\frac{3x}{4}}, x^2 e^{\frac{3x}{4}}\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{\frac{3x}{4}} + A_2 x^2 e^{\frac{3x}{4}}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{\frac{3x}{4}} + 8A_2 e^{\frac{3x}{4}} + 6A_2 x\,e^{\frac{3x}{4}} = x\,e^{\frac{3x}{4}}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{9}, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4x\,e^{\frac{3x}{4}}}{9} + \frac{x^2 e^{\frac{3x}{4}}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{3x}{4}} + c_2\right) + \left(-\frac{4x\,e^{\frac{3x}{4}}}{9} + \frac{x^2 e^{\frac{3x}{4}}}{6}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{\frac{3x}{4}} + c_2 - \frac{4x\,e^{\frac{3x}{4}}}{9} + \frac{x^2 e^{\frac{3x}{4}}}{6} \tag{1}$$

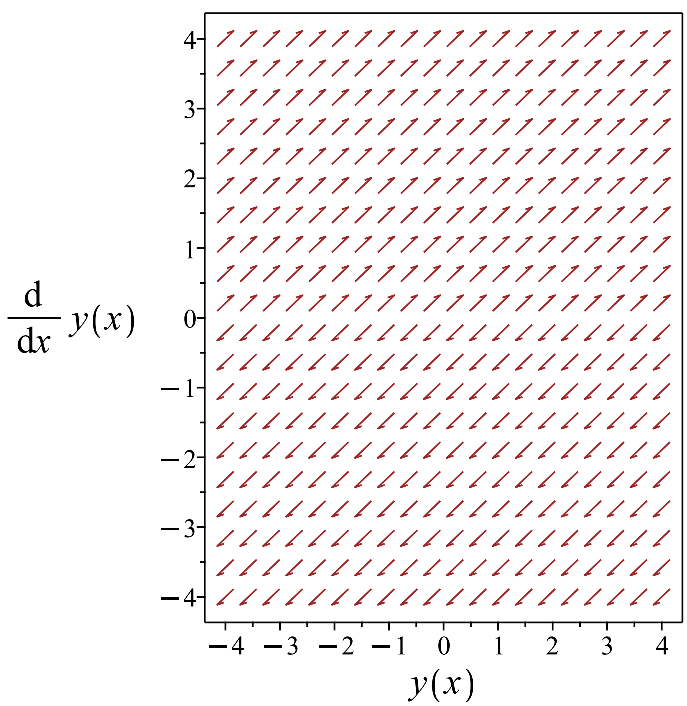


Figure 521: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{\frac{3x}{4}} + c_2 - \frac{4x\,e^{\frac{3x}{4}}}{9} + \frac{x^2 e^{\frac{3x}{4}}}{6}$$

Verified OK.

16.7.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (4y'' - 3y')\,dx = \int x\,\mathrm{e}^{\frac{3x}{4}}\,dx$$

$$4y' - 3y = \frac{4(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{4}$$
$$q(x) = \frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}$$

Hence the ode is

$$y' - \frac{3y}{4} = \frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{3}{4}dx}$$
$$= \mathrm{e}^{-\frac{3x}{4}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{3x}{4}}y\right) = \left(\mathrm{e}^{-\frac{3x}{4}}\right)\left(\frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\right)$$

$$\mathrm{d}\left(\mathrm{e}^{-\frac{3x}{4}}y\right) = \left(\frac{x}{3} - \frac{4}{9} + \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{4}\right)\,\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{-\frac{3x}{4}}y = \int \frac{x}{3} - \frac{4}{9} + \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{4}\,\mathrm{d}x$$

$$\mathrm{e}^{-\frac{3x}{4}}y = \frac{x^2}{6} - \frac{4x}{9} - \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{3x}{4}}$ results in

$$y = e^{\frac{3x}{4}} \left(\frac{x^2}{6} - \frac{4x}{9} - \frac{c_1 e^{-\frac{3x}{4}}}{3} \right) + c_2 e^{\frac{3x}{4}}$$

which simplifies to

$$y = \frac{(3x^2 + 18c_2 - 8x) e^{\frac{3x}{4}}}{18} - \frac{c_1}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(3x^2 + 18c_2 - 8x) e^{\frac{3x}{4}}}{18} - \frac{c_1}{3} \tag{1}$$

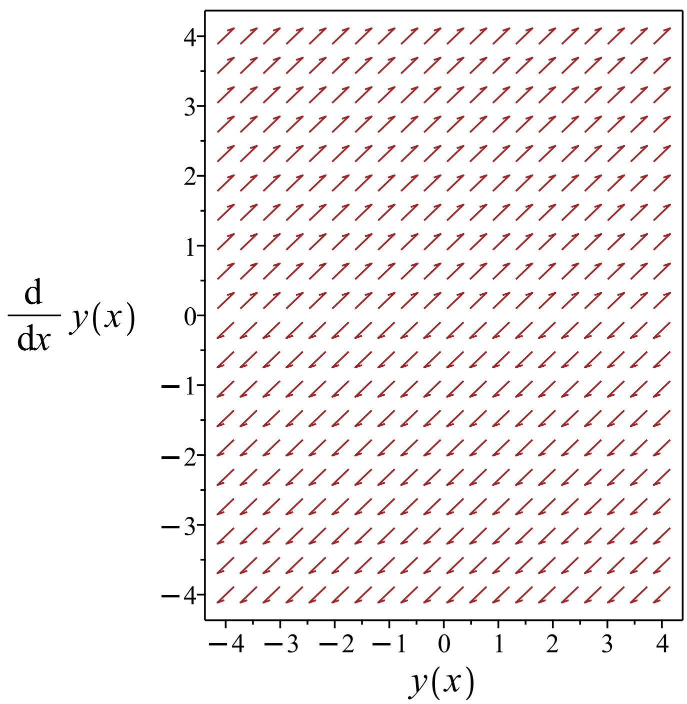


Figure 522: Slope field plot

Verification of solutions

$$y = \frac{(3x^2 + 18c_2 - 8x) e^{\frac{3x}{4}}}{18} - \frac{c_1}{3}$$

Verified OK.

16.7.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$4p'(x) - 3p(x) - x\,e^{\frac{3x}{4}} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{3}{4}$$
$$q(x) = \frac{x\,e^{\frac{3x}{4}}}{4}$$

Hence the ode is

$$p'(x) - \frac{3p(x)}{4} = \frac{x\,e^{\frac{3x}{4}}}{4}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{4}dx}$$
$$= e^{-\frac{3x}{4}}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)\left(\frac{x\,e^{\frac{3x}{4}}}{4}\right)$$
$$\frac{d}{dx}\left(e^{-\frac{3x}{4}}p\right) = \left(e^{-\frac{3x}{4}}\right)\left(\frac{x\,e^{\frac{3x}{4}}}{4}\right)$$
$$d\left(e^{-\frac{3x}{4}}p\right) = \left(\frac{x}{4}\right)\,dx$$

Integrating gives

$$e^{-\frac{3x}{4}} p = \int \frac{x}{4} \, dx$$
$$e^{-\frac{3x}{4}} p = \frac{x^2}{8} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{3x}{4}}$ results in

$$p(x) = \frac{x^2 e^{\frac{3x}{4}}}{8} + c_1 e^{\frac{3x}{4}}$$

which simplifies to

$$p(x) = e^{\frac{3x}{4}} \left(\frac{x^2}{8} + c_1\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{\frac{3x}{4}} \left(\frac{x^2}{8} + c_1\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{e^{\frac{3x}{4}} (x^2 + 8c_1)}{8} \, dx \\ &= \frac{(9x^2 + 72c_1 - 24x + 32) e^{\frac{3x}{4}}}{54} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(9x^2 + 72c_1 - 24x + 32) e^{\frac{3x}{4}}}{54} + c_2 \tag{1}$$

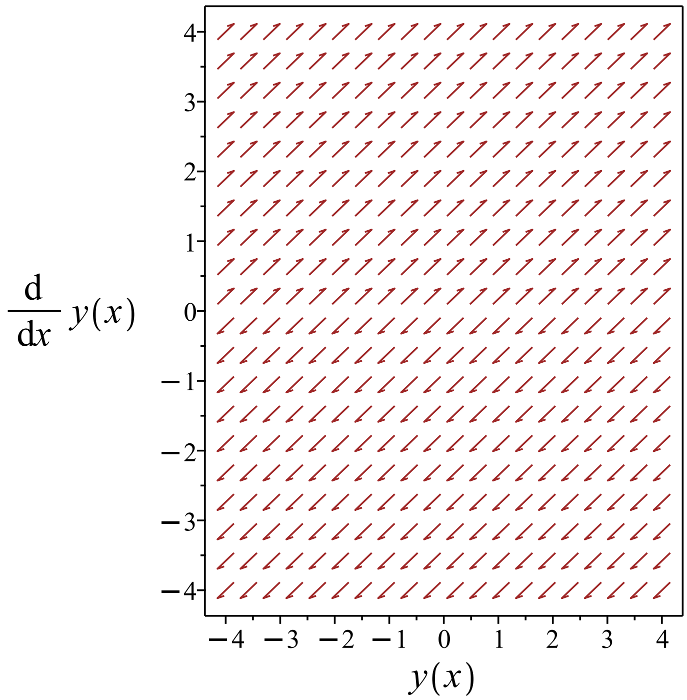


Figure 523: Slope field plot

Verification of solutions

$$y = \frac{(9x^2 + 72c_1 - 24x + 32)\,e^{\frac{3x}{4}}}{54} + c_2$$

Verified OK.

16.7.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$4y'' - 3y' = x\,e^{\frac{3x}{4}}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (4y'' - 3y')\,dx = \int x\,e^{\frac{3x}{4}}\,dx$$

$$4y' - 3y = \frac{4(3x-4)\,e^{\frac{3x}{4}}}{9} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{3}{4}\\ q(x) &= \frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\end{aligned}$$

Hence the ode is

$$y' - \frac{3y}{4} = \frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{3}{4}dx}\\ &= \mathrm{e}^{-\frac{3x}{4}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{3x}{4}}y\right) &= \left(\mathrm{e}^{-\frac{3x}{4}}\right)\left(\frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\right)\\ \mathrm{d}\left(\mathrm{e}^{-\frac{3x}{4}}y\right) &= \left(\frac{x}{3} - \frac{4}{9} + \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{4}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{3x}{4}}y &= \int \frac{x}{3} - \frac{4}{9} + \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{4}\,\mathrm{d}x\\ \mathrm{e}^{-\frac{3x}{4}}y &= \frac{x^2}{6} - \frac{4x}{9} - \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{3} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{3x}{4}}$ results in

$$y = \mathrm{e}^{\frac{3x}{4}}\left(\frac{x^2}{6} - \frac{4x}{9} - \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{3}\right) + c_2\mathrm{e}^{\frac{3x}{4}}$$

which simplifies to

$$y = \frac{(3x^2 + 18c_2 - 8x)\,\mathrm{e}^{\frac{3x}{4}}}{18} - \frac{c_1}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(3x^2 + 18c_2 - 8x)\,\mathrm{e}^{\frac{3x}{4}}}{18} - \frac{c_1}{3} \tag{1}$$

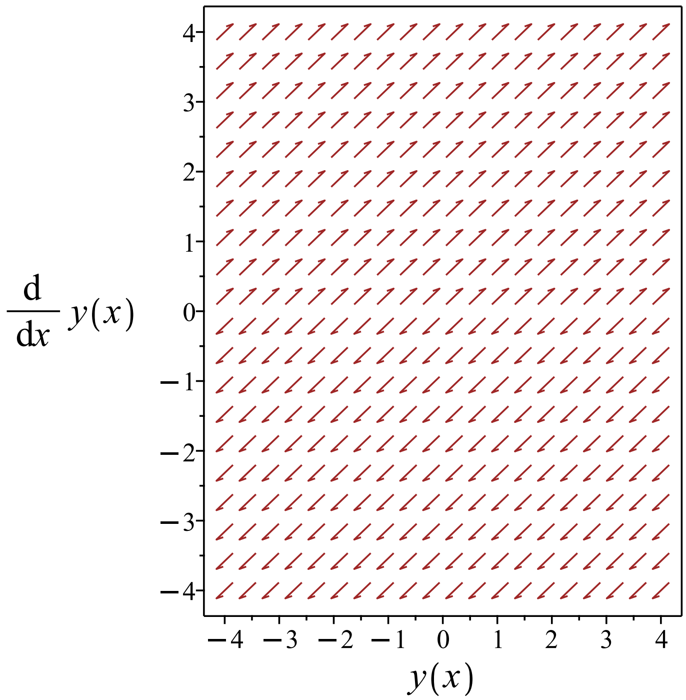


Figure 524: Slope field plot

Verification of solutions

$$y = \frac{(3x^2 + 18c_2 - 8x)\,\mathrm{e}^{\frac{3x}{4}}}{18} - \frac{c_1}{3}$$

Verified OK.

16.7.5 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 3y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 4 \\
B &= -3 \\
C &= 0
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{64} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 64
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{64} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{64}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{8}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{4}\,dx} \\ &= z_1 e^{\frac{3x}{8}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{8}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{4}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{3x}{4}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$4y'' - 3y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{4c_2\mathrm{e}^{\frac{3x}{4}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3} \\ 0 & \mathrm{e}^{\frac{3x}{4}} \end{vmatrix}$$

Therefore

$$W = (1)\left(\mathrm{e}^{\frac{3x}{4}}\right) - \left(\frac{4\,\mathrm{e}^{\frac{3x}{4}}}{3}\right)(0)$$

Which simplifies to

$$W = e^{\frac{3x}{4}}$$

Which simplifies to

$$W = e^{\frac{3x}{4}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4 e^{\frac{3x}{2}} x}{3}}{4 e^{\frac{3x}{4}}} dx$$

Which simplifies to

$$u_1 = -\int \frac{x e^{\frac{3x}{4}}}{3} dx$$

Hence

$$u_1 = -\frac{4(3x-4) e^{\frac{3x}{4}}}{27}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x e^{\frac{3x}{4}}}{4 e^{\frac{3x}{4}}} dx$$

Which simplifies to

$$u_2 = \int \frac{x}{4} dx$$

Hence

$$u_2 = \frac{x^2}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4(3x-4) e^{\frac{3x}{4}}}{27} + \frac{x^2 e^{\frac{3x}{4}}}{6}$$

Which simplifies to

$$y_p(x) = \frac{e^{\frac{3x}{4}} (9x^2 - 24x + 32)}{54}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \frac{4c_2 e^{\frac{3x}{4}}}{3}\right) + \left(\frac{e^{\frac{3x}{4}}(9x^2 - 24x + 32)}{54}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \frac{4c_2 e^{\frac{3x}{4}}}{3} + \frac{e^{\frac{3x}{4}}(9x^2 - 24x + 32)}{54} \tag{1}
$$

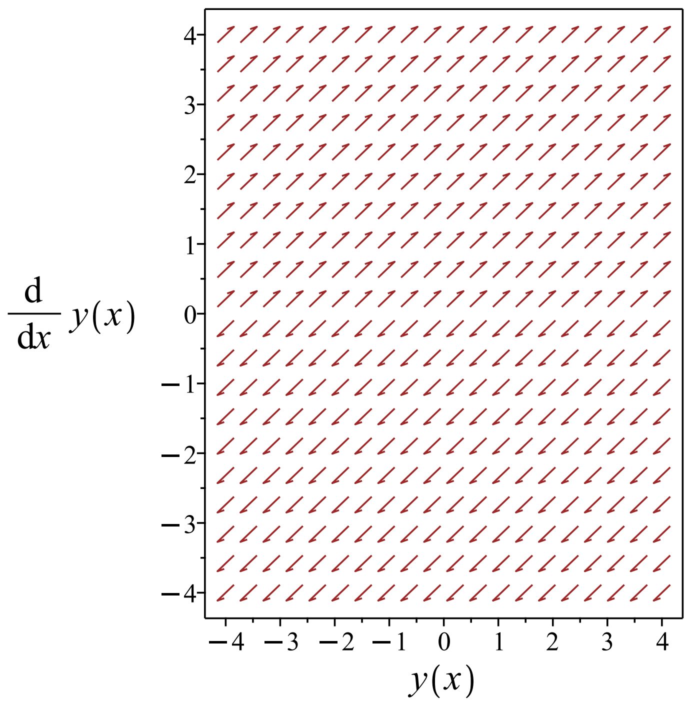


Figure 525: Slope field plot

Verification of solutions

$$
y = c_1 + \frac{4c_2 e^{\frac{3x}{4}}}{3} + \frac{e^{\frac{3x}{4}}(9x^2 - 24x + 32)}{54}
$$

Verified OK.

16.7.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 4\\
q(x) &= -3\\
r(x) &= 0\\
s(x) &= x\,\mathrm{e}^{\frac{3x}{4}}
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$4y' - 3y = \int x\,\mathrm{e}^{\frac{3x}{4}}\,dx$$

We now have a first order ode to solve which is

$$4y' - 3y = \frac{4(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{4} \\ q(x) &= \frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4} \end{aligned}$$

Hence the ode is

$$y' - \frac{3y}{4} = \frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{3}{4}dx} \\ &= \mathrm{e}^{-\frac{3x}{4}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{3x}{4}}y\right) &= \left(\mathrm{e}^{-\frac{3x}{4}}\right)\left(\frac{(3x-4)\,\mathrm{e}^{\frac{3x}{4}}}{9} + \frac{c_1}{4}\right) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{3x}{4}}y\right) &= \left(\frac{x}{3} - \frac{4}{9} + \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{4}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-\frac{3x}{4}}y &= \int \frac{x}{3} - \frac{4}{9} + \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{4}\,\mathrm{d}x \\ \mathrm{e}^{-\frac{3x}{4}}y &= \frac{x^2}{6} - \frac{4x}{9} - \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{3x}{4}}$ results in

$$y = \mathrm{e}^{\frac{3x}{4}}\left(\frac{x^2}{6} - \frac{4x}{9} - \frac{c_1\mathrm{e}^{-\frac{3x}{4}}}{3}\right) + c_2\mathrm{e}^{\frac{3x}{4}}$$

which simplifies to

$$y = \frac{(3x^2 + 18c_2 - 8x)\,\mathrm{e}^{\frac{3x}{4}}}{18} - \frac{c_1}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{(3x^2 + 18c_2 - 8x)\, e^{\frac{3x}{4}}}{18} - \frac{c_1}{3} \tag{1}$$

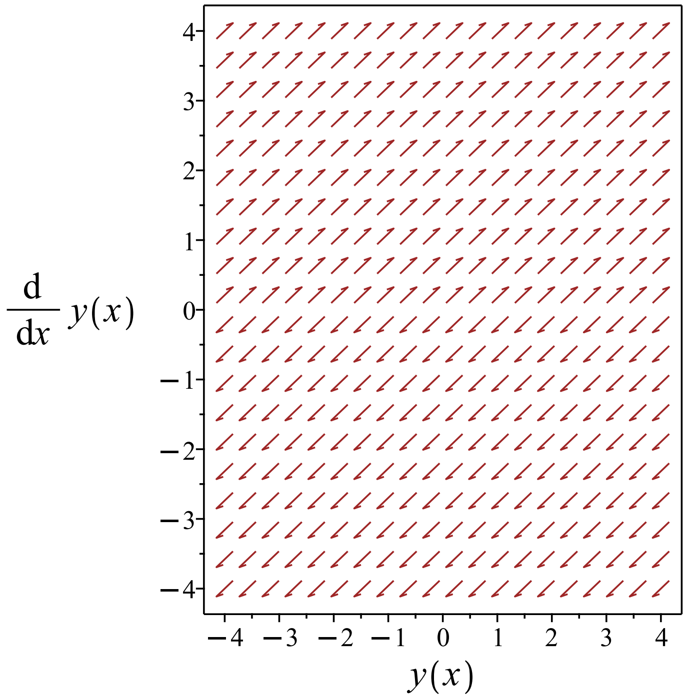


Figure 526: Slope field plot

Verification of solutions

$$y = \frac{(3x^2 + 18c_2 - 8x)\, e^{\frac{3x}{4}}}{18} - \frac{c_1}{3}$$

Verified OK.

16.7.7 Maple step by step solution

Let's solve

$4y'' - 3y' = x\, e^{\frac{3x}{4}}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{3y'}{4} + \frac{x\,e^{\frac{3x}{4}}}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{4} = \frac{x\,e^{\frac{3x}{4}}}{4}$

- Characteristic polynomial of homogeneous ODE

  $r^2 - \frac{3}{4}r = 0$

- Factor the characteristic polynomial

  $\frac{r(4r-3)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(0, \frac{3}{4}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{\frac{3x}{4}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{\frac{3x}{4}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{x\,e^{\frac{3x}{4}}}{4}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{\frac{3x}{4}} \\ 0 & \frac{3\,e^{\frac{3x}{4}}}{4} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{3\,e^{\frac{3x}{4}}}{4}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\left(\int x\,e^{\frac{3x}{4}}dx\right)}{3} + \frac{e^{\frac{3x}{4}}\left(\int x dx\right)}{3}$

- ○ Compute integrals

$$y_p(x) = \frac{e^{\frac{3x}{4}}(9x^2-24x+32)}{54}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 + c_2 e^{\frac{3x}{4}} + \frac{e^{\frac{3x}{4}}(9x^2-24x+32)}{54}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/4)*_a*exp((3/4)*_a)+(3/4)*_b(_a), _b
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(4*diff(y(x),x$2)-3*diff(y(x),x)=x*exp(3/4*x),y(x), singsol=all)
```

$$y(x) = \frac{(9x^2 + 72c_1 - 24x + 32)\, e^{\frac{3x}{4}}}{54} + c_2$$

✓ Solution by Mathematica

Time used: 0.155 (sec). Leaf size: 35

```
DSolve[y''[x]-3*y'[x]==x*Exp[3/4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{16}{243} e^{3x/4}(9x-8) + \frac{1}{3} c_1 e^{3x} + c_2$$

16.8 problem 481

Internal problem ID [15251]
Internal file name [`OUTPUT/15251_Wednesday_May_08_2024_03_54_10_PM_42495512/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 481.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 4y' = e^{4x}x}$$

16.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 0, f(x) = e^{4x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(0)}\\ &= 2 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 2\\ \lambda_2 &= 2 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$y = c_1 e^{4x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{4x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{4x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{4x} x, e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{4x}\}$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{4x}, e^{4x} x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{4x} + A_2 e^{4x} x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{4x} + 8A_1 x\, e^{4x} + 4A_2 e^{4x} = e^{4x} x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = -\frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{4x}}{8} - \frac{e^{4x} x}{16}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 e^{4x} + c_2) + \left(\frac{x^2 e^{4x}}{8} - \frac{e^{4x} x}{16}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 + \frac{x^2 e^{4x}}{8} - \frac{e^{4x} x}{16} \tag{1}$$

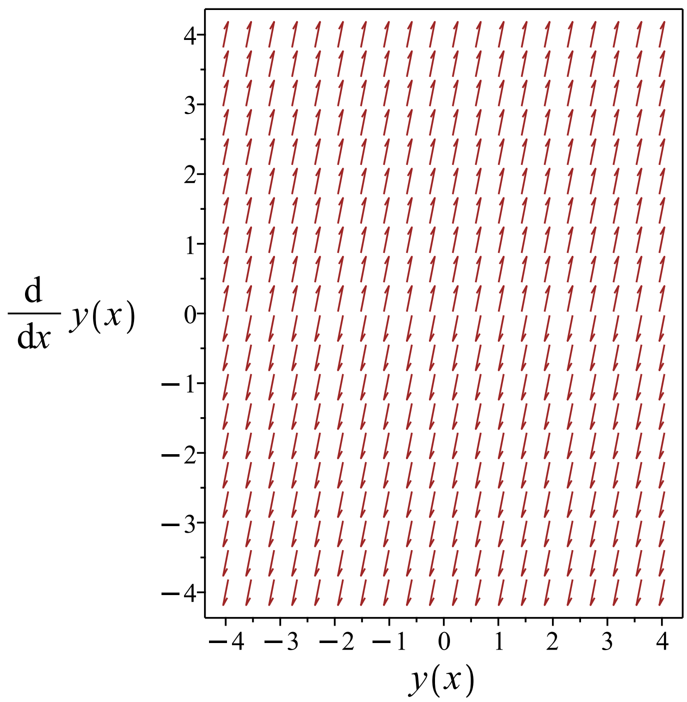


Figure 527: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 + \frac{x^2 e^{4x}}{8} - \frac{e^{4x} x}{16}$$

Verified OK.

16.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y')\, dx = \int \mathrm{e}^{4x} x dx$$
$$-4y + y' = \frac{(4x-1)\,\mathrm{e}^{4x}}{16} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = \frac{(4x-1)\,\mathrm{e}^{4x}}{16} + c_1$$

Hence the ode is

$$-4y + y' = \frac{(4x-1)\,\mathrm{e}^{4x}}{16} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int (-4)dx}$$
$$= \mathrm{e}^{-4x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{(4x-1)\,\mathrm{e}^{4x}}{16} + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-4x} y\right) = \left(\mathrm{e}^{-4x}\right)\left(\frac{(4x-1)\,\mathrm{e}^{4x}}{16} + c_1\right)$$
$$\mathrm{d}\left(\mathrm{e}^{-4x} y\right) = \left(\frac{x}{4} - \frac{1}{16} + c_1 \mathrm{e}^{-4x}\right) \mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{-4x} y = \int \frac{x}{4} - \frac{1}{16} + c_1 \mathrm{e}^{-4x}\, \mathrm{d}x$$
$$\mathrm{e}^{-4x} y = \frac{x^2}{8} - \frac{x}{16} - \frac{c_1 \mathrm{e}^{-4x}}{4} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = e^{4x}\left(\frac{x^2}{8} - \frac{x}{16} - \frac{c_1 e^{-4x}}{4}\right) + c_2 e^{4x}$$

which simplifies to

$$y = \frac{(2x^2 + 16c_2 - x)\, e^{4x}}{16} - \frac{c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x^2 + 16c_2 - x)\, e^{4x}}{16} - \frac{c_1}{4} \tag{1}$$

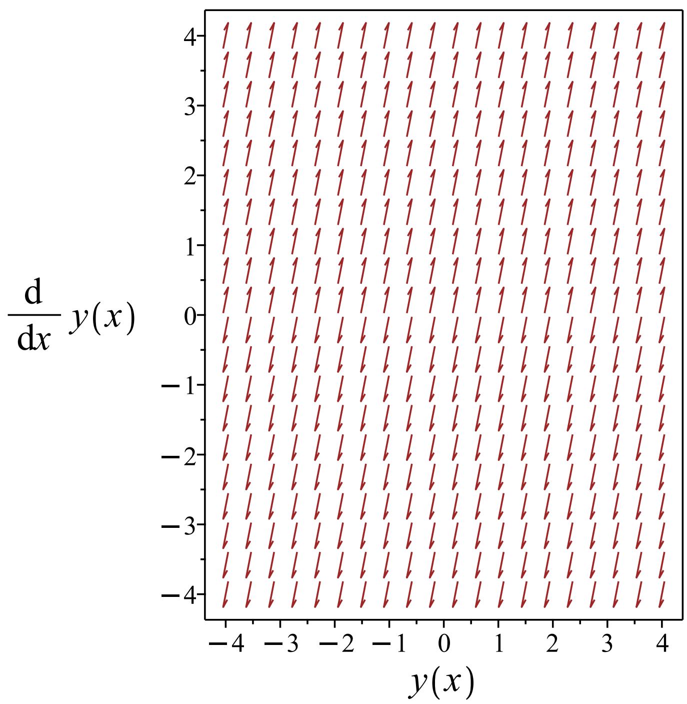


Figure 528: Slope field plot

Verification of solutions

$$y = \frac{(2x^2 + 16c_2 - x)\, e^{4x}}{16} - \frac{c_1}{4}$$

Verified OK.

16.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 4p(x) - e^{4x}x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= e^{4x}x \end{aligned}$$

Hence the ode is

$$p'(x) - 4p(x) = e^{4x}x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-4)dx} \\ &= e^{-4x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(e^{4x}x\right) \\ \frac{d}{dx}\left(e^{-4x}p\right) &= \left(e^{-4x}\right)\left(e^{4x}x\right) \\ d\left(e^{-4x}p\right) &= x\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-4x}p &= \int x\,dx \\ e^{-4x}p &= \frac{x^2}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$p(x) = \frac{x^2 e^{4x}}{2} + c_1 e^{4x}$$

which simplifies to

$$p(x) = e^{4x}\left(\frac{x^2}{2} + c_1\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{4x}\left(\frac{x^2}{2} + c_1\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{e^{4x}(x^2 + 2c_1)}{2}\, dx \\ &= \frac{(8x^2 + 16c_1 - 4x + 1)\, e^{4x}}{64} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(8x^2 + 16c_1 - 4x + 1)\, e^{4x}}{64} + c_2 \tag{1}$$

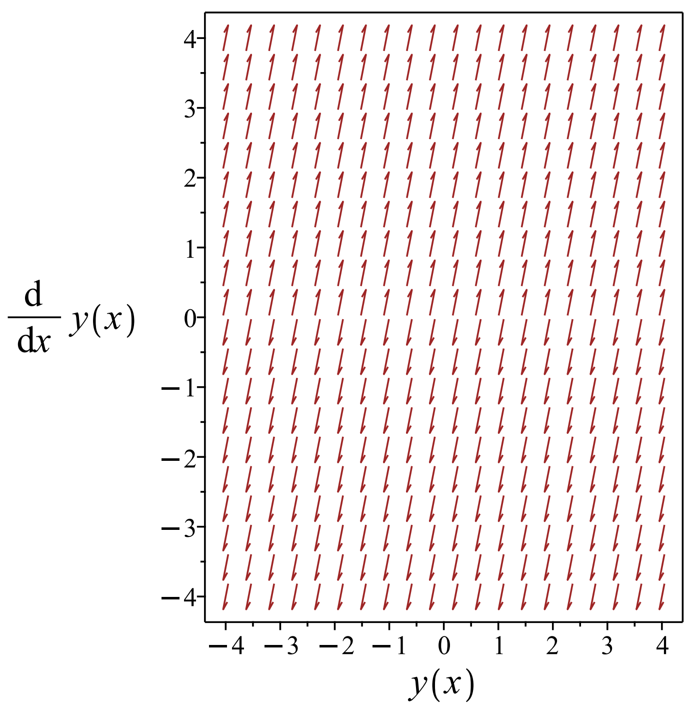


Figure 529: Slope field plot

Verification of solutions

$$y = \frac{(8x^2 + 16c_1 - 4x + 1)\,\mathrm{e}^{4x}}{64} + c_2$$

Verified OK.

16.8.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 4y' = \mathrm{e}^{4x} x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y')\, dx = \int \mathrm{e}^{4x} x dx$$

$$-4y + y' = \frac{(4x - 1)\,\mathrm{e}^{4x}}{16} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = \frac{(4x - 1)\,\mathrm{e}^{4x}}{16} + c_1$$

Hence the ode is

$$-4y + y' = \frac{(4x - 1)\,\mathrm{e}^{4x}}{16} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int (-4) dx}$$
$$= \mathrm{e}^{-4x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{(4x - 1)\,\mathrm{e}^{4x}}{16} + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-4x} y) = (\mathrm{e}^{-4x})\left(\frac{(4x - 1)\,\mathrm{e}^{4x}}{16} + c_1\right)$$
$$\mathrm{d}(\mathrm{e}^{-4x} y) = \left(\frac{x}{4} - \frac{1}{16} + c_1 \mathrm{e}^{-4x}\right)\,\mathrm{d}x$$

Integrating gives

$$e^{-4x}y = \int \frac{x}{4} - \frac{1}{16} + c_1 e^{-4x}\, dx$$
$$e^{-4x}y = \frac{x^2}{8} - \frac{x}{16} - \frac{c_1 e^{-4x}}{4} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = e^{4x}\left(\frac{x^2}{8} - \frac{x}{16} - \frac{c_1 e^{-4x}}{4}\right) + c_2 e^{4x}$$

which simplifies to

$$y = \frac{(2x^2 + 16c_2 - x)\, e^{4x}}{16} - \frac{c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x^2 + 16c_2 - x)\, e^{4x}}{16} - \frac{c_1}{4} \tag{1}$$

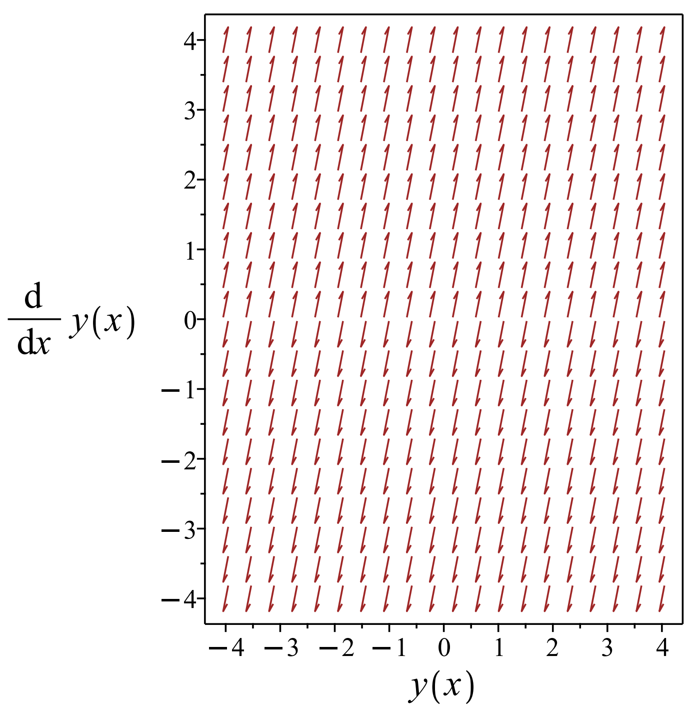


Figure 530: Slope field plot

Verification of solutions

$$y = \frac{(2x^2 + 16c_2 - x)\,e^{4x}}{16} - \frac{c_1}{4}$$

Verified OK.

16.8.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{4x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 4y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 \mathrm{e}^{4x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{\mathrm{e}^{4x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{\mathrm{e}^{4x}}{4} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{\mathrm{e}^{4x}}{4} \\ 0 & \mathrm{e}^{4x} \end{vmatrix}$$

Therefore

$$W = (1)\left(e^{4x}\right) - \left(\frac{e^{4x}}{4}\right)(0)$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{8x}x}{4}}{e^{4x}}dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{4x}x}{4}dx$$

Hence

$$u_1 = -\frac{(4x-1)\,e^{4x}}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{4x}x}{e^{4x}}dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(4x-1)\,e^{4x}}{64} + \frac{x^2 e^{4x}}{8}$$

Which simplifies to

$$y_p(x) = \frac{e^{4x}(8x^2 - 4x + 1)}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{4x}}{4}\right) + \left(\frac{e^{4x}(8x^2 - 4x + 1)}{64}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{4x}}{4} + \frac{e^{4x}(8x^2 - 4x + 1)}{64} \tag{1}$$

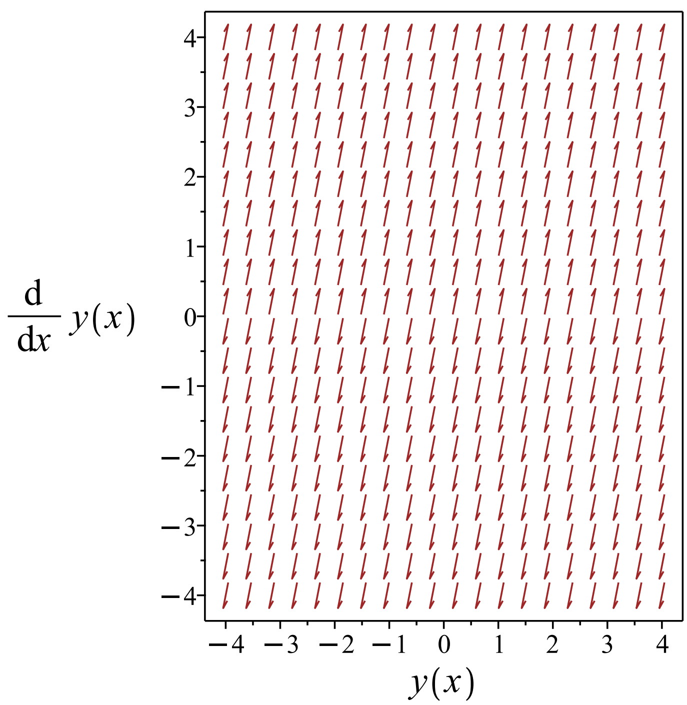


Figure 531: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{4x}}{4} + \frac{e^{4x}(8x^2 - 4x + 1)}{64}$$

Verified OK.

16.8.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= -4 \\
r(x) &= 0 \\
s(x) &= \mathrm{e}^{4x} x
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$-4y + y' = \int \mathrm{e}^{4x} x\,dx$$

We now have a first order ode to solve which is

$$-4y + y' = \frac{(4x-1)\,\mathrm{e}^{4x}}{16} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= \frac{(4x-1)\,e^{4x}}{16} + c_1 \end{aligned}$$

Hence the ode is

$$-4y + y' = \frac{(4x-1)\,e^{4x}}{16} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-4)dx} \\ &= e^{-4x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{(4x-1)\,e^{4x}}{16} + c_1\right) \\ \frac{d}{dx}\left(e^{-4x}y\right) &= \left(e^{-4x}\right)\left(\frac{(4x-1)\,e^{4x}}{16} + c_1\right) \\ d\left(e^{-4x}y\right) &= \left(\frac{x}{4} - \frac{1}{16} + c_1 e^{-4x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-4x}y &= \int \frac{x}{4} - \frac{1}{16} + c_1 e^{-4x}\, dx \\ e^{-4x}y &= \frac{x^2}{8} - \frac{x}{16} - \frac{c_1 e^{-4x}}{4} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = e^{4x}\left(\frac{x^2}{8} - \frac{x}{16} - \frac{c_1 e^{-4x}}{4}\right) + c_2 e^{4x}$$

which simplifies to

$$y = \frac{(2x^2 + 16c_2 - x)\,e^{4x}}{16} - \frac{c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x^2 + 16c_2 - x)\,e^{4x}}{16} - \frac{c_1}{4} \tag{1}$$

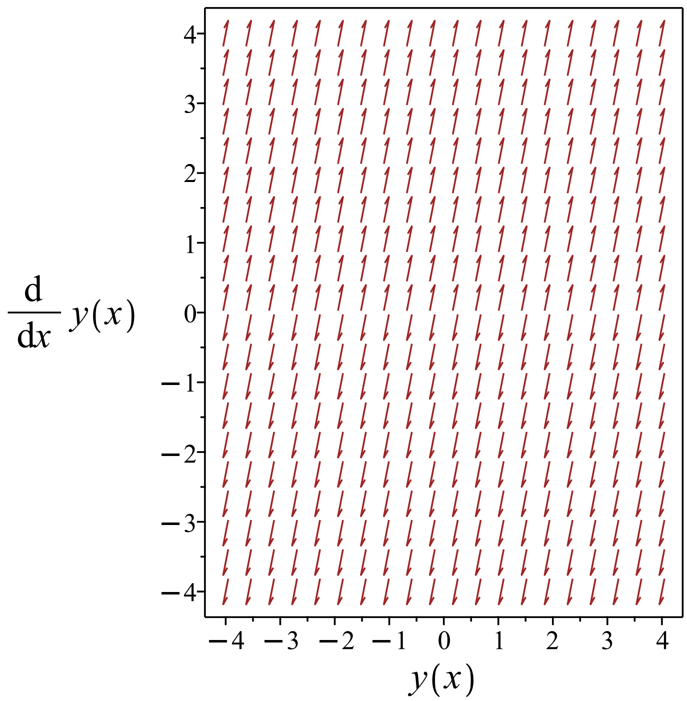


Figure 532: Slope field plot

Verification of solutions

$$y = \frac{(2x^2 + 16c_2 - x)\,e^{4x}}{16} - \frac{c_1}{4}$$

Verified OK.

16.8.7 Maple step by step solution

Let's solve

$y'' - 4y' = e^{4x}x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4r = 0$

- Factor the characteristic polynomial

  $r(r-4) = 0$
- Roots of the characteristic polynomial

  $r = (0, 4)$
- 1st solution of the homogeneous ODE

  $y_1(x) = 1$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{4x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{4x}x\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{4x} \\ 0 & 4e^{4x} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 4e^{4x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\left(\int e^{4x}x dx\right)}{4} + \frac{e^{4x}\left(\int x dx\right)}{4}$
- Compute integrals

  $y_p(x) = \frac{e^{4x}(8x^2-4x+1)}{64}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{4x} + \frac{e^{4x}(8x^2-4x+1)}{64}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(4*_a)*_a+4*_b(_a), _b(_a)`   *** Su
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)=x*exp(4*x),y(x), singsol=all)
```

$$y(x) = \frac{(8x^2 + 16c_1 - 4x + 1)\,e^{4x}}{64} + c_2$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 31

```
DSolve[y''[x]-4*y'[x]==x*Exp[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64}e^{4x}\left(8x^2 - 4x + 1 + 16c_1\right) + c_2$$

16.9 problem 482

Internal problem ID [15252]
Internal file name [`OUTPUT/15252_Wednesday_May_08_2024_03_54_11_PM_96654171/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 482.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 25y = \cos(5x)}$$

16.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 25, f(x) = \cos(5x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 25y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=25$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(25)}\\ &= \pm 5i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +5i\\ \lambda_2 &= -5i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 5i\\ \lambda_2 &= -5i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(5x) + c_2 \sin(5x))$$

Or

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(5x) + c_2 \sin(5x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(5x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5x), \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(5x), \sin(5x)\}$$

Since $\cos(5x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(5x), x\sin(5x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(5x) + A_2 x \sin(5x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-10A_1 \sin(5x) + 10A_2 \cos(5x) = \cos(5x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \sin(5x)}{10}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(5x) + c_2 \sin(5x)) + \left(\frac{x \sin(5x)}{10}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(5x) + c_2 \sin(5x) + \frac{x \sin(5x)}{10} \tag{1}
$$

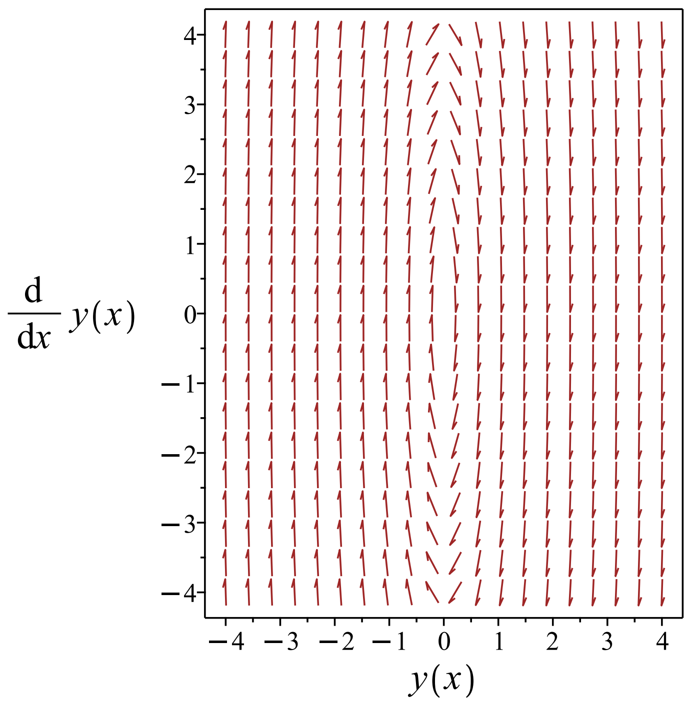


Figure 533: Slope field plot

Verification of solutions

$$
y = c_1 \cos(5x) + c_2 \sin(5x) + \frac{x \sin(5x)}{10}
$$

Verified OK.

16.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(5x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(5x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(5x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(5x) \int \frac{1}{\cos(5x)^2}\,dx \\ &= \cos(5x)\left(\frac{\tan(5x)}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(5x)) + c_2\left(\cos(5x)\left(\frac{\tan(5x)}{5}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 25y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(5x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5x), \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(5x)}{5}, \cos(5x)\right\}$$

Since $\cos(5x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(5x), x\sin(5x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(5x) + A_2 x \sin(5x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-10A_1 \sin(5x) + 10A_2 \cos(5x) = \cos(5x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \sin(5x)}{10}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5}\right) + \left(\frac{x \sin(5x)}{10}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5} + \frac{x \sin(5x)}{10} \tag{1}
$$

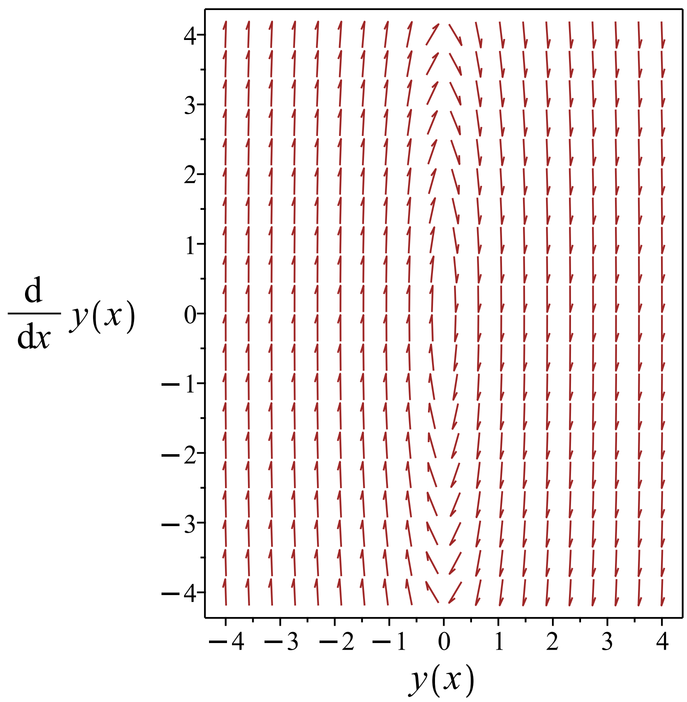


Figure 534: Slope field plot

Verification of solutions

$$
y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5} + \frac{x \sin(5x)}{10}
$$

Verified OK.

16.9.3 Maple step by step solution

Let's solve

$y'' + 25y = \cos(5x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (-5\,\mathrm{I}, 5\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(5x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(5x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(5x) + c_2 \sin(5x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(5x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(5x) & \sin(5x) \\ -5\sin(5x) & 5\cos(5x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 5$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(5x)\left(\int \sin(10x)dx\right)}{10} + \frac{\sin(5x)\left(\int \cos(5x)^2 dx\right)}{5}$$

- Compute integrals

$$y_p(x) = \frac{\cos(5x)}{100} + \frac{x\sin(5x)}{10}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(5x) + c_2 \sin(5x) + \frac{\cos(5x)}{100} + \frac{x\sin(5x)}{10}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+25*y(x)=cos(5*x),y(x), singsol=all)
```

$$y(x) = \frac{(50c_1 + 1)\cos(5x)}{50} + \frac{\sin(5x)(x + 10c_2)}{10}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 31

```
DSolve[y''[x]+25*y[x]==Cos[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{1}{100} + c_1\right)\cos(5x) + \frac{1}{10}(x + 10c_2)\sin(5x)$$

16.10 problem 483

Internal problem ID [15253]
Internal file name [`OUTPUT/15253_Wednesday_May_08_2024_03_54_12_PM_13540318/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 483.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = -\cos(x) + \sin(x)}$$

16.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = -\cos(x) + \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos(x) + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = -\cos(x) + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(x)}{2} - \frac{x\sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{x\cos(x)}{2} - \frac{x\sin(x)}{2}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - \frac{x\cos(x)}{2} - \frac{x\sin(x)}{2} \tag{1}
$$

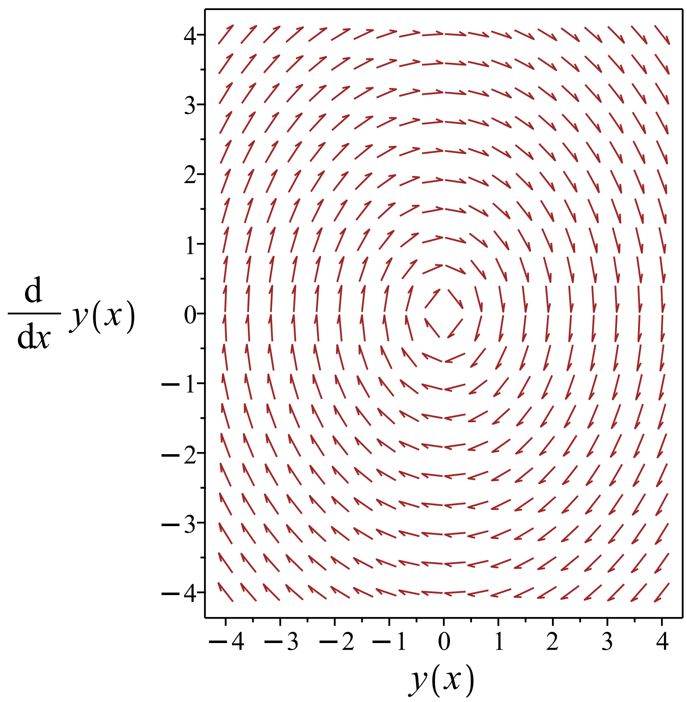


Figure 535: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - \frac{x\cos(x)}{2} - \frac{x\sin(x)}{2}
$$

Verified OK.

16.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 417: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-\cos(x) + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x) + A_2 x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = -\cos(x) + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(x)}{2} - \frac{x\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{x\cos(x)}{2} - \frac{x\sin(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \cos(x)}{2} - \frac{x \sin(x)}{2} \tag{1}$$

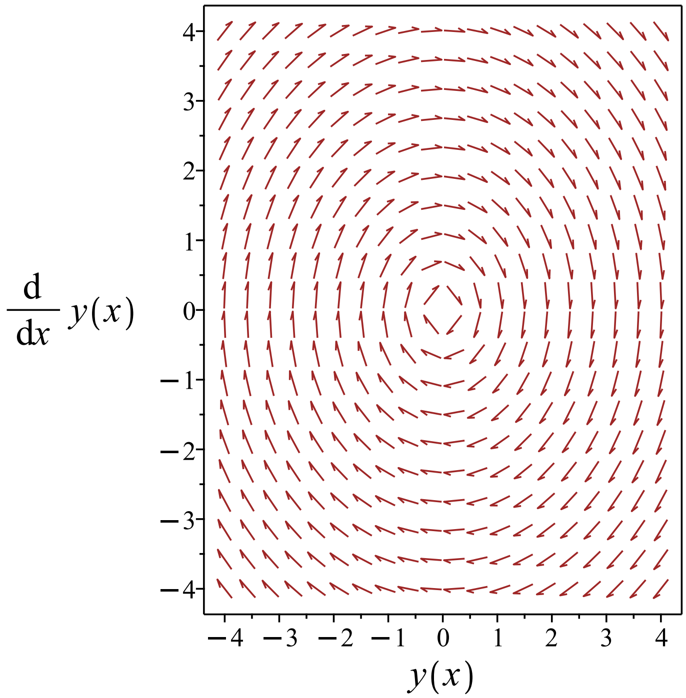


Figure 536: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \frac{x \cos(x)}{2} - \frac{x \sin(x)}{2}$$

Verified OK.

16.10.3 Maple step by step solution

Let's solve

$y'' + y = -\cos(x) + \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+1=0$

- Use quadratic formula to solve for r

$r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x)=\cos(x)$

- 2nd solution of the homogeneous ODE

$y_2(x)=\sin(x)$

- General solution of the ODE

$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y=c_1\cos(x)+c_2\sin(x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=-\cos(x)+\sin(x)\right]$

- Wronskian of solutions of the homogeneous equation

$W(y_1(x),y_2(x))=\begin{bmatrix}\cos(x)&\sin(x)\\-\sin(x)&\cos(x)\end{bmatrix}$

- Compute Wronskian

$W(y_1(x),y_2(x))=1$

- Substitute functions into equation for $y_p(x)$

$y_p(x)=\cos(x)\left(\int\sin(x)(\cos(x)-\sin(x))dx\right)-\sin(x)\left(\int\cos(x)(\cos(x)-\sin(x))dx\right)$

- Compute integrals

$y_p(x)=\frac{(-x-1)\cos(x)}{2}-\frac{x\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

$y=c_1\cos(x)+c_2\sin(x)+\frac{(-x-1)\cos(x)}{2}-\frac{x\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+y(x)=sin(x)-cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(2c_1 - x - 1)\cos(x)}{2} - \frac{\sin(x)(x - 2c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 31

```
DSolve[y''[x]+y[x]==Sin[x]-Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}(-((x + 1 - 2c_1)\cos(x)) - (x - 2c_2)\sin(x))$$

16.11 problem 484

Internal problem ID [15254]
Internal file name [`OUTPUT/15254_Wednesday_May_08_2024_03_54_13_PM_55399215/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 484.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = \sin(4x + \alpha)}$$

16.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 16, f(x) = \sin(4x + \alpha)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 16\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(16)} \\ &= \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4i \\ \lambda_2 &= -4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4i \\ \lambda_2 &= -4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(4x) + c_2 \sin(4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(4x) + c_2 \sin(4x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(4x) \\ y_2 &= \sin(4x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(4x) & \sin(4x) \\ \frac{d}{dx}(\cos(4x)) & \frac{d}{dx}(\sin(4x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{vmatrix}$$

Therefore

$$W = (\cos(4x))(4\cos(4x)) - (\sin(4x))(-4\sin(4x))$$

Which simplifies to

$$W = 4\cos(4x)^2 + 4\sin(4x)^2$$

Which simplifies to

$$W = 4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(4x)\sin(4x+\alpha)}{4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(4x)\sin(4x+\alpha)}{4}dx$$

Hence

$$u_1 = -\frac{x\cos(\alpha)}{8} + \frac{\sin(8x+\alpha)}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(4x)\sin(4x+\alpha)}{4}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(4x)\sin(4x+\alpha)}{4}dx$$

Hence

$$u_2 = \frac{x\sin(\alpha)}{8} - \frac{\cos(8x+\alpha)}{64}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x\cos(\alpha)}{8} + \frac{\sin(8x+\alpha)}{64}\right)\cos(4x) + \left(\frac{x\sin(\alpha)}{8} - \frac{\cos(8x+\alpha)}{64}\right)\sin(4x)$$

Which simplifies to

$$y_p(x) = -\frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(4x) + c_2 \sin(4x)) + \left(-\frac{x \cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(4x) + c_2 \sin(4x) - \frac{x \cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64} \tag{1}
$$

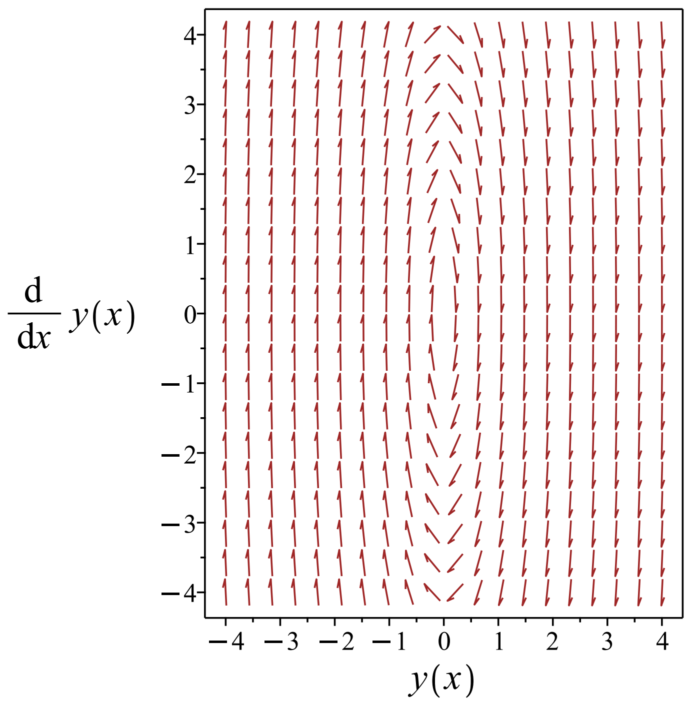


Figure 537: Slope field plot

Verification of solutions

$$
y = c_1 \cos(4x) + c_2 \sin(4x) - \frac{x \cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}
$$

Verified OK.

16.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 419: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(4x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(4x) \int \frac{1}{\cos(4x)^2} \, dx \\ &= \cos(4x) \left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(4x)) + c_2 \left(\cos(4x) \left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(4x)$$

$$y_2 = \frac{\sin(4x)}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(4x) & \frac{\sin(4x)}{4} \\ \frac{d}{dx}(\cos(4x)) & \frac{d}{dx}\left(\frac{\sin(4x)}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(4x) & \frac{\sin(4x)}{4} \\ -4\sin(4x) & \cos(4x) \end{vmatrix}$$

Therefore

$$W = (\cos(4x))(\cos(4x)) - \left(\frac{\sin(4x)}{4}\right)(-4\sin(4x))$$

Which simplifies to

$$W = \cos(4x)^2 + \sin(4x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(4x)\sin(4x+\alpha)}{4}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(4x)\sin(4x+\alpha)}{4} dx$$

Hence

$$u_1 = -\frac{x\cos(\alpha)}{8} + \frac{\sin(8x+\alpha)}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(4x)\sin(4x+\alpha)}{1} dx$$

Which simplifies to

$$u_2 = \int \cos(4x)\sin(4x+\alpha)\, dx$$

Hence

$$u_2 = \frac{x\sin(\alpha)}{2} - \frac{\cos(8x+\alpha)}{16}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x\cos(\alpha)}{8} + \frac{\sin(8x+\alpha)}{64}\right)\cos(4x) + \frac{\left(\frac{x\sin(\alpha)}{2} - \frac{\cos(8x+\alpha)}{16}\right)\sin(4x)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(4x) + \frac{c_2\sin(4x)}{4}\right) + \left(-\frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(4x) + \frac{c_2\sin(4x)}{4} - \frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64} \tag{1}$$

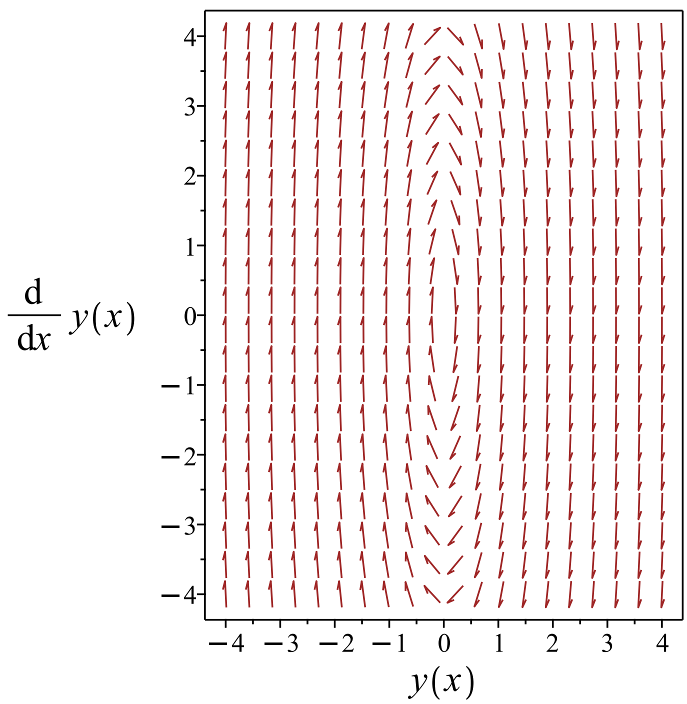


Figure 538: Slope field plot

Verification of solutions

$$y = c_1\cos(4x) + \frac{c_2\sin(4x)}{4} - \frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}$$

Verified OK.

16.11.3 Maple step by step solution

Let's solve

$y'' + 16y = \sin(4x + \alpha)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(4x + \alpha)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4$

- ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(4x)\left(\int \sin(4x)\sin(4x+\alpha)dx\right)}{4} + \frac{\sin(4x)\left(\int \cos(4x)\sin(4x+\alpha)dx\right)}{4}$$

- Compute integrals

$$y_p(x) = -\frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(4x) + c_2 \sin(4x) - \frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+16*y(x)=sin(4*x+alpha),y(x), singsol=all)
```

$$y(x) = \sin(4x) c_2 + \cos(4x) c_1 - \frac{x\cos(4x+\alpha)}{8} + \frac{\sin(4x+\alpha)}{64}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 41

```
DSolve[y''[x]+16*y[x]==Sin[4*x+\[Alpha]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64}\sin(\alpha+4x) - \frac{1}{8}x\cos(\alpha+4x) + c_1\cos(4x) + c_2\sin(4x)$$

16.12 problem 485

Internal problem ID [15255]
Internal file name [`OUTPUT/15255_Wednesday_May_08_2024_03_54_14_PM_24196050/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 485.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''+4y'+8y = e^{2x}(\sin(2x)+\cos(2x))}$$

16.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 8, f(x) = e^{2x}(\sin(2x)+\cos(2x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y''+4y'+8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x) = 0$$

Where in the above $A=1, B=4, C=8$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+4\lambda\,\mathrm{e}^{\lambda x}+8\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+4\lambda+8=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=4, C=8$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-4}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{4^2-(4)\,(1)\,(8)}\\&=-2\pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-2+2i\\\lambda_2&=-2-2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=-2+2i\\\lambda_2&=-2-2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=-2$ and $\beta=2$. Therefore the final solution, when using Euler relation, can be written as

$$y=e^{\alpha x}(c_1\cos(\beta x)+c_2\sin(\beta x))$$

Which becomes

$$y=e^{-2x}(c_1\cos\,(2x)+c_2\sin\,(2x))$$

Therefore the homogeneous solution y_h is

$$y_h=\mathrm{e}^{-2x}(c_1\cos\,(2x)+c_2\sin\,(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(\sin(2x) + \cos(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(2x), e^{2x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(2x), e^{-2x}\sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}\cos(2x) + A_2 e^{2x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-16A_1 e^{2x}\sin(2x) + 16A_2 e^{2x}\cos(2x) + 16A_1 e^{2x}\cos(2x) + 16A_2 e^{2x}\sin(2x)\\&= e^{2x}(\sin(2x) + \cos(2x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}\sin(2x)}{16}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(e^{-2x}(c_1\cos(2x) + c_2\sin(2x))\right) + \left(\frac{e^{2x}\sin(2x)}{16}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{e^{2x} \sin(2x)}{16} \tag{1}$$

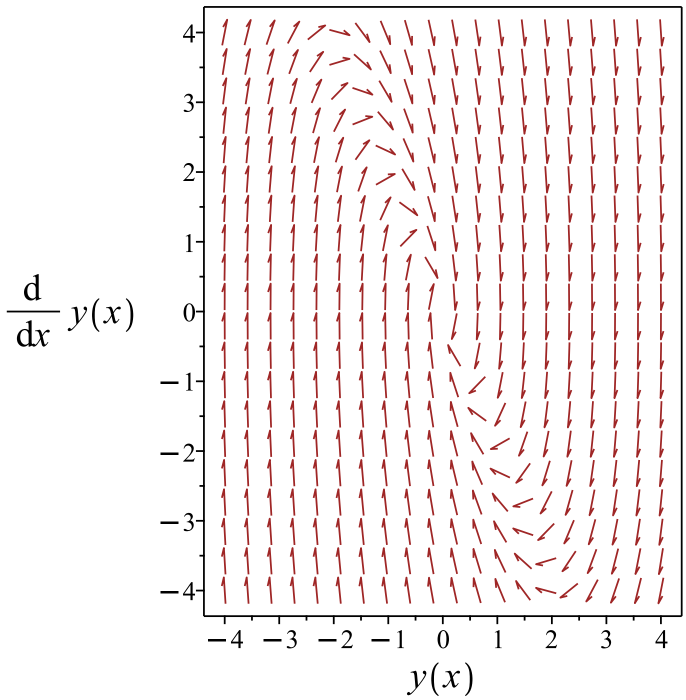


Figure 539: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{e^{2x} \sin(2x)}{16}$$

Verified OK.

16.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-2x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(2x)\, \mathrm{e}^{-2x} c_1 + \frac{\sin(2x)\, \mathrm{e}^{-2x} c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2x} (\sin(2x) + \cos(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(2x), e^{2x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2x}\cos(2x), \frac{e^{-2x}\sin(2x)}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}\cos(2x) + A_2 e^{2x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-16A_1 e^{2x}\sin(2x) + 16A_2 e^{2x}\cos(2x) + 16A_1 e^{2x}\cos(2x) + 16A_2 e^{2x}\sin(2x)\\&= e^{2x}(\sin(2x) + \cos(2x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}\sin(2x)}{16}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(\cos(2x)\,e^{-2x}c_1 + \frac{\sin(2x)\,e^{-2x}c_2}{2}\right) + \left(\frac{e^{2x}\sin(2x)}{16}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(2x)\,e^{-2x}c_1 + \frac{\sin(2x)\,e^{-2x}c_2}{2} + \frac{e^{2x}\sin(2x)}{16} \tag{1}$$

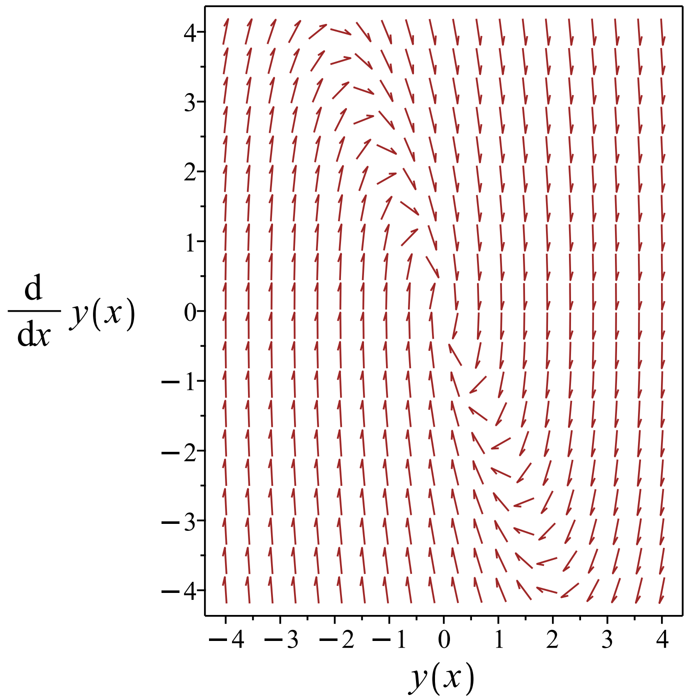


Figure 540: Slope field plot

Verification of solutions

$$y = \cos(2x)\,e^{-2x}c_1 + \frac{\sin(2x)\,e^{-2x}c_2}{2} + \frac{e^{2x}\sin(2x)}{16}$$

Verified OK.

16.12.3 Maple step by step solution

Let's solve

$y'' + 4y' + 8y = e^{2x}(\sin(2x) + \cos(2x))$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 8 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - 2\,\mathrm{I}, -2 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(2x)\,\mathrm{e}^{-2x} c_1 + \sin(2x)\,\mathrm{e}^{-2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{2x}(\sin(2x) + \cos(2x)\right)$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x}\cos(2x) & \mathrm{e}^{-2x}\sin(2x) \\ -2\,\mathrm{e}^{-2x}\cos(2x) - 2\,\mathrm{e}^{-2x}\sin(2x) & -2\,\mathrm{e}^{-2x}\sin(2x) + 2\,\mathrm{e}^{-2x}\cos(2x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^{-2x}\left(\cos(2x)\left(\int(-1+\cos(4x)-\sin(4x))\mathrm{e}^{4x} dx\right)+\sin(2x)\left(\int(1+\sin(4x)+\cos(4x))\mathrm{e}^{4x} dx\right)\right)}{4}$

  ◦ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{2x}\sin(2x)}{16}$

- Substitute particular solution into general solution to ODE

  $y = \sin(2x)\,\mathrm{e}^{-2x} c_2 + \cos(2x)\,\mathrm{e}^{-2x} c_1 + \frac{\mathrm{e}^{2x}\sin(2x)}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+8*y(x)=exp(2*x)*(sin(2*x)+cos(2*x)),y(x), singsol=all)
```

$$y(x) = \frac{(16c_2 \mathrm{e}^{-2x} + \mathrm{e}^{2x}) \sin(2x)}{16} + \mathrm{e}^{-2x} \cos(2x) c_1$$

✓ Solution by Mathematica

Time used: 0.287 (sec). Leaf size: 38

```
DSolve[y''[x]+4*y'[x]+8*y[x]==Exp[2*x]*(Sin[2*x]+Cos[2*x]),y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{16} e^{-2x} \left(16c_2 \cos(2x) + \left(e^{4x} + 16c_1\right) \sin(2x)\right)$$

16.13 problem 486

Internal problem ID [15256]
Internal file name [`OUTPUT/15256_Wednesday_May_08_2024_03_54_15_PM_73818280/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 486.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 8y = \mathrm{e}^{2x}(-\cos(2x) + \sin(2x))}$$

16.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 8, f(x) = \mathrm{e}^{2x}(-\cos(2x) + \sin(2x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 8\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(8)}\\ &= 2 \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = 2 + 2i$$
$$\lambda_2 = 2 - 2i$$

Which simplifies to

$$\lambda_1 = 2 + 2i$$
$$\lambda_2 = 2 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(-\cos(2x) + \sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(2x), e^{2x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(2x), e^{2x}\sin(2x)\}$$

Since $e^{2x}\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}\cos(2x), x\,e^{2x}\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x}\cos(2x) + A_2 x\,e^{2x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1e^{2x}\sin(2x) + 4A_2e^{2x}\cos(2x) = e^{2x}(-\cos(2x) + \sin(2x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{2x}\cos(2x)}{4} - \frac{x\,e^{2x}\sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{2x}(c_1\cos(2x) + c_2\sin(2x))\right) + \left(-\frac{x\,e^{2x}\cos(2x)}{4} - \frac{x\,e^{2x}\sin(2x)}{4}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{x e^{2x} \cos(2x)}{4} - \frac{x e^{2x} \sin(2x)}{4} \tag{1}$$

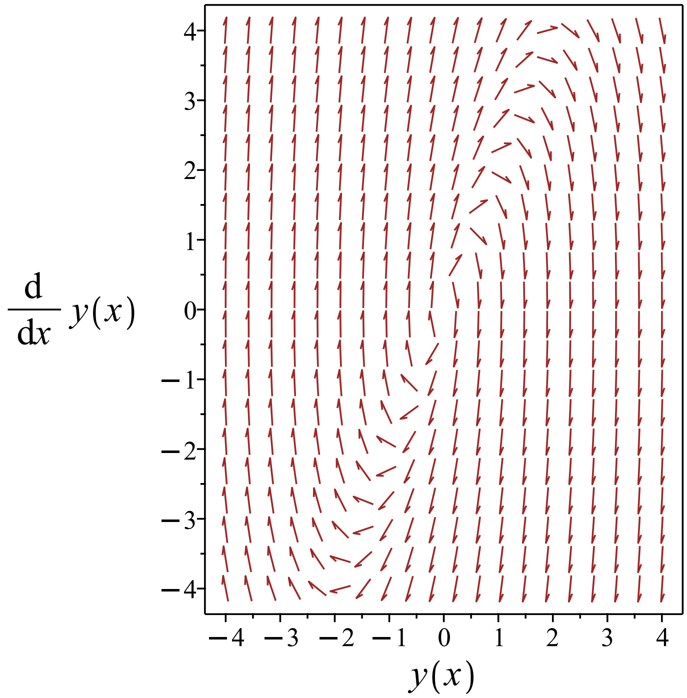


Figure 541: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{x e^{2x} \cos(2x)}{4} - \frac{x e^{2x} \sin(2x)}{4}$$

Verified OK.

16.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -4 \\
C &= 8
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 423: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{2x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{2x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{2x} \cos(2x)\, c_1 + \frac{\mathrm{e}^{2x} \sin(2x)\, c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-e^{2x}(\cos(2x) - \sin(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(2x), e^{2x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{2x}\cos(2x), \frac{e^{2x}\sin(2x)}{2}\right\}$$

Since $e^{2x}\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}\cos(2x), x\,e^{2x}\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x}\cos(2x) + A_2 x\,e^{2x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1e^{2x}\sin(2x) + 4A_2e^{2x}\cos(2x) = e^{2x}(-\cos(2x) + \sin(2x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{2x}\cos(2x)}{4} - \frac{x\,e^{2x}\sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}\cos(2x)\,c_1 + \frac{e^{2x}\sin(2x)\,c_2}{2}\right) + \left(-\frac{x\,e^{2x}\cos(2x)}{4} - \frac{x\,e^{2x}\sin(2x)}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x} \cos(2x) c_1 + \frac{e^{2x} \sin(2x) c_2}{2} - \frac{x e^{2x} \cos(2x)}{4} - \frac{x e^{2x} \sin(2x)}{4} \tag{1}$$

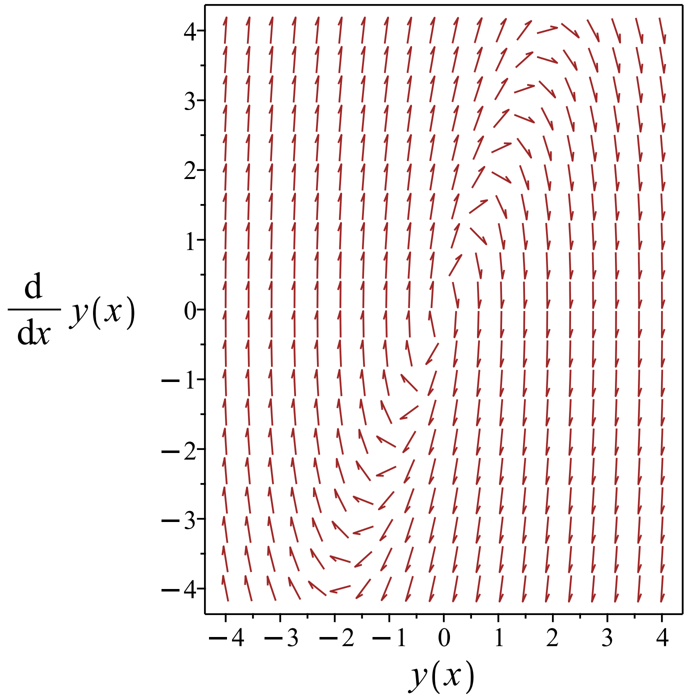


Figure 542: Slope field plot

Verification of solutions

$$y = e^{2x} \cos(2x) c_1 + \frac{e^{2x} \sin(2x) c_2}{2} - \frac{x e^{2x} \cos(2x)}{4} - \frac{x e^{2x} \sin(2x)}{4}$$

Verified OK.

16.13.3 Maple step by step solution

Let's solve

$y'' - 4y' + 8y = e^{2x}(-\cos(2x) + \sin(2x))$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -e^{2x} \cos(2x) + e^{2x} \sin(2x) + 4y' - 8y$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - 4y' + 8y = -e^{2x}(\cos(2x) - \sin(2x))$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 4r + 8 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{4 \pm (\sqrt{-16})}{2}$$

- Roots of the characteristic polynomial

$$r = (2 - 2\mathrm{I}, 2 + 2\mathrm{I})$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{2x} \cos(2x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{2x} \sin(2x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = e^{2x} \cos(2x) c_1 + e^{2x} \sin(2x) c_2 + y_p(x)$$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -e^{2x}(\cos(2x) - \sin(2x\right.$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} \cos(2x) & e^{2x} \sin(2x) \\ 2e^{2x} \cos(2x) - 2e^{2x} \sin(2x) & 2e^{2x} \sin(2x) + 2e^{2x} \cos(2x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2e^{4x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{2x}\left(\cos(2x)\left(\int(-\sin(4x)+1-\cos(4x))dx\right)-\sin(2x)\left(\int(-\cos(4x)-1+\sin(4x))dx\right)\right)}{4}$$

- Compute integrals

$$y_p(x) = -\frac{e^{2x}(-\sin(2x)+\cos(2x)+4x\cos(2x)+4x\sin(2x))}{16}$$

- Substitute particular solution into general solution to ODE

$$y = e^{2x}\cos(2x)\, c_1 + e^{2x}\sin(2x)\, c_2 - \frac{e^{2x}(-\sin(2x)+\cos(2x)+4x\cos(2x)+4x\sin(2x))}{16}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+8*y(x)=exp(2*x)*(sin(2*x)-cos(2*x)),y(x), singsol=all)
```

$$y(x) = -\frac{e^{2x}\left(\left(x - 4c_1 + \frac{1}{2}\right)\cos(2x) + \sin(2x)(x - 4c_2)\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.187 (sec). Leaf size: 41

```
DSolve[y''[x]-4*y'[x]+8*y[x]==Exp[2*x]*(Sin[2*x]-Cos[2*x]),y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -\frac{1}{8}e^{2x}((2x + 1 - 8c_2)\cos(2x) + 2(x - 4c_1)\sin(2x))$$

16.14 problem 487

Internal problem ID [15257]
Internal file name [`OUTPUT/15257_Wednesday_May_08_2024_03_54_17_PM_36949524/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 487.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 6y' + 13y = e^{-3x} \cos(2x)}$$

16.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 13, f(x) = e^{-3x} \cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 13 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(13)} \\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-3x} \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x} \cos(2x), e^{-3x} \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x} \cos(2x), e^{-3x} \sin(2x)\}$$

Since $e^{-3x} \cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{-3x} \cos(2x), x e^{-3x} \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{-3x} \cos(2x) + A_2 x e^{-3x} \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-3x} \sin(2x) + 4A_2 e^{-3x} \cos(2x) = e^{-3x} \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x e^{-3x} \sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(\frac{x e^{-3x} \sin(2x)}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{x e^{-3x} \sin(2x)}{4} \tag{1}$$

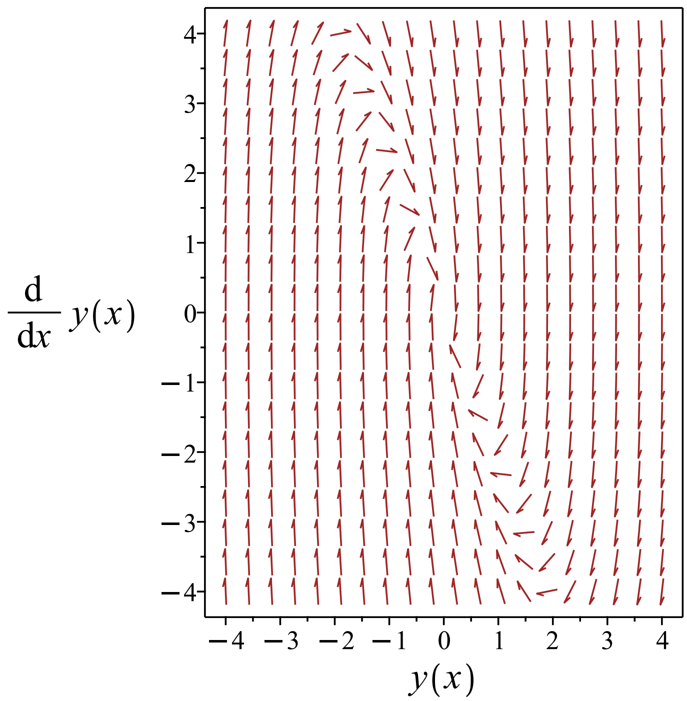


Figure 543: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{x e^{-3x} \sin(2x)}{4}$$

Verified OK.

16.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 13y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 6 \\
C &= 13
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{6}{1} \, dx} \\ &= z_1 e^{-3x} \\ &= z_1 \left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 13y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-3x} \cos(2x)\, c_1 + \frac{\mathrm{e}^{-3x} \sin(2x)\, c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-3x} \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\cos(2x), e^{-3x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-3x}\cos(2x), \frac{e^{-3x}\sin(2x)}{2}\right\}$$

Since $e^{-3x}\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-3x}\cos(2x), x\,e^{-3x}\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-3x}\cos(2x) + A_2 x\,e^{-3x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-3x}\sin(2x) + 4A_2 e^{-3x}\cos(2x) = e^{-3x}\cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^{-3x}\sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3x}\cos(2x)\,c_1 + \frac{e^{-3x}\sin(2x)\,c_2}{2}\right) + \left(\frac{x\,e^{-3x}\sin(2x)}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-3x} \cos(2x) c_1 + \frac{e^{-3x} \sin(2x) c_2}{2} + \frac{x e^{-3x} \sin(2x)}{4} \tag{1}$$

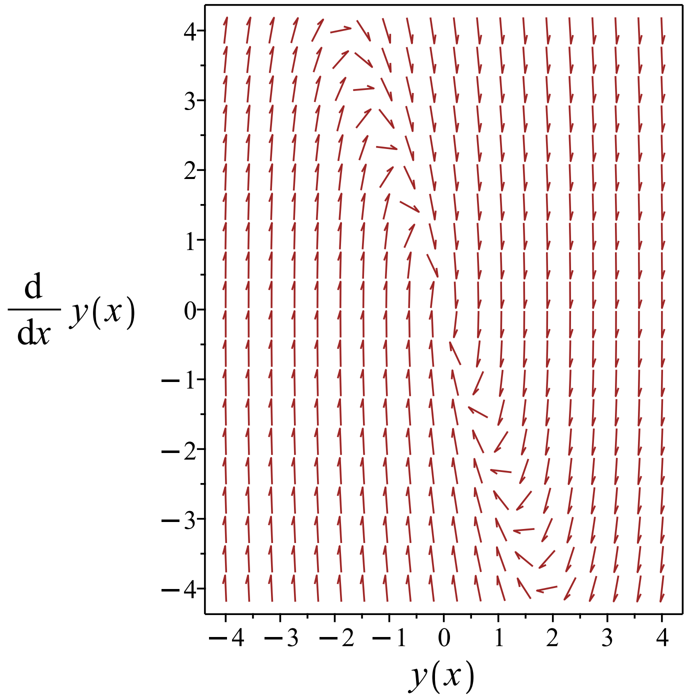


Figure 544: Slope field plot

Verification of solutions

$$y = e^{-3x} \cos(2x) c_1 + \frac{e^{-3x} \sin(2x) c_2}{2} + \frac{x e^{-3x} \sin(2x)}{4}$$

Verified OK.

16.14.3 Maple step by step solution

Let's solve

$y'' + 6y' + 13y = e^{-3x} \cos(2x)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-3x} \cos(2x) c_1 + \mathrm{e}^{-3x} \sin(2x) c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{-3x}\cos(2x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x}\cos(2x) & \mathrm{e}^{-3x}\sin(2x) \\ -3\,\mathrm{e}^{-3x}\cos(2x) - 2\,\mathrm{e}^{-3x}\sin(2x) & -3\,\mathrm{e}^{-3x}\sin(2x) + 2\,\mathrm{e}^{-3x}\cos(2x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-6x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-3x}\left(\cos(2x)(\int \sin(4x)dx) - 2\sin(2x)\left(\int \cos(2x)^2 dx\right)\right)}{4}$

  ○ Compute integrals

  $y_p(x) = \frac{(4x\sin(2x)+\cos(2x))\mathrm{e}^{-3x}}{16}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-3x}\cos(2x) c_1 + \mathrm{e}^{-3x}\sin(2x) c_2 + \frac{(4x\sin(2x)+\cos(2x))\mathrm{e}^{-3x}}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+13*y(x)=exp(-3*x)*cos(2*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-3x}\left(\sin(2x)(x+4c_2)+4\cos(2x)\left(c_1+\frac{1}{8}\right)\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 38

```
DSolve[y''[x]+6*y'[x]+13*y[x]==Exp[-3*x]*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}e^{-3x}((1+16c_2)\cos(2x)+4(x+4c_1)\sin(2x))$$

16.15 problem 488

Internal problem ID [15258]
Internal file name [`OUTPUT/15258_Wednesday_May_08_2024_03_54_18_PM_39573394/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 488.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + k^2 y = k \sin(kx + \alpha)}$$

16.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = k^2, f(x) = k \sin(kx + \alpha)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + k^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=k^2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + k^2 \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$k^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=k^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(k^2)} \\ &= \pm\sqrt{-k^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-k^2} \\ \lambda_2 &= -\sqrt{-k^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-k^2} \\ \lambda_2 &= -\sqrt{-k^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{-k^2}\right)x} + c_2 e^{\left(-\sqrt{-k^2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-k^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-k^2}\,x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\sqrt{-k^2}\,x} + c_2 \mathrm{e}^{-\sqrt{-k^2}\,x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{\sqrt{-k^2}\, x} \\ y_2 &= \mathrm{e}^{-\sqrt{-k^2}\, x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-k^2}\, x} & \mathrm{e}^{-\sqrt{-k^2}\, x} \\ \frac{d}{dx}\left(\mathrm{e}^{\sqrt{-k^2}\, x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-\sqrt{-k^2}\, x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-k^2}\, x} & \mathrm{e}^{-\sqrt{-k^2}\, x} \\ \sqrt{-k^2}\, \mathrm{e}^{\sqrt{-k^2}\, x} & -\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-k^2}\, x}\right)\left(-\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x}\right) - \left(\mathrm{e}^{-\sqrt{-k^2}\, x}\right)\left(\sqrt{-k^2}\, \mathrm{e}^{\sqrt{-k^2}\, x}\right)$$

Which simplifies to

$$W = -2\, \mathrm{e}^{\sqrt{-k^2}\, x} \sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x}$$

Which simplifies to

$$W = -2\sqrt{-k^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\sqrt{-k^2}\, x} k \sin(kx+\alpha)}{-2\sqrt{-k^2}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\sqrt{-k^2}\, x} k \sin(kx+\alpha)}{2\sqrt{-k^2}} dx$$

Hence

$$u_1 = -\frac{-\frac{x\, e^{-\sqrt{-k^2}\, x}}{4} - \frac{x\, e^{-\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{4} - \frac{\sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{4k^2} + \frac{\sqrt{-k^2}\, x\, e^{-\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)}{2k} + \frac{\sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{4k^2}}{1+\tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-k^2}\, x} k \sin(kx+\alpha)}{-2\sqrt{-k^2}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{\sqrt{-k^2}\, x} k \sin(kx+\alpha)}{2\sqrt{-k^2}} dx$$

Hence

$$u_2 = \frac{-\frac{x\, e^{\sqrt{-k^2}\, x}}{4} + \frac{x\, e^{\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{4} - \frac{\sqrt{-k^2}\, e^{\sqrt{-k^2}\, x}}{4k^2} + \frac{\sqrt{-k^2}\, x\, e^{\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)}{2k} + \frac{\sqrt{-k^2}\, e^{\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{4k^2}}{1+\tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}$$

Which simplifies to

$$u_1 = -\frac{\left(\left(\sin(kx+\alpha)\, kx - \cos(kx+\alpha)\right)\sqrt{-k^2} + x\, k^2 \cos(kx+\alpha)\right) e^{-\sqrt{-k^2}\, x}}{4k^2}$$

$$u_2 = -\frac{e^{\sqrt{-k^2}\, x}\left(\left(-\sin(kx+\alpha)\, kx + \cos(kx+\alpha)\right)\sqrt{-k^2} + x\, k^2 \cos(kx+\alpha)\right)}{4k^2}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
& y_p(x) \\
& = -\frac{\left(\left(\sin \left(k x+\alpha\right) k x-\cos \left(k x+\alpha\right)\right) \sqrt{-k^2}+x\, k^2 \cos \left(k x+\alpha\right)\right) \mathrm{e}^{-\sqrt{-k^2}\, x} \mathrm{e}^{\sqrt{-k^2}\, x}}{4 k^2} \\
& \quad -\frac{\mathrm{e}^{\sqrt{-k^2}\, x}\left(\left(-\sin \left(k x+\alpha\right) k x+\cos \left(k x+\alpha\right)\right) \sqrt{-k^2}+x\, k^2 \cos \left(k x+\alpha\right)\right) \mathrm{e}^{-\sqrt{-k^2}\, x}}{4 k^2}
\end{aligned}
$$

Which simplifies to

$$
y_p(x)=-\frac{x \cos \left(k x+\alpha\right)}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y & =y_h+y_p \\
& =\left(c_1 \mathrm{e}^{\sqrt{-k^2}\, x}+c_2 \mathrm{e}^{-\sqrt{-k^2}\, x}\right)+\left(-\frac{x \cos \left(k x+\alpha\right)}{2}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y=c_1 \mathrm{e}^{\sqrt{-k^2}\, x}+c_2 \mathrm{e}^{-\sqrt{-k^2}\, x}-\frac{x \cos \left(k x+\alpha\right)}{2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y=c_1 \mathrm{e}^{\sqrt{-k^2}\, x}+c_2 \mathrm{e}^{-\sqrt{-k^2}\, x}-\frac{x \cos \left(k x+\alpha\right)}{2}
$$

Verified OK.

16.15.2 Solving using Kovacic algorithm

Writing the ode as

$$
y''+k^2 y=0 \tag{1}
$$

$$
A y''+B y'+C y=0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
& A=1 \\
& B=0 \\
& C=k^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-k^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -k^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-k^2)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -k^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-k^2}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \mathrm{e}^{\sqrt{-k^2}\,x}\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-k^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{-k^2}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-k^2}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{-k^2}\, x} \left(\frac{\sqrt{-k^2}\, \mathrm{e}^{-2\sqrt{-k^2}\, x}}{2k^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{-k^2}\, x} \right) + c_2 \left(\mathrm{e}^{\sqrt{-k^2}\, x} \left(\frac{\sqrt{-k^2}\, \mathrm{e}^{-2\sqrt{-k^2}\, x}}{2k^2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + k^2 y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{\sqrt{-k^2}\, x} + \frac{c_2 \sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x}}{2k^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{\sqrt{-k^2}\,x}$$
$$y_2 = \frac{\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-k^2}\,x} & \frac{\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2} \\ \frac{d}{dx}\left(\mathrm{e}^{\sqrt{-k^2}\,x}\right) & \frac{d}{dx}\left(\frac{\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-k^2}\,x} & \frac{\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2} \\ \sqrt{-k^2}\,\mathrm{e}^{\sqrt{-k^2}\,x} & \frac{\mathrm{e}^{-\sqrt{-k^2}\,x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-k^2}\,x}\right)\left(\frac{\mathrm{e}^{-\sqrt{-k^2}\,x}}{2}\right) - \left(\frac{\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2}\right)\left(\sqrt{-k^2}\,\mathrm{e}^{\sqrt{-k^2}\,x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{\sqrt{-k^2}\, x} \mathrm{e}^{-\sqrt{-k^2}\, x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x} \sin(kx+\alpha)}{2k}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x} \sin(kx+\alpha)}{2k} dx$$

Hence

$$u_1 = -\frac{\frac{x\,\mathrm{e}^{-\sqrt{-k^2}\, x}}{4} - \frac{x\,\mathrm{e}^{-\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{4} - \frac{\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x}}{4k^2} + \frac{\sqrt{-k^2}\, x\, \mathrm{e}^{-\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)}{2k} + \frac{\sqrt{-k^2}\, \mathrm{e}^{-\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{4k^2}}{1+\tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\sqrt{-k^2}\, x} k \sin(kx+\alpha)}{1}\, dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{\sqrt{-k^2}\, x} k \sin(kx+\alpha)\, dx$$

Hence

$$u_2 = \frac{xk\,\mathrm{e}^{\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right) - \frac{\sqrt{-k^2}\, \mathrm{e}^{\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)}{k} + \frac{\sqrt{-k^2}\, x\, \mathrm{e}^{\sqrt{-k^2}\, x}}{2} - \frac{\sqrt{-k^2}\, x\, \mathrm{e}^{\sqrt{-k^2}\, x} \tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}{2}}{1+\tan\left(\frac{kx}{2}+\frac{\alpha}{2}\right)^2}$$

Which simplifies to

$$u_1 = -\frac{\left(\left(\sin\left(kx+\alpha\right)kx-\cos\left(kx+\alpha\right)\right)\sqrt{-k^2}+x\,k^2\cos\left(kx+\alpha\right)\right)\mathrm{e}^{-\sqrt{-k^2}\,x}}{4k^2}$$

$$u_2 = \frac{\mathrm{e}^{\sqrt{-k^2}\,x}\left(\left(\cos\left(kx+\alpha\right)kx-\sin\left(kx+\alpha\right)\right)\sqrt{-k^2}+k^2x\sin\left(kx+\alpha\right)\right)}{2k}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) \\ &= -\frac{\left(\left(\sin\left(kx+\alpha\right)kx-\cos\left(kx+\alpha\right)\right)\sqrt{-k^2}+x\,k^2\cos\left(kx+\alpha\right)\right)\mathrm{e}^{-\sqrt{-k^2}\,x}\mathrm{e}^{\sqrt{-k^2}\,x}}{4k^2} \\ &+\frac{\mathrm{e}^{\sqrt{-k^2}\,x}\left(\left(\cos\left(kx+\alpha\right)kx-\sin\left(kx+\alpha\right)\right)\sqrt{-k^2}+k^2x\sin\left(kx+\alpha\right)\right)\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{4k^3} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{-2x\,k^2\cos\left(kx+\alpha\right)+\sqrt{-k^2}\,\cos\left(kx+\alpha\right)+k\sin\left(kx+\alpha\right)}{4k^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{\sqrt{-k^2}\,x}+\frac{c_2\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2}\right) \\ &\quad+\left(\frac{-2x\,k^2\cos\left(kx+\alpha\right)+\sqrt{-k^2}\,\cos\left(kx+\alpha\right)+k\sin\left(kx+\alpha\right)}{4k^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1\mathrm{e}^{\sqrt{-k^2}\,x}+\frac{c_2\sqrt{-k^2}\,\mathrm{e}^{-\sqrt{-k^2}\,x}}{2k^2} \\ +\frac{-2x\,k^2\cos\left(kx+\alpha\right)+\sqrt{-k^2}\,\cos\left(kx+\alpha\right)+k\sin\left(kx+\alpha\right)}{4k^2} \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\sqrt{-k^2}\, x} + \frac{c_2 \sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{2k^2} + \frac{-2x\, k^2 \cos(kx+\alpha) + \sqrt{-k^2}\, \cos(kx+\alpha) + k \sin(kx+\alpha)}{4k^2}$$

Verified OK.

16.15.3 Maple step by step solution

Let's solve

$y'' + k^2 y = k \sin(kx+\alpha)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $k^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4k^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-k^2}, -\sqrt{-k^2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\sqrt{-k^2}\, x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\sqrt{-k^2}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = k \sin(kx+\alpha)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\sqrt{-k^2}\,x} & e^{-\sqrt{-k^2}\,x} \\ \sqrt{-k^2}\, e^{\sqrt{-k^2}\,x} & -\sqrt{-k^2}\, e^{-\sqrt{-k^2}\,x} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = -2\sqrt{-k^2}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = \frac{k\left(e^{\sqrt{-k^2}\,x}\left(\int e^{-\sqrt{-k^2}\,x}\sin(kx+\alpha)dx\right) - e^{-\sqrt{-k^2}\,x}\left(\int e^{\sqrt{-k^2}\,x}\sin(kx+\alpha)dx\right)\right)}{2\sqrt{-k^2}}$$

- Compute integrals

  $$y_p(x) = \frac{-\cos(kx+\alpha)kx + \sin(kx+\alpha)}{2k}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{\sqrt{-k^2}\,x} + c_2 e^{-\sqrt{-k^2}\,x} + \frac{-\cos(kx+\alpha)kx + \sin(kx+\alpha)}{2k}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+k^2*y(x)=k*sin(k*x+alpha),y(x), singsol=all)
```

$$y(x) = \frac{-2kx\cos(kx+\alpha) + 4\sin(kx)\, c_2 k + 4\cos(kx)\, c_1 k + \sin(kx+\alpha)}{4k}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 44

```
DSolve[y''[x]+k^2*y[x]==k*Sin[k*x+\[Alpha]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(\alpha + kx)}{4k} - \frac{1}{2}x\cos(\alpha + kx) + c_1\cos(kx) + c_2\sin(kx)$$

16.16 problem 489

Internal problem ID [15259]
Internal file name [OUTPUT/15259_Wednesday_May_08_2024_03_54_20_PM_46226927/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 489.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + k^2 y = k}$$

16.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = k^2, f(x) = k$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + k^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = k^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + k^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$k^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = k^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(k^2)}\\ &= \pm\sqrt{-k^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-k^2}\\ \lambda_2 &= -\sqrt{-k^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-k^2}\\ \lambda_2 &= -\sqrt{-k^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(\sqrt{-k^2}\right)x} + c_2 e^{\left(-\sqrt{-k^2}\right)x}\end{aligned}$$

Or

$$y = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\sqrt{-k^2}\, x}, e^{-\sqrt{-k^2}\, x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$k^2 A_1 = k$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{k}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{k}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x}\right) + \left(\frac{1}{k}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x} + \frac{1}{k} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x} + \frac{1}{k}$$

Verified OK.

16.16.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + k^2 y'y - y'k = 0$$

Integrating the above w.r.t x gives

$$\int \left(y'y'' + k^2 y'y - y'k\right) dx = 0$$
$$\frac{y'^2}{2} + \frac{k^2 y^2}{2} - yk = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-k^2 y^2 + 2yk + 2c_1} \tag{1}$$
$$y' = -\sqrt{-k^2 y^2 + 2yk + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-k^2 y^2 + 2ky + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{k^2}\,\left(y - \frac{1}{k}\right)}{\sqrt{-k^2 y^2 + 2yk + 2c_1}}\right)}{\sqrt{k^2}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-k^2y^2+2ky+2c_1}}dy = \int dx$$

$$-\frac{\arctan\left(\frac{\sqrt{k^2}\left(y-\frac{1}{k}\right)}{\sqrt{-k^2y^2+2yk+2c_1}}\right)}{\sqrt{k^2}} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{\sqrt{k^2}\left(y-\frac{1}{k}\right)}{\sqrt{-k^2y^2+2yk+2c_1}}\right)}{\sqrt{k^2}} = x + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{\sqrt{k^2}\left(y-\frac{1}{k}\right)}{\sqrt{-k^2y^2+2yk+2c_1}}\right)}{\sqrt{k^2}} = x + c_3 \tag{2}$$

Verification of solutions

$$\frac{\arctan\left(\frac{\sqrt{k^2}\left(y-\frac{1}{k}\right)}{\sqrt{-k^2y^2+2yk+2c_1}}\right)}{\sqrt{k^2}} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{\sqrt{k^2}\left(y-\frac{1}{k}\right)}{\sqrt{-k^2y^2+2yk+2c_1}}\right)}{\sqrt{k^2}} = x + c_3$$

Verified OK.

16.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + k^2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= k^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-k^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -k^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-k^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -k^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-k^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-k^2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{\sqrt{-k^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= e^{\sqrt{-k^2}\, x} \int \frac{1}{e^{2\sqrt{-k^2}\, x}}\, dx \\
&= e^{\sqrt{-k^2}\, x} \left(\frac{\sqrt{-k^2}\, e^{-2\sqrt{-k^2}\, x}}{2k^2} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{\sqrt{-k^2}\, x} \right) + c_2 \left(e^{\sqrt{-k^2}\, x} \left(\frac{\sqrt{-k^2}\, e^{-2\sqrt{-k^2}\, x}}{2k^2} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + k^2 y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\sqrt{-k^2}\, x} + \frac{c_2 \sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{2k^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{2k^2}, e^{\sqrt{-k^2}\, x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$k^2 A_1 = k$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{k}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{k}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-k^2}\, x} + \frac{c_2 \sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{2k^2}\right) + \left(\frac{1}{k}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\sqrt{-k^2}\, x} + \frac{c_2 \sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{2k^2} + \frac{1}{k} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\sqrt{-k^2}\, x} + \frac{c_2 \sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x}}{2k^2} + \frac{1}{k}$$

Verified OK.

16.16.4 Maple step by step solution

Let's solve

$y'' + k^2 y = k$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $k^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4k^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-k^2}, -\sqrt{-k^2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\sqrt{-k^2}\, x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\sqrt{-k^2}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = k\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\sqrt{-k^2}\, x} & e^{-\sqrt{-k^2}\, x} \\ \sqrt{-k^2}\, e^{\sqrt{-k^2}\, x} & -\sqrt{-k^2}\, e^{-\sqrt{-k^2}\, x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = -2\sqrt{-k^2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{k\left(e^{\sqrt{-k^2}\, x}\left(\int e^{-\sqrt{-k^2}\, x} dx\right) - e^{-\sqrt{-k^2}\, x}\left(\int e^{\sqrt{-k^2}\, x} dx\right)\right)}{2\sqrt{-k^2}}$$

- Compute integrals

$$y_p(x) = \frac{1}{k}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{\sqrt{-k^2}\, x} + c_2 e^{-\sqrt{-k^2}\, x} + \frac{1}{k}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+k^2*y(x)=k,y(x), singsol=all)
```

$$y(x) = \sin(kx)\, c_2 + \cos(kx)\, c_1 + \frac{1}{k}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 23

```
DSolve[y''[x]+k^2*y[x]==k,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(kx) + c_2 \sin(kx) + \frac{1}{k}$$

16.17 problem 490

Internal problem ID [15260]
Internal file name [OUTPUT/15260_Wednesday_May_08_2024_03_54_21_PM_11627360/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 490.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' + y = x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y = 0$$

The characteristic equation is

$$\lambda^3 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' + y = x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}, \mathrm{e}^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3\right) + (x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + x \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + x$$

Verified OK.

16.17.1 Maple step by step solution

Let's solve

$y''' + y = x$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x - y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x - y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ x \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ x \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{x}{2}} \cdot \left(\cos\left(\frac{x\sqrt{3}}{2}\right) - \mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)-\mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)-\mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right)}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) - \mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}, \vec{y}_3(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ -\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & e^{\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ -e^{-x} & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{-x} & e^{\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -e^{\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \phi(0)^{-1}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & e^{\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ -e^{-x} & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{-x} & e^{\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -e^{\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2e^{\frac{3x}{2}}e^{-x}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-x}}{3} & \frac{\left(e^{\frac{3x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}+e^{\frac{3x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)-1\right)e^{-x}}{3} & \frac{\left(e^{\frac{3x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{}\right.}{} \\ -\frac{\left(e^{\frac{3x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}-e^{\frac{3x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)+1\right)e^{-x}}{3} & \frac{2e^{\frac{3x}{2}}e^{-x}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-x}}{3} & \frac{\left(e^{\frac{3x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{}\right.}{} \\ -\frac{\left(e^{\frac{3x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}+e^{\frac{3x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)-1\right)e^{-x}}{3} & -\frac{\left(e^{\frac{3x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}-e^{\frac{3x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)+1\right)e^{-x}}{3} & \frac{2e^{\frac{3x}{2}}e^{-x}\cdots}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \Phi(x)^{-1} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{e^{-x}\left(-e^{\frac{3x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}+3x\, e^{x}-e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)+1\right)}{3} \\ \frac{e^{-x}\left(-2\, e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)+3\, e^{x}-1\right)}{3} \\ \frac{\left(e^{\frac{3x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}-e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)+1\right)e^{-x}}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{e^{-x}\left(-e^{\frac{3x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}+3x\, e^{x}-e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)+1\right)}{3} \\ \frac{e^{-x}\left(-2\, e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)+3\, e^{x}-1\right)}{3} \\ \frac{\left(e^{\frac{3x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}-e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)+1\right)e^{-x}}{3} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = e^{-x}\left(-\frac{e^{\frac{3x}{2}}\left(-c_3\sqrt{3}+c_2+\frac{2}{3}\right)\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{e^{\frac{3x}{2}}\left(\left(c_2-\frac{2}{3}\right)\sqrt{3}+c_3\right)\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} + x\, e^{x} + c_1 + \frac{1}{3}\right)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)+y(x)=x,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) + c_3 e^{\frac{3x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) + e^x x + c_1\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 57

```
DSolve[y'''[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^{-x} + c_3 e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

16.18 problem 491

Internal problem ID [15261]
Internal file name [`OUTPUT/15261_Wednesday_May_08_2024_03_54_24_PM_81871046/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 491.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 6y'' + 11y' + 6y = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 6y'' + 11y' + 6y = 0$$

The characteristic equation is

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{-2x} c_2 + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{-3x}$$

Now the particular solution to the given ODE is found

$$y''' + 6y'' + 11y' + 6y = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + e^{-2x} c_2 + e^{-3x} c_3\right) + \left(\frac{1}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + e^{-2x} c_2 + e^{-3x} c_3 + \frac{1}{6} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{-2x} c_2 + e^{-3x} c_3 + \frac{1}{6}$$

Verified OK.

16.18.1 Maple step by step solution

Let's solve

$y''' + 6y'' + 11y' + 6y = 1$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 1 - 6y_3(x) - 11y_2(x) - 6y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 1 - 6y_3(x) - 11y_2(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^{-3x} \cdot \left[\begin{array}{c} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{-2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & e^{-x} \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & -e^{-x} \\ e^{-3x} & e^{-2x} & e^{-x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \phi(0)^{-1}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & e^{-x} \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & -e^{-x} \\ e^{-3x} & e^{-2x} & e^{-x} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{9} & \frac{1}{4} & 1 \\ -\frac{1}{3} & -\frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} e^{-3x} - 3e^{-2x} + 3e^{-x} & \frac{3e^{-3x}}{2} - 4e^{-2x} + \frac{5e^{-x}}{2} & \frac{e^{-3x}}{2} - e^{-2x} + \frac{e^{-x}}{2} \\ -3e^{-3x} + 6e^{-2x} - 3e^{-x} & -\frac{9e^{-3x}}{2} + 8e^{-2x} - \frac{5e^{-x}}{2} & -\frac{3e^{-3x}}{2} + 2e^{-2x} - \frac{e^{-x}}{2} \\ 9e^{-3x} - 12e^{-2x} + 3e^{-x} & \frac{27e^{-3x}}{2} - 16e^{-2x} + \frac{5e^{-x}}{2} & \frac{9e^{-3x}}{2} - 4e^{-2x} + \frac{e^{-x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \Phi(x)^{-1} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

$\vec{v}(x) = \int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{e^{-x}}{2} + \frac{e^{-2x}}{2} - \frac{e^{-3x}}{6} + \frac{1}{6} \\ \frac{e^{-3x}}{2} - e^{-2x} + \frac{e^{-x}}{2} \\ -\frac{3\,e^{-3x}}{2} + 2\,e^{-2x} - \frac{e^{-x}}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} -\frac{e^{-x}}{2} + \frac{e^{-2x}}{2} - \frac{e^{-3x}}{6} + \frac{1}{6} \\ \frac{e^{-3x}}{2} - e^{-2x} + \frac{e^{-x}}{2} \\ -\frac{3\,e^{-3x}}{2} + 2\,e^{-2x} - \frac{e^{-x}}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{1}{6} + \frac{(-3+2c_1)e^{-3x}}{18} + \frac{(2+c_2)e^{-2x}}{4} + \frac{(-1+2c_3)e^{-x}}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$3)+6*diff(y(x),x$2)+11*diff(y(x),x)+6*y(x)=1,y(x), singsol=all)
```

$$y(x) = \frac{1}{6} + c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 33

```
DSolve[y'''[x]+6*y''[x]+11*y'[x]+6*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^{-x} + \frac{1}{6}$$

16.19 problem 492

Internal problem ID [15262]
Internal file name [`OUTPUT/15262_Wednesday_May_08_2024_03_54_25_PM_66393549/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 492.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y' = 2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{ix}c_2 + e^{-ix}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{ix}$$
$$y_3 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y''' + y' = 2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{ix}, e^{-ix}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{ix} c_2 + e^{-ix} c_3\right) + (2x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + e^{ix} c_2 + e^{-ix} c_3 + 2x \tag{1}$$

Verification of solutions

$$y = c_1 + e^{ix} c_2 + e^{-ix} c_3 + 2x$$

Verified OK.

16.19.1 Maple step by step solution

Let's solve

$y''' + y' = 2$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 2 - y_2(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2 - y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \phi(0)^{-1}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} 1 & \sin(x) & 1-\cos(x) \\ 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

    $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - Cancel like terms

    $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

  - Multiply by the inverse of the fundamental matrix

$\vec{v}'(x) = \Phi(x)^{-1} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \begin{bmatrix} 2x - 2\sin(x) \\ 2 - 2\cos(x) \\ 2\sin(x) \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \begin{bmatrix} 2x - 2\sin(x) \\ 2 - 2\cos(x) \\ 2\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = (c_3 - 2)\sin(x) - c_2\cos(x) + 2x + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -_b(_a)+2, _b(_a)`   *** Subl
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$3)+diff(y(x),x)=2,y(x), singsol=all)
```

$$y(x) = \sin(x) c_1 - \cos(x) c_2 + 2x + c_3$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 22

```
DSolve[y'''[x]+y'[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x - c_2 \cos(x) + c_1 \sin(x) + c_3$$

16.20 problem 493

Internal problem ID [15263]
Internal file name [`OUTPUT/15263_Wednesday_May_08_2024_03_54_26_PM_99242600/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 493.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' = 3}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= x
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + y'' = 3
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x, \mathrm{e}^{-x}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\}]
$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x^2
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 = 3
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 + c_3 x) + \left(\frac{3x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + c_3 x + \frac{3x^2}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + c_3 x + \frac{3x^2}{2}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+3, _b(_a)`   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)=3,y(x), singsol=all)
```

$$y(x) = \frac{3x^2}{2} + c_1 e^{-x} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 27

```
DSolve[y'''[x]+y''[x]==3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x^2}{2} + c_3 x + c_1 e^{-x} + c_2$$

16.21 problem 494

Internal problem ID [15264]
Internal file name [OUTPUT/15264_Wednesday_May_08_2024_03_54_26_PM_77979546/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 494.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{x} c_2 + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - y = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{ix}, e^{-x}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4\right) + (-1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4 - 1 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2 + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4 - 1$$

Verified OK.

16.21.1 Maple step by step solution

Let's solve

$y'''' - y = 1$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

$y_4'(x) = 1 + y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 1 + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}-1\\1\\-1\\1\end{array}\right]\right], \left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c}-\mathrm{I}\\-1\\\mathrm{I}\\1\end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c}\mathrm{I}\\-1\\-\mathrm{I}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}-1\\1\\-1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}-1\\1\\-1\\1\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

☐ Fundamental matrix

  ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \phi(0)^{-1}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\cos(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(x) = \Phi(x)^{-1} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\cos(x)}{2} + \frac{e^x}{4} - 1 + \frac{e^{-x}}{4} \\ -\frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\sin(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\cos(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^x}{4} + \frac{\sin(x)}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \begin{bmatrix} \frac{\cos(x)}{2} + \frac{e^x}{4} - 1 + \frac{e^{-x}}{4} \\ -\frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\sin(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\cos(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^x}{4} + \frac{\sin(x)}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -c_1 e^{-x} + e^x c_2 + \frac{\cos(x)}{2} + \frac{e^x}{4} - 1 + \frac{e^{-x}}{4} - \cos(x)\, c_4 - c_3 \sin(x)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$4)-y(x)=1,y(x), singsol=all)
```

$$y(x) = -1 + \cos(x) c_1 + c_2 e^x + c_3 \sin(x) + c_4 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 31

```
DSolve[y''''[x]-y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_3 e^{-x} + c_2 \cos(x) + c_4 \sin(x) - 1$$

16.22 problem 495

Internal problem ID [15265]
Internal file name [OUTPUT/15265_Wednesday_May_08_2024_03_54_28_PM_60277047/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 495.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y' = 2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y' = 0$$

The characteristic equation is

$$\lambda^4 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_4 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^x c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$
$$y_4 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y'''' - y' = 2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^x c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_4\right) + (-2x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + e^x c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_4 - 2x \tag{1}$$

Verification of solutions

$$y = c_1 + e^x c_2 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_4 - 2x$$

Verified OK.

16.22.1 Maple step by step solution

Let's solve

$y'''' - y' = 2$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Define new variable $y_4(x)$

  $$y_4(x) = y'''$$

- Isolate for $y_4'(x)$ using original ODE

  $$y_4'(x) = 2 + y_2(x)$$

  Convert linear ODE into a system of first order ODEs

  $$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 2 + y_2(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{1} = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_{2} = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{I\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} \cos\left(\frac{x\sqrt{3}}{2}\right) \\ -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}, \vec{y}_4(x) = \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\sin\left(\frac{x\sqrt{3}}{2}\right) \\ -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ -\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($
  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \\ 0 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ 0 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ 0 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t
  $\Phi(x) = \phi(x) \cdot \phi(0)^{-1}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \\ 0 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ 0 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & \mathrm{e}^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ 0 & \mathrm{e}^x & \mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & -1 + \frac{e}{3} \\ 0 & \frac{e^x}{3} + \frac{2\,e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos(}{3} \\ 0 & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^x}{3} + \frac{2\,e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos(}{3} \\ 0 & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^x}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^x}{3} + \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

    $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
  - Take the derivative of the particular solution

    $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(x) = \Phi(x)^{-1} \cdot \vec{f}(x)$
  - Integrate to solve for $\vec{v}(x)$

    $\vec{v}(x) = \int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\,ds$
  - Plug $\vec{v}(x)$ into the equation for the particular solution

    $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \Phi(s)^{-1} \cdot \vec{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{2\,e^{x}}{3} - 2x + \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \frac{2\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{2\,e^{x}}{3} - 2 + \frac{4\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ -\frac{2\,e^{-\frac{x}{2}}\left(\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}+\cos\left(\frac{x\sqrt{3}}{2}\right)-e^{\frac{3x}{2}}\right)}{3} \\ \frac{2\,e^{-\frac{x}{2}}\left(e^{\frac{3x}{2}}+\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}-\cos\left(\frac{x\sqrt{3}}{2}\right)\right)}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{2\,e^{x}}{3} - 2x + \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \frac{2\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{2\,e^{x}}{3} - 2 + \frac{4\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ -\frac{2\,e^{-\frac{x}{2}}\left(\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}+\cos\left(\frac{x\sqrt{3}}{2}\right)-e^{\frac{3x}{2}}\right)}{3} \\ \frac{2\,e^{-\frac{x}{2}}\left(e^{\frac{3x}{2}}+\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}-\cos\left(\frac{x\sqrt{3}}{2}\right)\right)}{3} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-\frac{x}{2}}(3c_3-2)\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - e^{-\frac{x}{2}}\left(c_4 - \frac{2\sqrt{3}}{3}\right)\sin\left(\frac{x\sqrt{3}}{2}\right) + \frac{(3c_2+2)e^{x}}{3} - 2x + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = _b(_a)+2, _b(_a)`
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(diff(y(x),x$4)-diff(y(x),x)=2,y(x), singsol=all)
```

$$y(x) = -\frac{e^{-\frac{x}{2}}\left(\sqrt{3}\, c_3 + c_2\right)\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\left(\sqrt{3}\, c_2 - c_3\right)\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} + c_1 e^{x} - 2x + c_4$$

✓ Solution by Mathematica

Time used: 0.219 (sec). Leaf size: 85

```
DSolve[y''''[x]-y'[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x + c_1 e^{x} - \frac{1}{2}\left(c_2 + \sqrt{3}c_3\right)e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + \frac{1}{2}\left(\sqrt{3}c_2 - c_3\right)e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right) + c_4$$

16.23 problem 496

Internal problem ID [15266]
Internal file name [`OUTPUT/15266_Wednesday_May_08_2024_03_54_31_PM_65931517/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 496.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - y'' = 3}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y'' = 0$$

The characteristic equation is

$$\lambda^4 - \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x + e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= 1 \\
y_3 &= x \\
y_4 &= e^x
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - y'' = 3$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^x, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 = 3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \mathrm{e}^{x} c_4\right) + \left(-\frac{3x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \mathrm{e}^{x} c_4 - \frac{3x^2}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \mathrm{e}^{x} c_4 - \frac{3x^2}{2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _b(_a)+3, _b(_a)`   *** Suble
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$4)-diff(y(x),x$2)=3,y(x), singsol=all)
```

$$y(x) = c_2 e^x - \frac{3x^2}{2} + c_1 e^{-x} + c_3 x + c_4$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 33

```
DSolve[y''''[x]-y''[x]==3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3x^2}{2} + c_4 x + c_1 e^x + c_2 e^{-x} + c_3$$

16.24 problem 497

Internal problem ID [15267]
Internal file name [`OUTPUT/15267_Wednesday_May_08_2024_03_54_32_PM_27709327/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 497.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - y''' = 4}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y''' = 0$$

The characteristic equation is

$$\lambda^4 - \lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = e^x$$

Now the particular solution to the given ODE is found

$$y'''' - y''' = 4$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, e^x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_3 x^2 + c_2 x + c_1 + e^x c_4\right) + \left(-\frac{2x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3 x^2 + c_2 x + c_1 + e^x c_4 - \frac{2x^3}{3} \tag{1}$$

Verification of solutions

$$y = c_3 x^2 + c_2 x + c_1 + e^x c_4 - \frac{2x^3}{3}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)+4, _b(_a)`   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$4)-diff(y(x),x$3)=4,y(x), singsol=all)
```

$$y(x) = c_1 e^x + \frac{c_2 x^2}{2} - \frac{2x^3}{3} + c_3 x + c_4$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 31

```
DSolve[y''''[x]-y'''[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2x^3}{3} + c_4 x^2 + c_3 x + c_1 e^x + c_2$$

16.25 problem 498

Internal problem ID [15268]
Internal file name [`OUTPUT/15268_Wednesday_May_08_2024_03_54_32_PM_53691888/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 498.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y''' + 4y'' = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y''' + 4y'' = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= -2 \\ \lambda_4 &= -2 \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{-2x} c_3 + x\,\mathrm{e}^{-2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{-2x} \\
y_4 &= x\,\mathrm{e}^{-2x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y''' + 4y'' = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x + c_1 + \mathrm{e}^{-2x} c_3 + x\,\mathrm{e}^{-2x} c_4\right) + \left(\frac{x^2}{8}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4 x + c_3)\,\mathrm{e}^{-2x} + c_2 x + c_1 + \frac{x^2}{8}$$

Summary

The solution(s) found are the following

$$y = (c_4 x + c_3)\,\mathrm{e}^{-2x} + c_2 x + c_1 + \frac{x^2}{8} \tag{1}$$

Verification of solutions

$$y = (c_4 x + c_3)\,\mathrm{e}^{-2x} + c_2 x + c_1 + \frac{x^2}{8}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -4*(diff(_b(_a), _a))-4*_b(_a
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$3)+4*diff(y(x),x$2)=1,y(x), singsol=all)
```

$$y(x) = \frac{(2c_1x + 2c_1 + 2c_2)\,\mathrm{e}^{-2x}}{8} + \frac{x^2}{8} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 37

```
DSolve[y''''[x]+4*y'''[x]+4*y''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{8} + c_4x + \frac{1}{4}e^{-2x}(c_2(x+1) + c_1) + c_3$$

16.26 problem 499

Internal problem ID [15269]
Internal file name [`OUTPUT/15269_Wednesday_May_08_2024_03_54_33_PM_89636470/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 499.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y'''' + 2y''' + y'' = e^{4x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' + y'' = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 x\, e^{-x} + c_3 + c_4 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\, e^{-x}$$
$$y_3 = 1$$
$$y_4 = x$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' + y'' = e^{4x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x\, e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{4x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$400A_1 e^{4x} = e^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{400}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{4x}}{400}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 x\, e^{-x} + c_3 + c_4 x) + \left(\frac{e^{4x}}{400}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + c_4 x + c_3 + \frac{e^{4x}}{400}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + c_4 x + c_3 + \frac{e^{4x}}{400} \tag{1}$$

<u>Verification of solutions</u>

$$y = e^{-x}(c_2 x + c_1) + c_4 x + c_3 + \frac{e^{4x}}{400}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -2*(diff(_b(_a), _a))-_b(_a)+
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+diff(y(x),x$2)=exp(4*x),y(x), singsol=all)
```

$$y(x) = (c_1(x+2)+c_2)\,\mathrm{e}^{-x}+c_3x+c_4+\frac{\mathrm{e}^{4x}}{400}$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 36

```
DSolve[y''''[x]+2*y'''[x]+y''[x]==Exp[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{4x}}{400}+e^{-x}(c_2(x+2)+c_1)+c_4x+c_3$$

16.27 problem 500

Internal problem ID [15270]
Internal file name [`OUTPUT/15270_Wednesday_May_08_2024_03_54_33_PM_1048459/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 500.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 2y''' + y'' = \mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' + y'' = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 0\\ \lambda_3 &= -1\\ \lambda_4 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 + c_4 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x e^{-x}$$
$$y_3 = 1$$
$$y_4 = x$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' + y'' = e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x e^{-x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{-x}\}]$$

Since $x e^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-x} = e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{-x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 x\, e^{-x} + c_3 + c_4 x\right) + \left(\frac{x^2 e^{-x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + c_4 x + c_3 + \frac{x^2 e^{-x}}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + c_4 x + c_3 + \frac{x^2 e^{-x}}{2} \tag{1}$$

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + c_4 x + c_3 + \frac{x^2 e^{-x}}{2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -2*(diff(_b(_a), _a))-_b(_a)+
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+diff(y(x),x$2)=exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + (2c_1 + 4)\,x + 4c_1 + 2c_2 + 6)\,e^{-x}}{2} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 47

```
DSolve[y''''[x]+2*y'''[x]+y''[x]==Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(x^2 + 2x(c_4e^x + 2 + c_2) + 2(c_3e^x + 3 + c_1 + 2c_2)\right)$$

16.28 problem 501

Internal problem ID [15271]
Internal file name [`OUTPUT/15271_Wednesday_May_08_2024_03_54_34_PM_94584042/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 501.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y'''' + 2y''' + y'' = x\,\mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' + y'' = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 + c_4 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= x e^{-x} \\ y_3 &= 1 \\ y_4 &= x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' + y'' = x e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x e^{-x}, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{-x}, x^2 e^{-x}\}]$$

Since $x e^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{-x}, x^3 e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{-x} + A_2 x^3 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{-x} - 12A_2\mathrm{e}^{-x} + 6A_2x\,\mathrm{e}^{-x} = x\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2\mathrm{e}^{-x} + \frac{x^3\mathrm{e}^{-x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2x\,\mathrm{e}^{-x} + c_3 + c_4x\right) + \left(x^2\mathrm{e}^{-x} + \frac{x^3\mathrm{e}^{-x}}{6}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2x + c_1) + c_4x + c_3 + x^2\mathrm{e}^{-x} + \frac{x^3\mathrm{e}^{-x}}{6}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2x + c_1) + c_4x + c_3 + x^2\mathrm{e}^{-x} + \frac{x^3\mathrm{e}^{-x}}{6} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}(c_2x + c_1) + c_4x + c_3 + x^2\mathrm{e}^{-x} + \frac{x^3\mathrm{e}^{-x}}{6}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = exp(-_a)*_a-_b(_a)-2*(diff(_b
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+diff(y(x),x$2)=x*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(24 + x^3 + 6x^2 + 6(3 + c_1)\,x + 12c_1 + 6c_2)\,\mathrm{e}^{-x}}{6} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 52

```
DSolve[y''''[x]+2*y'''[x]+y''[x]==x*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-x}\left(x^3 + 6x^2 + 6x(c_4e^x + 3 + c_2) + 6(c_3e^x + 4 + c_1 + 2c_2)\right)$$

16.29 problem 502

Internal problem ID [15272]
Internal file name [`OUTPUT/15272_Wednesday_May_08_2024_03_54_35_PM_96868665/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 502.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' + 4y'' + 4y = \sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$
$$\lambda_3 = i\sqrt{2}$$
$$\lambda_4 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{i\sqrt{2}\,x} c_1 + x\,\mathrm{e}^{i\sqrt{2}\,x} c_2 + \mathrm{e}^{-i\sqrt{2}\,x} c_3 + x\,\mathrm{e}^{-i\sqrt{2}\,x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{i\sqrt{2}\,x}$$
$$y_2 = x\,\mathrm{e}^{i\sqrt{2}\,x}$$
$$y_3 = \mathrm{e}^{-i\sqrt{2}\,x}$$
$$y_4 = x\,\mathrm{e}^{-i\sqrt{2}\,x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y'' + 4y = \sin(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x\,\mathrm{e}^{i\sqrt{2}\,x}, x\,\mathrm{e}^{-i\sqrt{2}\,x}, \mathrm{e}^{i\sqrt{2}\,x}, \mathrm{e}^{-i\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \cos(2x) + 4A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{i\sqrt{2}\,x} c_1 + x\,e^{i\sqrt{2}\,x} c_2 + e^{-i\sqrt{2}\,x} c_3 + x\,e^{-i\sqrt{2}\,x} c_4\right) + \left(\frac{\sin(2x)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = (c_2 x + c_1)\,e^{i\sqrt{2}\,x} + (c_4 x + c_3)\,e^{-i\sqrt{2}\,x} + \frac{\sin(2x)}{4}$$

Summary

The solution(s) found are the following

$$y = (c_2 x + c_1)\,e^{i\sqrt{2}\,x} + (c_4 x + c_3)\,e^{-i\sqrt{2}\,x} + \frac{\sin(2x)}{4} \tag{1}$$

Verification of solutions

$$y = (c_2 x + c_1)\,e^{i\sqrt{2}\,x} + (c_4 x + c_3)\,e^{-i\sqrt{2}\,x} + \frac{\sin(2x)}{4}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)+4*y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = (c_3x + c_1)\cos\left(\sqrt{2}\,x\right) + (c_4x + c_2)\sin\left(\sqrt{2}\,x\right) + \frac{\sin(2x)}{4}$$

✓ Solution by Mathematica

Time used: 0.521 (sec). Leaf size: 46

```
DSolve[y''''[x]+4*y''[x]+4*y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\sin(2x) + (c_2x + c_1)\cos\left(\sqrt{2}x\right) + (c_4x + c_3)\sin\left(\sqrt{2}x\right)$$

16.30 problem 503

Internal problem ID [15273]
Internal file name [`OUTPUT/15273_Wednesday_May_08_2024_03_54_35_PM_17876674/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 503.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y'' + 4y = \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$
$$\lambda_3 = i\sqrt{2}$$
$$\lambda_4 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{i\sqrt{2}\,x}c_1 + x\,e^{i\sqrt{2}\,x}c_2 + e^{-i\sqrt{2}\,x}c_3 + x\,e^{-i\sqrt{2}\,x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{i\sqrt{2}\,x}$$
$$y_2 = x\,e^{i\sqrt{2}\,x}$$
$$y_3 = e^{-i\sqrt{2}\,x}$$
$$y_4 = x\,e^{-i\sqrt{2}\,x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y'' + 4y = \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x\,e^{i\sqrt{2}\,x}, x\,e^{-i\sqrt{2}\,x}, e^{i\sqrt{2}\,x}, e^{-i\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(x) + A_2 \sin(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{i\sqrt{2}\,x}c_1 + x\,e^{i\sqrt{2}\,x}c_2 + e^{-i\sqrt{2}\,x}c_3 + x\,e^{-i\sqrt{2}\,x}c_4\right) + (\cos(x)) \end{aligned}$$

Which simplifies to

$$y = (c_2x + c_1)\,e^{i\sqrt{2}\,x} + (c_4x + c_3)\,e^{-i\sqrt{2}\,x} + \cos(x)$$

Summary
The solution(s) found are the following

$$y = (c_2x + c_1)\,e^{i\sqrt{2}\,x} + (c_4x + c_3)\,e^{-i\sqrt{2}\,x} + \cos(x) \tag{1}$$

Verification of solutions

$$y = (c_2x + c_1)\,e^{i\sqrt{2}\,x} + (c_4x + c_3)\,e^{-i\sqrt{2}\,x} + \cos(x)$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)+4*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = (c_3x + c_1)\cos\left(\sqrt{2}\,x\right) + (c_4x + c_2)\sin\left(\sqrt{2}\,x\right) + \cos(x)$$

✓ Solution by Mathematica

Time used: 0.288 (sec). Leaf size: 40

```
DSolve[y''''[x]+4*y''[x]+4*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x) + (c_2x + c_1)\cos\left(\sqrt{2}x\right) + (c_4x + c_3)\sin\left(\sqrt{2}x\right)$$

16.31 problem 504

Internal problem ID [15274]
Internal file name [`OUTPUT/15274_Wednesday_May_08_2024_03_54_36_PM_62930676/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 504.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y'' + 4y = x \sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y'' + 4y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$
$$\lambda_3 = i\sqrt{2}$$
$$\lambda_4 = -i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{i\sqrt{2}\,x}c_1 + x\,e^{i\sqrt{2}\,x}c_2 + e^{-i\sqrt{2}\,x}c_3 + x\,e^{-i\sqrt{2}\,x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{i\sqrt{2}\,x}$$
$$y_2 = x\,e^{i\sqrt{2}\,x}$$
$$y_3 = e^{-i\sqrt{2}\,x}$$
$$y_4 = x\,e^{-i\sqrt{2}\,x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y'' + 4y = x\sin(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(2x), x\sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x\,e^{i\sqrt{2}\,x}, x\,e^{-i\sqrt{2}\,x}, e^{i\sqrt{2}\,x}, e^{-i\sqrt{2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1x\cos(2x) + A_2x\sin(2x) + A_3\cos(2x) + A_4\sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1\sin(2x) + 4A_1x\cos(2x) - 16A_2\cos(2x) + 4A_2x\sin(2x) + 4A_3\cos(2x) + 4A_4\sin(2x) = x\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = 1, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \sin(2x)}{4} + \cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{i\sqrt{2}\,x} c_1 + x\, e^{i\sqrt{2}\,x} c_2 + e^{-i\sqrt{2}\,x} c_3 + x\, e^{-i\sqrt{2}\,x} c_4\right) + \left(\frac{x \sin(2x)}{4} + \cos(2x)\right) \end{aligned}$$

Which simplifies to

$$y = (c_2 x + c_1)\, e^{i\sqrt{2}\,x} + (c_4 x + c_3)\, e^{-i\sqrt{2}\,x} + \frac{x \sin(2x)}{4} + \cos(2x)$$

Summary

The solution(s) found are the following

$$y = (c_2 x + c_1)\, e^{i\sqrt{2}\,x} + (c_4 x + c_3)\, e^{-i\sqrt{2}\,x} + \frac{x \sin(2x)}{4} + \cos(2x) \tag{1}$$

Verification of solutions

$$y = (c_2 x + c_1)\, e^{i\sqrt{2}\,x} + (c_4 x + c_3)\, e^{-i\sqrt{2}\,x} + \frac{x \sin(2x)}{4} + \cos(2x)$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)+4*y(x)=x*sin(2*x),y(x), singsol=all)
```

$$y(x) = (c_3x + c_1)\cos\left(\sqrt{2}\,x\right) + (c_4x + c_2)\sin\left(\sqrt{2}\,x\right) + \frac{x\sin(2x)}{4} + \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 58

```
DSolve[y''''[x]+4*y''[x]+4*y[x]==x*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}x\sin(2x) + \cos(2x) + (c_2x + c_1)\cos\left(\sqrt{2}x\right) + c_3\sin\left(\sqrt{2}x\right) + c_4x\sin\left(\sqrt{2}x\right)$$

16.32 problem 505

Internal problem ID [15275]
Internal file name [`OUTPUT/15275_Wednesday_May_08_2024_03_54_37_PM_67705512/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 505.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' + 2n^2 y'' + n^4 y = a \sin(nx + \alpha)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2n^2 y'' + n^4 y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 n^2 + n^4 = 0$$

The roots of the above equation are

$$\lambda_1 = in$$
$$\lambda_2 = -in$$
$$\lambda_3 = in$$
$$\lambda_4 = -in$$

Therefore the homogeneous solution is

$$y_h(x) = e^{inx}c_1 + x\,e^{inx}c_2 + e^{-inx}c_3 + x\,e^{-inx}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{inx}$$
$$y_2 = x\,e^{inx}$$
$$y_3 = e^{-inx}$$
$$y_4 = x\,e^{-inx}$$

Now the particular solution to the given ODE is found

$$y'''' + 2n^2y'' + n^4y = a\sin(nx+\alpha)$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i}\int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{inx} & x\,e^{inx} & e^{-inx} & x\,e^{-inx} \\ in\,e^{inx} & e^{inx}(inx+1) & -in\,e^{-inx} & e^{-inx}(-inx+1) \\ -n^2e^{inx} & e^{inx}n(-nx+2i) & -n^2e^{-inx} & -2n\,e^{-inx}\left(\frac{nx}{2}+i\right) \\ -in^3e^{inx} & -e^{inx}n^2(inx+3) & in^3e^{-inx} & e^{-inx}n^2(inx-3) \end{bmatrix}$$

$$|W| = 16\,e^{2inx}e^{-2inx}n^4$$

The determinant simplifies to

$$|W| = 16n^4$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x\,e^{inx} & e^{-inx} & x\,e^{-inx} \\ e^{inx}(inx+1) & -in\,e^{-inx} & e^{-inx}(-inx+1) \\ e^{inx}n(-nx+2i) & -n^2e^{-inx} & -2n\,e^{-inx}\left(\frac{nx}{2}+i\right) \end{bmatrix}$$
$$= -4\,e^{-inx}n(nx-i)$$

$$W_2(x) = \det \begin{bmatrix} e^{inx} & e^{-inx} & x\,e^{-inx} \\ in\,e^{inx} & -in\,e^{-inx} & e^{-inx}(-inx+1) \\ -n^2e^{inx} & -n^2e^{-inx} & -2n\,e^{-inx}\left(\frac{nx}{2}+i\right) \end{bmatrix}$$
$$= -4n^2e^{-inx}$$

$$W_3(x) = \det \begin{bmatrix} e^{inx} & x\,e^{inx} & x\,e^{-inx} \\ in\,e^{inx} & e^{inx}(inx+1) & e^{-inx}(-inx+1) \\ -n^2e^{inx} & e^{inx}n(-nx+2i) & -2n\,e^{-inx}\left(\frac{nx}{2}+i\right) \end{bmatrix}$$
$$= -4\,e^{inx}n(nx+i)$$

$$W_4(x) = \det \begin{bmatrix} e^{inx} & x\,e^{inx} & e^{-inx} \\ in\,e^{inx} & e^{inx}(inx+1) & -in\,e^{-inx} \\ -n^2e^{inx} & e^{inx}n(-nx+2i) & -n^2e^{-inx} \end{bmatrix}$$
$$= -4n^2e^{inx}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3\int \frac{(a\sin(nx+\alpha))\left(-4\,e^{-inx}n(nx-i)\right)}{(1)\left(16n^4\right)}\,dx \\ &= -\int \frac{-4a\sin(nx+\alpha)\,e^{-inx}n(nx-i)}{16n^4}\,dx \\ &= -\int \left(-\frac{a\sin(nx+\alpha)\,e^{-inx}(nx-i)}{4n^3}\right)dx \\ &= -\left(\int -\frac{a\sin(nx+\alpha)\,e^{-inx}(nx-i)}{4n^3}dx\right) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(a\sin(nx+\alpha))\left(-4n^2\mathrm{e}^{-inx}\right)}{(1)\left(16n^4\right)}\,dx \\
&= \int \frac{-4a\sin(nx+\alpha)\,n^2\mathrm{e}^{-inx}}{16n^4}\,dx \\
&= \int \left(-\frac{a\sin(nx+\alpha)\,\mathrm{e}^{-inx}}{4n^2}\right)dx \\
&= \frac{-\frac{ax\,\mathrm{e}^{-inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n} + \frac{iax\,\mathrm{e}^{-inx}}{8n} - \frac{ia\,\mathrm{e}^{-inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n^2} - \frac{iax\,\mathrm{e}^{-inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2}{8n}}{n\left(1+\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2\right)}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(a\sin(nx+\alpha))\left(-4\,\mathrm{e}^{inx}n(nx+i)\right)}{(1)\left(16n^4\right)}\,dx \\
&= -\int \frac{-4a\sin(nx+\alpha)\,\mathrm{e}^{inx}n(nx+i)}{16n^4}\,dx \\
&= -\int \left(-\frac{a\sin(nx+\alpha)\,\mathrm{e}^{inx}(nx+i)}{4n^3}\right)dx \\
&= -\left(\int -\frac{a\sin(nx+\alpha)\,\mathrm{e}^{inx}(nx+i)}{4n^3}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(a\sin(nx+\alpha))\left(-4n^2\mathrm{e}^{inx}\right)}{(1)\left(16n^4\right)}\,dx \\
&= \int \frac{-4a\sin(nx+\alpha)\,n^2\mathrm{e}^{inx}}{16n^4}\,dx \\
&= \int \left(-\frac{a\sin(nx+\alpha)\,\mathrm{e}^{inx}}{4n^2}\right)dx \\
&= \frac{-\frac{ax\,\mathrm{e}^{inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n} - \frac{iax\,\mathrm{e}^{inx}}{8n} + \frac{ia\,\mathrm{e}^{inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n^2} + \frac{iax\,\mathrm{e}^{inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2}{8n}}{n\left(1+\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2\right)}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4
$$

Hence

$$y_p = \left(-\left(\int -\frac{a \sin(nx+\alpha)\,\mathrm{e}^{-inx}(nx-i)}{4n^3}dx\right)\right)\left(\mathrm{e}^{inx}\right)$$
$$+\left(\frac{-\frac{ax\,\mathrm{e}^{-inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n}+\frac{iax\,\mathrm{e}^{-inx}}{8n}-\frac{ia\,\mathrm{e}^{-inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n^2}-\frac{iax\,\mathrm{e}^{-inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2}{8n}}{n\left(1+\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2\right)}\right)\left(x\,\mathrm{e}^{inx}\right)$$
$$+\left(-\left(\int -\frac{a \sin(nx+\alpha)\,\mathrm{e}^{inx}(nx+i)}{4n^3}dx\right)\right)\left(\mathrm{e}^{-inx}\right)$$
$$+\left(\frac{-\frac{ax\,\mathrm{e}^{inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n}-\frac{iax\,\mathrm{e}^{inx}}{8n}+\frac{ia\,\mathrm{e}^{inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)}{4n^2}+\frac{iax\,\mathrm{e}^{inx}\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2}{8n}}{n\left(1+\tan\left(\frac{nx}{2}+\frac{\alpha}{2}\right)^2\right)}\right)\left(x\,\mathrm{e}^{-inx}\right)$$

Therefore the particular solution is

$$y_p = -\frac{\left(n\,x^2\sin(nx+\alpha)-\left(\int \sin(nx+\alpha)\,\mathrm{e}^{inx}(nx+i)\,dx\right)\mathrm{e}^{-inx}+\left(\int \sin(nx+\alpha)\,\mathrm{e}^{-inx}(-nx+i)\,dx\right)\mathrm{e}^{i}}{4n^3}$$

Which simplifies to

$$y_p = -\frac{\left((i\sin(nx)+\cos(nx))\left(\int(-nx\sin(2nx+\alpha)-nx\sin(\alpha)+\cos(\alpha)-\cos(2nx+\alpha))\,dx\right)+(i\sin}{}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(\mathrm{e}^{inx}c_1 + x\,\mathrm{e}^{inx}c_2 + \mathrm{e}^{-inx}c_3 + x\,\mathrm{e}^{-inx}c_4\right)$$
$$+\left(-\frac{\left((i\sin(nx)+\cos(nx))\left(\int(-nx\sin(2nx+\alpha)-nx\sin(\alpha)+\cos(\alpha)-\cos(2nx+\alpha))\,dx\right)+(i\,\mathrm{s}}{}\right.$$

Which simplifies to

$$y = (c_4x+c_3)\,\mathrm{e}^{-inx}+(c_2x+c_1)\,\mathrm{e}^{inx}$$
$$-\frac{\left((i\sin(nx)+\cos(nx))\left(\int(-nx\sin(2nx+\alpha)-nx\sin(\alpha)+\cos(\alpha)-\cos(2nx+\alpha))\,dx\right)+(i\sin(n}{}$$

Summary

The solution(s) found are the following

$$y = (c_4x+c_3)\,\mathrm{e}^{-inx}+(c_2x+c_1)\,\mathrm{e}^{inx} \tag{1}$$
$$-\frac{\left((i\sin(nx)+\cos(nx))\left(\int(-nx\sin(2nx+\alpha)-nx\sin(\alpha)+\cos(\alpha)-\cos(2nx+\alpha))\,dx\right)+(i\sin(n}{}$$

Verification of solutions

$$y = (c_4x + c_3)\,\mathrm{e}^{-inx} + (c_2x + c_1)\,\mathrm{e}^{inx}$$
$$- \frac{((i\sin(nx) + \cos(nx))\left(\int(-nx\sin(2nx+\alpha) - nx\sin(\alpha) + \cos(\alpha) - \cos(2nx+\alpha))\,dx\right) + (i\sin(n$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 67

```
dsolve(diff(y(x),x$4)+2*n^2*diff(y(x),x$2)+n^4*y(x)=a*sin(n*x+alpha),y(x), singsol=all)
```

$$y(x) = \frac{a(-n^2x^2+2)\sin(nx+\alpha) - 2(ax\cos(nx+\alpha) - 4((c_3x+c_1)\cos(nx) + \sin(nx)(c_4x+c_2))\,n^3)\,n}{8n^4}$$

✓ Solution by Mathematica

Time used: 0.193 (sec). Leaf size: 79

```
DSolve[y''''[x]+2*n^2*y''[x]+n^4*y[x]==a*Sin[n*x+\[Alpha]],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{3a\sin(\alpha+nx)}{16n^4} - \frac{ax\cos(\alpha+nx)}{4n^3} - \frac{ax^2\sin(\alpha+nx)}{8n^2} + (c_2x+c_1)\cos(nx) + c_4x\sin(nx) + c_3\sin(nx)$$

16.33 problem 506

Internal problem ID [15276]
Internal file name [`OUTPUT/15276_Wednesday_May_08_2024_03_54_39_PM_20973667/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 506.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 2n^2y'' + n^4y = \cos(nx + \alpha)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2n^2y'' + n^4y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^2n^2 + n^4 = 0$$

The roots of the above equation are

$$\lambda_1 = n$$
$$\lambda_2 = n$$
$$\lambda_3 = -n$$
$$\lambda_4 = -n$$

Therefore the homogeneous solution is

$$y_h(x) = e^{nx}c_1 + x\,e^{nx}c_2 + e^{-nx}c_3 + x\,e^{-nx}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{nx} \\ y_2 &= x\,e^{nx} \\ y_3 &= e^{-nx} \\ y_4 &= x\,e^{-nx} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 2n^2y'' + n^4y = \cos(nx + \alpha)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(nx + \alpha)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(nx + \alpha), \sin(nx + \alpha)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{nx}, x\,e^{-nx}, e^{nx}, e^{-nx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(nx + \alpha) + A_2\sin(nx + \alpha)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1n^4\cos(nx+\alpha) + A_2n^4\sin(nx+\alpha) - 2n^2(-A_1n^2\cos(nx+\alpha) - A_2n^2\sin(nx+\alpha)) + n^4(A_1\cos(nx + \alpha) + A_2\sin(nx + \alpha)) = \cos(nx + \alpha)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4n^4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(nx+\alpha)}{4n^4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{nx}c_1 + x\,e^{nx}c_2 + e^{-nx}c_3 + x\,e^{-nx}c_4\right) + \left(\frac{\cos(nx+\alpha)}{4n^4}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4x + c_3)\,e^{-nx} + e^{nx}(c_2x + c_1) + \frac{\cos(nx+\alpha)}{4n^4}$$

Summary

The solution(s) found are the following

$$y = (c_4x + c_3)\,e^{-nx} + e^{nx}(c_2x + c_1) + \frac{\cos(nx+\alpha)}{4n^4} \tag{1}$$

Verification of solutions

$$y = (c_4x + c_3)\,e^{-nx} + e^{nx}(c_2x + c_1) + \frac{\cos(nx+\alpha)}{4n^4}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$4)-2*n^2*diff(y(x),x$2)+n^4*y(x)=cos(n*x+alpha),y(x), singsol=all)
```

$$y(x) = \frac{\cos(nx+\alpha) + (4c_4x + 4c_2)\, n^4 e^{-nx} + (4c_3x + 4c_1)\, n^4 e^{nx}}{4n^4}$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 52

```
DSolve[y''''[x]-2*n^2*y''[x]+n^4*y[x]==Cos[n*x+\[Alpha]],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\cos(\alpha + nx)}{4n^4} + e^{-nx}\left(c_3e^{2nx} + c_4xe^{2nx} + c_2x + c_1\right)$$

16.34 problem 507

Internal problem ID [15277]
Internal file name [`OUTPUT/15277_Wednesday_May_08_2024_03_54_40_PM_43711821/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 507.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 4y''' + 6y'' + 4y' + y = \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y''' + 6y'' + 4y' + y = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} + x^2 \mathrm{e}^{-x} c_3 + x^3 \mathrm{e}^{-x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x \,\mathrm{e}^{-x}$$
$$y_3 = x^2 \mathrm{e}^{-x}$$
$$y_4 = x^3 \mathrm{e}^{-x}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y''' + 6y'' + 4y' + y = \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \,\mathrm{e}^{-x}, x^2 \mathrm{e}^{-x}, x^3 \mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \cos(x) - 4A_2 \sin(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\sin(x)}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + x^2 \mathrm{e}^{-x} c_3 + x^3 \mathrm{e}^{-x} c_4\right) + \left(-\frac{\sin(x)}{4}\right)\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}\left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) - \frac{\sin(x)}{4}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}\left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) - \frac{\sin(x)}{4} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}\left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) - \frac{\sin(x)}{4}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$3)+6*diff(y(x),x$2)+4*diff(y(x),x)+y(x)=sin(x),y(x), sin
```

$$y(x) = \left(c_3x^3 + c_2x^2 + c_4x + c_1\right) e^{-x} - \frac{\sin\left(x\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 35

```
DSolve[y''''[x]+4*y'''[x]+6*y''[x]+4*y'[x]+y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\sin(x)}{4} + e^{-x}(x(x(c_4x + c_3) + c_2) + c_1)$$

16.35 problem 508

Internal problem ID [15278]
Internal file name [`OUTPUT/15278_Wednesday_May_08_2024_03_54_41_PM_87940267/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 508.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' - 4y''' + 6y'' - 4y' + y = e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 4y''' + 6y'' - 4y' + y = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + x\, e^x c_2 + x^2 e^x c_3 + x^3 e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^x \\ y_2 &= x\, e^x \\ y_3 &= x^2 e^x \\ y_4 &= x^3 e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 4y''' + 6y'' - 4y' + y = e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, x^2 e^x, x^3 e^x, e^x\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x\}]$$

Since $x\, e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x\}]$$

Since $x^2 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3 e^x\}]$$

Since $x^3 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^4 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^4 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4 e^x}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + x e^x c_2 + x^2 e^x c_3 + x^3 e^x c_4\right) + \left(\frac{x^4 e^x}{24}\right) \end{aligned}$$

Which simplifies to

$$y = e^x \left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) + \frac{x^4 e^x}{24}$$

Summary

The solution(s) found are the following

$$y = e^x \left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) + \frac{x^4 e^x}{24} \tag{1}$$

Verification of solutions

$$y = e^x \left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) + \frac{x^4 e^x}{24}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+6*diff(y(x),x$2)-4*diff(y(x),x)+y(x)=exp(x),y(x), sin
```

$$y(x) = e^x\left(\frac{1}{24}x^4 + c_1 + c_2x + c_3x^2 + c_4x^3\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 39

```
DSolve[y''''[x]-4*y'''[x]+6*y''[x]-4*y'[x]+y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{24}e^x\left(x^4 + 24c_4x^3 + 24c_3x^2 + 24c_2x + 24c_1\right)$$

16.36 problem 509

Internal problem ID [15279]
Internal file name [`OUTPUT/15279_Wednesday_May_08_2024_03_54_41_PM_21800782/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 509.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 4y''' + 6y'' - 4y' + y = x\,\mathrm{e}^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 4y''' + 6y'' - 4y' + y = 0$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + x e^x c_2 + x^2 e^x c_3 + x^3 e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^x \\ y_2 &= x e^x \\ y_3 &= x^2 e^x \\ y_4 &= x^3 e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 4y''' + 6y'' - 4y' + y = x e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^x, x^2 e^x, x^3 e^x, e^x\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x, x^2 e^x\}]$$

Since $x e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x, x^3 e^x\}]$$

Since $x^2 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3 e^x, x^4 e^x\}]$$

Since $x^3 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^4 e^x, x^5 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^4 e^x + A_2 x^5 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$120 A_2 x\, e^x + 24 A_1 e^x = x\, e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{120}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^5 e^x}{120}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + x\, e^x c_2 + x^2 e^x c_3 + x^3 e^x c_4\right) + \left(\frac{x^5 e^x}{120}\right) \end{aligned}$$

Which simplifies to

$$y = e^x \left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) + \frac{x^5 e^x}{120}$$

Summary

The solution(s) found are the following

$$y = e^x \left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) + \frac{x^5 e^x}{120} \tag{1}$$

Verification of solutions

$$y = e^x \left(c_4 x^3 + c_3 x^2 + c_2 x + c_1\right) + \frac{x^5 e^x}{120}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+6*diff(y(x),x$2)-4*diff(y(x),x)+y(x)=x*exp(x),y(x), s
```

$$y(x) = \mathrm{e}^{x}\left(\frac{1}{120}x^{5} + c_{1} + c_{2}x + c_{3}x^{2} + c_{4}x^{3}\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 39

```
DSolve[y''''[x]-4*y'''[x]+6*y''[x]-4*y'[x]+y[x]==x*Exp[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{120}e^{x}\left(x^{5} + 120c_{4}x^{3} + 120c_{3}x^{2} + 120c_{2}x + 120c_{1}\right)$$

16.37 problem 510

Internal problem ID [15280]
Internal file name [`OUTPUT/15280_Wednesday_May_08_2024_03_54_42_PM_48254448/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 510.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = -2}$$

16.37.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = -2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 x\, e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = -2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x}\right) + (-2) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) - 2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) - 2 \tag{1}$$

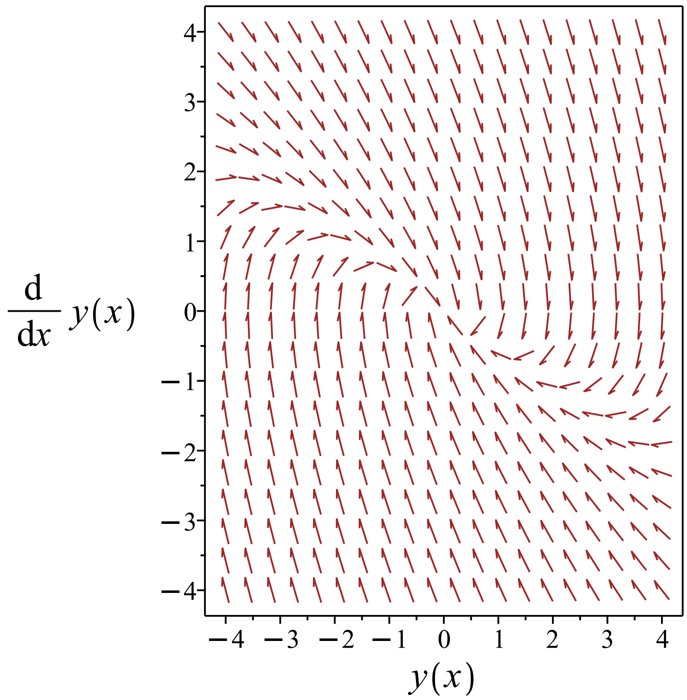


Figure 545: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) - 2$$

Verified OK.

16.37.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\, dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= -2\,\mathrm{e}^{x} \\
(\mathrm{e}^{x}y)'' &= -2\,\mathrm{e}^{x}
\end{aligned}
$$

Integrating once gives

$$
(\mathrm{e}^{x}y)' = -2\,\mathrm{e}^{x} + c_1
$$

Integrating again gives

$$
(\mathrm{e}^{x}y) = c_1x - 2\,\mathrm{e}^{x} + c_2
$$

Hence the solution is

$$
y = \frac{c_1x - 2\,\mathrm{e}^{x} + c_2}{\mathrm{e}^{x}}
$$

Or

$$
y = c_1x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}c_2 - 2
$$

Summary

The solution(s) found are the following

$$
y = c_1x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}c_2 - 2 \tag{1}
$$

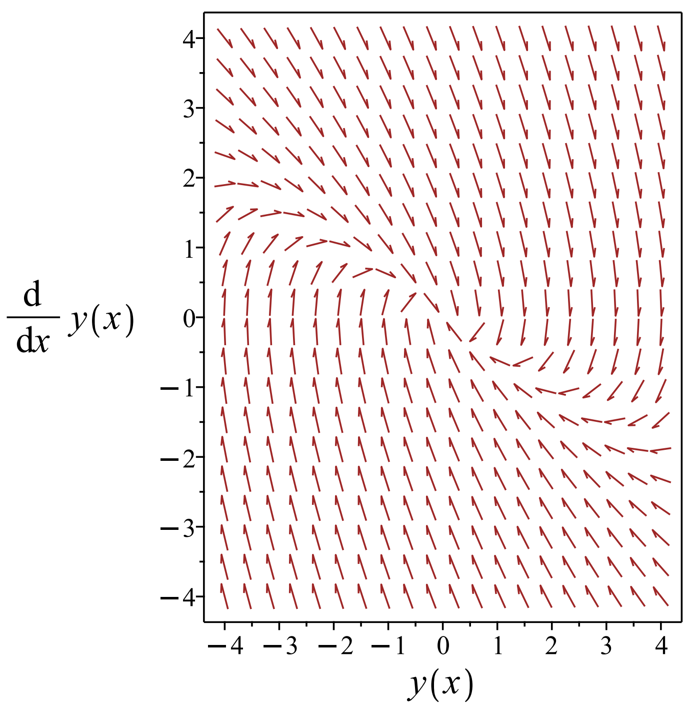


Figure 546: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} c_2 - 2$$

Verified OK.

16.37.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 436: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2 x \, e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \, e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = -2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 x \, e^{-x}\right) + (-2) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) - 2$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) - 2 \tag{1}$$

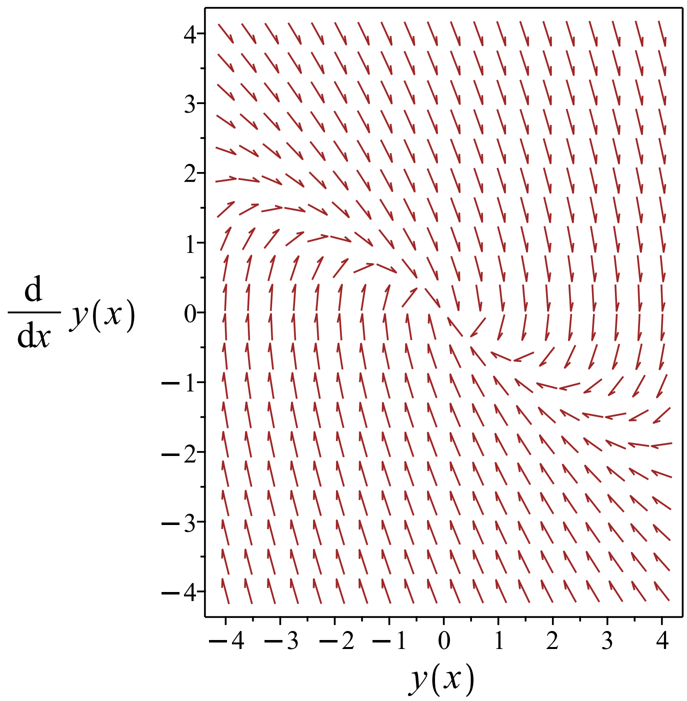


Figure 547: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) - 2$$

Verified OK.

16.37.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = -2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\,\mathrm{e}^{-x}\left(\int x\,\mathrm{e}^{x}dx - \left(\int \mathrm{e}^{x}dx\right)x\right)$

- Compute integrals

  $y_p(x) = -2$

- Substitute particular solution into general solution to ODE

  $y = c_2 x \,\mathrm{e}^{-x} + c_1 \mathrm{e}^{-x} - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=-2,y(x), singsol=all)
```

$$y(x) = -2 + (c_1 x + c_2)\,\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 23

```
DSolve[y''[x]+2*y'[x]+y[x]==-2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(-2e^x + c_2 x + c_1)$$

16.38 problem 511

Internal problem ID [15281]
Internal file name [`OUTPUT/15281_Wednesday_May_08_2024_03_54_43_PM_38289872/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 511.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' = -2}$$

16.38.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 0, f(x) = -2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+2y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=2, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+2\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+2\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=2, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-2}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{2^2-(4)\,(1)\,(0)}\\&=-1\pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-1+1\\\lambda_2&=-1-1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 + \mathrm{e}^{-2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + \mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{-2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = -2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \mathrm{e}^{-2x} c_2\right) + (-x) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 + \mathrm{e}^{-2x} c_2 - x \tag{1}$$

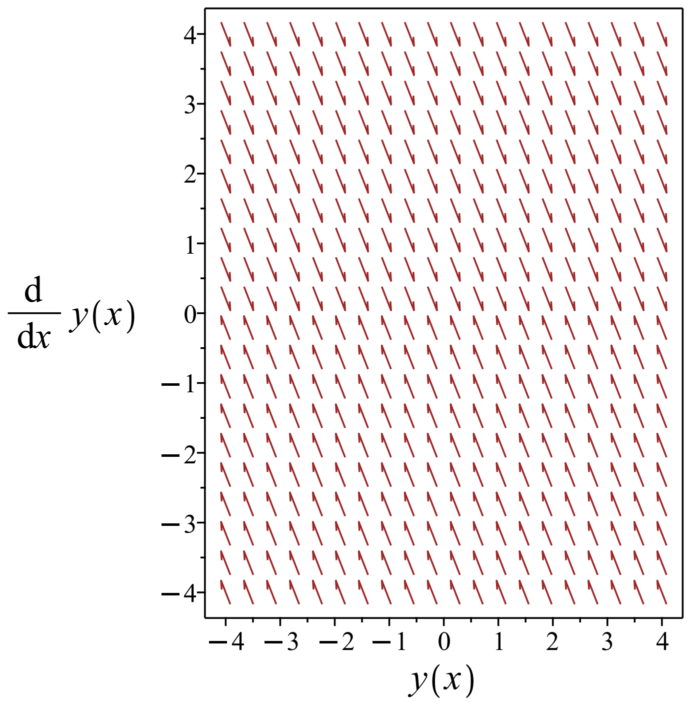


Figure 548: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 + \mathrm{e}^{-2x} c_2 - x$$

Verified OK.

16.38.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = \int (-2)\, dx$$
$$2y + y' = -2x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = -2x + c_1$$

Hence the ode is

$$2y + y' = -2x + c_1$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-2x + c_1)$$
$$\frac{d}{dx}(e^{2x}y) = (e^{2x})(-2x + c_1)$$
$$d(e^{2x}y) = ((-2x + c_1) e^{2x})\, dx$$

Integrating gives

$$e^{2x}y = \int (-2x + c_1) e^{2x}\, dx$$
$$e^{2x}y = -\frac{e^{2x}(2x - c_1 - 1)}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = -\frac{e^{-2x}e^{2x}(2x - c_1 - 1)}{2} + e^{-2x}c_2$$

which simplifies to

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x} c_2$$

Summary

The solution(s) found are the following

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x} c_2 \tag{1}$$

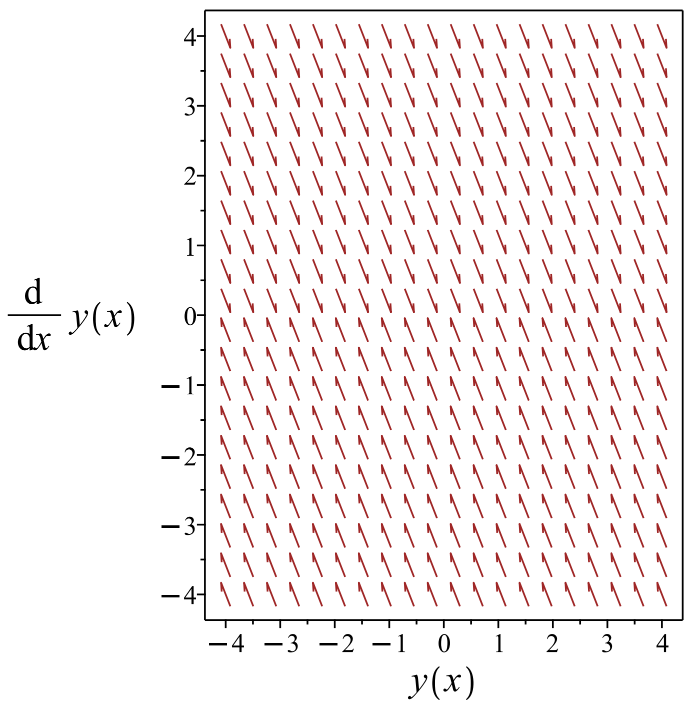


Figure 549: Slope field plot

Verification of solutions

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x} c_2$$

Verified OK.

16.38.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) + 2 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-2p-2}dp = \int dx$$
$$-\frac{\ln(-p-1)}{2} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-p-1}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{-p-1}} = e^{x}c_2$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{e^{-2x}}{c_2^2} - 1$$

Integrating both sides gives

$$y = \int -\frac{(e^{2x}c_2^2 + 1)e^{-2x}}{c_2^2}\, dx$$
$$= \frac{e^{-2x}}{2c_2^2} - \ln(e^x) + c_3$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-2x}}{2c_2^2} - \ln(e^x) + c_3 \tag{1}$$

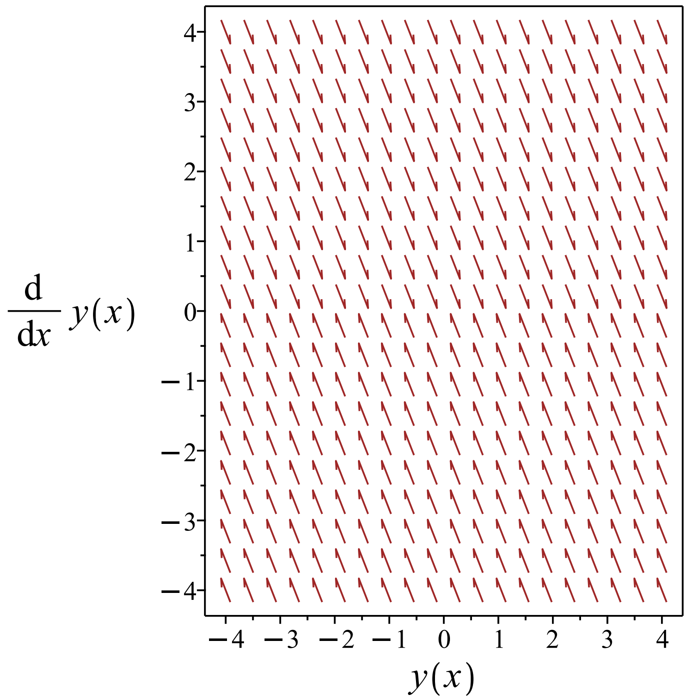


Figure 550: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}}{2c_2^2} - \ln\left(e^x\right) + c_3$$

Verified OK.

16.38.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = -2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = \int (-2)\, dx$$
$$2y + y' = -2x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= -2x + c_1 \end{aligned}$$

Hence the ode is

$$2y + y' = -2x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 2dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(-2x + c_1) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x} y\right) &= \left(\mathrm{e}^{2x}\right)(-2x + c_1) \\ \mathrm{d}\left(\mathrm{e}^{2x} y\right) &= \left((-2x + c_1)\,\mathrm{e}^{2x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{2x} y &= \int (-2x + c_1)\,\mathrm{e}^{2x}\,\mathrm{d}x \\ \mathrm{e}^{2x} y &= -\frac{\mathrm{e}^{2x}(2x - c_1 - 1)}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = -\frac{\mathrm{e}^{-2x}\mathrm{e}^{2x}(2x - c_1 - 1)}{2} + \mathrm{e}^{-2x} c_2$$

which simplifies to

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + \mathrm{e}^{-2x} c_2$$

<u>Summary</u>
The solution(s) found are the following

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + \mathrm{e}^{-2x} c_2 \tag{1}$$

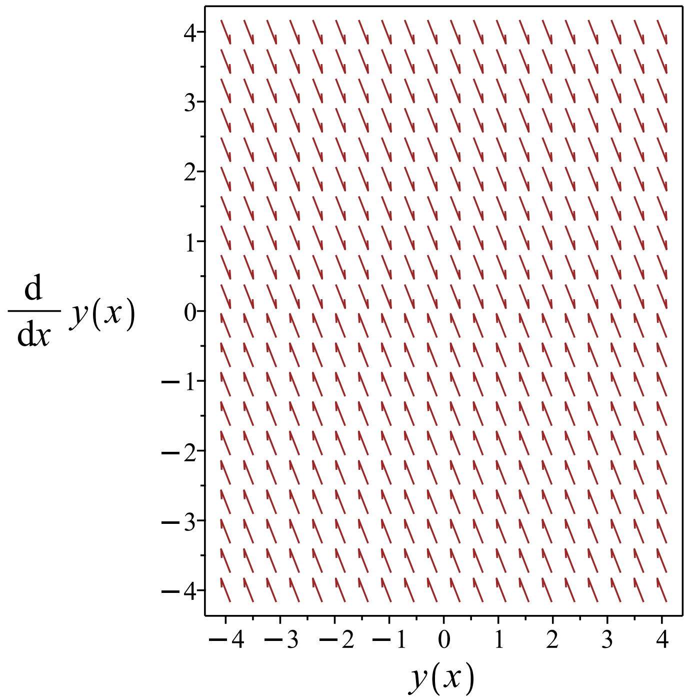


Figure 551: Slope field plot

<u>Verification of solutions</u>

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x} c_2$$

Verified OK.

16.38.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 438: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}
$$

Now we need to find the particular solution y_p to

$$
y'' + 2y' = -2
$$

Since the RHS of the ode $f(x)$ is a constant, because it does not depend on x, then let the particular solution be

$$
y_p = kx
$$

where k is a constant to be determined. Substituting $y = kx$ in the ODE gives

$$
2k = -2
$$

Therefore

$$
k = -x
$$

Hence $y_p = -x^2$. Therefore the complete solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} + (-x)
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{-2x} + \frac{c_2}{2}\right) + (-x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-2x} + \frac{c_2}{2} - x \tag{1}
$$

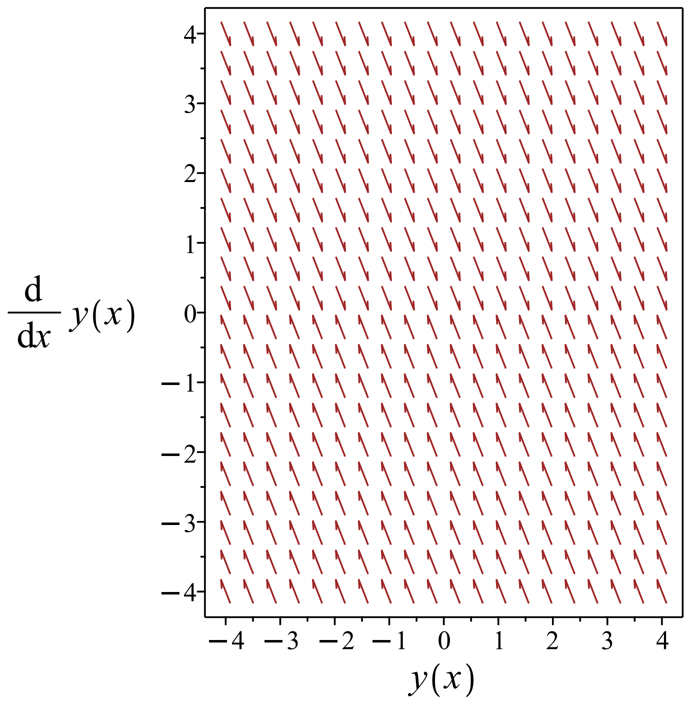


Figure 552: Slope field plot

Verification of solutions

$$
y = c_1 e^{-2x} + \frac{c_2}{2} - x
$$

Verified OK.

16.38.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= 2\\ r(x) &= 0\\ s(x) &= -2\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= 0\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$2y + y' = \int -2\,dx$$

We now have a first order ode to solve which is

$$2y + y' = -2x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= -2x + c_1 \end{aligned}$$

Hence the ode is

$$2y + y' = -2x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2dx} \\ &= e^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(-2x + c_1) \\ \frac{d}{dx}(e^{2x}y) &= (e^{2x})(-2x + c_1) \\ d(e^{2x}y) &= ((-2x + c_1)e^{2x})\ dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{2x}y &= \int (-2x + c_1)e^{2x}\,dx \\ e^{2x}y &= -\frac{e^{2x}(2x - c_1 - 1)}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = -\frac{e^{-2x}e^{2x}(2x - c_1 - 1)}{2} + e^{-2x}c_2$$

which simplifies to

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x}c_2$$

Summary

The solution(s) found are the following

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x}c_2 \tag{1}$$

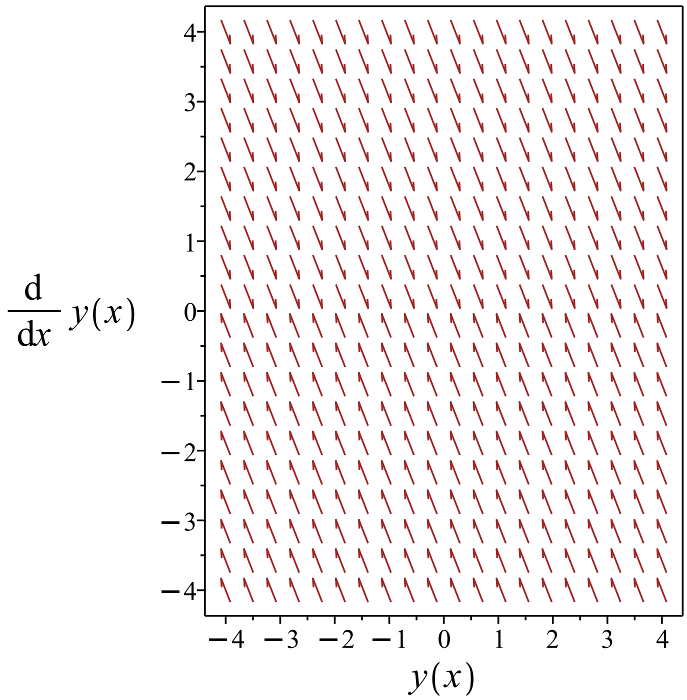


Figure 553: Slope field plot

Verification of solutions

$$y = -x + \frac{c_1}{2} + \frac{1}{2} + e^{-2x} c_2$$

Verified OK.

16.38.7 Maple step by step solution

Let's solve

$y'' + 2y' = -2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r + 2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & 1 \\ -2\,\mathrm{e}^{-2x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-2x} \left(\int \mathrm{e}^{2x} dx\right) - \left(\int 1 dx\right)$

  - Compute integrals

    $y_p(x) = \frac{1}{2} - x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + \frac{1}{2} - x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*_b(_a)-2, _b(_a)`   *** Sublevel 2 *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=-2,y(x), singsol=all)
```

$$y(x) = -\frac{e^{-2x}c_1}{2} - x + c_2$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]==-2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \frac{1}{2}c_1 e^{-2x} + c_2$$

16.39 problem 512

Internal problem ID [15282]
Internal file name [`OUTPUT/15282_Wednesday_May_08_2024_03_54_45_PM_22115679/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 512.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 9y = 9}$$

16.39.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 9$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 = 9$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + (1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + 1 \tag{1}$$

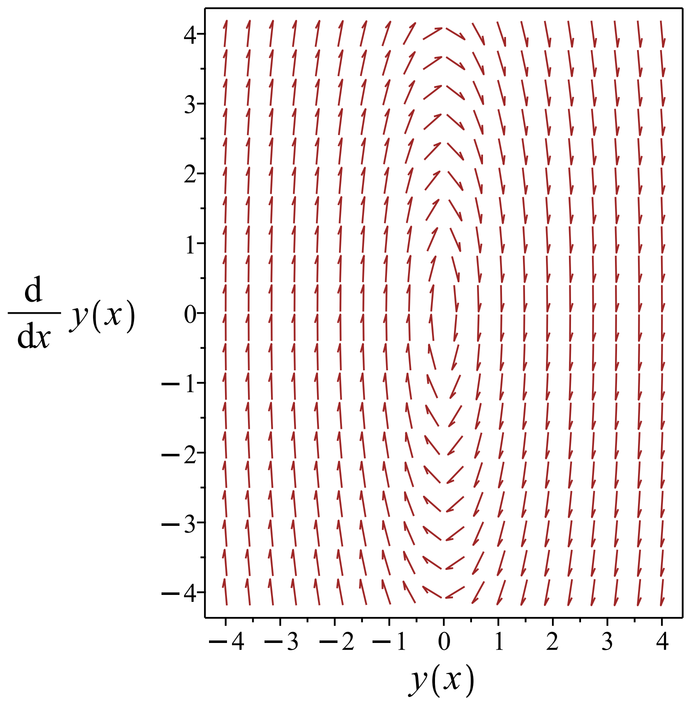


Figure 554: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + 1$$

Verified OK.

16.39.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 9yy' - 9y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 9yy' - 9y')\, dx = 0$$
$$\frac{y'^2}{2} + \frac{9y^2}{2} - 9y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-9y^2 + 18y + 2c_1} \tag{1}$$

$$y' = -\sqrt{-9y^2 + 18y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-9y^2 + 2c_1 + 18y}} dy = \int dx$$

$$\frac{\arctan\left(\frac{3y-3}{\sqrt{-9y^2+18y+2c_1}}\right)}{3} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-9y^2 + 2c_1 + 18y}} dy = \int dx$$

$$-\frac{\arctan\left(\frac{3y-3}{\sqrt{-9y^2+18y+2c_1}}\right)}{3} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{3y-3}{\sqrt{-9y^2+18y+2c_1}}\right)}{3} = x + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{3y-3}{\sqrt{-9y^2+18y+2c_1}}\right)}{3} = x + c_3 \tag{2}$$

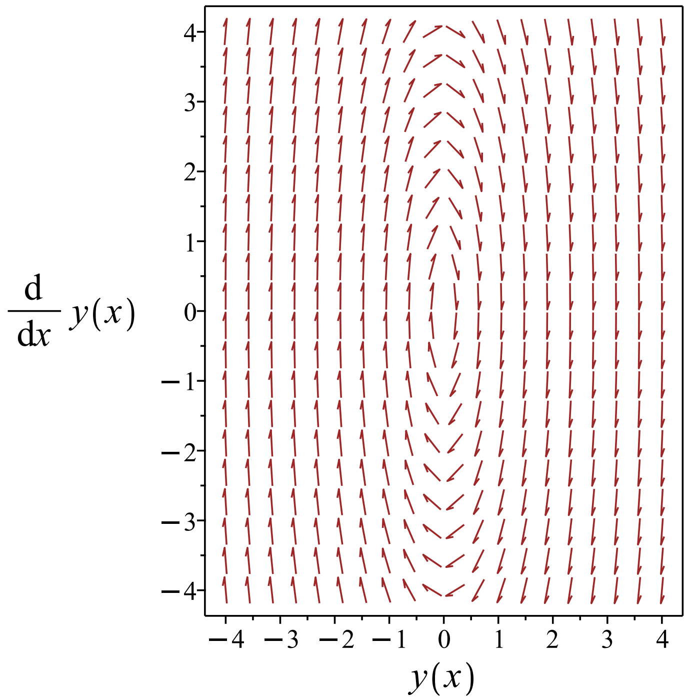


Figure 555: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{3y-3}{\sqrt{-9y^2+18y+2c_1}}\right)}{3} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{3y-3}{\sqrt{-9y^2+18y+2c_1}}\right)}{3} = x + c_3$$

Verified OK.

16.39.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 440: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(3x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x)\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 = 9$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + (1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + 1 \tag{1}$$

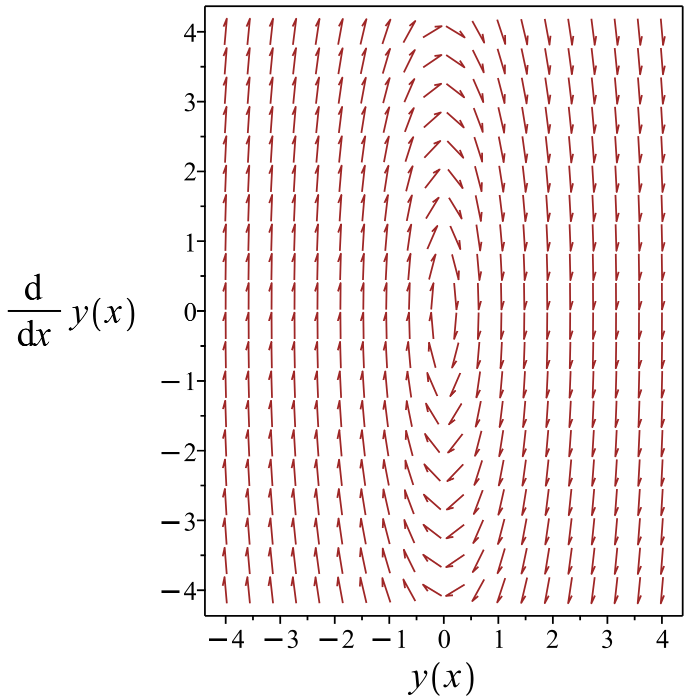


Figure 556: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + 1$$

Verified OK.

16.39.4 Maple step by step solution

Let's solve

$y'' + 9y = 9$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 9\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -3\cos(3x)\left(\int \sin(3x)\, dx\right) + 3\sin(3x)\left(\int \cos(3x)\, dx\right)$

  - Compute integrals

    $y_p(x) = 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+9*y(x)=9,y(x), singsol=all)
```

$$y(x) = \sin(3x)\, c_2 + \cos(3x)\, c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 21

```
DSolve[y''[x]+9*y[x]==9,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(3x) + c_2 \sin(3x) + 1$$

16.40 problem 513

Internal problem ID [15283]
Internal file name [`OUTPUT/15283_Wednesday_May_08_2024_03_54_46_PM_29340798/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 513.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= 1 \\
y_3 &= x
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$y''' + y'' = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + c_3 x\right) + \left(\frac{x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \frac{x^2}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + \frac{x^2}{2}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+1, _b(_a)`   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)=1,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + c_1 e^{-x} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 27

```
DSolve[y'''[x]+y''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + c_3 x + c_1 e^{-x} + c_2$$

16.41 problem 514

Internal problem ID [15284]
Internal file name [`OUTPUT/15284_Wednesday_May_08_2024_03_54_46_PM_64461918/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 514.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{5y''' - 7y'' = 3}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$5y''' - 7y'' = 0$$

The characteristic equation is

$$5\lambda^3 - 7\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{7}{5}$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + e^{\frac{7x}{5}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = e^{\frac{7x}{5}}$$

Now the particular solution to the given ODE is found

$$5y''' - 7y'' = 3$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, x, e^{\frac{7x}{5}}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-14A_1 = 3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{14}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3x^2}{14}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2x + c_1 + \mathrm{e}^{\frac{7x}{5}}c_3\right) + \left(-\frac{3x^2}{14}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2x + c_1 + \mathrm{e}^{\frac{7x}{5}}c_3 - \frac{3x^2}{14} \tag{1}$$

Verification of solutions

$$y = c_2x + c_1 + \mathrm{e}^{\frac{7x}{5}}c_3 - \frac{3x^2}{14}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (7/5)*_b(_a)+3/5, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(5*diff(y(x),x$3)-7*diff(y(x),x$2)=3,y(x), singsol=all)
```

$$y(x) = -\frac{3x^2}{14} + \frac{25\,\mathrm{e}^{\frac{7x}{5}}\,c_1}{49} + c_2x + c_3$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 30

```
DSolve[y'''[x]-7*y''[x]==3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3x^2}{14} + c_3x + \frac{1}{49}c_1e^{7x} + c_2$$

16.42 problem 515

Internal problem ID [15285]
Internal file name [OUTPUT/15285_Wednesday_May_08_2024_03_54_47_PM_13510882/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 515.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - 6y''' = -6}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 6y''' = 0$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 6$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^{6x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= x^2 \\ y_4 &= e^{6x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 6y''' = -6$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, e^{6x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-36A_1 = -6$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_3 x^2 + c_2 x + c_1 + e^{6x} c_4\right) + \left(\frac{x^3}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3 x^2 + c_2 x + c_1 + e^{6x} c_4 + \frac{x^3}{6} \tag{1}$$

Verification of solutions

$$y = c_3 x^2 + c_2 x + c_1 + e^{6x} c_4 + \frac{x^3}{6}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 6*_b(_a)-6, _b(_a)`   *** Sublevel 2 **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)=-6,y(x), singsol=all)
```

$$y(x) = \frac{e^{6x}c_1}{216} + \frac{x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 36

```
DSolve[y''''[x]-6*y'''[x]==-6,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + c_4x^2 + c_3x + \frac{1}{216}c_1e^{6x} + c_2$$

16.43 problem 516

Internal problem ID [15286]
Internal file name [`OUTPUT/15286_Wednesday_May_08_2024_03_54_47_PM_65729883/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 516.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{3y'''' + y''' = 2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$3y'''' + y''' = 0$$

The characteristic equation is

$$3\lambda^4 + \lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{3}$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^{-\frac{x}{3}}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = e^{-\frac{x}{3}}$$

Now the particular solution to the given ODE is found

$$3y'''' + y''' = 2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, x, x^2, e^{-\frac{x}{3}}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_3 x^2 + c_2 x + c_1 + e^{-\frac{x}{3}} c_4\right) + \left(\frac{x^3}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_3 x^2 + c_2 x + c_1 + e^{-\frac{x}{3}} c_4 + \frac{x^3}{3} \tag{1}$$

Verification of solutions

$$y = c_3 x^2 + c_2 x + c_1 + e^{-\frac{x}{3}} c_4 + \frac{x^3}{3}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(1/3)*_b(_a)+2/3, _b(_a)`   *** Sublev
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(3*diff(y(x),x$4)+diff(y(x),x$3)=2,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{3} + \frac{c_2 x^2}{2} - 27\,\mathrm{e}^{-\frac{x}{3}} c_1 + c_3 x + c_4$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 36

```
DSolve[3*y''''[x]+y'''[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} + c_4 x^2 + c_3 x - 27 c_1 e^{-x/3} + c_2$$

16.44 problem 517

Internal problem ID [15287]
Internal file name [`OUTPUT/15287_Wednesday_May_08_2024_03_54_48_PM_79180729/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 517.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 2y''' + 2y'' - 2y' + y = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' + 2y'' - 2y' + y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + x e^x c_2 + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x e^x$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' + 2y'' - 2y' + y = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^x, e^x, e^{ix}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + x\, e^x c_2 + e^{ix} c_3 + e^{-ix} c_4\right) + (1) \end{aligned}$$

Which simplifies to

$$y = e^{-ix} c_4 + e^{ix} c_3 + e^x (c_2 x + c_1) + 1$$

Summary

The solution(s) found are the following

$$y = e^{-ix} c_4 + e^{ix} c_3 + e^x (c_2 x + c_1) + 1 \tag{1}$$

Verification of solutions

$$y = e^{-ix} c_4 + e^{ix} c_3 + e^x (c_2 x + c_1) + 1$$

Verified OK.

16.44.1 Maple step by step solution

Let's solve

$y'''' - 2y''' + 2y'' - 2y' + y = 1$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 1 + 2y_4(x) - 2y_3(x) + 2y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 1 + 2y_4(x) - 2y_3(x) + 2y_2(x) - y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right], \left[1, \left[\begin{array}{c}0\\0\\0\\0\end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c}-\mathrm{I}\\-1\\\mathrm{I}\\1\end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c}\mathrm{I}\\-1\\-\mathrm{I}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right]$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{array}\right]-1\cdot\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x)=\mathrm{e}^{x}\cdot\left(x\cdot\left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]+\left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I},\left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x}\cdot\left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^x & (x-1)e^x & -\sin(x) & -\cos(x) \\ e^x & x e^x & -\cos(x) & \sin(x) \\ e^x & x e^x & \sin(x) & \cos(x) \\ e^x & x e^x & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & (x-1)\,e^{x} & -\sin(x) & -\cos(x) \\ e^{x} & x\,e^{x} & -\cos(x) & \sin(x) \\ e^{x} & x\,e^{x} & \sin(x) & \cos(x) \\ e^{x} & x\,e^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -(x-1)\,e^{x} & x\,e^{x} - \frac{e^{x}}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & -x\,e^{x} + e^{x} - \cos(x) & x\,e^{x} - \frac{e^{x}}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -x\,e^{x} & \frac{e^{x}}{2} + x\,e^{x} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -x\,e^{x} + \sin(x) & \frac{e^{x}}{2} + x\,e^{x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -x\,e^{x} & \frac{e^{x}}{2} + x\,e^{x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -x\,e^{x} + \cos(x) & \frac{e^{x}}{2} + x\,e^{x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ -x\,e^{x} & \frac{e^{x}}{2} + x\,e^{x} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} & -x\,e^{x} - \sin(x) & \frac{e^{x}}{2} + x\,e^{x} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} x\,e^x - \frac{3\,e^x}{2} + 1 + \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ x\,e^x - \frac{e^x}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ x\,e^x - \frac{e^x}{2} + 1 - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ x\,e^x - \frac{e^x}{2} + 1 + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} x\,e^x - \frac{3\,e^x}{2} + 1 + \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ x\,e^x - \frac{e^x}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ x\,e^x - \frac{e^x}{2} + 1 - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ x\,e^x - \frac{e^x}{2} + 1 + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((2x-2)c_2+2x+2c_1-3)e^x}{2} + \frac{(-2c_4+1)\cos(x)}{2} + 1 + \frac{(-2c_3+1)\sin(x)}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+2*diff(y(x),x$2)-2*diff(y(x),x)+y(x)=1,y(x), singsol=
```

$$y(x) = (c_4x + c_2)\,e^x + \cos(x)\,c_1 + c_3\sin(x) + 1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
DSolve[y''''[x]-2*y'''[x]+2*y''[x]-2*y'[x]+y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^x + c_4 e^x x + c_1 \cos(x) + c_2 \sin(x) + 1$$

16.45 problem 518

Internal problem ID [15288]
Internal file name [`OUTPUT/15288_Wednesday_May_08_2024_03_54_51_PM_96272005/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 518.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 4y = x^2}$$

16.45.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x\, e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 x\, e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3x^2 + 4A_2x - 8xA_3 + 4A_1 - 4A_2 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{8}, A_2 = \frac{1}{2}, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + c_2x\,\mathrm{e}^{2x}\right) + \left(\frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2x + c_1) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_2x + c_1) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} \tag{1}$$

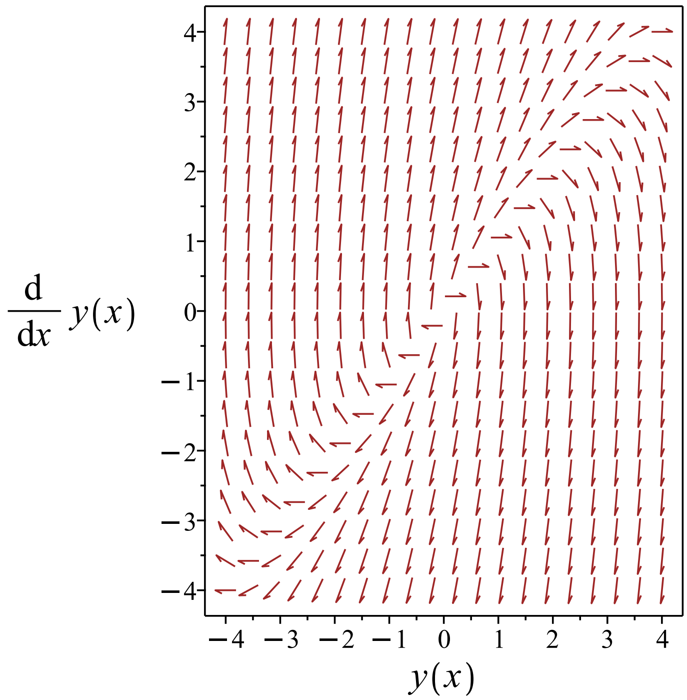


Figure 557: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

Verified OK.

16.45.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4\, dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \mathrm{e}^{-2x}x^2$$
$$\left(\mathrm{e}^{-2x}y\right)'' = \mathrm{e}^{-2x}x^2$$

Integrating once gives

$$\left(\mathrm{e}^{-2x}y\right)' = -\frac{(2x^2+2x+1)\,\mathrm{e}^{-2x}}{4} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2x}y\right) = \frac{(2x^2+4x+3)\,\mathrm{e}^{-2x}}{8} + c_1x + c_2$$

Hence the solution is

$$y = \frac{\frac{(2x^2+4x+3)\mathrm{e}^{-2x}}{8} + c_1x + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}c_2 + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{2x} + \mathrm{e}^{2x}c_2 + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} \tag{1}$$

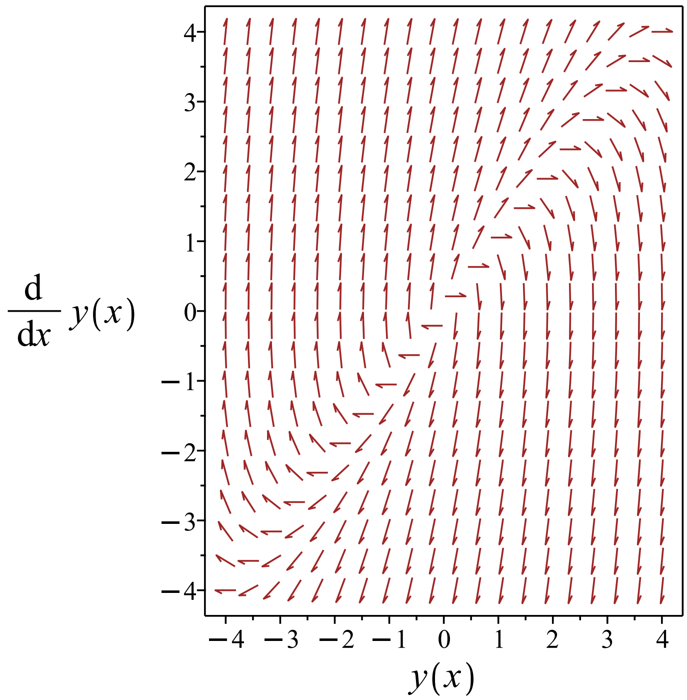


Figure 558: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

Verified OK.

16.45.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\mathrm{e}^{2x}) + c_2(\mathrm{e}^{2x}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{2x}c_1 + c_2x\,\mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{2x}, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3x^2 + A_2x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_3x^2 + 4A_2x - 8xA_3 + 4A_1 - 4A_2 + 2A_3 = x^2
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{8}, A_2 = \frac{1}{2}, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}c_1 + c_2 x\, e^{2x}\right) + \left(\frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} \tag{1}$$

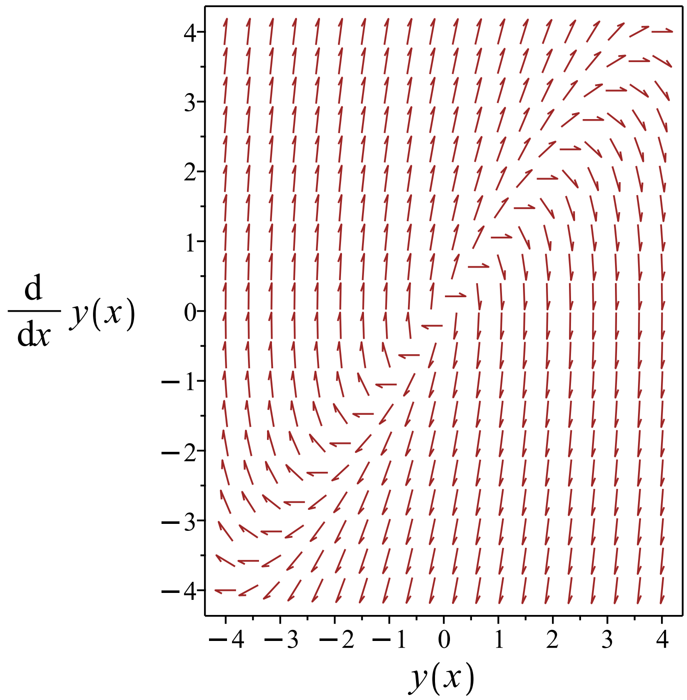


Figure 559: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$$

Verified OK.

16.45.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)^2 = 0$

- Root of the characteristic polynomial

$r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{2x} c_1 + c_2 x\,\mathrm{e}^{2x} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & \mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{2x}\left(-\left(\int x^3 \mathrm{e}^{-2x} dx\right) + \left(\int \mathrm{e}^{-2x} x^2 dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{2x} + \mathrm{e}^{2x} c_1 + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{3}{8} + (c_1 x + c_2)\, \mathrm{e}^{2x} + \frac{x^2}{4} + \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 37

```
DSolve[y''[x]-4*y'[x]+4*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{8}\left(2x^2 + 4x + 3\right) + c_1 e^{2x} + c_2 e^{2x} x$$

16.46 problem 519

Internal problem ID [15289]
Internal file name [`OUTPUT/15289_Wednesday_May_08_2024_03_54_52_PM_50072634/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 519.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 8y' = 8x}$$

16.46.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 8, C = 0, f(x) = 8x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+8y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=8, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+8\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+8\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=8, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-8}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{8^2-(4)\,(1)\,(0)}\\&=-4\pm 4\end{aligned}$$

Hence

$$\lambda_1=-4+4$$
$$\lambda_2=-4-4$$

Which simplifies to

$$\lambda_1=0$$
$$\lambda_2=-8$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$$
$$y=c_1e^{(0)x}+c_2e^{(-8)x}$$

Or

$$y = c_1 + c_2 e^{-8x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-8x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-8x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16xA_2 + 8A_1 + 2A_2 = 8x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 - \frac{1}{8}x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + c_2 e^{-8x}\right) + \left(\frac{1}{2}x^2 - \frac{1}{8}x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-8x} + \frac{x^2}{2} - \frac{x}{8} \tag{1}$$

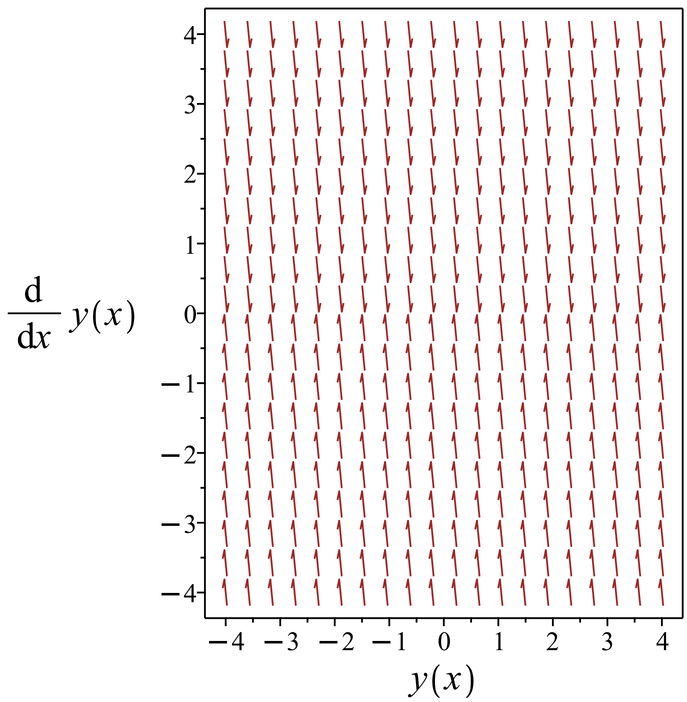


Figure 560: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-8x} + \frac{x^2}{2} - \frac{x}{8}$$

Verified OK.

16.46.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 8y')\,dx = \int 8x dx$$
$$8y + y' = 4x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 8$$
$$q(x) = 4x^2 + c_1$$

Hence the ode is

$$8y + y' = 4x^2 + c_1$$

The integrating factor μ is

$$\mu = e^{\int 8dx}$$
$$= e^{8x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(4x^2 + c_1\right)$$
$$\frac{d}{dx}\left(e^{8x} y\right) = \left(e^{8x}\right)\left(4x^2 + c_1\right)$$
$$d\left(e^{8x} y\right) = \left(\left(4x^2 + c_1\right) e^{8x}\right) dx$$

Integrating gives

$$e^{8x} y = \int \left(4x^2 + c_1\right) e^{8x}\, dx$$
$$e^{8x} y = \frac{\left(32x^2 + 8c_1 - 8x + 1\right) e^{8x}}{64} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{8x}$ results in

$$y = \frac{e^{-8x}\left(32x^2 + 8c_1 - 8x + 1\right) e^{8x}}{64} + c_2 e^{-8x}$$

which simplifies to

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x} \tag{1}$$

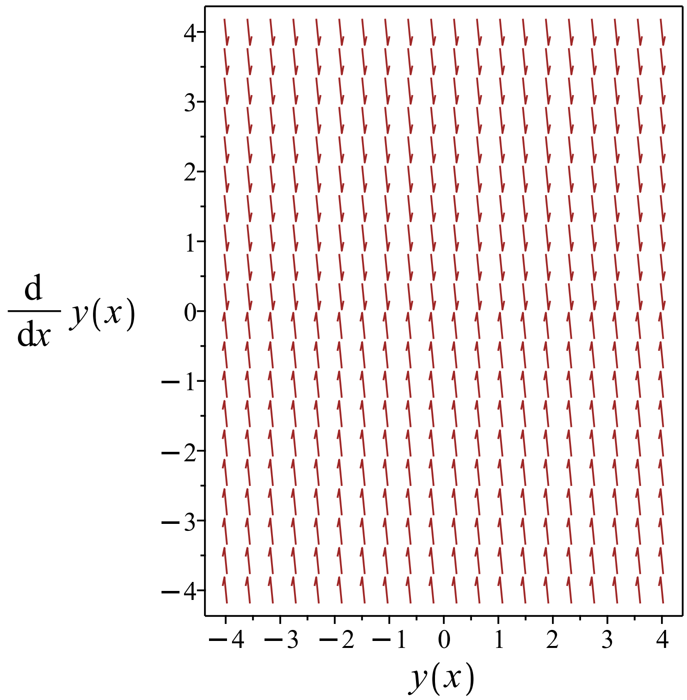


Figure 561: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x}$$

Verified OK.

16.46.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 8p(x) - 8x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 8 \\ q(x) &= 8x \end{aligned}$$

Hence the ode is

$$p'(x) + 8p(x) = 8x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 8dx} \\ &= \mathrm{e}^{8x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)(8x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{8x} p\right) &= \left(\mathrm{e}^{8x}\right)(8x) \\ \mathrm{d}\left(\mathrm{e}^{8x} p\right) &= \left(8x\,\mathrm{e}^{8x}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{8x} p &= \int 8x\,\mathrm{e}^{8x}\,\mathrm{d}x \\ \mathrm{e}^{8x} p &= \frac{(8x-1)\,\mathrm{e}^{8x}}{8} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{8x}$ results in

$$p(x) = \frac{e^{-8x}(8x-1)\,e^{8x}}{8} + c_1 e^{-8x}$$

which simplifies to

$$p(x) = x - \frac{1}{8} + c_1 e^{-8x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x - \frac{1}{8} + c_1 e^{-8x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x - \frac{1}{8} + c_1 e^{-8x}\, dx \\ &= \frac{x^2}{2} - \frac{x}{8} - \frac{c_1 e^{-8x}}{8} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - \frac{x}{8} - \frac{c_1 e^{-8x}}{8} + c_2 \tag{1}$$

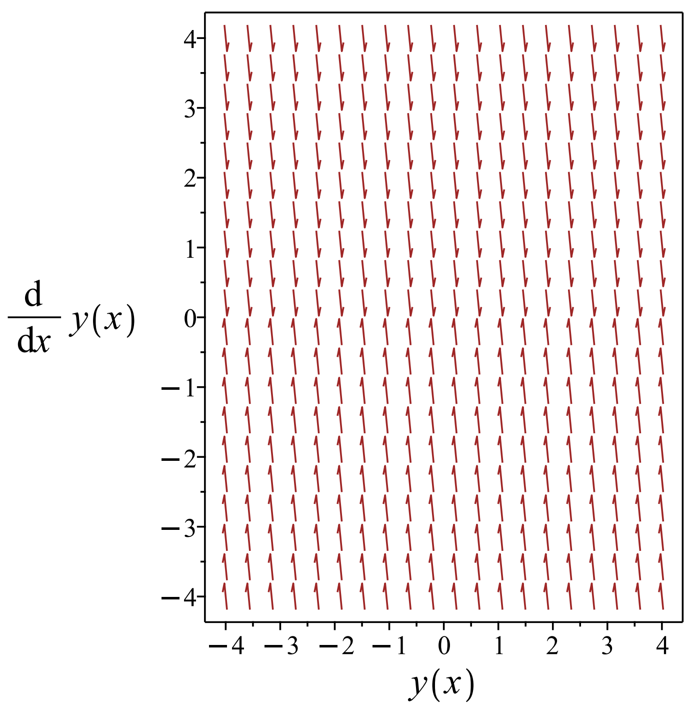


Figure 562: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{x}{8} - \frac{c_1 \mathrm{e}^{-8x}}{8} + c_2$$

Verified OK.

16.46.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 8y' = 8x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 8y')\, dx = \int 8x dx$$
$$8y + y' = 4x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 8$$
$$q(x) = 4x^2 + c_1$$

Hence the ode is

$$8y + y' = 4x^2 + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 8dx}$$
$$= \mathrm{e}^{8x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(4x^2 + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{8x} y\right) = \left(\mathrm{e}^{8x}\right)\left(4x^2 + c_1\right)$$
$$\mathrm{d}\left(\mathrm{e}^{8x} y\right) = \left(\left(4x^2 + c_1\right)\mathrm{e}^{8x}\right)\,\mathrm{d}x$$

Integrating gives

$$e^{8x}y = \int \left(4x^2 + c_1\right) e^{8x}\, dx$$

$$e^{8x}y = \frac{(32x^2 + 8c_1 - 8x + 1)\, e^{8x}}{64} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{8x}$ results in

$$y = \frac{e^{-8x}(32x^2 + 8c_1 - 8x + 1)\, e^{8x}}{64} + c_2 e^{-8x}$$

which simplifies to

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x} \tag{1}$$

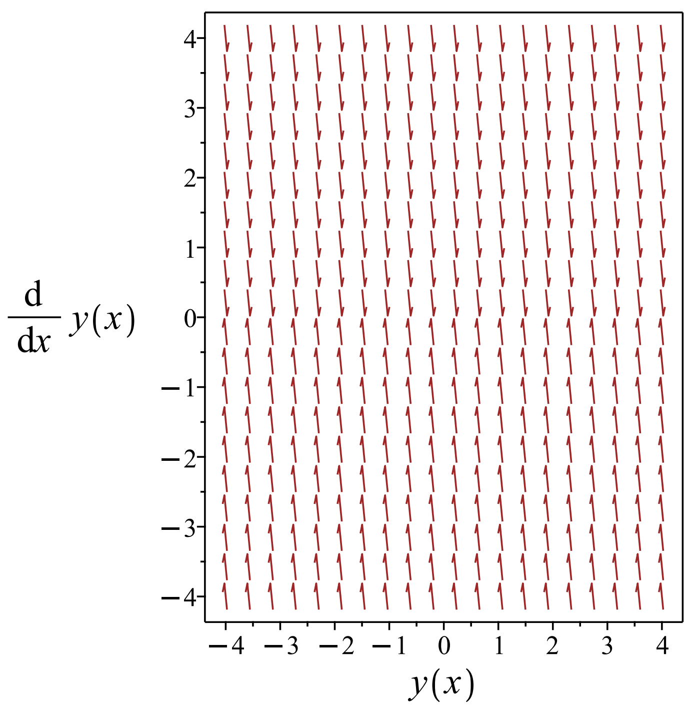


Figure 563: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x}$$

Verified OK.

16.46.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 8y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 16 z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-4x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\,dx} \\
&= z_1 e^{-4x} \\
&= z_1 \left(\mathrm{e}^{-4x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-8x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-8x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{8x}}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-8x}\right) + c_2 \left(\mathrm{e}^{-8x} \left(\frac{\mathrm{e}^{8x}}{8}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 8y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-8x} + \frac{c_2}{8}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-8x} \\ y_2 &= \frac{1}{8} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-8x} & \frac{1}{8} \\ \frac{d}{dx}(\mathrm{e}^{-8x}) & \frac{d}{dx}\left(\frac{1}{8}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-8x} & \frac{1}{8} \\ -8\,e^{-8x} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{-8x}\right)(0) - \left(\frac{1}{8}\right)\left(-8\,e^{-8x}\right)$$

Which simplifies to

$$W = e^{-8x}$$

Which simplifies to

$$W = e^{-8x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x}{e^{-8x}}\,dx$$

Which simplifies to

$$u_1 = -\int x\,e^{8x} dx$$

Hence

$$u_1 = -\frac{(8x-1)\,e^{8x}}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,e^{-8x}x}{e^{-8x}}\,dx$$

Which simplifies to

$$u_2 = \int 8x dx$$

Hence

$$u_2 = 4x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-8x}(8x-1)\,e^{8x}}{64} + \frac{x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{1}{8}x + \frac{1}{64} + \frac{1}{2}x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-8x} + \frac{c_2}{8}\right) + \left(-\frac{1}{8}x + \frac{1}{64} + \frac{1}{2}x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-8x} + \frac{c_2}{8} - \frac{x}{8} + \frac{1}{64} + \frac{x^2}{2} \tag{1}$$

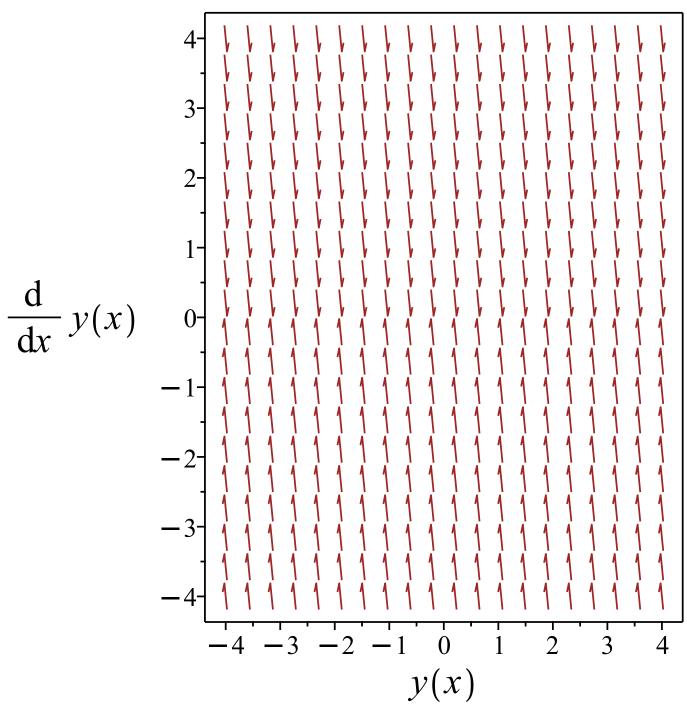


Figure 564: Slope field plot

Verification of solutions

$$y = c_1 e^{-8x} + \frac{c_2}{8} - \frac{x}{8} + \frac{1}{64} + \frac{x^2}{2}$$

Verified OK.

16.46.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 8 \\ r(x) &= 0 \\ s(x) &= 8x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$8y + y' = \int 8x\, dx$$

We now have a first order ode to solve which is

$$8y + y' = 4x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 8 \\ q(x) &= 4x^2 + c_1 \end{aligned}$$

Hence the ode is

$$8y + y' = 4x^2 + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 8dx} \\ &= \mathrm{e}^{8x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(4x^2 + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{8x} y\right) &= \left(\mathrm{e}^{8x}\right)\left(4x^2 + c_1\right) \\ \mathrm{d}\left(\mathrm{e}^{8x} y\right) &= \left(\left(4x^2 + c_1\right)\mathrm{e}^{8x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{8x} y &= \int \left(4x^2 + c_1\right)\mathrm{e}^{8x}\,\mathrm{d}x \\ \mathrm{e}^{8x} y &= \frac{\left(32x^2 + 8c_1 - 8x + 1\right)\mathrm{e}^{8x}}{64} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{8x}$ results in

$$y = \frac{\mathrm{e}^{-8x}\left(32x^2 + 8c_1 - 8x + 1\right)\mathrm{e}^{8x}}{64} + c_2\mathrm{e}^{-8x}$$

which simplifies to

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2\mathrm{e}^{-8x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x} \tag{1}$$

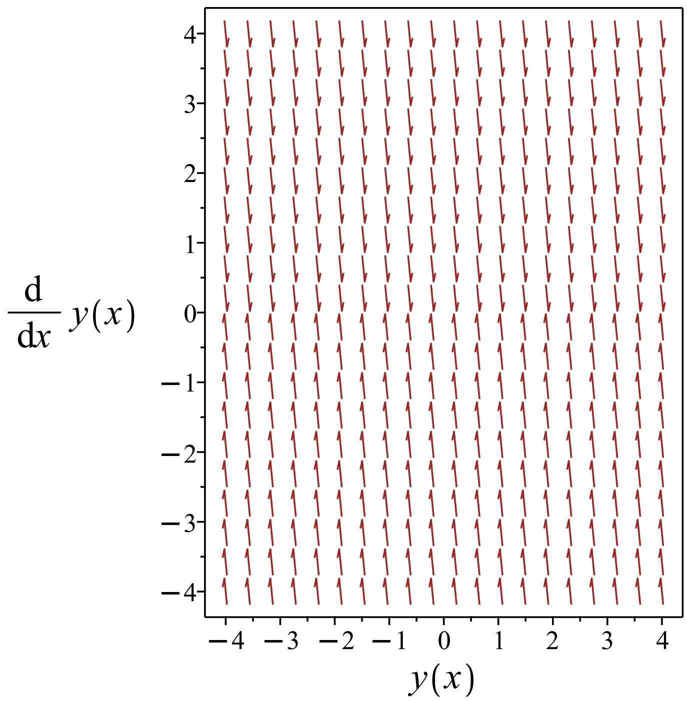


Figure 565: Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + \frac{c_1}{8} - \frac{x}{8} + \frac{1}{64} + c_2 e^{-8x}$$

Verified OK.

16.46.7 Maple step by step solution

Let's solve

$y'' + 8y' = 8x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 8r = 0$

- Factor the characteristic polynomial

  $r(r+8) = 0$

- Roots of the characteristic polynomial

  $r = (-8, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-8x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-8x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-8x} & 1 \\ -8\,e^{-8x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 8\,e^{-8x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^{-8x}\left(\int x\,e^{8x}dx\right) + \int x dx$

  - Compute integrals

    $y_p(x) = -\frac{1}{8}x + \frac{1}{64} + \frac{1}{2}x^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-8x} + c_2 - \frac{x}{8} + \frac{1}{64} + \frac{x^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -8*_b(_a)+8*_a, _b(_a)`   *** Sublevel
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+8*diff(y(x),x)=8*x,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - \frac{e^{-8x}c_1}{8} - \frac{x}{8} + c_2$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 31

```
DSolve[y''[x]+8*y'[x]==8*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - \frac{x}{8} - \frac{1}{8}c_1e^{-8x} + c_2$$

16.47 problem 520

Internal problem ID [15290]
Internal file name [`OUTPUT/15290_Wednesday_May_08_2024_03_54_54_PM_60428976/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 520.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y'k + k^2y = e^x}$$

16.47.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2k, C = k^2, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y'k + k^2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2k, C = k^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2k\lambda\, e^{\lambda x} + k^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$k^2 - 2k\lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2k, C = k^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2k}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2k)^2 - (4)\,(1)\,(k^2)}\\ &= k\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -k$. Therefore the solution is

$$y = c_1 e^{kx} + c_2 x\, e^{kx} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{kx} + c_2 x\, e^{kx}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{kx}, e^{kx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^x - 2A_1 \mathrm{e}^x k + k^2 A_1 \mathrm{e}^x = \mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{(k-1)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\mathrm{e}^x}{(k-1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{kx} + c_2 x\, \mathrm{e}^{kx}\right) + \left(\frac{\mathrm{e}^x}{(k-1)^2}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{kx}(c_2 x + c_1) + \frac{\mathrm{e}^x}{(k-1)^2}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{kx}(c_2 x + c_1) + \frac{\mathrm{e}^x}{(k-1)^2} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{kx}(c_2 x + c_1) + \frac{\mathrm{e}^x}{(k-1)^2}$$

Verified OK.

16.47.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2k$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2k\, dx} \\ &= \mathrm{e}^{-kx} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-kx}\mathrm{e}^{x} \\ \left(\mathrm{e}^{-kx}y\right)'' &= \mathrm{e}^{-kx}\mathrm{e}^{x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-kx}y\right)' = -\frac{\mathrm{e}^{-x(k-1)}}{k-1} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-kx}y\right) = \frac{\mathrm{e}^{-x(k-1)} + x(k^2 - 2k + 1)\, c_1}{(k-1)^2} + c_2$$

Hence the solution is

$$y = \frac{\frac{\mathrm{e}^{-x(k-1)} + x(k^2-2k+1)c_1}{(k-1)^2} + c_2}{\mathrm{e}^{-kx}}$$

Or

$$y = c_2\mathrm{e}^{kx} + \left(\frac{k^2 x\, \mathrm{e}^{kx}}{(k-1)^2} - \frac{2kx\, \mathrm{e}^{kx}}{(k-1)^2} + \frac{x\, \mathrm{e}^{kx}}{(k-1)^2}\right) c_1 + \frac{\mathrm{e}^{x}}{(k-1)^2}$$

Summary

The solution(s) found are the following

$$y = c_2\mathrm{e}^{kx} + \left(\frac{k^2 x\, \mathrm{e}^{kx}}{(k-1)^2} - \frac{2kx\, \mathrm{e}^{kx}}{(k-1)^2} + \frac{x\, \mathrm{e}^{kx}}{(k-1)^2}\right) c_1 + \frac{\mathrm{e}^{x}}{(k-1)^2} \tag{1}$$

Verification of solutions

$$y = c_2 e^{kx} + \left(\frac{k^2 x\, e^{kx}}{(k-1)^2} - \frac{2kx\, e^{kx}}{(k-1)^2} + \frac{x\, e^{kx}}{(k-1)^2}\right) c_1 + \frac{e^x}{(k-1)^2}$$

Verified OK.

16.47.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y'k + k^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2k \\ C &= k^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2k}{1}\,dx} \\
&= z_1 e^{kx} \\
&= z_1 \left(\mathrm{e}^{kx}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{kx}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2k}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2kx}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{kx}\right) + c_2 \left(\mathrm{e}^{kx}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y'k + k^2 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{kx} + c_2 x\, e^{kx}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{kx}, e^{kx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{x} - 2A_1 e^{x} k + k^2 A_1 e^{x} = e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{(k-1)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{x}}{(k-1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{kx} + c_2 x\, e^{kx}\right) + \left(\frac{e^{x}}{(k-1)^2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{kx}(c_2x + c_1) + \frac{e^x}{(k-1)^2}$$

Summary
The solution(s) found are the following

$$y = e^{kx}(c_2x + c_1) + \frac{e^x}{(k-1)^2} \tag{1}$$

Verification of solutions

$$y = e^{kx}(c_2x + c_1) + \frac{e^x}{(k-1)^2}$$

Verified OK.

16.47.4 Maple step by step solution

Let's solve

$y'' - 2y'k + k^2y = e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $k^2 - 2rk + r^2 = 0$

- Factor the characteristic polynomial

  $(k-r)^2 = 0$

- Root of the characteristic polynomial

  $r = k$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{kx}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{kx}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{kx} + c_2 x\,e^{kx} + y_p(x)$$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = e^x\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{kx} & x\,e^{kx} \\ k\,e^{kx} & e^{kx} + kx\,e^{kx} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{2kx}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = e^{kx}\left(-\left(\int x\,e^{-x(k-1)}dx\right) + \left(\int e^{-x(k-1)}dx\right)x\right)$$

- Compute integrals

$$y_p(x) = \frac{e^x}{(k-1)^2}$$

• Substitute particular solution into general solution to ODE

$$y = c_1 e^{kx} + c_2 x\,e^{kx} + \frac{e^x}{(k-1)^2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-2*k*diff(y(x),x)+k^2*y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(k-1)^2 (c_1 x + c_2) e^{kx} + e^x}{(k-1)^2}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 28

```
DSolve[y''[x]-2*k*y'[x]+k^2*y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{(k-1)^2} + (c_2 x + c_1) e^{kx}$$

16.48 problem 521

Internal problem ID [15291]
Internal file name [`OUTPUT/15291_Wednesday_May_08_2024_03_54_55_PM_49721009/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 521.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 4y' + 4y = 8\,\mathrm{e}^{-2x}}$$

16.48.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = 8\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 x\, e^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + c_2 x\, e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\, e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-2x}, e^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\}]$$

Since $x\,e^{-2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-2x} = 8\,e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4\,e^{-2x} x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 x\,e^{-2x}\right) + \left(4\,e^{-2x} x^2\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_2 x + c_1) + 4\,e^{-2x} x^2$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_2 x + c_1) + 4\,e^{-2x} x^2 \tag{1}$$

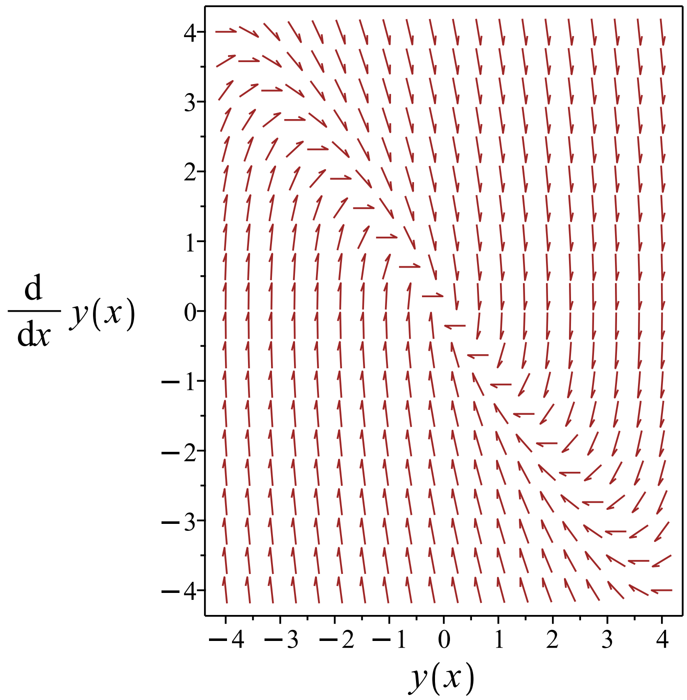


Figure 566: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + 4\,e^{-2x}x^2$$

Verified OK.

16.48.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 8\,e^{2x}e^{-2x}$$
$$\left(e^{2x}y\right)'' = 8\,e^{2x}e^{-2x}$$

Integrating once gives

$$\left(e^{2x}y\right)' = 8x + c_1$$

Integrating again gives

$$\left(e^{2x}y\right) = x(c_1 + 4x) + c_2$$

Hence the solution is

$$y = \frac{x(c_1 + 4x) + c_2}{e^{2x}}$$

Or

$$y = c_1 x\,e^{-2x} + 4\,e^{-2x}x^2 + e^{-2x}c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x\,e^{-2x} + 4\,e^{-2x}x^2 + e^{-2x}c_2 \tag{1}$$

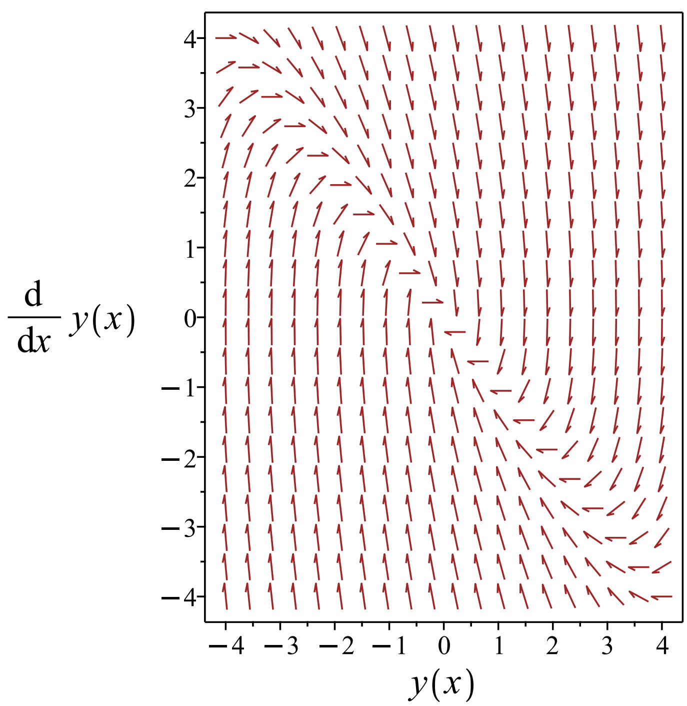


Figure 567: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-2x} + 4\,\mathrm{e}^{-2x} x^2 + \mathrm{e}^{-2x} c_2$$

Verified OK.

16.48.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + c_2 x \, e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \, e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \, e^{-2x}, e^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \, e^{-2x}\}]$$

Since $x \, e^{-2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x} x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-2x} = 8 \, e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 \, e^{-2x} x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + c_2 x\, \mathrm{e}^{-2x}\right) + \left(4\, \mathrm{e}^{-2x} x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + 4\, \mathrm{e}^{-2x} x^2
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + 4\, \mathrm{e}^{-2x} x^2 \tag{1}
$$

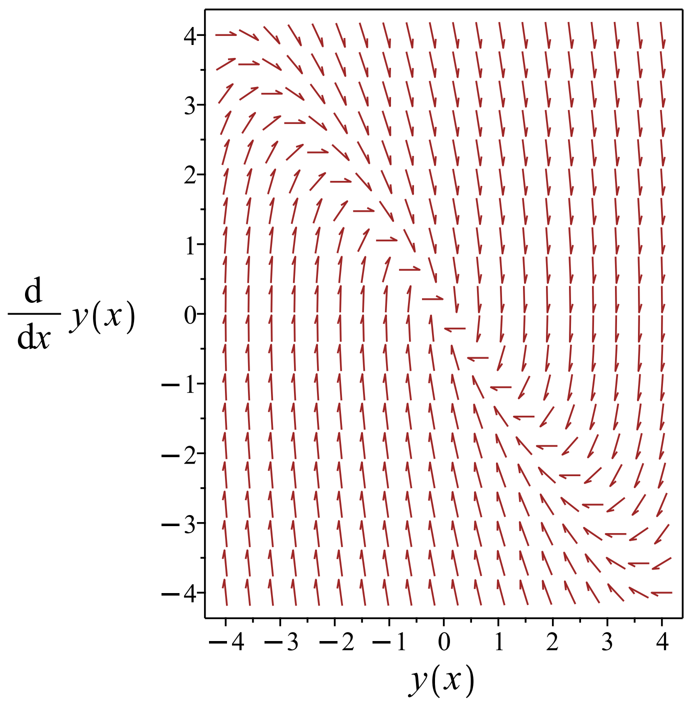


Figure 568: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + 4\, \mathrm{e}^{-2x} x^2
$$

Verified OK.

16.48.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = 8\,\mathrm{e}^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 8\,\mathrm{e}^{-2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -8\,\mathrm{e}^{-2x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = 4\,e^{-2x}x^2$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^{-2x} + 4\,e^{-2x}x^2 + c_1 e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=8*exp(-2*x),y(x), singsol=all)
```

$$y(x) = e^{-2x}(c_1 x + 4x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 23

```
DSolve[y''[x]+4*y'[x]+4*y[x]==8*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(4x^2 + c_2 x + c_1)$$

16.49 problem 522

Internal problem ID [15292]
Internal file name [`OUTPUT/15292_Wednesday_May_08_2024_03_54_56_PM_89602240/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 522.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 3y = 9\,\mathrm{e}^{-3x}}$$

16.49.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 3, f(x) = 9\,\mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(3)} \\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1 \\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,\mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}, \mathrm{e}^{-x}\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-3x} = 9\,\mathrm{e}^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{9}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{9x\,\mathrm{e}^{-3x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-3x}\right) + \left(-\frac{9x\,\mathrm{e}^{-3x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{9x\,e^{-3x}}{2} \tag{1}$$

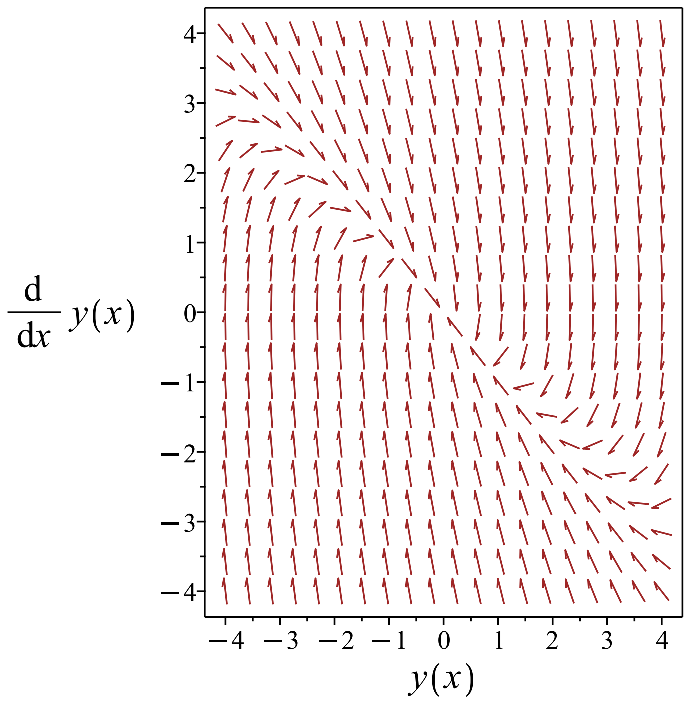


Figure 569: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{9x\,e^{-3x}}{2}$$

Verified OK.

16.49.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 4 \\
C &= 3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1\left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{\mathrm{e}^{-x} c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,\mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{-x}}{2}, \mathrm{e}^{-3x}\right\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-3x} = 9\,\mathrm{e}^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{9}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{9x\,\mathrm{e}^{-3x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-3x} + \frac{\mathrm{e}^{-x}c_2}{2}\right) + \left(-\frac{9x\,\mathrm{e}^{-3x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + \frac{\mathrm{e}^{-x} c_2}{2} - \frac{9x \, \mathrm{e}^{-3x}}{2} \tag{1}$$

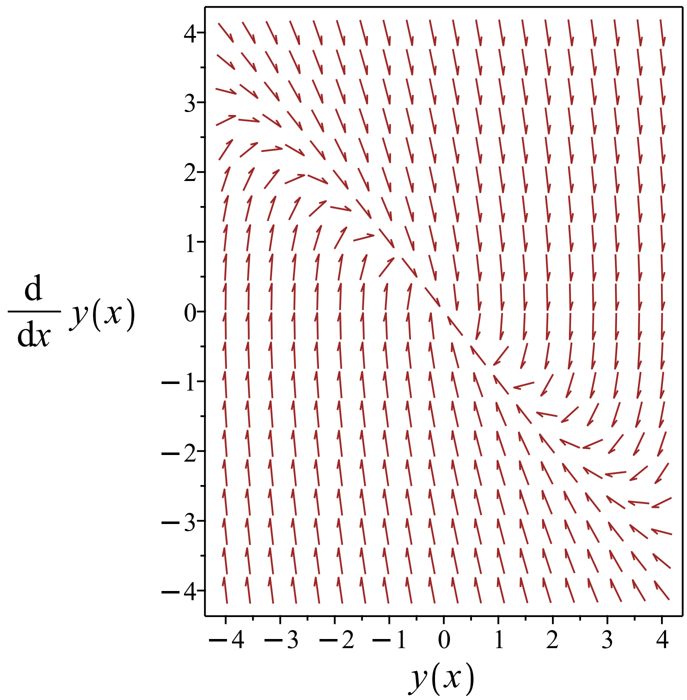


Figure 570: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-3x} + \frac{\mathrm{e}^{-x} c_2}{2} - \frac{9x \, \mathrm{e}^{-3x}}{2}$$

Verified OK.

16.49.3 Maple step by step solution

Let's solve

$y'' + 4y' + 3y = 9 \, \mathrm{e}^{-3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-3, -1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 9\,\mathrm{e}^{-3x}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{-x} \\ -3\,\mathrm{e}^{-3x} & -\mathrm{e}^{-x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-4x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{9\,\mathrm{e}^{-3x}\left(\int 1 dx\right)}{2} + \frac{9\,\mathrm{e}^{-x}\left(\int \mathrm{e}^{-2x} dx\right)}{2}$
  - Compute integrals

    $y_p(x) = -\frac{9\,\mathrm{e}^{-3x}(1+2x)}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{-x} c_2 - \frac{9\,\mathrm{e}^{-3x}(1+2x)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+3*y(x)=9*exp(-3*x),y(x), singsol=all)
```

$$y(x) = \frac{(-9x + 2c_2)\,\mathrm{e}^{-3x}}{2} + c_1\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 32

```
DSolve[y''[x]+4*y'[x]+3*y[x]==9*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-3x}\left(-18x + 4c_2e^{2x} - 9 + 4c_1\right)$$

16.50 problem 523

Internal problem ID [15293]
Internal file name [`OUTPUT/15293_Wednesday_May_08_2024_03_54_57_PM_6482856/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 523.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{7y'' - y' = 14x}$$

16.50.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 7, B = -1, C = 0, f(x) = 14x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$7y''-y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=7, B=-1, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$7\lambda^2 e^{\lambda x}-\lambda e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$7\lambda^2-\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=7, B=-1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{1}{(2)(7)}\pm\frac{1}{(2)(7)}\sqrt{-1^2-(4)(7)(0)}\\&=\frac{1}{14}\pm\frac{1}{14}\end{aligned}$$

Hence

$$\lambda_1=\frac{1}{14}+\frac{1}{14}$$
$$\lambda_2=\frac{1}{14}-\frac{1}{14}$$

Which simplifies to

$$\lambda_1=\frac{1}{7}$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$$
$$y=c_1e^{\left(\frac{1}{7}\right)x}+c_2e^{(0)x}$$

Or

$$y = c_1 e^{\frac{x}{7}} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\frac{x}{7}} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{\frac{x}{7}}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2xA_2 - A_1 + 14A_2 = 14x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -98, A_2 = -7]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -7x^2 - 98x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\frac{x}{7}} + c_2\right) + \left(-7x^2 - 98x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x}{7}} + c_2 - 7x^2 - 98x \tag{1}$$

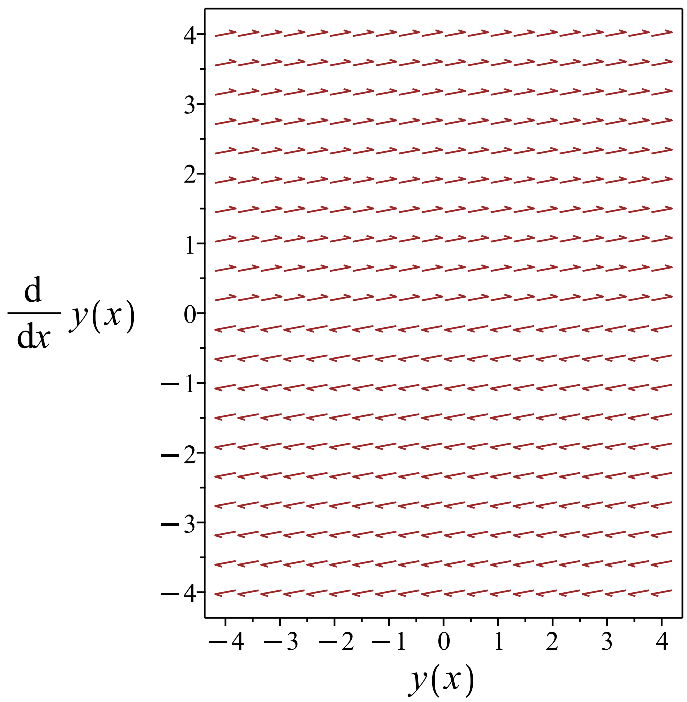


Figure 571: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{x}{7}} + c_2 - 7x^2 - 98x$$

Verified OK.

16.50.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (7y'' - y')\,dx = \int 14x dx$$
$$7y' - y = 7x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{7}$$
$$q(x) = x^2 + \frac{c_1}{7}$$

Hence the ode is

$$y' - \frac{y}{7} = x^2 + \frac{c_1}{7}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{7}dx}$$
$$= e^{-\frac{x}{7}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2 + \frac{c_1}{7}\right)$$
$$\frac{d}{dx}\left(e^{-\frac{x}{7}} y\right) = \left(e^{-\frac{x}{7}}\right)\left(x^2 + \frac{c_1}{7}\right)$$
$$d\left(e^{-\frac{x}{7}} y\right) = \left(\frac{(7x^2 + c_1)\,e^{-\frac{x}{7}}}{7}\right) dx$$

Integrating gives

$$e^{-\frac{x}{7}} y = \int \frac{(7x^2 + c_1)\,e^{-\frac{x}{7}}}{7}\, dx$$
$$e^{-\frac{x}{7}} y = -\left(7x^2 + c_1 + 98x + 686\right) e^{-\frac{x}{7}} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x}{7}}$ results in

$$y = -e^{\frac{x}{7}} \left(7x^2 + c_1 + 98x + 686\right) e^{-\frac{x}{7}} + c_2 e^{\frac{x}{7}}$$

which simplifies to

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}}$$

Summary

The solution(s) found are the following

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}} \tag{1}$$

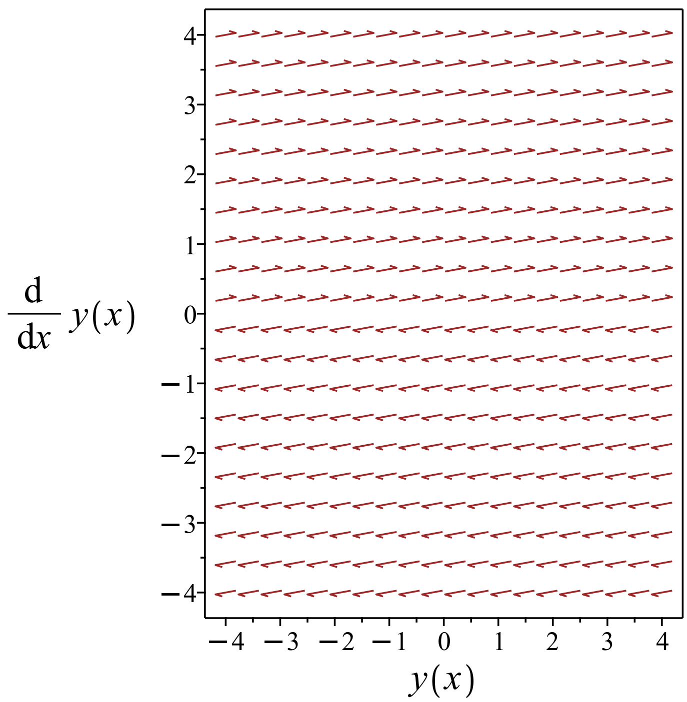


Figure 572: Slope field plot

Verification of solutions

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}}$$

Verified OK.

16.50.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$7p'(x) - p(x) - 14x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{7} \\ q(x) &= 2x \end{aligned}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{7} = 2x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{7}dx} \\ &= e^{-\frac{x}{7}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(2x) \\ \frac{d}{dx}\left(e^{-\frac{x}{7}} p\right) &= \left(e^{-\frac{x}{7}}\right)(2x) \\ d\left(e^{-\frac{x}{7}} p\right) &= \left(2x\, e^{-\frac{x}{7}}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-\frac{x}{7}} p &= \int 2x\, e^{-\frac{x}{7}}\, dx \\ e^{-\frac{x}{7}} p &= -14(x+7)\, e^{-\frac{x}{7}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x}{7}}$ results in

$$p(x) = -14\, e^{\frac{x}{7}} (x+7)\, e^{-\frac{x}{7}} + c_1 e^{\frac{x}{7}}$$

which simplifies to

$$p(x) = -14x - 98 + c_1 e^{\frac{x}{7}}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -14x - 98 + c_1 e^{\frac{x}{7}}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -14x - 98 + c_1 e^{\frac{x}{7}}\, dx \\ &= -98x + 7c_1 e^{\frac{x}{7}} - 7x^2 + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -98x + 7c_1 e^{\frac{x}{7}} - 7x^2 + c_2 \tag{1}$$

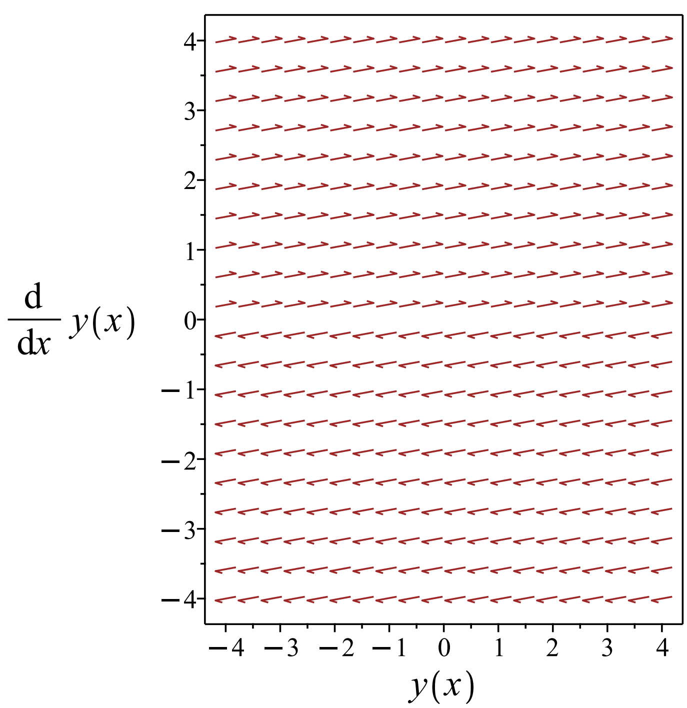


Figure 573: Slope field plot

Verification of solutions

$$y = -98x + 7c_1 e^{\frac{x}{7}} - 7x^2 + c_2$$

Verified OK.

16.50.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$7y'' - y' = 14x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (7y'' - y')\, dx = \int 14x dx$$
$$7y' - y = 7x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{7}$$
$$q(x) = x^2 + \frac{c_1}{7}$$

Hence the ode is

$$y' - \frac{y}{7} = x^2 + \frac{c_1}{7}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{7} dx}$$
$$= e^{-\frac{x}{7}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2 + \frac{c_1}{7}\right)$$
$$\frac{d}{dx}\left(e^{-\frac{x}{7}} y\right) = \left(e^{-\frac{x}{7}}\right)\left(x^2 + \frac{c_1}{7}\right)$$
$$d\left(e^{-\frac{x}{7}} y\right) = \left(\frac{(7x^2 + c_1)\, e^{-\frac{x}{7}}}{7}\right) dx$$

Integrating gives

$$e^{-\frac{x}{7}} y = \int \frac{(7x^2 + c_1) e^{-\frac{x}{7}}}{7} dx$$
$$e^{-\frac{x}{7}} y = -(7x^2 + c_1 + 98x + 686) e^{-\frac{x}{7}} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x}{7}}$ results in

$$y = -e^{\frac{x}{7}} (7x^2 + c_1 + 98x + 686) e^{-\frac{x}{7}} + c_2 e^{\frac{x}{7}}$$

which simplifies to

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}}$$

Summary

The solution(s) found are the following

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}} \tag{1}$$

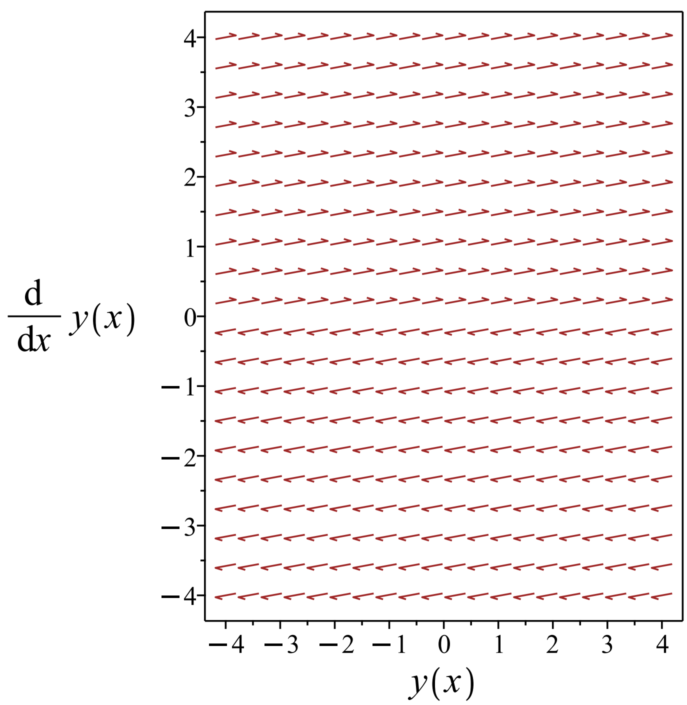


Figure 574: Slope field plot

Verification of solutions

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}}$$

Verified OK.

16.50.5 Solving using Kovacic algorithm

Writing the ode as

$$7y'' - y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 7 \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{196} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 196 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{196} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{196}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{14}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{7}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\frac{x}{14}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{14}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{7}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x}{7}}}{(y_1)^2}\, dx \\
&= y_1 \left(7\, \mathrm{e}^{\frac{x}{7}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (1) + c_2 \left(1 \left(7\, \mathrm{e}^{\frac{x}{7}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
7y'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + 7 c_2 \mathrm{e}^{\frac{x}{7}}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, 7\,\mathrm{e}^{\frac{x}{7}}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2xA_2 - A_1 + 14A_2 = 14x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -98, A_2 = -7]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -7x^2 - 98x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + 7c_2\mathrm{e}^{\frac{x}{7}}\right) + \left(-7x^2 - 98x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + 7c_2 e^{\frac{x}{7}} - 7x^2 - 98x \tag{1}$$

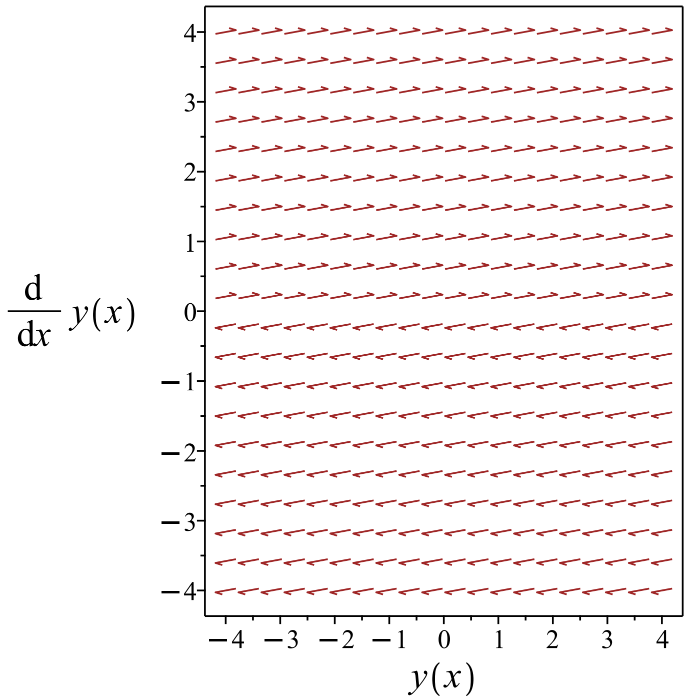


Figure 575: Slope field plot

Verification of solutions

$$y = c_1 + 7c_2 e^{\frac{x}{7}} - 7x^2 - 98x$$

Verified OK.

16.50.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 7 \\
q(x) &= -1 \\
r(x) &= 0 \\
s(x) &= 14x
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx
$$

Substituting the above values for p, q, r, s gives

$$
7y' - y = \int 14x\, dx
$$

We now have a first order ode to solve which is

$$
7y' - y = 7x^2 + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{7} \\
q(x) &= x^2 + \frac{c_1}{7}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{7} = x^2 + \frac{c_1}{7}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{7}dx} \\ &= e^{-\frac{x}{7}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^2 + \frac{c_1}{7}\right) \\ \frac{d}{dx}\left(e^{-\frac{x}{7}} y\right) &= \left(e^{-\frac{x}{7}}\right)\left(x^2 + \frac{c_1}{7}\right) \\ d\left(e^{-\frac{x}{7}} y\right) &= \left(\frac{(7x^2 + c_1)\, e^{-\frac{x}{7}}}{7}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-\frac{x}{7}} y &= \int \frac{(7x^2 + c_1)\, e^{-\frac{x}{7}}}{7}\, dx \\ e^{-\frac{x}{7}} y &= -\left(7x^2 + c_1 + 98x + 686\right) e^{-\frac{x}{7}} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x}{7}}$ results in

$$y = -e^{\frac{x}{7}}\left(7x^2 + c_1 + 98x + 686\right) e^{-\frac{x}{7}} + c_2 e^{\frac{x}{7}}$$

which simplifies to

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}}$$

Summary

The solution(s) found are the following

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}} \tag{1}$$

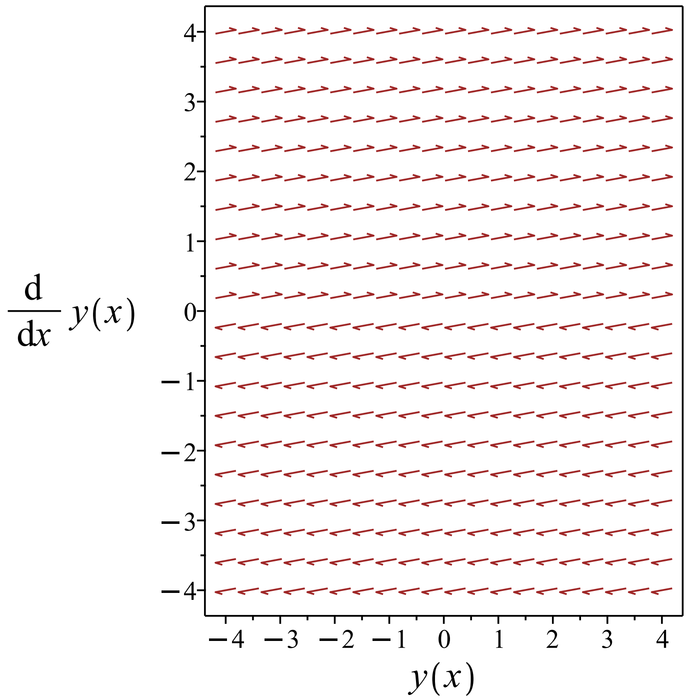


Figure 576: Slope field plot

Verification of solutions

$$y = -7x^2 - c_1 - 98x - 686 + c_2 e^{\frac{x}{7}}$$

Verified OK.

16.50.7 Maple step by step solution

Let's solve

$7y'' - y' = 14x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{7} + 2x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{7} = 2x$

- Characteristic polynomial of homogeneous ODE

$r^2 - \frac{1}{7}r = 0$

- Factor the characteristic polynomial

  $\frac{r(7r-1)}{7} = 0$

- Roots of the characteristic polynomial

  $r = \left(0, \frac{1}{7}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{7}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 \mathrm{e}^{\frac{x}{7}} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & \mathrm{e}^{\frac{x}{7}} \\ 0 & \frac{\mathrm{e}^{\frac{x}{7}}}{7} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \frac{\mathrm{e}^{\frac{x}{7}}}{7}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -14\left(\int x dx\right) + 14\,\mathrm{e}^{\frac{x}{7}}\left(\int x\,\mathrm{e}^{-\frac{x}{7}}dx\right)$

  - Compute integrals

    $y_p(x) = -7x^2 - 98x - 686$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 \mathrm{e}^{\frac{x}{7}} - 7x^2 - 98x - 686$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/7)*_b(_a)+2*_a, _b(_a)`   *** Sublev
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(7*diff(y(x),x$2)-diff(y(x),x)=14*x,y(x), singsol=all)
```

$$y(x) = 7\,e^{\frac{x}{7}} c_1 - 7x^2 - 98x + c_2$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 27

```
DSolve[7*y''[x]-y'[x]==14*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -7x^2 - 98x + 7c_1 e^{x/7} + c_2$$

16.51 problem 524

Internal problem ID [15294]
Internal file name [`OUTPUT/15294_Wednesday_May_08_2024_03_54_59_PM_96758476/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 524.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 3y' = 3x\,\mathrm{e}^{-3x}}$$

16.51.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 0, f(x) = 3x\,\mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda \,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(0)} \\ &= -\frac{3}{2} \pm \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= -\frac{3}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{3}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -3 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(0)x} + c_2 e^{(-3)x} \end{aligned}$$

Or

$$y = c_1 + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x\, e^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{-3x}, e^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-3x}\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{-3x}, x^2 e^{-3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{-3x} + A_2 x^2 e^{-3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 e^{-3x} + 2A_2 e^{-3x} - 6A_2 x\, e^{-3x} = 3x\, e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\, e^{-3x}}{3} - \frac{x^2 e^{-3x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-3x}) + \left(-\frac{x\,e^{-3x}}{3} - \frac{x^2 e^{-3x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-3x} - \frac{x\,e^{-3x}}{3} - \frac{x^2 e^{-3x}}{2} \tag{1}$$

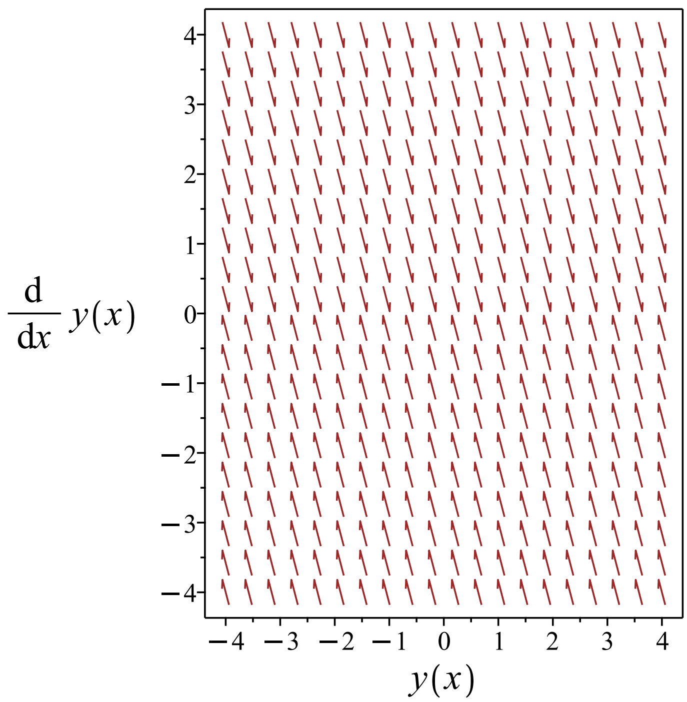


Figure 577: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-3x} - \frac{x\,e^{-3x}}{3} - \frac{x^2 e^{-3x}}{2}$$

Verified OK.

16.51.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 3y')\, dx = \int 3x\,\mathrm{e}^{-3x} dx$$
$$3y + y' = -\frac{(3x+1)\,\mathrm{e}^{-3x}}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 3$$
$$q(x) = -x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1$$

Hence the ode is

$$3y + y' = -x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 3dx}$$
$$= \mathrm{e}^{3x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{3x} y\right) = \left(\mathrm{e}^{3x}\right)\left(-x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1\right)$$
$$\mathrm{d}\left(\mathrm{e}^{3x} y\right) = \left(c_1 \mathrm{e}^{3x} - x - \frac{1}{3}\right)\,\mathrm{d}x$$

Integrating gives

$$\mathrm{e}^{3x} y = \int c_1 \mathrm{e}^{3x} - x - \frac{1}{3}\,\mathrm{d}x$$
$$\mathrm{e}^{3x} y = -\frac{x}{3} - \frac{x^2}{2} + \frac{c_1 \mathrm{e}^{3x}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = e^{-3x}\left(-\frac{x}{3} - \frac{x^2}{2} + \frac{c_1 e^{3x}}{3}\right) + c_2 e^{-3x}$$

which simplifies to

$$y = \frac{(-3x^2 + 6c_2 - 2x)\, e^{-3x}}{6} + \frac{c_1}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(-3x^2 + 6c_2 - 2x)\, e^{-3x}}{6} + \frac{c_1}{3} \tag{1}$$

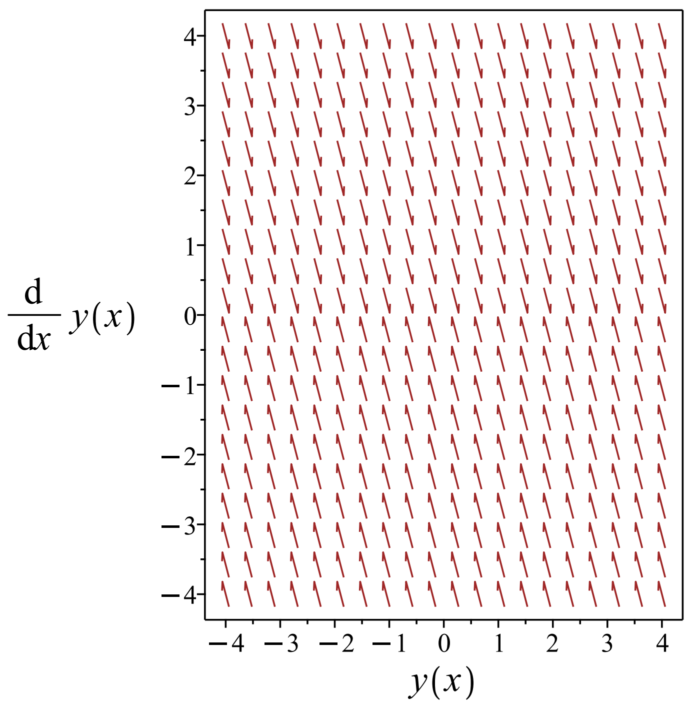


Figure 578: Slope field plot

Verification of solutions

$$y = \frac{(-3x^2 + 6c_2 - 2x)\, e^{-3x}}{6} + \frac{c_1}{3}$$

Verified OK.

16.51.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 3p(x) - 3x\,\mathrm{e}^{-3x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 3x\,\mathrm{e}^{-3x} \end{aligned}$$

Hence the ode is

$$p'(x) + 3p(x) = 3x\,\mathrm{e}^{-3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 3dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(3x\,\mathrm{e}^{-3x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{3x} p\right) &= \left(\mathrm{e}^{3x}\right)\left(3x\,\mathrm{e}^{-3x}\right) \\ \mathrm{d}\left(\mathrm{e}^{3x} p\right) &= (3x)\;\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{3x} p &= \int 3x\,\mathrm{d}x \\ \mathrm{e}^{3x} p &= \frac{3x^2}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$p(x) = \frac{3x^2 e^{-3x}}{2} + c_1 e^{-3x}$$

which simplifies to

$$p(x) = e^{-3x}\left(\frac{3x^2}{2} + c_1\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{-3x}\left(\frac{3x^2}{2} + c_1\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{e^{-3x}(3x^2 + 2c_1)}{2}\, dx \\ &= -\frac{(9x^2 + 6c_1 + 6x + 2)\, e^{-3x}}{18} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{(9x^2 + 6c_1 + 6x + 2)\, e^{-3x}}{18} + c_2 \tag{1}$$

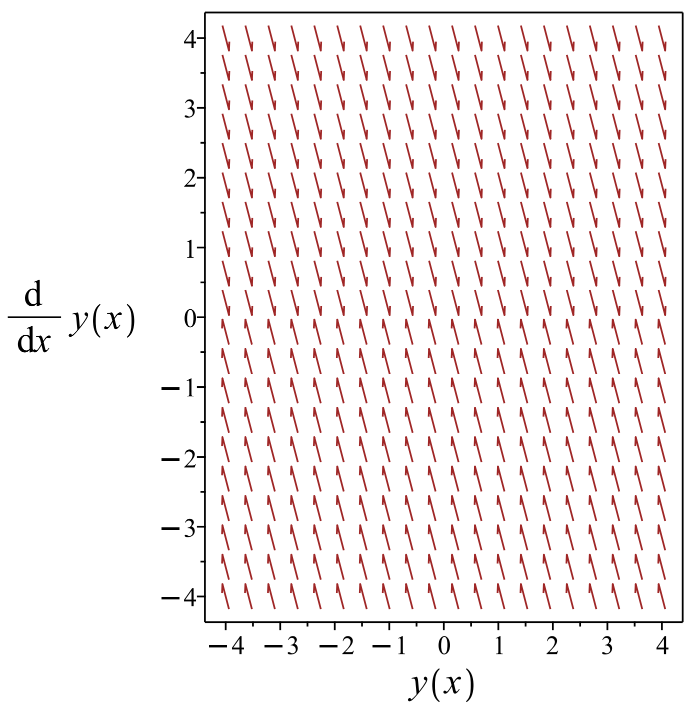


Figure 579: Slope field plot

Verification of solutions

$$y = -\frac{(9x^2 + 6c_1 + 6x + 2)\, e^{-3x}}{18} + c_2$$

Verified OK.

16.51.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 3y' = 3x\, e^{-3x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 3y')\, dx = \int 3x\, e^{-3x} dx$$

$$3y + y' = -\frac{(3x + 1)\, e^{-3x}}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= -x\, e^{-3x} - \frac{e^{-3x}}{3} + c_1 \end{aligned}$$

Hence the ode is

$$3y + y' = -x\, e^{-3x} - \frac{e^{-3x}}{3} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 3dx} \\ &= e^{3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu) \left(-x\, e^{-3x} - \frac{e^{-3x}}{3} + c_1\right) \\ \frac{d}{dx}\left(e^{3x} y\right) &= \left(e^{3x}\right) \left(-x\, e^{-3x} - \frac{e^{-3x}}{3} + c_1\right) \\ d\left(e^{3x} y\right) &= \left(c_1 e^{3x} - x - \frac{1}{3}\right) dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{3x}y &= \int c_1 e^{3x} - x - \frac{1}{3}\, dx \\
e^{3x}y &= -\frac{x}{3} - \frac{x^2}{2} + \frac{c_1 e^{3x}}{3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$
y = e^{-3x}\left(-\frac{x}{3} - \frac{x^2}{2} + \frac{c_1 e^{3x}}{3}\right) + c_2 e^{-3x}
$$

which simplifies to

$$
y = \frac{(-3x^2 + 6c_2 - 2x)\, e^{-3x}}{6} + \frac{c_1}{3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(-3x^2 + 6c_2 - 2x)\, e^{-3x}}{6} + \frac{c_1}{3} \tag{1}
$$

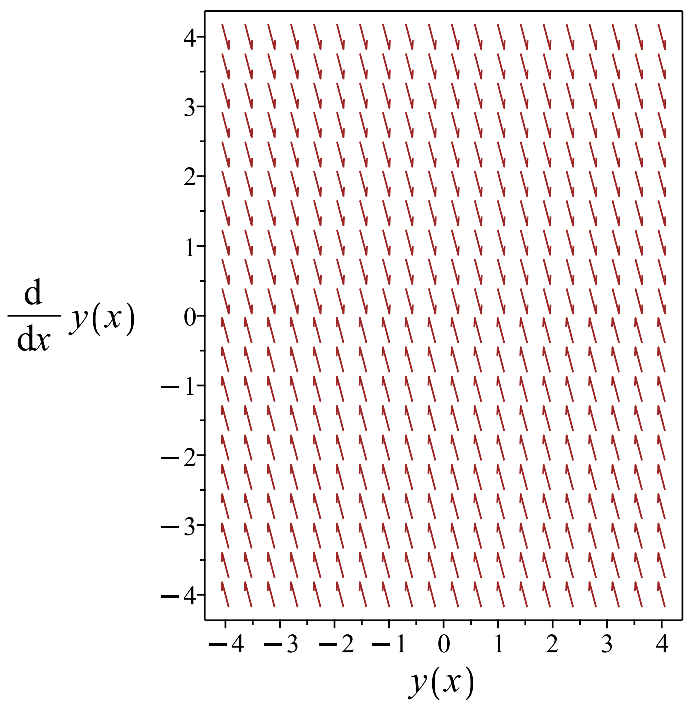


Figure 580: Slope field plot

Verification of solutions

$$y = \frac{(-3x^2 + 6c_2 - 2x)\,\mathrm{e}^{-3x}}{6} + \frac{c_1}{3}$$

Verified OK.

16.51.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x\,\mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{3}, \mathrm{e}^{-3x}\right\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-3x}, x^2\mathrm{e}^{-3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-3x} + A_2 x^2 \mathrm{e}^{-3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1\mathrm{e}^{-3x} + 2A_2\mathrm{e}^{-3x} - 6A_2 x\,\mathrm{e}^{-3x} = 3x\,\mathrm{e}^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,\mathrm{e}^{-3x}}{3} - \frac{x^2 \mathrm{e}^{-3x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + \frac{c_2}{3}\right) + \left(-\frac{x\,\mathrm{e}^{-3x}}{3} - \frac{x^2 \mathrm{e}^{-3x}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2}{3} - \frac{x\,\mathrm{e}^{-3x}}{3} - \frac{x^2 \mathrm{e}^{-3x}}{2} \tag{1}$$

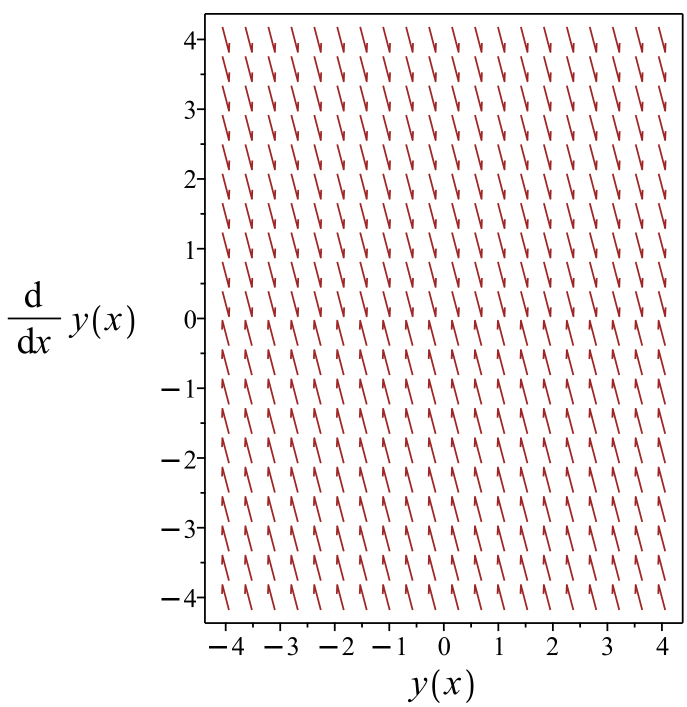


Figure 581: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2}{3} - \frac{x\,e^{-3x}}{3} - \frac{x^2 e^{-3x}}{2}$$

Verified OK.

16.51.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 3 \\ r(x) &= 0 \\ s(x) &= 3x\,e^{-3x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$3y + y' = \int 3x\,e^{-3x}\,dx$$

We now have a first order ode to solve which is

$$3y + y' = -\frac{(3x+1)\,\mathrm{e}^{-3x}}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= -x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1 \end{aligned}$$

Hence the ode is

$$3y + y' = -x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 3dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{3x} y\right) &= \left(\mathrm{e}^{3x}\right)\left(-x\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{-3x}}{3} + c_1\right) \\ \mathrm{d}\left(\mathrm{e}^{3x} y\right) &= \left(c_1 \mathrm{e}^{3x} - x - \frac{1}{3}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{3x} y &= \int c_1 \mathrm{e}^{3x} - x - \frac{1}{3}\,\mathrm{d}x \\ \mathrm{e}^{3x} y &= -\frac{x}{3} - \frac{x^2}{2} + \frac{c_1 \mathrm{e}^{3x}}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{3x}$ results in

$$y = \mathrm{e}^{-3x}\left(-\frac{x}{3} - \frac{x^2}{2} + \frac{c_1 \mathrm{e}^{3x}}{3}\right) + c_2 \mathrm{e}^{-3x}$$

which simplifies to

$$y = \frac{(-3x^2 + 6c_2 - 2x)\,e^{-3x}}{6} + \frac{c_1}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{(-3x^2 + 6c_2 - 2x)\,e^{-3x}}{6} + \frac{c_1}{3} \tag{1}$$

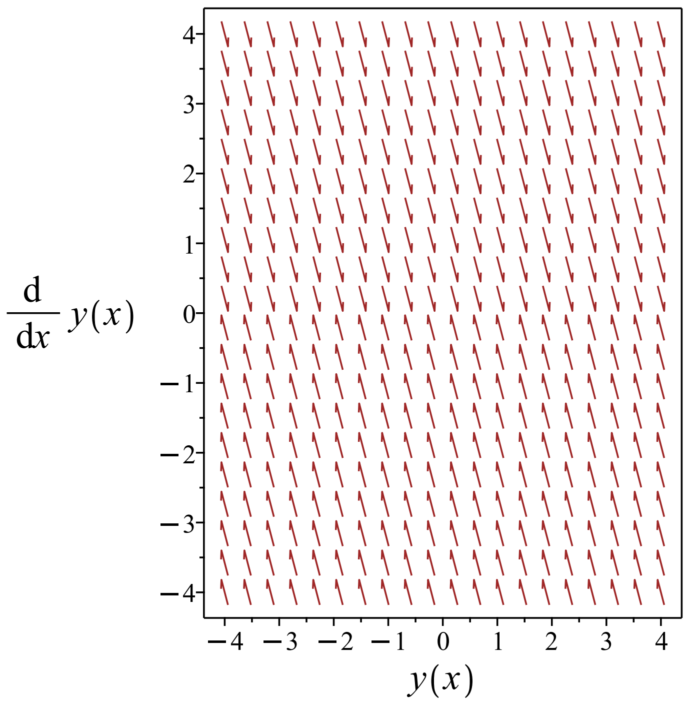


Figure 582: Slope field plot

Verification of solutions

$$y = \frac{(-3x^2 + 6c_2 - 2x)\,e^{-3x}}{6} + \frac{c_1}{3}$$

Verified OK.

16.51.7 Maple step by step solution

Let's solve

$y'' + 3y' = 3x\,e^{-3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r = 0$

- Factor the characteristic polynomial

  $r(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = 3x\,e^{-3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & 1 \\ -3\,e^{-3x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,e^{-3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^{-3x}\left(\int x dx\right) + \int x\,e^{-3x}dx$

- Compute integrals

  $y_p(x) = -\frac{e^{-3x}(9x^2+6x+2)}{18}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3x} + c_2 - \frac{e^{-3x}(9x^2+6x+2)}{18}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 3*_a*exp(-3*_a)-3*_b(_a), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)=3*x*exp(-3*x),y(x), singsol=all)
```

$$y(x) = \frac{(-9x^2 - 6c_1 - 6x - 2)\, e^{-3x}}{18} + c_2$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 31

```
DSolve[y''[x]+3*y'[x]==3*x*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{18} e^{-3x} (9x^2 + 6x + 2 + 6c_1)$$

16.52 problem 525

Internal problem ID [15295]
Internal file name [`OUTPUT/15295_Wednesday_May_08_2024_03_55_01_PM_46963766/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 525.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 6y = 10(1 - x)\,\mathrm{e}^{-2x}}$$

16.52.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 6, f(x) = -10(x - 1)\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 5\lambda\,\mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(6)} \\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-2)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-10(x-1)\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}, \mathrm{e}^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x} + A_2 \mathrm{e}^{-2x} x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{-2x} + 2A_2 \mathrm{e}^{-2x} x + 2A_2 \mathrm{e}^{-2x} = -10(x-1)\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 20, A_2 = -5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 20x\,\mathrm{e}^{-2x} - 5\,\mathrm{e}^{-2x} x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}\right) + \left(20x\,\mathrm{e}^{-2x} - 5\,\mathrm{e}^{-2x} x^2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x} + 20x\,\mathrm{e}^{-2x} - 5\,\mathrm{e}^{-2x} x^2 \tag{1}$$

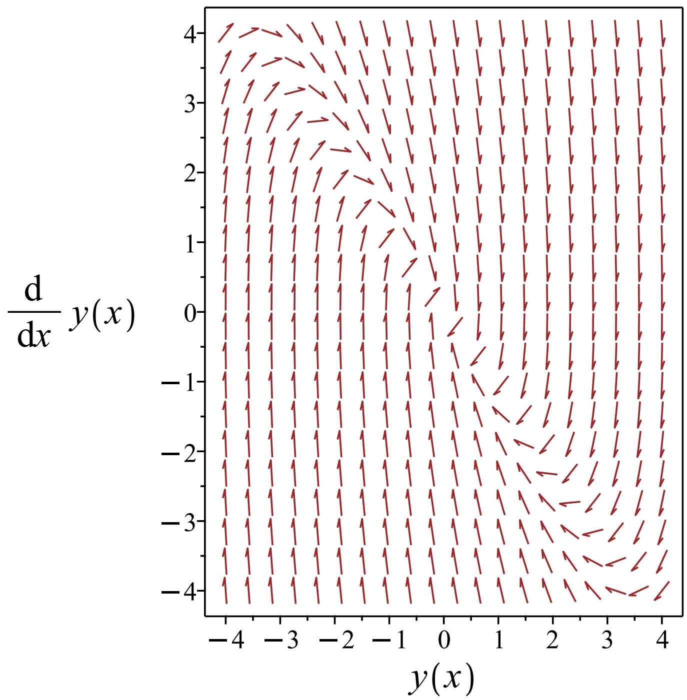


Figure 583: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x} + 20x\,\mathrm{e}^{-2x} - 5\,\mathrm{e}^{-2x} x^2$$

Verified OK.

16.52.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\ &= z_1 e^{-\frac{5x}{2}} \\ &= z_1 \left(e^{-\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-10(x-1)\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-2x}, e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{-2x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}, e^{-2x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-2x} + A_2 e^{-2x} x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-2x} + 2A_2 e^{-2x} x + 2A_2 e^{-2x} = -10(x-1)\,e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 20, A_2 = -5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 20x\,e^{-2x} - 5\,e^{-2x} x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + e^{-2x} c_2\right) + \left(20x\,e^{-2x} - 5\,e^{-2x} x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + e^{-2x} c_2 + 20x\,e^{-2x} - 5\,e^{-2x} x^2 \tag{1}$$

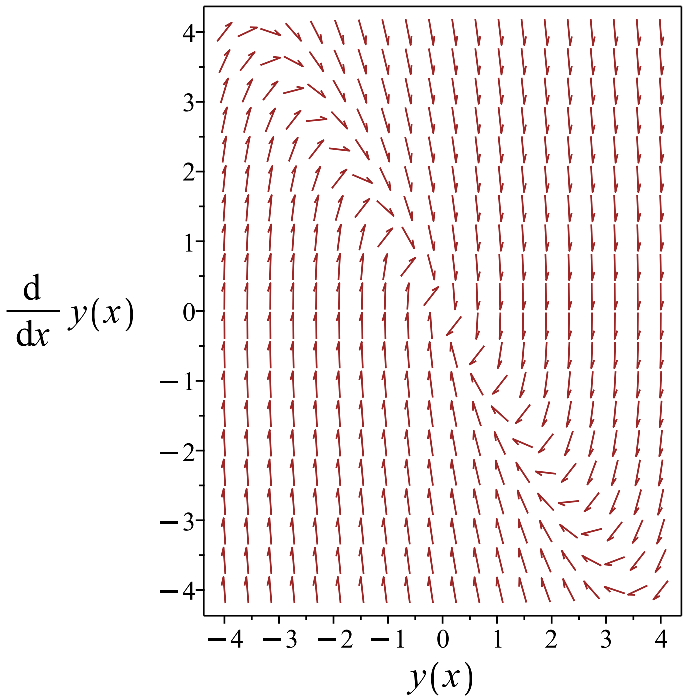


Figure 584: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + e^{-2x} c_2 + 20x\, e^{-2x} - 5\, e^{-2x} x^2$$

Verified OK.

16.52.3 Maple step by step solution

Let's solve

$y'' + 5y' + 6y = -10(x-1)\, e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-3, -2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{-2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -10(x-1)\,\mathrm{e}^{-2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{-2x} \\ -3\,\mathrm{e}^{-3x} & -2\,\mathrm{e}^{-2x} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-5x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = 10\,\mathrm{e}^{-3x}\left(\int (x-1)\,\mathrm{e}^{x} dx\right) - 10\,\mathrm{e}^{-2x}\left(\int (x-1)\,dx\right)$

  ◦ Compute integrals

  $y_p(x) = -5\,\mathrm{e}^{-2x}(x-2)^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{-2x} c_2 - 5\,\mathrm{e}^{-2x}(x-2)^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+5*diff(y(x),x)+6*y(x)=10*(1-x)*exp(-2*x),y(x), singsol=all)
```

$$y(x) = \left(-5x^2 + c_1 + 20x\right) e^{-2x} + e^{-3x} c_2$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 30

```
DSolve[y''[x]+5*y'[x]+6*y[x]==10*(1-x)*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(e^x\left(-5x^2 + 20x - 20 + c_2\right) + c_1\right)$$

16.53 problem 526

Internal problem ID [15296]
Internal file name [`OUTPUT/15296_Wednesday_May_08_2024_03_55_02_PM_20518114/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 526.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + 2y = x + 1}$$

16.53.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)}\\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x+1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(x), e^{-x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 + 2A_2 = x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1\cos(x) + c_2\sin(x))\right) + \left(\frac{x}{2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-x}(c_1\cos(x) + c_2\sin(x)) + \frac{x}{2} \tag{1}$$

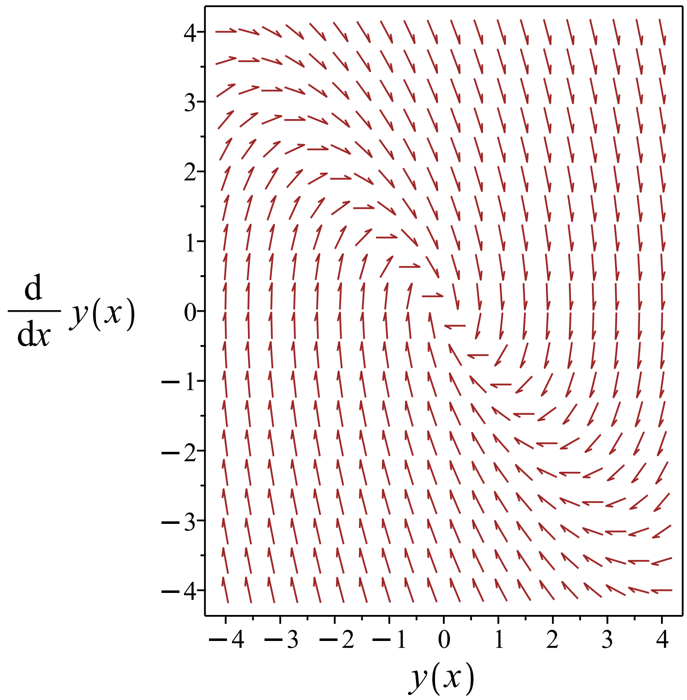


Figure 585: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2}$$

Verified OK.

16.53.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(x) (\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-x} \cos(x) c_1 + \mathrm{e}^{-x} \sin(x) c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x} \cos(x), \mathrm{e}^{-x} \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 + 2A_2 = x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x} \cos(x) c_1 + e^{-x} \sin(x) c_2\right) + \left(\frac{x}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2} \tag{1}$$

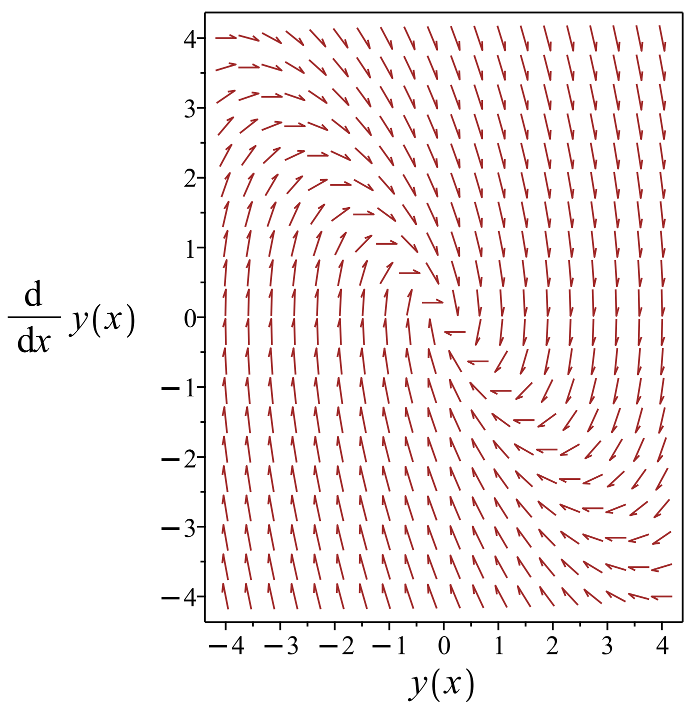


Figure 586: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2}$$

Verified OK.

16.53.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-x} \cos(x) c_1 + e^{-x} \sin(x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x + 1\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} \cos(x) & e^{-x} \sin(x) \\ -e^{-x} \cos(x) - e^{-x} \sin(x) & -e^{-x} \sin(x) + e^{-x} \cos(x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-2x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-x}\left(\cos(x)\left(\int(x+1)\sin(x)e^x dx\right) - \sin(x)\left(\int(x+1)\cos(x)e^x dx\right)\right)$
- Compute integrals

  $y_p(x) = \frac{x}{2}$

- Substitute particular solution into general solution to ODE

  $y = e^{-x}\cos(x)c_1 + e^{-x}\sin(x)c_2 + \frac{x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=1+x,y(x), singsol=all)
```

$$y(x) = e^{-x}\sin(x)c_2 + e^{-x}\cos(x)c_1 + \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 32

```
DSolve[y''[x]+2*y'[x]+2*y[x]==1+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}(e^x x + 2c_2\cos(x) + 2c_1\sin(x))$$

16.54 problem 527

Internal problem ID [15297]
Internal file name [`OUTPUT/15297_Wednesday_May_08_2024_03_55_03_PM_41321901/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 527.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = (x^2 + x)\,\mathrm{e}^x}$$

16.54.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = x(x+1)\,\mathrm{e}^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x(x+1)\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}x^2, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x} x^2 + A_3 \mathrm{e}^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\mathrm{e}^{x} + 3A_1 x\,\mathrm{e}^{x} + 3A_2\mathrm{e}^{x}x^2 + 6A_2\mathrm{e}^{x}x + 2A_2\mathrm{e}^{x} + 3A_3\mathrm{e}^{x} = x(x+1)\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = \frac{1}{3}, A_3 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,\mathrm{e}^x}{3} + \frac{\mathrm{e}^x x^2}{3} + \frac{\mathrm{e}^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)\right) + \left(-\frac{x\,\mathrm{e}^x}{3} + \frac{\mathrm{e}^x x^2}{3} + \frac{\mathrm{e}^x}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right) - \frac{x\,\mathrm{e}^x}{3} + \frac{\mathrm{e}^x x^2}{3} + \frac{\mathrm{e}^x}{9} \tag{1}$$

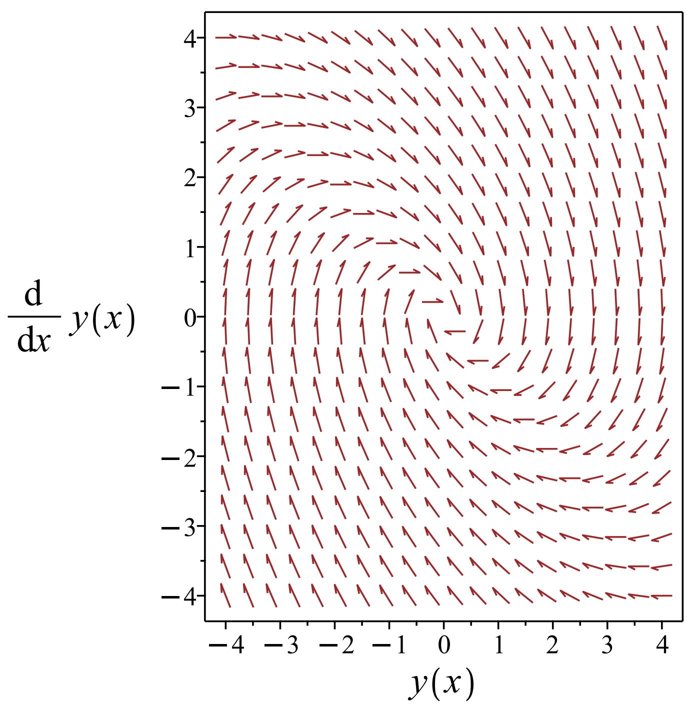


Figure 587: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right) - \frac{x\,e^x}{3} + \frac{e^x x^2}{3} + \frac{e^x}{9}$$

Verified OK.

16.54.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x\sqrt{3}}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{x\sqrt{3}}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{x\sqrt{3}}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x(x+1)\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}x^{2}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right), \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x} x^{2} + A_3 e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{x} + 3A_1 x\,e^{x} + 3A_2 e^{x} x^{2} + 6A_2 e^{x} x + 2A_2 e^{x} + 3A_3 e^{x} = x(x+1)\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = \frac{1}{3}, A_3 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^x}{3} + \frac{e^x x^2}{3} + \frac{e^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3}\right) + \left(-\frac{x\,e^x}{3} + \frac{e^x x^2}{3} + \frac{e^x}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \frac{x\,e^x}{3} + \frac{e^x x^2}{3} + \frac{e^x}{9} \tag{1}$$

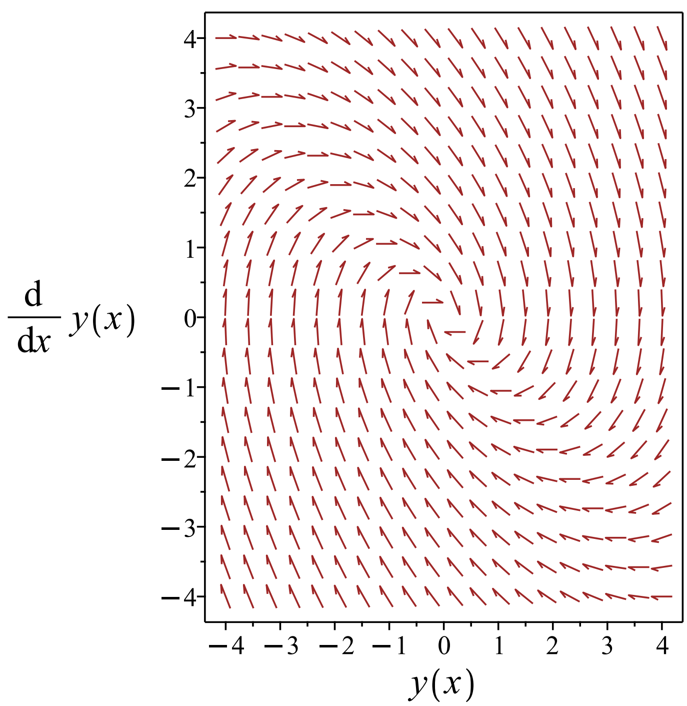


Figure 588: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \frac{x\,e^x}{3} + \frac{e^x x^2}{3} + \frac{e^x}{9}$$

Verified OK.

16.54.3 Maple step by step solution

Let's solve

$y'' + y' + y = x(x+1)\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = x(x+1)\,e^x\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\, e^{-\frac{x}{2}} \sqrt{3} \left(\cos\left(\frac{x\sqrt{3}}{2}\right)\left(\int x(x+1) e^{\frac{3x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) dx\right) - \sin\left(\frac{x\sqrt{3}}{2}\right)\left(\int x(x+1) e^{\frac{3x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = \frac{(3x^2-3x+1)e^x}{9}$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_2 + \frac{(3x^2-3x+1)e^x}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=(x+x^2)*exp(x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) c_2 + \frac{e^x\left(x^2 - x + \frac{1}{3}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 65

```
DSolve[y''[x]+y'[x]+y[x]==(x+x^2)*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}e^{-x/2}\left(e^{3x/2}(3x^2-3x+1)+9c_2\cos\left(\frac{\sqrt{3}x}{2}\right)+9c_1\sin\left(\frac{\sqrt{3}x}{2}\right)\right)$$

16.55 problem 528

Internal problem ID [15298]
Internal file name [`OUTPUT/15298_Wednesday_May_08_2024_03_55_04_PM_9502799/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 528.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' - 2y = 8\sin(2x)}$$

16.55.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = -2, f(x) = 8\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=4, C=-2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+4\lambda\,\mathrm{e}^{\lambda x}-2\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+4\lambda-2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=4, C=-2$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-4}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{4^2-(4)(1)(-2)}\\&=-2\pm\sqrt{6}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-2+\sqrt{6}\\\lambda_2&=-2-\sqrt{6}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=-2+\sqrt{6}\\\lambda_2&=-2-\sqrt{6}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{\left(-2+\sqrt{6}\right)x}+c_2e^{\left(-2-\sqrt{6}\right)x}\end{aligned}$$

Or

$$y=c_1\mathrm{e}^{\left(-2+\sqrt{6}\right)x}+c_2\mathrm{e}^{\left(-2-\sqrt{6}\right)x}$$

Therefore the homogeneous solution y_h is

$$y_h=c_1\mathrm{e}^{\left(-2+\sqrt{6}\right)x}+c_2\mathrm{e}^{\left(-2-\sqrt{6}\right)x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(-2-\sqrt{6}\right)x}, e^{\left(-2+\sqrt{6}\right)x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \cos(2x) - 6A_2 \sin(2x) - 8A_1 \sin(2x) + 8A_2 \cos(2x) = 8 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{16}{25}, A_2 = -\frac{12}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\left(-2+\sqrt{6}\right)x} + c_2 e^{\left(-2-\sqrt{6}\right)x}\right) + \left(-\frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{\left(-2+\sqrt{6}\right)x} + c_2 e^{-\left(2+\sqrt{6}\right)x} - \frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\left(-2+\sqrt{6}\right)x} + c_2 e^{-\left(2+\sqrt{6}\right)x} - \frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25} \tag{1}$$

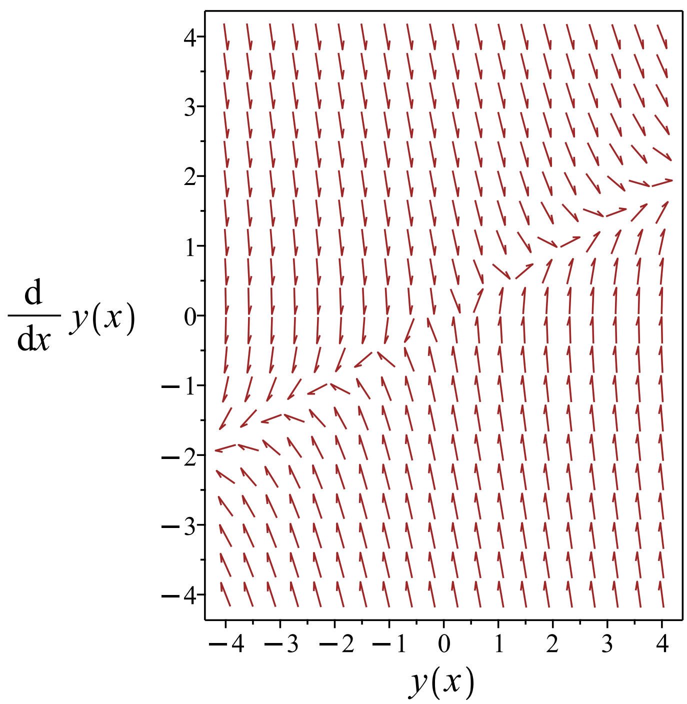


Figure 589: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(-2+\sqrt{6}\right)x} + c_2 e^{-\left(2+\sqrt{6}\right)x} - \frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

Verified OK.

16.55.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 6z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 6$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x\sqrt{6}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\left(2+\sqrt{6}\right)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{6}\, \mathrm{e}^{2x\sqrt{6}}}{12}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\left(2+\sqrt{6}\right)x}\right) + c_2 \left(\mathrm{e}^{-\left(2+\sqrt{6}\right)x} \left(\frac{\sqrt{6}\, \mathrm{e}^{2x\sqrt{6}}}{12}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-\left(2+\sqrt{6}\right)x} + \frac{c_2\sqrt{6}\, e^{\left(-2+\sqrt{6}\right)x}}{12}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{6}\, e^{\left(-2+\sqrt{6}\right)x}}{12}, e^{-\left(2+\sqrt{6}\right)x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2x) + A_2\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1\cos(2x) - 6A_2\sin(2x) - 8A_1\sin(2x) + 8A_2\cos(2x) = 8\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{16}{25}, A_2 = -\frac{12}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\left(2+\sqrt{6}\right)x} + \frac{c_2\sqrt{6}\,\mathrm{e}^{\left(-2+\sqrt{6}\right)x}}{12}\right) + \left(-\frac{16\cos\left(2x\right)}{25} - \frac{12\sin\left(2x\right)}{25}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\left(2+\sqrt{6}\right)x} + \frac{c_2\sqrt{6}\,\mathrm{e}^{\left(-2+\sqrt{6}\right)x}}{12} - \frac{16\cos\left(2x\right)}{25} - \frac{12\sin\left(2x\right)}{25} \tag{1}
$$

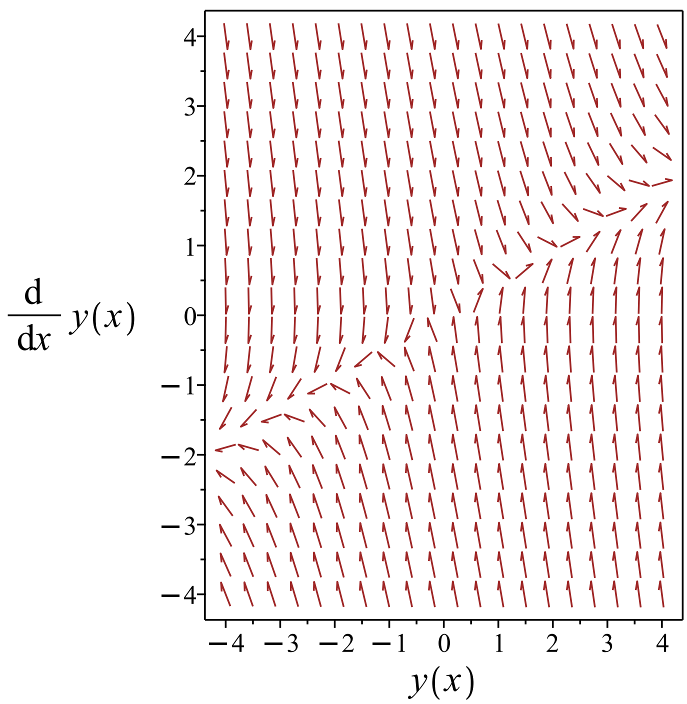


Figure 590: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-\left(2+\sqrt{6}\right)x} + \frac{c_2\sqrt{6}\,\mathrm{e}^{\left(-2+\sqrt{6}\right)x}}{12} - \frac{16\cos\left(2x\right)}{25} - \frac{12\sin\left(2x\right)}{25}
$$

Verified OK.

16.55.3 Maple step by step solution

Let's solve

$y'' + 4y' - 2y = 8\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r - 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm\left(\sqrt{24}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2-\sqrt{6}, -2+\sqrt{6}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{\left(-2-\sqrt{6}\right)x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{\left(-2+\sqrt{6}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{\left(-2-\sqrt{6}\right)x} + c_2 \mathrm{e}^{\left(-2+\sqrt{6}\right)x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{\left(-2-\sqrt{6}\right)x} & \mathrm{e}^{\left(-2+\sqrt{6}\right)x} \\ \left(-2-\sqrt{6}\right)\mathrm{e}^{\left(-2-\sqrt{6}\right)x} & \left(-2+\sqrt{6}\right)\mathrm{e}^{\left(-2+\sqrt{6}\right)x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\sqrt{6}\,\mathrm{e}^{-4x}$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{2\sqrt{6}\left(e^{\left(-2+\sqrt{6}\right)x}\left(\int e^{-\left(-2+\sqrt{6}\right)x}\sin(2x)dx\right)-e^{-\left(2+\sqrt{6}\right)x}\left(\int e^{\left(2+\sqrt{6}\right)x}\sin(2x)dx\right)\right)}{3}$$

- Compute integrals

$$y_p(x) = -\frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{\left(-2-\sqrt{6}\right)x} + c_2 e^{\left(-2+\sqrt{6}\right)x} - \frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)-2*y(x)=8*sin(2*x),y(x), singsol=all)
```

$$y(x) = e^{\left(-2+\sqrt{6}\right)x} c_2 + e^{-\left(2+\sqrt{6}\right)x} c_1 - \frac{16\cos(2x)}{25} - \frac{12\sin(2x)}{25}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 52

```
DSolve[y''[x]+4*y'[x]-2*y[x]==8*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\left(\left(2+\sqrt{6}\right)x\right)} + c_2 e^{\left(\sqrt{6}-2\right)x} - \frac{4}{25}(3\sin(2x) + 4\cos(2x))$$

16.56 problem 529

Internal problem ID [15299]
Internal file name [`OUTPUT/15299_Wednesday_May_08_2024_03_55_06_PM_42049133/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 529.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 4\cos(x)\, x}$$

16.56.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4\cos(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \cos(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \cos(x), x^2 \sin(x), \cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \cos(x) + A_2 x^2 \sin(x) + A_3 \cos(x) x + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) - 4A_1 x \sin(x) + 2A_2 \sin(x) + 4A_2 x \cos(x) - 2A_3 \sin(x) + 2A_4 \cos(x) = 4 \cos(x) x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 \sin(x) + \cos(x) x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(x^2 \sin(x) + \cos(x)\, x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x)\, x \tag{1}
$$

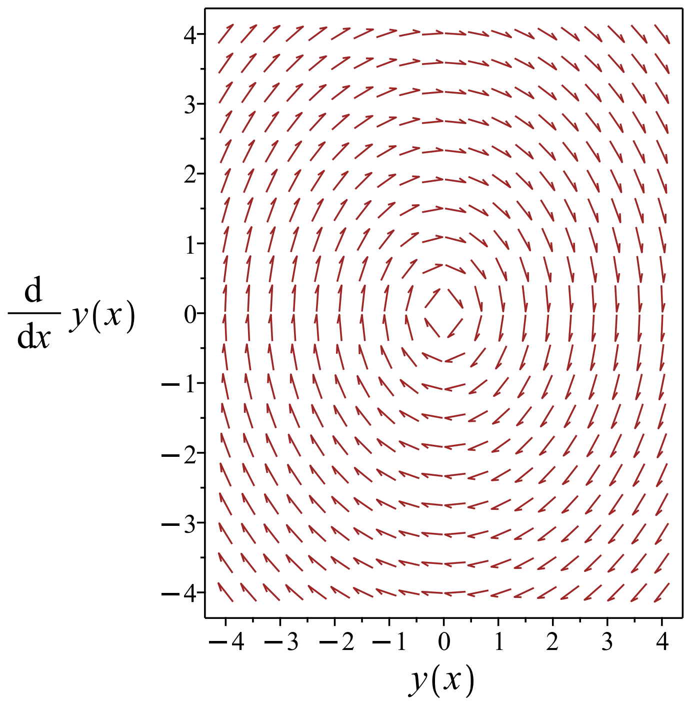


Figure 591: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x)\, x
$$

Verified OK.

16.56.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \cos(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \cos(x), x^2 \sin(x), \cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \cos(x) + A_2 x^2 \sin(x) + A_3 \cos(x) x + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &2A_1 \cos(x) - 4A_1 x \sin(x) + 2A_2 \sin(x) + 4A_2 x \cos(x) - 2A_3 \sin(x) + 2A_4 \cos(x) \\ &= 4 \cos(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 \sin(x) + \cos(x) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (x^2 \sin(x) + \cos(x) x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x)\, x \tag{1}$$

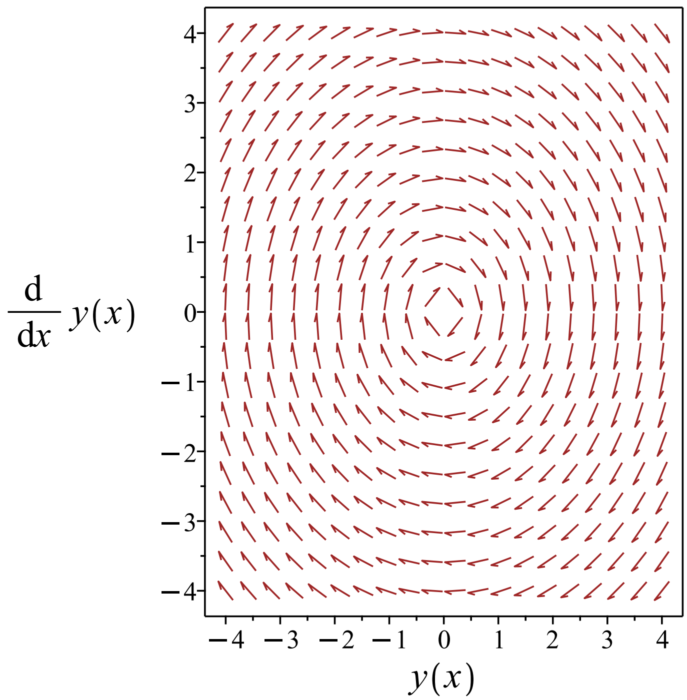


Figure 592: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x)\, x$$

Verified OK.

16.56.3 Maple step by step solution

Let's solve

$y'' + y = 4\cos(x)\, x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\cos(x)\, x\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\cos(x) \left(\int x \sin(2x)\, dx\right) + 4\sin(x) \left(\int \cos(x)^2 x dx\right)$
  - Compute integrals

    $y_p(x) = x^2 \sin(x) + \cos(x)\, x - \sin(x)$
- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x)\, x - \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+y(x)=4*x*cos(x),y(x), singsol=all)
```

$$y(x) = \left(x^2 + c_2 - 1\right) \sin\left(x\right) + \cos\left(x\right) \left(x + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 30

```
DSolve[y''[x]+y[x]==4*x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(2x^2 - 1 + 2c_2\right) \sin(x) + (x + c_1) \cos(x)$$

16.57 problem 530

Internal problem ID [15300]
Internal file name [`OUTPUT/15300_Wednesday_May_08_2024_03_55_07_PM_82893363/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 530.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 2my' + m^2y = \sin(nx)}$$

16.57.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2m, C = m^2, f(x) = \sin(nx)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2my' + m^2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2m, C = m^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2m\lambda\, e^{\lambda x} + m^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2m\lambda + m^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2m, C = m^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2m}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2m)^2 - (4)(1)(m^2)}\\ &= m\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -m$. Therefore the solution is

$$y = c_1 e^{mx} + c_2 x\, e^{mx} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{mx} + c_2 x\, e^{mx}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(nx)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(nx), \sin(nx)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{mx}, e^{mx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(nx) + A_2 \sin(nx)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 n^2 \cos(nx) - A_2 n^2 \sin(nx) - 2m(-A_1 n \sin(nx) + A_2 n \cos(nx)) \\ + m^2(A_1 \cos(nx) + A_2 \sin(nx)) = \sin(nx)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2mn}{(m^2+n^2)^2}, A_2 = \frac{m^2-n^2}{(m^2+n^2)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{mx} + c_2 x\, e^{mx}) + \left(\frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{mx}(c_2 x + c_1) + \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}$$

Summary

The solution(s) found are the following

$$y = e^{mx}(c_2 x + c_1) + \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2} \tag{1}$$

Verification of solutions

$$y = e^{mx}(c_2 x + c_1) + \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}$$

Verified OK.

16.57.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2m$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2m\,dx} \\ &= \mathrm{e}^{-mx} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-mx} \sin(nx) \\ \left(\mathrm{e}^{-mx} y\right)'' &= \mathrm{e}^{-mx} \sin(nx) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-mx} y\right)' = -\frac{\mathrm{e}^{-mx}(\sin(nx)\,m + n\cos(nx))}{m^2 + n^2} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-mx} y\right) = \frac{\left((m^2 - n^2)\sin(nx) + 2mn\cos(nx)\right)\mathrm{e}^{-mx} + c_1 x (m^2 + n^2)^2}{(m^2 + n^2)^2} + c_2$$

Hence the solution is

$$y = \frac{\frac{\left((m^2 - n^2)\sin(nx) + 2mn\cos(nx)\right)\mathrm{e}^{-mx} + c_1 x (m^2 + n^2)^2}{(m^2 + n^2)^2} + c_2}{\mathrm{e}^{-mx}}$$

Or

$$\begin{aligned} y = {} & \frac{m^2 \sin(nx)}{(m^2 + n^2)^2} + \frac{2mn\cos(nx)}{(m^2 + n^2)^2} - \frac{n^2 \sin(nx)}{(m^2 + n^2)^2} + c_2 \mathrm{e}^{mx} \\ & + \left(\frac{m^4 x\,\mathrm{e}^{mx}}{(m^2 + n^2)^2} + \frac{2m^2 n^2 x\,\mathrm{e}^{mx}}{(m^2 + n^2)^2} + \frac{n^4 x\,\mathrm{e}^{mx}}{(m^2 + n^2)^2}\right) c_1 \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{m^2 \sin(nx)}{(m^2+n^2)^2} + \frac{2mn\cos(nx)}{(m^2+n^2)^2} - \frac{n^2 \sin(nx)}{(m^2+n^2)^2} + c_2 e^{mx} \\
& + \left(\frac{m^4 x\, e^{mx}}{(m^2+n^2)^2} + \frac{2m^2 n^2 x\, e^{mx}}{(m^2+n^2)^2} + \frac{n^4 x\, e^{mx}}{(m^2+n^2)^2}\right) c_1
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \frac{m^2 \sin(nx)}{(m^2+n^2)^2} + \frac{2mn\cos(nx)}{(m^2+n^2)^2} - \frac{n^2 \sin(nx)}{(m^2+n^2)^2} + c_2 e^{mx} \\
& + \left(\frac{m^4 x\, e^{mx}}{(m^2+n^2)^2} + \frac{2m^2 n^2 x\, e^{mx}}{(m^2+n^2)^2} + \frac{n^4 x\, e^{mx}}{(m^2+n^2)^2}\right) c_1
\end{aligned}
$$

Verified OK.

16.57.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2my' + m^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2m \\
C &= m^2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2m}{1}\,dx} \\ &= z_1 e^{mx} \\ &= z_1(\mathrm{e}^{mx}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{mx}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2m}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2mx}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{mx}) + c_2(\mathrm{e}^{mx}(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2my' + m^2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{mx} + c_2 x\, e^{mx}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(nx)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(nx), \sin(nx)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{mx}, e^{mx}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(nx) + A_2 \sin(nx)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1n^2 \cos(nx) - A_2n^2 \sin(nx) - 2m(-A_1n \sin(nx) + A_2n \cos(nx)) \\ + m^2(A_1 \cos(nx) + A_2 \sin(nx)) = \sin(nx)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2mn}{(m^2+n^2)^2}, A_2 = \frac{m^2-n^2}{(m^2+n^2)^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{mx} + c_2 x\, e^{mx}) + \left(\frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{mx}(c_2 x + c_1) + \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}$$

Summary

The solution(s) found are the following

$$y = e^{mx}(c_2 x + c_1) + \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2} \tag{1}$$

Verification of solutions

$$y = e^{mx}(c_2 x + c_1) + \frac{2mn\cos(nx)}{(m^2+n^2)^2} + \frac{(m^2-n^2)\sin(nx)}{(m^2+n^2)^2}$$

Verified OK.

16.57.4 Maple step by step solution

Let's solve

$y'' - 2my' + m^2 y = \sin(nx)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $m^2 - 2mr + r^2 = 0$

- Factor the characteristic polynomial

$(m-r)^2=0$

- Root of the characteristic polynomial

  $r=m$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{mx}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=x\,\mathrm{e}^{mx}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{mx}+c_2x\,\mathrm{e}^{mx}+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\sin(nx)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\begin{bmatrix} \mathrm{e}^{mx} & x\,\mathrm{e}^{mx} \\ m\,\mathrm{e}^{mx} & \mathrm{e}^{mx}+xm\,\mathrm{e}^{mx} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x),y_2(x))=\mathrm{e}^{2mx}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\mathrm{e}^{mx}\left(-\left(\int \sin(nx)\,x\,\mathrm{e}^{-mx}dx\right)+x\left(\int \mathrm{e}^{-mx}\sin(nx)\,dx\right)\right)$

  - Compute integrals

    $y_p(x)=\frac{(m^2-n^2)\sin(nx)+2mn\cos(nx)}{(m^2+n^2)^2}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{mx}+c_2x\,\mathrm{e}^{mx}+\frac{(m^2-n^2)\sin(nx)+2mn\cos(nx)}{(m^2+n^2)^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$2)-2*m*diff(y(x),x)+m^2*y(x)=sin(n*x),y(x), singsol=all)
```

$$y(x) = \frac{(m^2+n^2)^2 (c_1 x + c_2) e^{mx} + (m^2-n^2) \sin(nx) + 2\cos(nx) mn}{(m^2+n^2)^2}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 56

```
DSolve[y''[x]-2*m*y'[x]+m^2*y[x]==Sin[n*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(m^2-n^2)\sin(nx) + 2mn\cos(nx)}{(m^2+n^2)^2} + c_1 e^{mx} + c_2 x e^{mx}$$

16.58 problem 531

Internal problem ID [15301]
Internal file name [`OUTPUT/15301_Wednesday_May_08_2024_03_55_08_PM_68723871/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 531.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = \mathrm{e}^{-x} \sin(2x)}$$

16.58.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 5, f(x) = \mathrm{e}^{-x} \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x} \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x} \cos(2x), e^{-x} \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(2x), e^{-x} \sin(2x)\}$$

Since $e^{-x} \cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{-x} \cos(2x), x e^{-x} \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{-x} \cos(2x) + A_2 x e^{-x} \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x} \sin(2x) + 4A_2 e^{-x} \cos(2x) = e^{-x} \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x e^{-x} \cos(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(-\frac{x e^{-x} \cos(2x)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{x e^{-x} \cos(2x)}{4} \tag{1}$$

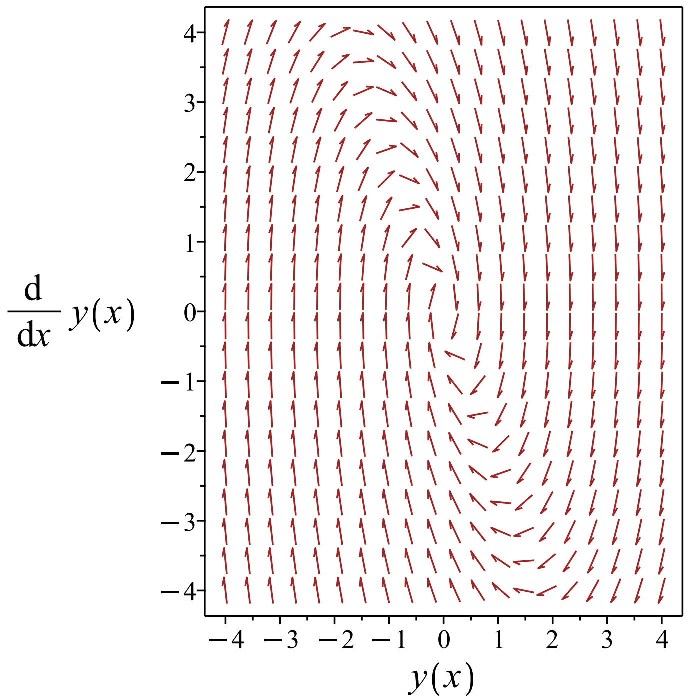


Figure 593: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{x e^{-x} \cos(2x)}{4}$$

Verified OK.

16.58.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 5
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-x} \cos(2x)\, c_1 + \frac{\mathrm{e}^{-x} \sin(2x)\, c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-x} \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\cos(2x), e^{-x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-x}\cos(2x), \frac{e^{-x}\sin(2x)}{2}\right\}$$

Since $e^{-x}\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\cos(2x), x\,e^{-x}\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x}\cos(2x) + A_2 x\,e^{-x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x}\sin(2x) + 4A_2 e^{-x}\cos(2x) = e^{-x}\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{-x}\cos(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}\cos(2x)\,c_1 + \frac{e^{-x}\sin(2x)\,c_2}{2}\right) + \left(-\frac{x\,e^{-x}\cos(2x)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x} \cos(2x) c_1 + \frac{e^{-x} \sin(2x) c_2}{2} - \frac{x e^{-x} \cos(2x)}{4} \tag{1}$$

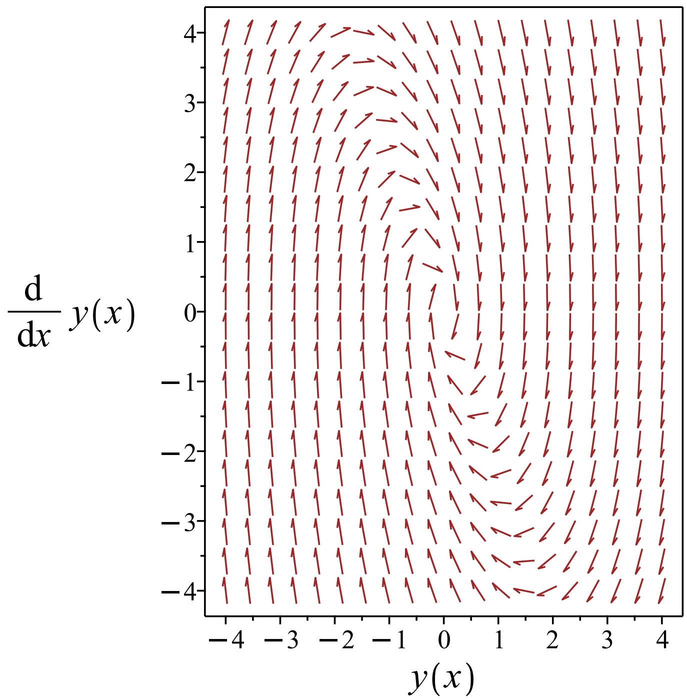


Figure 594: Slope field plot

Verification of solutions

$$y = e^{-x} \cos(2x) c_1 + \frac{e^{-x} \sin(2x) c_2}{2} - \frac{x e^{-x} \cos(2x)}{4}$$

Verified OK.

16.58.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = e^{-x} \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+2r+5=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-1-2\,\mathrm{I},-1+2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-x}\cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{-x}\sin(2x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=\mathrm{e}^{-x}\cos(2x)\,c_1+\mathrm{e}^{-x}\sin(2x)\,c_2+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\mathrm{e}^{-x}\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\begin{bmatrix}\mathrm{e}^{-x}\cos(2x) & \mathrm{e}^{-x}\sin(2x)\\ -\mathrm{e}^{-x}\cos(2x)-2\,\mathrm{e}^{-x}\sin(2x) & -\mathrm{e}^{-x}\sin(2x)+2\,\mathrm{e}^{-x}\cos(2x)\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x),y_2(x))=2\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\frac{\mathrm{e}^{-x}\left(\sin(2x)\left(\int\sin(4x)dx\right)-2\cos(2x)\left(\int\sin(2x)^2dx\right)\right)}{4}$

  - Compute integrals

    $y_p(x)=\frac{(\sin(2x)-4x\cos(2x))\mathrm{e}^{-x}}{16}$

- Substitute particular solution into general solution to ODE

  $y=\mathrm{e}^{-x}\cos(2x)\,c_1+\mathrm{e}^{-x}\sin(2x)\,c_2+\frac{(\sin(2x)-4x\cos(2x))\mathrm{e}^{-x}}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=exp(-x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-x}((x-4c_1)\cos(2x)-4\sin(2x)\,c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 38

```
DSolve[y''[x]+2*y'[x]+5*y[x]==Exp[-x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}e^{-x}((1+16c_1)\sin(2x)-4(x-4c_2)\cos(2x))$$

16.59 problem 532

Internal problem ID [15302]
Internal file name [`OUTPUT/15302_Wednesday_May_08_2024_03_55_09_PM_87584905/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 532.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + a^2 y = 2\cos(mx) + 3\sin(mx)}$$

16.59.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = a^2, f(x) = 2\cos(mx) + 3\sin(mx)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + a^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = a^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + a^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$a^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = a^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(a^2)} \\ &= \pm\sqrt{-a^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-a^2} \\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-a^2} \\ \lambda_2 &= -\sqrt{-a^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{-a^2}\right)x} + c_2 e^{\left(-\sqrt{-a^2}\right)x}\end{aligned}$$

Or

$$y = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(mx)+3\sin(mx)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(mx),\sin(mx)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{\sqrt{-a^2}\,x},\mathrm{e}^{-\sqrt{-a^2}\,x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(mx)+A_2\sin(mx)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_1m^2\cos(mx)-A_2m^2\sin(mx)+a^2(A_1\cos(mx)+A_2\sin(mx))\\&=2\cos(mx)+3\sin(mx)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{2}{a^2-m^2},A_2=\frac{3}{a^2-m^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{2\cos(mx)}{a^2-m^2}+\frac{3\sin(mx)}{a^2-m^2}$$

Therefore the general solution is

$$\begin{aligned}y&=y_h+y_p\\&=\left(c_1\mathrm{e}^{\sqrt{-a^2}\,x}+c_2\mathrm{e}^{-\sqrt{-a^2}\,x}\right)+\left(\frac{2\cos(mx)}{a^2-m^2}+\frac{3\sin(mx)}{a^2-m^2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x} + \frac{2\cos(mx)}{a^2 - m^2} + \frac{3\sin(mx)}{a^2 - m^2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x} + \frac{2\cos(mx)}{a^2 - m^2} + \frac{3\sin(mx)}{a^2 - m^2}$$

Verified OK.

16.59.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + a^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= a^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-a^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -a^2$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(-a^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r=-a^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z''=rz$ as one solution is

$$z_1(x)=\mathrm{e}^{\sqrt{-a^2}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}y_1&=z_1\\&=\mathrm{e}^{\sqrt{-a^2}\,x}\end{aligned}$$

Which simplifies to

$$y_1=\mathrm{e}^{\sqrt{-a^2}\,x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B=0$ then the above becomes

$$\begin{aligned}y_2&=y_1\int \frac{1}{y_1^2}\,dx\\&=\mathrm{e}^{\sqrt{-a^2}\,x}\int \frac{1}{\mathrm{e}^{2\sqrt{-a^2}\,x}}\,dx\\&=\mathrm{e}^{\sqrt{-a^2}\,x}\left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(\mathrm{e}^{\sqrt{-a^2}\,x}\right)+c_2\left(\mathrm{e}^{\sqrt{-a^2}\,x}\left(\frac{\sqrt{-a^2}\,\mathrm{e}^{-2\sqrt{-a^2}\,x}}{2a^2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + a^2 y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + \frac{c_2 \sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 \cos (mx) + 3 \sin (mx)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos (mx) , \sin (mx)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2}, \mathrm{e}^{\sqrt{-a^2}\, x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos (mx) + A_2 \sin (mx)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_1 m^2 \cos (mx) - A_2 m^2 \sin (mx) + a^2 (A_1 \cos (mx) + A_2 \sin (mx)) \\ &= 2 \cos (mx) + 3 \sin (mx)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{a^2 - m^2}, A_2 = \frac{3}{a^2 - m^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\cos(mx)}{a^2 - m^2} + \frac{3\sin(mx)}{a^2 - m^2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + \frac{c_2\sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2}\right) + \left(\frac{2\cos(mx)}{a^2 - m^2} + \frac{3\sin(mx)}{a^2 - m^2}\right)
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + \frac{c_2\sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2} + \frac{2\cos(mx)}{a^2 - m^2} + \frac{3\sin(mx)}{a^2 - m^2} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{\sqrt{-a^2}\, x} + \frac{c_2\sqrt{-a^2}\, \mathrm{e}^{-\sqrt{-a^2}\, x}}{2a^2} + \frac{2\cos(mx)}{a^2 - m^2} + \frac{3\sin(mx)}{a^2 - m^2}$$

Verified OK.

16.59.3 Maple step by step solution

Let's solve

$y'' + a^2 y = 2\cos(mx) + 3\sin(mx)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $a^2 + r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4a^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-a^2}, -\sqrt{-a^2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\sqrt{-a^2}\, x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\sqrt{-a^2}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\cos(mx) + 3\sin(mx)\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\sqrt{-a^2}\, x} & e^{-\sqrt{-a^2}\, x} \\ \sqrt{-a^2}\, e^{\sqrt{-a^2}\, x} & -\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = -2\sqrt{-a^2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{e^{\sqrt{-a^2}\, x}\left(\int (2\cos(mx)+3\sin(mx)) e^{-\sqrt{-a^2}\, x} dx\right) - e^{-\sqrt{-a^2}\, x}\left(\int (2\cos(mx)+3\sin(mx)) e^{\sqrt{-a^2}\, x} dx\right)}{2\sqrt{-a^2}}$

  - Compute integrals

    $y_p(x) = \frac{2\cos(mx)+3\sin(mx)}{a^2-m^2}$

- Substitute particular solution into general solution to ODE

  $y = \frac{2\cos(mx)+3\sin(mx)}{a^2-m^2} + c_1 e^{\sqrt{-a^2}\, x} + c_2 e^{-\sqrt{-a^2}\, x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+a^2*y(x)=2*cos(m*x)+3*sin(m*x),y(x), singsol=all)
```

$$y(x) = \sin(ax)\, c_2 + \cos(ax)\, c_1 + \frac{2\cos(mx) + 3\sin(mx)}{a^2 - m^2}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 45

```
DSolve[y''[x]+a^2*y[x]==2*Cos[m*x]+3*Sin[m*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3\sin(mx) + 2\cos(mx)}{a^2 - m^2} + c_1\cos(ax) + c_2\sin(ax)$$

16.60 problem 533

Internal problem ID [15303]
Internal file name [`OUTPUT/15303_Wednesday_May_08_2024_03_55_11_PM_49692873/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 533.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - y' = e^{x} \sin (x)}$$

16.60.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$A y''(x) + B y'(x) + C y(x) = f(x)$$

Where $A = 1, B = -1, C = 0, f(x) = e^{x} \sin (x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(0)x}$$

Or

$$y = e^x c_1 + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(x) + A_2 e^x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^x \sin(x) + A_2 e^x \cos(x) - A_1 e^x \cos(x) - A_2 e^x \sin(x) = e^x \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (e^x c_1 + c_2) + \left(-\frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = e^x c_1 + c_2 - \frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2} \tag{1}
$$

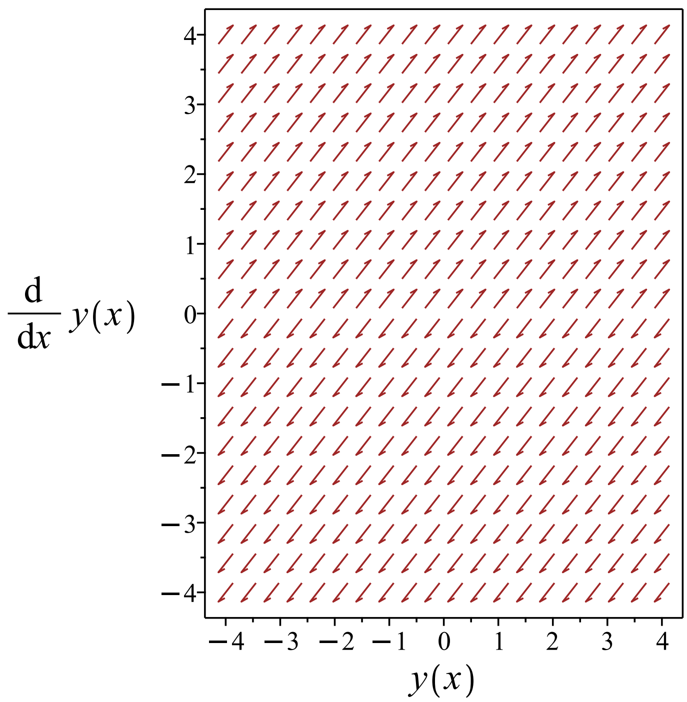


Figure 595: Slope field plot

<u>Verification of solutions</u>

$$
y = e^x c_1 + c_2 - \frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2}
$$

Verified OK.

16.60.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\, dx = \int e^x \sin(x)\, dx$$

$$-y + y' = -\frac{e^x \cos(x)}{2} + \frac{e^x \sin(x)}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2} \end{aligned}$$

Hence the ode is

$$-y + y' = c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2}\right) \\ \frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2}\right) \\ d\left(e^{-x} y\right) &= \left(\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x} y &= \int \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}\, dx \\ e^{-x} y &= -c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x}\left(-c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}\right) + e^{x} c_2$$

which simplifies to

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\, e^{x}}{2} - c_1$$

Summary

The solution(s) found are the following

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\, e^{x}}{2} - c_1 \tag{1}$$

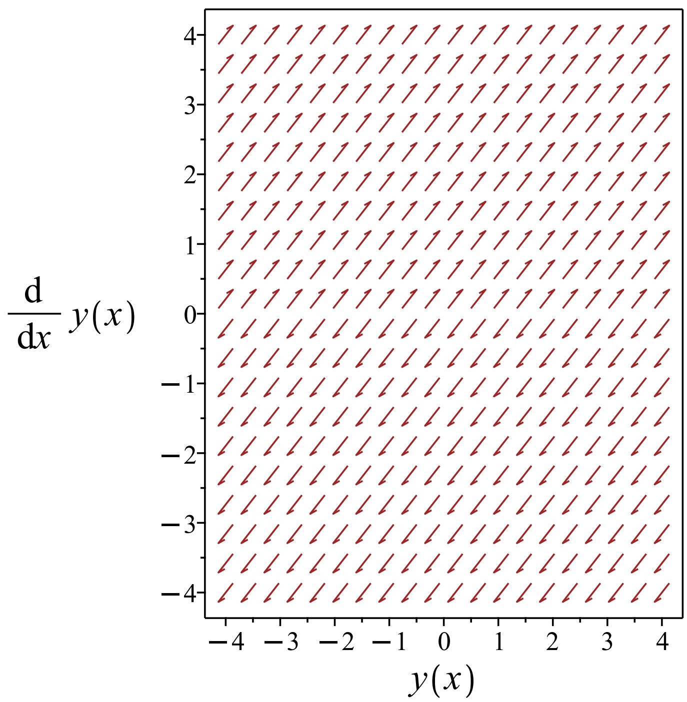


Figure 596: Slope field plot

Verification of solutions

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\, e^{x}}{2} - c_1$$

Verified OK.

16.60.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x) - e^x \sin(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= e^x \sin(x) \end{aligned}$$

Hence the ode is

$$p'(x) - p(x) = e^x \sin(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(e^x \sin(x)\right) \\ \frac{d}{dx}\left(e^{-x} p\right) &= \left(e^{-x}\right)\left(e^x \sin(x)\right) \\ d\left(e^{-x} p\right) &= \sin(x)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x} p &= \int \sin(x)\, dx \\ e^{-x} p &= -\cos(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$p(x) = -e^{x} \cos(x) + e^{x} c_1$$

which simplifies to

$$p(x) = e^{x}(-\cos(x) + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{x}(-\cos(x) + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -e^{x}(\cos(x) - c_1)\, dx \\ &= e^{x} c_1 + c_2 - \frac{e^{x}\cos(x)}{2} - \frac{e^{x}\sin(x)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{x} c_1 + c_2 - \frac{e^{x}\cos(x)}{2} - \frac{e^{x}\sin(x)}{2} \tag{1}$$

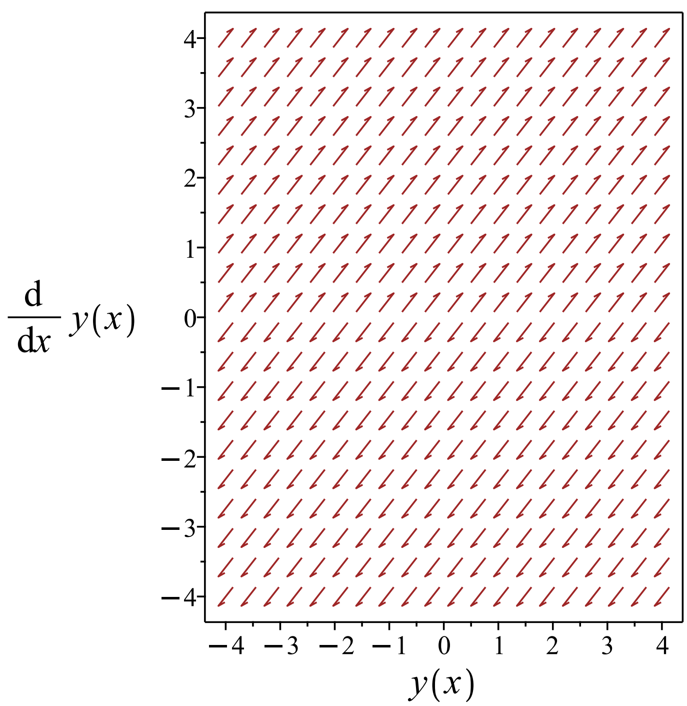


Figure 597: Slope field plot

Verification of solutions

$$y = e^{x} c_1 + c_2 - \frac{e^{x} \cos(x)}{2} - \frac{e^{x} \sin(x)}{2}$$

Verified OK.

16.60.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - y' = e^{x} \sin(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\, dx = \int e^{x} \sin(x)\, dx$$

$$-y + y' = -\frac{e^{x} \cos(x)}{2} + \frac{e^{x} \sin(x)}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x) y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= c_1 + \frac{e^{x}(-\cos(x) + \sin(x))}{2} \end{aligned}$$

Hence the ode is

$$-y + y' = c_1 + \frac{e^{x}(-\cos(x) + \sin(x))}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1) dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu) \left(c_1 + \frac{e^{x}(-\cos(x) + \sin(x))}{2}\right) \\ \frac{d}{dx}(e^{-x} y) &= (e^{-x}) \left(c_1 + \frac{e^{x}(-\cos(x) + \sin(x))}{2}\right) \\ d(e^{-x} y) &= \left(\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}\right) dx \end{aligned}$$

Integrating gives

$$e^{-x}y = \int \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}\, dx$$
$$e^{-x}y = -c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x}\left(-c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}\right) + e^{x} c_2$$

which simplifies to

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\, e^{x}}{2} - c_1$$

Summary

The solution(s) found are the following

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\, e^{x}}{2} - c_1 \tag{1}$$

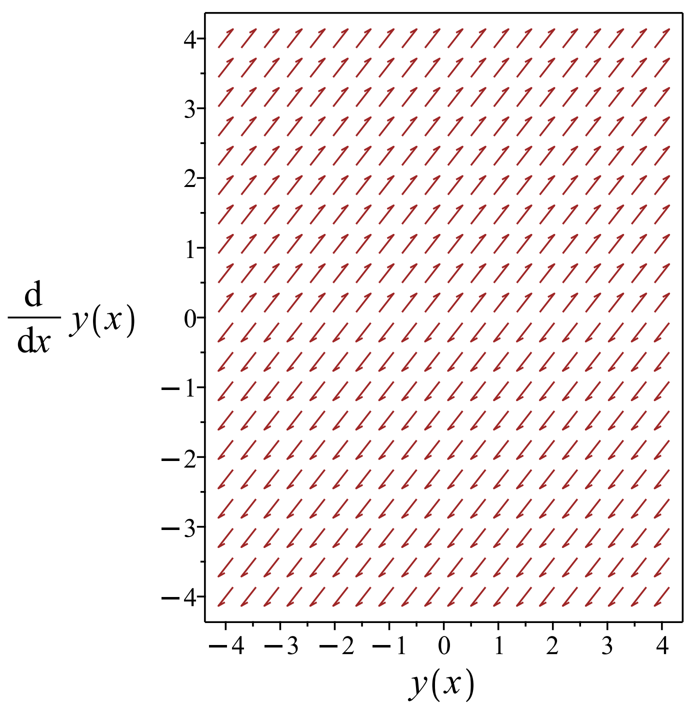


Figure 598: Slope field plot

Verification of solutions

$$
y = \frac{(2c_2 - \cos(x) - \sin(x))\,\mathrm{e}^{x}}{2} - c_1
$$

Verified OK.

16.60.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - y' &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= -1 \tag{3} \\
C &= 0
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1 \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^x}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(\mathrm{e}^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + e^x c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(x) + A_2 e^x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^x \sin(x) + A_2 e^x \cos(x) - A_1 e^x \cos(x) - A_2 e^x \sin(x) = e^x \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + e^x c_2) + \left(-\frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + e^x c_2 - \frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2} \tag{1}
$$

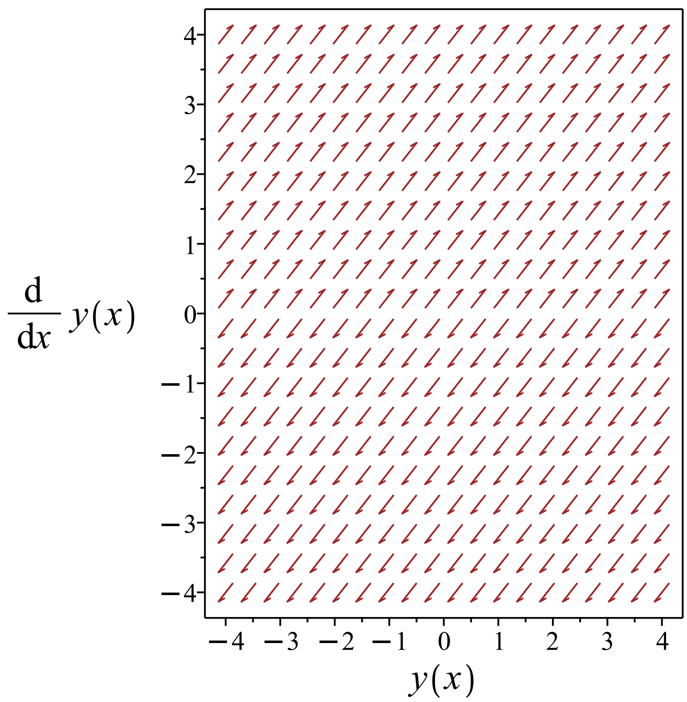


Figure 599: Slope field plot

Verification of solutions

$$
y = c_1 + e^x c_2 - \frac{e^x \cos(x)}{2} - \frac{e^x \sin(x)}{2}
$$

Verified OK.

16.60.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= -1 \\
r(x) &= 0 \\
s(x) &= \mathrm{e}^{x} \sin(x)
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$-y + y' = \int \mathrm{e}^{x} \sin(x)\, dx$$

We now have a first order ode to solve which is

$$-y + y' = -\frac{\mathrm{e}^{x} \cos(x)}{2} + \frac{\mathrm{e}^{x} \sin(x)}{2} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2} \end{aligned}$$

Hence the ode is

$$-y + y' = c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-1)dx} \\ &= e^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2}\right) \\ \frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)\left(c_1 + \frac{e^x(-\cos(x) + \sin(x))}{2}\right) \\ d\left(e^{-x} y\right) &= \left(\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x} y &= \int \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{-x} \, dx \\ e^{-x} y &= -c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^x\left(-c_1 e^{-x} - \frac{\sin(x)}{2} - \frac{\cos(x)}{2}\right) + e^x c_2$$

which simplifies to

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\, e^x}{2} - c_1$$

Summary

The solution(s) found are the following

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\,e^x}{2} - c_1 \tag{1}$$

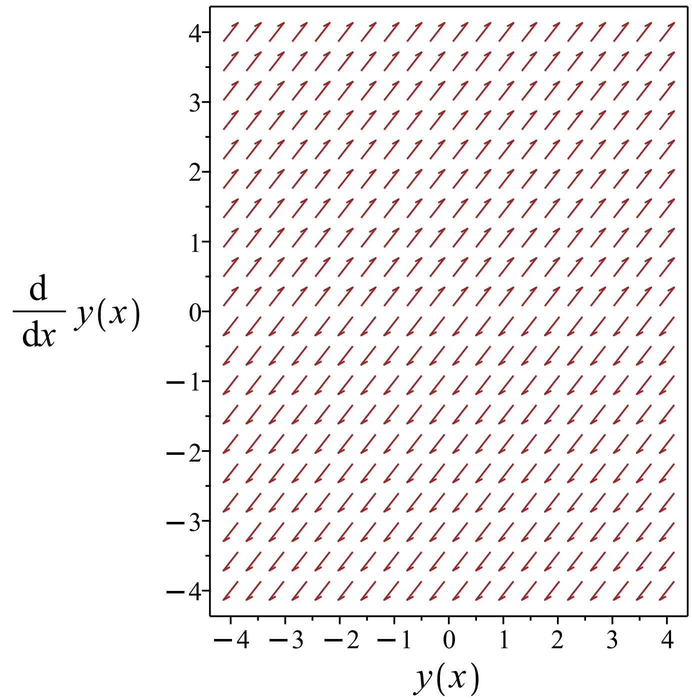


Figure 600: Slope field plot

Verification of solutions

$$y = \frac{(2c_2 - \cos(x) - \sin(x))\,e^x}{2} - c_1$$

Verified OK.

16.60.7 Maple step by step solution

Let's solve

$y'' - y' = e^x \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r = 0$

- Factor the characteristic polynomial

  $r(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (0, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + e^x c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^x \sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^x \\ 0 & e^x \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^x$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\left(\int e^x \sin(x)\, dx\right) + e^x\left(\int \sin(x)\, dx\right)$

  - Compute integrals

    $y_p(x) = -\frac{e^x(\sin(x) + \cos(x))}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + e^x c_2 - \frac{e^x(\sin(x) + \cos(x))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(_a)*sin(_a)+_b(_a), _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-diff(y(x),x)=exp(x)*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{(2c_1 - \cos(x) - \sin(x))\,e^x}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 24

```
DSolve[y''[x]-y'[x]==Exp[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{2}e^x(\sin(x) + \cos(x) - 2c_1)$$

16.61 problem 534

Internal problem ID [15304]
Internal file name [`OUTPUT/15304_Wednesday_May_08_2024_03_55_14_PM_22157174/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 534.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2y' = 4\,\mathrm{e}^{x}(\sin(x) + \cos(x))}$$

16.61.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 0, f(x) = (4\sin(x) + 4\cos(x))\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(0)}\\ &= -1 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = -1 + 1$$
$$\lambda_2 = -1 - 1$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 + e^{-2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,e^{x}(\sin(x) + \cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x} \cos(x), e^{x} \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} \cos(x) + A_2 e^{x} \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{x} \sin(x) + 4A_2 e^{x} \cos(x) + 2A_1 e^{x} \cos(x) + 2A_2 e^{x} \sin(x) = (4\sin(x) + 4\cos(x))\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{6}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\,e^{x} \cos(x)}{5} + \frac{6\,e^{x} \sin(x)}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + e^{-2x} c_2\right) + \left(-\frac{2 e^{x} \cos(x)}{5} + \frac{6 e^{x} \sin(x)}{5}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 + e^{-2x} c_2 - \frac{2 e^{x} \cos(x)}{5} + \frac{6 e^{x} \sin(x)}{5} \tag{1}
$$

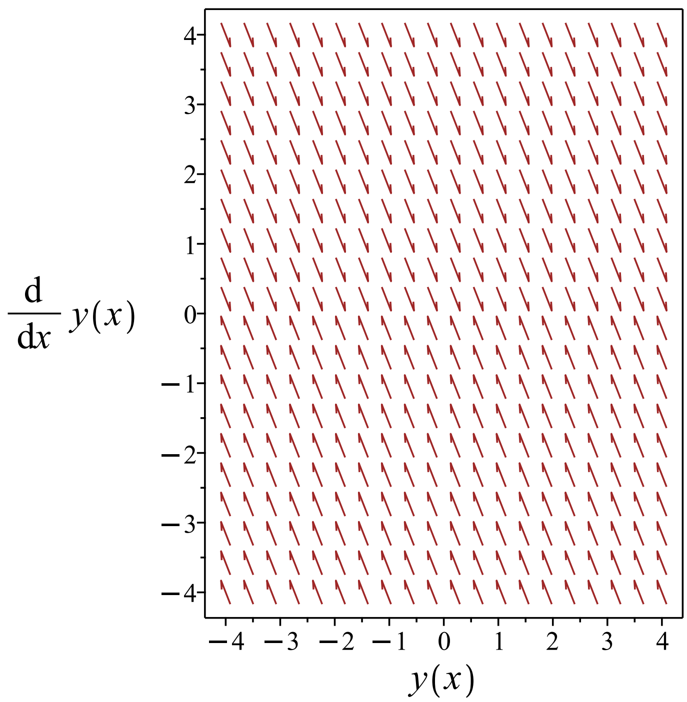


Figure 601: Slope field plot

Verification of solutions

$$
y = c_1 + e^{-2x} c_2 - \frac{2 e^{x} \cos(x)}{5} + \frac{6 e^{x} \sin(x)}{5}
$$

Verified OK.

16.61.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\,dx = \int (4\sin(x) + 4\cos(x))\,e^{x}dx$$
$$2y + y' = 4\,e^{x}\sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 4\,e^{x}\sin(x) + c_1 \end{aligned}$$

Hence the ode is

$$2y + y' = 4\,e^{x}\sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2dx} \\ &= e^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(4\,e^{x}\sin(x) + c_1\right) \\ \frac{d}{dx}\left(e^{2x}y\right) &= \left(e^{2x}\right)\left(4\,e^{x}\sin(x) + c_1\right) \\ d\left(e^{2x}y\right) &= \left(\left(4\,e^{x}\sin(x) + c_1\right)e^{2x}\right)\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{2x}y &= \int \left(4\,e^{x}\sin(x) + c_1\right)e^{2x}\,dx \\ e^{2x}y &= -\frac{2\,e^{3x}\cos(x)}{5} + \frac{6\,e^{3x}\sin(x)}{5} + \frac{e^{2x}c_1}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{2\,e^{3x}\cos(x)}{5} + \frac{6\,e^{3x}\sin(x)}{5} + \frac{e^{2x}c_1}{2}\right) + e^{-2x}c_2$$

which simplifies to

$$y = \frac{e^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5} + e^{2x}c_1 + 2c_2\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5} + e^{2x}c_1 + 2c_2\right)}{2} \tag{1}$$

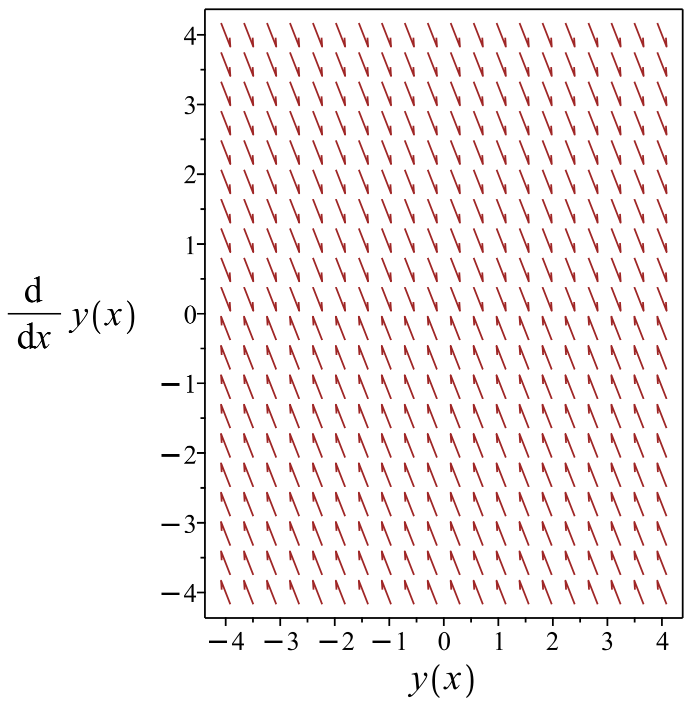


Figure 602: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5} + e^{2x}c_1 + 2c_2\right)}{2}$$

Verified OK.

16.61.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) - (4\sin(x) + 4\cos(x))\,\mathrm{e}^{x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 4\,\mathrm{e}^{x}(\sin(x) + \cos(x)) \end{aligned}$$

Hence the ode is

$$p'(x) + 2p(x) = 4\,\mathrm{e}^{x}(\sin(x) + \cos(x))$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 2dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(4\,\mathrm{e}^{x}(\sin(x) + \cos(x))\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}p\right) &= \left(\mathrm{e}^{2x}\right)\left(4\,\mathrm{e}^{x}(\sin(x) + \cos(x))\right) \\ \mathrm{d}\left(\mathrm{e}^{2x}p\right) &= \left(4(\sin(x) + \cos(x))\,\mathrm{e}^{3x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{2x}p &= \int 4(\sin(x) + \cos(x))\,\mathrm{e}^{3x}\,\mathrm{d}x \\ \mathrm{e}^{2x}p &= \frac{4\,\mathrm{e}^{3x}\cos(x)}{5} + \frac{8\,\mathrm{e}^{3x}\sin(x)}{5} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$p(x) = e^{-2x}\left(\frac{4\,e^{3x}\cos(x)}{5} + \frac{8\,e^{3x}\sin(x)}{5}\right) + c_1 e^{-2x}$$

which simplifies to

$$p(x) = e^{-2x}\left(\frac{4\,e^{3x}(\cos(x) + 2\sin(x))}{5} + c_1\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{-2x}\left(\frac{4\,e^{3x}(\cos(x) + 2\sin(x))}{5} + c_1\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{e^{-2x}(8\,e^{3x}\sin(x) + 4\,e^{3x}\cos(x) + 5c_1)}{5}\, dx \\ &= -\frac{2\,e^{x}\cos(x)}{5} + \frac{6\,e^{x}\sin(x)}{5} - \frac{c_1 e^{-2x}}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\,e^{x}\cos(x)}{5} + \frac{6\,e^{x}\sin(x)}{5} - \frac{c_1 e^{-2x}}{2} + c_2 \tag{1}$$

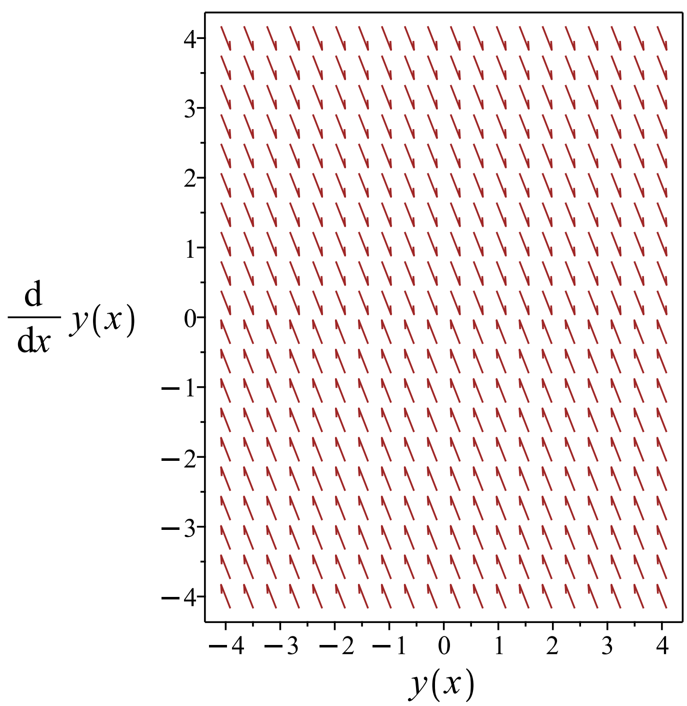


Figure 603: Slope field plot

Verification of solutions

$$y = -\frac{2\,\mathrm{e}^{x} \cos(x)}{5} + \frac{6\,\mathrm{e}^{x} \sin(x)}{5} - \frac{c_1 \mathrm{e}^{-2x}}{2} + c_2$$

Verified OK.

16.61.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = (4\sin(x) + 4\cos(x))\,\mathrm{e}^{x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\,dx = \int (4\sin(x) + 4\cos(x))\,\mathrm{e}^{x}\,dx$$

$$2y + y' = 4\,\mathrm{e}^{x} \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 4\,\mathrm{e}^{x} \sin(x) + c_1 \end{aligned}$$

Hence the ode is

$$2y + y' = 4\,\mathrm{e}^{x} \sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 2dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(4\,\mathrm{e}^{x} \sin(x) + c_1\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x} y\right) &= \left(\mathrm{e}^{2x}\right)\left(4\,\mathrm{e}^{x} \sin(x) + c_1\right) \\ \mathrm{d}\left(\mathrm{e}^{2x} y\right) &= \left(\left(4\,\mathrm{e}^{x} \sin(x) + c_1\right) \mathrm{e}^{2x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$e^{2x}y = \int (4\,e^{x}\sin(x) + c_1)\,e^{2x}\,dx$$

$$e^{2x}y = -\frac{2\,e^{3x}\cos(x)}{5} + \frac{6\,e^{3x}\sin(x)}{5} + \frac{e^{2x}c_1}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{2\,e^{3x}\cos(x)}{5} + \frac{6\,e^{3x}\sin(x)}{5} + \frac{e^{2x}c_1}{2}\right) + e^{-2x}c_2$$

which simplifies to

$$y = \frac{e^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5} + e^{2x}c_1 + 2c_2\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5} + e^{2x}c_1 + 2c_2\right)}{2} \tag{1}$$

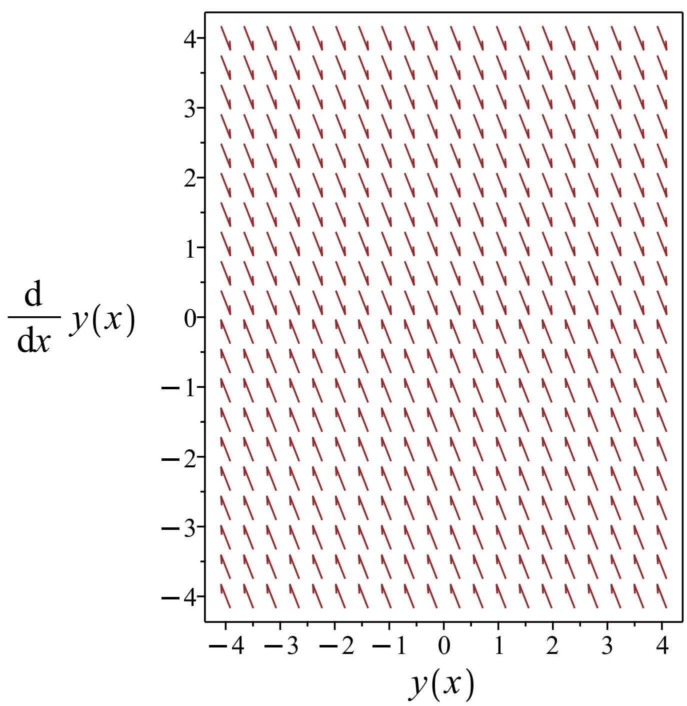


Figure 604: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5}+e^{2x}c_1+2c_2\right)}{2}$$

Verified OK.

16.61.5 Solving using Kovacic algorithm

Writing the ode as

$$y''+2y'=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1\\ B &= 2\\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t}\\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1\\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 e^{x} (\sin(x) + \cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x} \cos(x), e^{x} \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{2}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} \cos(x) + A_2 e^{x} \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{x} \sin(x) + 4A_2 e^{x} \cos(x) + 2A_1 e^{x} \cos(x) + 2A_2 e^{x} \sin(x) = (4\sin(x) + 4\cos(x)) e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{6}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2 e^{x} \cos(x)}{5} + \frac{6 e^{x} \sin(x)}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}\right) + \left(-\frac{2\,\mathrm{e}^{x} \cos(x)}{5} + \frac{6\,\mathrm{e}^{x} \sin(x)}{5}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} - \frac{2\,\mathrm{e}^{x} \cos(x)}{5} + \frac{6\,\mathrm{e}^{x} \sin(x)}{5} \tag{1}
$$

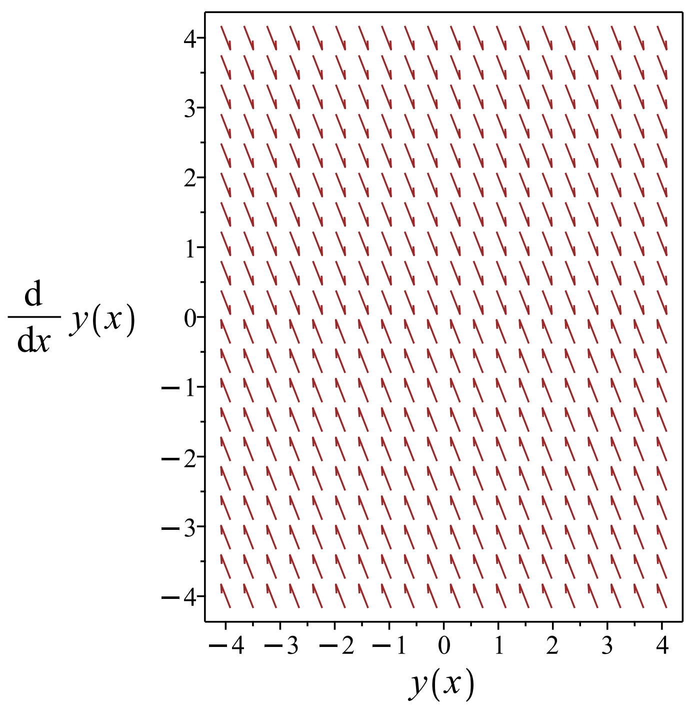


Figure 605: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} - \frac{2\,\mathrm{e}^{x} \cos(x)}{5} + \frac{6\,\mathrm{e}^{x} \sin(x)}{5}
$$

Verified OK.

16.61.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2 \\ r(x) &= 0 \\ s(x) &= (4\sin(x) + 4\cos(x))\, e^{x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$2y + y' = \int (4\sin(x) + 4\cos(x))\, e^{x}\, dx$$

We now have a first order ode to solve which is

$$2y + y' = 4\, e^{x} \sin(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 2\\ q(x) &= 4\,\mathrm{e}^{x} \sin(x) + c_1\end{aligned}$$

Hence the ode is

$$2y + y' = 4\,\mathrm{e}^{x} \sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2dx}\\ &= \mathrm{e}^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(4\,\mathrm{e}^{x} \sin(x) + c_1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x} y\right) &= \left(\mathrm{e}^{2x}\right)\left(4\,\mathrm{e}^{x} \sin(x) + c_1\right)\\ \mathrm{d}\left(\mathrm{e}^{2x} y\right) &= \left(\left(4\,\mathrm{e}^{x} \sin(x) + c_1\right)\mathrm{e}^{2x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{2x} y &= \int \left(4\,\mathrm{e}^{x} \sin(x) + c_1\right)\mathrm{e}^{2x}\,\mathrm{d}x\\ \mathrm{e}^{2x} y &= -\frac{2\,\mathrm{e}^{3x} \cos(x)}{5} + \frac{6\,\mathrm{e}^{3x} \sin(x)}{5} + \frac{\mathrm{e}^{2x} c_1}{2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = \mathrm{e}^{-2x}\left(-\frac{2\,\mathrm{e}^{3x} \cos(x)}{5} + \frac{6\,\mathrm{e}^{3x} \sin(x)}{5} + \frac{\mathrm{e}^{2x} c_1}{2}\right) + \mathrm{e}^{-2x} c_2$$

which simplifies to

$$y = \frac{\mathrm{e}^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))\mathrm{e}^{3x}}{5} + \mathrm{e}^{2x} c_1 + 2c_2\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-2x}\left(\frac{4(-\cos(x)+3\sin(x))\mathrm{e}^{3x}}{5} + \mathrm{e}^{2x} c_1 + 2c_2\right)}{2} \tag{1}$$

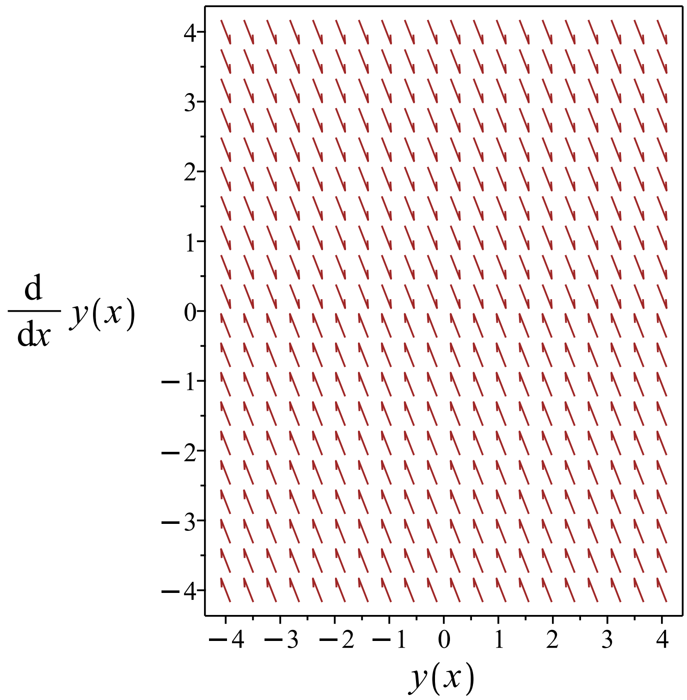


Figure 606: Slope field plot

Verification of solutions

$$y = \frac{e^{-2x} \left(\frac{4(-\cos(x)+3\sin(x))e^{3x}}{5} + e^{2x}c_1 + 2c_2\right)}{2}$$

Verified OK.

16.61.7 Maple step by step solution

Let's solve

$y'' + 2y' = (4\sin(x) + 4\cos(x))\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 4\,e^x \sin(x) + 4\,e^x \cos(x) - 2y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' = 4\,e^x(\sin(x) + \cos(x))$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r = 0$
- Factor the characteristic polynomial

  $r(r+2) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 0)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 + y_p(x)$
- ☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\,e^x(\sin(x) + \cos(x))\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & 1 \\ -2\,e^{-2x} & 0 \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,e^{-2x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,e^{-2x}\left(\int (\sin(x) + \cos(x))\,e^{3x}dx\right) + 2\left(\int e^x(\sin(x) + \cos(x))\,dx\right)$
  - Compute integrals

    $y_p(x) = -\frac{2\,e^x(\cos(x) - 3\sin(x))}{5}$
- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 - \frac{2\,e^x(\cos(x) - 3\sin(x))}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 4*exp(_a)*sin(_a)+4*exp(_a)*cos(_a)-2*_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)=4*exp(x)*(sin(x)+cos(x)),y(x), singsol=all)
```

$$y(x) = -\frac{\left(\frac{4(\cos(x)-3\sin(x))\mathrm{e}^{3x}}{5} - 2c_2\mathrm{e}^{2x} + c_1\right)\mathrm{e}^{-2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 37

```
DSolve[y''[x]+2*y'[x]==4*Exp[x]*(Sin[x]+Cos[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{6}{5}e^x \sin(x) - \frac{2}{5}e^x \cos(x) - \frac{1}{2}c_1e^{-2x} + c_2$$

16.62 problem 535

Internal problem ID [15305]
Internal file name [`OUTPUT/15305_Wednesday_May_08_2024_03_55_17_PM_74653418/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 535.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = 10\,\mathrm{e}^{-2x} \cos(x)}$$

16.62.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = 10\,\mathrm{e}^{-2x} \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\,\mathrm{e}^{-2x}\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\cos(x), \mathrm{e}^{-2x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}\cos(x), \mathrm{e}^{-2x}\sin(x)\}$$

Since $\mathrm{e}^{-2x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}\cos(x), x\,\mathrm{e}^{-2x}\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x}\cos(x) + A_2 x\,\mathrm{e}^{-2x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-2x}\sin(x) + 2A_2\mathrm{e}^{-2x}\cos(x) = 10\,\mathrm{e}^{-2x}\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 5x\,\mathrm{e}^{-2x}\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2x}(c_1\cos(x) + c_2\sin(x))\right) + \left(5x\,\mathrm{e}^{-2x}\sin(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + 5x e^{-2x} \sin(x) \tag{1}$$

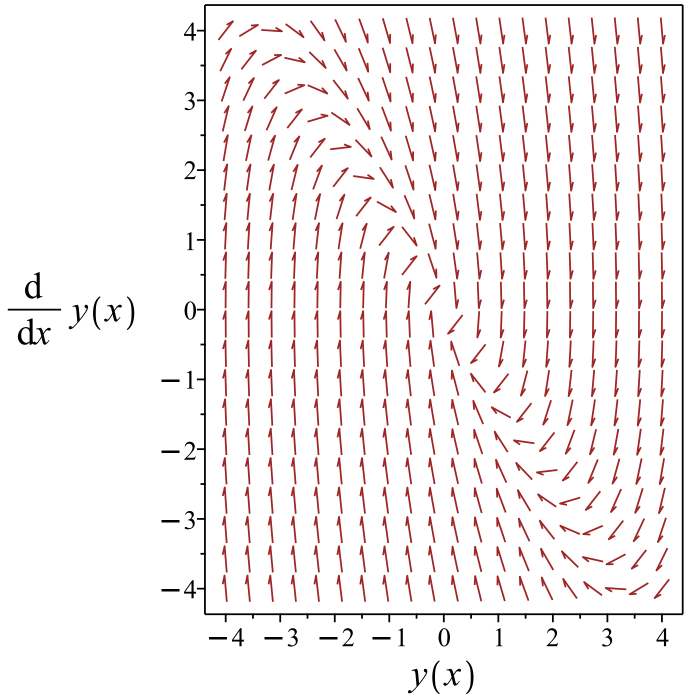


Figure 607: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + 5x e^{-2x} \sin(x)$$

Verified OK.

16.62.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 477: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1\left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-2x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-2x} \cos(x)\right) + c_2\left(e^{-2x} \cos(x)\,(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{-2x} \cos(x)\, c_1 + e^{-2x} \sin(x)\, c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\, e^{-2x} \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\cos(x), e^{-2x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), e^{-2x}\sin(x)\}$$

Since $e^{-2x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\cos(x), x\,e^{-2x}\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-2x}\cos(x) + A_2 x\,e^{-2x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-2x}\sin(x) + 2A_2 e^{-2x}\cos(x) = 10\,e^{-2x}\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 5x\,e^{-2x}\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}\cos(x)\,c_1 + e^{-2x}\sin(x)\,c_2\right) + \left(5x\,e^{-2x}\sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_1\cos(x) + c_2\sin(x)) + 5x\,e^{-2x}\sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + 5x\,e^{-2x} \sin(x) \tag{1}$$

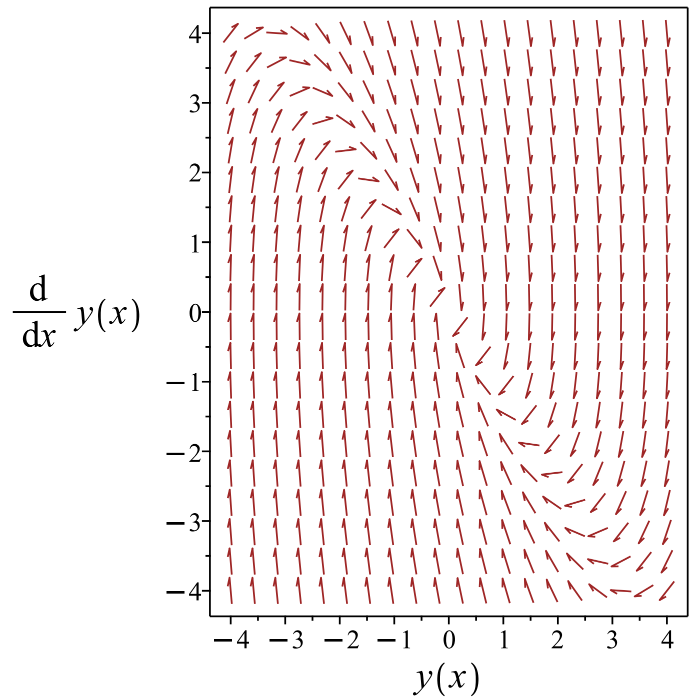


Figure 608: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + 5x\,e^{-2x} \sin(x)$$

Verified OK.

16.62.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = 10\,e^{-2x} \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r
  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-2-\mathrm{I}, -2+\mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \mathrm{e}^{-2x}\cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \mathrm{e}^{-2x}\sin(x)$
- General solution of the ODE
  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y = \mathrm{e}^{-2x}\cos(x)\, c_1 + \mathrm{e}^{-2x}\sin(x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 10\,\mathrm{e}^{-2x}\cos(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x}\cos(x) & \mathrm{e}^{-2x}\sin(x) \\ -2\,\mathrm{e}^{-2x}\cos(x) - \mathrm{e}^{-2x}\sin(x) & -2\,\mathrm{e}^{-2x}\sin(x) + \mathrm{e}^{-2x}\cos(x) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -5\,\mathrm{e}^{-2x}\left(\cos(x)\left(\int \sin(2x)\,dx\right) - 2\sin(x)\left(\int \cos(x)^2\,dx\right)\right)$
  - Compute integrals
    $y_p(x) = \frac{5(2\sin(x)x+\cos(x))\mathrm{e}^{-2x}}{2}$

- Substitute particular solution into general solution to ODE
  $y = \mathrm{e}^{-2x}\cos(x)\, c_1 + \mathrm{e}^{-2x}\sin(x)\, c_2 + \frac{5(2\sin(x)x+\cos(x))\mathrm{e}^{-2x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=10*exp(-2*x)*cos(x),y(x), singsol=all)
```

$$y(x) = ((c_2 + 5x)\sin(x) + \cos(x) c_1) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 34

```
DSolve[y''[x]+4*y'[x]+5*y[x]==10*Exp[-2*x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2x}((5 + 2c_2)\cos(x) + 2(5x + c_1)\sin(x))$$

16.63 problem 536

Internal problem ID [15306]
Internal file name [`OUTPUT/15306_Wednesday_May_08_2024_03_55_18_PM_6844121/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 536.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{4y'' + 8y' = \sin(x)\, x}$$

16.63.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4, B = 8, C = 0, f(x) = \sin(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$4y''+8y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=4, B=8, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x}+8\lambda\, e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2+8\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=4, B=8, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-8}{(2)\,(4)}\pm\frac{1}{(2)\,(4)}\sqrt{8^2-(4)\,(4)\,(0)}\\&=-1\pm 1\end{aligned}$$

Hence

$$\lambda_1=-1+1$$
$$\lambda_2=-1-1$$

Which simplifies to

$$\lambda_1=0$$
$$\lambda_2=-2$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$$
$$y=c_1e^{(0)x}+c_2e^{(-2)x}$$

Or

$$y = c_1 + e^{-2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) x + A_2 \sin(x) x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-4A_1 \cos(x) x - 8A_1 \sin(x) - 4A_2 \sin(x) x + 8A_2 \cos(x) \\
&- 4A_3 \cos(x) - 4A_4 \sin(x) - 8A_1 \sin(x) x + 8A_1 \cos(x) \\
&+ 8A_2 \cos(x) x + 8A_2 \sin(x) - 8A_3 \sin(x) + 8A_4 \cos(x) = \sin(x) x
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = -\frac{1}{20}, A_3 = -\frac{1}{50}, A_4 = \frac{7}{50}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x) x}{10} - \frac{\sin(x) x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + \mathrm{e}^{-2x} c_2) + \left(-\frac{\cos(x)\, x}{10} - \frac{\sin(x)\, x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \mathrm{e}^{-2x} c_2 - \frac{\cos(x)\, x}{10} - \frac{\sin(x)\, x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50} \tag{1}
$$

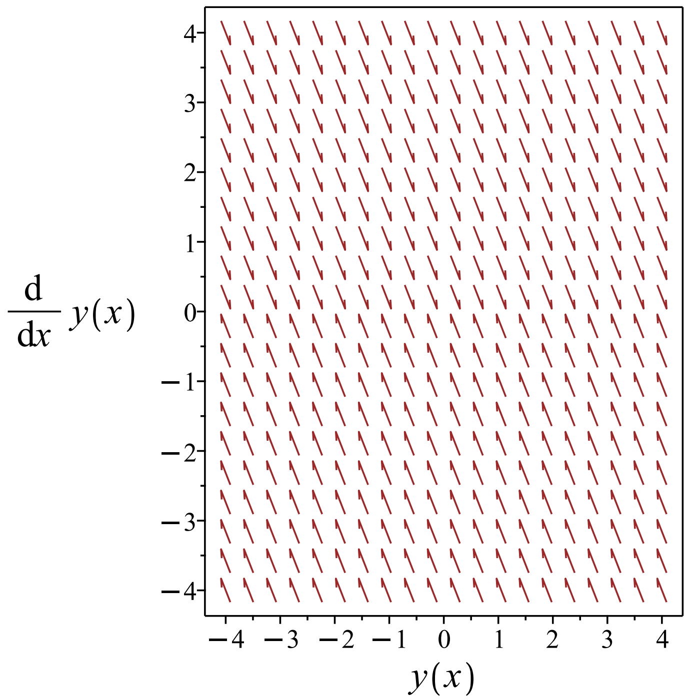


Figure 609: Slope field plot

Verification of solutions

$$
y = c_1 + \mathrm{e}^{-2x} c_2 - \frac{\cos(x)\, x}{10} - \frac{\sin(x)\, x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50}
$$

Verified OK.

16.63.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (4y'' + 8y')\, dx = \int \sin(x)\, x dx$$
$$8y + 4y' = \sin(x) - \cos(x)\, x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = \frac{\sin(x)}{4} - \frac{\cos(x)\, x}{4} + \frac{c_1}{4}$$

Hence the ode is

$$2y + y' = \frac{\sin(x)}{4} - \frac{\cos(x)\, x}{4} + \frac{c_1}{4}$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\sin(x)}{4} - \frac{\cos(x)\, x}{4} + \frac{c_1}{4}\right)$$
$$\frac{d}{dx}(e^{2x} y) = (e^{2x})\left(\frac{\sin(x)}{4} - \frac{\cos(x)\, x}{4} + \frac{c_1}{4}\right)$$
$$d(e^{2x} y) = \left(\frac{(\sin(x) - \cos(x)\, x + c_1)\, e^{2x}}{4}\right) dx$$

Integrating gives

$$e^{2x} y = \int \frac{(\sin(x) - \cos(x)\, x + c_1)\, e^{2x}}{4}\, dx$$
$$e^{2x} y = -\frac{e^{2x}\cos(x)}{20} + \frac{e^{2x}\sin(x)}{10} - \frac{\left(\frac{2x}{5} - \frac{3}{25}\right) e^{2x}\cos(x)}{4} + \frac{\left(-\frac{x}{5} + \frac{4}{25}\right) e^{2x}\sin(x)}{4} + \frac{e^{2x} c_1}{8} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{e^{2x}\cos(x)}{20} + \frac{e^{2x}\sin(x)}{10} - \frac{\left(\frac{2x}{5} - \frac{3}{25}\right)e^{2x}\cos(x)}{4} + \frac{\left(-\frac{x}{5} + \frac{4}{25}\right)e^{2x}\sin(x)}{4} + \frac{e^{2x}c_1}{8}\right) + e^{-2x}c_2$$

which simplifies to

$$y = e^{-2x}c_2 + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100} + \frac{c_1}{8}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}c_2 + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100} + \frac{c_1}{8} \tag{1}$$

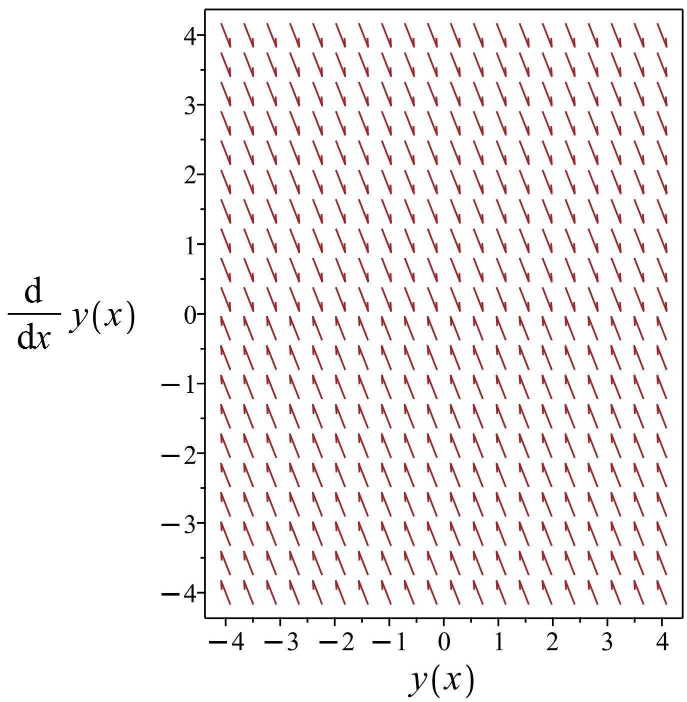


Figure 610: Slope field plot

Verification of solutions

$$y = e^{-2x}c_2 + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100} + \frac{c_1}{8}$$

Verified OK.

16.63.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$4p'(x) + 8p(x) - \sin(x)\, x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= \frac{\sin(x)\, x}{4} \end{aligned}$$

Hence the ode is

$$p'(x) + 2p(x) = \frac{\sin(x)\, x}{4}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2dx} \\ &= e^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(\frac{\sin(x)\, x}{4}\right) \\ \frac{d}{dx}\left(e^{2x} p\right) &= \left(e^{2x}\right)\left(\frac{\sin(x)\, x}{4}\right) \\ d\left(e^{2x} p\right) &= \left(\frac{\sin(x)\, x\, e^{2x}}{4}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{2x}p &= \int \frac{\sin(x)\, x\, e^{2x}}{4}\, dx \\ e^{2x}p &= \frac{\left(-\frac{x}{5}+\frac{4}{25}\right) e^{2x} \cos(x)}{4} + \frac{\left(\frac{2x}{5}-\frac{3}{25}\right) e^{2x} \sin(x)}{4} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$p(x) = e^{-2x}\left(\frac{\left(-\frac{x}{5}+\frac{4}{25}\right) e^{2x} \cos(x)}{4} + \frac{\left(\frac{2x}{5}-\frac{3}{25}\right) e^{2x} \sin(x)}{4}\right) + c_1 e^{-2x}$$

which simplifies to

$$p(x) = c_1 e^{-2x} + \frac{(4-5x)\cos(x)}{100} + \frac{(-3+10x)\sin(x)}{100}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 e^{-2x} + \frac{(4-5x)\cos(x)}{100} + \frac{(-3+10x)\sin(x)}{100}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sin(x)\, x}{10} - \frac{\cos(x)\, x}{20} - \frac{3\sin(x)}{100} + \frac{\cos(x)}{25} + c_1 e^{-2x}\, dx \\ &= -\frac{c_1 e^{-2x}}{2} - \frac{\cos(x)}{50} - \frac{\sin(x)\, x}{20} + \frac{7\sin(x)}{50} - \frac{\cos(x)\, x}{10} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1 e^{-2x}}{2} - \frac{\cos(x)}{50} - \frac{\sin(x)\, x}{20} + \frac{7\sin(x)}{50} - \frac{\cos(x)\, x}{10} + c_2 \tag{1}$$

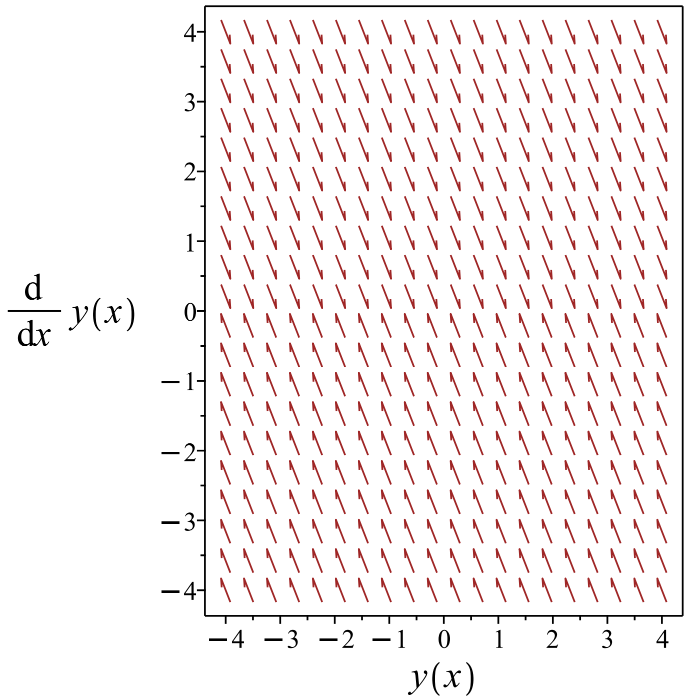


Figure 611: Slope field plot

Verification of solutions

$$y = -\frac{c_1 e^{-2x}}{2} - \frac{\cos(x)}{50} - \frac{\sin(x)\,x}{20} + \frac{7\sin(x)}{50} - \frac{\cos(x)\,x}{10} + c_2$$

Verified OK.

16.63.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$4y'' + 8y' = \sin(x)\,x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (4y'' + 8y')\,dx = \int \sin(x)\,x dx$$

$$8y + 4y' = \sin(x) - \cos(x)\,x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= \frac{\sin(x)}{4} - \frac{\cos(x)\,x}{4} + \frac{c_1}{4}
\end{aligned}
$$

Hence the ode is

$$
2y + y' = \frac{\sin(x)}{4} - \frac{\cos(x)\,x}{4} + \frac{c_1}{4}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2dx} \\
&= e^{2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sin(x)}{4} - \frac{\cos(x)\,x}{4} + \frac{c_1}{4}\right) \\
\frac{d}{dx}\left(e^{2x} y\right) &= \left(e^{2x}\right)\left(\frac{\sin(x)}{4} - \frac{\cos(x)\,x}{4} + \frac{c_1}{4}\right) \\
d\left(e^{2x} y\right) &= \left(\frac{(\sin(x) - \cos(x)\,x + c_1)\,e^{2x}}{4}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2x} y &= \int \frac{(\sin(x) - \cos(x)\,x + c_1)\,e^{2x}}{4}\,dx \\
e^{2x} y &= -\frac{e^{2x}\cos(x)}{20} + \frac{e^{2x}\sin(x)}{10} - \frac{\left(\frac{2x}{5} - \frac{3}{25}\right)e^{2x}\cos(x)}{4} + \frac{\left(-\frac{x}{5} + \frac{4}{25}\right)e^{2x}\sin(x)}{4} + \frac{e^{2x} c_1}{8} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$
y = e^{-2x}\left(-\frac{e^{2x}\cos(x)}{20} + \frac{e^{2x}\sin(x)}{10} - \frac{\left(\frac{2x}{5} - \frac{3}{25}\right)e^{2x}\cos(x)}{4} + \frac{\left(-\frac{x}{5} + \frac{4}{25}\right)e^{2x}\sin(x)}{4} + \frac{e^{2x} c_1}{8}\right) + e^{-2x} c_2
$$

which simplifies to

$$
y = e^{-2x} c_2 + \frac{(-1 - 5x)\cos(x)}{50} + \frac{(-5x + 14)\sin(x)}{100} + \frac{c_1}{8}
$$

Summary

The solution(s) found are the following

$$
y = e^{-2x} c_2 + \frac{(-1 - 5x)\cos(x)}{50} + \frac{(-5x + 14)\sin(x)}{100} + \frac{c_1}{8} \tag{1}
$$

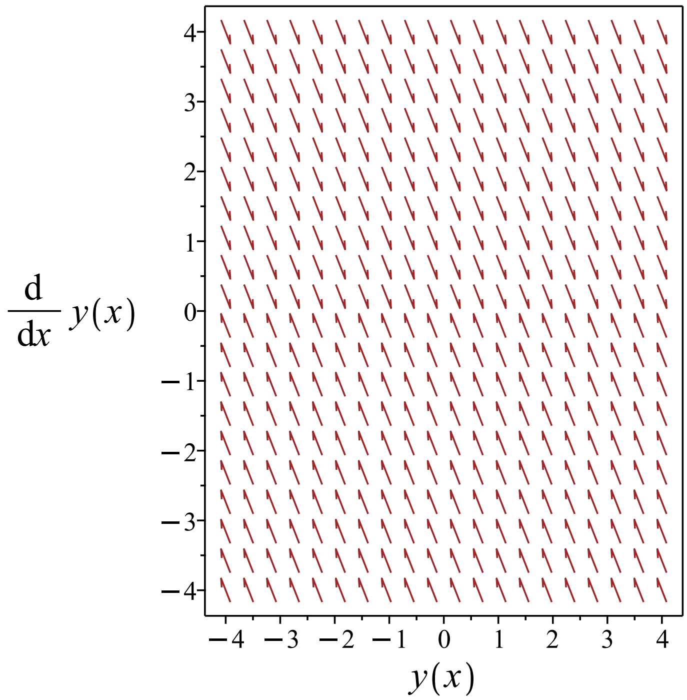


Figure 612: Slope field plot

Verification of solutions

$$y = e^{-2x}c_2 + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100} + \frac{c_1}{8}$$

Verified OK.

16.63.5 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 8y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 8 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 479: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8}{4}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{8}{4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4y'' + 8y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-2x} + \frac{c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)\, x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x)\, x, \sin(x)\, x, \cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{1}{2}, e^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x + A_3 \cos(x) + A_4 \sin(x)
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&-4A_1 \cos(x) x - 8A_1 \sin(x) - 4A_2 \sin(x) x + 8A_2 \cos(x) \\
&- 4A_3 \cos(x) - 4A_4 \sin(x) - 8A_1 \sin(x) x + 8A_1 \cos(x) \\
&+ 8A_2 \cos(x) x + 8A_2 \sin(x) - 8A_3 \sin(x) + 8A_4 \cos(x) = \sin(x) x
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = -\frac{1}{20}, A_3 = -\frac{1}{50}, A_4 = \frac{7}{50}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x) x}{10} - \frac{\sin(x) x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2}{2}\right) + \left(-\frac{\cos(x) x}{10} - \frac{\sin(x) x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2}{2} - \frac{\cos(x) x}{10} - \frac{\sin(x) x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50} \tag{1}$$

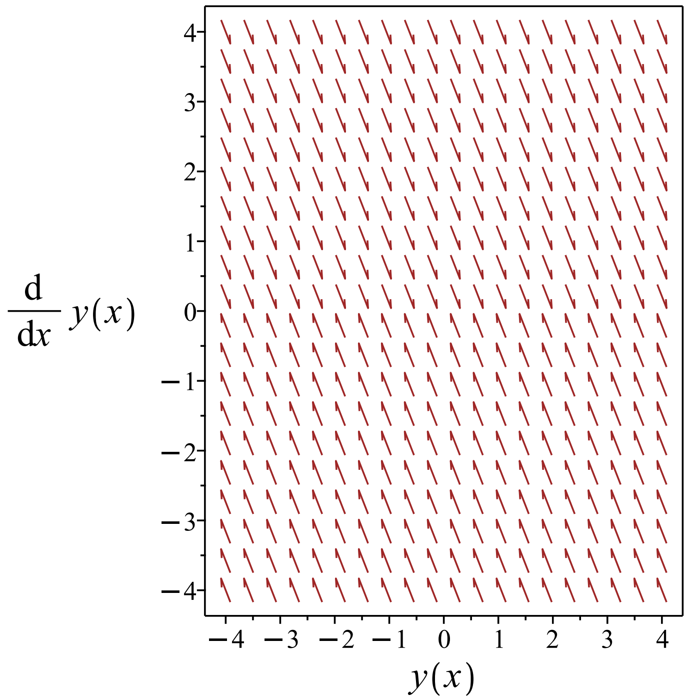


Figure 613: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2}{2} - \frac{\cos(x)\, x}{10} - \frac{\sin(x)\, x}{20} - \frac{\cos(x)}{50} + \frac{7\sin(x)}{50}$$

Verified OK.

16.63.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 8 \\ r(x) &= 0 \\ s(x) &= \sin(x)\, x \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
8y + 4y' = \int \sin(x)\, x\, dx
$$

We now have a first order ode to solve which is

$$
8y + 4y' = \sin(x) - \cos(x)\, x + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= \frac{\sin(x)}{4} - \frac{\cos(x)\, x}{4} + \frac{c_1}{4}
\end{aligned}
$$

Hence the ode is

$$
2y + y' = \frac{\sin(x)}{4} - \frac{\cos(x)\, x}{4} + \frac{c_1}{4}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 2dx} \\
&= e^{2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\sin(x)}{4}-\frac{\cos(x)\,x}{4}+\frac{c_1}{4}\right)\\
\frac{d}{dx}\left(e^{2x}y\right) &= \left(e^{2x}\right)\left(\frac{\sin(x)}{4}-\frac{\cos(x)\,x}{4}+\frac{c_1}{4}\right)\\
d\left(e^{2x}y\right) &= \left(\frac{(\sin(x)-\cos(x)\,x+c_1)\,e^{2x}}{4}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{2x}y &= \int \frac{(\sin(x)-\cos(x)\,x+c_1)\,e^{2x}}{4}\,dx\\
e^{2x}y &= -\frac{e^{2x}\cos(x)}{20}+\frac{e^{2x}\sin(x)}{10}-\frac{\left(\frac{2x}{5}-\frac{3}{25}\right)e^{2x}\cos(x)}{4}+\frac{\left(-\frac{x}{5}+\frac{4}{25}\right)e^{2x}\sin(x)}{4}+\frac{e^{2x}c_1}{8}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$
y = e^{-2x}\left(-\frac{e^{2x}\cos(x)}{20}+\frac{e^{2x}\sin(x)}{10}-\frac{\left(\frac{2x}{5}-\frac{3}{25}\right)e^{2x}\cos(x)}{4}+\frac{\left(-\frac{x}{5}+\frac{4}{25}\right)e^{2x}\sin(x)}{4}+\frac{e^{2x}c_1}{8}\right)+e^{-2x}c_2
$$

which simplifies to

$$
y = e^{-2x}c_2+\frac{(-1-5x)\cos(x)}{50}+\frac{(-5x+14)\sin(x)}{100}+\frac{c_1}{8}
$$

Summary

The solution(s) found are the following

$$
y = e^{-2x}c_2+\frac{(-1-5x)\cos(x)}{50}+\frac{(-5x+14)\sin(x)}{100}+\frac{c_1}{8} \tag{1}
$$

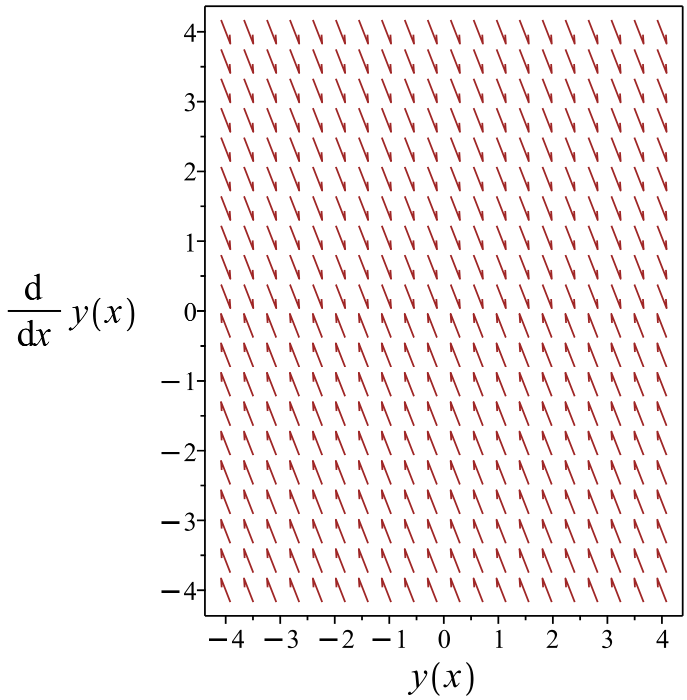


Figure 614: Slope field plot

Verification of solutions

$$y = e^{-2x}c_2 + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100} + \frac{c_1}{8}$$

Verified OK.

16.63.7 Maple step by step solution

Let's solve

$4y'' + 8y' = \sin(x)\,x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -2y' + \frac{\sin(x)x}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' = \frac{\sin(x)x}{4}$

- Characteristic polynomial of homogeneous ODE

$r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{\sin(x) x}{4}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & 1 \\ -2\,\mathrm{e}^{-2x} & 0 \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-2x} \left(\int \sin(x) x\, \mathrm{e}^{2x} dx\right)}{8} + \frac{\left(\int \sin(x) x dx\right)}{8}$

  ◦ Compute integrals

  $y_p(x) = \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/4)*sin(_a)*_a-2*_b(_a), _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(4*diff(y(x),x$2)+8*diff(y(x),x)=x*sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-2x}c_1}{2} + \frac{(-1-5x)\cos(x)}{50} + \frac{(-5x+14)\sin(x)}{100} + c_2$$

✓ Solution by Mathematica

Time used: 0.266 (sec). Leaf size: 42

```
DSolve[4*y''[x]+8*y'[x]==x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{7}{50} - \frac{x}{20}\right)\sin(x) - \frac{1}{50}(5x+1)\cos(x) - \frac{1}{2}c_1e^{-2x} + c_2$$

16.64 problem 537

Internal problem ID [15307]
Internal file name [`OUTPUT/15307_Wednesday_May_08_2024_03_55_21_PM_53385904/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 537.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = x\,\mathrm{e}^{x}}$$

16.64.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = x\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda \,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = \mathrm{e}^{2x} c_1 + \mathrm{e}^{x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x} c_1 + \mathrm{e}^{x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}, e^{x}x^{2}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x} x^{2}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{x} - 2A_2 e^{x} x + 2A_2 e^{x} = x\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^{x} - \frac{e^{x}x^{2}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}c_1 + e^{x}c_2\right) + \left(-x\,e^{x} - \frac{e^{x}x^{2}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x} c_1 + e^x c_2 - x\,e^x - \frac{e^x x^2}{2} \tag{1}$$

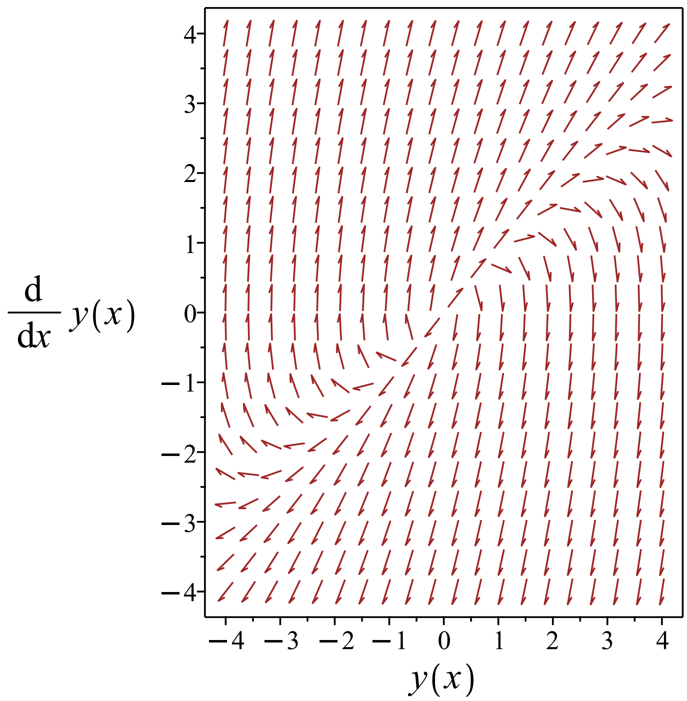


Figure 615: Slope field plot

Verification of solutions

$$y = e^{2x} c_1 + e^x c_2 - x\,e^x - \frac{e^x x^2}{2}$$

Verified OK.

16.64.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} \, dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(e^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^x c_1 + e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\, e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{2x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}x^{2}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x} x^{2}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\mathrm{e}^{x} - 2A_2\mathrm{e}^{x}x + 2A_2\mathrm{e}^{x} = x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^{x} - \frac{\mathrm{e}^{x}x^{2}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{x}c_1 + \mathrm{e}^{2x}c_2\right) + \left(-x\,\mathrm{e}^{x} - \frac{\mathrm{e}^{x}x^{2}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}c_1 + \mathrm{e}^{2x}c_2 - x\,\mathrm{e}^{x} - \frac{\mathrm{e}^{x}x^{2}}{2} \tag{1}$$

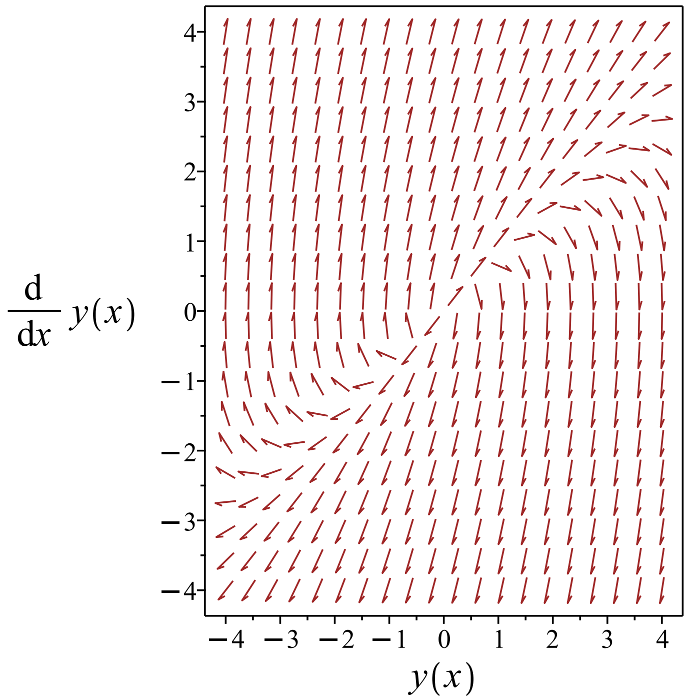


Figure 616: Slope field plot

<u>Verification of solutions</u>

$$y = e^x c_1 + e^{2x} c_2 - x e^x - \frac{e^x x^2}{2}$$

Verified OK.

16.64.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = x e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + e^{2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,e^x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{3x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^x\left(\int x dx\right) + e^{2x}\left(\int x\,e^{-x} dx\right)$

  ◦ Compute integrals

    $y_p(x) = e^x\left(-1 - \frac{1}{2}x^2 - x\right)$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + e^{2x} c_2 + e^x\left(-1 - \frac{1}{2}x^2 - x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=x*exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{(-2c_1 e^x + x^2 - 2c_2 + 2x) e^x}{2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 33

```
DSolve[y''[x]-3*y'[x]+2*y[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^x\left(-x^2 - 2x + 2(c_2e^x - 1 + c_1)\right)$$

16.65 problem 538

Internal problem ID [15308]
Internal file name [`OUTPUT/15308_Wednesday_May_08_2024_03_55_22_PM_23406949/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 538.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' - 2y = x^2 \mathrm{e}^{4x}}$$

16.65.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = x^2 \mathrm{e}^{4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda\, \mathrm{e}^{\lambda x} - 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{x} c_1 + \mathrm{e}^{-2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x} c_1 + \mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 e^{4x}, e^{4x} x, e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x^2 e^{4x} + A_2 e^{4x} x + A_3 e^{4x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{4x} + 18A_1 x\, e^{4x} + 18A_1 x^2 e^{4x} + 18A_2 e^{4x} x + 9A_2 e^{4x} + 18A_3 e^{4x} = x^2 e^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = -\frac{1}{18}, A_3 = \frac{7}{324}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7\, e^{4x}}{324}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{-2x} c_2\right) + \left(\frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7\, e^{4x}}{324}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{x} c_1 + e^{-2x} c_2 + \frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7 e^{4x}}{324} \tag{1}$$

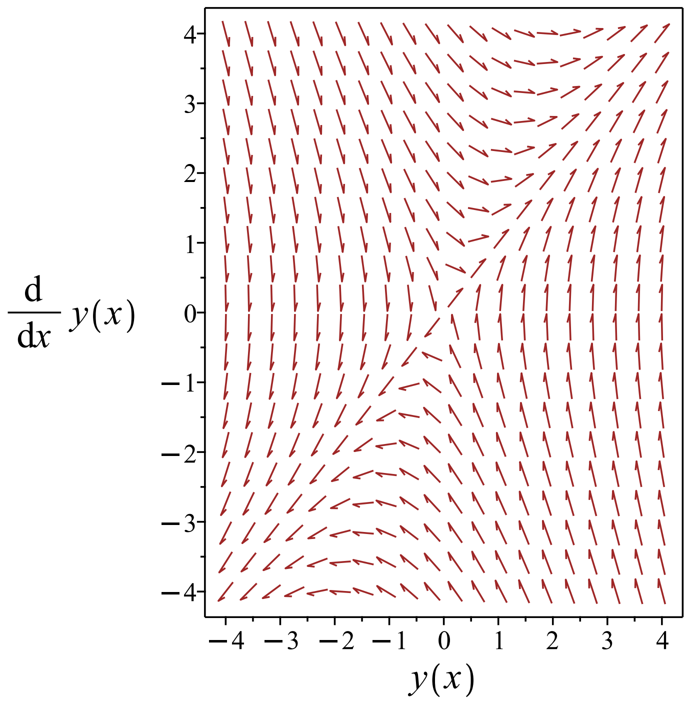


Figure 617: Slope field plot

Verification of solutions

$$y = e^{x} c_1 + e^{-2x} c_2 + \frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7 e^{4x}}{324}$$

Verified OK.

16.65.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -2
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{\mathrm{e}^{x} c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \mathrm{e}^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 e^{4x}, e^{4x} x, e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^x}{3}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x^2 e^{4x} + A_2 e^{4x} x + A_3 e^{4x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{4x} + 18A_1 x\, e^{4x} + 18A_1 x^2 e^{4x} + 18A_2 e^{4x} x + 9A_2 e^{4x} + 18A_3 e^{4x} = x^2 e^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = -\frac{1}{18}, A_3 = \frac{7}{324}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7\, e^{4x}}{324}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{e^x c_2}{3}\right) + \left(\frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7\, e^{4x}}{324}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{e^x c_2}{3} + \frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7\, e^{4x}}{324} \tag{1}$$

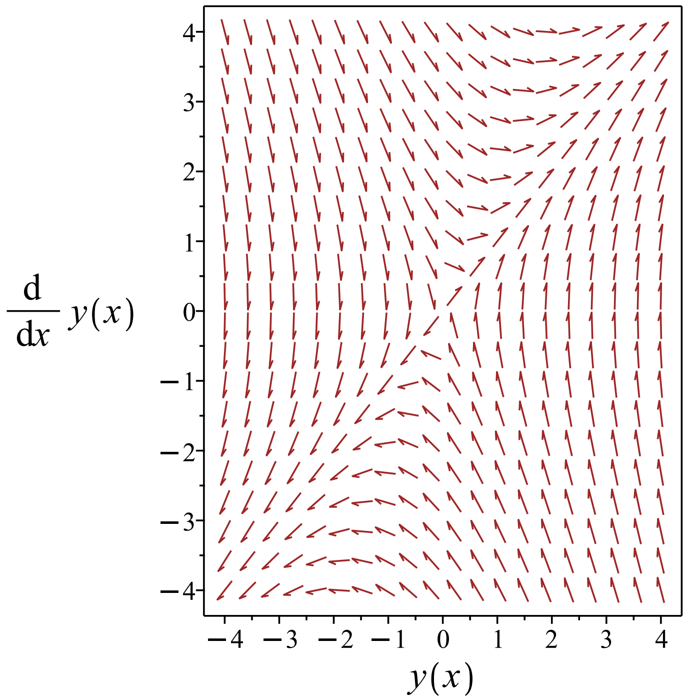


Figure 618: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{e^x c_2}{3} + \frac{x^2 e^{4x}}{18} - \frac{e^{4x} x}{18} + \frac{7 e^{4x}}{324}$$

Verified OK.

16.65.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = x^2 e^{4x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + e^{x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 e^{4x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\left(e^{3x}\left(\int x^2 e^{3x} dx\right) - \left(\int x^2 e^{6x} dx\right)\right) e^{-2x}}{3}$

  ◦ Compute integrals

  $y_p(x) = \frac{e^{4x}(18x^2 - 18x + 7)}{324}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + e^{x} c_2 + \frac{e^{4x}(18x^2 - 18x + 7)}{324}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=x^2*exp(4*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(\frac{7}{18} + x^2 - x\right) e^{6x} + 18c_1 e^{3x} + 18c_2\right) e^{-2x}}{18}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 39

```
DSolve[y''[x]+y'[x]-2*y[x]==x^2*Exp[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{324} e^{4x} \left(18x^2 - 18x + 7\right) + c_1 e^{-2x} + c_2 e^x$$

16.66 problem 539

Internal problem ID [15309]
Internal file name [`OUTPUT/15309_Wednesday_May_08_2024_03_55_23_PM_97603693/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 539.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = (x^2 + x)\, e^{3x}}$$

16.66.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = x(x+1)\, e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda \,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = \mathrm{e}^{2x} c_1 + \mathrm{e}^{x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x}c_1 + e^{x}c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x(x+1)\,e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 e^{3x}, e^{3x}x, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x^2 e^{3x} + A_2 e^{3x} x + A_3 e^{3x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{3x} + 6A_1 x\,e^{3x} + 2A_1 x^2 e^{3x} + 2A_2 e^{3x} x + 3A_2 e^{3x} + 2A_3 e^{3x} = x(x+1)\,e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -1, A_3 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{3x}}{2} - e^{3x} x + e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}c_1 + e^{x}c_2\right) + \left(\frac{x^2 e^{3x}}{2} - e^{3x} x + e^{3x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2x}c_1 + e^x c_2 + \frac{x^2 e^{3x}}{2} - e^{3x}x + e^{3x} \tag{1}$$

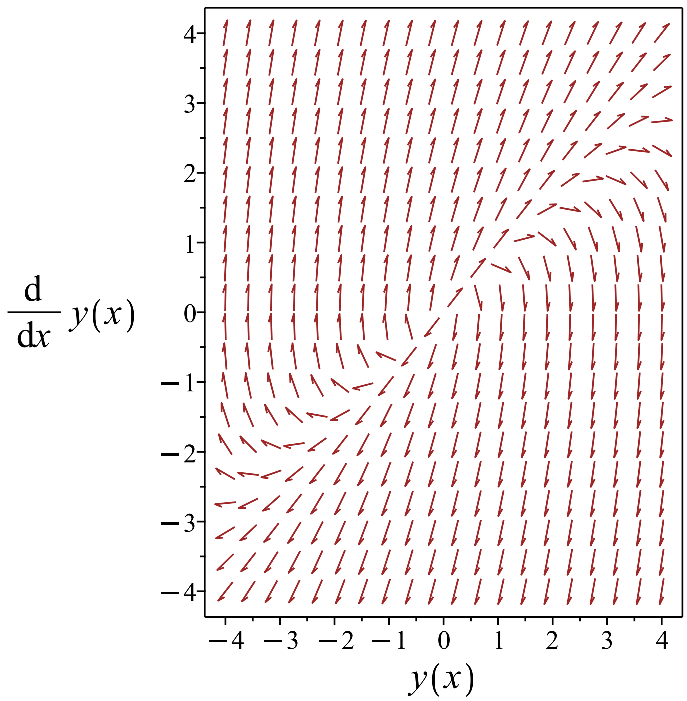


Figure 619: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + e^x c_2 + \frac{x^2 e^{3x}}{2} - e^{3x}x + e^{3x}$$

Verified OK.

16.66.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} \, dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2} dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(e^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^x c_1 + e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x(x + 1) e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 e^{3x}, e^{3x} x, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x^2 e^{3x} + A_2 e^{3x} x + A_3 e^{3x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{3x} + 6A_1 x\, e^{3x} + 2A_1 x^2 e^{3x} + 2A_2 e^{3x} x + 3A_2 e^{3x} + 2A_3 e^{3x} = x(x+1)\, e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -1, A_3 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 e^{3x}}{2} - e^{3x} x + e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + e^{2x} c_2) + \left(\frac{x^2 e^{3x}}{2} - e^{3x} x + e^{3x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + e^{2x} c_2 + \frac{x^2 e^{3x}}{2} - e^{3x} x + e^{3x} \tag{1}$$

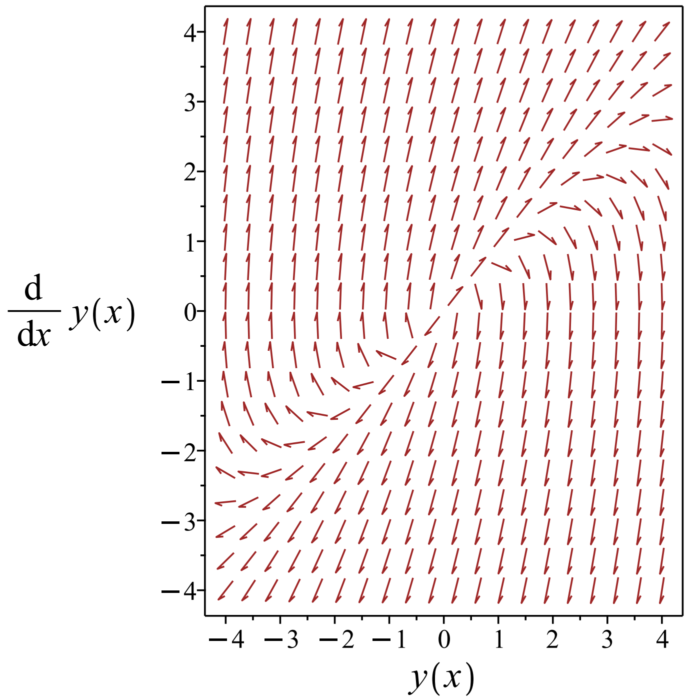


Figure 620: Slope field plot

Verification of solutions

$$y = e^x c_1 + e^{2x} c_2 + \frac{x^2 e^{3x}}{2} - e^{3x} x + e^{3x}$$

Verified OK.

16.66.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = x(x+1)\,e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + e^{2x} c_2 + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x(x+1)\,e^{3x}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{3x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -e^x\left(\int e^{2x} x(x+1)\,dx\right) + e^{2x}\left(\int x(x+1)\,e^x dx\right)$
  - Compute integrals

    $y_p(x) = e^{3x}\left(\frac{1}{2}x^2 - x + 1\right)$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + e^{2x} c_2 + e^{3x}\left(\frac{1}{2}x^2 - x + 1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=(x+x^2)*exp(3*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{x}\left(\left(x^{2}-2x+2\right)e^{2x}+2c_{1}e^{x}+2c_{2}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 37

```
DSolve[y''[x]-3*y'[x]+2*y[x]==(x+x^2)*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{3x}\left(x^{2}-2x+2\right)+c_{1}e^{x}+c_{2}e^{2x}$$

16.67 problem 540

Internal problem ID [15310]
Internal file name [`OUTPUT/15310_Wednesday_May_08_2024_03_55_24_PM_25925056/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 540.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - y'' + y' - y = x^2 + x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + y' - y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{ix} c_2 + e^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{ix}$$
$$y_3 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y''' - y'' + y' - y = x^2 + x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{ix}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3 x^2 - A_2 x + 2x A_3 - A_1 + A_2 - 2A_3 = x^2 + x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = -3, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 - 3x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{ix} c_2 + e^{-ix} c_3\right) + \left(-x^2 - 3x - 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + e^{ix} c_2 + e^{-ix} c_3 - x^2 - 3x - 1 \tag{1}$$

Verification of solutions

$$y = e^x c_1 + e^{ix} c_2 + e^{-ix} c_3 - x^2 - 3x - 1$$

Verified OK.

16.67.1 Maple step by step solution

Let's solve

$y''' - y'' + y' - y = x^2 + x$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x^2 + y_3(x) - y_2(x) + y_1(x) + x$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 + y_3(x) - y_2(x) + y_1(x) + x]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 + x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 + x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \overrightarrow{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} e^{x} & -\cos(x) & \sin(x) \\ e^{x} & \sin(x) & \cos(x) \\ e^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t
  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$
  $$\Phi(x)=\begin{bmatrix} e^{x} & -\cos(x) & \sin(x) \\ e^{x} & \sin(x) & \cos(x) \\ e^{x} & \cos(x) & -\sin(x) \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix
  $$\Phi(x)=\begin{bmatrix} \frac{e^{x}}{2}+\frac{\cos(x)}{2}-\frac{\sin(x)}{2} & \sin(x) & \frac{e^{x}}{2}-\frac{\cos(x)}{2}-\frac{\sin(x)}{2} \\ \frac{e^{x}}{2}-\frac{\cos(x)}{2}-\frac{\sin(x)}{2} & \cos(x) & \frac{e^{x}}{2}+\frac{\sin(x)}{2}-\frac{\cos(x)}{2} \\ \frac{e^{x}}{2}+\frac{\sin(x)}{2}-\frac{\cos(x)}{2} & -\sin(x) & \frac{e^{x}}{2}+\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}'_p(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- Cancel like terms
  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -x^2 + \frac{3\,\mathrm{e}^x}{2} - 3x - 1 - \frac{\cos(x)}{2} + \frac{3\sin(x)}{2} \\ \frac{3\,\mathrm{e}^x}{2} - 2x - 3 + \frac{3\cos(x)}{2} + \frac{\sin(x)}{2} \\ \frac{3\,\mathrm{e}^x}{2} - 2 + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \left[\begin{array}{c} -x^2 + \frac{3\,\mathrm{e}^x}{2} - 3x - 1 - \frac{\cos(x)}{2} + \frac{3\sin(x)}{2} \\ \frac{3\,\mathrm{e}^x}{2} - 2x - 3 + \frac{3\cos(x)}{2} + \frac{\sin(x)}{2} \\ \frac{3\,\mathrm{e}^x}{2} - 2 + \frac{\cos(x)}{2} - \frac{3\sin(x)}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-2c_2-1)\cos(x)}{2} + \frac{(2c_1+3)\mathrm{e}^x}{2} + \frac{(2c_3+3)\sin(x)}{2} - x^2 - 3x - 1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+diff(y(x),x)-y(x)=x+x^2,y(x), singsol=all)
```

$$y(x) = -x^2 - 3x - 1 + \cos(x) c_1 + c_2 e^x + c_3 \sin(x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 31

```
DSolve[y'''[x]-y''[x]+y'[x]-y[x]==x+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 - 3x + c_3 e^x + c_1 \cos(x) + c_2 \sin(x) - 1$$

16.68 problem 541

Internal problem ID [15311]
Internal file name [`OUTPUT/15311_Wednesday_May_08_2024_03_55_26_PM_48680824/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 541.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _with_linear_symmetries]]
```

$$\boxed{y'''' - 2y''' + 2y'' - 2y' + y = e^{x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y''' + 2y'' - 2y' + y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i \\
\lambda_3 &= 1 \\
\lambda_4 &= 1
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + x e^x c_2 + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x e^x$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - 2y''' + 2y'' - 2y' + y = e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^x, e^x, e^{ix}, e^{-ix}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x\}]$$

Since $x e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x x^2}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + x e^x c_2 + e^{ix} c_3 + e^{-ix} c_4\right) + \left(\frac{e^x x^2}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-ix} c_4 + e^{ix} c_3 + e^x (c_2 x + c_1) + \frac{e^x x^2}{4}$$

Summary

The solution(s) found are the following

$$y = e^{-ix} c_4 + e^{ix} c_3 + e^x (c_2 x + c_1) + \frac{e^x x^2}{4} \tag{1}$$

Verification of solutions

$$y = e^{-ix} c_4 + e^{ix} c_3 + e^x (c_2 x + c_1) + \frac{e^x x^2}{4}$$

Verified OK.

16.68.1 Maple step by step solution

Let's solve

$y'''' - 2y''' + 2y'' - 2y' + y = e^{x}$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = e^{x} + 2y_4(x) - 2y_3(x) + 2y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = e^{x} + 2y_4(x) - 2y_3(x) + 2y_2(x) - y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}\right], \left[I, \begin{bmatrix} I \\ -1 \\ -I \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

  $\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^{x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

□ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^x & (x-1)e^x & -\sin(x) & -\cos(x) \\ e^x & x e^x & -\cos(x) & \sin(x) \\ e^x & x e^x & \sin(x) & \cos(x) \\ e^x & x e^x & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^x & (x-1)e^x & -\sin(x) & -\cos(x) \\ e^x & x e^x & -\cos(x) & \sin(x) \\ e^x & x e^x & \sin(x) & \cos(x) \\ e^x & x e^x & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -(x-1)e^x & x e^x - \frac{e^x}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & -x e^x + e^x - \cos(x) & x e^x - \frac{e^x}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -x e^x & \frac{e^x}{2} + x e^x + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -x e^x + \sin(x) & \frac{e^x}{2} + x e^x - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ -x e^x & \frac{e^x}{2} + x e^x - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -x e^x + \cos(x) & \frac{e^x}{2} + x e^x + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ -x e^x & \frac{e^x}{2} + x e^x + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} & -x e^x - \sin(x) & \frac{e^x}{2} + x e^x + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

$\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{(x^2-x)\mathrm{e}^x}{2} + \frac{\sin(x)}{2} \\ \frac{(x^2+x-1)\mathrm{e}^x}{2} + \frac{\cos(x)}{2} \\ \frac{(x^2+x)\mathrm{e}^x}{2} - \frac{\sin(x)}{2} \\ \frac{(x^2+x+1)\mathrm{e}^x}{2} - \frac{\cos(x)}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + c_4 \overrightarrow{y}_4(x) + \left[\begin{array}{c} \frac{(x^2-x)\mathrm{e}^x}{2} + \frac{\sin(x)}{2} \\ \frac{(x^2+x-1)\mathrm{e}^x}{2} + \frac{\cos(x)}{2} \\ \frac{(x^2+x)\mathrm{e}^x}{2} - \frac{\sin(x)}{2} \\ \frac{(x^2+x+1)\mathrm{e}^x}{2} - \frac{\cos(x)}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(x^2+(2c_2-1)x+2c_1-2c_2)\mathrm{e}^x}{2} + \frac{(-2c_3+1)\sin(x)}{2} - \cos(x)\, c_4$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+2*diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(x),y(x), sin
```

$$y(x) = \frac{(4c_4x + x^2 + 4c_2)\,\mathrm{e}^x}{4} + \cos(x)\,c_1 + c_3 \sin(x)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 40

```
DSolve[y''''[x]-2*y'''[x]+2*y''[x]-2*y'[x]+y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{4}e^x\left(x^2 - 2x + 4c_4x + 1 + 4c_3\right) + c_1\cos(x) + c_2\sin(x)$$

16.69 problem 542

Internal problem ID [15312]
Internal file name [`OUTPUT/15312_Wednesday_May_08_2024_03_55_28_PM_77479980/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 542.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = x^3}$$

16.69.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x} c_1 + x e^{x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{x}, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_4x^3 + A_3x^2 - 6x^2A_4 + A_2x - 4xA_3 + 6xA_4 + A_1 - 2A_2 + 2A_3 = x^3$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 24, A_2 = 18, A_3 = 6, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + 6x^2 + 18x + 24$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + x e^x c_2) + \left(x^3 + 6x^2 + 18x + 24\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2x + c_1) + x^3 + 6x^2 + 18x + 24$$

Summary

The solution(s) found are the following

$$y = e^x(c_2x + c_1) + x^3 + 6x^2 + 18x + 24 \tag{1}$$

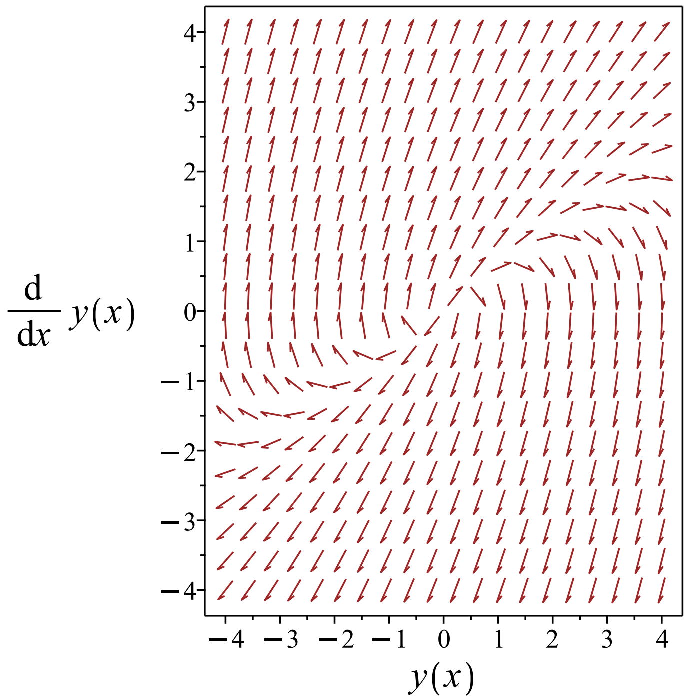


Figure 621: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + x^3 + 6x^2 + 18x + 24$$

Verified OK.

16.69.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{-x}x^3 \\
(e^{-x}y)'' &= e^{-x}x^3
\end{aligned}
$$

Integrating once gives

$$
(e^{-x}y)' = -(x^3 + 3x^2 + 6x + 6)\,e^{-x} + c_1
$$

Integrating again gives

$$
(e^{-x}y) = (x^3 + 6x^2 + 18x + 24)\,e^{-x} + c_1x + c_2
$$

Hence the solution is

$$
y = \frac{(x^3 + 6x^2 + 18x + 24)\,e^{-x} + c_1x + c_2}{e^{-x}}
$$

Or

$$
y = c_1x\,e^{x} + x^3 + e^{x}c_2 + 6x^2 + 18x + 24
$$

Summary

The solution(s) found are the following

$$
y = c_1x\,e^{x} + x^3 + e^{x}c_2 + 6x^2 + 18x + 24 \tag{1}
$$

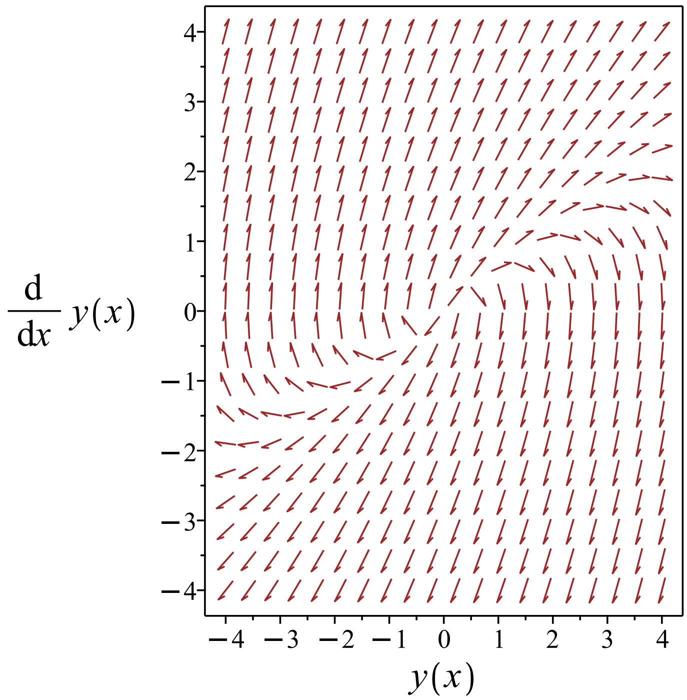


Figure 622: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^x + x^3 + \mathrm{e}^x c_2 + 6x^2 + 18x + 24$$

Verified OK.

16.69.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + x\,\mathrm{e}^x c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^3
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2, x^3\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^x, \mathrm{e}^x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_4 x^3 + A_3 x^2 - 6x^2 A_4 + A_2 x - 4x A_3 + 6x A_4 + A_1 - 2A_2 + 2A_3 = x^3
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 24, A_2 = 18, A_3 = 6, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + 6x^2 + 18x + 24$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + x\,e^x c_2) + \left(x^3 + 6x^2 + 18x + 24\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2 x + c_1) + x^3 + 6x^2 + 18x + 24$$

Summary
The solution(s) found are the following

$$y = e^x(c_2 x + c_1) + x^3 + 6x^2 + 18x + 24 \tag{1}$$

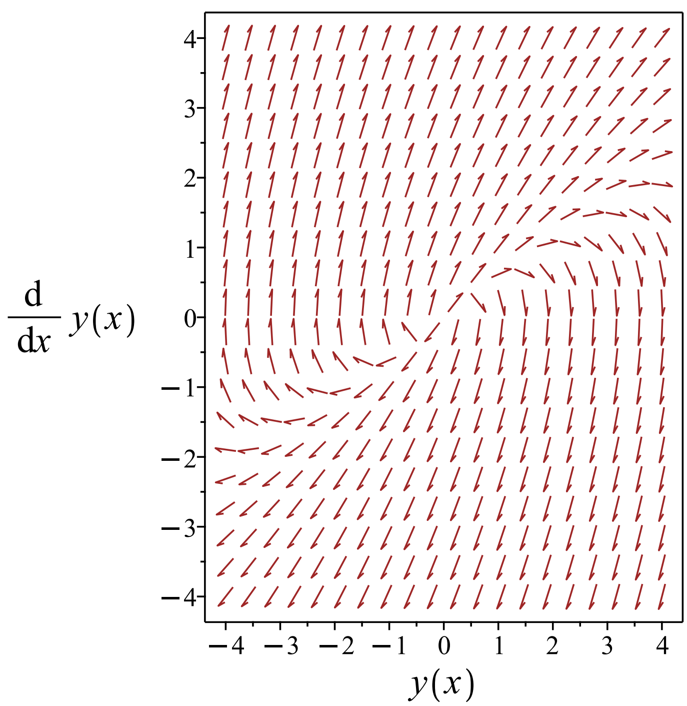


Figure 623: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + x^3 + 6x^2 + 18x + 24$$

Verified OK.

16.69.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = x^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{x} c_1 + x\,e^{x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^3\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{x} & x\,e^{x} \\ e^{x} & x\,e^{x} + e^{x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^x \left(-\left(\int x^4 e^{-x} dx\right) + \left(\int e^{-x} x^3 dx\right) x\right)$
- Compute integrals

  $y_p(x) = x^3 + 6x^2 + 18x + 24$

- Substitute particular solution into general solution to ODE

  $y = x\, e^x c_2 + x^3 + e^x c_1 + 6x^2 + 18x + 24$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=x^3,y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2) e^x + x^3 + 6x^2 + 18x + 24$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 31

```
DSolve[y''[x]-2*y'[x]+y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 + 6x^2 + x(18 + c_2 e^x) + c_1 e^x + 24$$

16.70 problem 543

Internal problem ID [15313]
Internal file name [`OUTPUT/15313_Wednesday_May_08_2024_03_55_29_PM_46704334/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 543.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + y'' = x^2 + x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + y'' = 0$$

The characteristic equation is

$$\lambda^4 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + e^{ix}c_3 + e^{-ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' + y'' = x^2 + x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^{ix}, e^{-ix}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^4 + A_2x^3 + A_1x^2$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12x^2A_3 + 6xA_2 + 2A_1 + 24A_3 = x^2 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{6}, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{12}x^4 + \frac{1}{6}x^3 - x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1 + e^{ix}c_3 + e^{-ix}c_4) + \left(\frac{1}{12}x^4 + \frac{1}{6}x^3 - x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2x + c_1 + e^{ix}c_3 + e^{-ix}c_4 + \frac{x^4}{12} + \frac{x^3}{6} - x^2 \tag{1}$$

Verification of solutions

$$y = c_2x + c_1 + e^{ix}c_3 + e^{-ix}c_4 + \frac{x^4}{12} + \frac{x^3}{6} - x^2$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2-_b(_a)+_a, _b(_a)`   ***
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$4)+diff(y(x),x$2)=x^2+x,y(x), singsol=all)
```

$$y(x) = \frac{x^3}{6} - x^2 + \frac{x^4}{12} - \cos(x) c_1 - \sin(x) c_2 + c_3 x + c_4$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 43

```
DSolve[y''''[x]+y''[x]==x^2+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{12} + \frac{x^3}{6} - x^2 + c_4 x - c_1 \cos(x) - c_2 \sin(x) + c_3$$

16.71 problem 544

Internal problem ID [15314]
Internal file name [`OUTPUT/15314_Wednesday_May_08_2024_03_55_30_PM_17930692/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 544.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = x^2 \sin(x)}$$

16.71.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^2 \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 \cos(x), x^2 \sin(x), \cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \cos(x), x^2 \sin(x), x^3 \cos(x), x^3 \sin(x), \cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \cos(x) + A_2 x^2 \sin(x) + A_3 x^3 \cos(x) + A_4 x^3 \sin(x) + A_5 \cos(x) x + A_6 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &2A_1 \cos(x) - 4A_1 x \sin(x) + 2A_2 \sin(x) + 4A_2 x \cos(x) + 6A_3 x \cos(x) - 6A_3 x^2 \sin(x) \\ &+ 6A_4 x \sin(x) + 6A_4 x^2 \cos(x) - 2A_5 \sin(x) + 2A_6 \cos(x) = x^2 \sin(x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = -\frac{1}{6}, A_4 = 0, A_5 = \frac{1}{4}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x) x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x) x}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x) x}{4} \tag{1}$$

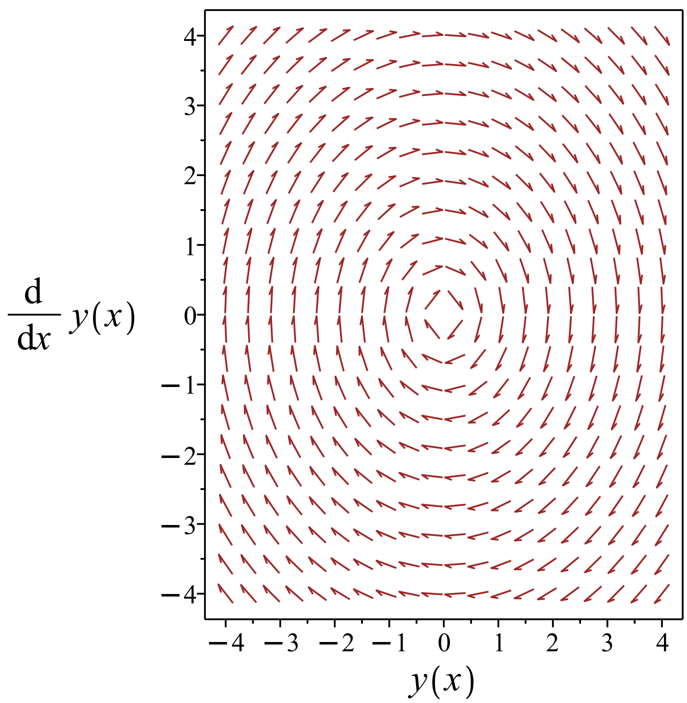


Figure 624: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x) x}{4}$$

Verified OK.

16.71.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 491: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B=0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x^2 \cos(x), x^2 \sin(x), \cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \cos(x), x^2 \sin(x), x^3 \cos(x), x^3 \sin(x), \cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \cos(x) + A_2 x^2 \sin(x) + A_3 x^3 \cos(x) + A_4 x^3 \sin(x) + A_5 \cos(x) x + A_6 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &2A_1 \cos(x) - 4A_1 x \sin(x) + 2A_2 \sin(x) + 4A_2 x \cos(x) + 6A_3 x \cos(x) - 6A_3 x^2 \sin(x) \\ &+ 6A_4 x \sin(x) + 6A_4 x^2 \cos(x) - 2A_5 \sin(x) + 2A_6 \cos(x) = x^2 \sin(x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{4}, A_3 = -\frac{1}{6}, A_4 = 0, A_5 = \frac{1}{4}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x) x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x)\, x}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x)\, x}{4} \tag{1}
$$

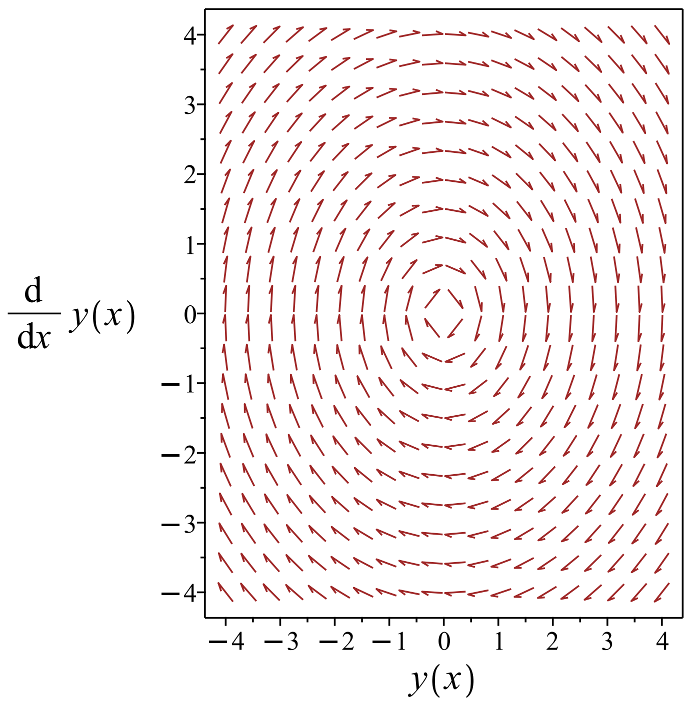


Figure 625: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x^2 \sin(x)}{4} - \frac{x^3 \cos(x)}{6} + \frac{\cos(x)\, x}{4}
$$

Verified OK.

16.71.3 Maple step by step solution

Let's solve

$y'' + y = x^2 \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 \sin(x)\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int \sin(x)^2 x^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x) x^2 dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{(-2x^3+3x)\cos(x)}{12} + \frac{\sin(x)(2x^2-1)}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{(-2x^3+3x)\cos(x)}{12} + \frac{\sin(x)(2x^2-1)}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+y(x)=x^2*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{(-2x^3 + 12c_1 + 3x)\cos(x)}{12} + \frac{\sin(x)(x^2 + 4c_2 - 1)}{4}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 41

```
DSolve[y''[x]+y[x]==x^2*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x^3}{6} + \frac{x}{4} + c_1\right)\cos(x) + \frac{1}{8}(2x^2 - 1 + 8c_2)\sin(x)$$

16.72 problem 545

Internal problem ID [15315]
Internal file name [`OUTPUT/15315_Wednesday_May_08_2024_03_55_31_PM_60385736/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 545.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 2y' + y = x^2 e^{-x} \cos(x)}$$

16.72.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = x^2 e^{-x} \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 x\, e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{-x} \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x} \cos(x), e^{-x} \sin(x), x \cos(x) e^{-x}, x \sin(x) e^{-x}, x^2 e^{-x} \cos(x), x^2 \sin(x) e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
\begin{aligned}
y_p = {} & A_1 \mathrm{e}^{-x} \cos(x) + A_2 \mathrm{e}^{-x} \sin(x) + A_3 x \cos(x) \mathrm{e}^{-x} \\
& + A_4 x \sin(x) \mathrm{e}^{-x} + A_5 x^2 \mathrm{e}^{-x} \cos(x) + A_6 x^2 \sin(x) \mathrm{e}^{-x}
\end{aligned}
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
& -A_4 x \sin(x) \mathrm{e}^{-x} - A_5 x^2 \mathrm{e}^{-x} \cos(x) - A_6 x^2 \sin(x) \mathrm{e}^{-x} - A_3 x \cos(x) \mathrm{e}^{-x} \\
& - 2A_3 \sin(x) \mathrm{e}^{-x} + 2A_4 \cos(x) \mathrm{e}^{-x} + 2A_5 \mathrm{e}^{-x} \cos(x) - 4A_5 x \mathrm{e}^{-x} \sin(x) \\
& + 2A_6 \sin(x) \mathrm{e}^{-x} + 4A_6 x \cos(x) \mathrm{e}^{-x} - A_1 \mathrm{e}^{-x} \cos(x) - A_2 \mathrm{e}^{-x} \sin(x) = x^2 \mathrm{e}^{-x} \cos(x)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = 6, A_2 = 0, A_3 = 0, A_4 = 4, A_5 = -1, A_6 = 0]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = 6\,\mathrm{e}^{-x} \cos(x) + 4x \sin(x) \mathrm{e}^{-x} - x^2 \mathrm{e}^{-x} \cos(x)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x}\right) + \left(6\,\mathrm{e}^{-x} \cos(x) + 4x \sin(x) \mathrm{e}^{-x} - x^2 \mathrm{e}^{-x} \cos(x)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-x}(c_2 x + c_1) + 6\,\mathrm{e}^{-x} \cos(x) + 4x \sin(x) \mathrm{e}^{-x} - x^2 \mathrm{e}^{-x} \cos(x)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x}(c_2 x + c_1) + 6\,\mathrm{e}^{-x} \cos(x) + 4x \sin(x) \mathrm{e}^{-x} - x^2 \mathrm{e}^{-x} \cos(x) \tag{1}
$$

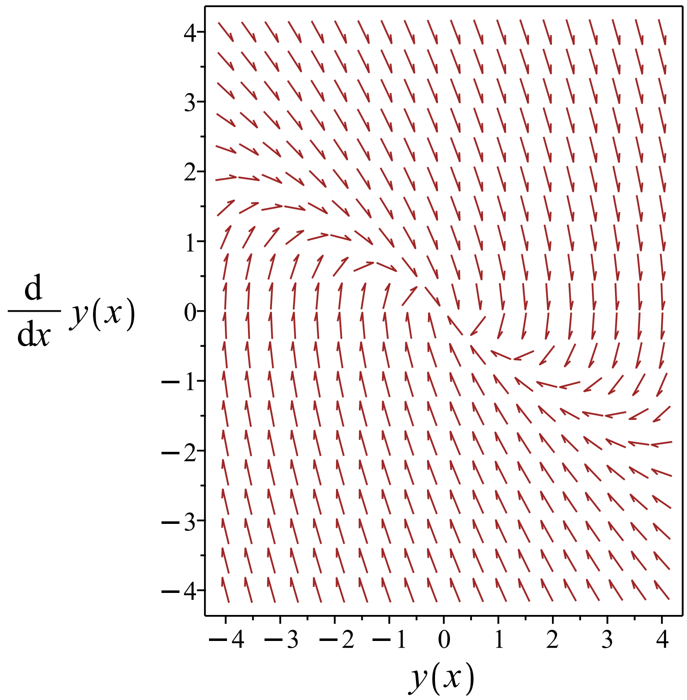


Figure 626: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + 6\,e^{-x}\cos(x) + 4x\sin(x)\,e^{-x} - x^2e^{-x}\cos(x)$$

Verified OK.

16.72.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{x}x^{2}e^{-x}\cos(x)\\
(e^{x}y)'' &= e^{x}x^{2}e^{-x}\cos(x)
\end{aligned}
$$

Integrating once gives

$$(e^{x}y)' = x^{2}\sin(x) - 2\sin(x) + 2\cos(x)\,x + c_1$$

Integrating again gives

$$(e^{x}y) = \left(-x^{2}+6\right)\cos(x) + x(c_1 + 4\sin(x)) + c_2$$

Hence the solution is

$$y = \frac{(-x^{2}+6)\cos(x) + x(c_1 + 4\sin(x)) + c_2}{e^{x}}$$

Or

$$y = -x^{2}e^{-x}\cos(x) + c_1 x\,e^{-x} + 4x\sin(x)\,e^{-x} + e^{-x}c_2 + 6\,e^{-x}\cos(x)$$

Summary

The solution(s) found are the following

$$y = -x^{2}e^{-x}\cos(x) + c_1 x\,e^{-x} + 4x\sin(x)\,e^{-x} + e^{-x}c_2 + 6\,e^{-x}\cos(x) \tag{1}$$

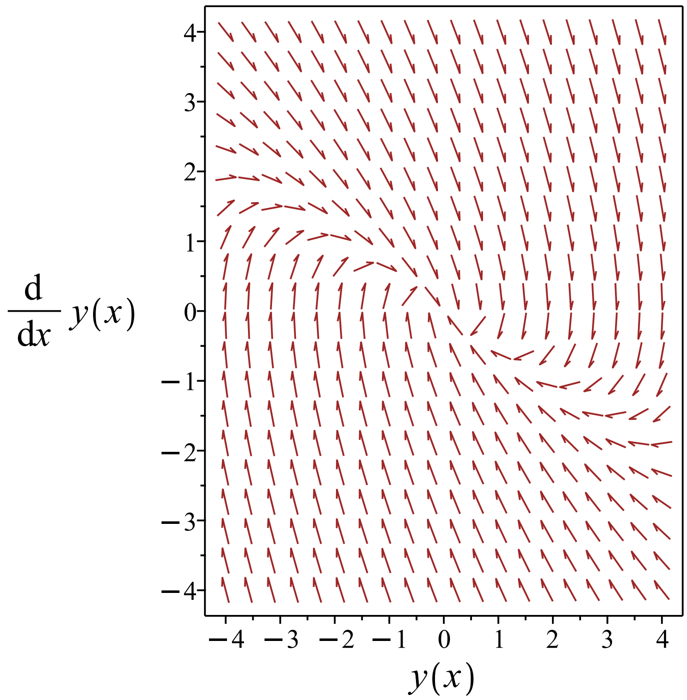


Figure 627: Slope field plot

Verification of solutions

$$y = -x^2 e^{-x} \cos(x) + c_1 x e^{-x} + 4x \sin(x) e^{-x} + e^{-x} c_2 + 6 e^{-x} \cos(x)$$

Verified OK.

16.72.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 493: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 \mathrm{e}^{-x} \cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-x}\cos(x), \mathrm{e}^{-x}\sin(x), x\cos(x)\,\mathrm{e}^{-x}, x\sin(x)\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}\cos(x), x^2\sin(x)\,\mathrm{e}^{-x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
\begin{aligned}
y_p = {} & A_1\mathrm{e}^{-x}\cos(x) + A_2\mathrm{e}^{-x}\sin(x) + A_3 x\cos(x)\,\mathrm{e}^{-x} \\
& + A_4 x\sin(x)\,\mathrm{e}^{-x} + A_5 x^2\mathrm{e}^{-x}\cos(x) + A_6 x^2\sin(x)\,\mathrm{e}^{-x}
\end{aligned}
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_4 x \sin\left(x\right) {\mathrm e}^{-x}-A_5 x^{2} {\mathrm e}^{-x} \cos\left(x\right)-A_6 x^{2} \sin\left(x\right) {\mathrm e}^{-x}+2 A_5 {\mathrm e}^{-x} \cos\left(x\right)\\ &-4 A_5 x \,{\mathrm e}^{-x} \sin\left(x\right)+2 A_6 \sin\left(x\right) {\mathrm e}^{-x}+4 A_6 x \cos\left(x\right) {\mathrm e}^{-x}-A_3 x \cos\left(x\right) {\mathrm e}^{-x}\\ &-2 A_3 \sin\left(x\right) {\mathrm e}^{-x}+2 A_4 \cos\left(x\right) {\mathrm e}^{-x}-A_1 {\mathrm e}^{-x} \cos\left(x\right)-A_2 {\mathrm e}^{-x} \sin\left(x\right) = x^{2} {\mathrm e}^{-x} \cos\left(x\right)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = 0, A_3 = 0, A_4 = 4, A_5 = -1, A_6 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6\,{\mathrm e}^{-x} \cos\left(x\right)+4 x \sin\left(x\right) {\mathrm e}^{-x}-x^{2} {\mathrm e}^{-x} \cos\left(x\right)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1 {\mathrm e}^{-x}+c_2 x \,{\mathrm e}^{-x}\right) + \left(6\,{\mathrm e}^{-x} \cos\left(x\right)+4 x \sin\left(x\right) {\mathrm e}^{-x}-x^{2} {\mathrm e}^{-x} \cos\left(x\right)\right)\end{aligned}$$

Which simplifies to

$$y = {\mathrm e}^{-x} \left(c_2 x +c_1 \right)+6\,{\mathrm e}^{-x} \cos\left(x\right)+4 x \sin\left(x\right) {\mathrm e}^{-x}-x^{2} {\mathrm e}^{-x} \cos\left(x\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = {\mathrm e}^{-x} \left(c_2 x +c_1 \right)+6\,{\mathrm e}^{-x} \cos\left(x\right)+4 x \sin\left(x\right) {\mathrm e}^{-x}-x^{2} {\mathrm e}^{-x} \cos\left(x\right) \tag{1}$$

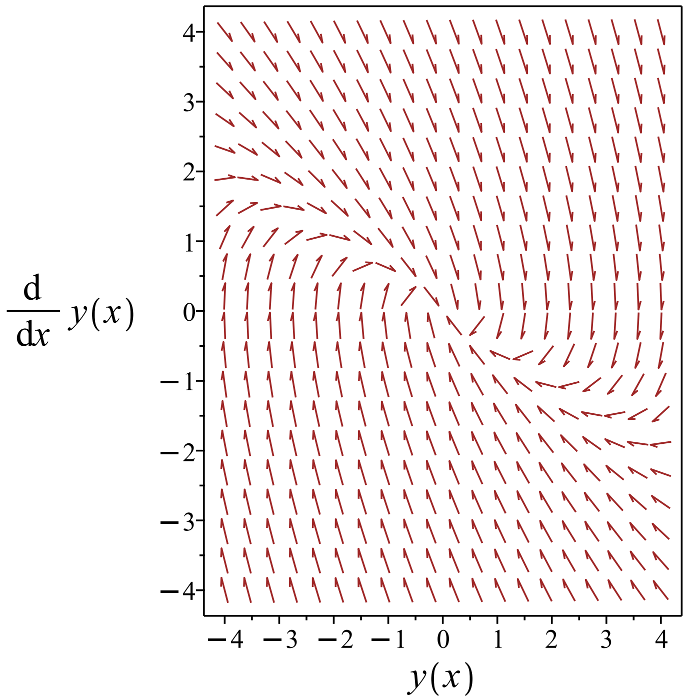


Figure 628: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + 6\,e^{-x}\cos(x) + 4x\sin(x)\,e^{-x} - x^2 e^{-x}\cos(x)$$

Verified OK.

16.72.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = x^2 e^{-x}\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 x\, e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2 e^{-x}\cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & x\, e^{-x} \\ -e^{-x} & e^{-x} - x\, e^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{-x}\left(-\left(\int x^3 \cos(x)\, dx\right) + x\left(\int x^2 \cos(x)\, dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = -(x^2 \cos(x) - 4\sin(x)\, x - 6\cos(x))\, e^{-x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 x\, e^{-x} - (x^2 \cos(x) - 4\sin(x)\, x - 6\cos(x))\, e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=x^2*exp(-x)*cos(x),y(x), singsol=all)
```

$$y(x) = -\left(\left(x^2-6\right)\cos\left(x\right)-c_1x-4\sin\left(x\right)x-c_2\right)\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 32

```
DSolve[y''[x]+2*y'[x]+y[x]==x^2*Exp[-x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(-\left(x^2-6\right)\cos(x)+4x\sin(x)+c_2x+c_1\right)$$

16.73 problem 546

Internal problem ID [15316]
Internal file name [OUTPUT/15316_Wednesday_May_08_2024_03_55_32_PM_44534583/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 546.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y = \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y = 0$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

$$y_3 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' - y = \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \sin(x) - A_2 \cos(x) - A_1 \cos(x) - A_2 \sin(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3\right) + \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}$$

Verification of solutions

$$y = e^x c_1 + e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

16.73.1 Maple step by step solution

Let's solve

$y''' - y = \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \sin(x) + y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \sin(x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)-I\sin\left(\frac{x\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)-I\sin\left(\frac{x\sqrt{3}}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{3}}{2}} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) - I\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right], \vec{y}_3(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ -\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{ccc} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{ccc} 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{array}\right.}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{ccc} \frac{e^{x}}{3} + \frac{2\,e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} + \frac{2\,e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} + \frac{2\,e}{} \end{array}\right.$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\cos(x)}{2} - \frac{\sin(x)}{2} + \frac{e^x}{6} - \frac{2\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} + \frac{e^x}{6} \\ -\frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} + \frac{e^x}{6} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \left[\begin{array}{c} \frac{\cos(x)}{2} - \frac{\sin(x)}{2} + \frac{e^x}{6} - \frac{2\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} + \frac{e^x}{6} \\ -\frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} + \frac{e^x}{6} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^{-\frac{x}{2}}\left(\sqrt{3}\,c_3 + c_2 + \frac{4}{3}\right)\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{e^{-\frac{x}{2}}\left(\sqrt{3}\,c_2 - c_3\right)\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{(6c_1+1)e^x}{6} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$3)-y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) + c_3 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) + c_1 e^{x} - \frac{\sin(x)}{2} + \frac{\cos(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.419 (sec). Leaf size: 66

```
DSolve[y'''[x]-y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{x} + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_3 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

16.74 problem 547

Internal problem ID [15317]
Internal file name [`OUTPUT/15317_Wednesday_May_08_2024_03_55_39_PM_33309750/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 547.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 2y'' + y = \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x} + e^x c_3 + x e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x e^{-x}$$
$$y_3 = e^x$$
$$y_4 = x e^x$$

Now the particular solution to the given ODE is found

$$y'''' - 2y'' + y = \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^x, x e^{-x}, e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \cos(x) + 4A_2 \sin(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} + \mathrm{e}^{x} c_3 + x \,\mathrm{e}^{x} c_4\right) + \left(\frac{\cos(x)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x}(c_4 x + c_3) + \frac{\cos(x)}{4}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x}(c_4 x + c_3) + \frac{\cos(x)}{4} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x}(c_4 x + c_3) + \frac{\cos(x)}{4}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$2)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = (c_4x + c_2)\,\mathrm{e}^{-x} + (c_3x + c_1)\,\mathrm{e}^{x} + \frac{\cos(x)}{4}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 42

```
DSolve[y''''[x]-2*y''[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cos(x)}{4} + e^{-x}\left(c_2x + c_3e^{2x} + c_4e^{2x}x + c_1\right)$$

16.75 problem 548

Internal problem ID [15318]
Internal file name [`OUTPUT/15318_Wednesday_May_08_2024_03_55_39_PM_18569722/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 548.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 3y'' + 3y' - y = e^{x} \cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + x e^x c_2 + x^2 e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x e^x$$
$$y_3 = e^x x^2$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 3y' - y = e^x \cos(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^x, e^x x^2, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(2x) + A_2 e^x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 e^x \sin(2x) - 8A_2 e^x \cos(2x) = e^x \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x \sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + x e^x c_2 + x^2 e^x c_3) + \left(-\frac{e^x \sin(2x)}{8}\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_3 x^2 + c_2 x + c_1) - \frac{e^x \sin(2x)}{8}$$

Summary

The solution(s) found are the following

$$y = e^x(c_3 x^2 + c_2 x + c_1) - \frac{e^x \sin(2x)}{8} \tag{1}$$

Verification of solutions

$$y = e^x(c_3 x^2 + c_2 x + c_1) - \frac{e^x \sin(2x)}{8}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+3*diff(y(x),x)-y(x)=exp(x)*cos(2*x),y(x), singsol=all
```

$$y(x) = -\frac{e^{x}\left(-8c_3x^2 - 8c_2x + \sin\left(2x\right) - 8c_1 - 2x\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 33

```
DSolve[y'''[x]-3*y''[x]+3*y'[x]-y[x]==Exp[x]*Cos[2*x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{8}e^x(-\sin(2x) + 8(x(c_3x + c_2) + c_1))$$

16.76 problem 549

Internal problem ID [15319]
Internal file name [`OUTPUT/15319_Wednesday_May_08_2024_03_55_40_PM_53495954/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Trial and error method. Exercises page 132
Problem number: 549.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 5y = e^{2x}(\sin(x) + 2\cos(x))}$$

16.76.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 5, f(x) = e^{2x}(\sin(x) + 2\cos(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(5)}\\ &= 2 \pm i\end{aligned}$$

Hence

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$

Which simplifies to

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}(\sin(x) + 2\cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\cos(x), e^{2x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(x), e^{2x}\sin(x)\}$$

Since $e^{2x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}\cos(x), \sin(x)\,x\,e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{2x}\cos(x) + A_2\sin(x)\,x\,e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1e^{2x}\sin(x) + 2A_2\cos(x)\,e^{2x} = e^{2x}(\sin(x) + 2\cos(x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{2x}\cos(x)}{2} + \sin(x)\,x\,e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}(c_1\cos(x) + c_2\sin(x))\right) + \left(-\frac{x\,e^{2x}\cos(x)}{2} + \sin(x)\,x\,e^{2x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x e^{2x} \cos(x)}{2} + \sin(x) x e^{2x} \tag{1}$$

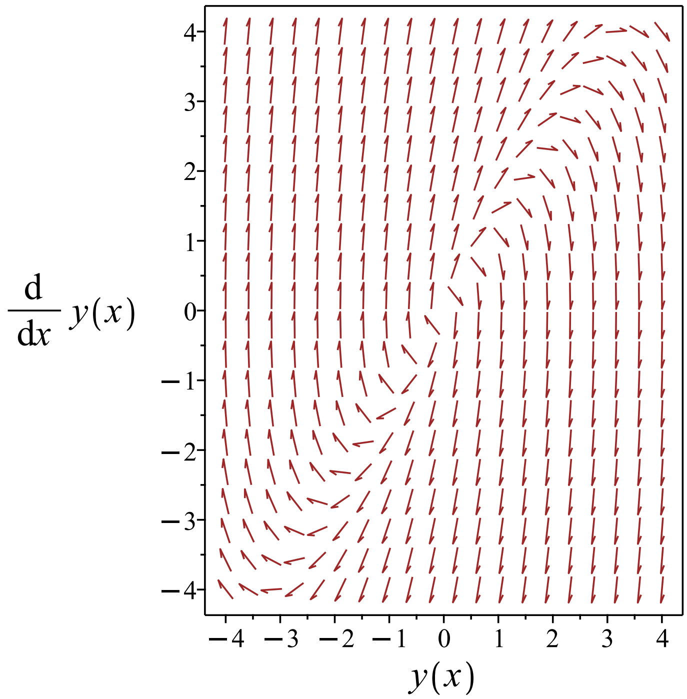


Figure 629: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x e^{2x} \cos(x)}{2} + \sin(x) x e^{2x}$$

Verified OK.

16.76.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 496: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} \, dx} \\ &= z_1 e^{2x} \\ &= z_1 \left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{2x}\cos(x)\right) + c_2\left(\mathrm{e}^{2x}\cos(x)(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{2x}\cos(x)\,c_1 + \mathrm{e}^{2x}\sin(x)\,c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2x}(\sin(x) + 2\cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x} \cos(x), e^{2x} \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} \cos(x), e^{2x} \sin(x)\}$$

Since $e^{2x} \cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{2x} \cos(x), \sin(x)\, x\, e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{2x} \cos(x) + A_2 \sin(x)\, x\, e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{2x} \sin(x) + 2A_2 \cos(x)\, e^{2x} = e^{2x}(\sin(x) + 2\cos(x))$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\, e^{2x} \cos(x)}{2} + \sin(x)\, x\, e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x} \cos(x)\, c_1 + e^{2x} \sin(x)\, c_2\right) + \left(-\frac{x\, e^{2x} \cos(x)}{2} + \sin(x)\, x\, e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x\, e^{2x} \cos(x)}{2} + \sin(x)\, x\, e^{2x}$$

Summary
The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x e^{2x} \cos(x)}{2} + \sin(x) x e^{2x} \tag{1}$$

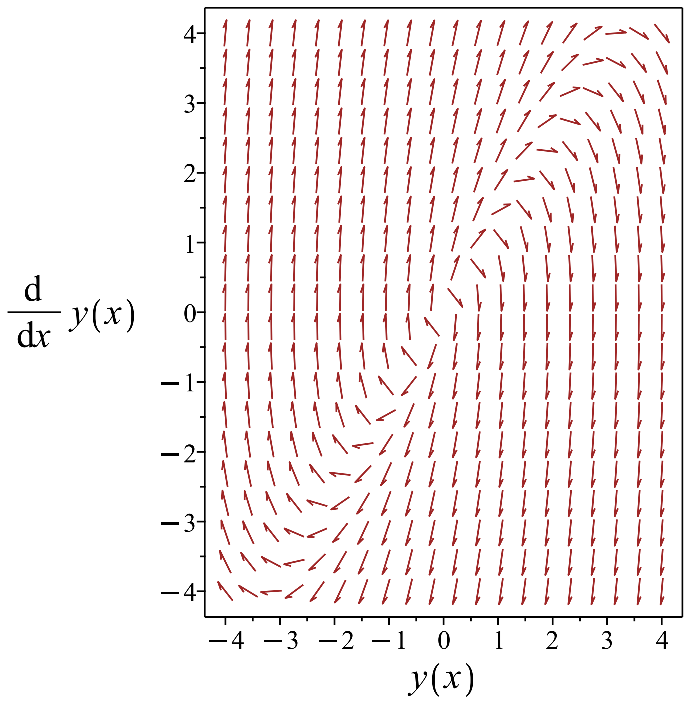


Figure 630: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x e^{2x} \cos(x)}{2} + \sin(x) x e^{2x}$$

Verified OK.

16.76.3 Maple step by step solution

Let's solve

$y'' - 4y' + 5y = e^{2x}(\sin(x) + 2\cos(x))$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{2x} \cos(x) c_1 + \mathrm{e}^{2x} \sin(x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{2x}(\sin(x) + 2\cos(x))\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} \cos(x) & \mathrm{e}^{2x} \sin(x) \\ 2\,\mathrm{e}^{2x} \cos(x) - \mathrm{e}^{2x} \sin(x) & 2\,\mathrm{e}^{2x} \sin(x) + \mathrm{e}^{2x} \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{2x}\left(-\cos(x) \left(\int \sin(x) (\sin(x) + 2\cos(x)) dx\right) + \sin(x) \left(\int \cos(x) (\sin(x) + 2\cos(x)) d\right.\right.$

  - Compute integrals

    $y_p(x) = -\frac{\mathrm{e}^{2x}((x-2)\cos(x) - 2\sin(x) x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{2x} \cos(x) c_1 + \mathrm{e}^{2x} \sin(x) c_2 - \frac{\mathrm{e}^{2x}((x-2)\cos(x) - 2\sin(x) x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=exp(2*x)*(sin(x)+2*cos(x)),y(x), singsol=all)
```

$$y(x) = -\frac{((x - 2c_1 - 2)\cos(x) - 2\sin(x)(c_2 + x))\, e^{2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 36

```
DSolve[y''[x]-4*y'[x]+5*y[x]==Exp[2*x]*(Sin[x]+Cos[x]),y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{2}e^{2x}((-x + 1 + 2c_2)\cos(x) + (x + 2c_1)\sin(x))$$

17 Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137

17.1 problem 551

Internal problem ID [15320]
Internal file name [`OUTPUT/15320_Wednesday_May_08_2024_03_55_41_PM_77436034/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 551.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 2y = \mathrm{e}^{x} + \mathrm{e}^{-2x}}$$

17.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = (\mathrm{e}^{3x} + 1)\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)}\\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = e^{2x} c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(e^{3x}+1)\,e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\},\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} + A_2 e^{-2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{x} + 4A_2 e^{-2x} = (e^{3x}+1)\,e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{x}}{2} + \frac{e^{-2x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{2x}c_1 + e^{-x}c_2) + \left(-\frac{e^{x}}{2} + \frac{e^{-2x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}c_1 + e^{-x}c_2 - \frac{e^{x}}{2} + \frac{e^{-2x}}{4} \tag{1}$$

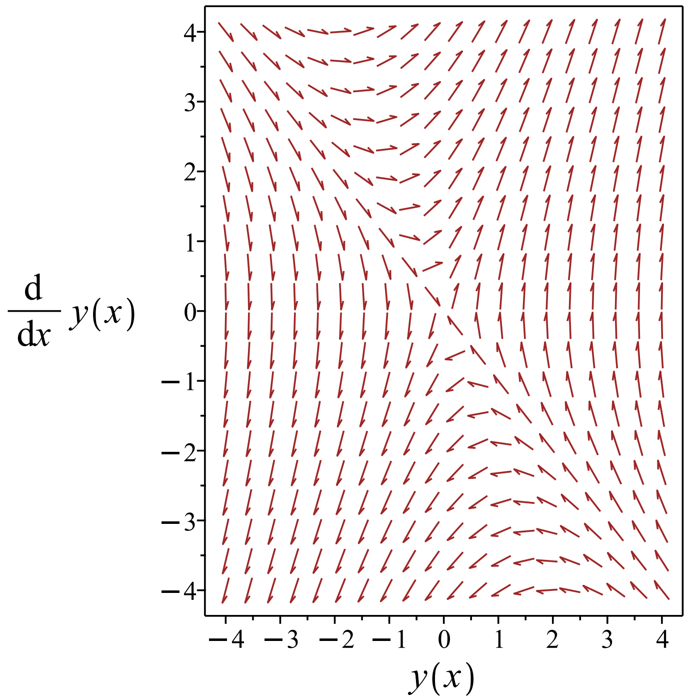


Figure 631: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + e^{-x}c_2 - \frac{e^x}{2} + \frac{e^{-2x}}{4}$$

Verified OK.

17.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 498: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(e^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{e^{3x}}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{\mathrm{e}^{2x} c_2}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\left(\mathrm{e}^{3x} + 1\right) \mathrm{e}^{-2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}, \{\mathrm{e}^{-2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{3}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x} + A_2 \mathrm{e}^{-2x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1e^x + 4A_2e^{-2x} = (e^{3x} + 1) e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x}{2} + \frac{e^{-2x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-x} + \frac{e^{2x}c_2}{3}\right) + \left(-\frac{e^x}{2} + \frac{e^{-2x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-x} + \frac{e^{2x}c_2}{3} - \frac{e^x}{2} + \frac{e^{-2x}}{4} \tag{1}$$

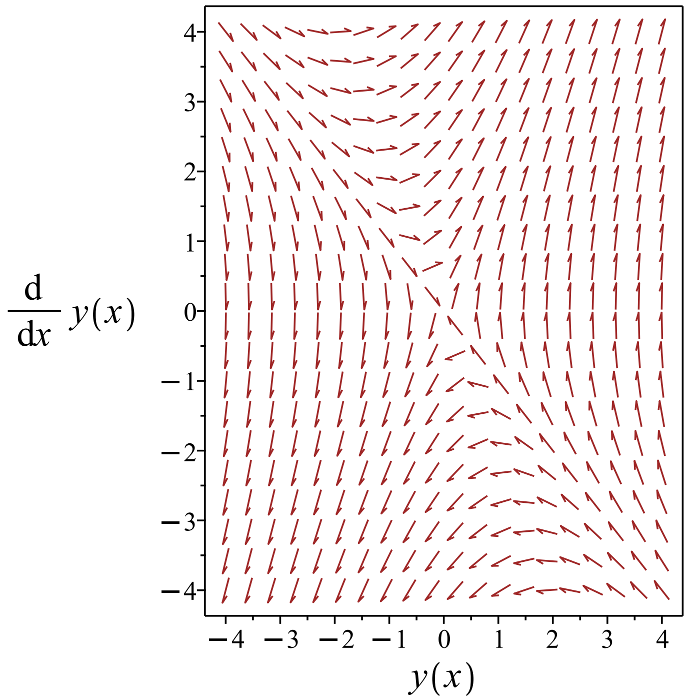


Figure 632: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{e^{2x} c_2}{3} - \frac{e^{x}}{2} + \frac{e^{-2x}}{4}$$

Verified OK.

17.1.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = (e^{3x} + 1) e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + e^{2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (e^{3x} + 1) e^{-2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2 e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3 e^{x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-x} \left(\int (e^{2x} + e^{-x}) dx\right)}{3} + \frac{e^{2x} \left(\int (e^{3x} + 1) e^{-4x} dx\right)}{3}$

  ◦ Compute integrals

  $y_p(x) = -\frac{(2 e^{3x} - 1) e^{-2x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + e^{2x} c_2 - \frac{(2 e^{3x} - 1) e^{-2x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=exp(x)+exp(-2*x),y(x), singsol=all)
```

$$y(x) = \frac{(4c_1 e^{4x} - 2\, e^{3x} + 4c_2 e^{x} + 1)\, e^{-2x}}{4}$$

✓ Solution by Mathematica

Time used: 0.114 (sec). Leaf size: 39

```
DSolve[y''[x]-y'[x]-2*y[x]==Exp[x]+Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-2x} \left(-2e^{3x} + 4c_1 e^{x} + 4c_2 e^{4x} + 1\right)$$

17.2 problem 552

Internal problem ID [15321]
Internal file name [`OUTPUT/15321_Wednesday_May_08_2024_03_55_42_PM_16454416/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 552.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 4y' = x + e^{-4x}}$$

17.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 0, f(x) = x + e^{-4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+4y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=4, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+4\lambda\, e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+4\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=4, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-4}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{4^2-(4)\,(1)\,(0)}\\&=-2\pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-2+2\\\lambda_2&=-2-2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-4)x}\end{aligned}$$

Or

$$y = c_1 + c_2 e^{-4x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 e^{-4x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{-4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-4x}\}, \{x, x^2\}]$$

Since e^{-4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{-4x}\}, \{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{-4x} + A_2 x + A_3 x^2$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-4x} + 2A_3 + 4A_2 + 8A_3 x = x + e^{-4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{16}, A_3 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{-4x}}{4} - \frac{x}{16} + \frac{x^2}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-4x}\right) + \left(-\frac{x\,e^{-4x}}{4} - \frac{x}{16} + \frac{x^2}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-4x} - \frac{x\,e^{-4x}}{4} - \frac{x}{16} + \frac{x^2}{8} \tag{1}$$

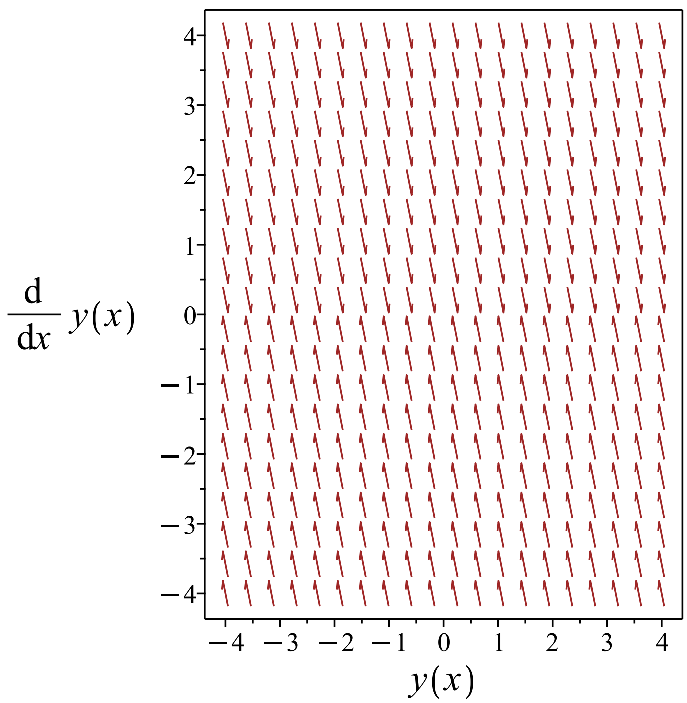


Figure 633: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-4x} - \frac{x\,e^{-4x}}{4} - \frac{x}{16} + \frac{x^2}{8}$$

Verified OK.

17.2.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'' + 4y'\right) dx = \int \left(x + e^{-4x}\right) dx$$
$$4y + y' = \frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 4$$
$$q(x) = \frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1$$

Hence the ode is

$$4y + y' = \frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1$$

The integrating factor μ is

$$\mu = e^{\int 4dx}$$
$$= e^{4x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1\right)$$
$$\frac{d}{dx}\left(e^{4x}y\right) = \left(e^{4x}\right)\left(\frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1\right)$$
$$d\left(e^{4x}y\right) = \left(-\frac{1}{4} + \frac{(2x^2 + 4c_1)\,e^{4x}}{4}\right) dx$$

Integrating gives

$$e^{4x}y = \int -\frac{1}{4} + \frac{(2x^2 + 4c_1)\,e^{4x}}{4}\, dx$$
$$e^{4x}y = -\frac{x}{4} + \frac{x^2 e^{4x}}{8} - \frac{e^{4x}x}{16} + \frac{e^{4x}}{64} + \frac{e^{4x}c_1}{4} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{4x}$ results in

$$y = e^{-4x}\left(-\frac{x}{4} + \frac{x^2 e^{4x}}{8} - \frac{e^{4x} x}{16} + \frac{e^{4x}}{64} + \frac{e^{4x} c_1}{4}\right) + c_2 e^{-4x}$$

which simplifies to

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\, e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\, e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4} \tag{1}$$

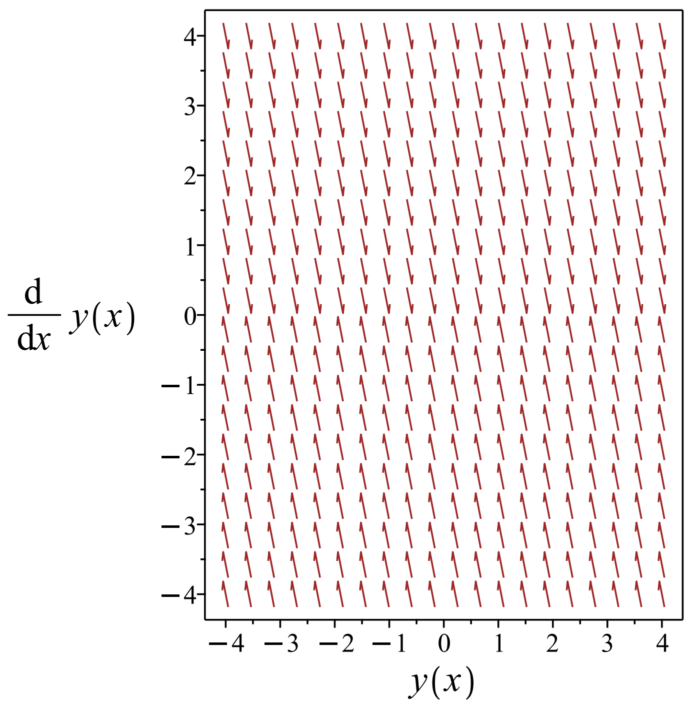


Figure 634: Slope field plot

Verification of solutions

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\, e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4}$$

Verified OK.

17.2.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 4p(x) - x - e^{-4x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= x + e^{-4x} \end{aligned}$$

Hence the ode is

$$p'(x) + 4p(x) = x + e^{-4x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 4dx} \\ &= e^{4x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(x + e^{-4x}\right) \\ \frac{d}{dx}\left(e^{4x} p\right) &= \left(e^{4x}\right)\left(x + e^{-4x}\right) \\ d\left(e^{4x} p\right) &= \left(e^{4x} x + 1\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{4x} p &= \int e^{4x} x + 1 \, dx \\ e^{4x} p &= x + \frac{e^{4x} x}{4} - \frac{e^{4x}}{16} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{4x}$ results in

$$p(x) = \mathrm{e}^{-4x}\left(x + \frac{\mathrm{e}^{4x}x}{4} - \frac{\mathrm{e}^{4x}}{16}\right) + c_1\mathrm{e}^{-4x}$$

which simplifies to

$$p(x) = -\frac{1}{16} + (x + c_1)\,\mathrm{e}^{-4x} + \frac{x}{4}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\frac{1}{16} + (x + c_1)\,\mathrm{e}^{-4x} + \frac{x}{4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{-4x} + \frac{x}{4} - \frac{1}{16} + c_1\mathrm{e}^{-4x}\,\mathrm{d}x \\ &= \frac{x^2}{8} - \frac{x}{16} - \frac{x\,\mathrm{e}^{-4x}}{4} - \frac{\mathrm{e}^{-4x}}{16} - \frac{c_1\mathrm{e}^{-4x}}{4} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{8} - \frac{x}{16} - \frac{x\,\mathrm{e}^{-4x}}{4} - \frac{\mathrm{e}^{-4x}}{16} - \frac{c_1\mathrm{e}^{-4x}}{4} + c_2 \tag{1}$$

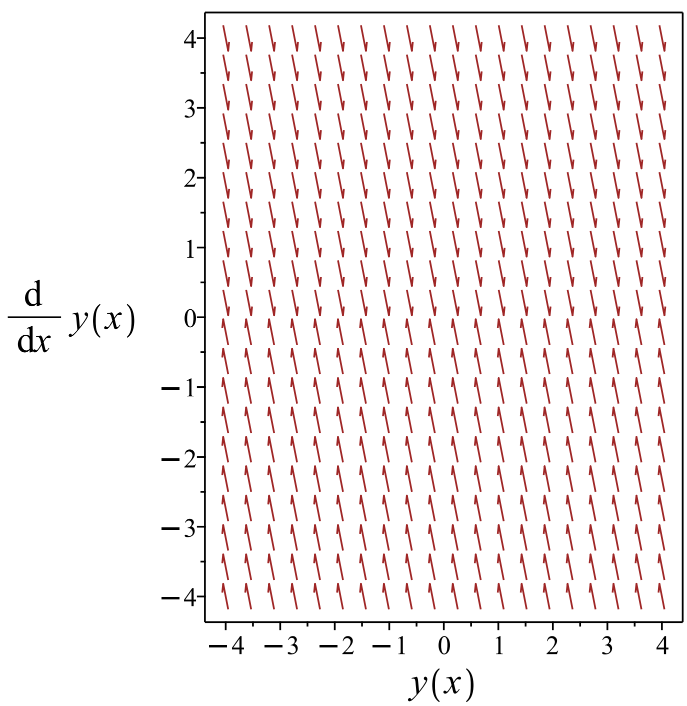


Figure 635: Slope field plot

Verification of solutions

$$y = \frac{x^2}{8} - \frac{x}{16} - \frac{x\,\mathrm{e}^{-4x}}{4} - \frac{\mathrm{e}^{-4x}}{16} - \frac{c_1 \mathrm{e}^{-4x}}{4} + c_2$$

Verified OK.

17.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 4y' = x + \mathrm{e}^{-4x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 4y')\, dx = \int \left(x + \mathrm{e}^{-4x}\right) dx$$

$$4y + y' = \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 4$$
$$q(x) = \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1$$

Hence the ode is

$$4y + y' = \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 4dx}$$
$$= \mathrm{e}^{4x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{4x} y\right) = \left(\mathrm{e}^{4x}\right)\left(\frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1\right)$$
$$\mathrm{d}\left(\mathrm{e}^{4x} y\right) = \left(-\frac{1}{4} + \frac{(2x^2 + 4c_1)\,\mathrm{e}^{4x}}{4}\right) \mathrm{d}x$$

Integrating gives

$$e^{4x}y = \int -\frac{1}{4} + \frac{(2x^2 + 4c_1)\, e^{4x}}{4}\, dx$$

$$e^{4x}y = -\frac{x}{4} + \frac{x^2 e^{4x}}{8} - \frac{e^{4x}x}{16} + \frac{e^{4x}}{64} + \frac{e^{4x}c_1}{4} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{4x}$ results in

$$y = e^{-4x}\left(-\frac{x}{4} + \frac{x^2 e^{4x}}{8} - \frac{e^{4x}x}{16} + \frac{e^{4x}}{64} + \frac{e^{4x}c_1}{4}\right) + c_2 e^{-4x}$$

which simplifies to

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\, e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\, e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4} \tag{1}$$

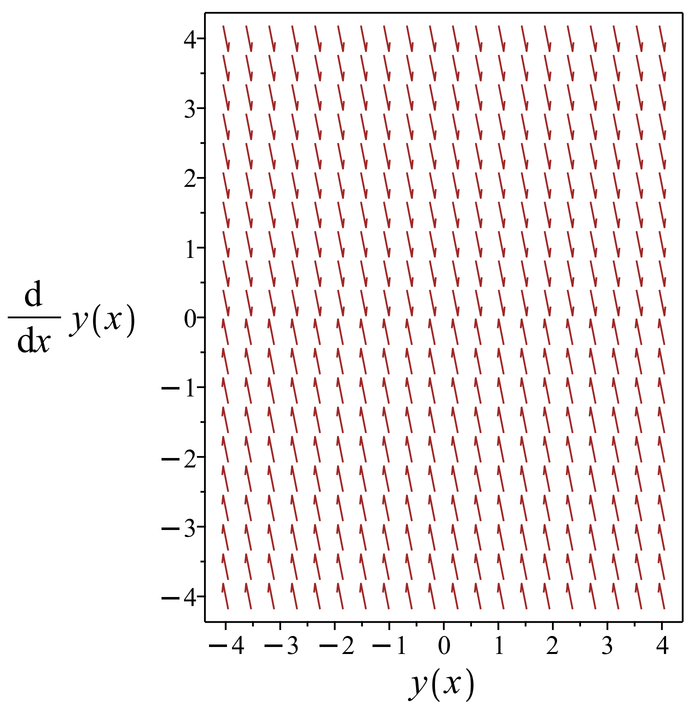


Figure 636: Slope field plot

Verification of solutions

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\,e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4}$$

Verified OK.

17.2.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 500: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-4x}\right) + c_2\left(\mathrm{e}^{-4x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-4x} + \frac{c_2}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-4x}$$
$$y_2 = \frac{1}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-4x} & \frac{1}{4} \\ \frac{d}{dx}(\mathrm{e}^{-4x}) & \frac{d}{dx}\left(\frac{1}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-4x} & \frac{1}{4} \\ -4\,e^{-4x} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{-4x}\right)(0) - \left(\frac{1}{4}\right)\left(-4\,e^{-4x}\right)$$

Which simplifies to

$$W = e^{-4x}$$

Which simplifies to

$$W = e^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x}{4} + \frac{e^{-4x}}{4}}{e^{-4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{e^{4x}x}{4} + \frac{1}{4}\right) dx$$

Hence

$$u_1 = -\frac{x}{4} - \frac{e^{4x}x}{16} + \frac{e^{4x}}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-4x}\left(x + e^{-4x}\right)}{e^{-4x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(x + e^{-4x}\right) dx$$

Hence

$$u_2 = \frac{x^2}{2} - \frac{e^{-4x}}{4}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{(-4x+1)\,\mathrm{e}^{4x}}{64} - \frac{x}{4} \\
u_2 &= \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \left(\frac{(-4x+1)\,\mathrm{e}^{4x}}{64} - \frac{x}{4}\right)\mathrm{e}^{-4x} + \frac{x^2}{8} - \frac{\mathrm{e}^{-4x}}{16}
$$

Which simplifies to

$$
y_p(x) = \frac{1}{64} + \frac{(-4x-1)\,\mathrm{e}^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{-4x} + \frac{c_2}{4}\right) + \left(\frac{1}{64} + \frac{(-4x-1)\,\mathrm{e}^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-4x} + \frac{c_2}{4} + \frac{1}{64} + \frac{(-4x-1)\,\mathrm{e}^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16} \tag{1}
$$

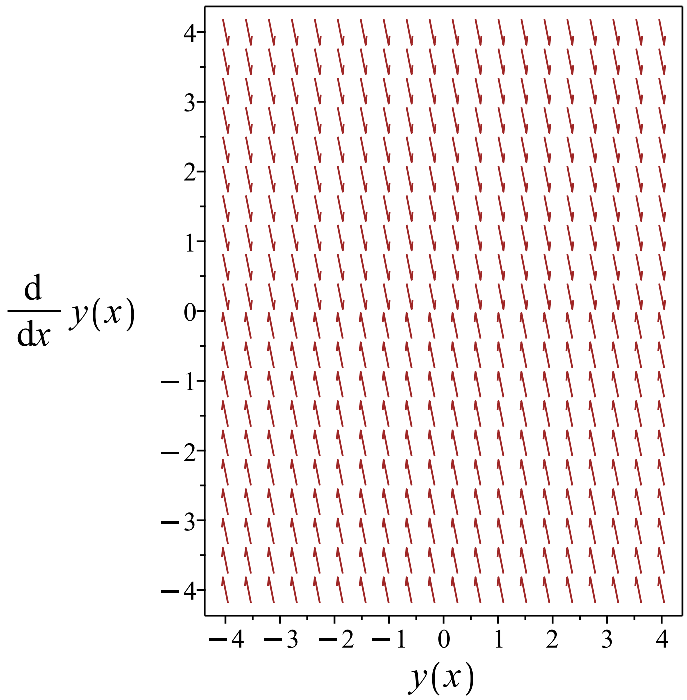


Figure 637: Slope field plot

Verification of solutions

$$y = c_1 e^{-4x} + \frac{c_2}{4} + \frac{1}{64} + \frac{(-4x-1)\, e^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16}$$

Verified OK.

17.2.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 4 \\ r(x) &= 0 \\ s(x) &= x + e^{-4x} \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$4y + y' = \int x + \mathrm{e}^{-4x}\, dx$$

We now have a first order ode to solve which is

$$4y + y' = \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 4$$
$$q(x) = \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1$$

Hence the ode is

$$4y + y' = \frac{x^2}{2} - \frac{\mathrm{e}^{-4x}}{4} + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 4dx}$$
$$= \mathrm{e}^{4x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1\right)$$
$$\frac{d}{dx}(e^{4x}y) = (e^{4x})\left(\frac{x^2}{2} - \frac{e^{-4x}}{4} + c_1\right)$$
$$d(e^{4x}y) = \left(-\frac{1}{4} + \frac{(2x^2 + 4c_1)\,e^{4x}}{4}\right) dx$$

Integrating gives

$$e^{4x}y = \int -\frac{1}{4} + \frac{(2x^2 + 4c_1)\,e^{4x}}{4}\,dx$$
$$e^{4x}y = -\frac{x}{4} + \frac{x^2e^{4x}}{8} - \frac{e^{4x}x}{16} + \frac{e^{4x}}{64} + \frac{e^{4x}c_1}{4} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{4x}$ results in

$$y = e^{-4x}\left(-\frac{x}{4} + \frac{x^2e^{4x}}{8} - \frac{e^{4x}x}{16} + \frac{e^{4x}}{64} + \frac{e^{4x}c_1}{4}\right) + c_2e^{-4x}$$

which simplifies to

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\,e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\,e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4} \tag{1}$$

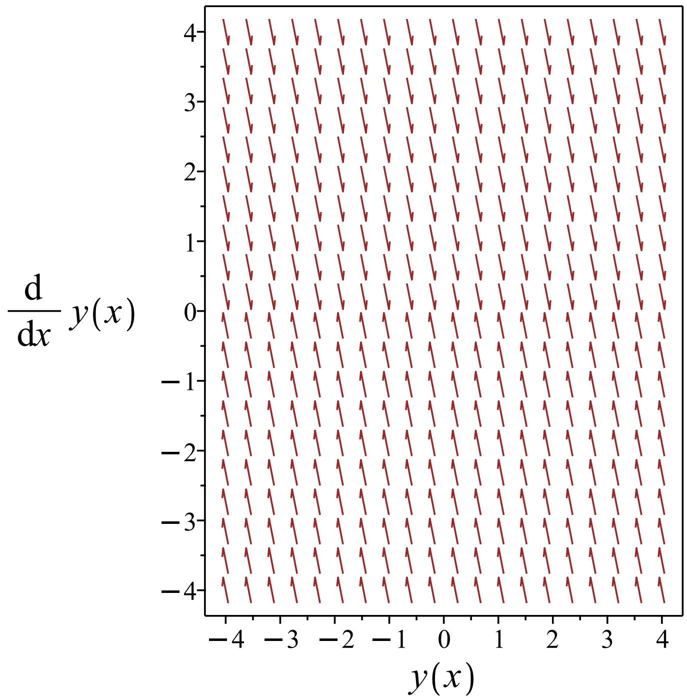


Figure 638: Slope field plot

Verification of solutions

$$y = \frac{1}{64} + \frac{(-x + 4c_2)\,e^{-4x}}{4} + \frac{x^2}{8} - \frac{x}{16} + \frac{c_1}{4}$$

Verified OK.

17.2.7 Maple step by step solution

Let's solve

$y'' + 4y' = x + e^{-4x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r = 0$

- Factor the characteristic polynomial

  $r(r + 4) = 0$

- Roots of the characteristic polynomial

$r = (-4, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-4x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-4x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x + \mathrm{e}^{-4x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-4x} & 1 \\ -4\,\mathrm{e}^{-4x} & 0 \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{-4x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-4x}\left(\int (\mathrm{e}^{4x} x + 1) dx\right)}{4} + \frac{\left(\int (x + \mathrm{e}^{-4x}) dx\right)}{4}$

- ○ Compute integrals

  $y_p(x) = \frac{1}{64} + \frac{(-4x-1)\mathrm{e}^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-4x} + c_2 + \frac{1}{64} + \frac{(-4x-1)\mathrm{e}^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -4*_b(_a)+_a+exp(-4*_a), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)=x+exp(-4*x),y(x), singsol=all)
```

$$y(x) = \frac{(-4x - 4c_1 - 1)\,\mathrm{e}^{-4x}}{16} + \frac{x^2}{8} - \frac{x}{16} + c_2$$

✓ Solution by Mathematica

Time used: 0.35 (sec). Leaf size: 38

```
DSolve[y''[x]+4*y'[x]==x+Exp[-4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{8} - \frac{x}{16} - \frac{1}{16}e^{-4x}(4x + 1 + 4c_1) + c_2$$

17.3 problem 553

Internal problem ID [15322]
Internal file name [`OUTPUT/15322_Wednesday_May_08_2024_03_55_45_PM_76296683/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 553.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = x + \sin(x)}$$

17.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x + \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^x c_1 + \mathrm{e}^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^x c_1 + \mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1 + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3 \cos(x) - 2A_4 \sin(x) - A_2 x - A_1 = x + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -1, A_3 = 0, A_4 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + e^{-x} c_2) + \left(-x - \frac{\sin(x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{x} c_1 + e^{-x} c_2 - x - \frac{\sin(x)}{2} \tag{1}$$

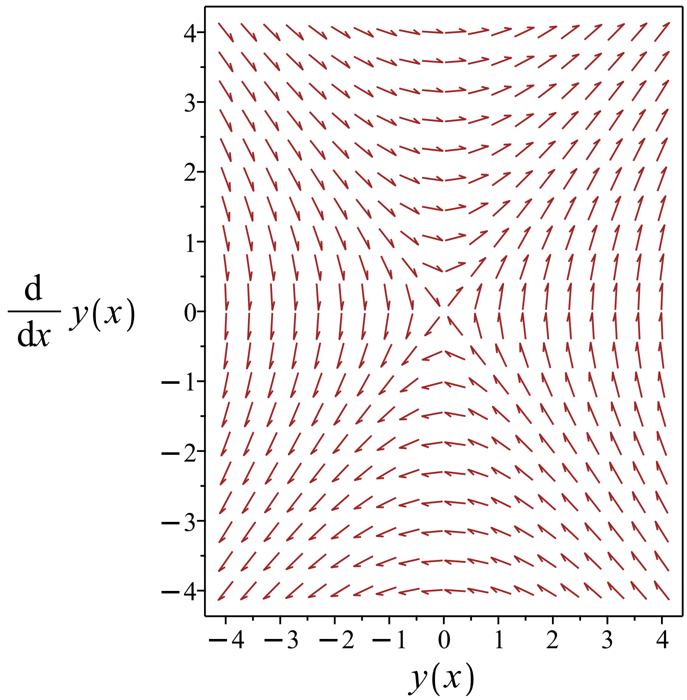


Figure 639: Slope field plot

Verification of solutions

$$y = e^{x} c_1 + e^{-x} c_2 - x - \frac{\sin(x)}{2}$$

Verified OK.

17.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 502: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-x} \int \frac{1}{e^{-2x}}\, dx \\ &= e^{-x} \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{e^{x} c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x+\sin (x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\},\{\cos (x), \sin (x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_{p}=A_{2} x+A_{1}+A_{3} \cos (x)+A_{4} \sin (x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2 A_{3} \cos (x)-2 A_{4} \sin (x)-A_{2} x-A_{1}=x+\sin (x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_{1}=0, A_{2}=-1, A_{3}=0, A_{4}=-\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_{p}=-x-\frac{\sin (x)}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_{h}+y_{p} \\
&= \left(c_{1} \mathrm{e}^{-x}+\frac{\mathrm{e}^{x} c_{2}}{2}\right)+\left(-x-\frac{\sin (x)}{2}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{e^x c_2}{2} - x - \frac{\sin(x)}{2} \tag{1}$$

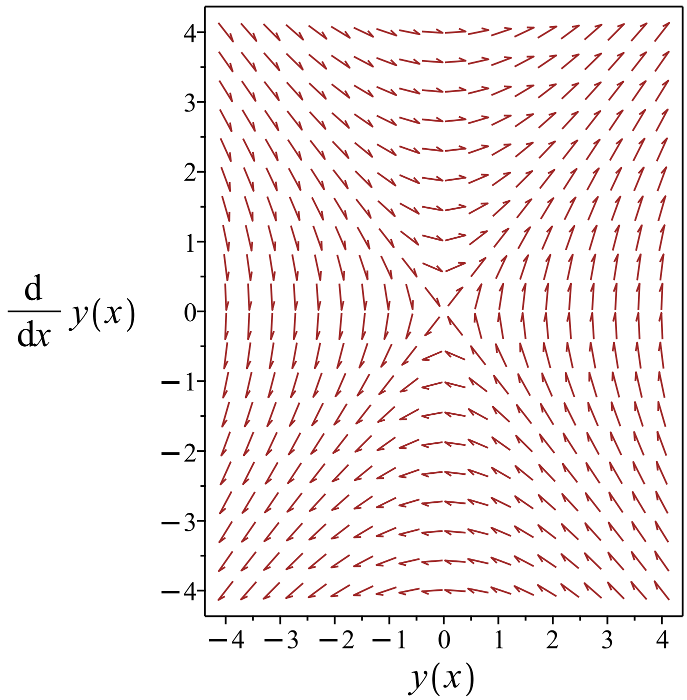


Figure 640: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{e^x c_2}{2} - x - \frac{\sin(x)}{2}$$

Verified OK.

17.3.3 Maple step by step solution

Let's solve

$y'' - y = x + \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + \sin(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-x}\left(\int e^{x}(x+\sin(x))dx\right)}{2} + \frac{e^{x}\left(\int e^{-x}(x+\sin(x))dx\right)}{2}$

- ○ Compute integrals

  $y_p(x) = -x - \frac{\sin(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{x} - x - \frac{\sin(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-y(x)=x+sin(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + c_1 e^{x} - \frac{\sin(x)}{2} - x$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 29

```
DSolve[y''[x]-y[x]==x+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \frac{\sin(x)}{2} + c_1 e^{x} + c_2 e^{-x}$$

17.4 problem 554

Internal problem ID [15323]
Internal file name [`OUTPUT/15323_Wednesday_May_08_2024_03_55_46_PM_19877591/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 554.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = (1 + \sin(x))\,e^{x}}$$

17.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = (1 + \sin(x))\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(1+\sin(x))\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\},\{e^{x}\cos(x)\,,e^{x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}\cos(x)\,,e^{x}\sin(x)\}$$

Since $e^{x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\},\{x\,e^{x}\cos(x)\,,x\,e^{x}\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{x} + A_2 x\,e^{x}\cos(x) + A_3 x\,e^{x}\sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{x} - 2A_2 e^{x}\sin(x) + 2A_3 e^{x}\cos(x) = (1+\sin(x))\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{1}{2}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{x} - \frac{x\,e^{x}\cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{x}(c_1\cos(x) + c_2\sin(x))) + \left(e^{x} - \frac{x\,e^{x}\cos(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x(c_1 \cos(x) + c_2 \sin(x)) + e^x - \frac{x e^x \cos(x)}{2} \tag{1}$$

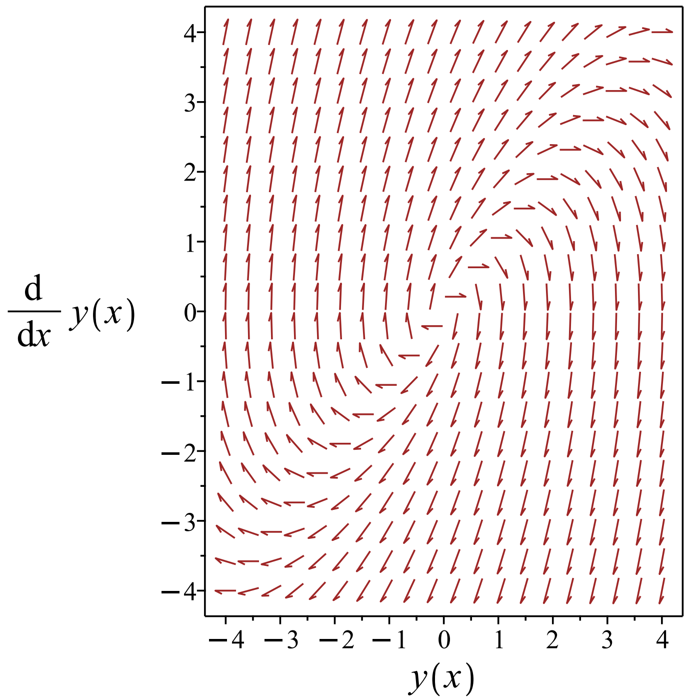


Figure 641: Slope field plot

Verification of solutions

$$y = e^x(c_1 \cos(x) + c_2 \sin(x)) + e^x - \frac{x e^x \cos(x)}{2}$$

Verified OK.

17.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$y_1 = e^x \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x \cos(x)) + c_2(e^x \cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^x \cos(x)\, c_1 + e^x \sin(x)\, c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(1 + \sin(x))\, e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(x), e^x \sin(x)\}$$

Since $e^x \cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x e^x \cos(x), x e^x \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x e^x \cos(x) + A_3 x e^x \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^x - 2A_2 e^x \sin(x) + 2A_3 e^x \cos(x) = (1 + \sin(x)) e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{1}{2}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x - \frac{x e^x \cos(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x \cos(x) c_1 + e^x \sin(x) c_2) + \left(e^x - \frac{x e^x \cos(x)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_1 \cos(x) + c_2 \sin(x)) + e^x - \frac{x e^x \cos(x)}{2}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x} - \frac{x e^{x} \cos(x)}{2} \tag{1}$$

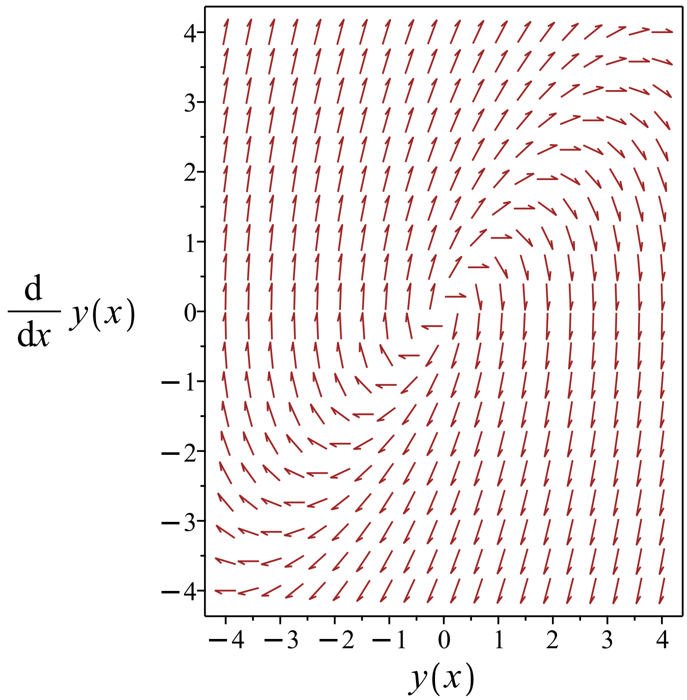


Figure 642: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x} - \frac{x e^{x} \cos(x)}{2}$$

Verified OK.

17.4.3 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = (1 + \sin(x)) e^{x}$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

$r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^x \cos(x)$

- 2nd solution of the homogeneous ODE

$y_2(x) = \mathrm{e}^x \sin(x)$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = \mathrm{e}^x \cos(x)\, c_1 + \mathrm{e}^x \sin(x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (1 + \sin(x))\, \mathrm{e}^x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x \cos(x) & \mathrm{e}^x \sin(x) \\ \mathrm{e}^x \cos(x) - \mathrm{e}^x \sin(x) & \mathrm{e}^x \sin(x) + \mathrm{e}^x \cos(x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^x\left(-\cos(x)\left(\int \left(\sin(x)^2 + \sin(x)\right) dx\right) + \sin(x)\left(\int \cos(x)\,(1 + \sin(x))\, dx\right)\right)$

- ○ Compute integrals

  $y_p(x) = \frac{(\sin(x) + 2 - \cos(x) x)\, \mathrm{e}^x}{2}$

- Substitute particular solution into general solution to ODE

$y = \mathrm{e}^x \cos(x)\, c_1 + \mathrm{e}^x \sin(x)\, c_2 + \frac{(\sin(x) + 2 - \cos(x) x)\, \mathrm{e}^x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=(1+sin(x))*exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{x}((x - 2c_1)\cos(x) - 2 + (-2c_2 - 1)\sin(x))}{2}$$

✓ Solution by Mathematica

Time used: 0.194 (sec). Leaf size: 32

```
DSolve[y''[x]-2*y'[x]+2*y[x]==(1+Sin[x])*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{x}(-((x - 2c_2)\cos(x)) + 2(1 + c_1)\sin(x) + 2)$$

17.5 problem 555

Internal problem ID [15324]
Internal file name [`OUTPUT/15324_Wednesday_May_08_2024_03_55_47_PM_84433538/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 555.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y'' = 1 + e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = e^x$$

Now the particular solution to the given ODE is found

$$y''' - y'' = 1 + e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{e^x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}, \{e^x\}]$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}, \{x\, e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 + A_2 x\, e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 e^x - 2A_1 = 1 + e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^2}{2} + x\,\mathrm{e}^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1 + \mathrm{e}^x c_3) + \left(-\frac{x^2}{2} + x\,\mathrm{e}^x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + \mathrm{e}^x c_3 - \frac{x^2}{2} + x\,\mathrm{e}^x \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + \mathrm{e}^x c_3 - \frac{x^2}{2} + x\,\mathrm{e}^x$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)+1+exp(_a), _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)=1+exp(x),y(x), singsol=all)
```

$$y(x) = (c_1 + x - 2)\,e^x - \frac{x^2}{2} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 28

```
DSolve[y'''[x]-y''[x]==1+Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{2} + c_3 x + e^x(x - 2 + c_1) + c_2$$

17.6 problem 556

Internal problem ID [15325]
Internal file name [`OUTPUT/15325_Wednesday_May_08_2024_03_55_47_PM_94148368/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 556.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 4y' = e^{2x} + \sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 4y' = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{2ix} c_2 + e^{-2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{2ix}$$
$$y_3 = e^{-2ix}$$

Now the particular solution to the given ODE is found

$$y''' + 4y' = e^{2x} + \sin(2x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & e^{2ix} & e^{-2ix} \\ 0 & 2i e^{2ix} & -2i e^{-2ix} \\ 0 & -4 e^{2ix} & -4 e^{-2ix} \end{bmatrix}$$

$$|W| = -16i e^{2ix} e^{-2ix}$$

The determinant simplifies to

$$|W| = -16i$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{2ix} & e^{-2ix} \\ 2ie^{2ix} & -2ie^{-2ix} \end{bmatrix} \\ &= -4i \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} 1 & e^{-2ix} \\ 0 & -2ie^{-2ix} \end{bmatrix} \\ &= -2ie^{-2ix} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} 1 & e^{2ix} \\ 0 & 2ie^{2ix} \end{bmatrix} \\ &= 2ie^{2ix} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\ &= (-1)^2 \int \frac{\left(e^{2x} + \sin(2x)\right)(-4i)}{(1)(-16i)}\, dx \\ &= \int \frac{-4i(e^{2x} + \sin(2x))}{-16i}\, dx \\ &= \int \left(\frac{e^{2x}}{4} + \frac{\sin(2x)}{4}\right) dx \\ &= \frac{e^{2x}}{8} - \frac{\cos(2x)}{8} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{\left(\mathrm{e}^{2x} + \sin(2x)\right)\left(-2i\mathrm{e}^{-2ix}\right)}{(1)(-16i)} \, dx \\
&= -\int \frac{-2i\left(\mathrm{e}^{2x} + \sin(2x)\right) \mathrm{e}^{-2ix}}{-16i} \, dx \\
&= -\int \left(\frac{\left(\mathrm{e}^{2x} + \sin(2x)\right) \mathrm{e}^{-2ix}}{8}\right) dx \\
&= -\frac{\mathrm{e}^{(2-2i)x}}{32} - \frac{i\mathrm{e}^{(2-2i)x}}{32} + \frac{ix}{16} + \frac{\mathrm{e}^{-4ix}}{64}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{\left(\mathrm{e}^{2x} + \sin(2x)\right)\left(2i\mathrm{e}^{2ix}\right)}{(1)(-16i)} \, dx \\
&= \int \frac{2i\left(\mathrm{e}^{2x} + \sin(2x)\right) \mathrm{e}^{2ix}}{-16i} \, dx \\
&= \int \left(-\frac{\left(\mathrm{e}^{2x} + \sin(2x)\right) \mathrm{e}^{2ix}}{8}\right) dx \\
&= -\frac{ix}{16} + \frac{\mathrm{e}^{4ix}}{64} - \frac{\mathrm{e}^{(2+2i)x}}{32} + \frac{i\mathrm{e}^{(2+2i)x}}{32}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(\frac{\mathrm{e}^{2x}}{8} - \frac{\cos(2x)}{8}\right) \\
& + \left(-\frac{\mathrm{e}^{(2-2i)x}}{32} - \frac{i\mathrm{e}^{(2-2i)x}}{32} + \frac{ix}{16} + \frac{\mathrm{e}^{-4ix}}{64}\right)\left(\mathrm{e}^{2ix}\right) \\
& + \left(-\frac{ix}{16} + \frac{\mathrm{e}^{4ix}}{64} - \frac{\mathrm{e}^{(2+2i)x}}{32} + \frac{i\mathrm{e}^{(2+2i)x}}{32}\right)\left(\mathrm{e}^{-2ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\mathrm{e}^{2x}}{16} - \frac{3\cos(2x)}{32} - \frac{x\sin(2x)}{8}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3\right) + \left(\frac{\mathrm{e}^{2x}}{16} - \frac{3\cos(2x)}{32} - \frac{x\sin(2x)}{8}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 + e^{2ix} c_2 + e^{-2ix} c_3 + \frac{e^{2x}}{16} - \frac{3\cos(2x)}{32} - \frac{x \sin(2x)}{8} \tag{1}$$

Verification of solutions

$$y = c_1 + e^{2ix} c_2 + e^{-2ix} c_3 + \frac{e^{2x}}{16} - \frac{3\cos(2x)}{32} - \frac{x \sin(2x)}{8}$$

Verified OK.

17.6.1 Maple step by step solution

Let's solve

$y''' + 4y' = e^{2x} + \sin(2x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = e^{2x} + \sin(2x) - 4y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^{2x} + \sin(2x) - 4y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^{2x} + \sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^{2x} + \sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x}\cdot\begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x))\cdot\begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$$

□ Fundamental matrix

 ◦ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ 0 & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ 0 & \cos(2x) & -\sin(2x) \end{bmatrix}$$

 ◦ The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ 0 & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ 0 & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{\sin(2x)}{2} & \frac{1}{4} - \frac{\cos(2x)}{4} \\ 0 & \cos(2x) & \frac{\sin(2x)}{2} \\ 0 & -2\sin(2x) & \cos(2x) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

$\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

$\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

$\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{(-2x-1)\sin(2x)}{16} - \frac{\cos(2x)}{16} + \frac{e^{2x}}{16} \\ \frac{\cos(2x)(-2x-1)}{8} + \frac{e^{2x}}{8} \\ \frac{(1+2x)\sin(2x)}{4} - \frac{\cos(2x)}{4} + \frac{e^{2x}}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{(-2x-1)\sin(2x)}{16} - \frac{\cos(2x)}{16} + \frac{e^{2x}}{16} \\ \frac{\cos(2x)(-2x-1)}{8} + \frac{e^{2x}}{8} \\ \frac{(1+2x)\sin(2x)}{4} - \frac{\cos(2x)}{4} + \frac{e^{2x}}{4} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3-2x-1)\sin(2x)}{16} + \frac{(-4c_2-1)\cos(2x)}{16} + c_1 + \frac{e^{2x}}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -4*_b(_a)+exp(2*_a)+sin(2*_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x)=exp(2*x)+sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-8c_2 - 1)\cos(2x)}{16} + \frac{(-x + 4c_1)\sin(2x)}{8} + c_3 + \frac{e^{2x}}{16}$$

✓ Solution by Mathematica

Time used: 0.836 (sec). Leaf size: 44

```
DSolve[y'''[x]+4*y'[x]==Exp[2*x]+Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32}\left(2e^{2x} - ((3 + 16c_2)\cos(2x)) - 4(x - 4c_1)\sin(2x)\right) + c_3$$

17.7 problem 557

Internal problem ID [15326]
Internal file name [`OUTPUT/15326_Wednesday_May_08_2024_03_55_50_PM_35132079/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 557.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(x)\sin(2x)}$$

17.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(x)\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(3x) + A_4 \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) - 5A_3 \cos(3x) - 5A_4 \sin(3x) = \sin(x)\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = 0, A_3 = \frac{1}{10}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{6} + \frac{\cos(3x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{\cos(x)}{6} + \frac{\cos(3x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\cos(x)}{6} + \frac{\cos(3x)}{10} \tag{1}$$

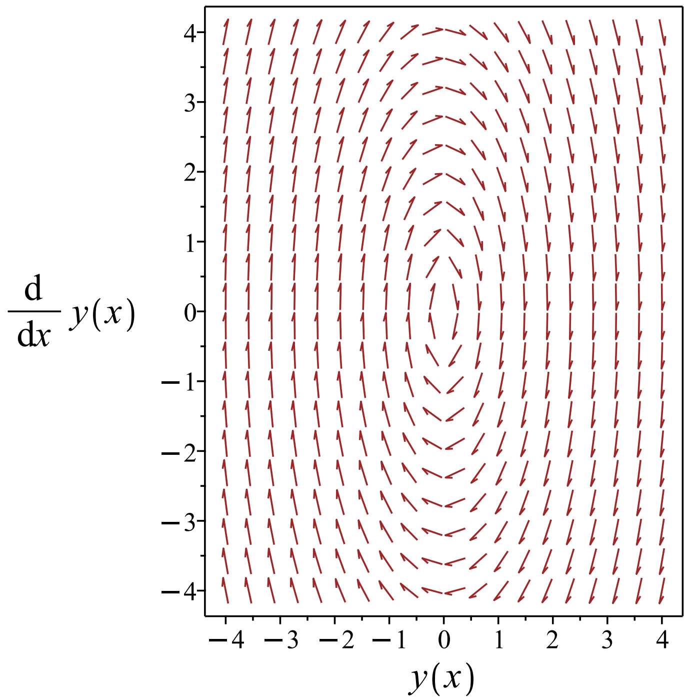


Figure 643: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\cos(x)}{6} + \frac{\cos(3x)}{10}$$

Verified OK.

17.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x) + A_3\cos(3x) + A_4\sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\cos(x) + 3A_2\sin(x) - 5A_3\cos(3x) - 5A_4\sin(3x) = \sin(x)\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = 0, A_3 = \frac{1}{10}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{6} + \frac{\cos(3x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{\cos(x)}{6} + \frac{\cos(3x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\cos(x)}{6} + \frac{\cos(3x)}{10} \tag{1}$$

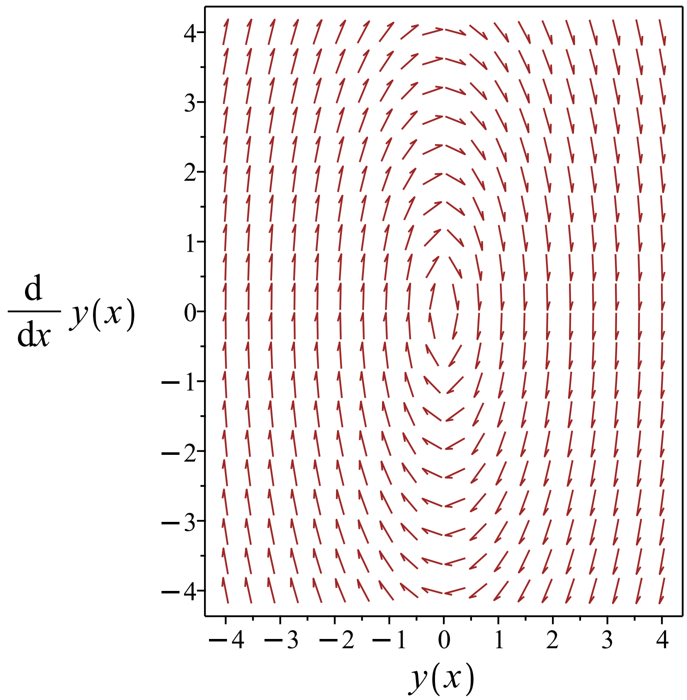


Figure 644: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\cos(x)}{6} + \frac{\cos(3x)}{10}$$

Verified OK.

17.7.3 Maple step by step solution

Let's solve

$y'' + 4y = \sin(x)\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+4=0$

- Use quadratic formula to solve for r

$r=\frac{0\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r=(-2\,\mathrm{I},2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x)=\cos(2x)$

- 2nd solution of the homogeneous ODE

$y_2(x)=\sin(2x)$

- General solution of the ODE

$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y=c_1\cos(2x)+c_2\sin(2x)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\sin(x)\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\begin{bmatrix}\cos(2x) & \sin(2x)\\ -2\sin(2x) & 2\cos(2x)\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x),y_2(x))=2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\frac{\cos(2x)\left(\int\sin(2x)^2\sin(x)dx\right)}{2}+\frac{\sin(2x)\left(\int(-\cos(5x)+\cos(3x))dx\right)}{8}$

  - Compute integrals

    $y_p(x)=\frac{2\cos(x)^3}{5}-\frac{2\cos(x)}{15}$

- Substitute particular solution into general solution to ODE

$y=c_1\cos(2x)+c_2\sin(2x)+\frac{2\cos(x)^3}{5}-\frac{2\cos(x)}{15}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+4*y(x)=sin(x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + \cos(2x) c_1 - \frac{2\cos(x)}{15} + \frac{2\cos(x)^3}{5}$$

✓ Solution by Mathematica

Time used: 0.178 (sec). Leaf size: 34

```
DSolve[y''[x]+4*y[x]==Sin[x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\cos(x)}{6} + \frac{1}{10}\cos(3x) + c_1\cos(2x) + c_2\sin(2x)$$

17.8 problem 558

Internal problem ID [15327]
Internal file name [`OUTPUT/15327_Wednesday_May_08_2024_03_55_52_PM_65050519/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 558.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 4y' = 2\cos(4x)^2}$$

17.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 0, f(x) = 1 + \cos(8x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-4y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=-4, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}-4\lambda\, \mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2-4\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-4, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{4}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{-4^2-(4)(1)(0)}\\&=2\pm 2\end{aligned}$$

Hence

$$\lambda_1=2+2$$
$$\lambda_2=2-2$$

Which simplifies to

$$\lambda_1=4$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$$
$$y=c_1e^{(4)x}+c_2e^{(0)x}$$

Or

$$y = e^{4x}c_1 + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^{4x}c_1 + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + \cos(8x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(8x), \sin(8x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{\cos(8x), \sin(8x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x + A_2 \cos(8x) + A_3 \sin(8x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-64A_2 \cos(8x) - 64A_3 \sin(8x) - 4A_1 + 32A_2 \sin(8x) - 32A_3 \cos(8x) = 1 + \cos(8x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{80}, A_3 = -\frac{1}{160}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{4x}c_1 + c_2) + \left(-\frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{4x}c_1 + c_2 - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160} \tag{1}$$

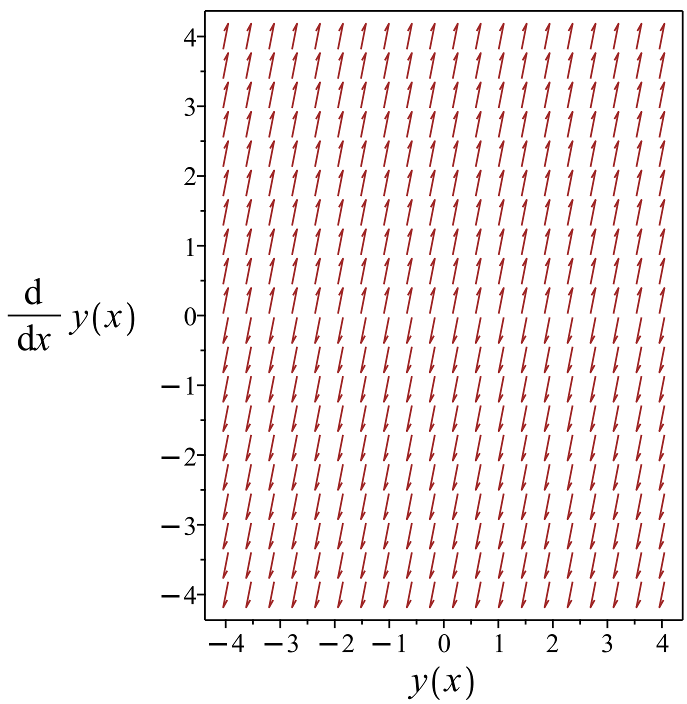


Figure 645: Slope field plot

Verification of solutions

$$y = e^{4x}c_1 + c_2 - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Verified OK.

17.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y''-4y')\,dx &= \int (1+\cos(8x))\,dx \\
-4y+y' &= x+\frac{\sin(8x)}{8}+c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -4 \\
q(x) &= x+\frac{\sin(8x)}{8}+c_1
\end{aligned}
$$

Hence the ode is

$$
-4y+y' = x+\frac{\sin(8x)}{8}+c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int(-4)dx} \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x+\frac{\sin(8x)}{8}+c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-4x}y\right) &= \left(\mathrm{e}^{-4x}\right)\left(x+\frac{\sin(8x)}{8}+c_1\right) \\
\mathrm{d}\left(\mathrm{e}^{-4x}y\right) &= \left(\frac{(\sin(8x)+8c_1+8x)\,\mathrm{e}^{-4x}}{8}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-4x}y &= \int \frac{(\sin(8x)+8c_1+8x)\,\mathrm{e}^{-4x}}{8}\,\mathrm{d}x \\
\mathrm{e}^{-4x}y &= -\frac{\mathrm{e}^{-4x}\cos(8x)}{80}-\frac{\mathrm{e}^{-4x}\sin(8x)}{160}-\frac{\mathrm{e}^{-4x}\cos(2x)}{20}-\frac{\mathrm{e}^{-4x}\sin(2x)}{10}-\frac{\mathrm{e}^{-4x}(-4\sin(2x)-2\cos(2x))}{40}-
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-4x}$ results in

$$y = \mathrm{e}^{4x}\left(-\frac{\mathrm{e}^{-4x}\cos(8x)}{80} - \frac{\mathrm{e}^{-4x}\sin(8x)}{160} - \frac{\mathrm{e}^{-4x}\cos(2x)}{20} - \frac{\mathrm{e}^{-4x}\sin(2x)}{10} - \frac{\mathrm{e}^{-4x}(-4\sin(2x) - 2\cos(2x))}{40} \right.$$

which simplifies to

$$y = -\frac{1}{16} + c_2\mathrm{e}^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{16} + c_2\mathrm{e}^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160} \tag{1}$$

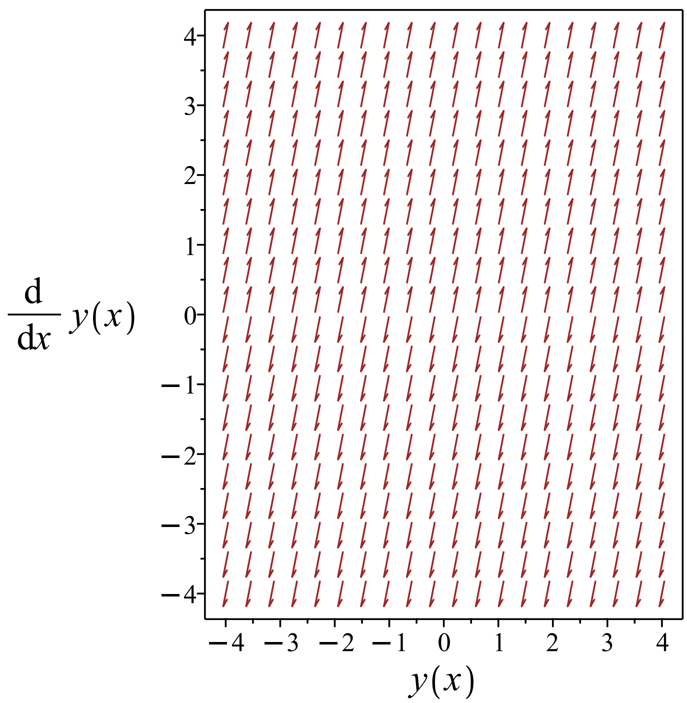


Figure 646: Slope field plot

Verification of solutions

$$y = -\frac{1}{16} + c_2\mathrm{e}^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Verified OK.

17.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 4p(x) - 1 - \cos(8x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 1 + \cos(8x) \end{aligned}$$

Hence the ode is

$$p'(x) - 4p(x) = 1 + \cos(8x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-4)dx} \\ &= e^{-4x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(1 + \cos(8x)) \\ \frac{d}{dx}\left(e^{-4x} p\right) &= \left(e^{-4x}\right)(1 + \cos(8x)) \\ d\left(e^{-4x} p\right) &= \left((1 + \cos(8x))\, e^{-4x}\right) dx \end{aligned}$$

Integrating gives

$$e^{-4x} p = \int (1 + \cos(8x))\, e^{-4x}\, dx$$

$$e^{-4x} p = -\frac{3\, e^{-4x}}{10} + \frac{8(-4\cos(x) + 8\sin(x))\, e^{-4x} \cos(x)^7}{5} - \frac{16(-4\cos(x) + 6\sin(x))\, e^{-4x} \cos(x)^5}{5} + 2(-$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$p(x) = e^{4x}\left(-\frac{3\,e^{-4x}}{10} + \frac{8(-4\cos(x) + 8\sin(x))\,e^{-4x}\cos(x)^7}{5} - \frac{16(-4\cos(x) + 6\sin(x))\,e^{-4x}\cos(x)^5}{5} + 2\right.$$

which simplifies to

$$p(x) = \frac{\sin(8x)}{10} - \frac{\cos(8x)}{20} - \frac{1}{4} + e^{4x}c_1$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\sin(8x)}{10} - \frac{\cos(8x)}{20} - \frac{1}{4} + e^{4x}c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sin(8x)}{10} - \frac{\cos(8x)}{20} - \frac{1}{4} + e^{4x}c_1\, dx \\ &= -\frac{x}{4} + \frac{e^{4x}c_1}{4} - \frac{\sin(8x)}{160} - \frac{\cos(8x)}{80} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{4} + \frac{e^{4x}c_1}{4} - \frac{\sin(8x)}{160} - \frac{\cos(8x)}{80} + c_2 \tag{1}$$

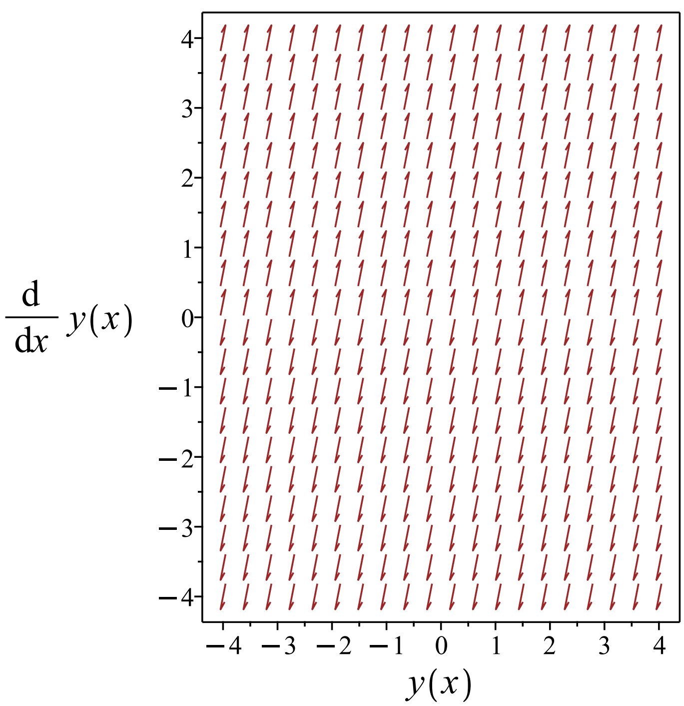


Figure 647: Slope field plot

Verification of solutions

$$y = -\frac{x}{4} + \frac{e^{4x} c_1}{4} - \frac{\sin(8x)}{160} - \frac{\cos(8x)}{80} + c_2$$

Verified OK.

17.8.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 4y' = 1 + \cos(8x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y')\, dx = \int (1 + \cos(8x))\, dx$$
$$-4y + y' = x + \frac{\sin(8x)}{8} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = x + \frac{\sin(8x)}{8} + c_1$$

Hence the ode is

$$-4y + y' = x + \frac{\sin(8x)}{8} + c_1$$

The integrating factor μ is

$$\mu = e^{\int(-4)dx}$$
$$= e^{-4x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x + \frac{\sin(8x)}{8} + c_1\right)$$
$$\frac{d}{dx}(e^{-4x} y) = (e^{-4x})\left(x + \frac{\sin(8x)}{8} + c_1\right)$$
$$d(e^{-4x} y) = \left(\frac{(\sin(8x) + 8c_1 + 8x)\, e^{-4x}}{8}\right) dx$$

Integrating gives

$$e^{-4x}y = \int \frac{(\sin(8x) + 8c_1 + 8x)\,e^{-4x}}{8}\,dx$$

$$e^{-4x}y = -\frac{e^{-4x}\cos(8x)}{80} - \frac{e^{-4x}\sin(8x)}{160} - \frac{e^{-4x}\cos(2x)}{20} - \frac{e^{-4x}\sin(2x)}{10} - \frac{e^{-4x}(-4\sin(2x) - 2\cos(2x))}{40} -$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = e^{4x}\left(-\frac{e^{-4x}\cos(8x)}{80} - \frac{e^{-4x}\sin(8x)}{160} - \frac{e^{-4x}\cos(2x)}{20} - \frac{e^{-4x}\sin(2x)}{10} - \frac{e^{-4x}(-4\sin(2x) - 2\cos(2x))}{40} -\right.$$

which simplifies to

$$y = -\frac{1}{16} + c_2 e^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{16} + c_2 e^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160} \tag{1}$$

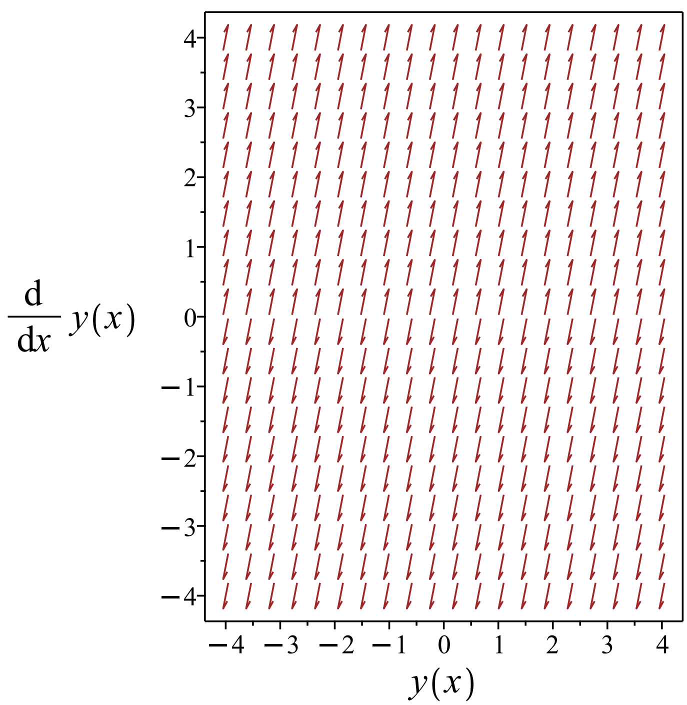


Figure 648: Slope field plot

Verification of solutions

$$y = -\frac{1}{16} + c_2 e^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Verified OK.

17.8.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 e^{4x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + \cos(8x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(8x), \sin(8x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, \frac{e^{4x}}{4}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{\cos(8x), \sin(8x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x + A_2 \cos(8x) + A_3 \sin(8x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-64A_2 \cos(8x) - 64A_3 \sin(8x) - 4A_1 + 32A_2 \sin(8x) - 32A_3 \cos(8x) = 1 + \cos(8x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{80}, A_3 = -\frac{1}{160}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \frac{c_2 e^{4x}}{4}\right) + \left(-\frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160} \tag{1}
$$

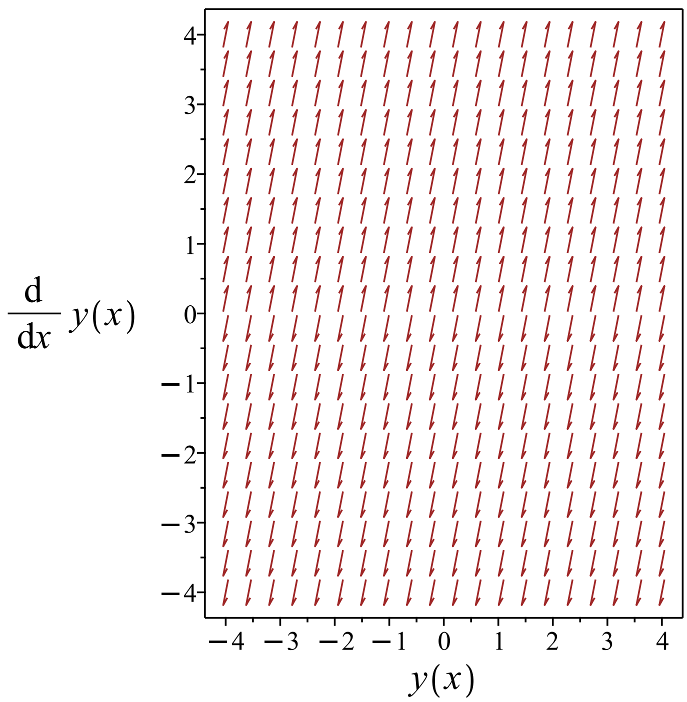


Figure 649: Slope field plot

Verification of solutions

$$
y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}
$$

Verified OK.

17.8.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -4 \\ r(x) &= 0 \\ s(x) &= 1 + \cos(8x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$-4y + y' = \int 1 + \cos(8x)\;dx$$

We now have a first order ode to solve which is

$$-4y + y' = x + \frac{\sin(8x)}{8} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= x + \frac{\sin(8x)}{8} + c_1 \end{aligned}$$

Hence the ode is

$$-4y + y' = x + \frac{\sin(8x)}{8} + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int(-4)dx} \\ &= e^{-4x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(x + \frac{\sin(8x)}{8} + c_1\right) \\ \frac{d}{dx}\left(e^{-4x} y\right) &= \left(e^{-4x}\right)\left(x + \frac{\sin(8x)}{8} + c_1\right) \\ d\left(e^{-4x} y\right) &= \left(\frac{(\sin(8x) + 8c_1 + 8x)\, e^{-4x}}{8}\right) dx \end{aligned}$$

Integrating gives

$$e^{-4x} y = \int \frac{(\sin(8x) + 8c_1 + 8x)\, e^{-4x}}{8}\, dx$$

$$e^{-4x} y = -\frac{e^{-4x}\cos(8x)}{80} - \frac{e^{-4x}\sin(8x)}{160} - \frac{e^{-4x}\cos(2x)}{20} - \frac{e^{-4x}\sin(2x)}{10} - \frac{e^{-4x}(-4\sin(2x) - 2\cos(2x))}{40} -$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = e^{4x}\left(-\frac{e^{-4x}\cos(8x)}{80} - \frac{e^{-4x}\sin(8x)}{160} - \frac{e^{-4x}\cos(2x)}{20} - \frac{e^{-4x}\sin(2x)}{10} - \frac{e^{-4x}(-4\sin(2x) - 2\cos(2x))}{40} -\right.$$

which simplifies to

$$y = -\frac{1}{16} + c_2 e^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{16} + c_2 e^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160} \tag{1}$$

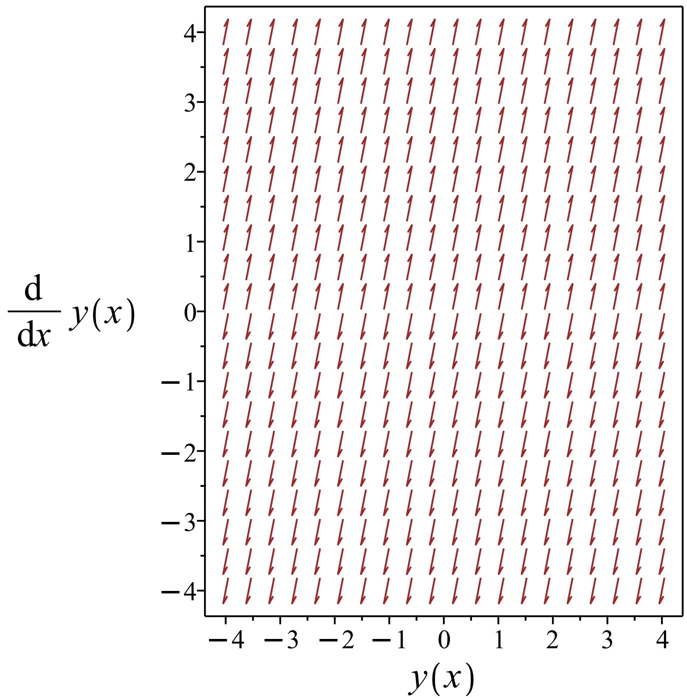


Figure 650: Slope field plot

Verification of solutions

$$y = -\frac{1}{16} + c_2 e^{4x} - \frac{c_1}{4} - \frac{x}{4} - \frac{\cos(8x)}{80} - \frac{\sin(8x)}{160}$$

Verified OK.

17.8.7 Maple step by step solution

Let's solve

$y'' - 4y' = 1 + \cos(8x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4r = 0$

- Factor the characteristic polynomial

  $r(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (0, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1 + \cos(8x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{4x} \\ 0 & 4\,e^{4x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,e^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\left(\int (1+\cos(8x))dx\right)}{4} + \frac{e^{4x}\left(\int (1+\cos(8x))e^{-4x}dx\right)}{4}$

  ◦ Compute integrals

  $y_p(x) = -\frac{\sin(8x)}{160} - \frac{\cos(8x)}{80} - \frac{1}{16} - \frac{x}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{4x} - \frac{\sin(8x)}{160} - \frac{\cos(8x)}{80} - \frac{1}{16} - \frac{x}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*cos(4*_a)^2+4*_b(_a), _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)=2*cos(4*x)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{4x}}{4} - \frac{\sin(8x)}{160} - \frac{\cos(8x)}{80} - \frac{x}{4} + c_2$$

✓ Solution by Mathematica

Time used: 0.243 (sec). Leaf size: 40

```
DSolve[y''[x]-4*y'[x]==2*Cos[4*x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{4} - \frac{1}{160}\sin(8x) - \frac{1}{80}\cos(8x) + \frac{1}{4}c_1 e^{4x} + c_2$$

17.9 problem 559

Internal problem ID [15328]
Internal file name [`OUTPUT/15328_Wednesday_May_08_2024_03_55_57_PM_69781155/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 559.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = 4x - 2\,e^{x}}$$

17.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 4x - 2\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{3}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 2$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(2)x} + c_2 e^{(-1)x}$$

Or

$$y = e^{2x} c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x - 2\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{x} - A_3 - 2A_2 - 2A_3 x = 4x - 2\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x} + 1 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + \mathrm{e}^{-x}c_2\right) + \left(\mathrm{e}^{x} + 1 - 2x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}c_1 + \mathrm{e}^{-x}c_2 + \mathrm{e}^{x} + 1 - 2x \tag{1}$$

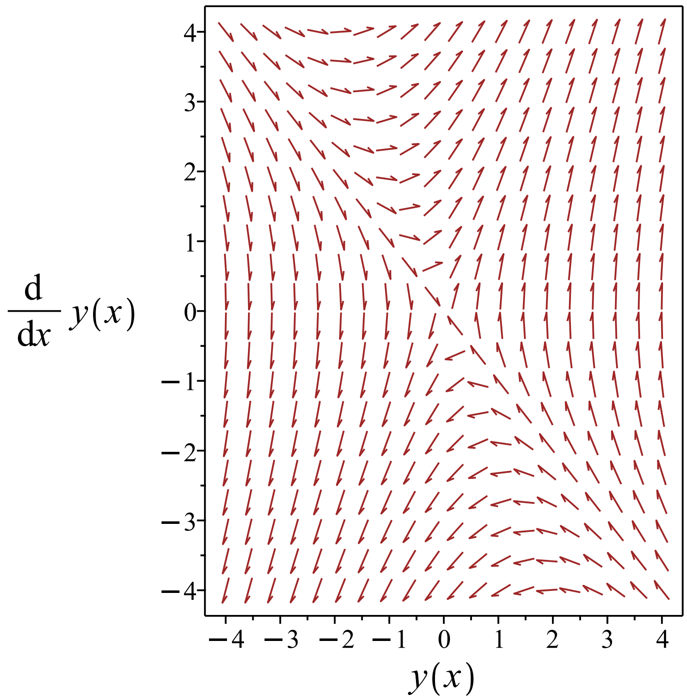


Figure 651: Slope field plot

Verification of solutions

$$y = e^{2x}c_1 + e^{-x}c_2 + e^x + 1 - 2x$$

Verified OK.

17.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 511: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx}\\ &= z_1 e^{\frac{x}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{\mathrm{e}^{2x} c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x - 2\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{2x}}{3}, \mathrm{e}^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^x - A_3 - 2A_2 - 2A_3 x = 4x - 2\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x + 1 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{e^{2x} c_2}{3}\right) + (e^x + 1 - 2x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{e^{2x} c_2}{3} + e^x + 1 - 2x \tag{1}$$

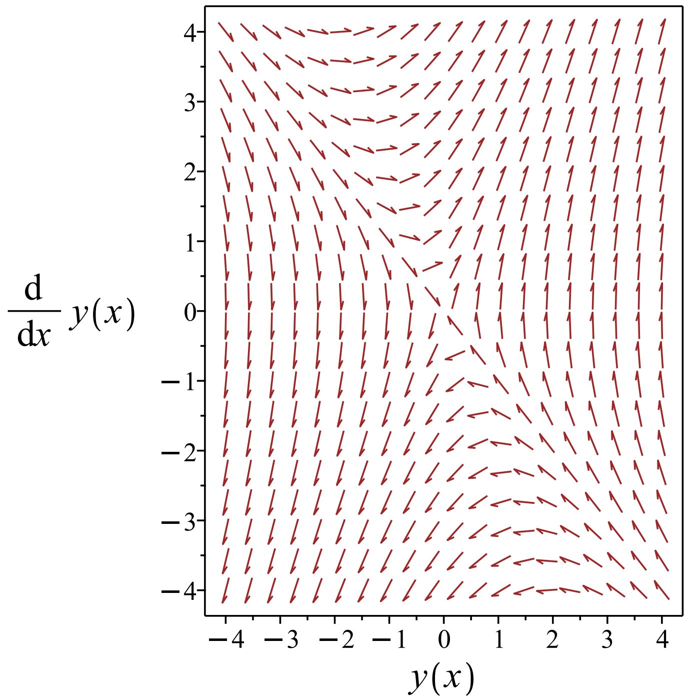


Figure 652: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{e^{2x} c_2}{3} + e^x + 1 - 2x$$

Verified OK.

17.9.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 4x - 2\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + e^{2x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 4x - 2\,e^{x}\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2\,e^{2x} \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{x}$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{2\,e^{-x}\left(\int (-2x+e^{x}) e^{x} dx\right)}{3} - \frac{2\,e^{2x}\left(\int (-2x+e^{x}) e^{-2x} dx\right)}{3}$
- ○ Compute integrals

  $y_p(x) = e^{x} + 1 - 2x$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + e^{2x} c_2 + e^{x} + 1 - 2x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=4*x-2*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + c_1 e^{2x} + e^x - 2x + 1$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 29

```
DSolve[y''[x]-y'[x]-2*y[x]==4*x-2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x + e^x + c_1 e^{-x} + c_2 e^{2x} + 1$$

17.10 problem 560

Internal problem ID [15329]
Internal file name [`OUTPUT/15329_Wednesday_May_08_2024_03_55_58_PM_14319520/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 560.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 3y' = 18x - 10\cos(x)}$$

17.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 0, f(x) = 18x - 10\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 0$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(0)} \\
&= \frac{3}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{3}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 0
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(0)x}
\end{aligned}$$

Or

$$y = c_1 e^{3x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18x - 10\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{3x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 - A_3 \cos(x) - A_4 \sin(x) - 6A_2 x - 3A_1 + 3A_3 \sin(x) - 3A_4 \cos(x) = 18x - 10\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -3, A_3 = 1, A_4 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3x^2 - 2x + \cos(x) + 3\sin(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{3x} + c_2\right) + \left(-3x^2 - 2x + \cos(x) + 3\sin(x)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{3x} + c_2 - 3x^2 - 2x + \cos(x) + 3\sin(x) \tag{1}
$$

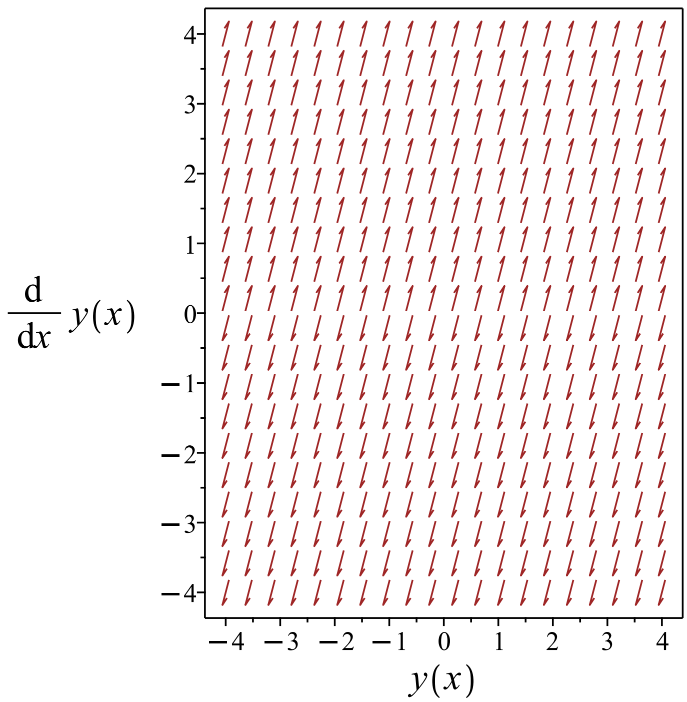


Figure 653: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{3x} + c_2 - 3x^2 - 2x + \cos(x) + 3\sin(x)
$$

Verified OK.

17.10.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = \int (18x - 10\cos(x))\, dx$$

$$-3y + y' = 9x^2 - 10\sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 9x^2 - 10\sin(x) + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = 9x^2 - 10\sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(9x^2 - 10\sin(x) + c_1\right) \\ \frac{d}{dx}\left(e^{-3x} y\right) &= \left(e^{-3x}\right)\left(9x^2 - 10\sin(x) + c_1\right) \\ d\left(e^{-3x} y\right) &= \left(\left(9x^2 - 10\sin(x) + c_1\right) e^{-3x}\right)\, dx \end{aligned}$$

Integrating gives

$$e^{-3x} y = \int \left(9x^2 - 10\sin(x) + c_1\right) e^{-3x}\, dx$$

$$e^{-3x} y = -3x^2 e^{-3x} - 2x\, e^{-3x} - \frac{2\, e^{-3x}}{3} + e^{-3x}\cos(x) + 3\, e^{-3x}\sin(x) - \frac{c_1 e^{-3x}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-3x^2 e^{-3x} - 2x\, e^{-3x} - \frac{2\, e^{-3x}}{3} + e^{-3x}\cos(x) + 3\, e^{-3x}\sin(x) - \frac{c_1 e^{-3x}}{3}\right) + c_2 e^{3x}$$

which simplifies to

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x) \tag{1}$$

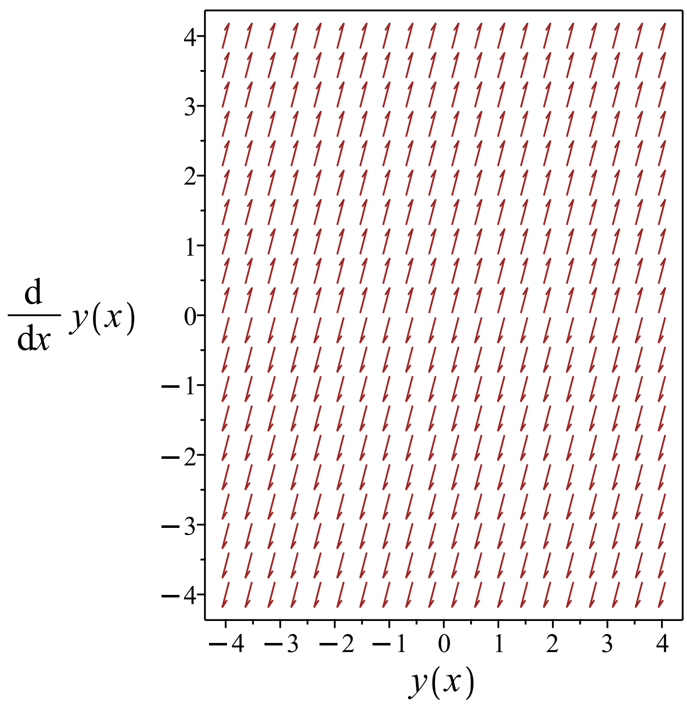


Figure 654: Slope field plot

Verification of solutions

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x)$$

Verified OK.

17.10.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 3p(x) - 18x + 10\cos(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 18x - 10\cos(x) \end{aligned}$$

Hence the ode is

$$p'(x) - 3p(x) = 18x - 10\cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\,(18x - 10\cos(x)) \\ \frac{d}{dx}\left(e^{-3x}p\right) &= \left(e^{-3x}\right)(18x - 10\cos(x)) \\ d\left(e^{-3x}p\right) &= \left((18x - 10\cos(x))\,e^{-3x}\right)\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-3x}p &= \int (18x - 10\cos(x))\,e^{-3x}\,dx \\ e^{-3x}p &= -6x\,e^{-3x} - 2\,e^{-3x} + 3\,e^{-3x}\cos(x) - e^{-3x}\sin(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$p(x) = e^{3x}\left(-6x\,e^{-3x} - 2\,e^{-3x} + 3\,e^{-3x}\cos(x) - e^{-3x}\sin(x)\right) + c_1 e^{3x}$$

which simplifies to

$$p(x) = c_1 e^{3x} - 6x - \sin(x) + 3\cos(x) - 2$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 e^{3x} - 6x - \sin(x) + 3\cos(x) - 2$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 e^{3x} - 6x - \sin(x) + 3\cos(x) - 2\,dx \\ &= -2x + \frac{c_1 e^{3x}}{3} - 3x^2 + 3\sin(x) + \cos(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2x + \frac{c_1 e^{3x}}{3} - 3x^2 + 3\sin(x) + \cos(x) + c_2 \tag{1}$$

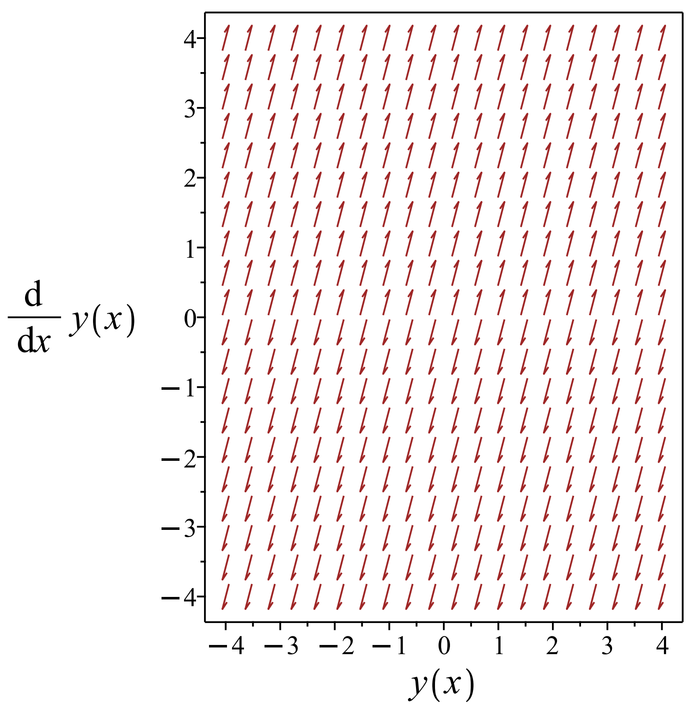


Figure 655: Slope field plot

Verification of solutions

$$y = -2x + \frac{c_1 e^{3x}}{3} - 3x^2 + 3\sin(x) + \cos(x) + c_2$$

Verified OK.

17.10.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 3y' = 18x - 10\cos(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = \int (18x - 10\cos(x))\, dx$$
$$-3y + y' = 9x^2 - 10\sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 9x^2 - 10\sin(x) + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = 9x^2 - 10\sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(9x^2 - 10\sin(x) + c_1\right) \\ \frac{d}{dx}\left(e^{-3x} y\right) &= \left(e^{-3x}\right)\left(9x^2 - 10\sin(x) + c_1\right) \\ d\left(e^{-3x} y\right) &= \left(\left(9x^2 - 10\sin(x) + c_1\right) e^{-3x}\right)\, dx \end{aligned}$$

Integrating gives

$$e^{-3x}y = \int \left(9x^2 - 10\sin(x) + c_1\right) e^{-3x}\, dx$$

$$e^{-3x}y = -3x^2 e^{-3x} - 2x\, e^{-3x} - \frac{2\, e^{-3x}}{3} + e^{-3x}\cos(x) + 3\, e^{-3x}\sin(x) - \frac{c_1 e^{-3x}}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-3x^2 e^{-3x} - 2x\, e^{-3x} - \frac{2\, e^{-3x}}{3} + e^{-3x}\cos(x) + 3\, e^{-3x}\sin(x) - \frac{c_1 e^{-3x}}{3}\right) + c_2 e^{3x}$$

which simplifies to

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x)$$

Summary
The solution(s) found are the following

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x) \tag{1}$$

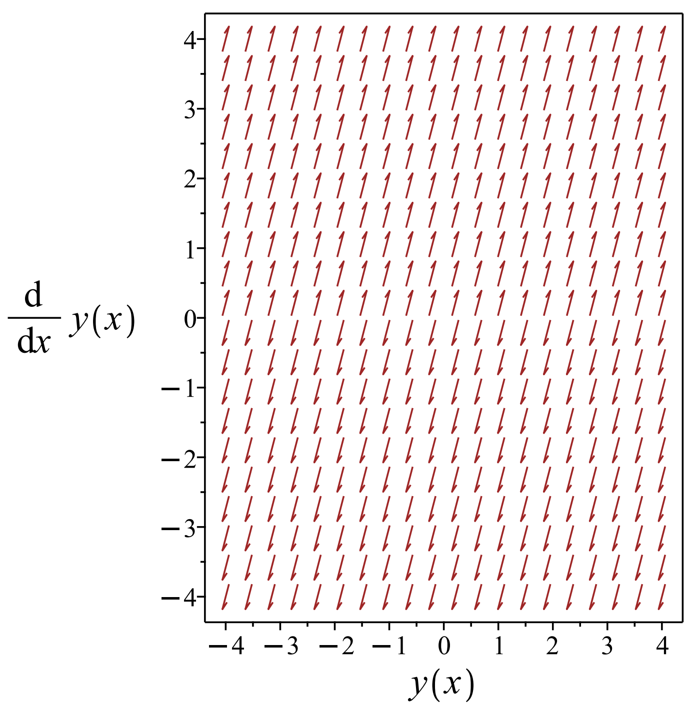


Figure 656: Slope field plot

Verification of solutions

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x)$$

Verified OK.

17.10.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 513: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 \mathrm{e}^{3x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18x - 10 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, \frac{\mathrm{e}^{3x}}{3}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 - A_3 \cos(x) - A_4 \sin(x) - 6A_2 x - 3A_1 + 3A_3 \sin(x) - 3A_4 \cos(x) = 18x - 10 \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -3, A_3 = 1, A_4 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3x^2 - 2x + \cos(x) + 3\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{3x}}{3}\right) + \left(-3x^2 - 2x + \cos(x) + 3\sin(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{3x}}{3} - 3x^2 - 2x + \cos(x) + 3\sin(x) \tag{1}$$

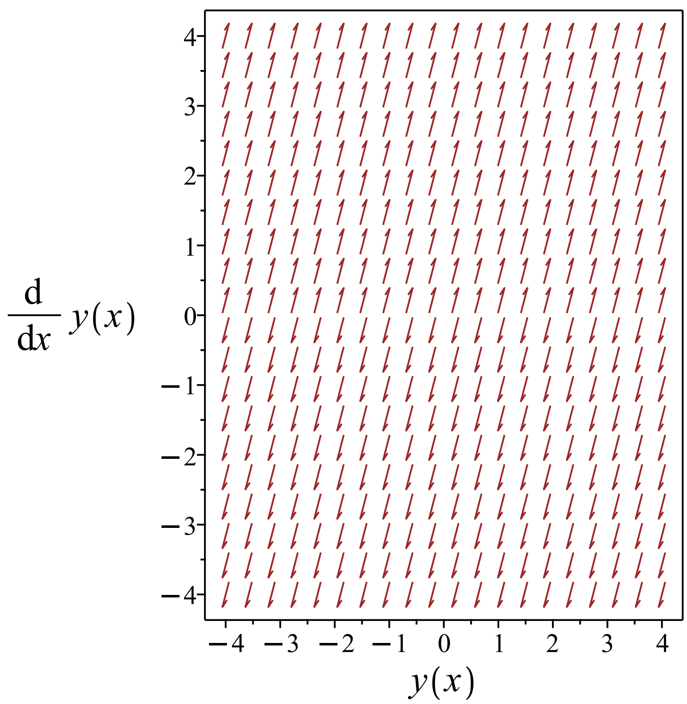


Figure 657: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{3x}}{3} - 3x^2 - 2x + \cos(x) + 3\sin(x)$$

Verified OK.

17.10.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -3 \\ r(x) &= 0 \\ s(x) &= 18x - 10\cos(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$-3y + y' = \int 18x - 10\cos(x)\, dx$$

We now have a first order ode to solve which is

$$-3y + y' = 9x^2 - 10\sin(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 9x^2 - 10\sin(x) + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = 9x^2 - 10\sin(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(9x^2 - 10\sin(x) + c_1\right) \\ \frac{d}{dx}\left(e^{-3x}y\right) &= \left(e^{-3x}\right)\left(9x^2 - 10\sin(x) + c_1\right) \\ d\left(e^{-3x}y\right) &= \left(\left(9x^2 - 10\sin(x) + c_1\right)e^{-3x}\right)\,dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-3x}y &= \int \left(9x^2 - 10\sin(x) + c_1\right)e^{-3x}\,dx \\ e^{-3x}y &= -3x^2e^{-3x} - 2x\,e^{-3x} - \frac{2\,e^{-3x}}{3} + e^{-3x}\cos(x) + 3\,e^{-3x}\sin(x) - \frac{c_1e^{-3x}}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-3x^2e^{-3x} - 2x\,e^{-3x} - \frac{2\,e^{-3x}}{3} + e^{-3x}\cos(x) + 3\,e^{-3x}\sin(x) - \frac{c_1e^{-3x}}{3}\right) + c_2e^{3x}$$

which simplifies to

$$y = -\frac{2}{3} + c_2e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{3} + c_2e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x) \tag{1}$$

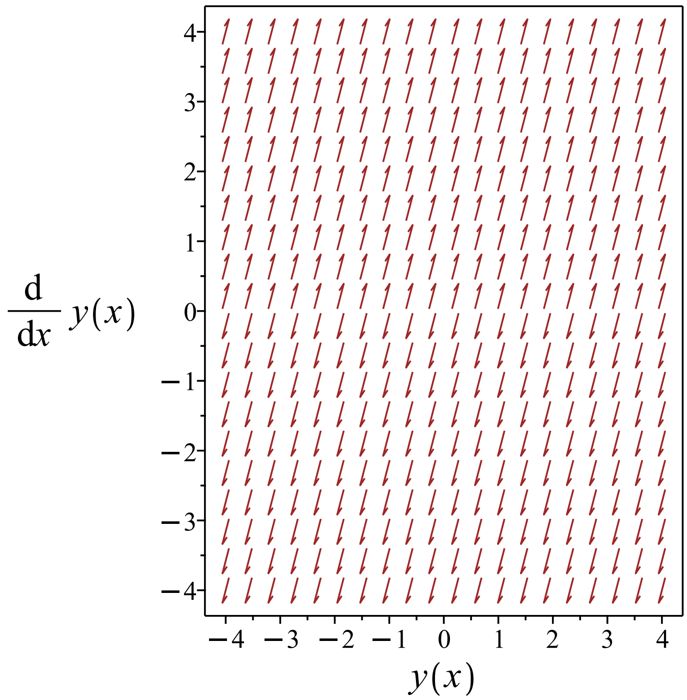


Figure 658: Slope field plot

Verification of solutions

$$y = -\frac{2}{3} + c_2 e^{3x} - 3x^2 - \frac{c_1}{3} - 2x + \cos(x) + 3\sin(x)$$

Verified OK.

17.10.7 Maple step by step solution

Let's solve

$y'' - 3y' = 18x - 10\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r = 0$

- Factor the characteristic polynomial

  $r(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (0, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 18x - 10\cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{3x} \\ 0 & 3\,e^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,e^{3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{2\left(\int (9x-5\cos(x)) dx\right)}{3} - \frac{2\,e^{3x}\left(\int (-9x+5\cos(x))e^{-3x} dx\right)}{3}$

  - Compute integrals

    $y_p(x) = -3x^2 + 3\sin(x) - 2x - \frac{2}{3} + \cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{3x} - 3x^2 + 3\sin(x) - 2x - \frac{2}{3} + \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 3*_b(_a)+18*_a-10*cos(_a), _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)=18*x-10*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{3x}}{3} - 3x^2 + 3\sin(x) + \cos(x) - 2x + c_2$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 33

```
DSolve[y''[x]-3*y'[x]==18*x-10*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3x^2 - 2x + 3\sin(x) + \cos(x) + \frac{1}{3}c_1 e^{3x} + c_2$$

17.11 problem 561

Internal problem ID [15330]
Internal file name [`OUTPUT/15330_Wednesday_May_08_2024_03_56_00_PM_33164822/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 561.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 2y' + y = 2 + e^{x} \sin(x)}$$

17.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 2 + e^{x} \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + x\, e^x c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 + e^x \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^x \cos(x) + A_3 e^x \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2 e^x \cos(x) - A_3 e^x \sin(x) + A_1 = 2 + e^x \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 - e^x \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + x e^x c_2) + (2 - e^x \sin(x)) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + 2 - e^x \sin(x)$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + 2 - e^x \sin(x) \tag{1}$$

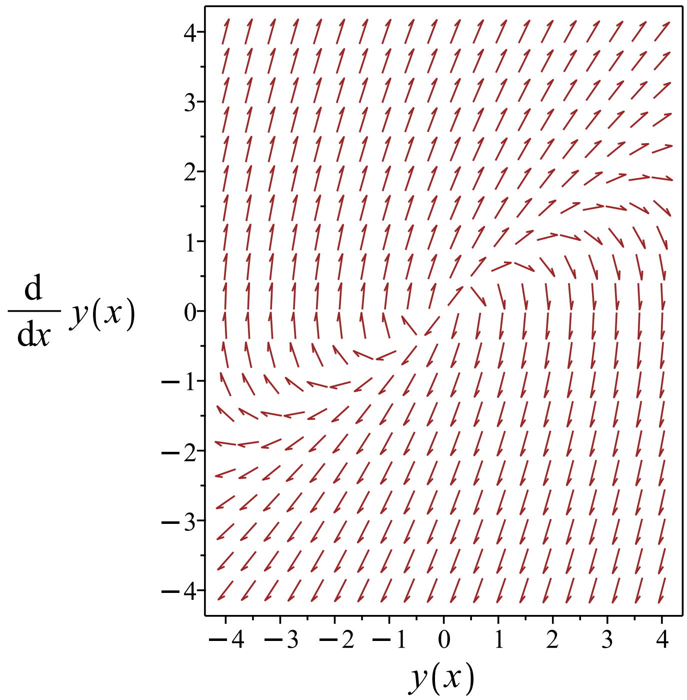


Figure 659: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + 2 - e^x \sin(x)$$

Verified OK.

17.11.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-x}(2+\mathrm{e}^{x}\sin(x)) \\
(\mathrm{e}^{-x}y)'' &= \mathrm{e}^{-x}(2+\mathrm{e}^{x}\sin(x))
\end{aligned}
$$

Integrating once gives

$$
(\mathrm{e}^{-x}y)' = -2\,\mathrm{e}^{-x}-\cos(x)+c_1
$$

Integrating again gives

$$
(\mathrm{e}^{-x}y) = c_1x-\sin(x)+2\,\mathrm{e}^{-x}+c_2
$$

Hence the solution is

$$
y = \frac{c_1x-\sin(x)+2\,\mathrm{e}^{-x}+c_2}{\mathrm{e}^{-x}}
$$

Or

$$
y = c_1x\,\mathrm{e}^{x}+\mathrm{e}^{x}c_2-\mathrm{e}^{x}\sin(x)+2
$$

Summary

The solution(s) found are the following

$$
y = c_1x\,\mathrm{e}^{x}+\mathrm{e}^{x}c_2-\mathrm{e}^{x}\sin(x)+2 \tag{1}
$$

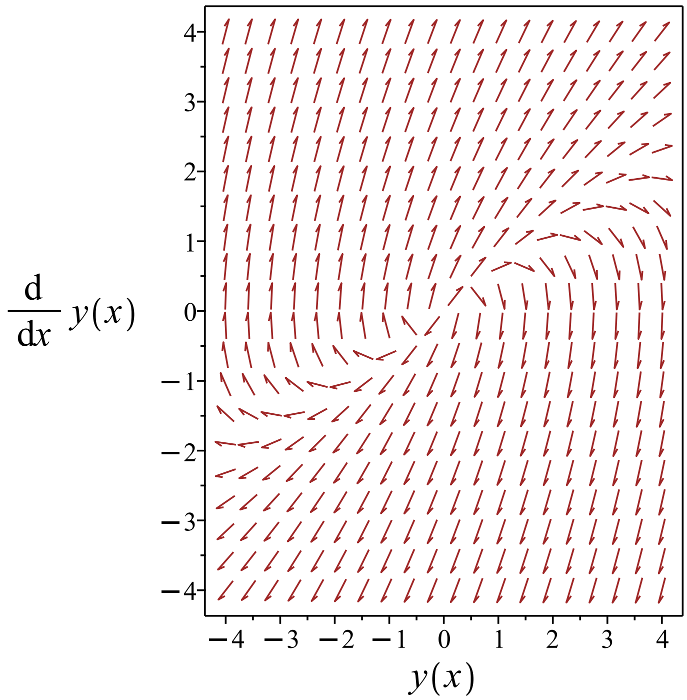


Figure 660: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{x} + \mathrm{e}^{x} c_2 - \mathrm{e}^{x} \sin(x) + 2$$

Verified OK.

17.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 515: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + x\,\mathrm{e}^x c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2 + \mathrm{e}^x \sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{\mathrm{e}^x \cos(x), \mathrm{e}^x \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^x, \mathrm{e}^x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 + A_2 \mathrm{e}^x \cos(x) + A_3 \mathrm{e}^x \sin(x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-A_2 \mathrm{e}^x \cos(x) - A_3 \mathrm{e}^x \sin(x) + A_1 = 2 + \mathrm{e}^x \sin(x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 - e^x \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + x\, e^x c_2) + (2 - e^x \sin(x)) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + 2 - e^x \sin(x)$$

Summary
The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + 2 - e^x \sin(x) \tag{1}$$

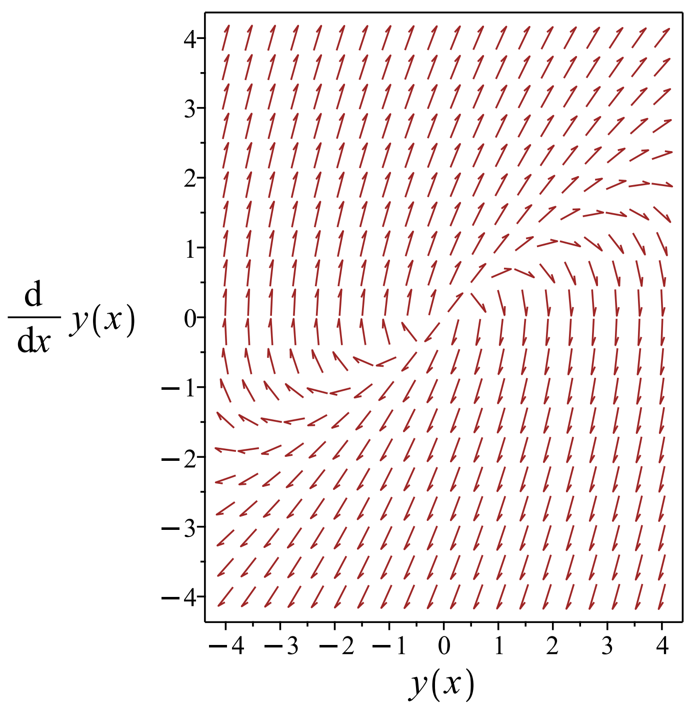


Figure 661: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + 2 - e^x \sin(x)$$

Verified OK.

17.11.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 2 + e^x \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + x\,e^x c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2 + e^x \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^x\left(-\left(\int x(2\,e^{-x} + \sin(x))\,dx\right) + x\left(\int (2\,e^{-x} + \sin(x))\,dx\right)\right)$
- Compute integrals

  $y_p(x) = 2 - e^x \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = x\,e^x c_2 - e^x \sin(x) + e^x c_1 + 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2+exp(x)*sin(x),y(x), singsol=all)
```

$$y(x) = 2 + (c_1 x + c_2 - \sin(x))\,e^x$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 25

```
DSolve[y''[x]-2*y'[x]+y[x]==2+Exp[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^x \sin(x) + e^x(c_2 x + c_1) + 2$$

17.12 problem 562

Internal problem ID [15331]
Internal file name [`OUTPUT/15331_Wednesday_May_08_2024_03_56_02_PM_68264857/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 562.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = (5x + 4)\,\mathrm{e}^{x} + \mathrm{e}^{-x}}$$

17.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = (5x + 4)\,\mathrm{e}^{x} + \mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(5x+4)\,\mathrm{e}^{x}+\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\},\{x\,\mathrm{e}^{x},\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}\cos(x),\mathrm{e}^{-x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-x}+A_2x\,\mathrm{e}^{x}+A_3\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^{-x}+4A_2\mathrm{e}^{x}+5A_2x\,\mathrm{e}^{x}+5A_3\mathrm{e}^{x}=(5x+4)\,\mathrm{e}^{x}+\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-x}+x\,\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(\mathrm{e}^{-x}(c_1\cos(x)+c_2\sin(x))\right)+\left(\mathrm{e}^{-x}+x\,\mathrm{e}^{x}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_1\cos(x)+c_2\sin(x))+\mathrm{e}^{-x}+x\,\mathrm{e}^{x} \tag{1}$$

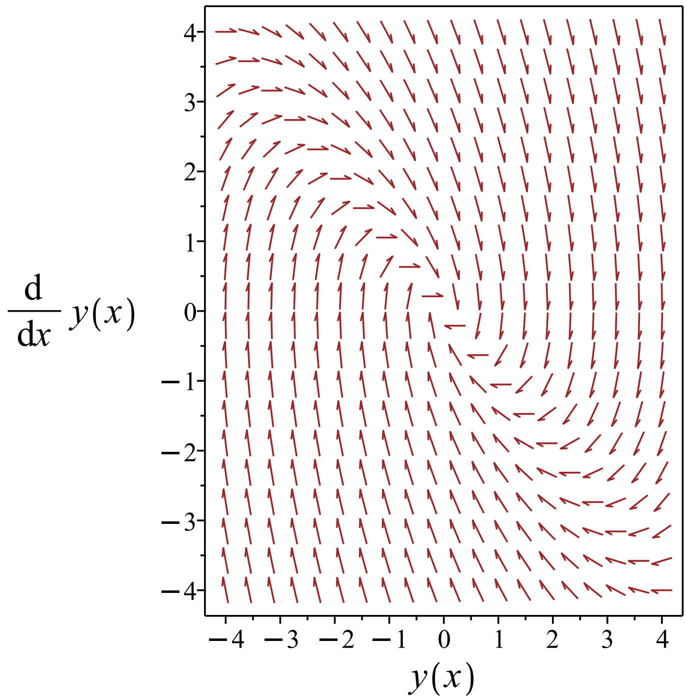


Figure 662: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-x} + x e^{x}$$

Verified OK.

17.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 517: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx\\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos\left(x\right)\right) + c_2 \left(\mathrm{e}^{-x} \cos\left(x\right) \left(\tan\left(x\right)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-x} \cos\left(x\right) c_1 + \mathrm{e}^{-x} \sin\left(x\right) c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
(5x + 4)\,\mathrm{e}^{x} + \mathrm{e}^{-x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-x}\}, \{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-x} \cos\left(x\right), \mathrm{e}^{-x} \sin\left(x\right)\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-x} + A_2 x\,\mathrm{e}^{x} + A_3 \mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 \mathrm{e}^{-x} + 4A_2 \mathrm{e}^{x} + 5A_2 x\,\mathrm{e}^{x} + 5A_3 \mathrm{e}^{x} = (5x + 4)\,\mathrm{e}^{x} + \mathrm{e}^{-x}
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-x} + x\,e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}\cos(x)\,c_1 + e^{-x}\sin(x)\,c_2\right) + \left(e^{-x} + x\,e^{x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_1\cos(x) + c_2\sin(x)) + e^{-x} + x\,e^{x}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_1\cos(x) + c_2\sin(x)) + e^{-x} + x\,e^{x} \tag{1}$$

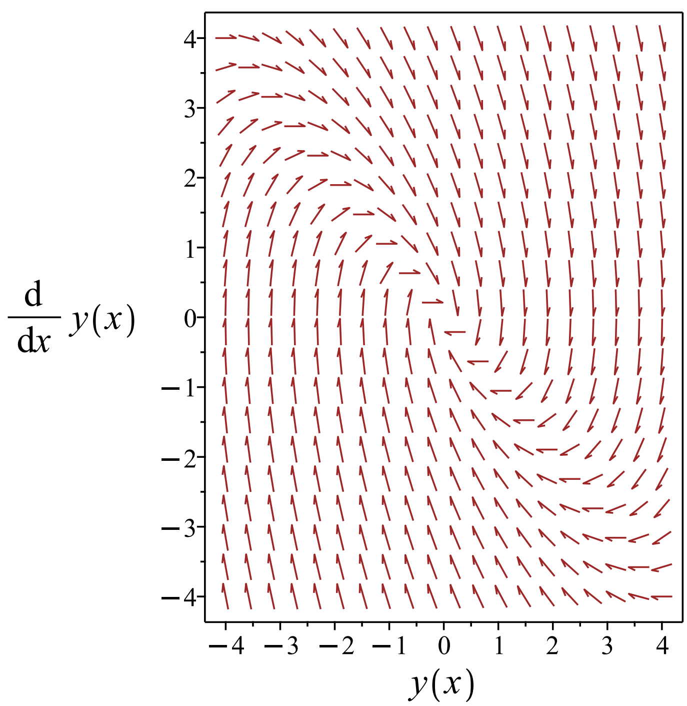


Figure 663: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-x} + x e^x$$

Verified OK.

17.12.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = (5x + 4) e^x + e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 5x e^x - 2y' - 2y + 4 e^x + e^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + 2y = 5x e^x + 4 e^x + e^{-x}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - I, -1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-x} \cos(x) c_1 + e^{-x} \sin(x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5x\,\mathrm{e}^{x} + 4\,\mathrm{e}^{x} + \mathrm{e}^{-x}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x}\cos(x) & \mathrm{e}^{-x}\sin(x) \\ -\mathrm{e}^{-x}\cos(x) - \mathrm{e}^{-x}\sin(x) & -\mathrm{e}^{-x}\sin(x) + \mathrm{e}^{-x}\cos(x) \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\mathrm{e}^{-x}\left(\cos(x)\left(\int ((5x+4)\sin(x)\,\mathrm{e}^{2x} + \sin(x))\,dx\right) - \sin(x)\left(\int ((5x+4)\cos(x)\,\mathrm{e}^{2x} + \mathrm{co}\right.\right.$$

- Compute integrals

  $$y_p(x) = \mathrm{e}^{-x} + x\,\mathrm{e}^{x}$$

- Substitute particular solution into general solution to ODE

  $$y = \mathrm{e}^{-x}\cos(x)\,c_1 + \mathrm{e}^{-x}\sin(x)\,c_2 + x\,\mathrm{e}^{x} + \mathrm{e}^{-x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=(5*x+4)*exp(x)+exp(-x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\sin(x)\,c_2 + \mathrm{e}^{-x}\cos(x)\,c_1 + \mathrm{e}^{x}x + \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.193 (sec). Leaf size: 30

```
DSolve[y''[x]+2*y'[x]+2*y[x]==(5*x+4)*Exp[x]+Exp[-x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-x}\left(e^{2x}x + c_2 \cos(x) + c_1 \sin(x) + 1\right)$$

17.13 problem 563

Internal problem ID [15332]
Internal file name [`OUTPUT/15332_Wednesday_May_08_2024_03_56_03_PM_8777090/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 563.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 4\,\mathrm{e}^{-x} + 17\sin(2x)}$$

17.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 5, f(x) = 4\,\mathrm{e}^{-x} + 17\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x}+17\sin\left(2x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\},\{\cos\left(2x\right),\sin\left(2x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}\cos\left(2x\right),\mathrm{e}^{-x}\sin\left(2x\right)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-x}+A_2\cos\left(2x\right)+A_3\sin\left(2x\right)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{-x}+A_2\cos\left(2x\right)+A_3\sin\left(2x\right)-4A_2\sin\left(2x\right)+4A_3\cos\left(2x\right)=4\,\mathrm{e}^{-x}+17\sin\left(2x\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -4, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-x}-4\cos\left(2x\right)+\sin\left(2x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x}(c_1\cos\left(2x\right)+c_2\sin\left(2x\right))\right)+\left(\mathrm{e}^{-x}-4\cos\left(2x\right)+\sin\left(2x\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_1\cos\left(2x\right)+c_2\sin\left(2x\right))+\mathrm{e}^{-x}-4\cos\left(2x\right)+\sin\left(2x\right) \tag{1}$$

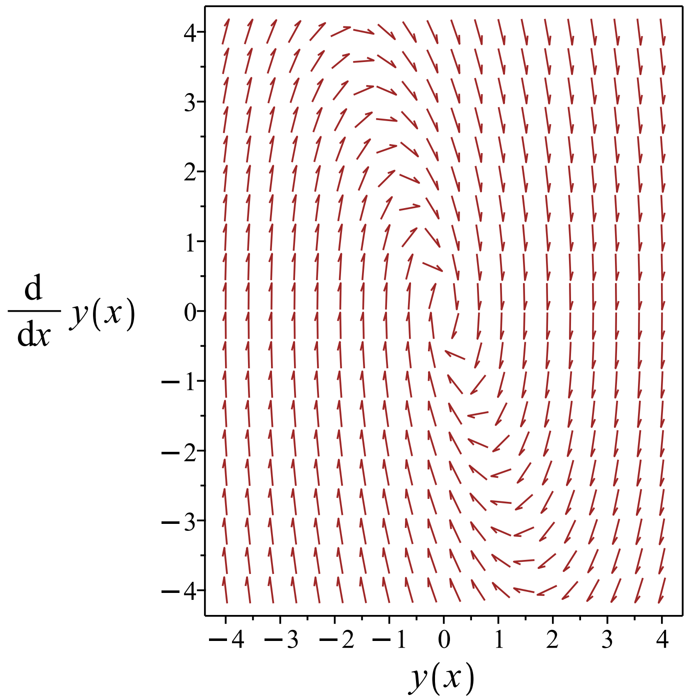


Figure 664: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{-x} - 4\cos(2x) + \sin(2x)$$

Verified OK.

17.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 519: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-x} \cos(2x)\, c_1 + \frac{\mathrm{e}^{-x} \sin(2x)\, c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x} + 17 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-x} \cos(2x), \frac{\mathrm{e}^{-x} \sin(2x)}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-x} + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{-x} + A_2 \cos(2x) + A_3 \sin(2x) - 4A_2 \sin(2x) + 4A_3 \cos(2x) = 4\,e^{-x} + 17\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -4, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-x} - 4\cos(2x) + \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x} \cos(2x)\, c_1 + \frac{e^{-x} \sin(2x)\, c_2}{2}\right) + \left(e^{-x} - 4\cos(2x) + \sin(2x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x} \cos(2x)\, c_1 + \frac{e^{-x} \sin(2x)\, c_2}{2} + e^{-x} - 4\cos(2x) + \sin(2x) \tag{1}$$

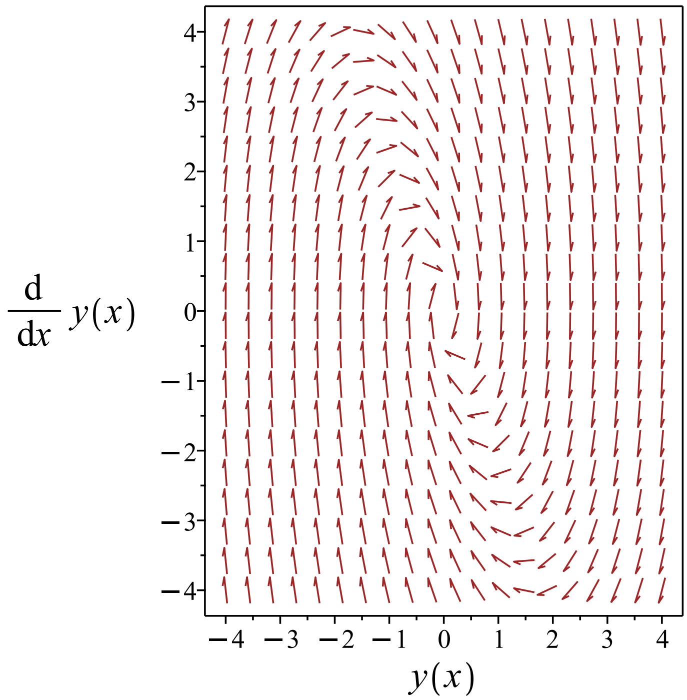


Figure 665: Slope field plot

Verification of solutions

$$y = e^{-x}\cos(2x)\,c_1 + \frac{e^{-x}\sin(2x)\,c_2}{2} + e^{-x} - 4\cos(2x) + \sin(2x)$$

Verified OK.

17.13.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = 4\,e^{-x} + 17\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-1-2\mathrm{I}, -1+2\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}\cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}\sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x}\cos(2x)\, c_1 + \mathrm{e}^{-x}\sin(2x)\, c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\,\mathrm{e}^{-x} + 17\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x}\cos(2x) & \mathrm{e}^{-x}\sin(2x) \\ -\mathrm{e}^{-x}\cos(2x) - 2\,\mathrm{e}^{-x}\sin(2x) & -\mathrm{e}^{-x}\sin(2x) + 2\,\mathrm{e}^{-x}\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\mathrm{e}^{-x}\left(-\cos(2x)\left(\int\left(17\sin(2x)^2\mathrm{e}^{x} + 4\sin(2x)\right)dx\right) + \sin(2x)\left(\int \cos(2x)(17\,\mathrm{e}^{x}\sin(2x)+4)dx\right)\right)}{2}$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{-x} - 4\cos(2x) + \sin(2x)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x}\sin(2x)\, c_2 + \mathrm{e}^{-x}\cos(2x)\, c_1 + \mathrm{e}^{-x} + \sin(2x) - 4\cos(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=4*exp(-x)+17*sin(2*x),y(x), singsol=all)
```

$$y(x) = ((c_1 + 1)\cos(2x) + \sin(2x)\, c_2 + 1)\, e^{-x} - 4\cos(2x) + \sin(2x)$$

✓ Solution by Mathematica

Time used: 0.335 (sec). Leaf size: 37

```
DSolve[y''[x]+2*y'[x]+5*y[x]==4*Exp[-x]+17*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}((-4e^x + c_2)\cos(2x) + (e^x + c_1)\sin(2x) + 1)$$

17.14 problem 564

Internal problem ID [15333]
Internal file name [`OUTPUT/15333_Wednesday_May_08_2024_03_56_04_PM_35508675/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 564.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2y'' - 3y' - 2y = 5\,\mathrm{e}^{x} \cosh(x)}$$

17.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = -3, C = -2, f(x) = 5\,\mathrm{e}^{x} \cosh(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' - 3y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=2, B=-3, C=-2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\,\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 3\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=2, B=-3, C=-2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{-3^2 - (4)\,(2)\,(-2)} \\
&= \frac{3}{4} \pm \frac{5}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{4} + \frac{5}{4} \\
\lambda_2 &= \frac{3}{4} - \frac{5}{4}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -\frac{1}{2}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{\left(-\frac{1}{2}\right)x}
\end{aligned}$$

Or

$$y = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{-\frac{x}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{-\frac{x}{2}} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(e^{-\frac{x}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{-\frac{x}{2}} \\ 2\,e^{2x} & -\frac{e^{-\frac{x}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(-\frac{e^{-\frac{x}{2}}}{2}\right) - \left(e^{-\frac{x}{2}}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -\frac{5\,e^{2x} e^{-\frac{x}{2}}}{2}$$

Which simplifies to

$$W = -\frac{5\,\mathrm{e}^{\frac{3x}{2}}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{5\,\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{x}\cosh\left(x\right)}{-5\,\mathrm{e}^{\frac{3x}{2}}}dx$$

Which simplifies to

$$u_1 = -\int -\cosh\left(x\right)\mathrm{e}^{-x}dx$$

Hence

$$u_1 = \frac{x}{2}+\frac{\sinh\left(2x\right)}{4}-\frac{\cosh\left(2x\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{5\,\mathrm{e}^{2x}\mathrm{e}^{x}\cosh\left(x\right)}{-5\,\mathrm{e}^{\frac{3x}{2}}}dx$$

Which simplifies to

$$u_2 = \int -\cosh\left(x\right)\mathrm{e}^{\frac{3x}{2}}dx$$

Hence

$$u_2 = -\sinh\left(\frac{x}{2}\right)-\frac{\sinh\left(\frac{5x}{2}\right)}{5}-\cosh\left(\frac{x}{2}\right)-\frac{\cosh\left(\frac{5x}{2}\right)}{5}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= \left(\frac{x}{2}+\frac{\sinh\left(2x\right)}{4}-\frac{\cosh\left(2x\right)}{4}\right)\mathrm{e}^{2x} \\ &\quad+\left(-\sinh\left(\frac{x}{2}\right)-\frac{\sinh\left(\frac{5x}{2}\right)}{5}-\cosh\left(\frac{x}{2}\right)-\frac{\cosh\left(\frac{5x}{2}\right)}{5}\right)\mathrm{e}^{-\frac{x}{2}} \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{5}{4}+\frac{\left(-\sinh\left(\frac{5x}{2}\right)-\cosh\left(\frac{5x}{2}\right)\right)\mathrm{e}^{-\frac{x}{2}}}{5}+\frac{x\,\mathrm{e}^{2x}}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{2x} c_1 + c_2 e^{-\frac{x}{2}}\right) + \left(-\frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\, e^{2x}}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{2x} c_1 + c_2 e^{-\frac{x}{2}} - \frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\, e^{2x}}{2} \tag{1}
$$

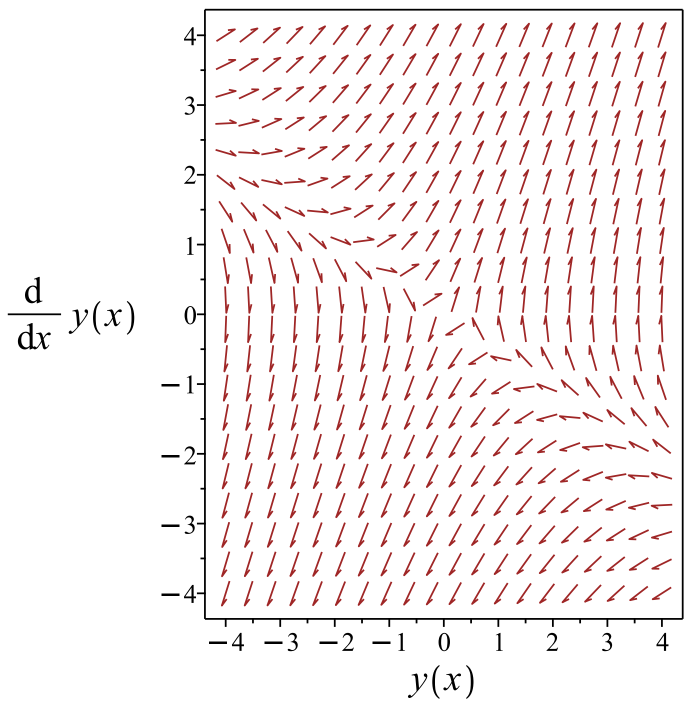


Figure 666: Slope field plot

Verification of solutions

$$
y = e^{2x} c_1 + c_2 e^{-\frac{x}{2}} - \frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\, e^{2x}}{2}
$$

Verified OK.

17.14.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 3y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -3 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 521: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{4}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{2}\,dx}
\end{aligned}
$$

$$= z_1 e^{\frac{3x}{4}}$$
$$= z_1 \left(\mathrm{e}^{\frac{3x}{4}}\right)$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-3}{2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{\frac{3x}{2}}}{(y_1)^2}\, dx$$
$$= y_1 \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)\right)$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$2y'' - 3y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{2\,\mathrm{e}^{2x} c_2}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$
$$y_2 = \frac{2\,\mathrm{e}^{2x}}{5}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x}{2}} & \frac{2\,\mathrm{e}^{2x}}{5} \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{x}{2}}\right) & \frac{d}{dx}\left(\frac{2\,\mathrm{e}^{2x}}{5}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x}{2}} & \frac{2\,\mathrm{e}^{2x}}{5} \\ -\frac{\mathrm{e}^{-\frac{x}{2}}}{2} & \frac{4\,\mathrm{e}^{2x}}{5} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-\frac{x}{2}}\right)\left(\frac{4\,\mathrm{e}^{2x}}{5}\right) - \left(\frac{2\,\mathrm{e}^{2x}}{5}\right)\left(-\frac{\mathrm{e}^{-\frac{x}{2}}}{2}\right)$$

Which simplifies to

$$W = \mathrm{e}^{2x}\mathrm{e}^{-\frac{x}{2}}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{3x}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{2x}\mathrm{e}^{x}\cosh(x)}{2\,\mathrm{e}^{\frac{3x}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int \cosh(x)\,\mathrm{e}^{\frac{3x}{2}}\,dx$$

Hence

$$u_1 = -\sinh\left(\frac{x}{2}\right) - \frac{\sinh\left(\frac{5x}{2}\right)}{5} - \cosh\left(\frac{x}{2}\right) - \frac{\cosh\left(\frac{5x}{2}\right)}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{5\,\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{x}\cosh(x)}{2\,\mathrm{e}^{\frac{3x}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{5\cosh(x)\,\mathrm{e}^{-x}}{2}\,dx$$

Hence

$$u_2 = \frac{5x}{4} + \frac{5\sinh(2x)}{8} - \frac{5\cosh(2x)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sinh\left(\frac{x}{2}\right) - \frac{\sinh\left(\frac{5x}{2}\right)}{5} - \cosh\left(\frac{x}{2}\right) - \frac{\cosh\left(\frac{5x}{2}\right)}{5}\right) e^{-\frac{x}{2}} + \frac{2\left(\frac{5x}{4} + \frac{5\sinh(2x)}{8} - \frac{5\cosh(2x)}{8}\right) e^{2x}}{5}$$

Which simplifies to

$$y_p(x) = -\frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\,e^{2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} + \frac{2\,e^{2x} c_2}{5}\right) + \left(-\frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\,e^{2x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + \frac{2\,e^{2x} c_2}{5} - \frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\,e^{2x}}{2} \tag{1}$$

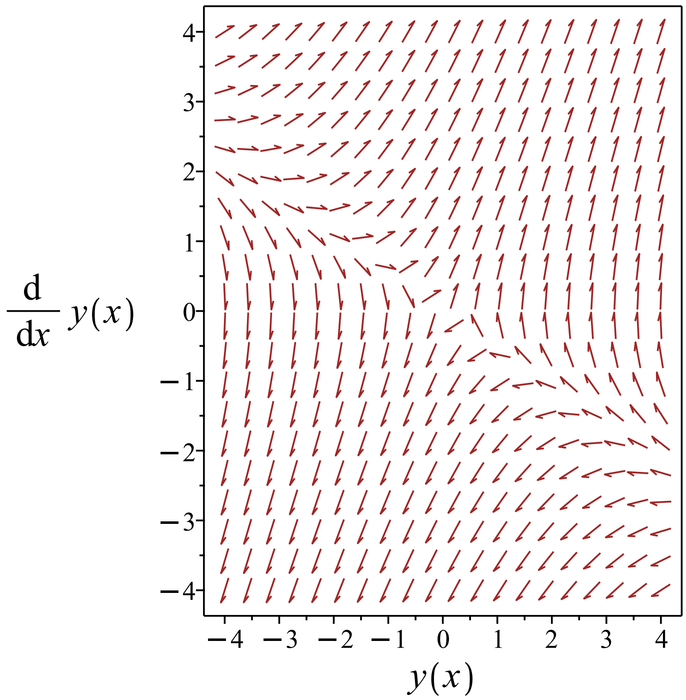


Figure 667: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + \frac{2\,e^{2x} c_2}{5} - \frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\,e^{2x}}{2}$$

Verified OK.

17.14.3 Maple step by step solution

Let's solve

$2y'' - 3y' - 2y = 5\,e^{x} \cosh(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2} + y + \frac{5\,e^{x}\cosh(x)}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2} - y = \frac{5\,e^{x}\cosh(x)}{2}$

- Characteristic polynomial of homogeneous ODE

$r^2 - \frac{3}{2}r - 1 = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)(r-2)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(2, -\frac{1}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + c_2 e^{-\frac{x}{2}} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{5\, e^{x} \cosh(x)}{2}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{-\frac{x}{2}} \\ 2\, e^{2x} & -\frac{e^{-\frac{x}{2}}}{2} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = -\frac{5\, e^{\frac{3x}{2}}}{2}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{2x}\left(\int \cosh(x)\, e^{-x} dx\right) - e^{-\frac{x}{2}}\left(\int \cosh(x)\, e^{\frac{3x}{2}} dx\right)$

- ○ Compute integrals

  $y_p(x) = -\frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\, e^{2x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = e^{2x} c_1 + c_2 e^{-\frac{x}{2}} - \frac{5}{4} + \frac{\left(-\sinh\left(\frac{5x}{2}\right) - \cosh\left(\frac{5x}{2}\right)\right) e^{-\frac{x}{2}}}{5} + \frac{x\, e^{2x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(2*diff(y(x),x$2)-3*diff(y(x),x)-2*y(x)=5*exp(x)*cosh(x),y(x), singsol=all)
```

$$y(x) = -\frac{5}{4} + \mathrm{e}^{-\frac{x}{2}} c_2 + \frac{(-2 + 5x + 10c_1)\,\mathrm{e}^{2x}}{10}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 36

```
DSolve[2*y''[x]-3*y'[x]-2*y[x]==5*Exp[x]*Cosh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/2} + e^{2x}\left(\frac{x}{2} - \frac{1}{5} + c_2\right) - \frac{5}{4}$$

17.15 problem 565

Internal problem ID [15334]
Internal file name [`OUTPUT/15334_Wednesday_May_08_2024_03_56_06_PM_28483645/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 565.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = x \sin(x)^2}$$

17.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = x \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos{(2x)} + c_2 \sin{(2x)})$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x\}, \{x \cos(2x), x \sin(2x), \cos(2x) x^2, \sin(2x) x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x + A_1 + A_3 x \cos(2x) + A_4 x \sin(2x) + A_5 \cos(2x) x^2 + A_6 \sin(2x) x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3 \sin(2x) + 4A_4 \cos(2x) - 8A_5 \sin(2x) x + 2A_5 \cos(2x) + 8A_6 \cos(2x) x + 2A_6 \sin(2x) + 4A_2 x + 4A_1 = x \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{8}, A_3 = -\frac{1}{32}, A_4 = 0, A_5 = 0, A_6 = -\frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2x) + c_2\sin(2x)) + \left(\frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16} \tag{1}$$

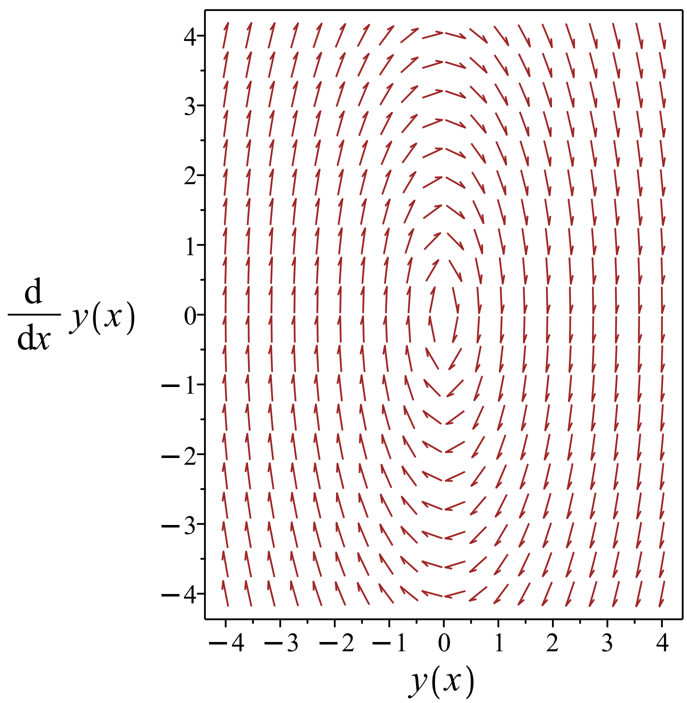


Figure 668: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16}$$

Verified OK.

17.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 523: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x\}, \{x \cos(2x), x \sin(2x), \cos(2x) x^2, \sin(2x) x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x + A_1 + A_3 x \cos(2x) + A_4 x \sin(2x) + A_5 \cos(2x) x^2 + A_6 \sin(2x) x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3 \sin(2x) + 4A_4 \cos(2x) - 8A_5 \sin(2x) x + 2A_5 \cos(2x) + 8A_6 \cos(2x) x + 2A_6 \sin(2x) + 4A_2 x + 4A_1 = x \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{8}, A_3 = -\frac{1}{32}, A_4 = 0, A_5 = 0, A_6 = -\frac{1}{16}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16} \tag{1}$$

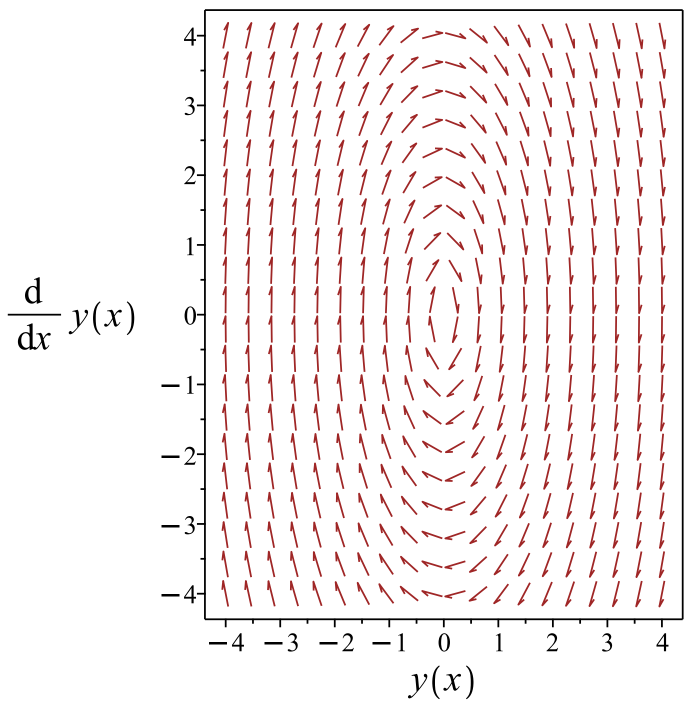


Figure 669: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{x}{8} - \frac{x\cos(2x)}{32} - \frac{\sin(2x)\,x^2}{16}$$

Verified OK.

17.15.3 Maple step by step solution

Let's solve

$y'' + 4y = x \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x \sin(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)x\sin(x)^2dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)x\sin(x)^2dx\right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{\sin(2x)x^2}{16} + \frac{\sin(2x)}{128} - \frac{x\cos(2x)}{32} + \frac{x}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) - \frac{\sin(2x)x^2}{16} + \frac{\sin(2x)}{128} - \frac{x\cos(2x)}{32} + \frac{x}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+4*y(x)=x*sin(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(-8x^2 + 128c_2 + 1)\sin(2x)}{128} + \frac{(-x + 32c_1)\cos(2x)}{32} + \frac{x}{8}$$

✓ Solution by Mathematica

Time used: 0.245 (sec). Leaf size: 41

```
DSolve[y''[x]+4*y[x]==x*Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{128}\left((-8x^2 + 1 + 128c_2)\sin(2x) + 16x - 4(x - 32c_1)\cos(2x)\right)$$

17.16 problem 566

Internal problem ID [15335]
Internal file name [`OUTPUT/15335_Wednesday_May_08_2024_03_56_07_PM_42007306/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 566.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2y''' + 2y'' + 2y' + y = x\,e^{x} + \frac{\cos(x)}{2}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' + 2y'' + 2y' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} + \mathrm{e}^{ix} c_3 + \mathrm{e}^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= x \,\mathrm{e}^{-x} \\ y_3 &= \mathrm{e}^{ix} \\ y_4 &= \mathrm{e}^{-ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' + 2y'' + 2y' + y = x \,\mathrm{e}^{x} + \frac{\cos(x)}{2}$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & x\,e^{-x} & e^{ix} & e^{-ix} \\ -e^{-x} & e^{-x}(1-x) & ie^{ix} & -ie^{-ix} \\ e^{-x} & e^{-x}(x-2) & -e^{ix} & -e^{-ix} \\ -e^{-x} & e^{-x}(3-x) & -ie^{ix} & ie^{-ix} \end{bmatrix}$$
$$|W| = -8ie^{-2x}e^{-ix}e^{ix}$$

The determinant simplifies to

$$|W| = -8ie^{-2x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x\,e^{-x} & e^{ix} & e^{-ix} \\ e^{-x}(1-x) & ie^{ix} & -ie^{-ix} \\ e^{-x}(x-2) & -e^{ix} & -e^{-ix} \end{bmatrix}$$
$$= -4ie^{-x}(x-1)$$

$$W_2(x) = \det \begin{bmatrix} e^{-x} & e^{ix} & e^{-ix} \\ -e^{-x} & ie^{ix} & -ie^{-ix} \\ e^{-x} & -e^{ix} & -e^{-ix} \end{bmatrix}$$
$$= -4ie^{-x}$$

$$W_3(x) = \det \begin{bmatrix} e^{-x} & x\,e^{-x} & e^{-ix} \\ -e^{-x} & e^{-x}(1-x) & -ie^{-ix} \\ e^{-x} & e^{-x}(x-2) & -e^{-ix} \end{bmatrix}$$
$$= -2ie^{(-2-i)x}$$

$$W_4(x) = \det \begin{bmatrix} e^{-x} & x\,e^{-x} & e^{ix} \\ -e^{-x} & e^{-x}(1-x) & ie^{ix} \\ e^{-x} & e^{-x}(x-2) & -e^{ix} \end{bmatrix}$$
$$= 2ie^{(-2+i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^3 \int \frac{\left(x\,\mathrm{e}^x + \frac{\cos(x)}{2}\right)\left(-4i\mathrm{e}^{-x}(x-1)\right)}{(1)\left(-8i\mathrm{e}^{-2x}\right)}\, dx \\
&= -\int \frac{-4i\left(x\,\mathrm{e}^x + \frac{\cos(x)}{2}\right)\mathrm{e}^{-x}(x-1)}{-8i\mathrm{e}^{-2x}}\, dx \\
&= -\int \left(\frac{(x-1)\left(2x\,\mathrm{e}^x + \cos(x)\right)\mathrm{e}^x}{4}\right) dx \\
&= -\frac{x^2\mathrm{e}^{2x}}{4} + \frac{x\,\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4} - \frac{x\,\mathrm{e}^x\cos(x)}{8} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^x\sin(x)}{4} + \frac{\mathrm{e}^x\cos(x)}{8} + \frac{\mathrm{e}^x\sin(x)}{8} \\
&= -\frac{x^2\mathrm{e}^{2x}}{4} + \frac{x\,\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4} - \frac{x\,\mathrm{e}^x\cos(x)}{8} + \frac{\left(-\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^x\sin(x)}{4} + \frac{\mathrm{e}^x\cos(x)}{8} + \frac{\mathrm{e}^x\sin(x)}{8}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{\left(x\,\mathrm{e}^x + \frac{\cos(x)}{2}\right)\left(-4i\mathrm{e}^{-x}\right)}{(1)\left(-8i\mathrm{e}^{-2x}\right)}\, dx \\
&= \int \frac{-4i\left(x\,\mathrm{e}^x + \frac{\cos(x)}{2}\right)\mathrm{e}^{-x}}{-8i\mathrm{e}^{-2x}}\, dx \\
&= \int \left(\frac{x\,\mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^x\cos(x)}{4}\right) dx \\
&= \frac{\mathrm{e}^x\cos(x)}{8} + \frac{\mathrm{e}^x\sin(x)}{8} + \frac{x\,\mathrm{e}^{2x}}{4} - \frac{\mathrm{e}^{2x}}{8} \\
&= \frac{\mathrm{e}^x\cos(x)}{8} + \frac{\mathrm{e}^x\sin(x)}{8} + \frac{x\,\mathrm{e}^{2x}}{4} - \frac{\mathrm{e}^{2x}}{8}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{1} \int \frac{\left(x\,\mathrm{e}^{x} + \frac{\cos(x)}{2}\right) \left(-2i \mathrm{e}^{(-2-i)x}\right)}{(1)\left(-8i \mathrm{e}^{-2x}\right)}\, dx \\
&= -\int \frac{-2i\left(x\,\mathrm{e}^{x} + \frac{\cos(x)}{2}\right) \mathrm{e}^{(-2-i)x}}{-8i \mathrm{e}^{-2x}}\, dx \\
&= -\int \left(\frac{(2x\,\mathrm{e}^{x} + \cos(x))\, \mathrm{e}^{-ix}}{8}\right) dx \\
&= -\left(\int \frac{(2x\,\mathrm{e}^{x} + \cos(x))\, \mathrm{e}^{-ix}}{8} dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{\left(x\,\mathrm{e}^{x} + \frac{\cos(x)}{2}\right) \left(2i \mathrm{e}^{(-2+i)x}\right)}{(1)\left(-8i \mathrm{e}^{-2x}\right)}\, dx \\
&= \int \frac{2i\left(x\,\mathrm{e}^{x} + \frac{\cos(x)}{2}\right) \mathrm{e}^{(-2+i)x}}{-8i \mathrm{e}^{-2x}}\, dx \\
&= \int \left(-\frac{(2x\,\mathrm{e}^{x} + \cos(x))\, \mathrm{e}^{ix}}{8}\right) dx \\
&= -\frac{x}{16} + \frac{i \mathrm{e}^{2ix}}{32} + \left(-\frac{1}{16} + \frac{i}{16}\right)(2x - 1 + i)\, \mathrm{e}^{(1+i)x} \\
&= -\frac{x}{16} + \frac{i \mathrm{e}^{2ix}}{32} + \left(-\frac{1}{16} + \frac{i}{16}\right)(2x - 1 + i)\, \mathrm{e}^{(1+i)x}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{x^2 \mathrm{e}^{2x}}{4} + \frac{x\,\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4} - \frac{x\,\mathrm{e}^{x} \cos(x)}{8} + \frac{\left(-\frac{x}{2} + \frac{1}{2}\right) \mathrm{e}^{x} \sin(x)}{4} + \frac{\mathrm{e}^{x} \cos(x)}{8} + \frac{\mathrm{e}^{x} \sin(x)}{8}\right) \left(\mathrm{e}^{-x}\right) \\
&+ \left(\frac{\mathrm{e}^{x} \cos(x)}{8} + \frac{\mathrm{e}^{x} \sin(x)}{8} + \frac{x\,\mathrm{e}^{2x}}{4} - \frac{\mathrm{e}^{2x}}{8}\right) \left(x\,\mathrm{e}^{-x}\right) \\
&+ \left(-\left(\int \frac{(2x\,\mathrm{e}^{x} + \cos(x))\, \mathrm{e}^{-ix}}{8} dx\right)\right) \left(\mathrm{e}^{ix}\right) \\
&+ \left(-\frac{x}{16} + \frac{i \mathrm{e}^{2ix}}{32} + \left(-\frac{1}{16} + \frac{i}{16}\right)(2x - 1 + i)\, \mathrm{e}^{(1+i)x}\right) \left(\mathrm{e}^{-ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = \frac{x\,\mathrm{e}^{x}}{8}-\frac{i\sin\left(x\right)x}{16}-\frac{\cos\left(x\right)x}{16}+\frac{3\sin\left(x\right)}{16}+\frac{i\mathrm{e}^{ix}}{32}-\frac{x\,\mathrm{e}^{-ix}}{16}-\frac{\mathrm{e}^{x}}{4}+\frac{\cos\left(x\right)}{8}$$

Which simplifies to

$$y_p = \frac{\left(4+i-4x\right)\cos\left(x\right)}{32}+\frac{\left(x-2\right)\mathrm{e}^{x}}{8}+\frac{5\sin\left(x\right)}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x}+c_2 x\,\mathrm{e}^{-x}+\mathrm{e}^{ix}c_3+\mathrm{e}^{-ix}c_4\right)+\left(\frac{\left(4+i-4x\right)\cos\left(x\right)}{32}+\frac{\left(x-2\right)\mathrm{e}^{x}}{8}+\frac{5\sin\left(x\right)}{32}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-ix}c_4+\mathrm{e}^{ix}c_3+\mathrm{e}^{-x}\left(c_2x+c_1\right)+\frac{\left(4+i-4x\right)\cos\left(x\right)}{32}+\frac{\left(x-2\right)\mathrm{e}^{x}}{8}+\frac{5\sin\left(x\right)}{32}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-ix}c_4+\mathrm{e}^{ix}c_3+\mathrm{e}^{-x}\left(c_2x+c_1\right)+\frac{\left(4+i-4x\right)\cos\left(x\right)}{32}+\frac{\left(x-2\right)\mathrm{e}^{x}}{8}+\frac{5\sin\left(x\right)}{32} \quad (1)$$

Verification of solutions

$$y = \mathrm{e}^{-ix}c_4+\mathrm{e}^{ix}c_3+\mathrm{e}^{-x}\left(c_2x+c_1\right)+\frac{\left(4+i-4x\right)\cos\left(x\right)}{32}+\frac{\left(x-2\right)\mathrm{e}^{x}}{8}+\frac{5\sin\left(x\right)}{32}$$

Verified OK.

17.16.1 Maple step by step solution

Let's solve

$y'''' + 2y''' + 2y'' + 2y' + y = x\,\mathrm{e}^{x} + \frac{\cos(x)}{2}$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

$y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = x\,e^x + \frac{\cos(x)}{2} - 2y_4(x) - 2y_3(x) - 2y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = x\,e^x + \frac{\cos(x)}{2} - 2y_4(x) - 2y_3(x) - 2y_2(x) -\right.$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x\,e^x + \frac{\cos(x)}{2} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x\,e^x + \frac{\cos(x)}{2} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\, \mathrm{e}^{\lambda x} \left(x \vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x} \vec{v} = \mathrm{e}^{\lambda x} \left(\lambda x \vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -2 & -2 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = \mathrm{e}^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{-x}(-x-1) & -\sin(x) & -\cos(x) \\ e^{-x} & x\,e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & -x\,e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & x\,e^{-x} & \cos(x) & -\sin(x) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \mathrm{e}^{-x}(-x-1) & -\sin(x) & -\cos(x) \\ \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} & -\cos(x) & \sin(x) \\ -\mathrm{e}^{-x} & -x\,\mathrm{e}^{-x} & \sin(x) & \cos(x) \\ \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (x+1)\,\mathrm{e}^{-x} & x\,\mathrm{e}^{-x} - \frac{\cos(x)}{2} + \frac{\mathrm{e}^{-x}}{2} + \frac{\sin(x)}{2} & x\,\mathrm{e}^{-x} - \cos(x) + \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} - \frac{\cos(x)}{2} + \frac{\mathrm{e}^{-}}{2} \\ -x\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}}{2} - x\,\mathrm{e}^{-x} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & -x\,\mathrm{e}^{-x} + \sin(x) & \frac{\mathrm{e}^{-x}}{2} - x\,\mathrm{e}^{-x} - \frac{\cos(}{2} \\ x\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}}{2} + x\,\mathrm{e}^{-x} - \frac{\sin(x)}{2} + \frac{\cos(x)}{2} & x\,\mathrm{e}^{-x} + \cos(x) & -\frac{\mathrm{e}^{-x}}{2} + x\,\mathrm{e}^{-x} + \frac{\sin(}{2} \\ -x\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}}{2} - x\,\mathrm{e}^{-x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -x\,\mathrm{e}^{-x} - \sin(x) & \frac{\mathrm{e}^{-x}}{2} - x\,\mathrm{e}^{-x} + \frac{\cos(}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{e^{-x}}{4}+\frac{(-x-1)\cos(x)}{8}+\frac{(-x+4)\sin(x)}{8}-\frac{e^{x}}{8} \\ -\frac{e^{-x}}{4}+\frac{(3-x)\cos(x)}{8}+\frac{\sin(x)x}{8}-\frac{e^{x}}{8} \\ \frac{e^{-x}}{4}+\frac{(x+1)\cos(x)}{8}+\frac{(4x-3)e^{x}}{8}+\frac{\sin(x)x}{8} \\ -\frac{e^{-x}}{4}+\frac{\cos(x)(x-1)}{8}+\frac{(-x-2)\sin(x)}{8}+\frac{3\,e^{x}}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{e^{-x}}{4}+\frac{(-x-1)\cos(x)}{8}+\frac{(-x+4)\sin(x)}{8}-\frac{e^{x}}{8} \\ -\frac{e^{-x}}{4}+\frac{(3-x)\cos(x)}{8}+\frac{\sin(x)x}{8}-\frac{e^{x}}{8} \\ \frac{e^{-x}}{4}+\frac{(x+1)\cos(x)}{8}+\frac{(4x-3)e^{x}}{8}+\frac{\sin(x)x}{8} \\ -\frac{e^{-x}}{4}+\frac{\cos(x)(x-1)}{8}+\frac{(-x-2)\sin(x)}{8}+\frac{3\,e^{x}}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-8c_2x-8c_1-8c_2+2)e^{-x}}{8} + \frac{(-x-8c_4-1)\cos(x)}{8} + \frac{(-x-8c_3+4)\sin(x)}{8} - \frac{e^{x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+2*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=x*exp(x)+1/2*cos
```

$$y(x) = (c_4x + c_3)\,e^{-x} + \frac{(-x + 8c_1 + 1)\cos(x)}{8} + \frac{(x-2)\,e^{x}}{8} + \frac{\sin(x)\,(4c_2 + 1)}{4}$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 52

```
DSolve[y''''[x]+2*y'''[x]+2*y''[x]+2*y'[x]+y[x]==x*Exp[x]+1/2*Cos[x],y[x],x,IncludeSingularS
```

$$y(x) \to \frac{1}{16}\left(2e^{x}(x-2) + 16e^{-x}(c_4x + c_3) - 2(x - 1 - 8c_1)\cos(x) + (3 + 16c_2)\sin(x)\right)$$

17.17 problem 567

Internal problem ID [15336]
Internal file name [`OUTPUT/15336_Wednesday_May_08_2024_03_56_13_PM_29861703/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 567.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = \cos(x)^2 + e^x + x^2}$$

17.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \cos(x)^2 + e^x + x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=1, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x}+\lambda e^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{1^2-(4)(1)(0)}\\&=-\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{1}{2}\\\lambda_2&=-\frac{1}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)^2 + e^x + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 \cos(2x) + A_3 \sin(2x) + A_4 x + A_5 x^2 + A_6 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x - 4A_2 \cos(2x) - 4A_3 \sin(2x) + 2A_5 + 6A_6 x - 2A_2 \sin(2x) \\ + 2A_3 \cos(2x) + A_4 + 2A_5 x + 3A_6 x^2 = \cos(x)^2 + e^x + x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{10}, A_3 = \frac{1}{20}, A_4 = \frac{5}{2}, A_5 = -1, A_6 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + e^{-x} c_2) + \left(\frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-x} c_2 + \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3} \tag{1}$$

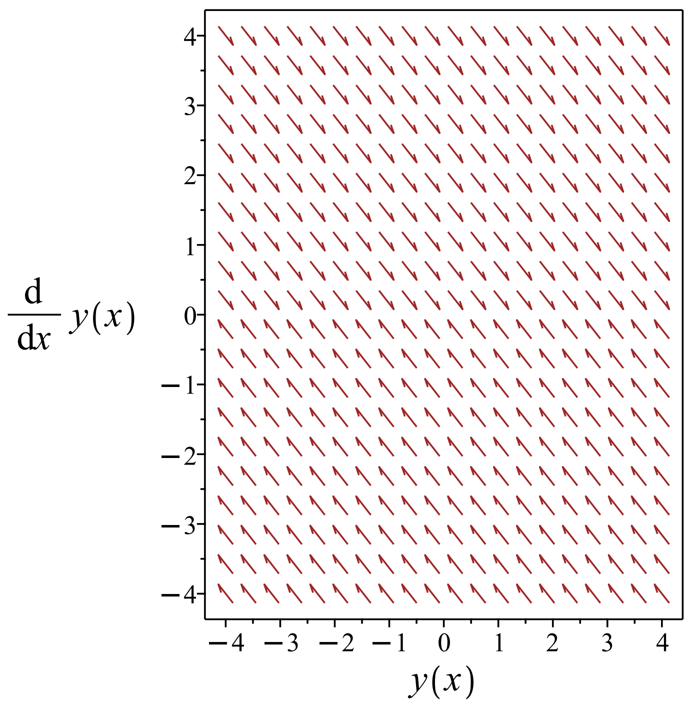


Figure 670: Slope field plot

Verification of solutions

$$y = c_1 + e^{-x} c_2 + \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3}$$

Verified OK.

17.17.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int \left(\cos(x)^2 + e^x + x^2\right) dx$$

$$y' + y = \frac{x^3}{3} + \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + e^x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1\right) \\ \frac{d}{dx}(e^x y) &= (e^x)\left(\frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1\right) \\ d(e^x y) &= \left(\frac{(4x^3 + 12\,e^x + 3\sin(2x) + 12c_1 + 6x)\,e^x}{12}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int \frac{(4x^3 + 12\,e^x + 3\sin(2x) + 12c_1 + 6x)\,e^x}{12}\, dx \\ e^x y &= \frac{x^3 e^x}{3} - e^x x^2 + \frac{5x\,e^x}{2} - \frac{5\,e^x}{2} + \frac{e^{2x}}{2} + \frac{e^x(\sin(2x) - 2\cos(2x))}{20} + e^x c_1 + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = \mathrm{e}^{-x}\left(\frac{x^{3}\mathrm{e}^{x}}{3} - \mathrm{e}^{x} x^{2} + \frac{5x\,\mathrm{e}^{x}}{2} - \frac{5\,\mathrm{e}^{x}}{2} + \frac{\mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{x}(\sin(2x) - 2\cos(2x))}{20} + \mathrm{e}^{x} c_1\right) + \mathrm{e}^{-x} c_2$$

which simplifies to

$$y = \frac{x^{3}}{3} - x^{2} + \frac{\mathrm{e}^{x}}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + \mathrm{e}^{-x} c_2$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^{3}}{3} - x^{2} + \frac{\mathrm{e}^{x}}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + \mathrm{e}^{-x} c_2 \tag{1}$$

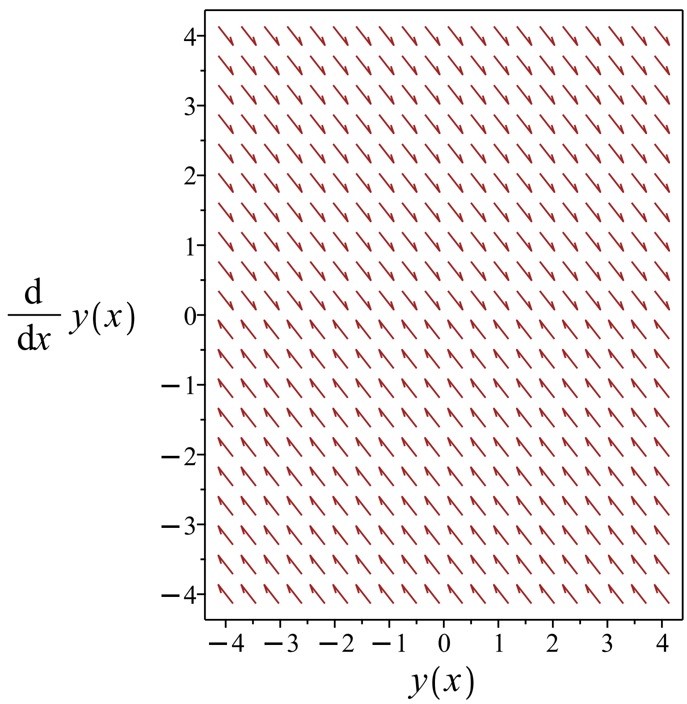


Figure 671: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^{3}}{3} - x^{2} + \frac{\mathrm{e}^{x}}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + \mathrm{e}^{-x} c_2$$

Verified OK.

17.17.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - \cos(x)^2 - e^x - x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \cos(x)^2 + e^x + x^2 \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = \cos(x)^2 + e^x + x^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(\cos(x)^2 + e^x + x^2\right) \\ \frac{d}{dx}(e^x p) &= (e^x)\left(\cos(x)^2 + e^x + x^2\right) \\ d(e^x p) &= \left(\left(\cos(x)^2 + e^x + x^2\right) e^x\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x p &= \int \left(\cos(x)^2 + e^x + x^2\right) e^x \, dx \\ e^x p &= \frac{(\cos(x) + 2\sin(x)) e^x \cos(x)}{5} + \frac{12 e^x}{5} + \frac{e^{2x}}{2} + e^x x^2 - 2x e^x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$p(x) = e^{-x}\left(\frac{(\cos(x) + 2\sin(x))\, e^{x} \cos(x)}{5} + \frac{12\, e^{x}}{5} + \frac{e^{2x}}{2} + e^{x} x^{2} - 2x\, e^{x}\right) + c_1 e^{-x}$$

which simplifies to

$$p(x) = x^{2} + \frac{e^{x}}{2} - 2x + \frac{\sin(2x)}{5} + \frac{\cos(2x)}{10} + \frac{5}{2} + c_1 e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x^{2} + \frac{e^{x}}{2} - 2x + \frac{\sin(2x)}{5} + \frac{\cos(2x)}{10} + \frac{5}{2} + c_1 e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int x^{2} + \frac{e^{x}}{2} - 2x + \frac{\sin(2x)}{5} + \frac{\cos(2x)}{10} + \frac{5}{2} + c_1 e^{-x}\, dx \\ &= -x^{2} + \frac{5x}{2} + \frac{x^{3}}{3} - c_1 e^{-x} + \frac{e^{x}}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x^{2} + \frac{5x}{2} + \frac{x^{3}}{3} - c_1 e^{-x} + \frac{e^{x}}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + c_2 \tag{1}$$

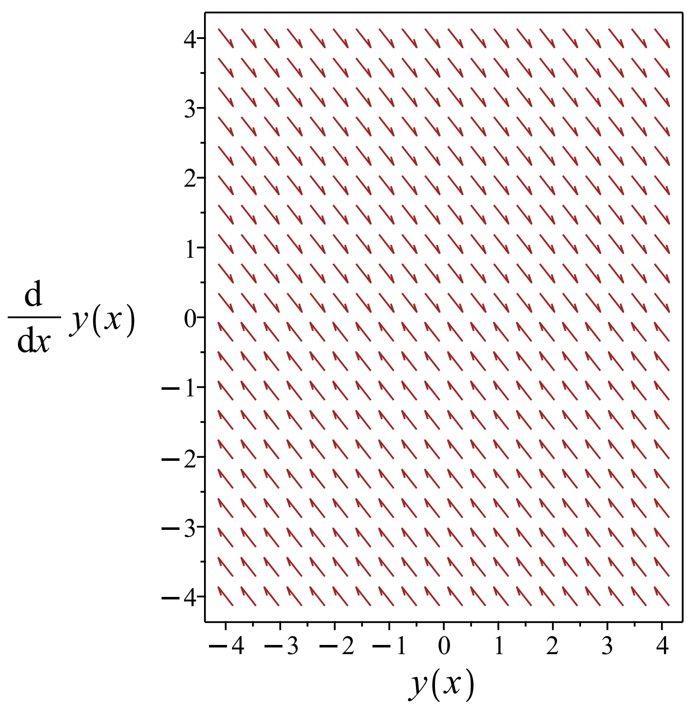


Figure 672: Slope field plot

Verification of solutions

$$y = -x^2 + \frac{5x}{2} + \frac{x^3}{3} - c_1 \mathrm{e}^{-x} + \frac{\mathrm{e}^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + c_2$$

Verified OK.

17.17.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = \cos(x)^2 + \mathrm{e}^x + x^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\, dx = \int (\cos(x)^2 + \mathrm{e}^x + x^2)\, dx$$

$$y' + y = \frac{x^3}{3} + \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + \mathrm{e}^x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + \mathrm{e}^x + c_1$$

Hence the ode is

$$y' + y = \frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + \mathrm{e}^x + c_1$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int 1 dx}$$
$$= \mathrm{e}^x$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + \mathrm{e}^x + c_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x y) = (\mathrm{e}^x)\left(\frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + \mathrm{e}^x + c_1\right)$$
$$\mathrm{d}(\mathrm{e}^x y) = \left(\frac{(4x^3 + 12\,\mathrm{e}^x + 3\sin(2x) + 12c_1 + 6x)\,\mathrm{e}^x}{12}\right) \mathrm{d}x$$

Integrating gives

$$e^x y = \int \frac{(4x^3 + 12\,e^x + 3\sin(2x) + 12c_1 + 6x)\,e^x}{12}\,dx$$

$$e^x y = \frac{x^3 e^x}{3} - e^x x^2 + \frac{5x\,e^x}{2} - \frac{5\,e^x}{2} + \frac{e^{2x}}{2} + \frac{e^x(\sin(2x) - 2\cos(2x))}{20} + e^x c_1 + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\frac{x^3 e^x}{3} - e^x x^2 + \frac{5x\,e^x}{2} - \frac{5\,e^x}{2} + \frac{e^{2x}}{2} + \frac{e^x(\sin(2x) - 2\cos(2x))}{20} + e^x c_1\right) + e^{-x} c_2$$

which simplifies to

$$y = \frac{x^3}{3} - x^2 + \frac{e^x}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + e^{-x} c_2$$

Summary

The solution(s) found are the following

$$y = \frac{x^3}{3} - x^2 + \frac{e^x}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + e^{-x} c_2 \tag{1}$$

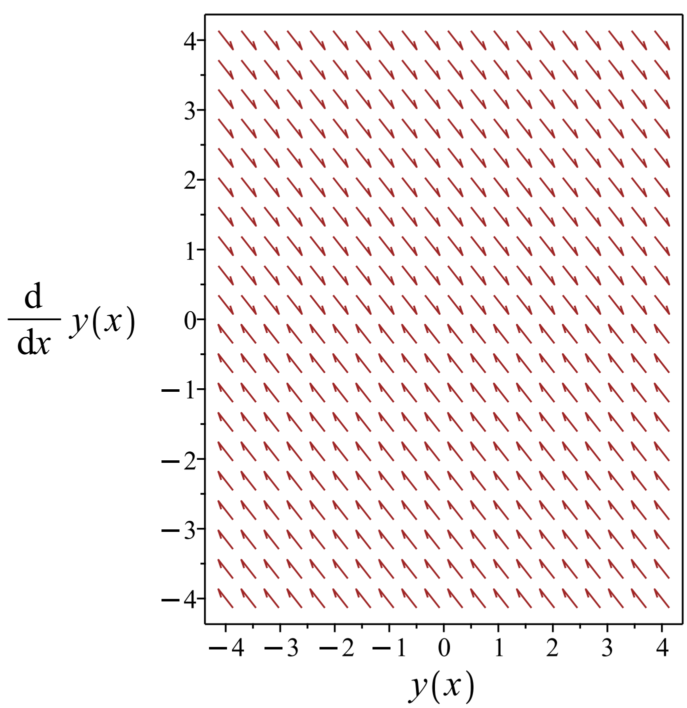


Figure 673: Slope field plot

Verification of solutions

$$
y = \frac{x^3}{3} - x^2 + \frac{e^x}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + e^{-x} c_2
$$

Verified OK.

17.17.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + y' &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 1 \tag{3} \\
C &= 0
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1 \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 526: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(e^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(e^x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x}\right) + c_2\left(e^{-x}(e^x)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)^2 + e^x + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 \cos(2x) + A_3 \sin(2x) + A_4 x + A_5 x^2 + A_6 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{gathered}2A_1 e^x - 4A_2 \cos(2x) - 4A_3 \sin(2x) + 2A_5 + 6A_6 x - 2A_2 \sin(2x) \\ + 2A_3 \cos(2x) + A_4 + 2A_5 x + 3A_6 x^2 = \cos(x)^2 + e^x + x^2\end{gathered}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{10}, A_3 = \frac{1}{20}, A_4 = \frac{5}{2}, A_5 = -1, A_6 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3} \tag{1}$$

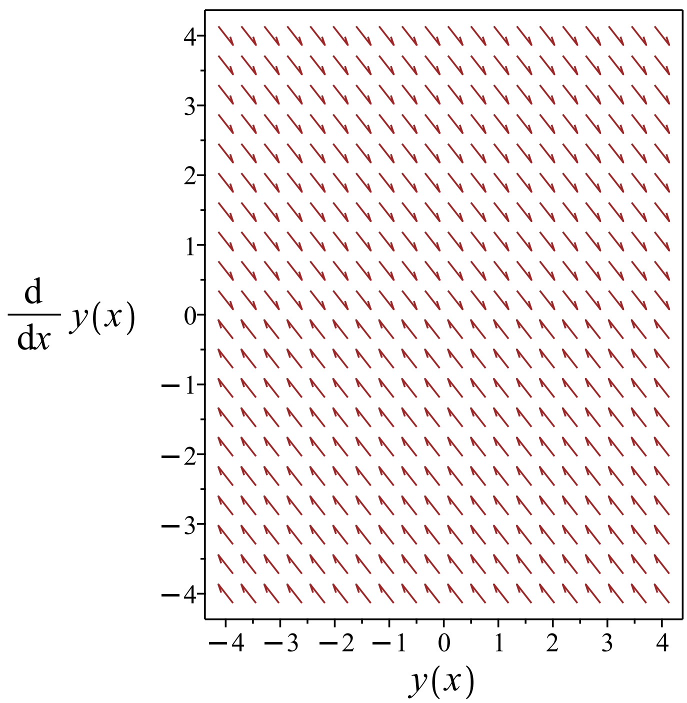


Figure 674: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} - x^2 + \frac{x^3}{3}$$

Verified OK.

17.17.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \cos(x)^2 + e^x + x^2 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int \cos(x)^2 + e^x + x^2\, dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{x^3}{3} + \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + e^x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 1\\ q(x) &= \frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1\end{aligned}$$

Hence the ode is

$$y' + y = \frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dx}\\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1\right)\\ \frac{d}{dx}(e^x y) &= (e^x)\left(\frac{x^3}{3} + \frac{\sin(2x)}{4} + \frac{x}{2} + e^x + c_1\right)\\ d(e^x y) &= \left(\frac{(4x^3 + 12e^x + 3\sin(2x) + 12c_1 + 6x)e^x}{12}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^x y &= \int \frac{(4x^3 + 12e^x + 3\sin(2x) + 12c_1 + 6x)e^x}{12} dx\\ e^x y &= \frac{x^3 e^x}{3} - e^x x^2 + \frac{5x e^x}{2} - \frac{5e^x}{2} + \frac{e^{2x}}{2} + \frac{e^x(\sin(2x) - 2\cos(2x))}{20} + e^x c_1 + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\frac{x^3 e^x}{3} - e^x x^2 + \frac{5x e^x}{2} - \frac{5e^x}{2} + \frac{e^{2x}}{2} + \frac{e^x(\sin(2x) - 2\cos(2x))}{20} + e^x c_1\right) + e^{-x} c_2$$

which simplifies to

$$y = \frac{x^3}{3} - x^2 + \frac{e^x}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + e^{-x} c_2$$

Summary

The solution(s) found are the following

$$y = \frac{x^3}{3} - x^2 + \frac{e^x}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + e^{-x} c_2 \tag{1}$$

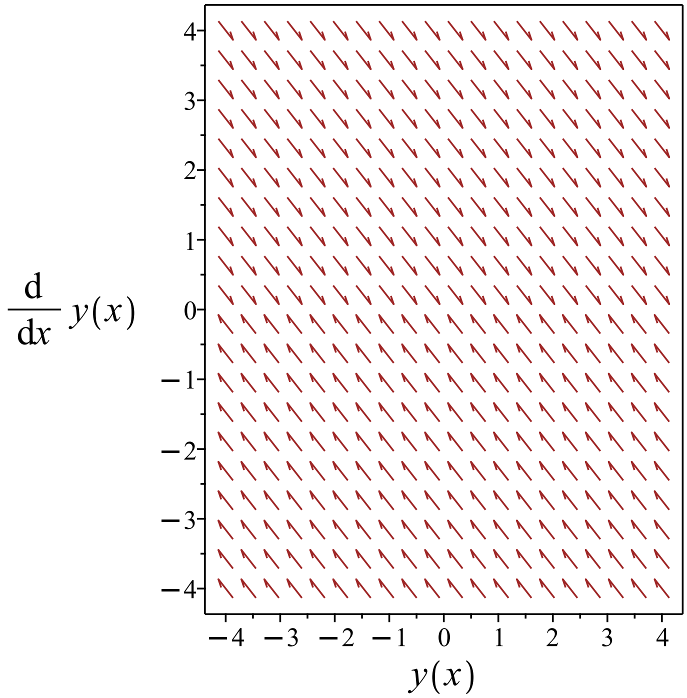


Figure 675: Slope field plot

Verification of solutions

$$y = \frac{x^3}{3} - x^2 + \frac{e^x}{2} + c_1 + \frac{5x}{2} + \frac{\sin(2x)}{20} - \frac{\cos(2x)}{10} - \frac{5}{2} + e^{-x} c_2$$

Verified OK.

17.17.7 Maple step by step solution

Let's solve

$y'' + y' = \cos(x)^2 + e^x + x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)^2 + \mathrm{e}^x + x^2\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{-x}\left(\int \left(\cos(x)^2 + \mathrm{e}^x + x^2\right) \mathrm{e}^x dx\right) + \int \left(\cos(x)^2 + \mathrm{e}^x + x^2\right) dx$
- ○ Compute integrals

  $y_p(x) = -\frac{5}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} - x^2 + \frac{5x}{2} + \frac{\mathrm{e}^x}{2} + \frac{x^3}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 - \frac{5}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} - x^2 + \frac{5x}{2} + \frac{\mathrm{e}^x}{2} + \frac{x^3}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = cos(_a)^2+_a^2-_b(_a)+exp(_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=cos(x)^2+exp(x)+x^2,y(x), singsol=all)
```

$$y(x) = -x^2 + \frac{x^3}{3} - c_1 e^{-x} + \frac{e^x}{2} - \frac{\cos(2x)}{10} + \frac{\sin(2x)}{20} + \frac{5x}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.529 (sec). Leaf size: 55

```
DSolve[y''[x]+y'[x]==Cos[x]^2+Exp[x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(x\left(2x^2 - 6x + 15\right) + 3e^x\right) + \frac{1}{20}\sin(2x) - \frac{1}{10}\cos(2x) - c_1 e^{-x} + c_2$$

17.18 problem 568

Internal problem ID [15337]
Internal file name [`OUTPUT/15337_Wednesday_May_08_2024_03_56_18_PM_45960724/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 568.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 4y''' = e^{x} + 3\sin(2x) + 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y''' = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = -4$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^{-4x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= x^2 \\ y_4 &= e^{-4x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 4y''' = e^x + 3\sin(2x) + 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + 3\sin(2x) + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, e^{-4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{e^x\}, \{\cos(2x), \sin(2x)\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}, \{e^x\}, \{\cos(2x), \sin(2x)\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\}, \{e^x\}, \{\cos(2x), \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^3 + A_2\mathrm{e}^x + A_3\cos(2x) + A_4\sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&5A_2\mathrm{e}^x + 16A_3\cos(2x) + 16A_4\sin(2x) + 24A_1 + 32A_3\sin(2x) - 32A_4\cos(2x)\\&= \mathrm{e}^x + 3\sin(2x) + 1\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = \frac{1}{5}, A_3 = \frac{3}{40}, A_4 = \frac{3}{80}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^3}{24} + \frac{\mathrm{e}^x}{5} + \frac{3\cos(2x)}{40} + \frac{3\sin(2x)}{80}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_3x^2 + c_2x + c_1 + \mathrm{e}^{-4x}c_4\right) + \left(\frac{x^3}{24} + \frac{\mathrm{e}^x}{5} + \frac{3\cos(2x)}{40} + \frac{3\sin(2x)}{80}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3x^2 + c_2x + c_1 + \mathrm{e}^{-4x}c_4 + \frac{x^3}{24} + \frac{\mathrm{e}^x}{5} + \frac{3\cos(2x)}{40} + \frac{3\sin(2x)}{80} \tag{1}$$

Verification of solutions

$$y = c_3x^2 + c_2x + c_1 + \mathrm{e}^{-4x}c_4 + \frac{x^3}{24} + \frac{\mathrm{e}^x}{5} + \frac{3\cos(2x)}{40} + \frac{3\sin(2x)}{80}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -4*_b(_a)+exp(_a)+3*sin(2*_a)+1, _b(_a)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 60

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$3)=exp(x)+3*sin(2*x)+1,y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(-\frac{18\sin(x)^2}{5} + \frac{9\sin(x)\cos(x)}{5} + x^3 + \left(12c_2 - \frac{18}{5}\right)x^2 + \left(24c_3 - \frac{9}{5}\right)x + 24c_4\right)e^{4x} + \frac{24\,e^{5x}}{5} - \frac{3c_1}{8}\right)e^{-4x}}{24}$$

✓ Solution by Mathematica

Time used: 0.877 (sec). Leaf size: 59

```
DSolve[y''''[x]+4*y'''[x]==Exp[x]+3*Sin[2*x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{24} + c_4x^2 + \frac{e^x}{5} + \frac{3}{80}\sin(2x) + \frac{3}{40}\cos(2x) + c_3x - \frac{1}{64}c_1e^{-4x} + c_2$$

17.19 problem 569

Internal problem ID [15338]
Internal file name [OUTPUT/15338_Wednesday_May_08_2024_03_56_18_PM_60835992/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 569.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 5y = 10\sin(x) + 17\sin(2x)}$$

17.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 5, f(x) = 10\sin(x) + 17\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + 5\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)}\\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\sin(x) + 17\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(2x), e^x \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(2x) + A_4 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&4A_1 \cos(x) + 4A_2 \sin(x) + A_3 \cos(2x) + A_4 \sin(2x) + 2A_1 \sin(x)\\&- 2A_2 \cos(x) + 4A_3 \sin(2x) - 4A_4 \cos(2x) = 10\sin(x) + 17\sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 4, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (e^x(c_1 \cos(2x) + c_2 \sin(2x))) + (\cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x(c_1 \cos(2x) + c_2 \sin(2x)) + \cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x) \tag{1}$$

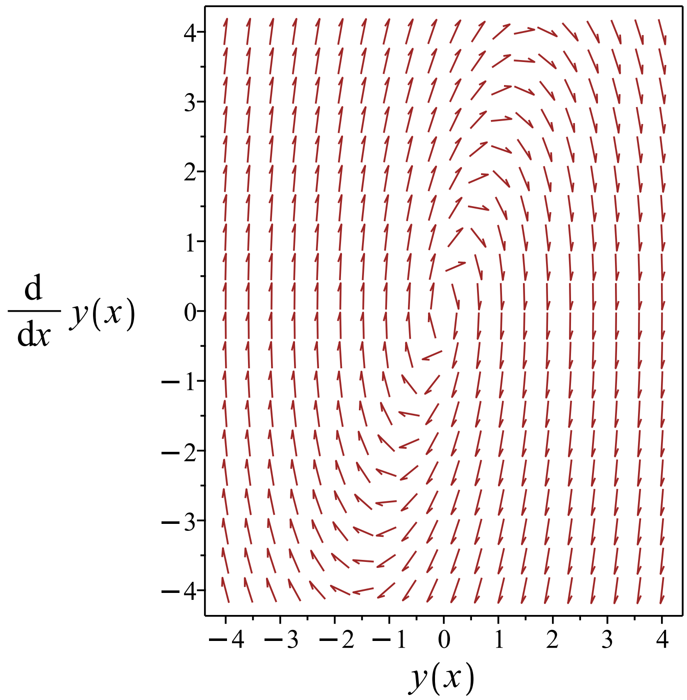


Figure 676: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x)) + \cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x)$$

Verified OK.

17.19.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 528: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x \cos(2x)) + c_2\left(\mathrm{e}^x \cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x \cos(2x)\, c_1 + \frac{\sin(2x)\, \mathrm{e}^x c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10\sin(x) + 17\sin(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^x \cos(2x), \frac{\mathrm{e}^x \sin(2x)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 \cos(2x) + A_4 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \cos(x) + 4A_2 \sin(x) + A_3 \cos(2x) + A_4 \sin(2x) + 2A_1 \sin(x) - 2A_2 \cos(x) + 4A_3 \sin(2x) - 4A_4 \cos(2x) = 10 \sin(x) + 17 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 4, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x \cos(2x) c_1 + \frac{\sin(2x) e^x c_2}{2}\right) + (\cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x \cos(2x) c_1 + \frac{\sin(2x) e^x c_2}{2} + \cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x) \tag{1}$$

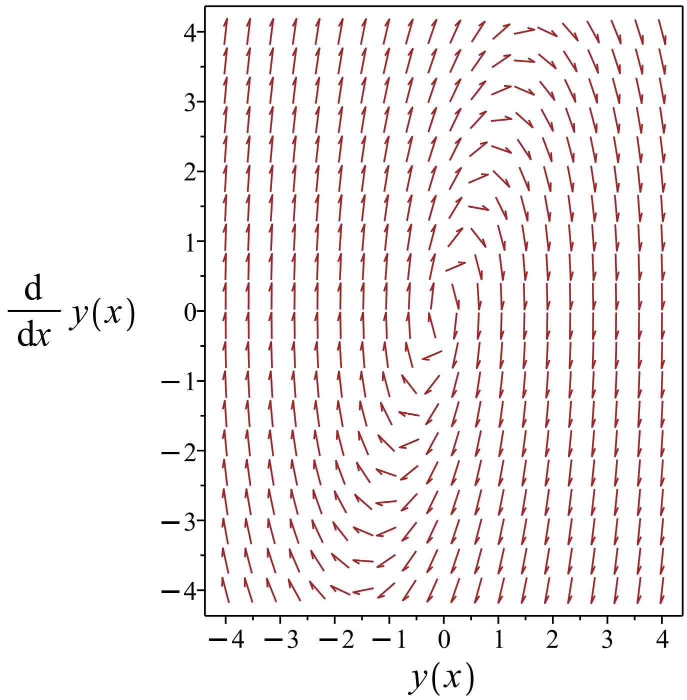


Figure 677: Slope field plot

Verification of solutions

$$y = e^{x} \cos(2x) c_1 + \frac{\sin(2x) e^{x} c_2}{2} + \cos(x) + 2\sin(x) + 4\cos(2x) + \sin(2x)$$

Verified OK.

17.19.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = 10\sin(x) + 17\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (1-2\mathrm{I}, 1+2\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^x \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^x \cos(2x) c_1 + \sin(2x) \mathrm{e}^x c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 10\sin(x) + 17\sin(2x)\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x \cos(2x) & \mathrm{e}^x \sin(2x) \\ \mathrm{e}^x \cos(2x) - 2\,\mathrm{e}^x \sin(2x) & \mathrm{e}^x \sin(2x) + 2\,\mathrm{e}^x \cos(2x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^x\left(-\cos(2x)\left(\int \sin(2x)(17\cos(x)+5)\sin(x)\,\mathrm{e}^{-x} dx\right) + \sin(2x)\left(\int \cos(2x)(17\cos(x)\right.\right.$

  ◦ Compute integrals

  $y_p(x) = 8\cos(x)^2 + (2\sin(x)+1)\cos(x) + 2\sin(x) - 4$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^x \cos(2x) c_1 + \sin(2x) \mathrm{e}^x c_2 + 8\cos(x)^2 + (2\sin(x)+1)\cos(x) + 2\sin(x) - 4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=10*sin(x)+17*sin(2*x),y(x), singsol=all)
```

$$y(x) = (c_1 e^x + 4) \cos (2x) + e^x \sin (2x) c_2 + \cos (x) + 2 \sin (x) + \sin (2x)$$

✓ Solution by Mathematica

Time used: 0.538 (sec). Leaf size: 37

```
DSolve[y''[x]-2*y'[x]+5*y[x]==10*Sin[x]+17*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x) + (4 + c_2 e^x) \cos(2x) + 2 \sin(x) (\cos(x) + c_1 e^x \cos(x) + 1)$$

17.20 problem 570

Internal problem ID [15339]
Internal file name [`OUTPUT/15339_Wednesday_May_08_2024_03_56_20_PM_56990095/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 570.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = x^2 - e^{-x} + e^x}$$

17.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = x^2 - e^{-x} + e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=1, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{1^2-(4)\,(1)\,(0)}\\&=-\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{1}{2}\\\lambda_2&=-\frac{1}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}\\y&=c_1e^{(0)x}+c_2e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 - e^{-x} + e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{-x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{e^{-x}\}, \{x, x^2, x^3\}]$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\, e^{-x}\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x\, e^{-x} + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x - A_2 e^{-x} + 2A_4 + 6A_5 x + A_3 + 2A_4 x + 3A_5 x^2 = x^2 - e^{-x} + e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 1, A_3 = 2, A_4 = -1, A_5 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + e^{-x}c_2) + \left(\frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-x}c_2 + \frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3} \tag{1}$$

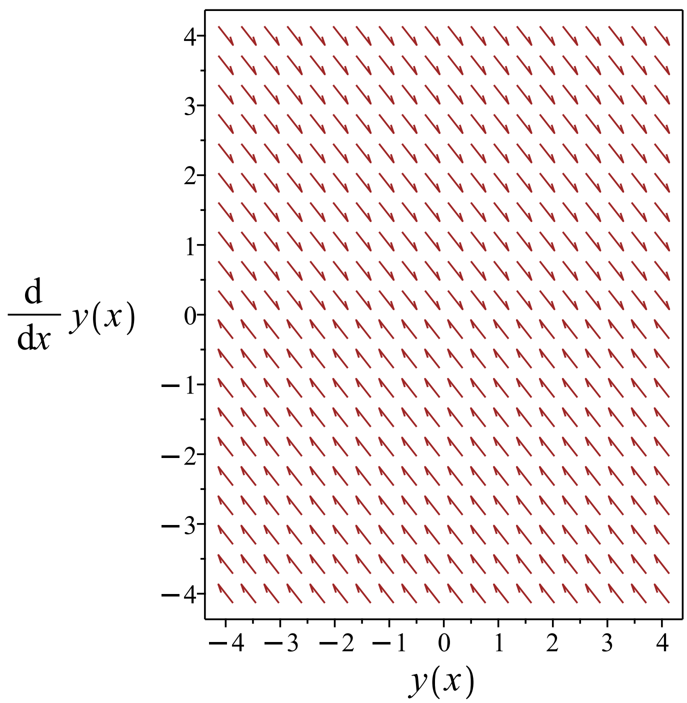


Figure 678: Slope field plot

Verification of solutions

$$y = c_1 + e^{-x}c_2 + \frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3}$$

Verified OK.

17.20.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y''+y')\,dx = \int (x^2-\mathrm{e}^{-x}+\mathrm{e}^{x})\,dx$$

$$y'+y = \frac{x^3}{3}+\mathrm{e}^{-x}+\mathrm{e}^{x}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1\\ q(x) &= \frac{x^3}{3}+\mathrm{e}^{-x}+\mathrm{e}^{x}+c_1 \end{aligned}$$

Hence the ode is

$$y'+y = \frac{x^3}{3}+\mathrm{e}^{-x}+\mathrm{e}^{x}+c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 1dx}\\ &= \mathrm{e}^{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^3}{3}+\mathrm{e}^{-x}+\mathrm{e}^{x}+c_1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\left(\frac{x^3}{3}+\mathrm{e}^{-x}+\mathrm{e}^{x}+c_1\right)\\ \mathrm{d}(\mathrm{e}^{x} y) &= \left(\mathrm{e}^{2x}+1+\frac{(x^3+3c_1)\,\mathrm{e}^{x}}{3}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{x} y &= \int \mathrm{e}^{2x}+1+\frac{(x^3+3c_1)\,\mathrm{e}^{x}}{3}\,\mathrm{d}x\\ \mathrm{e}^{x} y &= x+\frac{x^3\mathrm{e}^{x}}{3}-\mathrm{e}^{x}x^2+2x\,\mathrm{e}^{x}-2\,\mathrm{e}^{x}+\mathrm{e}^{x}c_1+\frac{\mathrm{e}^{2x}}{2}+c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(x + \frac{x^3 e^x}{3} - e^x x^2 + 2x\, e^x - 2\, e^x + e^x c_1 + \frac{e^{2x}}{2}\right) + e^{-x} c_2$$

which simplifies to

$$y = -2 + (x + c_2)\, e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^x}{2}$$

Summary

The solution(s) found are the following

$$y = -2 + (x + c_2)\, e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^x}{2} \tag{1}$$

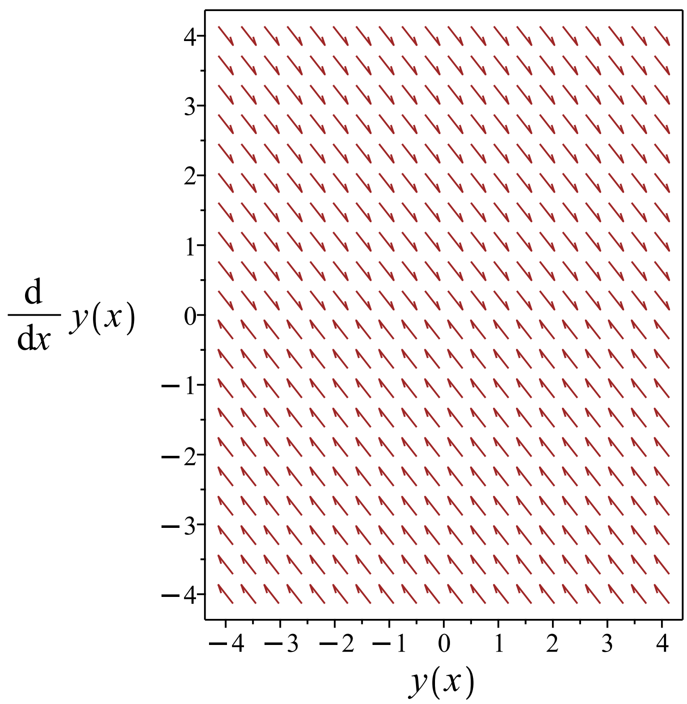


Figure 679: Slope field plot

Verification of solutions

$$y = -2 + (x + c_2)\, e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^x}{2}$$

Verified OK.

17.20.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - x^2 + \mathrm{e}^{-x} - \mathrm{e}^x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= x^2 - \mathrm{e}^{-x} + \mathrm{e}^x
\end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = x^2 - \mathrm{e}^{-x} + \mathrm{e}^x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^x
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(x^2 - \mathrm{e}^{-x} + \mathrm{e}^x\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(p\,\mathrm{e}^x) &= (\mathrm{e}^x)\left(x^2 - \mathrm{e}^{-x} + \mathrm{e}^x\right) \\
\mathrm{d}(p\,\mathrm{e}^x) &= \left(\mathrm{e}^x x^2 + \mathrm{e}^{2x} - 1\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
p\,\mathrm{e}^x &= \int \mathrm{e}^x x^2 + \mathrm{e}^{2x} - 1\,\mathrm{d}x \\
p\,\mathrm{e}^x &= -x + \mathrm{e}^x x^2 - 2x\,\mathrm{e}^x + 2\,\mathrm{e}^x + \frac{\mathrm{e}^{2x}}{2} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$p(x) = e^{-x}\left(-x + e^x x^2 - 2x\,e^x + 2\,e^x + \frac{e^{2x}}{2}\right) + c_1 e^{-x}$$

which simplifies to

$$p(x) = 2 + (-x + c_1)\,e^{-x} + x^2 - 2x + \frac{e^x}{2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 2 + (-x + c_1)\,e^{-x} + x^2 - 2x + \frac{e^x}{2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int -x\,e^{-x} + x^2 - 2x + 2 + \frac{e^x}{2} + c_1 e^{-x}\,dx \\ &= -x^2 + 2x + \frac{x^3}{3} - c_1 e^{-x} + x\,e^{-x} + e^{-x} + \frac{e^x}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -x^2 + 2x + \frac{x^3}{3} - c_1 e^{-x} + x\,e^{-x} + e^{-x} + \frac{e^x}{2} + c_2 \tag{1}$$

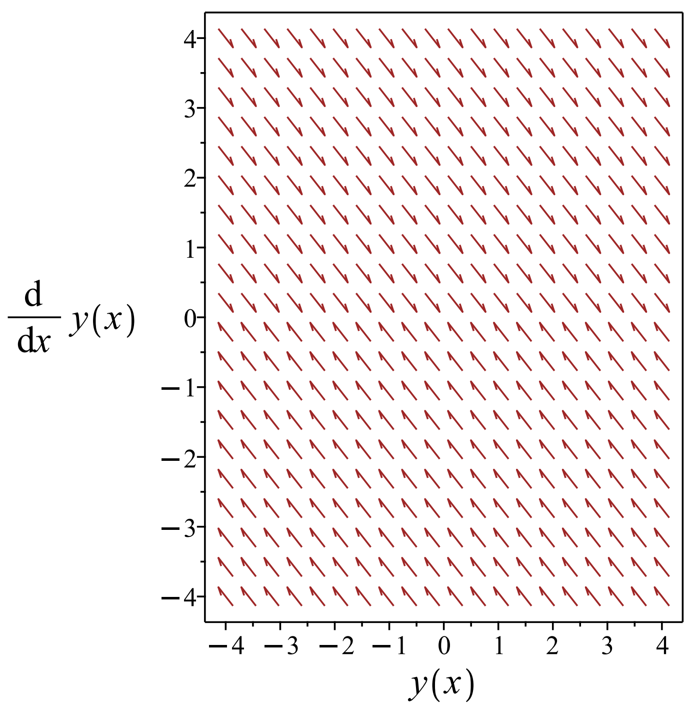


Figure 680: Slope field plot

Verification of solutions

$$y = -x^2 + 2x + \frac{x^3}{3} - c_1 e^{-x} + x\,e^{-x} + e^{-x} + \frac{e^x}{2} + c_2$$

Verified OK.

17.20.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = x^2 - e^{-x} + e^x$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int (x^2 - e^{-x} + e^x)\,dx$$
$$y' + y = \frac{x^3}{3} + e^{-x} + e^x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{x^3}{3} + e^{-x} + e^x + c_1$$

Hence the ode is

$$y' + y = \frac{x^3}{3} + e^{-x} + e^x + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^3}{3} + e^{-x} + e^x + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(\frac{x^3}{3} + e^{-x} + e^x + c_1\right)$$
$$d(e^x y) = \left(e^{2x} + 1 + \frac{(x^3 + 3c_1)\,e^x}{3}\right)\,dx$$

Integrating gives

$$e^{x}y = \int e^{2x} + 1 + \frac{(x^3 + 3c_1)e^{x}}{3}\, dx$$

$$e^{x}y = x + \frac{x^3 e^{x}}{3} - e^{x}x^2 + 2x\,e^{x} - 2\,e^{x} + e^{x}c_1 + \frac{e^{2x}}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}\left(x + \frac{x^3 e^{x}}{3} - e^{x}x^2 + 2x\,e^{x} - 2\,e^{x} + e^{x}c_1 + \frac{e^{2x}}{2}\right) + e^{-x}c_2$$

which simplifies to

$$y = -2 + (x + c_2)\,e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^{x}}{2}$$

Summary

The solution(s) found are the following

$$y = -2 + (x + c_2)\,e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^{x}}{2} \tag{1}$$

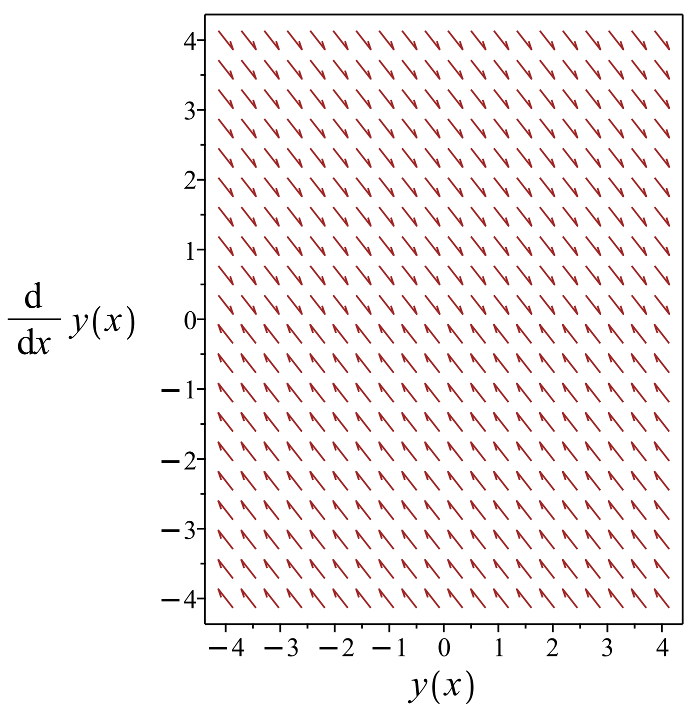


Figure 681: Slope field plot

Verification of solutions

$$y = -2 + (x + c_2)\,\mathrm{e}^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{\mathrm{e}^x}{2}$$

Verified OK.

17.20.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 530: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 - e^{-x} + e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{-x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x\,e^{-x}\}, \{1, x, x^2\}]$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x\,e^{-x}\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{x} + A_2 x\,e^{-x} + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} - A_2 e^{-x} + 2A_4 + 6A_5 x + A_3 + 2A_4 x + 3A_5 x^2 = x^2 - e^{-x} + e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 1, A_3 = 2, A_4 = -1, A_5 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2) + \left(\frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \frac{e^x}{2} + x\,e^{-x} + 2x - x^2 + \frac{x^3}{3} \tag{1}$$

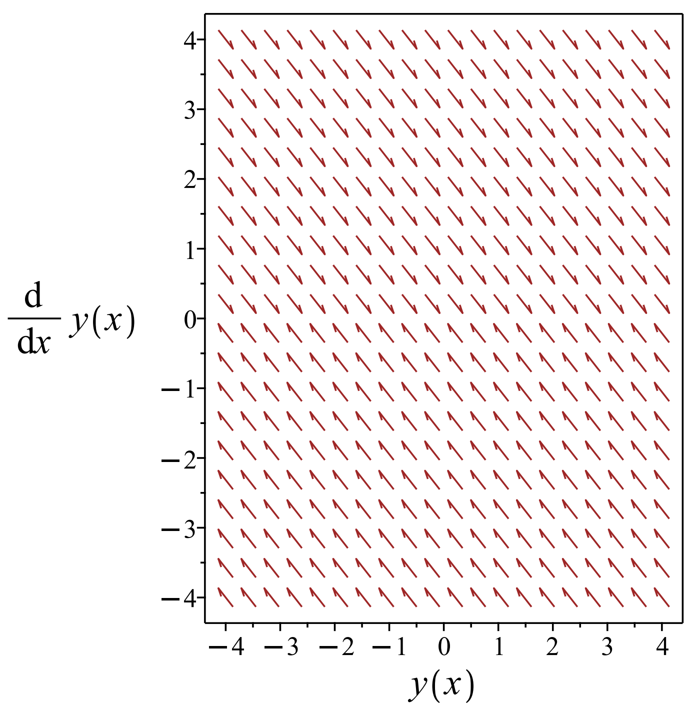


Figure 682: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \frac{e^x}{2} + x e^{-x} + 2x - x^2 + \frac{x^3}{3}$$

Verified OK.

17.20.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= x^2 - e^{-x} + e^x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x) y' + (q(x) - p'(x)) y)' = s(x)$$

Integrating gives

$$p(x) y' + (q(x) - p'(x)) y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int x^2 - e^{-x} + e^x \, dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{x^3}{3} + e^{-x} + e^x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{x^3}{3} + e^{-x} + e^x + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = \frac{x^3}{3} + e^{-x} + e^x + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x^3}{3} + e^{-x} + e^x + c_1\right) \\ \frac{d}{dx}(e^x y) &= (e^x)\left(\frac{x^3}{3} + e^{-x} + e^x + c_1\right) \\ d(e^x y) &= \left(e^{2x} + 1 + \frac{(x^3 + 3c_1)\, e^x}{3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int e^{2x} + 1 + \frac{(x^3 + 3c_1)\, e^x}{3}\, dx \\ e^x y &= x + \frac{x^3 e^x}{3} - e^x x^2 + 2x\, e^x - 2\, e^x + e^x c_1 + \frac{e^{2x}}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(x + \frac{x^3 e^x}{3} - e^x x^2 + 2x\, e^x - 2\, e^x + e^x c_1 + \frac{e^{2x}}{2}\right) + e^{-x} c_2$$

which simplifies to

$$y = -2 + (x + c_2)\,e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^x}{2}$$

Summary
The solution(s) found are the following

$$y = -2 + (x + c_2)\,e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^x}{2} \tag{1}$$

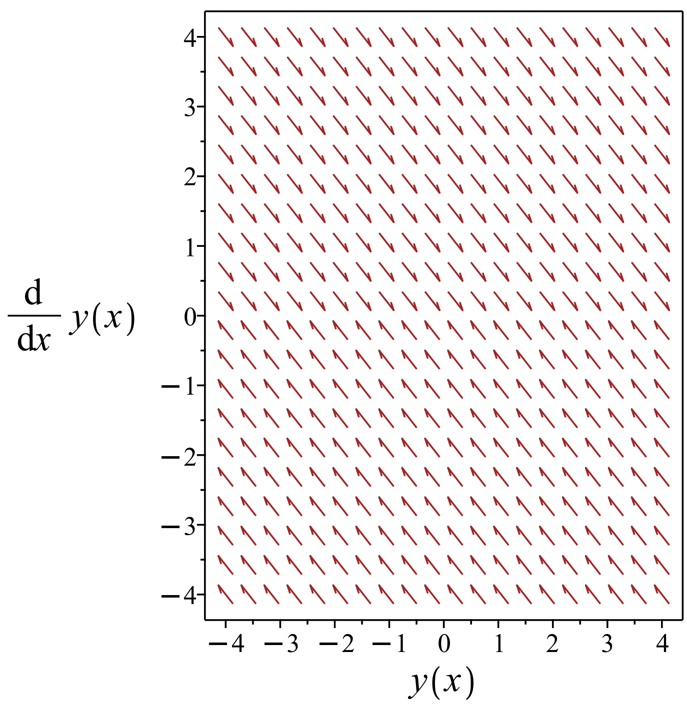


Figure 683: Slope field plot

Verification of solutions

$$y = -2 + (x + c_2)\,e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_1 + \frac{e^x}{2}$$

Verified OK.

17.20.7 Maple step by step solution

Let's solve

$y'' + y' = x^2 - e^{-x} + e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2 - e^{-x} + e^{x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-x}\left(\int (e^{x} x^2 + e^{2x} - 1) dx\right) + \int (x^2 - e^{-x} + e^{x}) dx$

- Compute integrals

  $y_p(x) = x\,e^{-x} - x^2 + 2x - 2 + \frac{e^x}{2} + \frac{x^3}{3} + e^{-x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 + x\,e^{-x} - x^2 + 2x - 2 + \frac{e^x}{2} + \frac{x^3}{3} + e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2+exp(_a)-_b(_a)-exp(-_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=x^2-exp(-x)+exp(x),y(x), singsol=all)
```

$$y(x) = (1 + x - c_1)\,e^{-x} + \frac{x^3}{3} - x^2 + 2x + c_2 + \frac{e^x}{2}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 43

```
DSolve[y''[x]+y'[x]==x^2-Exp[-x]+Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} - x^2 + 2x + \frac{e^x}{2} + e^{-x}(x + 1 - c_1) + c_2$$

17.21 problem 571

Internal problem ID [15340]
Internal file name [`OUTPUT/15340_Wednesday_May_08_2024_03_56_23_PM_2229027/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 571.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''-2y'-3y=2x+\mathrm{e}^{-x}-2\,\mathrm{e}^{3x}}$$

17.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1, B=-2, C=-3, f(x)=2x+\mathrm{e}^{-x}-2\,\mathrm{e}^{3x}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-2y'-3y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} - 3\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-3)} \\ &= 1 \pm 2 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= 1 + 2 \\ \lambda_2 &= 1 - 2 \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -1 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(-1)x} \end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + \mathrm{e}^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + \mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x + e^{-x} - 2\,e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{e^{3x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-x}\}, \{e^{3x}x\}, \{1, x\}]$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\}, \{e^{3x}x\}, \{1, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 e^{3x} x + A_3 + A_4 x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x} + 4A_2 e^{3x} - 2A_4 - 3A_3 - 3A_4 x = 2x + e^{-x} - 2\,e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = -\frac{1}{2}, A_3 = \frac{4}{9}, A_4 = -\frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{-x}}{4} - \frac{e^{3x}x}{2} + \frac{4}{9} - \frac{2x}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{3x} + e^{-x} c_2\right) + \left(-\frac{x\, e^{-x}}{4} - \frac{e^{3x} x}{2} + \frac{4}{9} - \frac{2x}{3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{3x} + e^{-x} c_2 - \frac{x\, e^{-x}}{4} - \frac{e^{3x} x}{2} + \frac{4}{9} - \frac{2x}{3} \tag{1}
$$

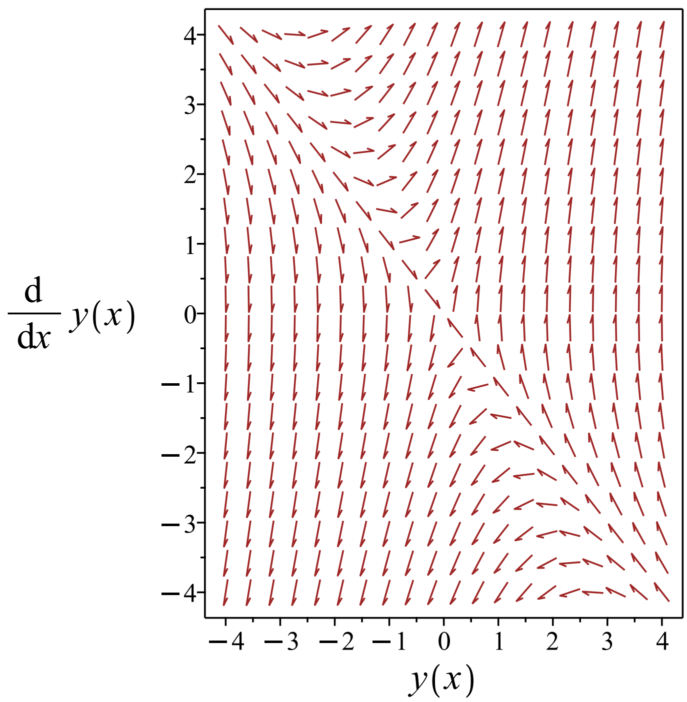


Figure 684: Slope field plot

Verification of solutions

$$
y = c_1 e^{3x} + e^{-x} c_2 - \frac{x\, e^{-x}}{4} - \frac{e^{3x} x}{2} + \frac{4}{9} - \frac{2x}{3}
$$

Verified OK.

17.21.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 532: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx}
\end{aligned}$$

$$= z_1 e^x$$
$$= z_1 (\mathrm{e}^x)$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\mathrm{e}^{-x}) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$
$$y_2 = \frac{e^{3x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^{3x}}{4} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(\frac{e^{3x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^{3x}}{4} \\ -e^{-x} & \frac{3\,e^{3x}}{4} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(\frac{3\,e^{3x}}{4}\right) - \left(\frac{e^{3x}}{4}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-x} e^{3x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{3x}(2x+e^{-x}-2e^{3x})}{4}}{e^{2x}} dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{e^{4x}}{2} + \frac{x e^{x}}{2} + \frac{1}{4}\right) dx$$

Hence

$$u_1 = -\frac{x}{4} - \frac{x e^{x}}{2} + \frac{e^{x}}{2} + \frac{e^{4x}}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x}(2x + e^{-x} - 2e^{3x})}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \left(2x e^{x} - 2e^{4x} + 1\right) e^{-4x} dx$$

Hence

$$u_2 = -2x - \frac{e^{-4x}}{4} - \frac{2x e^{-3x}}{3} - \frac{2e^{-3x}}{9}$$

Which simplifies to

$$u_1 = \frac{e^{4x}}{8} + \frac{(-4x+4)e^{x}}{8} - \frac{x}{4}$$
$$u_2 = \frac{2(-1-3x)e^{-3x}}{9} - 2x - \frac{e^{-4x}}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{e^{4x}}{8} + \frac{(-4x+4)e^{x}}{8} - \frac{x}{4}\right)e^{-x} + \frac{\left(\frac{2(-1-3x)e^{-3x}}{9} - 2x - \frac{e^{-4x}}{4}\right)e^{3x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{4}{9} + \frac{(-4x-1)\,\mathrm{e}^{-x}}{16} + \frac{(1-4x)\,\mathrm{e}^{3x}}{8} - \frac{2x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}\right) + \left(\frac{4}{9} + \frac{(-4x-1)\,\mathrm{e}^{-x}}{16} + \frac{(1-4x)\,\mathrm{e}^{3x}}{8} - \frac{2x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4} + \frac{4}{9} + \frac{(-4x-1)\,\mathrm{e}^{-x}}{16} + \frac{(1-4x)\,\mathrm{e}^{3x}}{8} - \frac{2x}{3} \tag{1}$$

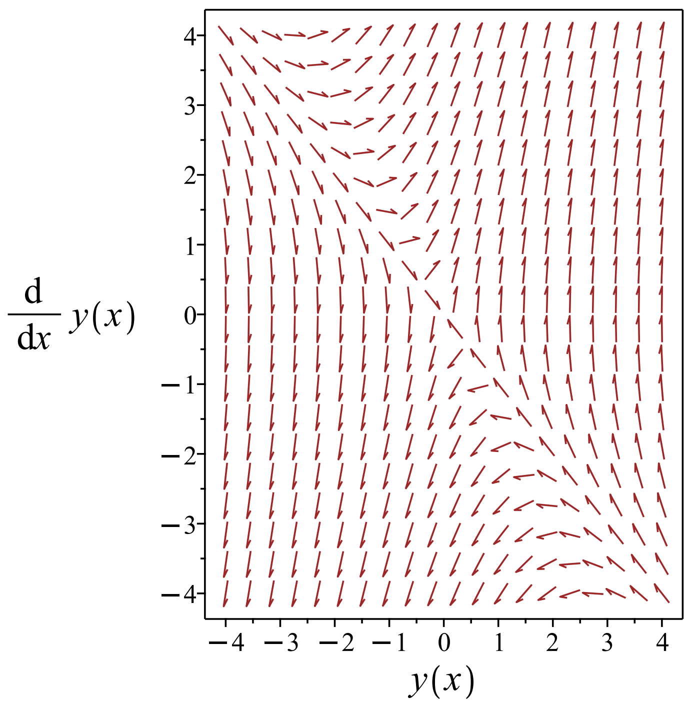


Figure 685: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4} + \frac{4}{9} + \frac{(-4x-1)\,\mathrm{e}^{-x}}{16} + \frac{(1-4x)\,\mathrm{e}^{3x}}{8} - \frac{2x}{3}$$

Verified OK.

17.21.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 2x + e^{-x} - 2\,e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2x + e^{-x} - 2\,e^{3x}\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3\,e^{3x} \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,e^{2x}$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x}\left(\int (2x e^x - 2e^{4x} + 1)\,dx\right)}{4} + \frac{e^{3x}\left(\int (2x e^x - 2e^{4x} + 1)e^{-4x}\,dx\right)}{4}$$

- Compute integrals

$$y_p(x) = \frac{4}{9} + \frac{(-4x-1)e^{-x}}{16} + \frac{(1-4x)e^{3x}}{8} - \frac{2x}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^{3x} + \frac{4}{9} + \frac{(-4x-1)e^{-x}}{16} + \frac{(1-4x)e^{3x}}{8} - \frac{2x}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=2*x+exp(-x)-2*exp(3*x),y(x), singsol=all)
```

$$y(x) = \frac{4}{9} + \frac{(-1 - 4x + 16c_1)e^{-x}}{16} + \frac{(1 - 4x + 8c_2)e^{3x}}{8} - \frac{2x}{3}$$

✓ Solution by Mathematica

Time used: 0.501 (sec). Leaf size: 51

```
DSolve[y''[x]-2*y'[x]-3*y[x]==2*x+Exp[-x]-2*Exp[3*x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{144}e^{-x}\left(e^x(64 - 96x) - 9(4x + 1 - 16c_1) - 18e^{4x}(4x - 1 - 8c_2)\right)$$

17.22 problem 572

Internal problem ID [15341]
Internal file name [`OUTPUT/15341_Wednesday_May_08_2024_03_56_24_PM_787114/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 572.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = e^{x} + 4\sin(2x) + 2\cos(x)^{2} - 1}$$

17.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = e^{x} + \cos(2x) + 4\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + \cos(2x) + 4\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x \cos(2x) + A_3 x \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^x - 4A_2 \sin(2x) + 4A_3 \cos(2x) = e^x + \cos(2x) + 4\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = -1, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{5} - x\cos(2x) + \frac{x\sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2x) + c_2\sin(2x)) + \left(\frac{e^x}{5} - x\cos(2x) + \frac{x\sin(2x)}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{e^x}{5} - x\cos(2x) + \frac{x\sin(2x)}{4} \tag{1}$$

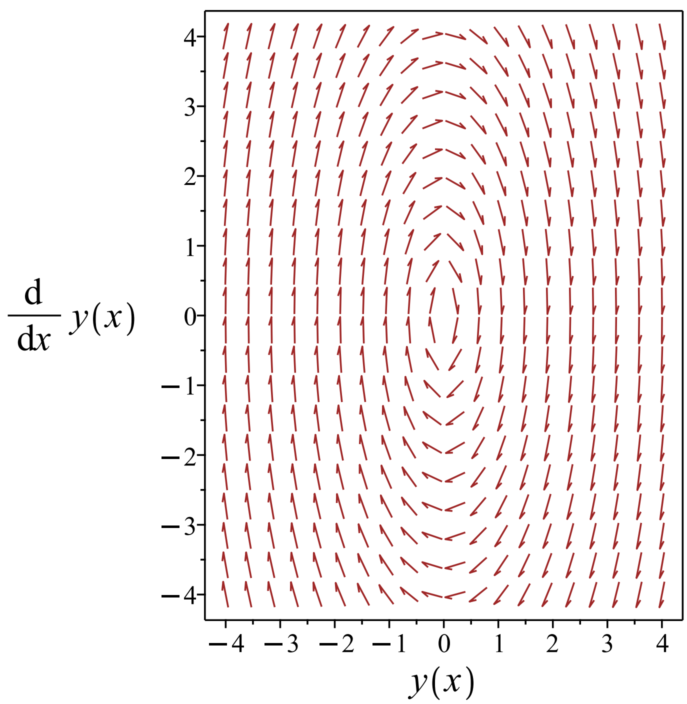


Figure 686: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{e^x}{5} - x\cos(2x) + \frac{x\sin(2x)}{4}$$

Verified OK.

17.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 534: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B=0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B=0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + \cos(2x) + 4\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x\cos(2x) + A_3 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^x - 4A_2 \sin(2x) + 4A_3 \cos(2x) = e^x + \cos(2x) + 4\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = -1, A_3 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{5} - x\cos(2x) + \frac{x\sin(2x)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{e^x}{5} - x \cos(2x) + \frac{x \sin(2x)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{e^x}{5} - x \cos(2x) + \frac{x \sin(2x)}{4} \tag{1}
$$

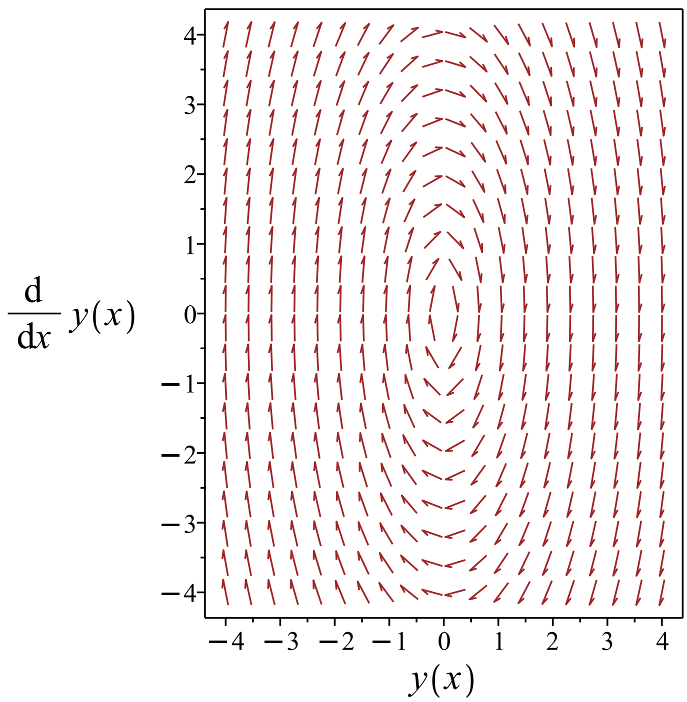


Figure 687: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{e^x}{5} - x \cos(2x) + \frac{x \sin(2x)}{4}
$$

Verified OK.

17.22.3 Maple step by step solution

Let's solve

$y'' + 4y = e^x + \cos(2x) + 4\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^x + \cos(2x) + 4\sin(2\right.$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\left(e^x+\cos(2x)+4\sin(2x)\right)dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\left(e^x+\cos(2x)+4\sin(2x)\right)dx\right)}{2}$$

- Compute integrals

$$y_p(x) = -2\cos(x)^2 x + \frac{\sin(x)(2+x)\cos(x)}{2} + x + \frac{e^x}{5}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) - 2\cos(x)^2 x + \frac{\sin(x)(2+x)\cos(x)}{2} + x + \frac{e^x}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+4*y(x)=exp(x)+4*sin(2*x)+2*cos(x)^2-1,y(x), singsol=all)
```

$$y(x) = \frac{(2 + x + 4c_2)\sin(2x)}{4} + (c_1 - x)\cos(2x) + \frac{e^x}{5}$$

✓ Solution by Mathematica

Time used: 0.435 (sec). Leaf size: 42

```
DSolve[y''[x]+4*y[x]==Exp[x]+4*Sin[2*x]+2*Cos[x]^2-1,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{e^x}{5} + \left(-x + \frac{1}{16} + c_1\right)\cos(2x) + \frac{1}{4}(x + 1 + 4c_2)\sin(2x)$$

17.23 problem 573

Internal problem ID [15342]
Internal file name [`OUTPUT/15342_Wednesday_May_08_2024_03_56_26_PM_45463235/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 573.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = 6x\,\mathrm{e}^{-x}\left(1 - \mathrm{e}^{-x}\right)}$$

17.23.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = -6x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-6x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x} x^2\}]$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x} x^2\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x} x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x} + A_2 \mathrm{e}^{-2x} x^2 + A_3 x\,\mathrm{e}^{-x} + A_4 \mathrm{e}^{-x} x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \mathrm{e}^{-2x} - 2A_2 \mathrm{e}^{-2x} x + 2A_2 \mathrm{e}^{-2x} + A_3 \mathrm{e}^{-x} + 2A_4 \mathrm{e}^{-x} x + 2A_4 \mathrm{e}^{-x} = -6x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = 3, A_3 = -6, A_4 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x} x^2 - 6x\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 e^{-x} + e^{-2x} c_2\right) + \left(6x\, e^{-2x} + 3\, e^{-2x} x^2 - 6x\, e^{-x} + 3\, e^{-x} x^2\right)$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{-2x} c_2 + 6x\, e^{-2x} + 3\, e^{-2x} x^2 - 6x\, e^{-x} + 3\, e^{-x} x^2 \quad (1)$$

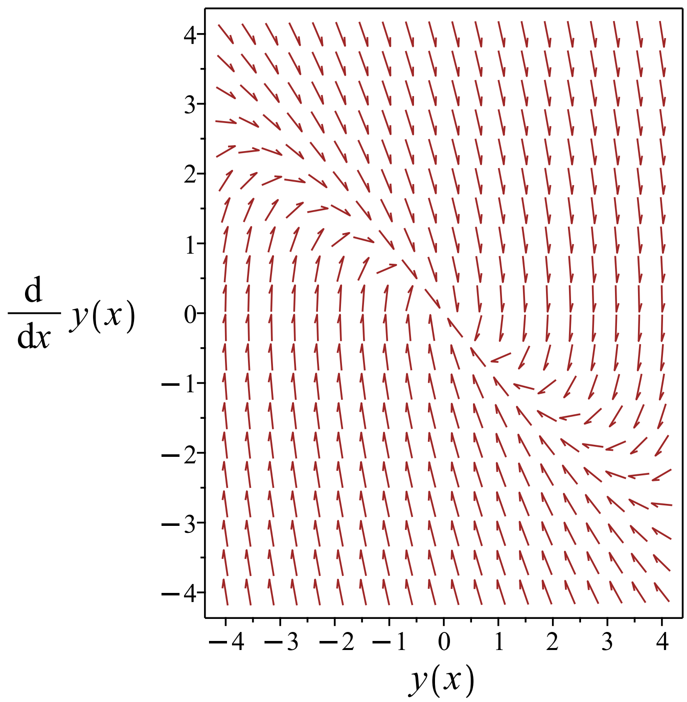


Figure 688: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + e^{-2x} c_2 + 6x\, e^{-2x} + 3\, e^{-2x} x^2 - 6x\, e^{-x} + 3\, e^{-x} x^2$$

Verified OK.

17.23.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 536: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = e^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-6x\,\mathrm{e}^{-x}\left(\mathrm{e}^{-x}-1\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x^2\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x} + A_2\mathrm{e}^{-2x}x^2 + A_3 x\,\mathrm{e}^{-x} + A_4\mathrm{e}^{-x}x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\mathrm{e}^{-2x} - 2A_2\mathrm{e}^{-2x}x + 2A_2\mathrm{e}^{-2x} + A_3\mathrm{e}^{-x} + 2A_4\mathrm{e}^{-x}x + 2A_4\mathrm{e}^{-x} = -6x\,\mathrm{e}^{-x}\left(\mathrm{e}^{-x}-1\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = 3, A_3 = -6, A_4 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x}x^2 - 6x\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x}x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2\right) + \left(6x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x} x^2 - 6x\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2 + 6x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x} x^2 - 6x\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2 \tag{1}
$$

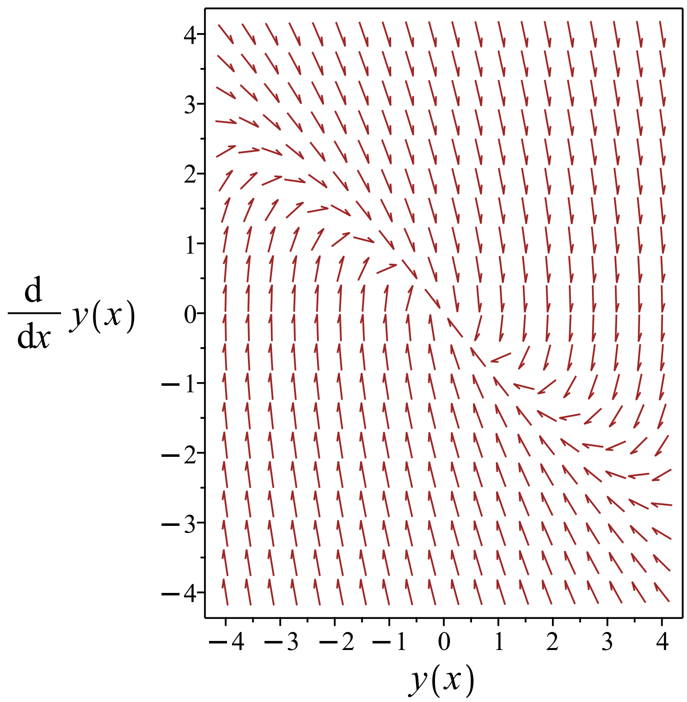


Figure 689: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{-x} c_2 + 6x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x} x^2 - 6x\,\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x} x^2
$$

Verified OK.

17.23.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = -6x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-2x} + \mathrm{e}^{-x}c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -6x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -6\,\mathrm{e}^{-2x}\left(\int x(-1 + \mathrm{e}^{x})\,dx\right) - 6\,\mathrm{e}^{-x}\left(\int (\mathrm{e}^{-x} - 1)\,x dx\right)$

- Compute integrals

  $y_p(x) = (3x^2+6x+6)\,e^{-2x} + 3\,e^{-x}(x^2-2x+2)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + e^{-x} c_2 + (3x^2+6x+6)\,e^{-2x} + 3\,e^{-x}(x^2-2x+2)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=6*x*exp(-x)*(1-exp(-x)),y(x), singsol=all)
```

$$y(x) = 3\left(\left(x^2+2x-\frac{1}{3}c_1+2\right)e^{-x} + x^2 - 2x + \frac{c_2}{3}\right)e^{-x}$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 39

```
DSolve[y''[x]+3*y'[x]+2*y[x]==6*x*Exp[-x]*(1-Exp[-x]),y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-2x}\left(3x^2 + e^x\left(3x^2-6x+6+c_2\right) + 6x + 6 + c_1\right)$$

17.24 problem 574

Internal problem ID [15343]
Internal file name [`OUTPUT/15343_Wednesday_May_08_2024_03_56_27_PM_63161318/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 574.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \cos(2x)^2 + \sin\left(\frac{x}{2}\right)^2}$$

17.24.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \frac{\cos(4x)}{2} + 1 - \frac{\cos(x)}{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\frac{\cos(4x)}{2} + 1 - \frac{\cos(x)}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(x), \sin(x)\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{\cos(x)\, x, \sin(x)\, x\}, \{\cos(4x), \sin(4x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 \cos(x)\, x + A_3 \sin(x)\, x + A_4 \cos(4x) + A_5 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2 \sin(x) + 2A_3 \cos(x) - 15A_4 \cos(4x) - 15A_5 \sin(4x) + A_1 = \frac{\cos(4x)}{2} + 1 - \frac{\cos(x)}{2}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = -\frac{1}{4}, A_4 = -\frac{1}{30}, A_5 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30} \tag{1}$$

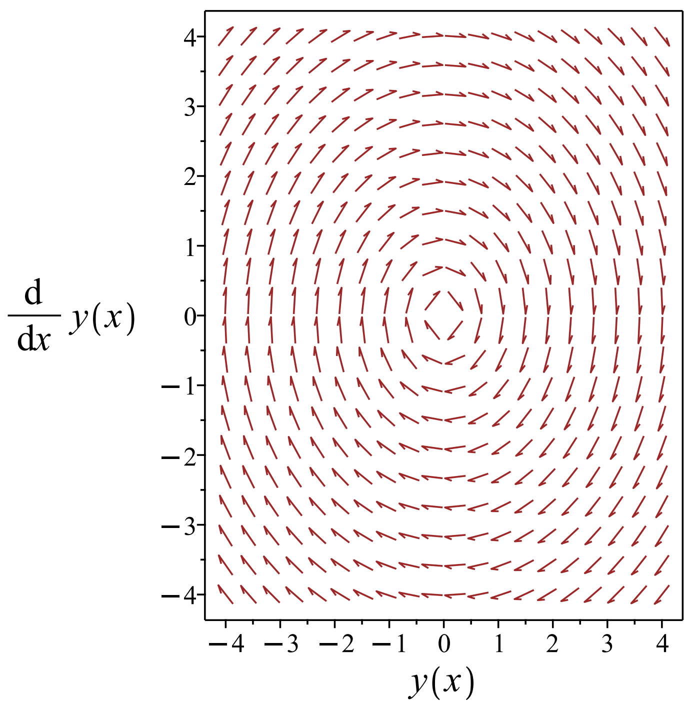


Figure 690: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30}$$

Verified OK.

17.24.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 538: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x)^4 - 4\cos(x)^2 + \frac{3}{2} - \frac{\cos(x)}{2}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{\cos(x)x, \sin(x)x\}, \{\cos(2x), \sin(2x)\}, \{\cos(4x), \sin(4x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2\cos(x)x + A_3\sin(x)x + A_4\cos(2x) + A_5\sin(2x) + A_6\cos(4x) + A_7\sin(4x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-2A_2\sin(x) + 2A_3\cos(x) - 3A_4\cos(2x) - 3A_5\sin(2x)\\&- 15A_6\cos(4x) - 15A_7\sin(4x) + A_1 = \frac{\cos(4x)}{2} + 1 - \frac{\cos(x)}{2}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = -\frac{1}{4}, A_4 = 0, A_5 = 0, A_6 = -\frac{1}{30}, A_7 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 - \frac{\sin(x)x}{4} - \frac{\cos(4x)}{30}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + 1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30} \tag{1}
$$

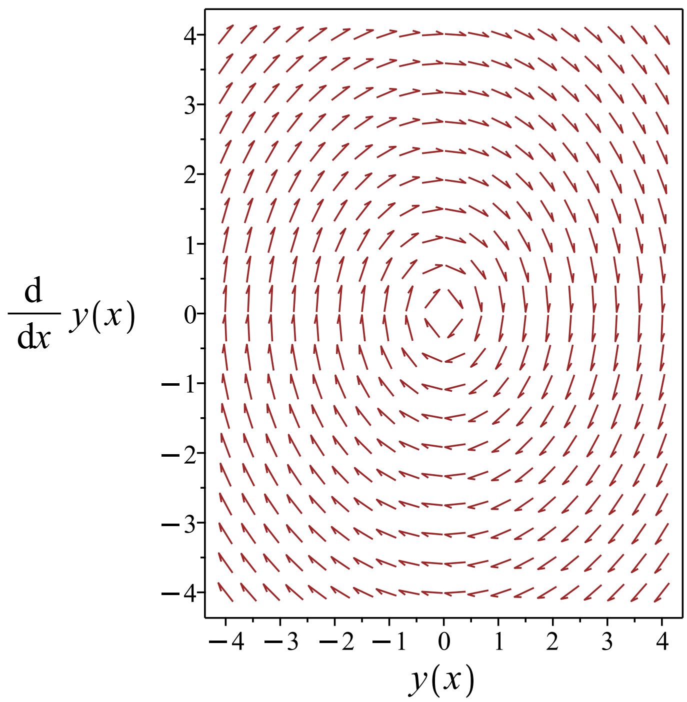


Figure 691: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + 1 - \frac{\sin(x)\, x}{4} - \frac{\cos(4x)}{30}
$$

Verified OK.

17.24.3 Maple step by step solution

Let's solve

$y'' + y = \frac{\cos(4x)}{2} + 1 - \frac{\cos(x)}{2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{\cos(4x)}{2} + 1 - \frac{\cos(x)}{2}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(x)\left(\int \sin(x)(\cos(4x)+2-\cos(x))dx\right)}{2} + \frac{\sin(x)\left(\int \left(8\cos(x)^5-8\cos(x)^3-\cos(x)^2+3\cos(x)\right)dx\right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{4\cos(x)^4}{15} + \frac{4\cos(x)^2}{15} - \frac{\sin(x)x}{4} - \frac{\cos(x)}{8} + \frac{29}{30}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(x) + c_2\sin(x) - \frac{4\cos(x)^4}{15} + \frac{4\cos(x)^2}{15} - \frac{\sin(x)x}{4} - \frac{\cos(x)}{8} + \frac{29}{30}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+y(x)=cos(2*x)^2+sin(x/2)^2,y(x), singsol=all)
```

$$y(x) = 1 - \frac{\cos(4x)}{30} + \frac{(-1+8c_1)\cos(x)}{8} + \frac{(-x+4c_2)\sin(x)}{4}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 36

```
DSolve[y''[x]+y[x]==Cos[2*x]^2+Sin[x/2]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{4}x\sin(x) - \frac{1}{30}\cos(4x) + \left(-\frac{1}{4}+c_1\right)\cos(x) + c_2\sin(x) + 1$$

17.25 problem 575

Internal problem ID [15344]
Internal file name [`OUTPUT/15344_Wednesday_May_08_2024_03_56_29_PM_99699687/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 575.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 5y = 1 + 8\cos(x) + e^{2x}}$$

17.25.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 5, f(x) = 1 + 8\cos(x) + e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 5\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(5)}\\ &= 2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + i\\ \lambda_2 &= 2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 + i\\ \lambda_2 &= 2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1+8\cos(x)+e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\},\{e^{2x}\},\{\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(x),e^{2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{2x} + A_3\cos(x) + A_4\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 e^{2x} + 4A_3\cos(x) + 4A_4\sin(x) + 4A_3\sin(x) - 4A_4\cos(x) + 5A_1 = 1 + 8\cos(x) + e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{5}, A_2=1, A_3=1, A_4=-1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{5} + e^{2x} + \cos(x) - \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}(c_1\cos(x) + c_2\sin(x))\right) + \left(\frac{1}{5} + e^{2x} + \cos(x) - \sin(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{5} + e^{2x} + \cos(x) - \sin(x) \tag{1}$$

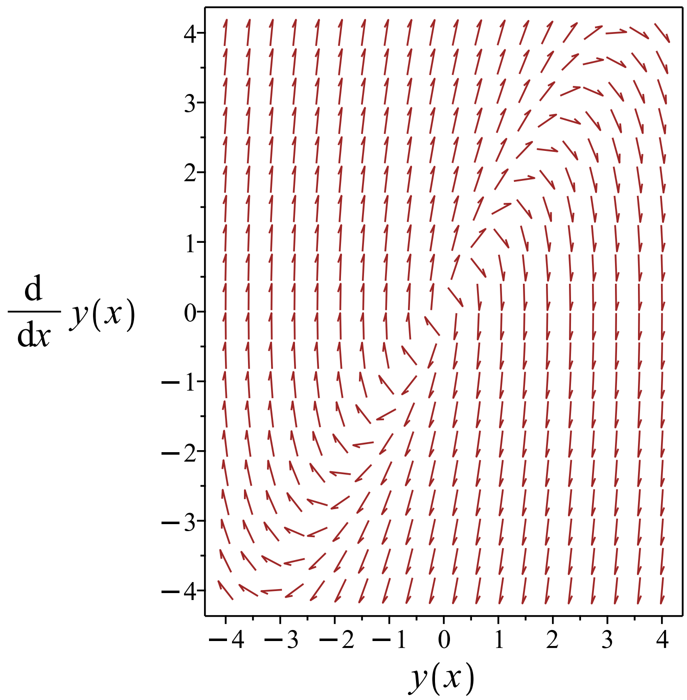


Figure 692: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{5} + e^{2x} + \cos(x) - \sin(x)$$

Verified OK.

17.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\ B &= -4\\ C &= 5\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -1\\ t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 540: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} \, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{2x} \cos(x)\right) + c_2 \left(e^{2x} \cos(x) (\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) e^{2x} c_1 + e^{2x} \sin(x) c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + 8 \cos(x) + e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{2x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(x), e^{2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{2x} + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 e^{2x} + 4A_3 \cos(x) + 4A_4 \sin(x) + 4A_3 \sin(x) - 4A_4 \cos(x) + 5A_1 = 1 + 8\cos(x) + e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{5}, A_2 = 1, A_3 = 1, A_4 = -1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{5} + e^{2x} + \cos(x) - \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x) e^{2x} c_1 + e^{2x} \sin(x) c_2\right) + \left(\frac{1}{5} + e^{2x} + \cos(x) - \sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{5} + e^{2x} + \cos(x) - \sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{5} + e^{2x} + \cos(x) - \sin(x) \tag{1}$$

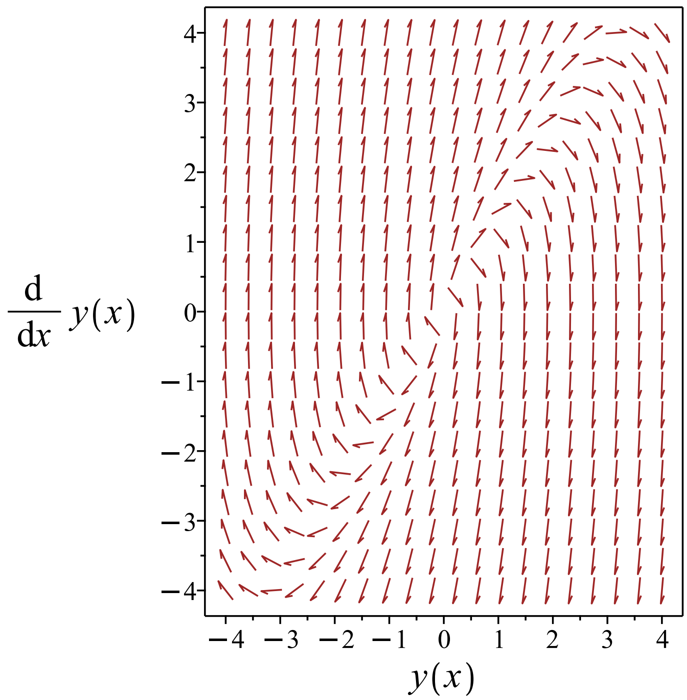


Figure 693: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{1}{5} + e^{2x} + \cos(x) - \sin(x)$$

Verified OK.

17.25.3 Maple step by step solution

Let's solve

$y'' - 4y' + 5y = 1 + 8\cos(x) + e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\,\mathrm{e}^{2x} c_1 + \mathrm{e}^{2x} \sin(x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1 + 8\cos(x) + \mathrm{e}^{2x}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} \cos(x) & \mathrm{e}^{2x} \sin(x) \\ 2\,\mathrm{e}^{2x} \cos(x) - \mathrm{e}^{2x} \sin(x) & 2\,\mathrm{e}^{2x} \sin(x) + \mathrm{e}^{2x} \cos(x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^{2x} \left(\cos(x) \left(\int \sin(x) \left(\mathrm{e}^{-2x} + 8\,\mathrm{e}^{-2x} \cos(x) + 1\right) dx\right) - \sin(x) \left(\int \left(\left(8\cos(x)^2 + \cos(x)\right)\right.\right.\right.$

  ○ Compute integrals

  $y_p(x) = \frac{1}{5} + \mathrm{e}^{2x} + \cos(x) - \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = \cos(x)\,\mathrm{e}^{2x} c_1 + \mathrm{e}^{2x} \sin(x)\, c_2 + \cos(x) + \mathrm{e}^{2x} - \sin(x) + \frac{1}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=1+8*cos(x)+exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{2x} \sin(x) c_2 + e^{2x} \cos(x) c_1 - \sin(x) + \cos(x) + \frac{1}{5} + e^{2x}$$

✓ Solution by Mathematica

Time used: 0.341 (sec). Leaf size: 40

```
DSolve[y''[x]-4*y'[x]+5*y[x]==1+8*Cos[x]+Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x} + (1 + c_2 e^{2x}) \cos(x) + (-1 + c_1 e^{2x}) \sin(x) + \frac{1}{5}$$

17.26 problem 576

Internal problem ID [15345]
Internal file name [`OUTPUT/15345_Wednesday_May_08_2024_03_56_30_PM_35597348/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 576.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = e^{x} \sin\left(\frac{x}{2}\right)^{2}}$$

17.26.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = e^{x} \sin\left(\frac{x}{2}\right)^{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^x (c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin\left(\frac{x}{2}\right)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(x), e^x \sin(x)\}$$

Since $e^x \cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\, e^x \cos(x), x\, e^x \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x\, e^x \cos(x) + A_3 x\, e^x \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^x - 2A_2 e^x \sin(x) + 2A_3 e^x \cos(x) = e^x \sin\left(\frac{x}{2}\right)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 0, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{2} - \frac{x\, e^x \sin(x)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^x (c_1 \cos(x) + c_2 \sin(x))\right) + \left(\frac{e^x}{2} - \frac{x\, e^x \sin(x)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^x (c_1 \cos(x) + c_2 \sin(x)) + \frac{e^x}{2} - \frac{x\, e^x \sin(x)}{4} \tag{1}
$$

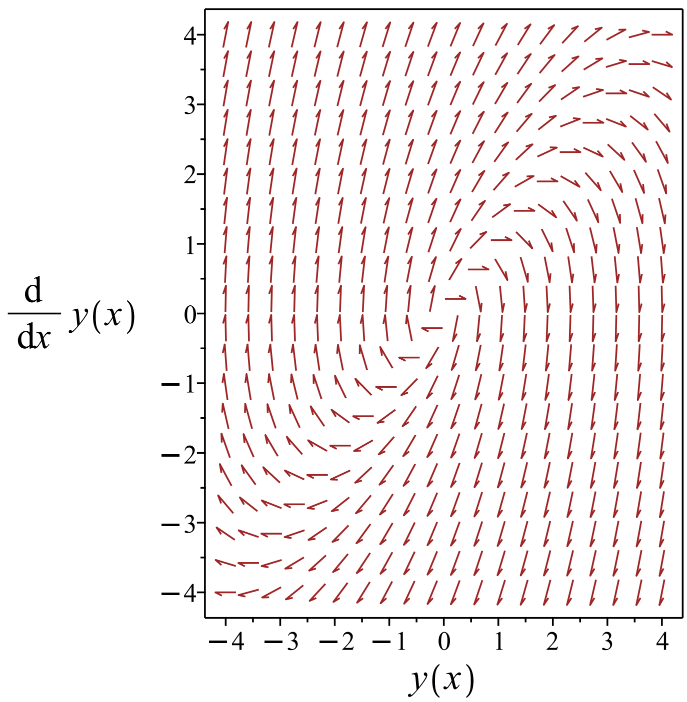


Figure 694: Slope field plot

Verification of solutions

$$
y = e^x (c_1 \cos(x) + c_2 \sin(x)) + \frac{e^x}{2} - \frac{x\, e^x \sin(x)}{4}
$$

Verified OK.

17.26.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 542: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x \cos(x)) + c_2(\mathrm{e}^x \cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x \cos(x)\, c_1 + c_2 \mathrm{e}^x \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin\left(\frac{x}{2}\right)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(x), e^x \sin(x)\}$$

Since $e^x \cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\, e^x \cos(x), x\, e^x \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x\, e^x \cos(x) + A_3 x\, e^x \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^x - 2A_2 e^x \sin(x) + 2A_3 e^x \cos(x) = e^x \sin\left(\frac{x}{2}\right)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 0, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{2} - \frac{x\, e^x \sin(x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x \cos(x)\, c_1 + c_2 e^x \sin(x)) + \left(\frac{e^x}{2} - \frac{x\, e^x \sin(x)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{2} - \frac{x\,e^{x} \sin(x)}{4}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{2} - \frac{x\,e^{x} \sin(x)}{4} \tag{1}$$

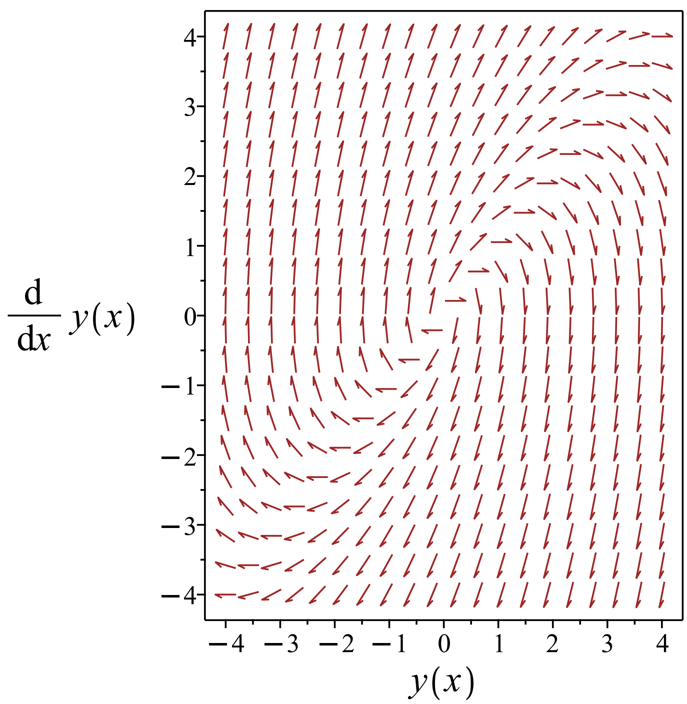


Figure 695: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{e^{x}}{2} - \frac{x\,e^{x} \sin(x)}{4}$$

Verified OK.

17.26.3 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = e^x \sin\left(\frac{x}{2}\right)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x \cos(x) c_1 + c_2 e^x \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^x \sin\left(\frac{x}{2}\right)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x \cos(x) & e^x \sin(x) \\ e^x \cos(x) - e^x \sin(x) & e^x \sin(x) + e^x \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2x}$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^x \left(\cos(x) \left(\int \sin(x) \sin\left(\frac{x}{2}\right)^2 dx\right) - \sin(x) \left(\int \cos(x) \sin\left(\frac{x}{2}\right)^2 dx\right)\right)$$

- Compute integrals

$$y_p(x) = -\frac{\left(\sin(x) x + \frac{\cos(x)}{2} - 2\right) e^x}{4}$$

- Substitute particular solution into general solution to ODE

$$y = e^x \cos(x) c_1 + c_2 e^x \sin(x) - \frac{\left(\sin(x) x + \frac{\cos(x)}{2} - 2\right) e^x}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=exp(x)*sin(x/2)^2,y(x), singsol=all)
```

$$y(x) = -\frac{\left(\left(-4c_1 + \frac{1}{2}\right) \cos(x) - 2 + (x - 4c_2) \sin(x)\right) e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 33

```
DSolve[y''[x]-2*y'[x]+2*y[x]==Exp[x]*Sin[x/2]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{8} e^x ((1 - 8c_2) \cos(x) + 2(x - 4c_1) \sin(x) - 4)$$

17.27 problem 577

Internal problem ID [15346]
Internal file name [`OUTPUT/15346_Wednesday_May_08_2024_03_56_32_PM_66462574/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 577.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 3y' = 1 + e^{x} + \cos(x) + \sin(x)}$$

17.27.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 0, f(x) = 1 + e^{x} + \cos(x) + \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-3y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=-3, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}-3\lambda\, \mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2-3\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-3, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{3}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{-3^2-(4)(1)(0)}\\&=\frac{3}{2}\pm\frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1=\frac{3}{2}+\frac{3}{2}$$
$$\lambda_2=\frac{3}{2}-\frac{3}{2}$$

Which simplifies to

$$\lambda_1=3$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$$
$$y=c_1e^{(3)x}+c_2e^{(0)x}$$

Or

$$y = c_1 e^{3x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + e^x + \cos(x) + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{3x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{e^x\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x + A_2 e^x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-2A_2 e^x - A_3 \cos(x) - A_4 \sin(x) - 3A_1 + 3A_3 \sin(x) - 3A_4 \cos(x) \\ &= 1 + e^x + \cos(x) + \sin(x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{1}{2}, A_3 = \frac{1}{5}, A_4 = -\frac{2}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{3x} + c_2) + \left(-\frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 - \frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5} \tag{1}$$

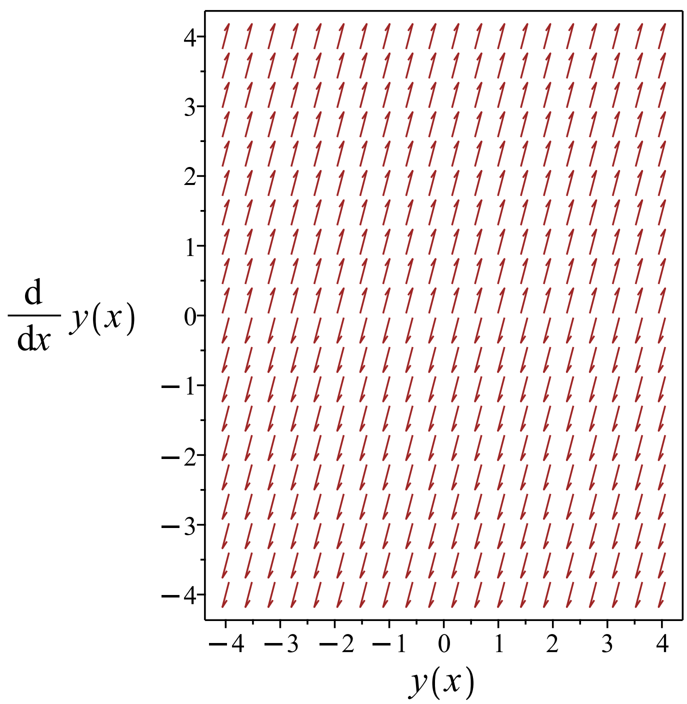


Figure 696: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 - \frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Verified OK.

17.27.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = \int (1 + e^x + \cos(x) + \sin(x))\, dx$$

$$-3y + y' = x + \sin(x) + e^x - \cos(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= x + \sin(x) + e^x - \cos(x) + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = x + \sin(x) + e^x - \cos(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\,(x + \sin(x) + e^x - \cos(x) + c_1) \\ \frac{d}{dx}\left(e^{-3x} y\right) &= \left(e^{-3x}\right)(x + \sin(x) + e^x - \cos(x) + c_1) \\ d\left(e^{-3x} y\right) &= \left((x + \sin(x) + e^x - \cos(x) + c_1)\, e^{-3x}\right)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-3x} y &= \int (x + \sin(x) + e^x - \cos(x) + c_1)\, e^{-3x}\, dx \\ e^{-3x} y &= -\frac{x\, e^{-3x}}{3} - \frac{e^{-3x}}{9} + \frac{e^{-3x}\cos(x)}{5} - \frac{2\, e^{-3x}\sin(x)}{5} - \frac{e^{-2x}}{2} - \frac{c_1 e^{-3x}}{3} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-\frac{x\, e^{-3x}}{3} - \frac{e^{-3x}}{9} + \frac{e^{-3x}\cos(x)}{5} - \frac{2\, e^{-3x}\sin(x)}{5} - \frac{e^{-2x}}{2} - \frac{c_1 e^{-3x}}{3}\right) + c_2 e^{3x}$$

which simplifies to

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5} \tag{1}$$

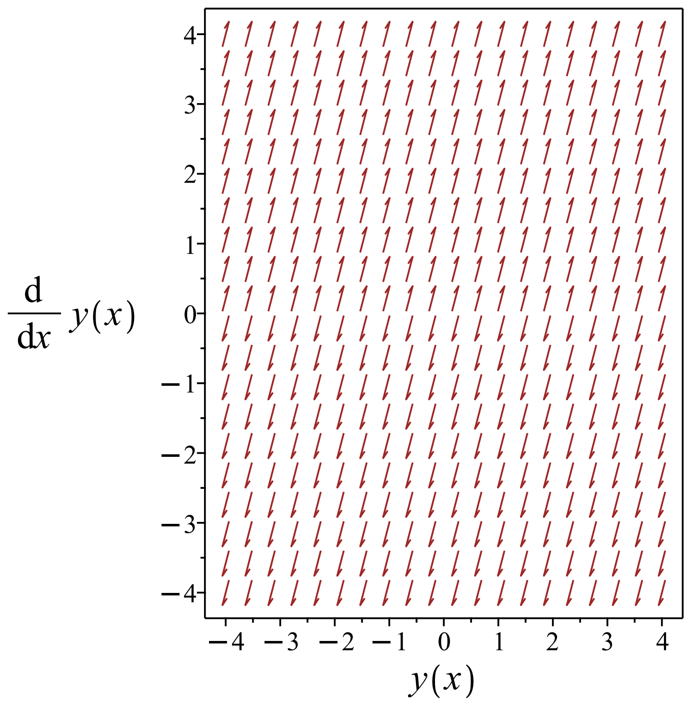


Figure 697: Slope field plot

Verification of solutions

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Verified OK.

17.27.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 3p(x) - 1 - e^{x} - \cos(x) - \sin(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -3 \\
q(x) &= 1 + e^{x} + \cos(x) + \sin(x)
\end{aligned}$$

Hence the ode is

$$p'(x) - 3p(x) = 1 + e^{x} + \cos(x) + \sin(x)$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int (-3)dx} \\
&= e^{-3x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu p) &= (\mu)\left(1 + e^{x} + \cos(x) + \sin(x)\right) \\
\frac{d}{dx}\left(e^{-3x} p\right) &= \left(e^{-3x}\right)\left(1 + e^{x} + \cos(x) + \sin(x)\right) \\
d\left(e^{-3x} p\right) &= \left(\left(1 + e^{x} + \cos(x) + \sin(x)\right) e^{-3x}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-3x} p &= \int \left(1 + e^{x} + \cos(x) + \sin(x)\right) e^{-3x} \, dx \\
e^{-3x} p &= -\frac{e^{-3x}}{3} - \frac{e^{-2x}}{2} - \frac{2 e^{-3x} \cos(x)}{5} - \frac{e^{-3x} \sin(x)}{5} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3x}$ results in

$$p(x) = \mathrm{e}^{3x}\left(-\frac{\mathrm{e}^{-3x}}{3} - \frac{\mathrm{e}^{-2x}}{2} - \frac{2\,\mathrm{e}^{-3x}\cos(x)}{5} - \frac{\mathrm{e}^{-3x}\sin(x)}{5}\right) + c_1\mathrm{e}^{3x}$$

which simplifies to

$$p(x) = c_1\mathrm{e}^{3x} - \frac{2\cos(x)}{5} - \frac{\sin(x)}{5} - \frac{1}{3} - \frac{\mathrm{e}^{x}}{2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1\mathrm{e}^{3x} - \frac{2\cos(x)}{5} - \frac{\sin(x)}{5} - \frac{1}{3} - \frac{\mathrm{e}^{x}}{2}$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1\mathrm{e}^{3x} - \frac{2\cos(x)}{5} - \frac{\sin(x)}{5} - \frac{1}{3} - \frac{\mathrm{e}^{x}}{2}\,\mathrm{d}x \\ &= -\frac{x}{3} + \frac{c_1\mathrm{e}^{3x}}{3} - \frac{2\sin(x)}{5} - \frac{\mathrm{e}^{x}}{2} + \frac{\cos(x)}{5} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{3} + \frac{c_1\mathrm{e}^{3x}}{3} - \frac{2\sin(x)}{5} - \frac{\mathrm{e}^{x}}{2} + \frac{\cos(x)}{5} + c_2 \tag{1}$$

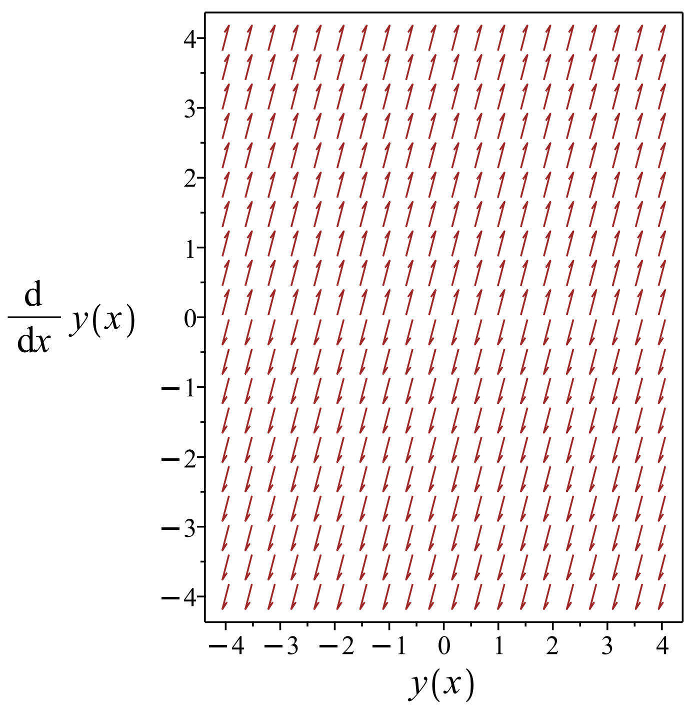


Figure 698: Slope field plot

Verification of solutions

$$y = -\frac{x}{3} + \frac{c_1 e^{3x}}{3} - \frac{2\sin(x)}{5} - \frac{e^x}{2} + \frac{\cos(x)}{5} + c_2$$

Verified OK.

17.27.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 3y' = 1 + e^x + \cos(x) + \sin(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = \int (1 + e^x + \cos(x) + \sin(x))\, dx$$

$$-3y + y' = x + \sin(x) + e^x - \cos(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= x + \sin(x) + e^x - \cos(x) + c_1 \end{aligned}$$

Hence the ode is

$$-3y + y' = x + \sin(x) + e^x - \cos(x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int (-3)dx} \\ &= e^{-3x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(x + \sin(x) + e^x - \cos(x) + c_1) \\ \frac{d}{dx}\left(e^{-3x} y\right) &= \left(e^{-3x}\right)(x + \sin(x) + e^x - \cos(x) + c_1) \\ d\left(e^{-3x} y\right) &= \left((x + \sin(x) + e^x - \cos(x) + c_1)\, e^{-3x}\right)\, dx \end{aligned}$$

Integrating gives

$$e^{-3x}y = \int \left(x+\sin(x)+e^{x}-\cos(x)+c_1\right)e^{-3x}\,dx$$
$$e^{-3x}y = -\frac{x\,e^{-3x}}{3}-\frac{e^{-3x}}{9}+\frac{e^{-3x}\cos(x)}{5}-\frac{2\,e^{-3x}\sin(x)}{5}-\frac{e^{-2x}}{2}-\frac{c_1e^{-3x}}{3}+c_2$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x}\left(-\frac{x\,e^{-3x}}{3}-\frac{e^{-3x}}{9}+\frac{e^{-3x}\cos(x)}{5}-\frac{2\,e^{-3x}\sin(x)}{5}-\frac{e^{-2x}}{2}-\frac{c_1e^{-3x}}{3}\right)+c_2e^{3x}$$

which simplifies to

$$y = -\frac{1}{9}+c_2e^{3x}-\frac{e^{x}}{2}-\frac{c_1}{3}-\frac{x}{3}+\frac{\cos(x)}{5}-\frac{2\sin(x)}{5}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{9}+c_2e^{3x}-\frac{e^{x}}{2}-\frac{c_1}{3}-\frac{x}{3}+\frac{\cos(x)}{5}-\frac{2\sin(x)}{5} \tag{1}$$

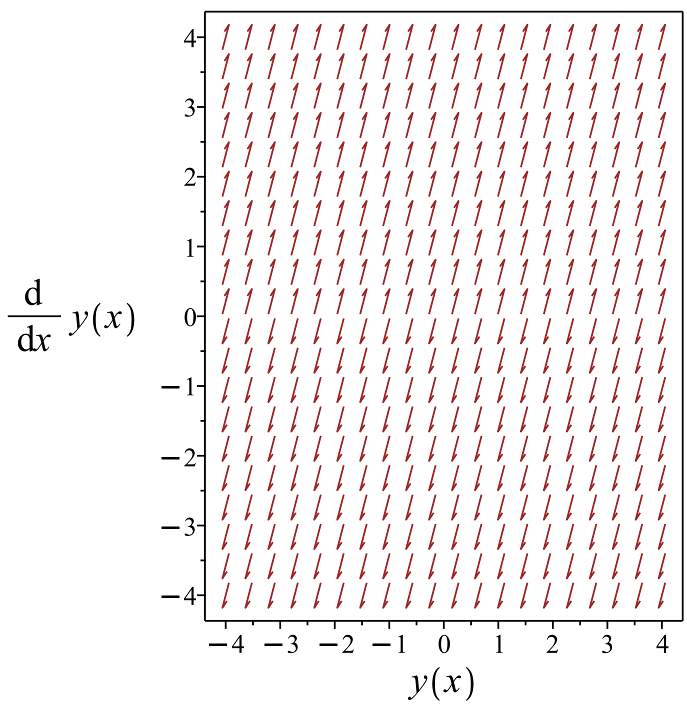


Figure 699: Slope field plot

Verification of solutions

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Verified OK.

17.27.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 544: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-3y'=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1+\frac{c_2 e^{3x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1+e^x+\cos(x)+\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\},\{e^x\},\{\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1,\frac{e^{3x}}{3}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\},\{e^x\},\{\cos(x),\sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p=A_1x+A_2e^x+A_3\cos(x)+A_4\sin(x)$$

The unknowns $\{A_1,A_2,A_3,A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2e^x-A_3\cos(x)-A_4\sin(x)-3A_1+3A_3\sin(x)-3A_4\cos(x) \\ =1+e^x+\cos(x)+\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{1}{2}, A_3 = \frac{1}{5}, A_4 = -\frac{2}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{3x}}{3}\right) + \left(-\frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{3x}}{3} - \frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5} \tag{1}$$

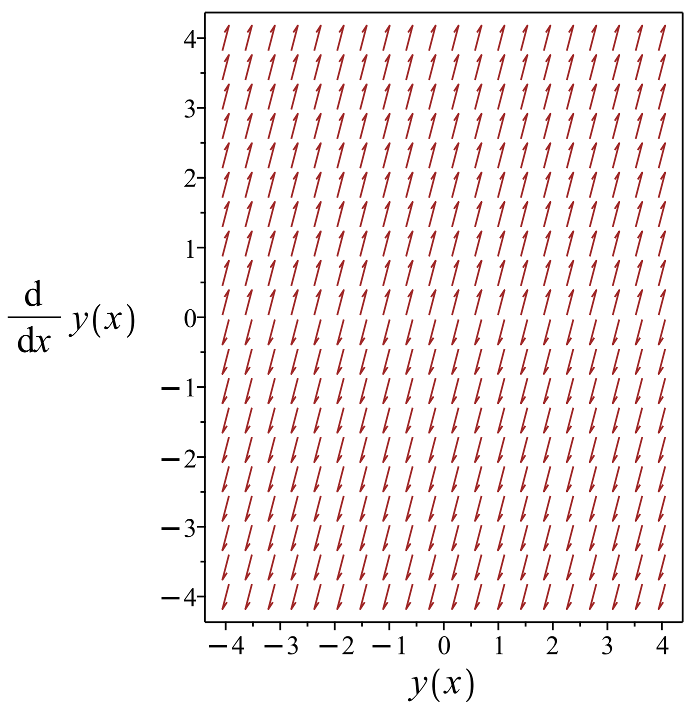


Figure 700: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{3x}}{3} - \frac{x}{3} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Verified OK.

17.27.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -3 \\ r(x) &= 0 \\ s(x) &= 1 + e^x + \cos(x) + \sin(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$-3y + y' = \int 1 + e^x + \cos(x) + \sin(x)\,dx$$

We now have a first order ode to solve which is

$$-3y + y' = x + \sin(x) + e^x - \cos(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -3 \\
q(x) &= x + \sin(x) + e^x - \cos(x) + c_1
\end{aligned}$$

Hence the ode is

$$-3y + y' = x + \sin(x) + e^x - \cos(x) + c_1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int (-3) dx} \\
&= e^{-3x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x + \sin(x) + e^x - \cos(x) + c_1) \\
\frac{d}{dx}(e^{-3x} y) &= (e^{-3x})(x + \sin(x) + e^x - \cos(x) + c_1) \\
d(e^{-3x} y) &= ((x + \sin(x) + e^x - \cos(x) + c_1) e^{-3x})\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{-3x} y &= \int (x + \sin(x) + e^x - \cos(x) + c_1) e^{-3x}\, dx \\
e^{-3x} y &= -\frac{x\, e^{-3x}}{3} - \frac{e^{-3x}}{9} + \frac{e^{-3x} \cos(x)}{5} - \frac{2\, e^{-3x} \sin(x)}{5} - \frac{e^{-2x}}{2} - \frac{c_1 e^{-3x}}{3} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-3x}$ results in

$$y = e^{3x} \left(-\frac{x\, e^{-3x}}{3} - \frac{e^{-3x}}{9} + \frac{e^{-3x} \cos(x)}{5} - \frac{2\, e^{-3x} \sin(x)}{5} - \frac{e^{-2x}}{2} - \frac{c_1 e^{-3x}}{3} \right) + c_2 e^{3x}$$

which simplifies to

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5} \tag{1}$$

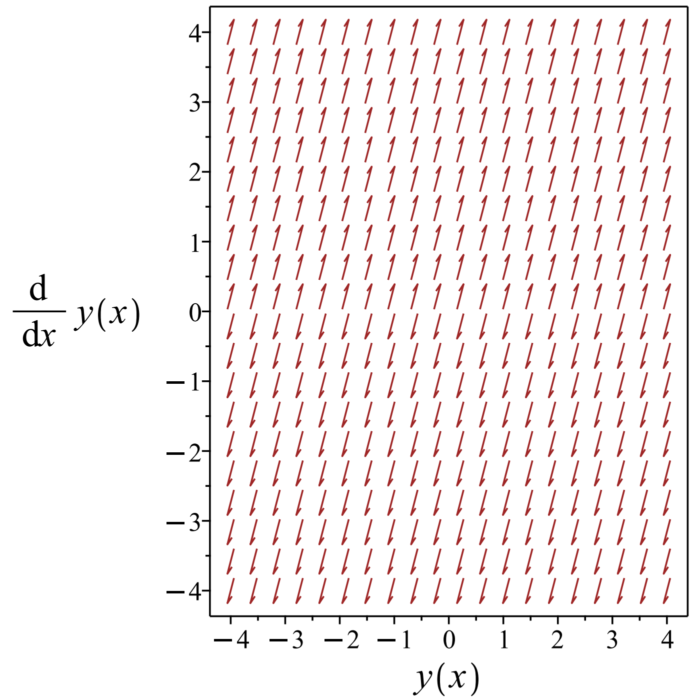


Figure 701: Slope field plot

Verification of solutions

$$y = -\frac{1}{9} + c_2 e^{3x} - \frac{e^x}{2} - \frac{c_1}{3} - \frac{x}{3} + \frac{\cos(x)}{5} - \frac{2\sin(x)}{5}$$

Verified OK.

17.27.7 Maple step by step solution

Let's solve

$y'' - 3y' = 1 + e^x + \cos(x) + \sin(x)$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 3r = 0$

- Factor the characteristic polynomial

  $r(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (0, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1 + e^x + \cos(x) + \sin(\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{3x} \\ 0 & 3 e^{3x} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3 e^{3x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\left(\int (1+e^x+\cos(x)+\sin(x))dx\right)}{3} + \frac{e^{3x}\left(\int (1+e^x+\cos(x)+\sin(x))e^{-3x}dx\right)}{3}$

  ◦ Compute integrals

  $y_p(x) = -\frac{x}{3} - \frac{2\sin(x)}{5} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{1}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{3x} - \frac{x}{3} - \frac{2\sin(x)}{5} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{1}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 3*_b(_a)+1+exp(_a)+cos(_a)+sin(_a), _b(
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)=1+exp(x)+cos(x)+sin(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{3x}}{3} - \frac{2\sin(x)}{5} - \frac{e^x}{2} + \frac{\cos(x)}{5} - \frac{x}{3} + c_2$$

✓ Solution by Mathematica
Time used: 0.261 (sec). Leaf size: 43

```
DSolve[y''[x]-3*y'[x]==1+Exp[x]+Cos[x]+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{3} - \frac{e^x}{2} - \frac{2\sin(x)}{5} + \frac{\cos(x)}{5} + \frac{1}{3}c_1 e^{3x} + c_2$$

17.28 problem 578

Internal problem ID [15347]
Internal file name [`OUTPUT/15347_Wednesday_May_08_2024_03_56_34_PM_51317879/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 578.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 5y = e^{x}\left(1 - 2\sin\left(x\right)^{2}\right) + 10x + 1}$$

17.28.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 5, f(x) = e^{x}\cos\left(2x\right) + 10x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda \, e^{\lambda x} + 5 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(5)} \\
&= 1 \pm 2i
\end{aligned}$$

Hence

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Which simplifies to

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} \cos(2x) + 10x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{e^{x} \cos(2x), e^{x} \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x} \cos(2x), e^{x} \sin(2x)\}$$

Since $e^{x} \cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x\}, \{x\, e^{x} \cos(2x), x\, e^{x} \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x + A_1 + A_3 x\, e^{x} \cos(2x) + A_4 x\, e^{x} \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3 e^{x} \sin(2x) + 4A_4 e^{x} \cos(2x) - 2A_2 + 5A_2 x + 5A_1 = e^{x} \cos(2x) + 10x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 2, A_3 = 0, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x + 1 + \frac{x\, e^{x} \sin(2x)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^x (c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(2x + 1 + \frac{x\,e^x \sin(2x)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^x (c_1 \cos(2x) + c_2 \sin(2x)) + 2x + 1 + \frac{x\,e^x \sin(2x)}{4} \tag{1}
$$

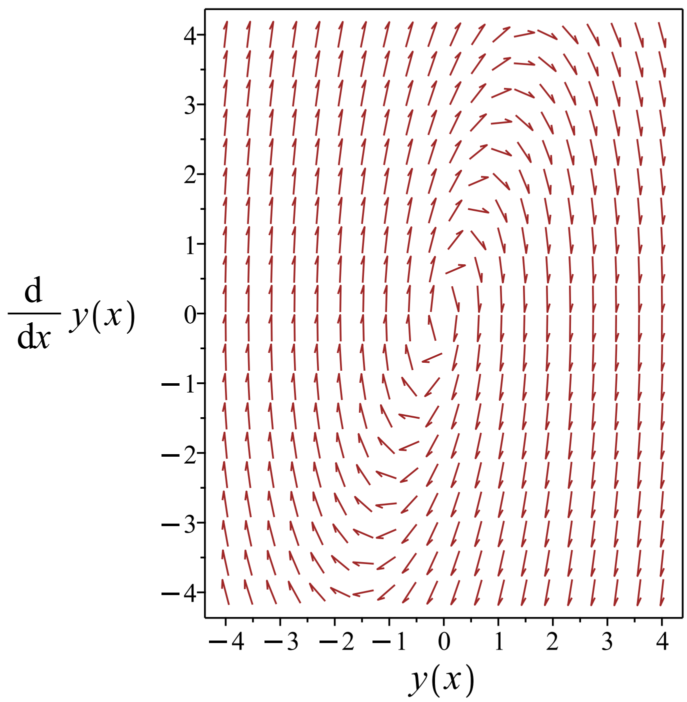


Figure 702: Slope field plot

Verification of solutions

$$
y = e^x (c_1 \cos(2x) + c_2 \sin(2x)) + 2x + 1 + \frac{x\,e^x \sin(2x)}{4}
$$

Verified OK.

17.28.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 546: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x \cos(2x)) + c_2\left(\mathrm{e}^x \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x \cos(2x)\, c_1 + \frac{\sin(2x)\, \mathrm{e}^x c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} \cos(2x) + 10x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{e^{x} \cos(2x), e^{x} \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{x} \cos(2x), \frac{e^{x} \sin(2x)}{2}\right\}$$

Since $e^{x} \cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x\}, \{x\, e^{x} \cos(2x), x\, e^{x} \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x + A_1 + A_3 x\, e^{x} \cos(2x) + A_4 x\, e^{x} \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_3 e^{x} \sin(2x) + 4A_4 e^{x} \cos(2x) - 2A_2 + 5A_2 x + 5A_1 = e^{x} \cos(2x) + 10x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 2, A_3 = 0, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x + 1 + \frac{x\, e^{x} \sin(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{x} \cos(2x)\, c_1 + \frac{\sin(2x)\, e^{x} c_2}{2}\right) + \left(2x + 1 + \frac{x\, e^{x} \sin(2x)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{x} \cos(2x) c_1 + \frac{\sin(2x) e^{x} c_2}{2} + 2x + 1 + \frac{x e^{x} \sin(2x)}{4} \tag{1}$$

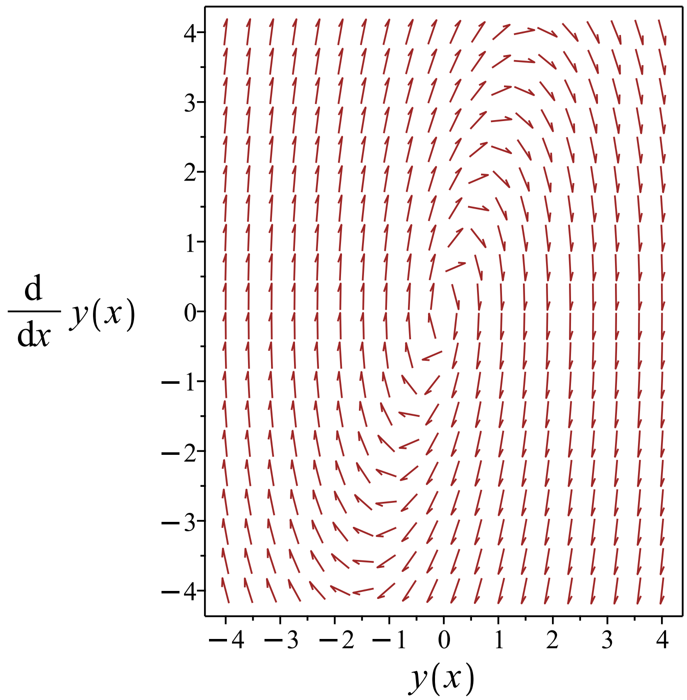


Figure 703: Slope field plot

Verification of solutions

$$y = e^{x} \cos(2x) c_1 + \frac{\sin(2x) e^{x} c_2}{2} + 2x + 1 + \frac{x e^{x} \sin(2x)}{4}$$

Verified OK.

17.28.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = e^{x} \cos(2x) + 10x + 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

$r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^x \cos(2x)$

- 2nd solution of the homogeneous ODE

$y_2(x) = \mathrm{e}^x \sin(2x)$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = \mathrm{e}^x \cos(2x)\, c_1 + \sin(2x)\, \mathrm{e}^x c_2 + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^x \cos(2x) + 10x + 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x \cos(2x) & \mathrm{e}^x \sin(2x) \\ \mathrm{e}^x \cos(2x) - 2\,\mathrm{e}^x \sin(2x) & \mathrm{e}^x \sin(2x) + 2\,\mathrm{e}^x \cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^x \left(\cos(2x) \left(\int \sin(2x) \left(\cos(2x) + 10x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}\right) dx\right) - \sin(2x) \left(\int \cos(2x) \left(\cos(2x) + 10x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}\right) dx\right)\right)}{2}$

  - Compute integrals

    $y_p(x) = 2x + 1 + \frac{x\,\mathrm{e}^x \sin(2x)}{4}$

- Substitute particular solution into general solution to ODE

$y = \mathrm{e}^x \cos(2x)\, c_1 + \sin(2x)\, \mathrm{e}^x c_2 + \frac{x\,\mathrm{e}^x \sin(2x)}{4} + 2x + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=exp(x)*(1-2*sin(x)^2)+10*x+1,y(x), singsol=all)
```

$$y(x) = \frac{e^{x}(x+4c_2)\sin(2x)}{4} + e^{x}\cos(2x)\,c_1 + 2x + 1$$

✓ Solution by Mathematica

Time used: 1.163 (sec). Leaf size: 44

```
DSolve[y''[x]-2*y'[x]+5*y[x]==Exp[x]*(1-2*Sin[x]^2)+10*x+1,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to 2x + \frac{1}{16}(1+16c_2)e^{x}\cos(2x) + \frac{1}{4}e^{x}(x+4c_1)\sin(2x) + 1$$

17.29 problem 579

Internal problem ID [15348]
Internal file name [`OUTPUT/15348_Wednesday_May_08_2024_03_56_36_PM_55756618/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 579.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = 4x + \sin(x) + \sin(2x)}$$

17.29.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 4x + \sin(x) + \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-4)^2 - (4)\,(1)\,(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x\, e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 x\, e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x + \sin(x) + \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1 + A_3 \cos(x) + A_4 \sin(x) + A_5 \cos(2x) + A_6 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&3A_3 \cos(x) + 3A_4 \sin(x) - 4A_2 + 4A_3 \sin(x) - 4A_4 \cos(x)\\&+ 8A_5 \sin(2x) - 8A_6 \cos(2x) + 4A_2 x + 4A_1 = 4x + \sin(x) + \sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 1, A_3 = \frac{4}{25}, A_4 = \frac{3}{25}, A_5 = \frac{1}{8}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(e^{2x} c_1 + c_2 x\, e^{2x}\right) + \left(x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}\right)\end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8} \tag{1}$$

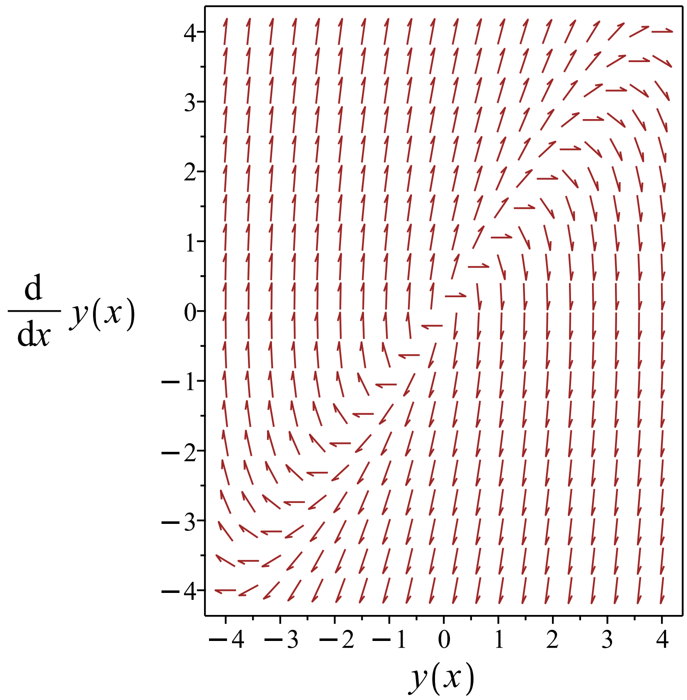


Figure 704: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

Verified OK.

17.29.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{-2x}(4x + \sin(x) + \sin(2x)) \\
\left(e^{-2x}y\right)'' &= e^{-2x}(4x + \sin(x) + \sin(2x))
\end{aligned}
$$

Integrating once gives

$$
\left(e^{-2x}y\right)' = \frac{\left(-10\cos(x)^2 + (-10\sin(x) - 4)\cos(x) - 40x - 8\sin(x) - 15\right)e^{-2x}}{20} + c_1
$$

Integrating again gives

$$
\left(e^{-2x}y\right) = \frac{\left(50\cos(x)^2 + 200x + 32\cos(x) + 24\sin(x) + 175\right)e^{-2x}}{200} + c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{\frac{\left(50\cos(x)^2+200x+32\cos(x)+24\sin(x)+175\right)e^{-2x}}{200} + c_1 x + c_2}{e^{-2x}}
$$

Or

$$
y = c_1 x\, e^{2x} + e^{2x} c_2 + \frac{\cos(x)^2}{4} + x + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{7}{8}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x\, e^{2x} + e^{2x} c_2 + \frac{\cos(x)^2}{4} + x + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{7}{8} \tag{1}
$$

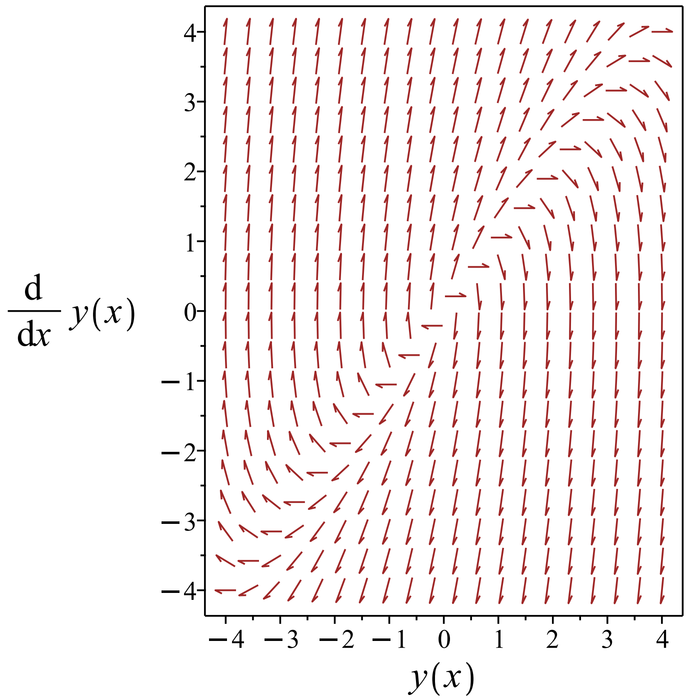


Figure 705: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \frac{\cos(x)^2}{4} + x + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{7}{8}$$

Verified OK.

17.29.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 548: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{2x} c_1 + c_2 x\,\mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
4x + \sin(x) + \sin(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{2x}, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1 + A_3 \cos(x) + A_4 \sin(x) + A_5 \cos(2x) + A_6 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3 \cos(x) + 3A_4 \sin(x) - 4A_2 + 4A_3 \sin(x) - 4A_4 \cos(x) \\ + 8A_5 \sin(2x) - 8A_6 \cos(2x) + 4A_2 x + 4A_1 = 4x + \sin(x) + \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 1, A_3 = \frac{4}{25}, A_4 = \frac{3}{25}, A_5 = \frac{1}{8}, A_6 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x} c_1 + c_2 x\, \mathrm{e}^{2x}\right) + \left(x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2 x + c_1) + x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_2 x + c_1) + x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8} \tag{1}$$

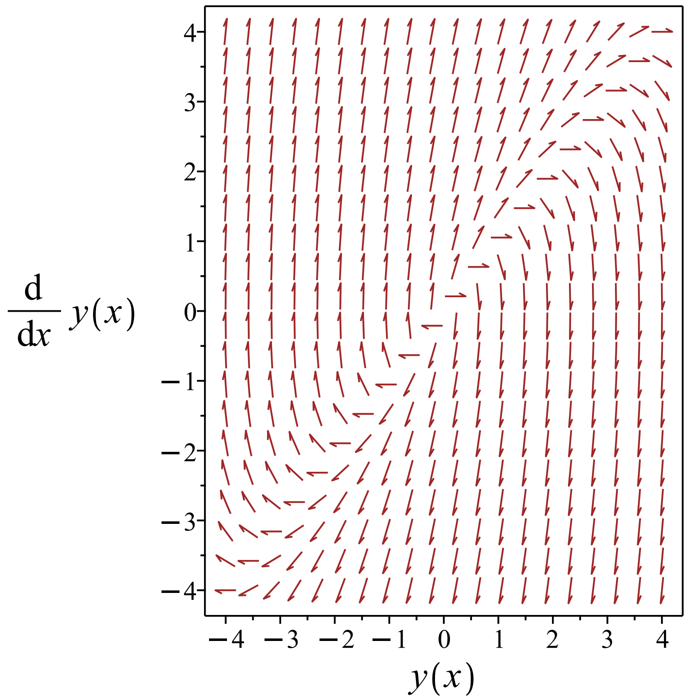


Figure 706: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + x + 1 + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

Verified OK.

17.29.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 4x + \sin(x) + \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

$r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + c_2 x\,e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4x + \sin(x) + \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{2x}\left(-\left(\int (4x + \sin(x) + \sin(2x))\,x\,e^{-2x}dx\right) + x\left(\int e^{-2x}(4x + \sin(x) + \sin(2x))\,dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{\cos(x)^2}{4} + x + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{7}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^{2x} + e^{2x} c_1 + \frac{3\sin(x)}{25} + \frac{4\cos(x)}{25} + \frac{\cos(x)^2}{4} + \frac{7}{8} + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=4*x+sin(x)+sin(2*x),y(x), singsol=all)
```

$$y(x) = 1 + (c_1 x + c_2) e^{2x} + x + \frac{4\cos(x)}{25} + \frac{3\sin(x)}{25} + \frac{\cos(2x)}{8}$$

✓ Solution by Mathematica

Time used: 0.324 (sec). Leaf size: 45

```
DSolve[y''[x]-4*y'[x]+4*y[x]==4*x+Sin[x]+Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{3\sin(x)}{25} + \frac{4\cos(x)}{25} + \frac{1}{8}\cos(2x) + c_2 e^{2x} x + c_1 e^{2x} + 1$$

17.30 problem 580

Internal problem ID [15349]
Internal file name [`OUTPUT/15349_Wednesday_May_08_2024_03_56_38_PM_31575215/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 580.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 2y' + y = 1 + 2\cos(x) + \cos(2x) - \sin(2x)}$$

17.30.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 1 + 2\cos(x) + \cos(2x) - \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + 2\cos(x) + \cos(2x) - \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(x) + A_3 \sin(x) + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_4 \cos(2x) - 3A_5 \sin(2x) - 2A_2 \sin(x) + 2A_3 \cos(x) - 4A_4 \sin(2x) \\ + 4A_5 \cos(2x) + A_1 = 1 + 2\cos(x) + \cos(2x) - \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = 1, A_4 = \frac{1}{25}, A_5 = \frac{7}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 x\, e^{-x}\right) + \left(1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25} \tag{1}$$

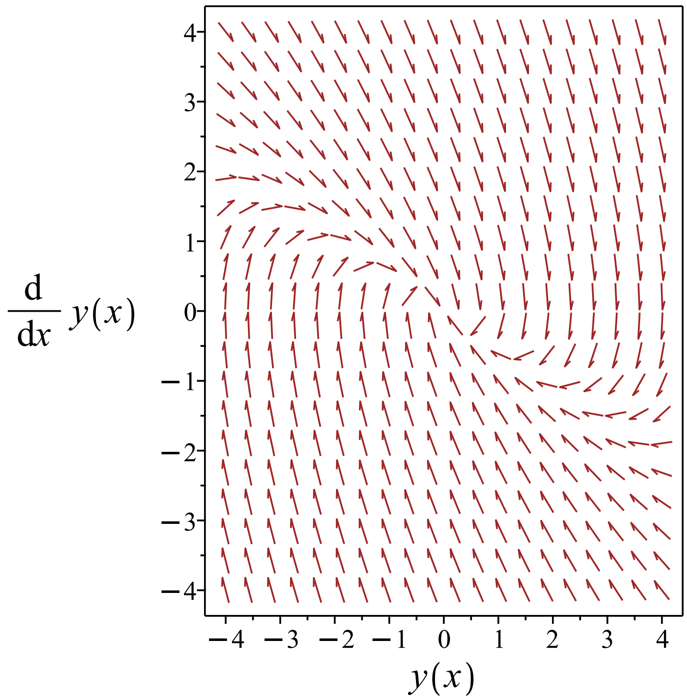


Figure 707: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

Verified OK.

17.30.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{x}(1 + 2\cos(x) + \cos(2x) - \sin(2x)) \\
(e^{x}y)'' &= e^{x}(1 + 2\cos(x) + \cos(2x) - \sin(2x))
\end{aligned}
$$

Integrating once gives

$$
(e^{x}y)' = \frac{\left(6\cos(x)^{2} + 2\sin(x)\cos(x) + 2 + 5\cos(x) + 5\sin(x)\right)e^{x}}{5} + c_1
$$

Integrating again gives

$$
(e^{x}y) = \frac{\left(2\cos(x)^{2} + 14\sin(x)\cos(x) + 25\sin(x) + 24\right)e^{x}}{25} + c_1x + c_2
$$

Hence the solution is

$$
y = \frac{\frac{\left(2\cos(x)^{2}+14\sin(x)\cos(x)+25\sin(x)+24\right)e^{x}}{25} + c_1x + c_2}{e^{x}}
$$

Or

$$
y = \frac{2\cos(x)^{2}}{25} + \frac{14\cos(x)\sin(x)}{25} + c_1x\,e^{-x} + \sin(x) + e^{-x}c_2 + \frac{24}{25}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\cos(x)^{2}}{25} + \frac{14\cos(x)\sin(x)}{25} + c_1x\,e^{-x} + \sin(x) + e^{-x}c_2 + \frac{24}{25} \tag{1}
$$

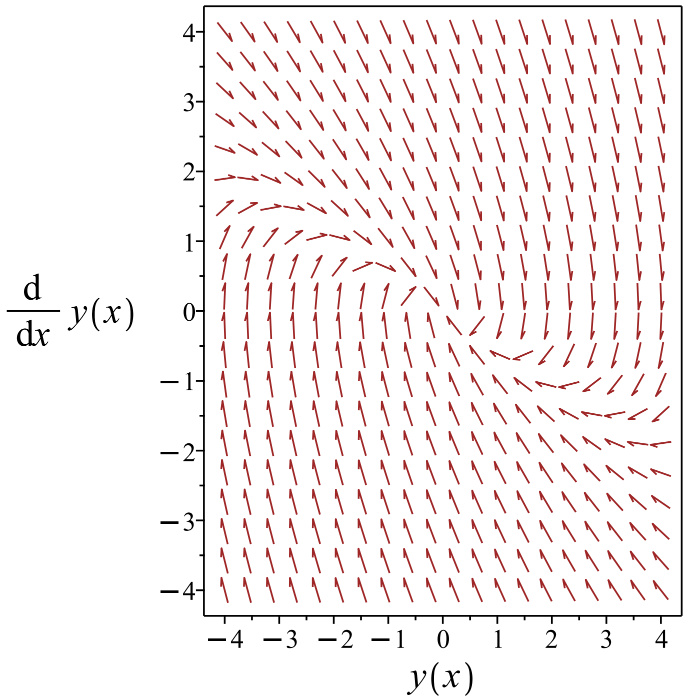


Figure 708: Slope field plot

Verification of solutions

$$y = \frac{2\cos(x)^2}{25} + \frac{14\cos(x)\sin(x)}{25} + c_1 x \,\mathrm{e}^{-x} + \sin(x) + \mathrm{e}^{-x} c_2 + \frac{24}{25}$$

Verified OK.

17.30.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 550: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\cos(x)\left(-\sin(x) + \cos(x) + 1\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x \,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 + A_2 \cos(x) + A_3 \sin(x) + A_4 \cos(2x) + A_5 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_4 \cos(2x) - 3A_5 \sin(2x) - 2A_2 \sin(x) + 2A_3 \cos(x) - 4A_4 \sin(2x) \\ + 4A_5 \cos(2x) + A_1 = 1 + 2\cos(x) + \cos(2x) - \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = 1, A_4 = \frac{1}{25}, A_5 = \frac{7}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}\right) + \left(1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25} \tag{1}$$

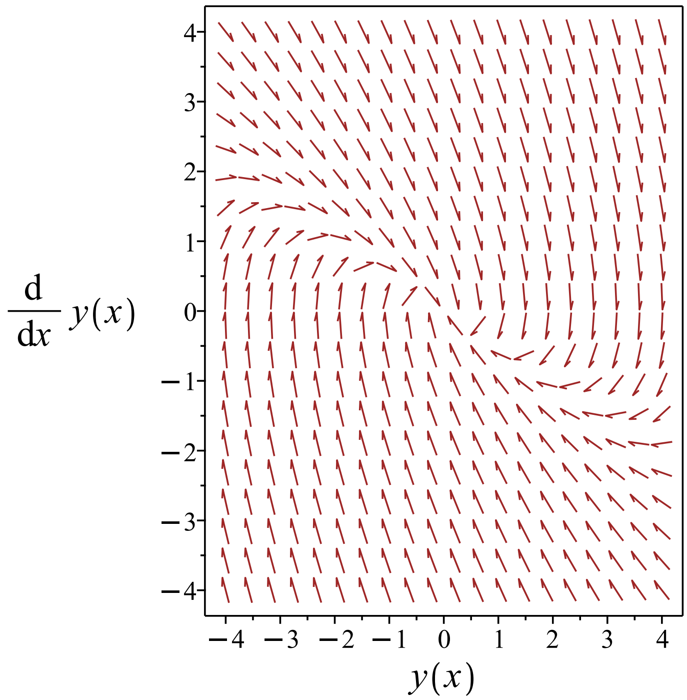


Figure 709: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + 1 + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

Verified OK.

17.30.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 1 + 2\cos(x) + \cos(2x) - \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r + 1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1 + 2\cos(x) + \cos(2x\right.$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 2\,\mathrm{e}^{-x}\left(-\left(\int x\cos(x)(-\sin(x) + \cos(x) + 1)\,\mathrm{e}^{x}dx\right) + x\left(\int \cos(x)(-\sin(x) + \cos(x) + 1\right.\right.$

  - Compute integrals

    $y_p(x) = \frac{(14\cos(x)+25)\sin(x)}{25} + \frac{2\cos(x)^2}{25} + \frac{24}{25}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} + \frac{(14\cos(x)+25)\sin(x)}{25} + \frac{2\cos(x)^2}{25} + \frac{24}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=1+2*cos(x)+cos(2*x)-sin(2*x),y(x), singsol=all)
```

$$y(x) = 1 + (c_1 x + c_2)\,\mathrm{e}^{-x} + \sin(x) + \frac{\cos(2x)}{25} + \frac{7\sin(2x)}{25}$$

✓ Solution by Mathematica

Time used: 1.184 (sec). Leaf size: 42

```
DSolve[y''[x]+2*y'[x]+y[x]==1+2*Cos[x]+Cos[2*x]-Sin[2*x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \sin(x) + \frac{7}{25}\sin(2x) + \frac{1}{25}\cos(2x) + c_1 e^{-x} + c_2 e^{-x} x + 1$$

17.31 problem 581

Internal problem ID [15350]
Internal file name [`OUTPUT/15350_Wednesday_May_08_2024_03_56_40_PM_30379150/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 581.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = -1 + \sin(x) + x + x^2}$$

17.31.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = -1 + \sin(x) + x + x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-1 + \sin(x) + x + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_5 - A_1 \sin(x) + A_2 \cos(x) + A_4 + 2A_5 x + A_3 + A_4 x + A_5 x^2 = -1 + \sin(x) + x + x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0, A_3 = -2, A_4 = -1, A_5 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x) - 2 - x + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)\right) + \left(-\cos(x) - 2 - x + x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{x\sqrt{3}}{2} \right) + c_2 \sin \left(\frac{x\sqrt{3}}{2} \right) \right) - \cos(x) - 2 - x + x^2 \tag{1}$$

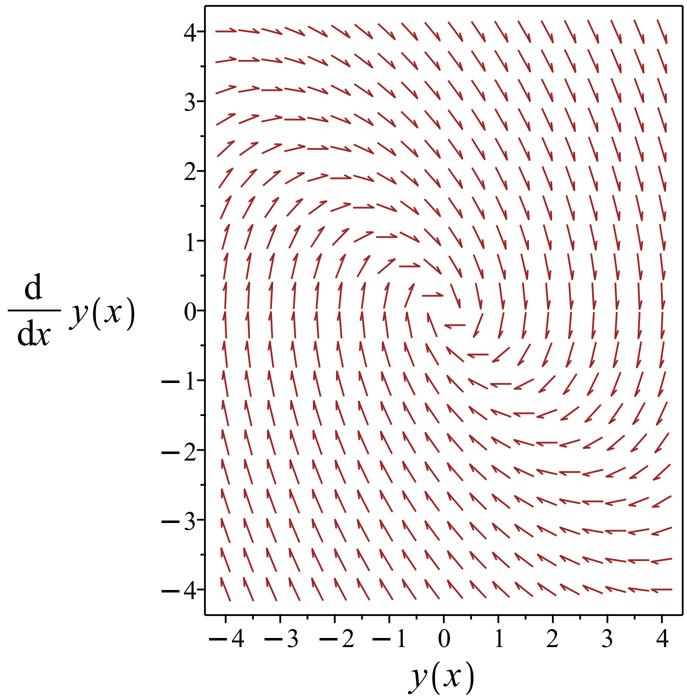


Figure 710: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{x\sqrt{3}}{2} \right) + c_2 \sin \left(\frac{x\sqrt{3}}{2} \right) \right) - \cos(x) - 2 - x + x^2$$

Verified OK.

17.31.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 552: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{x\sqrt{3}}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{x\sqrt{3}}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{x\sqrt{3}}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\frac{x\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \frac{2c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-1 + \sin(x) + x + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right), \frac{2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_5 - A_1 \sin(x) + A_2 \cos(x) + A_4 + 2A_5 x + A_3 + A_4 x + A_5 x^2 = -1 + \sin(x) + x + x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0, A_3 = -2, A_4 = -1, A_5 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x) - 2 - x + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3}\right) + \left(-\cos(x) - 2 - x + x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \cos(x) - 2 - x + x^2 \tag{1}$$

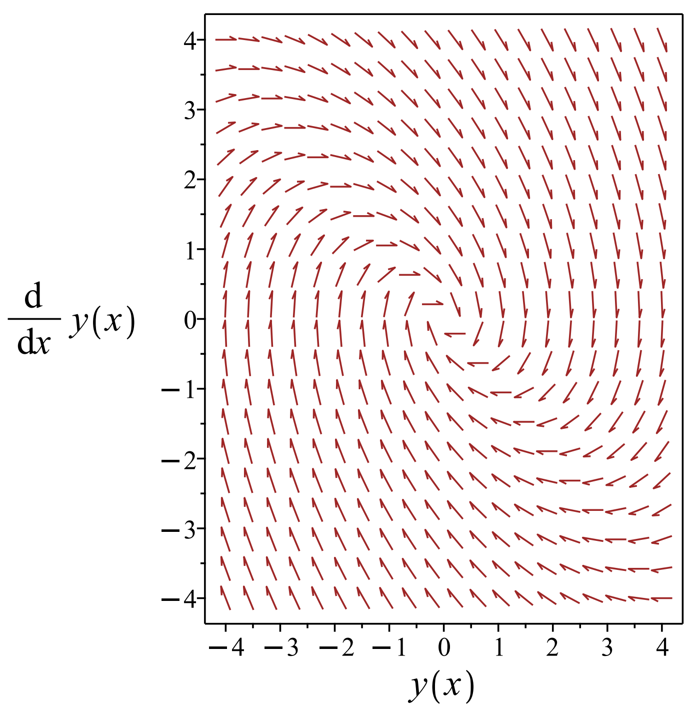


Figure 711: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \frac{2c_2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \cos(x) - 2 - x + x^2$$

Verified OK.

17.31.3 Maple step by step solution

Let's solve

$y'' + y' + y = -1 + \sin(x) + x + x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{x\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \sin\left(\frac{x\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -1 + \sin(x) + x + x^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2 e^{-\frac{x}{2}} \sqrt{3} \left(\cos\left(\frac{x\sqrt{3}}{2}\right)\left(\int e^{\frac{x}{2}} (-1+\sin(x)+x+x^2) \sin\left(\frac{x\sqrt{3}}{2}\right)dx\right) - \sin\left(\frac{x\sqrt{3}}{2}\right)\left(\int e^{\frac{x}{2}} (-1+\sin(x)+x+x^2) \cos\left(\frac{x\sqrt{3}}{2}\right)dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = -\cos(x) - 2 - x + x^2$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{x\sqrt{3}}{2}\right) e^{-\frac{x}{2}} c_2 + x^2 - \cos(x) - x - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)+1=sin(x)+x+x^2,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right) c_1 - 2 + x^2 - \cos(x) - x$$

✓ Solution by Mathematica

Time used: 2.943 (sec). Leaf size: 59

```
DSolve[y''[x]+y'[x]+y[x]+1==Sin[x]+x+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 - x - \cos(x) + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) - 2$$

17.32 problem 582

Internal problem ID [15351]
Internal file name [`OUTPUT/15351_Wednesday_May_08_2024_03_56_42_PM_92338412/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 582.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 9y = 18\,\mathrm{e}^{-3x} + 8\sin(x) + 6\cos(x)}$$

17.32.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 18\,\mathrm{e}^{-3x} + 8\sin(x) + 6\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + c_2 x\, e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18\, e^{-3x} + 8 \sin(x) + 6 \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-3x}, e^{-3x}\}$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-3x}\}, \{\cos(x), \sin(x)\}]$$

Since $x\,e^{-3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{-3x}\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{-3x} + A_2\cos(x) + A_3\sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1e^{-3x} + 8A_2\cos(x) + 8A_3\sin(x) - 6A_2\sin(x) + 6A_3\cos(x)\\&= 18\,e^{-3x} + 8\sin(x) + 6\cos(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 9, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 9x^2e^{-3x} + \sin(x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_1e^{-3x} + c_2x\,e^{-3x}\right) + \left(9x^2e^{-3x} + \sin(x)\right)\end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2x + c_1) + 9x^2e^{-3x} + \sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_2x + c_1) + 9x^2e^{-3x} + \sin(x) \tag{1}$$

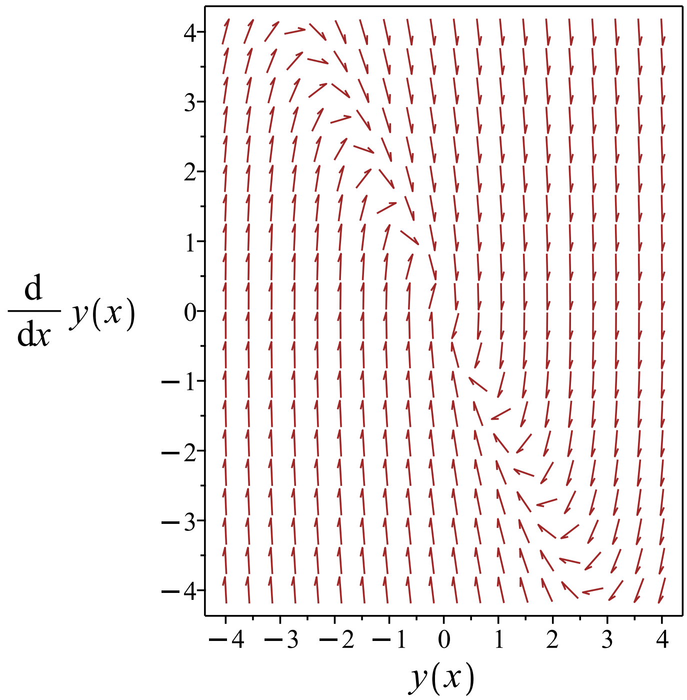


Figure 712: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2x + c_1) + 9x^2e^{-3x} + \sin(x)$$

Verified OK.

17.32.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{3x}\left(18\,e^{-3x} + 8\sin(x) + 6\cos(x)\right) \\
\left(e^{3x}y\right)'' &= e^{3x}\left(18\,e^{-3x} + 8\sin(x) + 6\cos(x)\right)
\end{aligned}
$$

Integrating once gives

$$
\left(e^{3x}y\right)' = 18x + (\cos(x) + 3\sin(x))\,e^{3x} + c_1
$$

Integrating again gives

$$
\left(e^{3x}y\right) = c_1x + e^{3x}\sin(x) + 9x^2 + c_2
$$

Hence the solution is

$$
y = \frac{c_1x + e^{3x}\sin(x) + 9x^2 + c_2}{e^{3x}}
$$

Or

$$
y = \sin(x) + c_1x\,e^{-3x} + 9x^2e^{-3x} + c_2e^{-3x}
$$

Summary

The solution(s) found are the following

$$
y = \sin(x) + c_1x\,e^{-3x} + 9x^2e^{-3x} + c_2e^{-3x} \tag{1}
$$

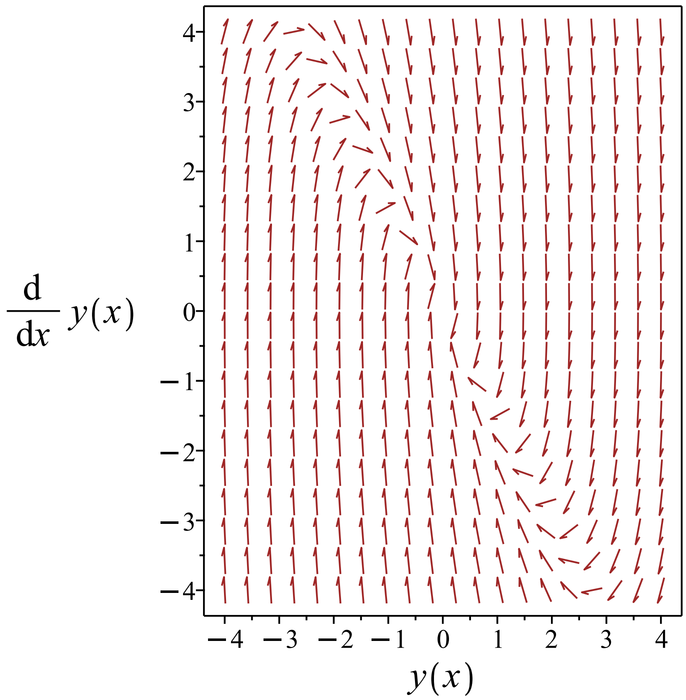


Figure 713: Slope field plot

Verification of solutions

$$y = \sin(x) + c_1 x\,\mathrm{e}^{-3x} + 9x^2\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x}$$

Verified OK.

17.32.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 554: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1 \left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
18\,\mathrm{e}^{-3x} + 8\sin(x) + 6\cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-3x}\}, \{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}
$$

Since e^{-3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^{-3x}\}, \{\cos(x), \sin(x)\}]
$$

Since $x\,\mathrm{e}^{-3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2 \mathrm{e}^{-3x}\}, \{\cos(x), \sin(x)\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{-3x} + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{-3x} + 8A_2 \cos(x) + 8A_3 \sin(x) - 6A_2 \sin(x) + 6A_3 \cos(x)$$
$$= 18\,e^{-3x} + 8\sin(x) + 6\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 9, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 9x^2 e^{-3x} + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 x\, e^{-3x}\right) + \left(9x^2 e^{-3x} + \sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2 x + c_1) + 9x^2 e^{-3x} + \sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_2 x + c_1) + 9x^2 e^{-3x} + \sin(x) \tag{1}$$

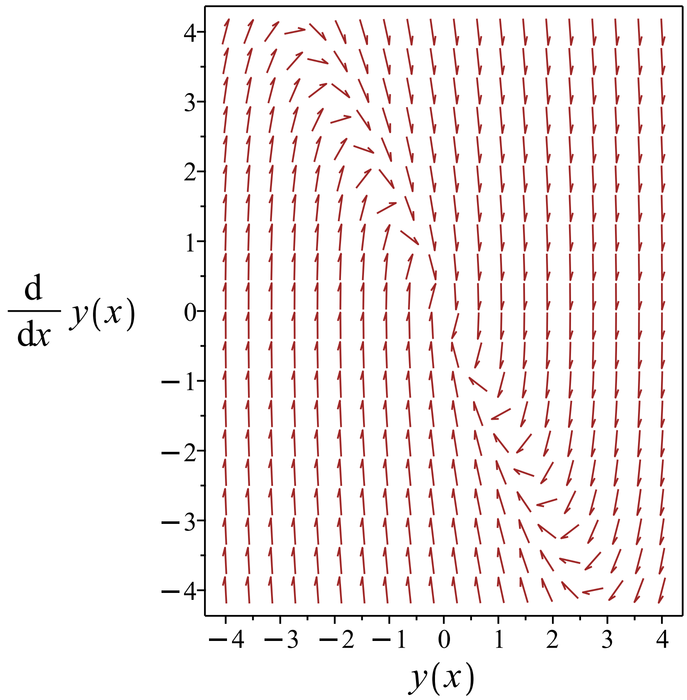


Figure 714: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2x + c_1) + 9x^2e^{-3x} + \sin(x)$$

Verified OK.

17.32.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 18\,e^{-3x} + 8\sin(x) + 6\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

$r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 x\, e^{-3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 18\, e^{-3x} + 8 \sin(x) + 6\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & x\, e^{-3x} \\ -3\, e^{-3x} & e^{-3x} - 3x\, e^{-3x} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-6x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\, e^{-3x}\left(-x\left(\int (9 + (4 \sin(x) + 3 \cos(x))\, e^{3x})\, dx\right) + \int 3\left(3 + \left(\cos(x) + \frac{4 \sin(x)}{3}\right) e^{3x}\right) x d$

  ◦ Compute integrals

  $y_p(x) = 9x^2 e^{-3x} + \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{-3x} + 9x^2 e^{-3x} + c_1 e^{-3x} + \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=18*exp(-3*x)+8*sin(x)+6*cos(x),y(x), singsol=all
```

$$y(x) = \left(c_1 x + 9x^2 + c_2\right) e^{-3x} + \sin(x)$$

✓ Solution by Mathematica

Time used: 0.257 (sec). Leaf size: 31

```
DSolve[y''[x]+6*y'[x]+9*y[x]==18*Exp[-3*x]+8*Sin[x]+6*Cos[x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to e^{-3x}\left(9x^2 + e^{3x} \sin(x) + c_2 x + c_1\right)$$

17.33 problem 583

Internal problem ID [15352]
Internal file name [`OUTPUT/15352_Wednesday_May_08_2024_03_56_43_PM_48545704/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 583.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2y' = -1 + 3\sin(2x) + \cos(x)}$$

17.33.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 0, f(x) = -1 + 3\sin(2x) + \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(0)}\\ &= -1 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = -1 + 1$$
$$\lambda_2 = -1 - 1$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 + e^{-2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-1 + 3\sin(2x) + \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{\cos(x), \sin(x)\}, \{\cos(2x), \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x + A_2 \cos(x) + A_3 \sin(x) + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2 \cos(x) - A_3 \sin(x) - 4A_4 \cos(2x) - 4A_5 \sin(2x) + 2A_1 - 2A_2 \sin(x) + 2A_3 \cos(x) - 4A_4 \sin(2x) + 4A_5 \cos(2x) = -1 + 3\sin(2x) + \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -\frac{1}{5}, A_3 = \frac{2}{5}, A_4 = -\frac{3}{8}, A_5 = -\frac{3}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{2} - \frac{\cos(x)}{5} + \frac{2\sin(x)}{5} - \frac{3\cos(2x)}{8} - \frac{3\sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + e^{-2x} c_2) + \left(-\frac{x}{2} - \frac{\cos(x)}{5} + \frac{2\sin(x)}{5} - \frac{3\cos(2x)}{8} - \frac{3\sin(2x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-2x} c_2 - \frac{x}{2} - \frac{\cos(x)}{5} + \frac{2\sin(x)}{5} - \frac{3\cos(2x)}{8} - \frac{3\sin(2x)}{8} \tag{1}$$

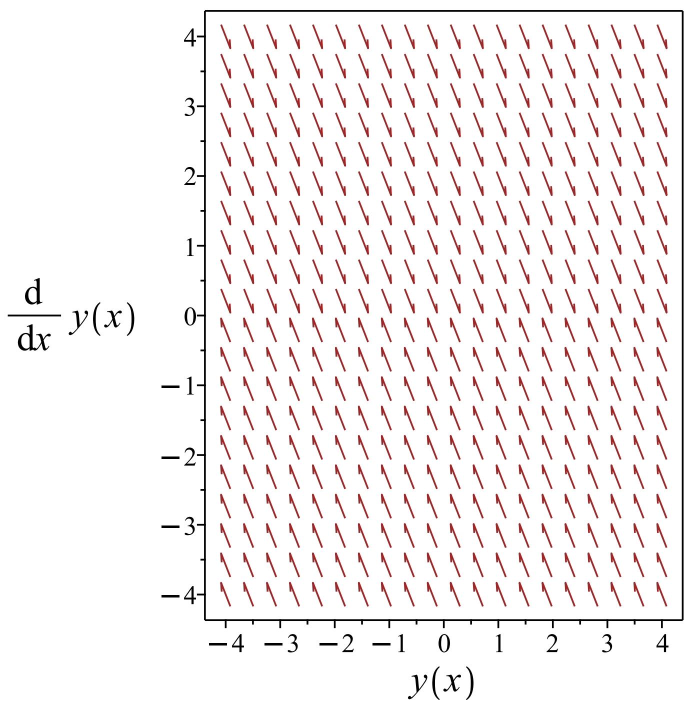


Figure 715: Slope field plot

Verification of solutions

$$y = c_1 + e^{-2x} c_2 - \frac{x}{2} - \frac{\cos(x)}{5} + \frac{2\sin(x)}{5} - \frac{3\cos(2x)}{8} - \frac{3\sin(2x)}{8}$$

Verified OK.

17.33.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = \int (-1 + 3\sin(2x) + \cos(x))\, dx$$

$$2y + y' = -x - \frac{3\cos(2x)}{2} + \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = -x - \frac{3\cos(2x)}{2} + \sin(x) + c_1$$

Hence the ode is

$$2y + y' = -x - \frac{3\cos(2x)}{2} + \sin(x) + c_1$$

The integrating factor μ is

$$\mu = e^{\int 2dx} = e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-x - \frac{3\cos(2x)}{2} + \sin(x) + c_1\right)$$
$$\frac{d}{dx}(e^{2x} y) = (e^{2x})\left(-x - \frac{3\cos(2x)}{2} + \sin(x) + c_1\right)$$
$$d(e^{2x} y) = \left(\frac{(2\sin(x) - 3\cos(2x) + 2c_1 - 2x)\, e^{2x}}{2}\right) dx$$

Integrating gives

$$e^{2x} y = \int \frac{(2\sin(x) - 3\cos(2x) + 2c_1 - 2x)\, e^{2x}}{2}\, dx$$
$$e^{2x} y = -\frac{3(2\sin(x) + 2\cos(x))\, e^{2x}\cos(x)}{8} + \frac{5\, e^{2x}}{8} - \frac{e^{2x}\cos(x)}{5} + \frac{2\, e^{2x}\sin(x)}{5} - \frac{x\, e^{2x}}{2} + \frac{e^{2x} c_1}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$y = \mathrm{e}^{-2x}\left(-\frac{3(2\sin(x)+2\cos(x))\,\mathrm{e}^{2x}\cos(x)}{8} + \frac{5\,\mathrm{e}^{2x}}{8} - \frac{\mathrm{e}^{2x}\cos(x)}{5} + \frac{2\,\mathrm{e}^{2x}\sin(x)}{5} - \frac{x\,\mathrm{e}^{2x}}{2} + \frac{\mathrm{e}^{2x}c_1}{2}\right) + \mathrm{e}^{-2}$$

which simplifies to

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + \mathrm{e}^{-2x}c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + \mathrm{e}^{-2x}c_2 \quad (1)$$

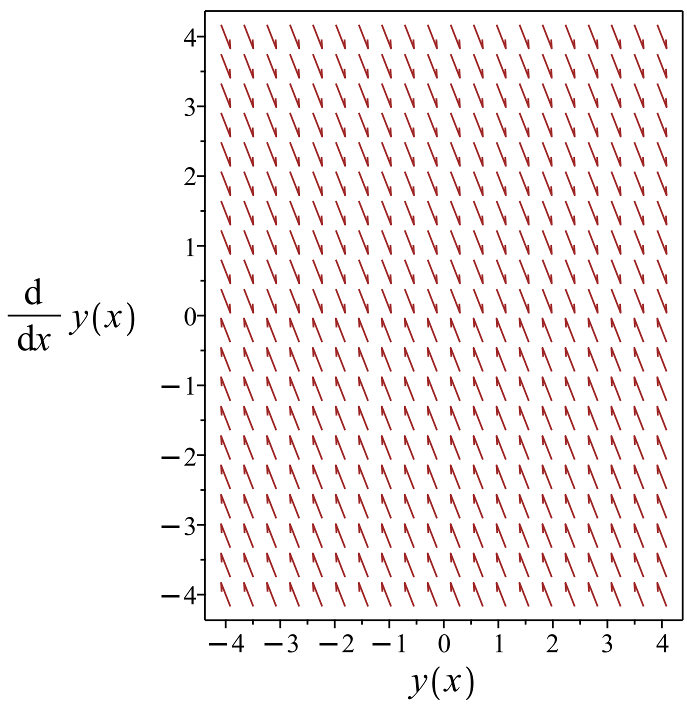


Figure 716: Slope field plot

Verification of solutions

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + \mathrm{e}^{-2x}c_2$$

Verified OK.

17.33.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2p(x) + 1 - 3\sin(2x) - \cos(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= -1 + 3\sin(2x) + \cos(x) \end{aligned}$$

Hence the ode is

$$p'(x) + 2p(x) = -1 + 3\sin(2x) + \cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 2dx} \\ &= e^{2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)(-1 + 3\sin(2x) + \cos(x)) \\ \frac{d}{dx}(e^{2x} p) &= (e^{2x})(-1 + 3\sin(2x) + \cos(x)) \\ d(e^{2x} p) &= ((-1 + 3\sin(2x) + \cos(x))e^{2x})\ dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{2x} p &= \int (-1 + 3\sin(2x) + \cos(x))e^{2x}\, dx \\ e^{2x} p &= -\frac{e^{2x}}{2} + \frac{3e^{2x}(2\sin(2x) - 2\cos(2x))}{8} + \frac{2e^{2x}\cos(x)}{5} + \frac{e^{2x}\sin(x)}{5} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$p(x) = e^{-2x}\left(-\frac{e^{2x}}{2} + \frac{3\,e^{2x}(2\sin(2x) - 2\cos(2x))}{8} + \frac{2\,e^{2x}\cos(x)}{5} + \frac{e^{2x}\sin(x)}{5}\right) + c_1 e^{-2x}$$

which simplifies to

$$p(x) = \frac{3\cos(x)\sin(x)}{2} - \frac{3\cos(x)^2}{2} + \frac{\sin(x)}{5} + \frac{2\cos(x)}{5} + \frac{1}{4} + c_1 e^{-2x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{3\cos(x)\sin(x)}{2} - \frac{3\cos(x)^2}{2} + \frac{\sin(x)}{5} + \frac{2\cos(x)}{5} + \frac{1}{4} + c_1 e^{-2x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{3\cos(x)\sin(x)}{2} - \frac{3\cos(x)^2}{2} + \frac{\sin(x)}{5} + \frac{2\cos(x)}{5} + \frac{1}{4} + c_1 e^{-2x}\, dx \\ &= -\frac{x}{2} - \frac{3\cos(x)\sin(x)}{4} - \frac{c_1 e^{-2x}}{2} + \frac{3\sin(x)^2}{4} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{2} - \frac{3\cos(x)\sin(x)}{4} - \frac{c_1 e^{-2x}}{2} + \frac{3\sin(x)^2}{4} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_2 \tag{1}$$

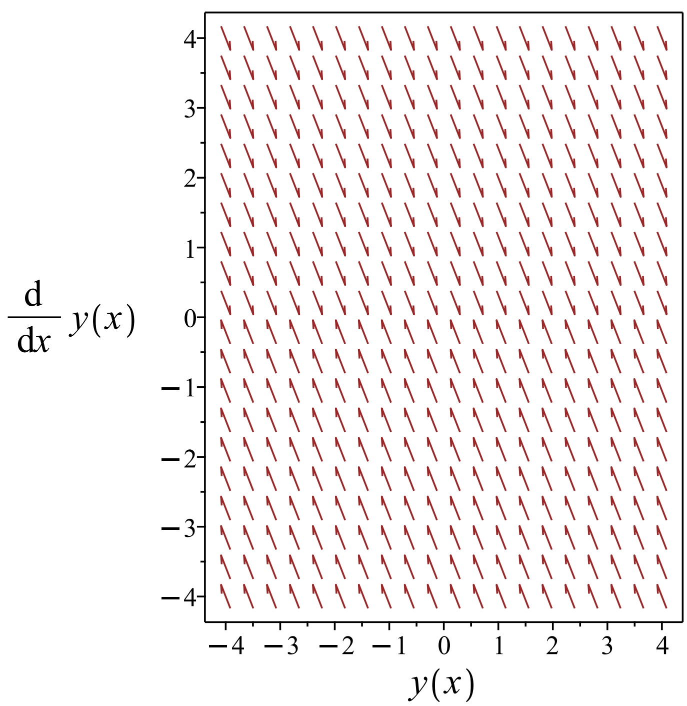


Figure 717: Slope field plot

Verification of solutions

$$y = -\frac{x}{2} - \frac{3\cos(x)\sin(x)}{4} - \frac{c_1 e^{-2x}}{2} + \frac{3\sin(x)^2}{4} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + c_2$$

Verified OK.

17.33.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y' = -1 + 3\sin(2x) + \cos(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 2y')\, dx = \int (-1 + 3\sin(2x) + \cos(x))\, dx$$

$$2y + y' = -x - \frac{3\cos(2x)}{2} + \sin(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = -x - \frac{3\cos(2x)}{2} + \sin(x) + c_1$$

Hence the ode is

$$2y + y' = -x - \frac{3\cos(2x)}{2} + \sin(x) + c_1$$

The integrating factor μ is

$$\mu = e^{\int 2dx}$$
$$= e^{2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-x - \frac{3\cos(2x)}{2} + \sin(x) + c_1\right)$$
$$\frac{d}{dx}(e^{2x} y) = (e^{2x})\left(-x - \frac{3\cos(2x)}{2} + \sin(x) + c_1\right)$$
$$d(e^{2x} y) = \left(\frac{(2\sin(x) - 3\cos(2x) + 2c_1 - 2x)\, e^{2x}}{2}\right) dx$$

Integrating gives

$$e^{2x}y = \int \frac{(2\sin(x) - 3\cos(2x) + 2c_1 - 2x)\, e^{2x}}{2}\, dx$$

$$e^{2x}y = -\frac{3(2\sin(x) + 2\cos(x))\, e^{2x}\cos(x)}{8} + \frac{5\, e^{2x}}{8} - \frac{e^{2x}\cos(x)}{5} + \frac{2\, e^{2x}\sin(x)}{5} - \frac{x\, e^{2x}}{2} + \frac{e^{2x}c_1}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{3(2\sin(x) + 2\cos(x))\, e^{2x}\cos(x)}{8} + \frac{5\, e^{2x}}{8} - \frac{e^{2x}\cos(x)}{5} + \frac{2\, e^{2x}\sin(x)}{5} - \frac{x\, e^{2x}}{2} + \frac{e^{2x}c_1}{2}\right) + e^{-2}$$

which simplifies to

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + e^{-2x}c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + e^{-2x}c_2 \quad (1)$$

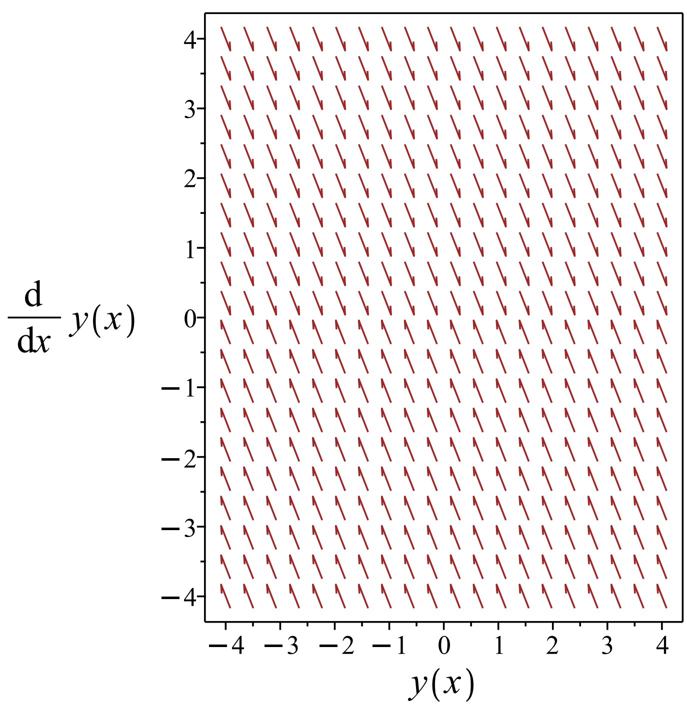


Figure 718: Slope field plot

Verification of solutions

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + \mathrm{e}^{-2x}c_2$$

Verified OK.

17.33.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 556: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$
$$y_2 = \frac{1}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & \frac{1}{2} \\ \frac{d}{dx}\left(e^{-2x}\right) & \frac{d}{dx}\left(\frac{1}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & \frac{1}{2} \\ -2\,e^{-2x} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)(0) - \left(\frac{1}{2}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{1}{2} + \frac{3\sin(2x)}{2} + \frac{\cos(x)}{2}}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(-1 + 3\sin(2x) + \cos(x)\right)e^{2x}}{2}dx$$

Hence

$$u_1 = \frac{e^{2x}}{4} - \frac{3\,e^{2x}\left(2\sin(2x) - 2\cos(2x)\right)}{16} - \frac{e^{2x}\cos(x)}{5} - \frac{e^{2x}\sin(x)}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}\left(-1 + 3\sin(2x) + \cos(x)\right)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(-1 + 3\sin(2x) + \cos(x)\right)dx$$

Hence

$$u_2 = -x - \frac{3\cos(2x)}{2} + \sin(x)$$

Which simplifies to

$$u_1 = \frac{\left(30\cos(x)^2+(-30\sin(x)-8)\cos(x)-4\sin(x)-5\right)e^{2x}}{40}$$
$$u_2 = -x-\frac{3\cos(2x)}{2}+\sin(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(30\cos(x)^2+(-30\sin(x)-8)\cos(x)-4\sin(x)-5\right)e^{2x}e^{-2x}}{40} - \frac{x}{2}-\frac{3\cos(2x)}{4}+\frac{\sin(x)}{2}$$

Which simplifies to

$$y_p(x) = -\frac{3\cos(x)\sin(x)}{4}+\frac{2\sin(x)}{5}-\frac{3\cos(x)^2}{4}-\frac{\cos(x)}{5}+\frac{5}{8}-\frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h+y_p \\ &= \left(c_1e^{-2x}+\frac{c_2}{2}\right)+\left(-\frac{3\cos(x)\sin(x)}{4}+\frac{2\sin(x)}{5}-\frac{3\cos(x)^2}{4}-\frac{\cos(x)}{5}+\frac{5}{8}-\frac{x}{2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-2x}+\frac{c_2}{2}-\frac{3\cos(x)\sin(x)}{4}+\frac{2\sin(x)}{5}-\frac{3\cos(x)^2}{4}-\frac{\cos(x)}{5}+\frac{5}{8}-\frac{x}{2} \quad (1)$$

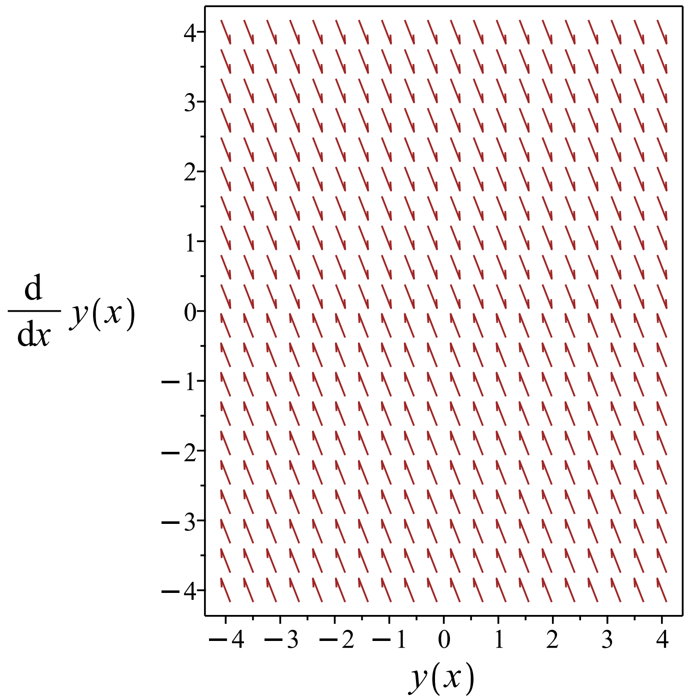


Figure 719: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2}{2} - \frac{3\cos(x)\sin(x)}{4} + \frac{2\sin(x)}{5} - \frac{3\cos(x)^2}{4} - \frac{\cos(x)}{5} + \frac{5}{8} - \frac{x}{2}$$

Verified OK.

17.33.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2 \\ r(x) &= 0 \\ s(x) &= -1 + 3\sin(2x) + \cos(x) \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\, y' + \left(q(x) - p'(x)\right) y\right)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + \left(q(x) - p'(x)\right) y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
2y + y' = \int -1 + 3 \sin \left(2x\right) + \cos \left(x\right) dx
$$

We now have a first order ode to solve which is

$$
2y + y' = -x - \frac{3 \cos \left(2x\right)}{2} + \sin \left(x\right) + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= -x - \frac{3 \cos \left(2x\right)}{2} + \sin \left(x\right) + c_1
\end{aligned}
$$

Hence the ode is

$$
2y + y' = -x - \frac{3 \cos \left(2x\right)}{2} + \sin \left(x\right) + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2dx} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-x - \frac{3\cos(2x)}{2} + \sin(x) + c_1\right)$$
$$\frac{d}{dx}(e^{2x} y) = (e^{2x})\left(-x - \frac{3\cos(2x)}{2} + \sin(x) + c_1\right)$$
$$d(e^{2x} y) = \left(\frac{(2\sin(x) - 3\cos(2x) + 2c_1 - 2x)\,e^{2x}}{2}\right) dx$$

Integrating gives

$$e^{2x} y = \int \frac{(2\sin(x) - 3\cos(2x) + 2c_1 - 2x)\,e^{2x}}{2}\, dx$$
$$e^{2x} y = -\frac{3(2\sin(x) + 2\cos(x))\,e^{2x}\cos(x)}{8} + \frac{5\,e^{2x}}{8} - \frac{e^{2x}\cos(x)}{5} + \frac{2\,e^{2x}\sin(x)}{5} - \frac{x\,e^{2x}}{2} + \frac{e^{2x} c_1}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = e^{-2x}\left(-\frac{3(2\sin(x) + 2\cos(x))\,e^{2x}\cos(x)}{8} + \frac{5\,e^{2x}}{8} - \frac{e^{2x}\cos(x)}{5} + \frac{2\,e^{2x}\sin(x)}{5} - \frac{x\,e^{2x}}{2} + \frac{e^{2x} c_1}{2}\right) + e^{-2}$$

which simplifies to

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + e^{-2x} c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + e^{-2x} c_2 \quad (1)$$

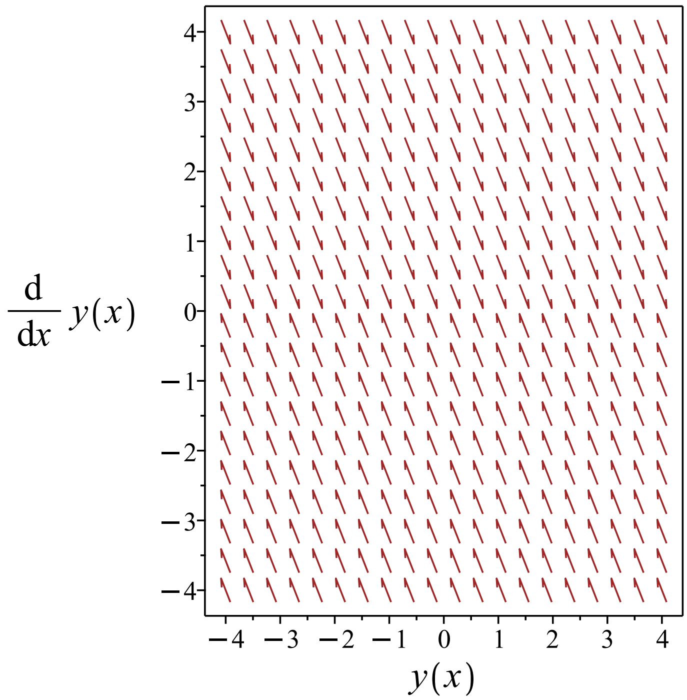


Figure 720: Slope field plot

Verification of solutions

$$y = -\frac{3\cos(x)\sin(x)}{4} - \frac{3\cos(x)^2}{4} + \frac{c_1}{2} - \frac{x}{2} + \frac{2\sin(x)}{5} - \frac{\cos(x)}{5} + \frac{5}{8} + e^{-2x}c_2$$

Verified OK.

17.33.7 Maple step by step solution

Let's solve

$y'' + 2y' = -1 + 3\sin(2x) + \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r = 0$

- Factor the characteristic polynomial

  $r(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -1 + 3\sin(2x) + \cos(\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & 1 \\ -2\,\mathrm{e}^{-2x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-2x}\left(\int (-1+3\sin(2x)+\cos(x))\mathrm{e}^{2x} dx\right)}{2} + \frac{\left(\int (-1+3\sin(2x)+\cos(x)) dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -\frac{3\cos(x)\sin(x)}{4} + \frac{2\sin(x)}{5} - \frac{3\cos(x)^2}{4} - \frac{\cos(x)}{5} + \frac{5}{8} - \frac{x}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 - \frac{3\cos(x)\sin(x)}{4} + \frac{2\sin(x)}{5} - \frac{3\cos(x)^2}{4} - \frac{\cos(x)}{5} + \frac{5}{8} - \frac{x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*_b(_a)-1+3*sin(2*_a)+cos(_a), _b(_a)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+1=3*sin(2*x)+cos(x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-2x}c_1}{2} + \frac{2\sin(x)}{5} - \frac{3\sin(2x)}{8} - \frac{\cos(x)}{5} - \frac{3\cos(2x)}{8} - \frac{x}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.316 (sec). Leaf size: 52

```
DSolve[y''[x]+2*y'[x]+1==3*Sin[2*x]+Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{2} + \frac{2\sin(x)}{5} - \frac{3}{8}\sin(2x) - \frac{\cos(x)}{5} - \frac{3}{8}\cos(2x) - \frac{1}{2}c_1 e^{-2x} + c_2$$

17.34 problem 584

Internal problem ID [15353]
Internal file name [`OUTPUT/15353_Wednesday_May_08_2024_03_56_47_PM_31495563/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 584.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 2y'' + y' = 2x + e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' + y' = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + x\,e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \mathrm{e}^x \\
y_3 &= x\,\mathrm{e}^x
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - 2y'' + y' = 2x + \mathrm{e}^x
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2x + \mathrm{e}^x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^x\}, \{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x\,\mathrm{e}^x, \mathrm{e}^x\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\mathrm{e}^x\}, \{x, x^2\}]
$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^x\}, \{x, x^2\}]
$$

Since $x\,\mathrm{e}^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\mathrm{e}^x x^2\}, \{x, x^2\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 \mathrm{e}^x x^2 + A_2 x + A_3 x^2
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 \mathrm{e}^x - 4A_3 + A_2 + 2A_3 x = 2x + \mathrm{e}^x
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = 4, A_3 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x x^2}{2} + 4x + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^x + x\, e^x c_3) + \left(\frac{e^x x^2}{2} + 4x + x^2\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\, e^x + c_1 + \frac{e^x x^2}{2} + 4x + x^2$$

Summary

The solution(s) found are the following

$$y = (c_3 x + c_2)\, e^x + c_1 + \frac{e^x x^2}{2} + 4x + x^2 \tag{1}$$

Verification of solutions

$$y = (c_3 x + c_2)\, e^x + c_1 + \frac{e^x x^2}{2} + 4x + x^2$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 2*(diff(_b(_a), _a))-_b(_a)+2
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)+diff(y(x),x)=2*x+exp(x),y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + (2c_1 - 2)\, x - 2c_1 + 2c_2 + 2)\, \mathrm{e}^x}{2} + x^2 + 4x + c_3$$

✓ Solution by Mathematica

Time used: 0.268 (sec). Leaf size: 39

```
DSolve[y'''[x]-2*y''[x]+y'[x]==2*x+Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + e^x \left(\frac{x^2}{2} + (-1 + c_2)x + 1 + c_1 - c_2\right) + 4x + c_3$$

17.35 problem 585

Internal problem ID [15354]
Internal file name [`OUTPUT/15354_Wednesday_May_08_2024_03_56_48_PM_29271787/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 585.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 2\sin(x)\sin(2x)}$$

17.35.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cos(x) - \cos(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x) - \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}, \{\cos(3x), \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x + A_3 \cos(3x) + A_4 \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) - 8A_3 \cos(3x) - 8A_4 \sin(3x) = \cos(x) - \cos(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = \frac{1}{8}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{2} + \frac{\cos(3x)}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\sin(x)\, x}{2} + \frac{\cos(3x)}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2} + \frac{\cos(3x)}{8} \tag{1}
$$

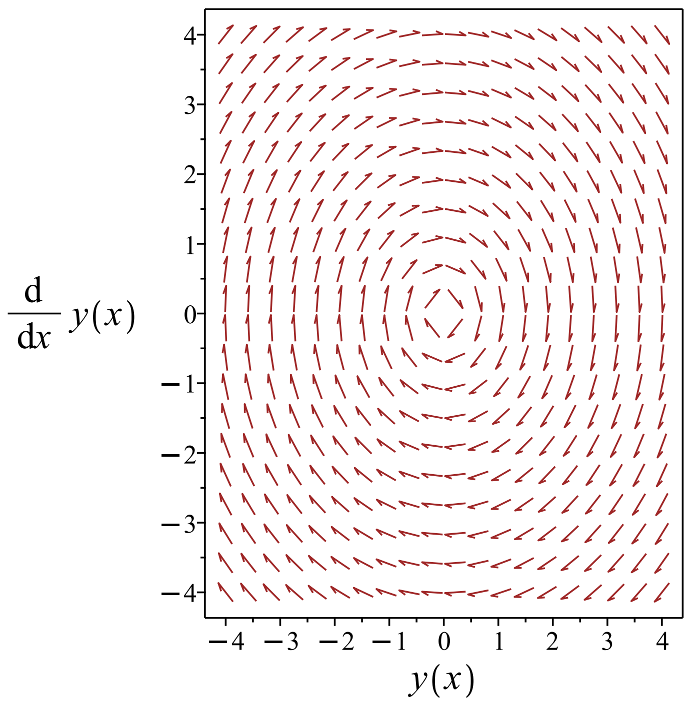


Figure 721: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2} + \frac{\cos(3x)}{8}
$$

Verified OK.

17.35.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 558: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B=0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\, dx\\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx\\ &= \cos(x)\,(\tan(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(x)^2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)x, \sin(x)x\}, \{\cos(3x), \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\cos(x)x + A_2\sin(x)x + A_3\cos(3x) + A_4\sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 2A_2\cos(x) - 8A_3\cos(3x) - 8A_4\sin(3x) = \cos(x) - \cos(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}, A_3 = \frac{1}{8}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)x}{2} + \frac{\cos(3x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{\sin(x)x}{2} + \frac{\cos(3x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x) x}{2} + \frac{\cos(3x)}{8} \tag{1}$$

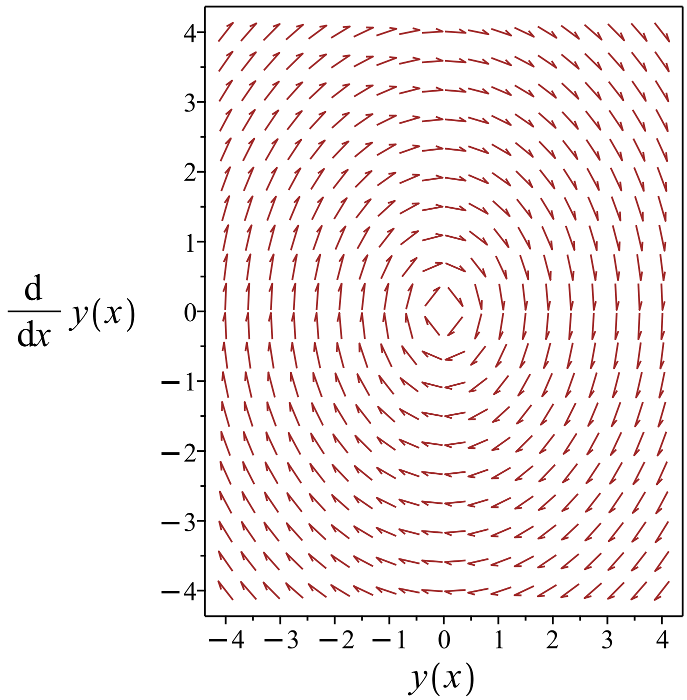


Figure 722: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x) x}{2} + \frac{\cos(3x)}{8}$$

Verified OK.

17.35.3 Maple step by step solution

Let's solve

$y'' + y = \cos(x) - \cos(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x) - \cos(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\cos(x)\left(\int \sin(x)^3 \cos(x)\, dx\right) + \frac{\sin(x)\left(\int (1-\cos(4x)) dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{\sin(x)(-\cos(x)\sin(x)+x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)(-\cos(x)\sin(x)+x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+y(x)=2*sin(x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{\cos(x)\sin(x)^2}{2} + \frac{(2c_2 + x)\sin(x)}{2} + \cos(x)\, c_1$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 33

```
DSolve[y''[x]+y[x]==2*Sin[x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}(\cos(3x) + (-1 + 8c_1)\cos(x) + 4(x + 2c_2)\sin(x))$$

17.36 problem 586

Internal problem ID [15355]
Internal file name [`OUTPUT/15355_Wednesday_May_08_2024_03_56_50_PM_93568330/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 586.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y'' - 2y' = 4x + 3\sin(x) + \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - 2y' = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - 2y' = 4x + 3\sin(x) + \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x + 3\sin(x) + \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3 \sin(x) - 3A_4 \cos(x) - 2A_2 + A_3 \cos(x) + A_4 \sin(x) - 4A_2 x - 2A_1 = 4x + 3\sin(x) + \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -1, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 + x + \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{2x} c_3\right) + \left(-x^2 + x + \cos(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{2x} c_3 - x^2 + x + \cos(x) \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{2x} c_3 - x^2 + x + \cos(x)$$

Verified OK.

17.36.1 Maple step by step solution

Let's solve

$y''' - y'' - 2y' = 4x + 3\sin(x) + \cos(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 4x + 3\sin(x) + \cos(x) + y_3(x) + 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4x + 3\sin(x) + \cos(x) + y_3(x) + 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 4x + 3\sin(x) + \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 4x + 3\sin(x) + \cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{2x}}{4} \\ -e^{-x} & 0 & \frac{e^{2x}}{2} \\ e^{-x} & 0 & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & \frac{e^{2x}}{4} \\ -e^{-x} & 0 & \frac{e^{2x}}{2} \\ e^{-x} & 0 & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ -1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} 1 & -\frac{2e^{-x}}{3} + \frac{1}{2} + \frac{e^{2x}}{6} & \frac{e^{-x}}{3} - \frac{1}{2} + \frac{e^{2x}}{6} \\ 0 & \frac{2e^{-x}}{3} + \frac{e^{2x}}{3} & -\frac{e^{-x}}{3} + \frac{e^{2x}}{3} \\ 0 & -\frac{2e^{-x}}{3} + \frac{2e^{2x}}{3} & \frac{e^{-x}}{3} + \frac{2e^{2x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -x^2 + x + \cos(x) + \frac{e^{2x}}{3} - 3 + \frac{5\,e^{-x}}{3} \\ \frac{2\,e^{2x}}{3} - 2x - \sin(x) + 1 - \frac{5\,e^{-x}}{3} \\ \frac{4\,e^{2x}}{3} - \cos(x) - 2 + \frac{5\,e^{-x}}{3} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \left[\begin{array}{c} -x^2 + x + \cos(x) + \frac{e^{2x}}{3} - 3 + \frac{5\,e^{-x}}{3} \\ \frac{2\,e^{2x}}{3} - 2x - \sin(x) + 1 - \frac{5\,e^{-x}}{3} \\ \frac{4\,e^{2x}}{3} - \cos(x) - 2 + \frac{5\,e^{-x}}{3} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(12c_1 + 20)e^{-x}}{12} + \frac{(3c_3 + 4)e^{2x}}{12} - x^2 + x + c_2 + \cos(x) - 3$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = diff(_b(_a), _a)+2*_b(_a)+4*_
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-2*diff(y(x),x)=4*x+3*sin(x)+cos(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{2x}}{2} - c_2 e^{-x} - x^2 + \cos(x) + x + c_3$$

✓ Solution by Mathematica

Time used: 0.338 (sec). Leaf size: 46

```
DSolve[y'''[x]-y''[x]-2*y'[x]==4*x+2*Sin[x]+Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + x - \frac{\sin(x)}{10} + \frac{7\cos(x)}{10} - c_1 e^{-x} + \frac{1}{2} c_2 e^{2x} + c_3$$

17.37 problem 587

Internal problem ID [15356]
Internal file name [`OUTPUT/15356_Wednesday_May_08_2024_03_56_51_PM_19858730/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 587.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 4y' = x\,\mathrm{e}^{2x} + \sin(x) + x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{-2x} c_2 + \mathrm{e}^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \mathrm{e}^{-2x} \\ y_3 &= \mathrm{e}^{2x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 4y' = x\,\mathrm{e}^{2x} + \sin(x) + x^2$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ 0 & -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \\ 0 & 4\,\mathrm{e}^{-2x} & 4\,\mathrm{e}^{2x} \end{bmatrix}$$

$$|W| = -16\,\mathrm{e}^{2x} \mathrm{e}^{-2x}$$

The determinant simplifies to

$$|W| = -16$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{-2x} & e^{2x} \\ -2\,e^{-2x} & 2\,e^{2x} \end{bmatrix}$$
$$= 4$$

$$W_2(x) = \det \begin{bmatrix} 1 & e^{2x} \\ 0 & 2\,e^{2x} \end{bmatrix}$$
$$= 2\,e^{2x}$$

$$W_3(x) = \det \begin{bmatrix} 1 & e^{-2x} \\ 0 & -2\,e^{-2x} \end{bmatrix}$$
$$= -2\,e^{-2x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(x\,e^{2x} + \sin(x) + x^2)(4)}{(1)(-16)}\,dx \\ &= \int \frac{4x\,e^{2x} + 4\sin(x) + 4x^2}{-16}\,dx \\ &= \int \left(-\frac{x\,e^{2x}}{4} - \frac{\sin(x)}{4} - \frac{x^2}{4}\right)dx \\ &= -\frac{x^3}{12} - \frac{x\,e^{2x}}{8} + \frac{e^{2x}}{16} + \frac{\cos(x)}{4} \end{aligned}$$

$$\begin{aligned} U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\ &= (-1)^1 \int \frac{(x\,e^{2x} + \sin(x) + x^2)(2\,e^{2x})}{(1)(-16)}\,dx \\ &= -\int \frac{2(x\,e^{2x} + \sin(x) + x^2)\,e^{2x}}{-16}\,dx \\ &= -\int \left(-\frac{(x\,e^{2x} + \sin(x) + x^2)\,e^{2x}}{8}\right)dx \\ &= \frac{e^{4x}x}{32} - \frac{e^{4x}}{128} - \frac{e^{2x}\cos(x)}{40} + \frac{e^{2x}\sin(x)}{20} + \frac{x^2 e^{2x}}{16} - \frac{x\,e^{2x}}{16} + \frac{e^{2x}}{32} \end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^{0} \int \frac{\left(x\,\mathrm{e}^{2x}+\sin\left(x\right)+x^{2}\right) \left(-2\,\mathrm{e}^{-2x}\right)}{\left(1\right) \left(-16\right)}\, dx \\
&= \int \frac{-2 (x\,\mathrm{e}^{2x}+\sin\left(x\right)+x^{2})\,\mathrm{e}^{-2x}}{-16}\, dx \\
&= \int \left(\frac{\left(x^{2}+\sin\left(x\right)\right) \mathrm{e}^{-2x}}{8}+\frac{x}{8}\right)\, dx \\
&= \frac{x^{2}}{16}-\frac{\mathrm{e}^{-2x} x^{2}}{16}-\frac{x\,\mathrm{e}^{-2x}}{16}-\frac{\mathrm{e}^{-2x}}{32}-\frac{\mathrm{e}^{-2x} \cos\left(x\right)}{40}-\frac{\mathrm{e}^{-2x} \sin\left(x\right)}{20} \\
&= \frac{x^{2}}{16}-\frac{\mathrm{e}^{-2x} x^{2}}{16}-\frac{x\,\mathrm{e}^{-2x}}{16}-\frac{\mathrm{e}^{-2x}}{32}-\frac{\mathrm{e}^{-2x} \cos\left(x\right)}{40}-\frac{\mathrm{e}^{-2x} \sin\left(x\right)}{20}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p &= \left(-\frac{x^{3}}{12}-\frac{x\,\mathrm{e}^{2x}}{8}+\frac{\mathrm{e}^{2x}}{16}+\frac{\cos\left(x\right)}{4}\right) \\
&+ \left(\frac{\mathrm{e}^{4x} x}{32}-\frac{\mathrm{e}^{4x}}{128}-\frac{\mathrm{e}^{2x} \cos\left(x\right)}{40}+\frac{\mathrm{e}^{2x} \sin\left(x\right)}{20}+\frac{x^{2} \mathrm{e}^{2x}}{16}-\frac{x\,\mathrm{e}^{2x}}{16}+\frac{\mathrm{e}^{2x}}{32}\right) \left(\mathrm{e}^{-2x}\right) \\
&+ \left(\frac{x^{2}}{16}-\frac{\mathrm{e}^{-2x} x^{2}}{16}-\frac{x\,\mathrm{e}^{-2x}}{16}-\frac{\mathrm{e}^{-2x}}{32}-\frac{\mathrm{e}^{-2x} \cos\left(x\right)}{40}-\frac{\mathrm{e}^{-2x} \sin\left(x\right)}{20}\right) \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\left(8 x^{2}-12 x+7\right) \mathrm{e}^{2x}}{128}-\frac{x^{3}}{12}-\frac{x}{8}+\frac{\cos\left(x\right)}{5}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1+\mathrm{e}^{-2x} c_2+\mathrm{e}^{2x} c_3\right) + \left(\frac{\left(8 x^{2}-12 x+7\right) \mathrm{e}^{2x}}{128}-\frac{x^{3}}{12}-\frac{x}{8}+\frac{\cos\left(x\right)}{5}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1+\mathrm{e}^{-2x} c_2+\mathrm{e}^{2x} c_3+\frac{\left(8 x^{2}-12 x+7\right) \mathrm{e}^{2x}}{128}-\frac{x^{3}}{12}-\frac{x}{8}+\frac{\cos\left(x\right)}{5} \tag{1}
$$

Verification of solutions

$$y = c_1 + e^{-2x}c_2 + e^{2x}c_3 + \frac{(8x^2 - 12x + 7)\,e^{2x}}{128} - \frac{x^3}{12} - \frac{x}{8} + \frac{\cos(x)}{5}$$

Verified OK.

17.37.1 Maple step by step solution

Let's solve

$y''' - 4y' = x\,e^{2x} + \sin(x) + x^2$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x\,e^{2x} + x^2 + 4y_2(x) + \sin(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x\,e^{2x} + x^2 + 4y_2(x) + \sin(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x\,e^{2x} + \sin(x) + x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x\,\mathrm{e}^{2x} + \sin(x) + x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \\ 0 & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} \\ 0 & -e^{-2x} + e^{2x} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} -\frac{5}{16} + \frac{(40x^2-60x+71)e^{2x}}{640} - \frac{x^3}{12} - \frac{x}{8} + \frac{\cos(x)}{5} + \frac{e^{-2x}}{640} \\ \frac{(40x^2-20x+41)e^{2x}}{320} - \frac{x^2}{4} - \frac{\sin(x)}{5} - \frac{e^{-2x}}{320} - \frac{1}{8} \\ \frac{(40x^2+20x+31)e^{2x}}{160} - \frac{x}{2} - \frac{\cos(x)}{5} + \frac{e^{-2x}}{160} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} -\frac{5}{16} + \frac{(40x^2-60x+71)e^{2x}}{640} - \frac{x^3}{12} - \frac{x}{8} + \frac{\cos(x)}{5} + \frac{e^{-2x}}{640} \\ \frac{(40x^2-20x+41)e^{2x}}{320} - \frac{x^2}{4} - \frac{\sin(x)}{5} - \frac{e^{-2x}}{320} - \frac{1}{8} \\ \frac{(40x^2+20x+31)e^{2x}}{160} - \frac{x}{2} - \frac{\cos(x)}{5} + \frac{e^{-2x}}{160} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{5}{16} + \frac{(40x^2+160c_3-60x+71)e^{2x}}{640} + \frac{(1+160c_1)e^{-2x}}{640} - \frac{x^3}{12} - \frac{x}{8} + c_2 + \frac{\cos(x)}{5}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = exp(2*_a)*_a+_a^2+sin(_a)+4*_
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x)=x*exp(2*x)+sin(x)+x^2,y(x), singsol=all)
```

$$y(x) = \frac{(8x^2 + 64c_1 - 12x + 7)\,e^{2x}}{128} - \frac{x^3}{12} - \frac{c_2e^{-2x}}{2} - \frac{x}{8} + c_3 + \frac{\cos(x)}{5}$$

✓ Solution by Mathematica

Time used: 0.617 (sec). Leaf size: 60

```
DSolve[y'''[x]-4*y'[x]==x*Exp[2*x]+Sin[x]+x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^3}{12} + \frac{1}{128}e^{2x}\left(8x^2 - 12x + 7 + 64c_1\right) - \frac{x}{8} + \frac{\cos(x)}{5} - \frac{1}{2}c_2e^{-2x} + c_3$$

17.38 problem 588

Internal problem ID [15357]
Internal file name [`OUTPUT/15357_Wednesday_May_08_2024_03_56_54_PM_31214681/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 588.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y^{(5)} - y'''' = x\,\mathrm{e}^{x} - 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y^{(5)} - y'''' = 0$$

The characteristic equation is

$$\lambda^5 - \lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$
$$\lambda_5 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_4x^3 + c_3x^2 + c_2x + c_1 + e^x c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= x^2 \\ y_4 &= x^3 \\ y_5 &= e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y^{(5)} - y'''' = x e^x - 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x e^x - 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{x e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, x^3, e^x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{x e^x, e^x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}, \{x e^x, e^x\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\}, \{x e^x, e^x\}]$$

Since x^3 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^4\}, \{x e^x, e^x\}]$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^4\}, \{x\,\mathrm{e}^{x}, \mathrm{e}^{x}x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^4 + A_2x\,\mathrm{e}^{x} + A_3\mathrm{e}^{x}x^2$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2\mathrm{e}^{x} + 2A_3\mathrm{e}^{x}x + 8A_3\mathrm{e}^{x} - 24A_1 = x\,\mathrm{e}^{x} - 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = -4, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4}{24} - 4x\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_4x^3 + c_3x^2 + c_2x + c_1 + \mathrm{e}^{x}c_5\right) + \left(\frac{x^4}{24} - 4x\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^2}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_4x^3 + c_3x^2 + c_2x + c_1 + \mathrm{e}^{x}c_5 + \frac{x^4}{24} - 4x\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^2}{2} \tag{1}$$

Verification of solutions

$$y = c_4x^3 + c_3x^2 + c_2x + c_1 + \mathrm{e}^{x}c_5 + \frac{x^4}{24} - 4x\,\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^2}{2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(_a)*_a+_b(_a)-1, _b(_a)`   *** Subl
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$5)-diff(y(x),x$4)=x*exp(x)-1,y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + 2c_1 - 8x + 20)\,\mathrm{e}^x}{2} + \frac{x^4}{24} + \frac{c_2 x^3}{6} + \frac{c_3 x^2}{2} + c_4 x + c_5$$

✓ Solution by Mathematica

Time used: 0.337 (sec). Leaf size: 49

```
DSolve[y'''''[x]-y''''[x]==x*Exp[x]-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{24} + c_5 x^3 + c_4 x^2 + e^x \left(\frac{x^2}{2} - 4x + 10 + c_1\right) + c_3 x + c_2$$

17.39 problem 589

Internal problem ID [15358]
Internal file name [`OUTPUT/15358_Wednesday_May_08_2024_03_56_55_PM_69386188/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Superposition principle. Exercises page 137
Problem number: 589.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y^{(5)} - y''' = x + 2\,\mathrm{e}^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y^{(5)} - y''' = 0$$

The characteristic equation is

$$\lambda^5 - \lambda^3 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 0\\ \lambda_3 &= 0\\ \lambda_4 &= 1\\ \lambda_5 &= -1\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x + c_4 x^2 + e^x c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= 1 \\ y_3 &= x \\ y_4 &= x^2 \\ y_5 &= e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y^{(5)} - y''' = x + 2\,e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 2\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, e^x, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\}, \{1, x\}]$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\}, \{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\}, \{x^2, x^3\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\}, \{x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x} + A_2 x^3 + A_3 x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-x} - 6A_2 - 24A_3 x = x + 2\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = -\frac{1}{24}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,\mathrm{e}^{-x} - \frac{x^4}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + c_3 x + c_4 x^2 + \mathrm{e}^{x} c_5\right) + \left(x\,\mathrm{e}^{-x} - \frac{x^4}{24}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + c_4 x^2 + \mathrm{e}^{x} c_5 + x\,\mathrm{e}^{-x} - \frac{x^4}{24} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x + c_4 x^2 + \mathrm{e}^{x} c_5 + x\,\mathrm{e}^{-x} - \frac{x^4}{24}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 5; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _b(_a)+_a+2*exp(-_a), _b(_a)`
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 5; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$5)-diff(y(x),x$3)=x+2*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(7 + 2x - 2c_1) \, \mathrm{e}^{-x}}{2} - \frac{x^4}{24} + \frac{c_3 x^2}{2} + c_4 x + c_2 \mathrm{e}^{x} + c_5$$

✓ Solution by Mathematica

Time used: 0.394 (sec). Leaf size: 46

```
DSolve[y'''''[x]-y'''[x]==x+2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^4}{24} + c_5 x^2 + c_4 x + c_1 e^{x} + e^{-x} \left(x + \frac{7}{2} - c_2 \right) + c_3$$

18 Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140

18.1 problem 590

Internal problem ID [15359]
Internal file name [`OUTPUT/15359_Wednesday_May_08_2024_03_56_56_PM_79411468/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 590.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + y = 2 - 2x}$$

With initial conditions

$$[y(0) = 2, y'(0) = -2]$$

18.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 2 - 2x \end{aligned}$$

Hence the ode is

$$y'' + y = 2 - 2x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2 - 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2 - 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2x + A_1 = 2 - 2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (2 - 2x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 2 - 2x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -2 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

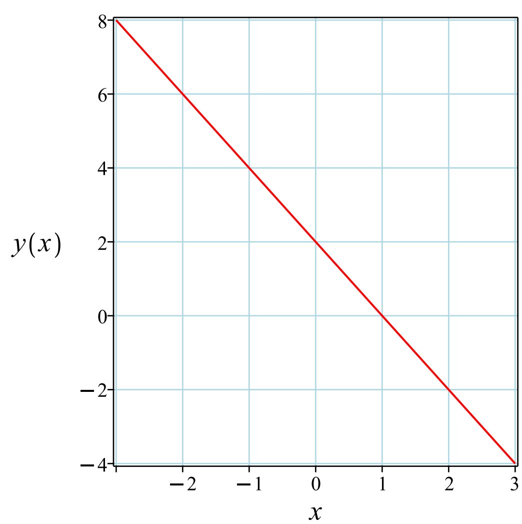
Substituting these values back in above solution results in

$$y = 2 - 2x$$

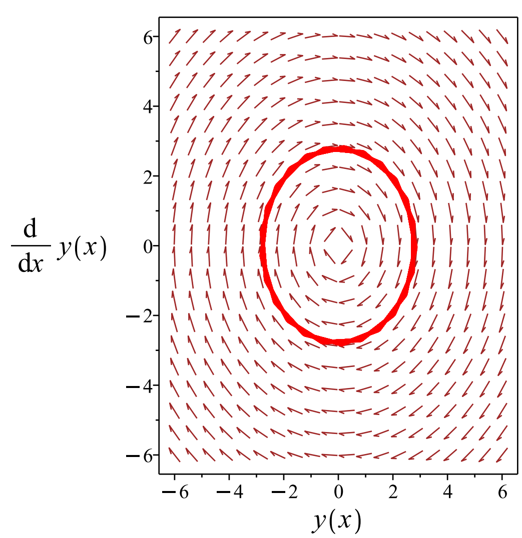
Summary

The solution(s) found are the following

$$y = 2 - 2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2 - 2x$$

Verified OK.

18.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 562: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 2 - 2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (2 - 2x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 2 - 2x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) - 2$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -2 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

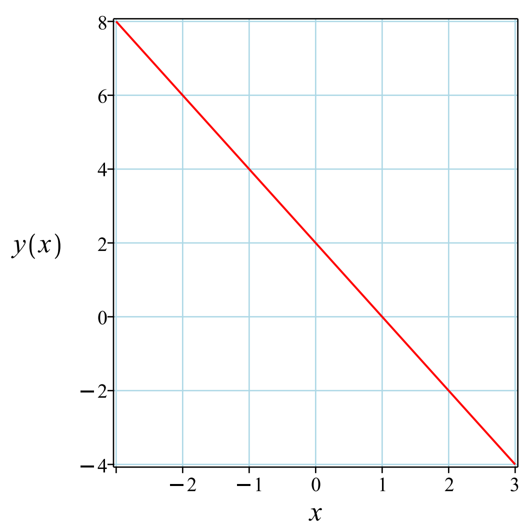
Substituting these values back in above solution results in

$$y = 2 - 2x$$

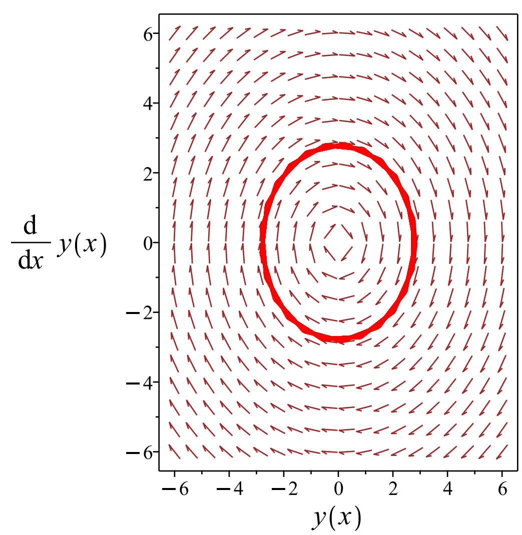
Summary

The solution(s) found are the following

$$y = 2 - 2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2 - 2x$$

Verified OK.

18.1.4 Maple step by step solution

Let's solve

$\left[y'' + y = 2 - 2x, y(0) = 2, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2 - 2x\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\cos(x)\left(\int(x-1)\sin(x)\,dx\right) - 2\sin(x)\left(\int\cos(x)(x-1)\,dx\right)$
- Compute integrals

  $y_p(x) = 2 - 2x$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(x) + c_2\sin(x) + 2 - 2x$

☐ Check validity of solution $y = c_1\cos(x) + c_2\sin(x) + 2 - 2x$

- Use initial condition $y(0) = 2$

  $2 = c_1 + 2$
- Compute derivative of the solution

  $y' = -c_1\sin(x) + c_2\cos(x) - 2$
- Use the initial condition $y'\Big|_{\{x=0\}} = -2$

  $-2 = -2 + c_2$
- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = 2 - 2x$

- Solution to the IVP

  $y = 2 - 2x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve([diff(y(x),x$2)+y(x)=2*(1-x),y(0) = 2, D(y)(0) = -2],y(x), singsol=all)
```

$$y(x) = 2 - 2x$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 10

```
DSolve[{y''[x]+y[x]==2*(1-x),{y[0]==2,y'[0]==-2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 - 2x$$

18.2 problem 591

Internal problem ID [15360]
Internal file name [`OUTPUT/15360_Wednesday_May_08_2024_03_56_57_PM_79611899/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 591.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = 9x^2 - 12x + 2}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3]$$

18.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -6 \\ q(x) &= 9 \\ F &= 9x^2 - 12x + 2 \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 9y = 9x^2 - 12x + 2$$

The domain of $p(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 9x^2 - 12x + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.2.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 9x^2 - 12x + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)} \\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 e^{3x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x} x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_3x^2 + 9A_2x - 12xA_3 + 9A_1 - 6A_2 + 2A_3 = 9x^2 - 12x + 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 x\, e^{3x}\right) + \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2x + c_1) + x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{3x}(c_2x + c_1) + x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3\,\mathrm{e}^{3x}(c_2x + c_1) + c_2\mathrm{e}^{3x} + 2x$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

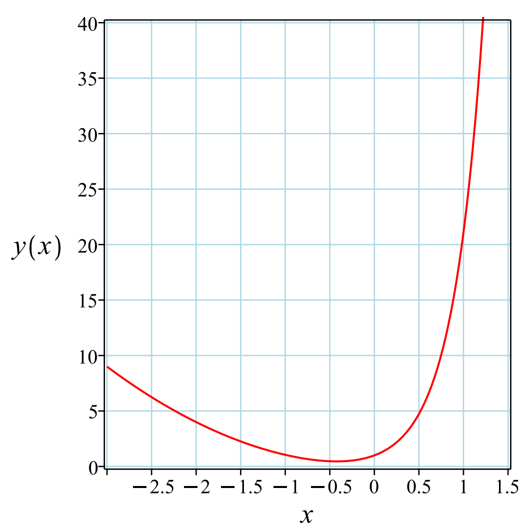
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

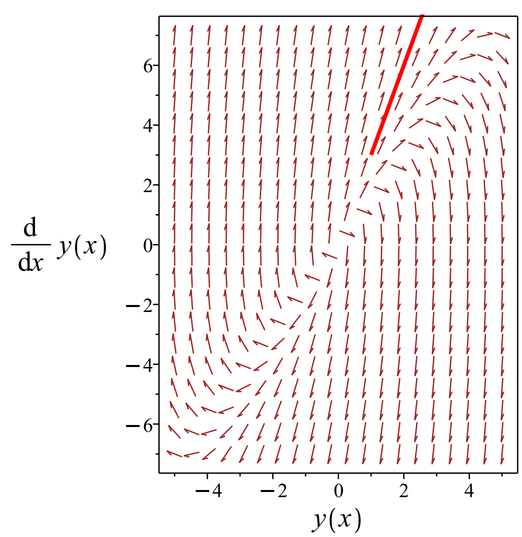
$$y = x^2 + \mathrm{e}^{3x}$$

Summary
The solution(s) found are the following

$$y = x^2 + \mathrm{e}^{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 + \mathrm{e}^{3x}$$

Verified OK.

18.2.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2}+p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -6\,dx} \\
&= \mathrm{e}^{-3x}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-3x}\left(9x^{2}-12x+2\right) \\
\left(\mathrm{e}^{-3x}y\right)'' &= \mathrm{e}^{-3x}\left(9x^{2}-12x+2\right)
\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x}y\right)' = \left(-3x^{2}+2x\right)\mathrm{e}^{-3x}+c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x}y\right) = x\left(x\,\mathrm{e}^{-3x}+c_1\right)+c_2$$

Hence the solution is

$$y = \frac{x\left(x\,\mathrm{e}^{-3x}+c_1\right)+c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} + x^{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\,\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} + x^{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{3x} + 3c_1 x\, e^{3x} + 3c_2 e^{3x} + 2x$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

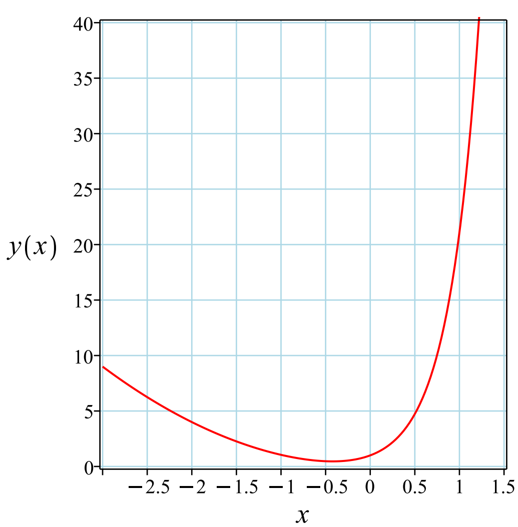
$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

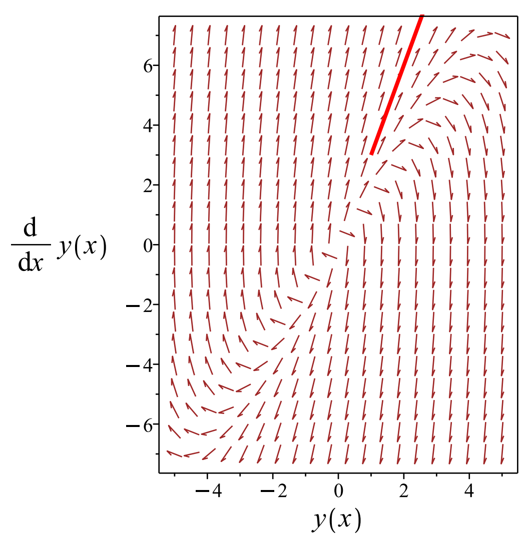
$$y = x^2 + e^{3x}$$

Summary
The solution(s) found are the following

$$y = x^2 + e^{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 + e^{3x}$$

Verified OK.

18.2.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 564: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-6}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{3x} \\
&= z_1\left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{3x} + c_2 x\, \mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x}x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_3x^2 + 9A_2x - 12xA_3 + 9A_1 - 6A_2 + 2A_3 = 9x^2 - 12x + 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 x\, e^{3x}\right) + \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{3x}(c_2x + c_1) + x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3\,e^{3x}(c_2x + c_1) + c_2e^{3x} + 2x$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

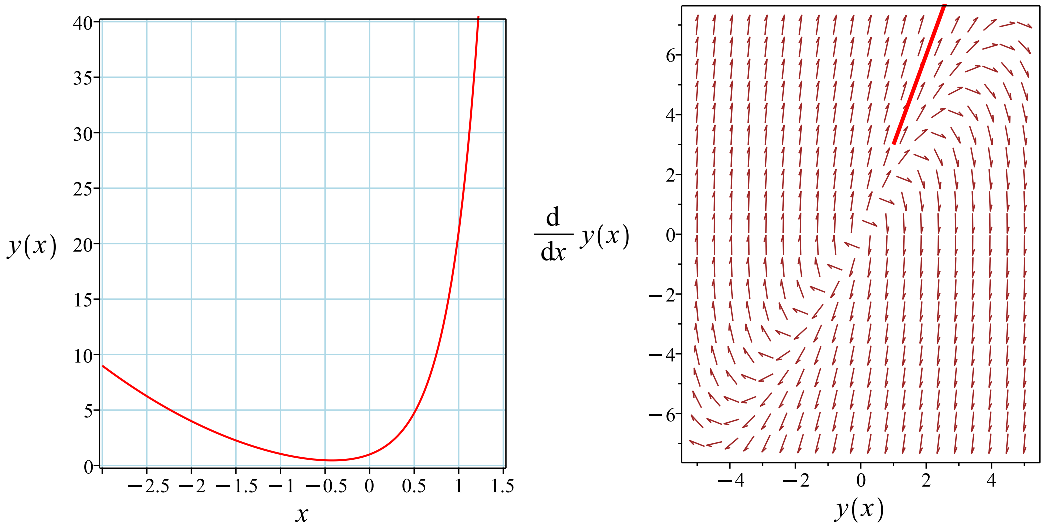
Substituting these values back in above solution results in

$$y = x^2 + e^{3x}$$

Summary

The solution(s) found are the following

$$y = x^2 + e^{3x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x^2 + e^{3x}$$

Verified OK.

18.2.5 Maple step by step solution

Let's solve

$$\left[y'' - 6y' + 9y = 9x^2 - 12x + 2, y(0) = 1, y'\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r - 3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{3x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 9x^2 - 12x + 2\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & e^{3x}x \\ 3\,e^{3x} & 3\,e^{3x}x + e^{3x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{6x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{3x}\left(-\left(\int x\, e^{-3x}(9x^2 - 12x + 2)\, dx\right) + \left(\int e^{-3x}(9x^2 - 12x + 2)\, dx\right) x\right)$

- Compute integrals

  $y_p(x) = x^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + x^2$

☐ Check validity of solution $y = c_1 e^{3x} + c_2 x e^{3x} + x^2$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x\, e^{3x} + 2x$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = 3c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = x^2 + e^{3x}$

- Solution to the IVP

  $y = x^2 + e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=9*x^2-12*x+2,y(0) = 1, D(y)(0) = 3],y(x), sings
```

$$y(x) = e^{3x} + x^2$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 14

```
DSolve[{y''[x]-6*y'[x]+9*y[x]==9*x^2-12*x+2,{y[0]==1,y'[0]==3}},y[x],x,IncludeSingularSoluti
```

$$y(x) \rightarrow x^2 + e^{3x}$$

18.3 problem 592

Internal problem ID [15361]
Internal file name [`OUTPUT/15361_Wednesday_May_08_2024_03_56_59_PM_49047121/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 592.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = 36\,\mathrm{e}^{3x}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 6]$$

18.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= 36\,\mathrm{e}^{3x} \end{aligned}$$

Hence the ode is

$$y'' + 9y = 36\,e^{3x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 36\,e^{3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.3.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 36\,e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$36\,\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$18 A_1 e^{3x} = 36 e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + (2 e^{3x}) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + c_2 \sin(3x) + 2 e^{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + 6\,e^{3x}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = 6 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

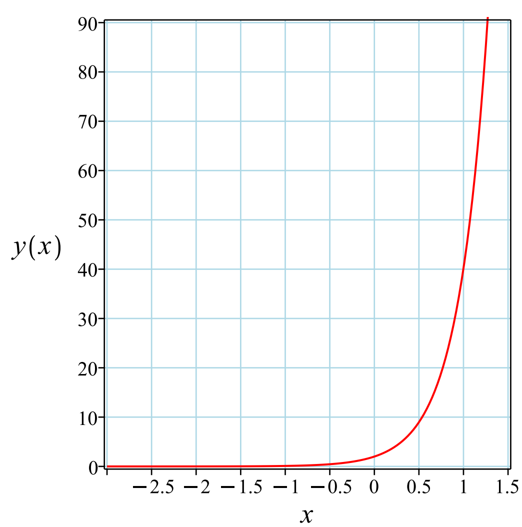
Substituting these values back in above solution results in

$$y = 2\,e^{3x}$$

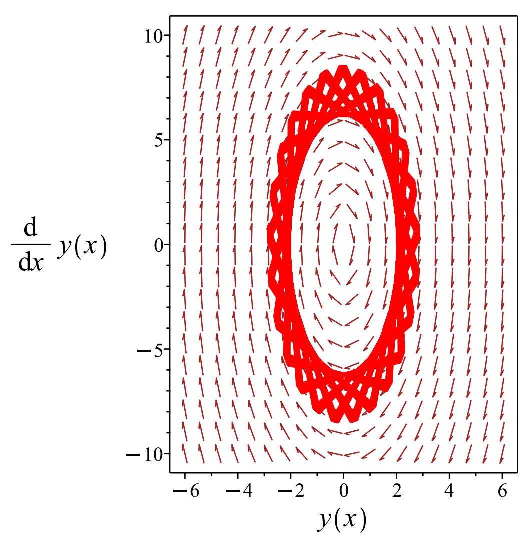
Summary

The solution(s) found are the following

$$y = 2\,e^{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{3x}$$

Verified OK.

18.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 566: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$36\,\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$18A_1 \mathrm{e}^{3x} = 36\,\mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(2\,\mathrm{e}^{3x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + 2\,e^{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + c_2 \cos(3x) + 6\,e^{3x}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = 6 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

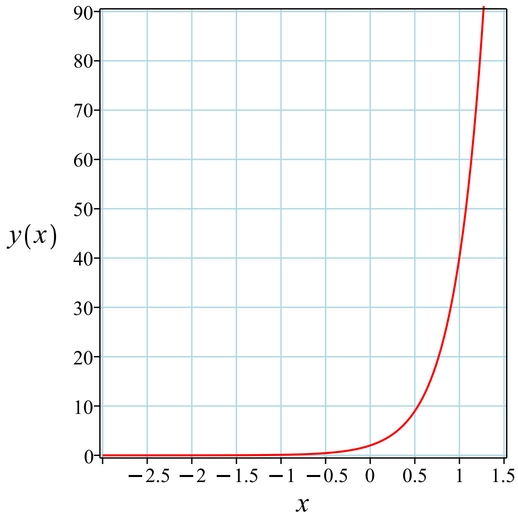
Substituting these values back in above solution results in

$$y = 2\,e^{3x}$$

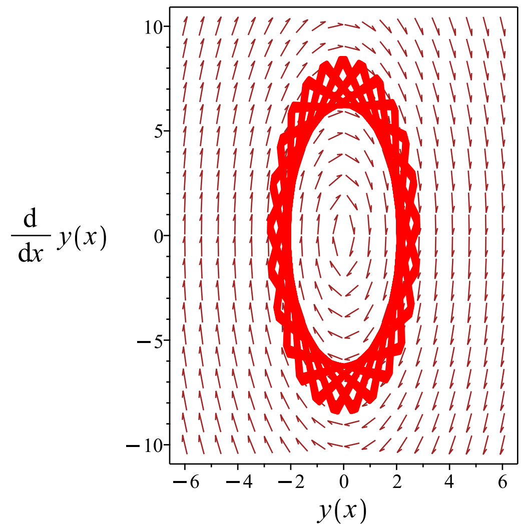
Summary

The solution(s) found are the following

$$y = 2\,e^{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{3x}$$

Verified OK.

18.3.4 Maple step by step solution

Let's solve

$$\left[y'' + 9y = 36\,\mathrm{e}^{3x}, y(0) = 2, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 36\,\mathrm{e}^{3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -12\cos(3x)\left(\int \sin(3x)\,\mathrm{e}^{3x}dx\right) + 12\sin(3x)\left(\int \cos(3x)\,\mathrm{e}^{3x}dx\right)$

  - Compute integrals

    $y_p(x) = 2\,\mathrm{e}^{3x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + 2\,\mathrm{e}^{3x}$

☐ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + 2\mathrm{e}^{3x}$

  - Use initial condition $y(0) = 2$

    $2 = c_1 + 2$

  - Compute derivative of the solution

    $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + 6\,\mathrm{e}^{3x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 6$

    $6 = 6 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = 2\,e^{3x}$

- Solution to the IVP

  $y = 2\,e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)+9*y(x)=36*exp(3*x),y(0) = 2, D(y)(0) = 6],y(x), singsol=all)
```

$$y(x) = 2\,e^{3x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 12

```
DSolve[{y''[x]+9*y[x]==36*Exp[3*x],{y[0]==2,y'[0]==6}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to 2e^{3x}$$

18.4 problem 593

Internal problem ID [15362]
Internal file name [`OUTPUT/15362_Wednesday_May_08_2024_03_57_00_PM_74964762/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 593.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 4y = 2\,e^{2x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

18.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 4 \\ F &= 2\,e^{2x} \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = 2\,e^{2x}$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\,e^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.4.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 2\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)} \\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 x e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{2x}\}]$$

Since $x\,\mathrm{e}^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{2x} = 2\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2\mathrm{e}^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + c_2 x\,\mathrm{e}^{2x}\right) + \left(x^2\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2 x + c_1) + x^2\mathrm{e}^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{2x}(c_2 x + c_1) + x^2\mathrm{e}^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,\mathrm{e}^{2x}(c_2 x + c_1) + \mathrm{e}^{2x} c_2 + 2x\,\mathrm{e}^{2x} + 2x^2 \mathrm{e}^{2x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

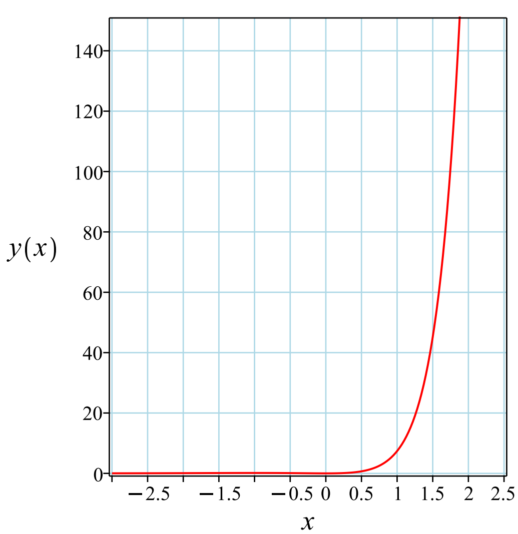
Substituting these values back in above solution results in

$$y = x^2 \mathrm{e}^{2x}$$

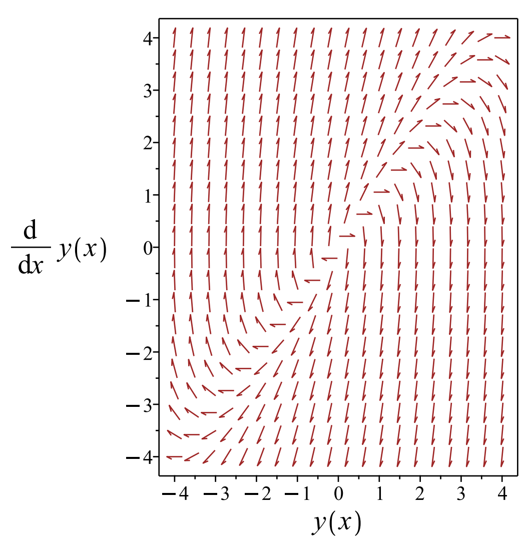
Summary

The solution(s) found are the following

$$y = x^2 \mathrm{e}^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 e^{2x}$$

Verified OK.

18.4.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4\, dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 2\, e^{2x} e^{-2x} \\ \left(e^{-2x} y\right)'' &= 2\, e^{2x} e^{-2x} \end{aligned}$$

Integrating once gives

$$\left(e^{-2x} y\right)' = 2x + c_1$$

Integrating again gives

$$\left(e^{-2x} y\right) = x(x + c_1) + c_2$$

Hence the solution is

$$y = \frac{x(x + c_1) + c_2}{e^{-2x}}$$

Or

$$y = c_1 x\, e^{2x} + x^2 e^{2x} + e^{2x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\, e^{2x} + x^2 e^{2x} + e^{2x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{2x} c_1 + 2c_1 x\, e^{2x} + 2x\, e^{2x} + 2x^2 e^{2x} + 2\, e^{2x} c_2$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

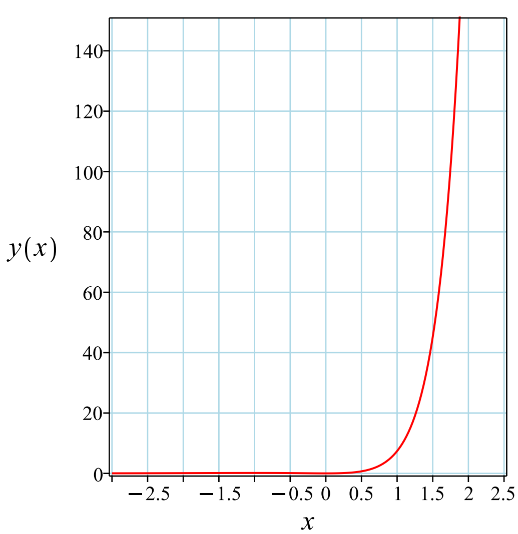
$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

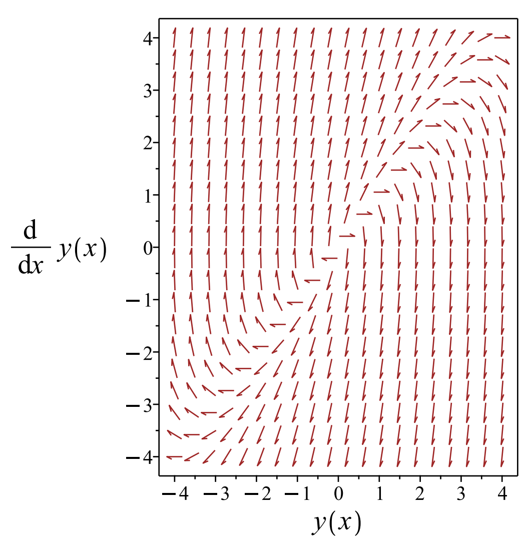
$$y = x^2 e^{2x}$$

Summary
The solution(s) found are the following

$$y = x^2 e^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 e^{2x}$$

Verified OK.

18.4.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 568: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{2x} c_1 + c_2 x\, e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{2x}\}]$$

Since $x\, e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2 A_1 e^{2x} = 2\, e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^{2x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{2x} c_1 + c_2 x\, e^{2x}\right) + \left(x^2 e^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{2x} (c_2 x + c_1) + x^2 e^{2x}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{2x} (c_2 x + c_1) + x^2 e^{2x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2\, e^{2x} (c_2 x + c_1) + e^{2x} c_2 + 2x\, e^{2x} + 2x^2 e^{2x}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = 2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

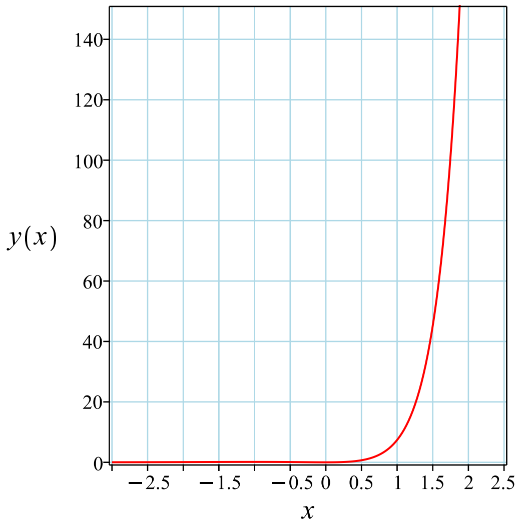
Substituting these values back in above solution results in

$$
y = x^2 e^{2x}
$$

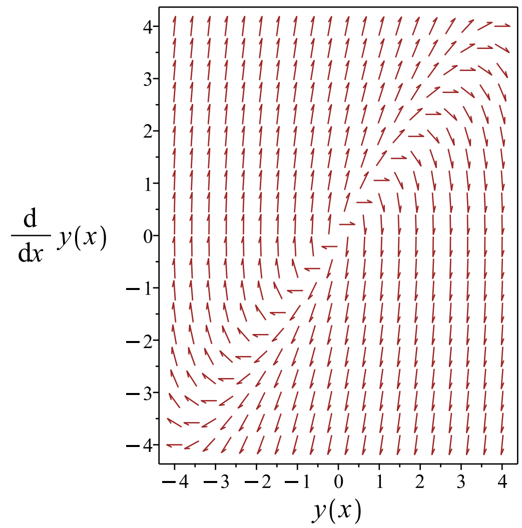
Summary

The solution(s) found are the following

$$
y = x^2 e^{2x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 e^{2x}$$

Verified OK.

18.4.5 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 4y = 2\,e^{2x}, y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + c_2 x\,e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,e^{2x}\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{4x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,e^{2x}\left(\int x dx - \left(\int 1 dx\right) x\right)$
- Compute integrals

  $y_p(x) = x^2 e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = e^{2x} c_1 + c_2 x\,e^{2x} + x^2 e^{2x}$

☐ Check validity of solution $y = e^{2x} c_1 + c_2 x e^{2x} + x^2 e^{2x}$

- Use initial condition $y(0) = 0$

  $0 = c_1$
- Compute derivative of the solution

  $y' = 2\,e^{2x} c_1 + e^{2x} c_2 + 2c_2 x\,e^{2x} + 2x\,e^{2x} + 2x^2 e^{2x}$
- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = 2c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = x^2 e^{2x}$

- Solution to the IVP

  $y = x^2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=2*exp(2*x),y(0) = 0, D(y)(0) = 0],y(x), singsol
```

$$y(x) = e^{2x} x^2$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[{y''[x]-4*y'[x]+4*y[x]==2*Exp[2*x],{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{2x} x^2$$

18.5 problem 594

Internal problem ID [15363]
Internal file name [`OUTPUT/15363_Wednesday_May_08_2024_03_57_02_PM_79938848/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 594.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 5y' + 6y = (12x - 7)\,\mathrm{e}^{-x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

18.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -5 \\ q(x) &= 6 \\ F &= (12x - 7)\,\mathrm{e}^{-x} \end{aligned}$$

Hence the ode is

$$y'' - 5y' + 6y = (12x - 7)\,e^{-x}$$

The domain of $p(x) = -5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = (12x - 7)\,e^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.5.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = (12x - 7)\,e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)} \\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(3)x} + c_2 e^{(2)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + \mathrm{e}^{2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + \mathrm{e}^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(12x - 7)\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 e^{-x} + 12A_1 x\,e^{-x} + 12A_2 e^{-x} = (12x - 7)\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + e^{2x} c_2\right) + \left(x\,e^{-x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + e^{2x} c_2 + x\,e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} + 2\,e^{2x} c_2 + e^{-x} - x\,e^{-x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 3c_1 + 2c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

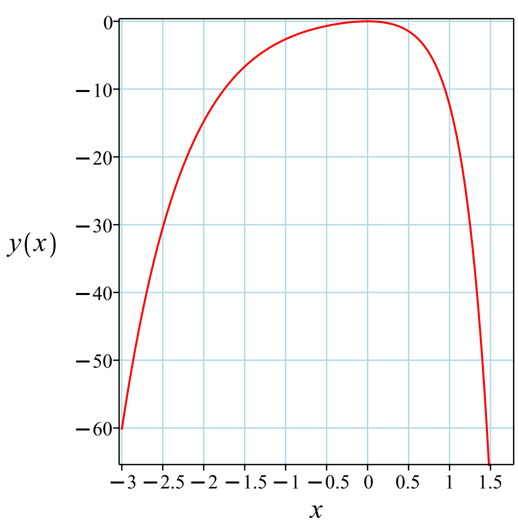
$$c_1 = -1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

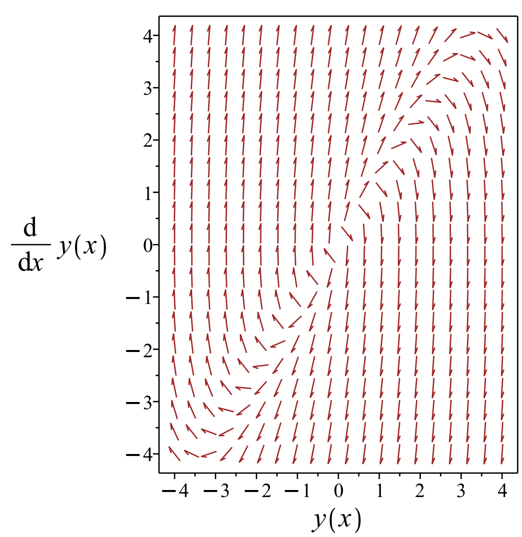
$$y = x\,e^{-x} - e^{3x} + e^{2x}$$

Summary
The solution(s) found are the following

$$y = x\,e^{-x} - e^{3x} + e^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x\,e^{-x} - e^{3x} + e^{2x}$$

Verified OK.

18.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 570: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(12x - 7)\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 e^{-x} + 12A_1 x\,e^{-x} + 12A_2 e^{-x} = (12x - 7)\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x} c_1 + c_2 e^{3x}\right) + \left(x\,e^{-x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x} c_1 + c_2 e^{3x} + x\,e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,e^{2x} c_1 + 3 c_2 e^{3x} + e^{-x} - x\,e^{-x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 + 3c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

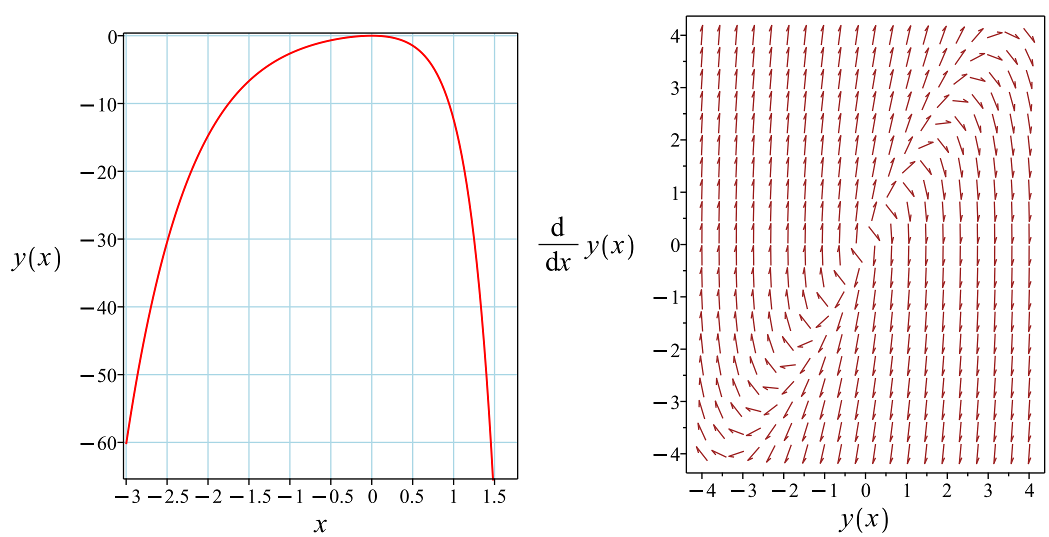
$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = x\,e^{-x} - e^{3x} + e^{2x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x\,e^{-x} - e^{3x} + e^{2x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x\,e^{-x} - e^{3x} + e^{2x}$$

Verified OK.

18.5.4 Maple step by step solution

Let's solve

$$\left[y'' - 5y' + 6y = (12x - 7)\,e^{-x}, y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r - 2)\,(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (12x - 7)\,e^{-x}\right]$$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{3x} \\ 2\,e^{2x} & 3\,e^{3x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{5x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{2x}\left(\int (12x-7)\,e^{-3x}dx\right) + e^{3x}\left(\int (12x-7)\,e^{-4x}dx\right)$

- Compute integrals

  $y_p(x) = x\,e^{-x}$

- Substitute particular solution into general solution to ODE

  $y = e^{2x}c_1 + c_2e^{3x} + x\,e^{-x}$

☐ Check validity of solution $y = e^{2x}c_1 + c_2e^{3x} + xe^{-x}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $y' = 2\,e^{2x}c_1 + 3c_2e^{3x} + e^{-x} - x\,e^{-x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = 2c_1 + 3c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $y = x\,e^{-x} - e^{3x} + e^{2x}$

- Solution to the IVP

  $y = x\,e^{-x} - e^{3x} + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=(12*x-7)*exp(-x),y(0) = 0, D(y)(0) = 0],y(x), s
```

$$y(x) = e^{2x} - e^{3x} + e^{-x}x$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 25

```
DSolve[{y''[x]-5*y'[x]+6*y[x]==(12*x-7)*Exp[-x],{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSo
```

$$y(x) \to e^{-x}\left(x + e^{3x} - e^{4x}\right)$$

18.6 problem 595

Internal problem ID [15364]
Internal file name [`OUTPUT/15364_Wednesday_May_08_2024_03_57_03_PM_75631738/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 595.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = e^{-x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

18.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ F &= \mathrm{e}^{-x} \end{aligned}$$

Hence the ode is

$$y'' + y' = \mathrm{e}^{-x}$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = \mathrm{e}^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.6.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(0)}\\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\, e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-x} = e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\, e^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{-x} c_2\right) + \left(-x\, e^{-x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + e^{-x} c_2 - x\,e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -e^{-x} c_2 - e^{-x} + x\,e^{-x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

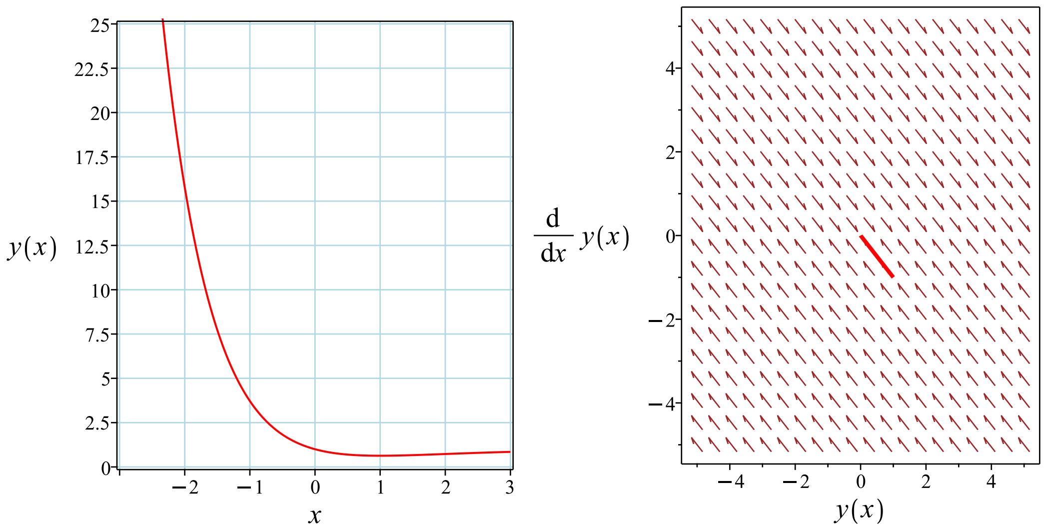
Substituting these values back in above solution results in

$$y = 1 - x\,e^{-x}$$

Summary

The solution(s) found are the following

$$y = 1 - x\,e^{-x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 - x\,\mathrm{e}^{-x}$$

Verified OK.

18.6.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int \mathrm{e}^{-x} dx$$
$$y' + y = -\mathrm{e}^{-x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = -\mathrm{e}^{-x} + c_1$$

Hence the ode is

$$y' + y = -\mathrm{e}^{-x} + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-e^{-x} + c_1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(-e^{-x} + c_1\right)$$
$$d(e^x y) = (e^x c_1 - 1)\, dx$$

Integrating gives

$$e^x y = \int e^x c_1 - 1\, dx$$
$$e^x y = -x + e^x c_1 + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}(-x + e^x c_1) + e^{-x} c_2$$

which simplifies to

$$y = (-x + c_2)\, e^{-x} + c_1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (-x + c_2)\, e^{-x} + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -e^{-x} - (-x + c_2)\, e^{-x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

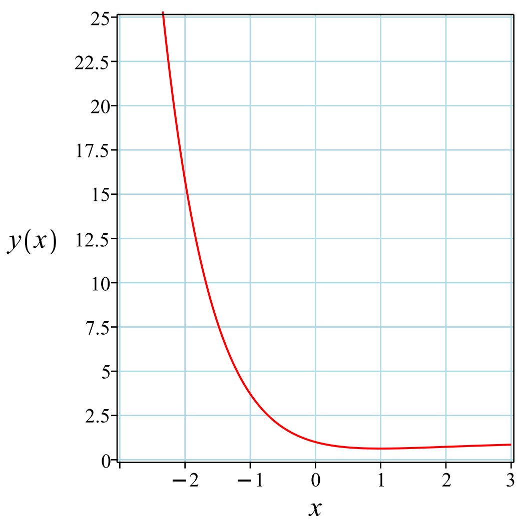
Substituting these values back in above solution results in

$$y = 1 - x\,\mathrm{e}^{-x}$$

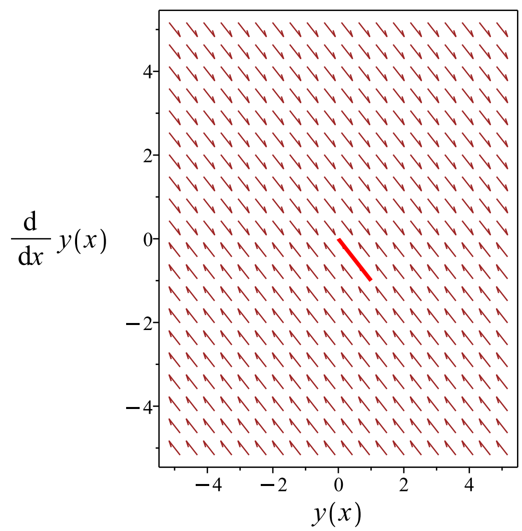
Summary

The solution(s) found are the following

$$y = 1 - x\,\mathrm{e}^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 - x\,\mathrm{e}^{-x}$$

Verified OK.

18.6.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - e^{-x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(e^{-x}\right) \\ \frac{d}{dx}(e^{x} p) &= (e^{x})\left(e^{-x}\right) \\ d(e^{x} p) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x} p &= \int dx \\ e^{x} p &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$p(x) = x\, e^{-x} + c_1 e^{-x}$$

which simplifies to

$$p(x) = (x + c_1)\, e^{-x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$p(x) = e^{-x}(x - 1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{-x}(x - 1)$$

Integrating both sides gives

$$\begin{aligned}
y &= \int e^{-x}(x - 1)\, dx \\
&= -x\, e^{-x} + c_2
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

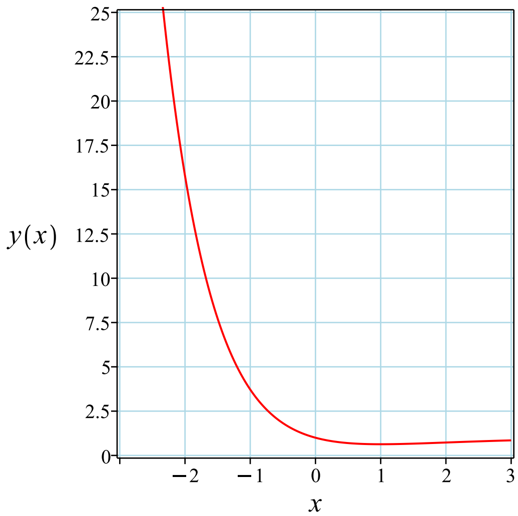
$$y = 1 - x\, e^{-x}$$

Initial conditions are used to solve for the constants of integration.

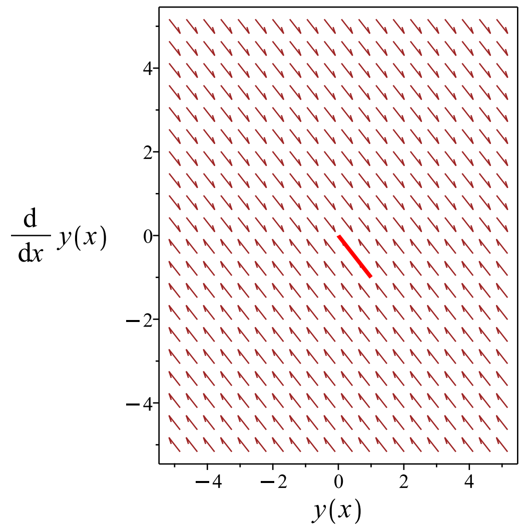
Summary

The solution(s) found are the following

$$y = 1 - x\, e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 1 - x\,\mathrm{e}^{-x}$$

Verified OK.

18.6.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = \mathrm{e}^{-x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int \mathrm{e}^{-x} dx$$
$$y' + y = -\mathrm{e}^{-x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = -\mathrm{e}^{-x} + c_1$$

Hence the ode is

$$y' + y = -e^{-x} + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx}\\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(-e^{-x} + c_1\right)\\ \frac{d}{dx}(e^{x} y) &= (e^{x})\left(-e^{-x} + c_1\right)\\ d(e^{x} y) &= (e^{x} c_1 - 1)\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x} y &= \int e^{x} c_1 - 1\, dx\\ e^{x} y &= -x + e^{x} c_1 + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}(-x + e^{x} c_1) + e^{-x} c_2$$

which simplifies to

$$y = (-x + c_2)\, e^{-x} + c_1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (-x + c_2)\, e^{-x} + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -e^{-x} - (-x + c_2)\, e^{-x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_2 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

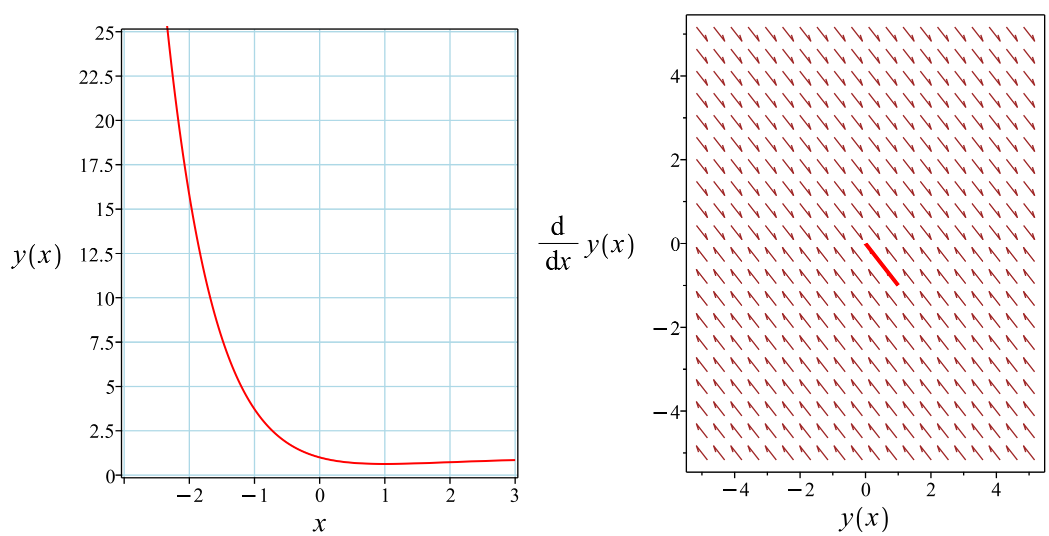
Substituting these values back in above solution results in

$$y = 1 - x\,\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = 1 - x\,\mathrm{e}^{-x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 - x\,\mathrm{e}^{-x}$$

Verified OK.

18.6.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 572: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx}
\end{aligned}
$$

$$= z_1 e^{-\frac{x}{2}}$$
$$= z_1 \left(e^{-\frac{x}{2}}\right)$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx$$
$$= y_1(e^x)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x}(e^x)\right)$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-x} = e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,e^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2\right) + \left(-x\,e^{-x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 - x\, e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} - e^{-x} + x\, e^{-x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_1 - 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

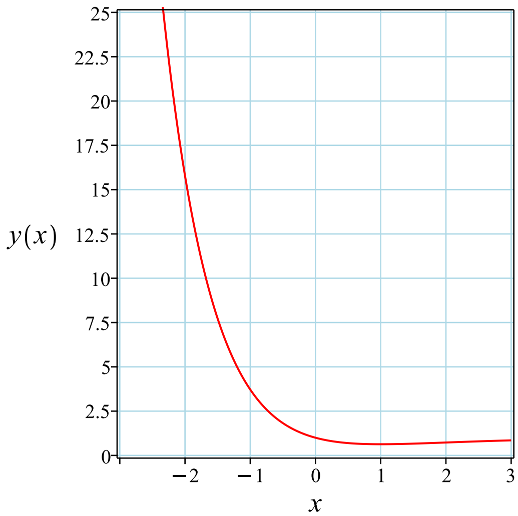
$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

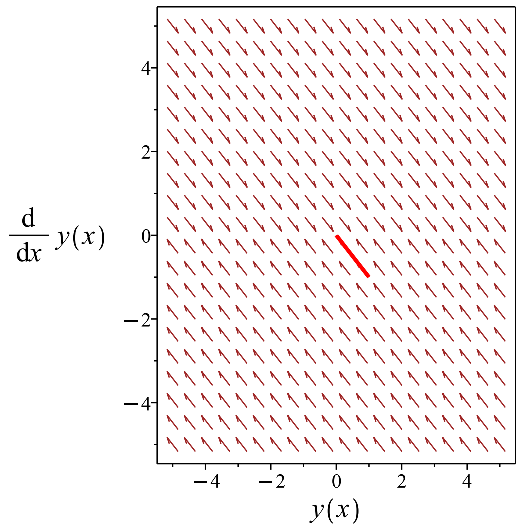
$$y = 1 - x\, e^{-x}$$

Summary
The solution(s) found are the following

$$y = 1 - x\, e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 - x\,\mathrm{e}^{-x}$$

Verified OK.

18.6.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \mathrm{e}^{-x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y'+y=\int \mathrm{e}^{-x}\;dx$$

We now have a first order ode to solve which is

$$y'+y=-\mathrm{e}^{-x}+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=1\\ q(x)&=-\mathrm{e}^{-x}+c_1\end{aligned}$$

Hence the ode is

$$y'+y=-\mathrm{e}^{-x}+c_1$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int 1dx}\\ &=\mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(-\mathrm{e}^{-x}+c_1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x}y)&=(\mathrm{e}^{x})\left(-\mathrm{e}^{-x}+c_1\right)\\ \mathrm{d}(\mathrm{e}^{x}y)&=(\mathrm{e}^{x}c_1-1)\;\mathrm{d}x\end{aligned}$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int e^{x} c_1 - 1 \,dx \\
e^{x} y &= -x + e^{x} c_1 + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = e^{-x}(-x + e^{x} c_1) + e^{-x} c_2
$$

which simplifies to

$$
y = (-x + c_2)\, e^{-x} + c_1
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = (-x + c_2)\, e^{-x} + c_1 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -e^{-x} - (-x + c_2)\, e^{-x}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = -c_2 - 1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

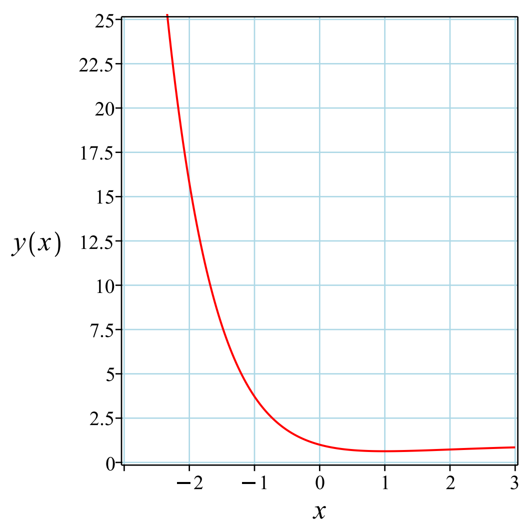
$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

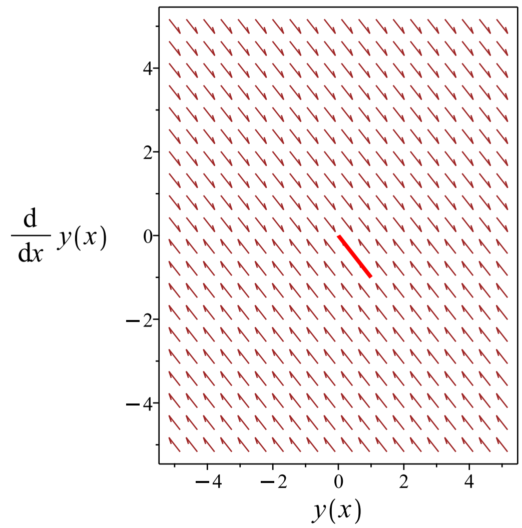
$$
y = 1 - x\, e^{-x}
$$

Summary
The solution(s) found are the following

$$
y = 1 - x\, e^{-x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 1 - x\,\mathrm{e}^{-x}$$

Verified OK.

18.6.8 Maple step by step solution

Let's solve

$$\left[y'' + y' = \mathrm{e}^{-x}, y(0) = 1, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{-x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int 1dx\right) + \int \mathrm{e}^{-x}dx$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{-x}(-x-1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{-x}(-x-1)$

□ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{-x}(-x-1)$

  - Use initial condition $y(0) = 1$

    $1 = -1 + c_1 + c_2$

  - Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} - \mathrm{e}^{-x}(-x-1) - \mathrm{e}^{-x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -1$

    $-1 = -c_1$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = 1 - x\,\mathrm{e}^{-x}$

- Solution to the IVP

  $y = 1 - x\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)+exp(-_a), _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)+diff(y(x),x)=exp(-x),y(0) = 1, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = -\mathrm{e}^{-x}x + 1$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 15

```
DSolve[{y''[x]+y'[x]==Exp[-x],{y[0]==1,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - e^{-x}x$$

18.7 problem 596

Internal problem ID [15365]
Internal file name [`OUTPUT/15365_Wednesday_May_08_2024_03_57_05_PM_93415704/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 596.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 9y = 10 \sin(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

18.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 9 \\ F &= 10\sin(x) \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 9y = 10\sin(x)$$

The domain of $p(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 10\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.7.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 10\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)} \\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + c_2 x e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x e^{-3x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) - 6A_1 \sin(x) + 6A_2 \cos(x) = 10 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{5}, A_2 = \frac{4}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 x\, e^{-3x}\right) + \left(-\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2 x + c_1) - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3x}(c_2 x + c_1) - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -\frac{3}{5} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,\mathrm{e}^{-3x}(c_2x + c_1) + c_2\mathrm{e}^{-3x} + \frac{3\sin(x)}{5} + \frac{4\cos(x)}{5}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{4}{5} - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{5}$$
$$c_2 = 1$$

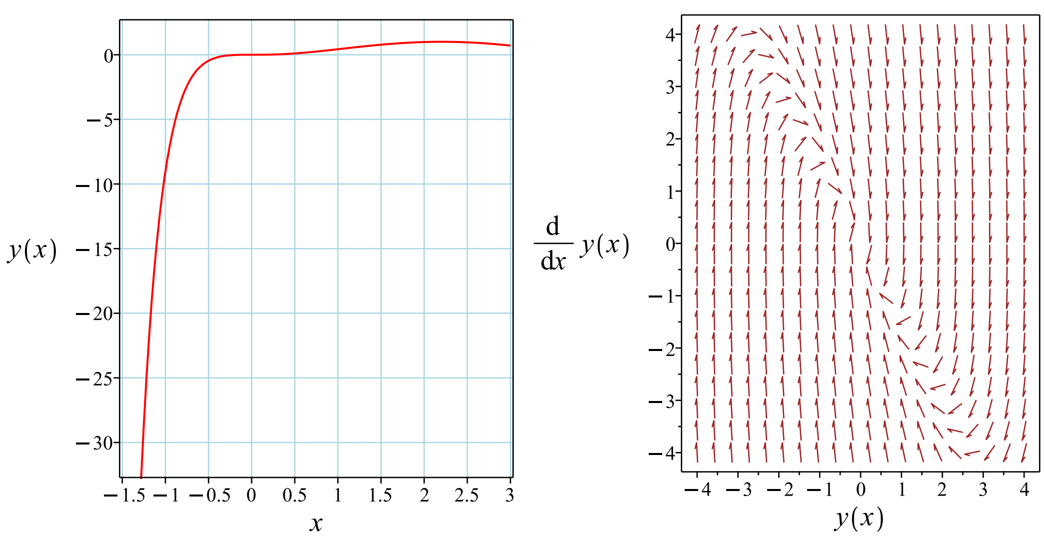
Substituting these values back in above solution results in

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Verified OK.

18.7.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 6\, dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 10\,\mathrm{e}^{3x} \sin(x) \\ \left(\mathrm{e}^{3x} y\right)'' &= 10\,\mathrm{e}^{3x} \sin(x) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{3x} y\right)' = -(\cos(x) - 3\sin(x))\,\mathrm{e}^{3x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{3x} y\right) = \frac{(-3\cos(x) + 4\sin(x))\,\mathrm{e}^{3x}}{5} + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{(-3\cos(x)+4\sin(x))\mathrm{e}^{3x}}{5} + c_1 x + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = -\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} + c_1 x\,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} + c_1 x\,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -\frac{3}{5} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{3\sin(x)}{5} + \frac{4\cos(x)}{5} + c_1 \mathrm{e}^{-3x} - 3c_1 x\,\mathrm{e}^{-3x} - 3c_2 \mathrm{e}^{-3x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{4}{5} + c_1 - 3c_2 \tag{2A}$$

Equations $\{1\mathrm{A},2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{3}{5}$$

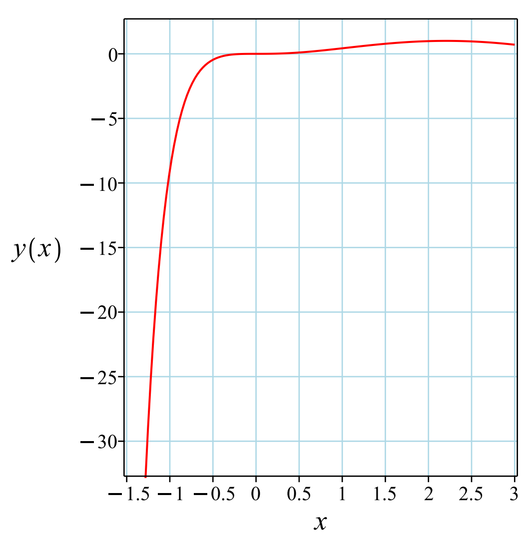
Substituting these values back in above solution results in

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

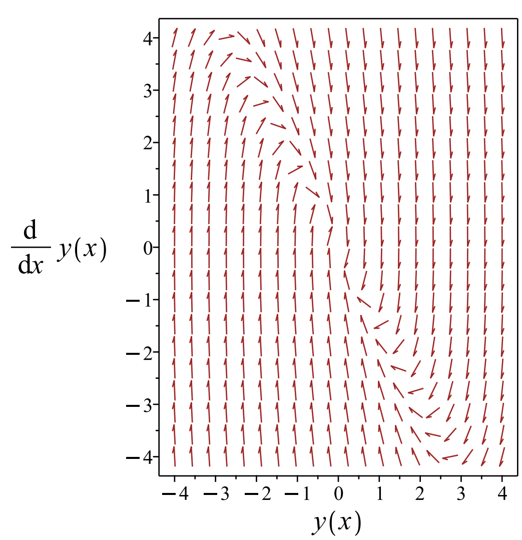
Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Verified OK.

18.7.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 574: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1 \left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + c_2 x \,\mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) - 6A_1 \sin(x) + 6A_2 \cos(x) = 10 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{5}, A_2 = \frac{4}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + c_2 x \,\mathrm{e}^{-3x}\right) + \left(-\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2x + c_1) - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3x}(c_2x + c_1) - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = -\frac{3}{5} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,e^{-3x}(c_2x + c_1) + c_2e^{-3x} + \frac{3\sin(x)}{5} + \frac{4\cos(x)}{5}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{4}{5} - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{3}{5}$$
$$c_2 = 1$$

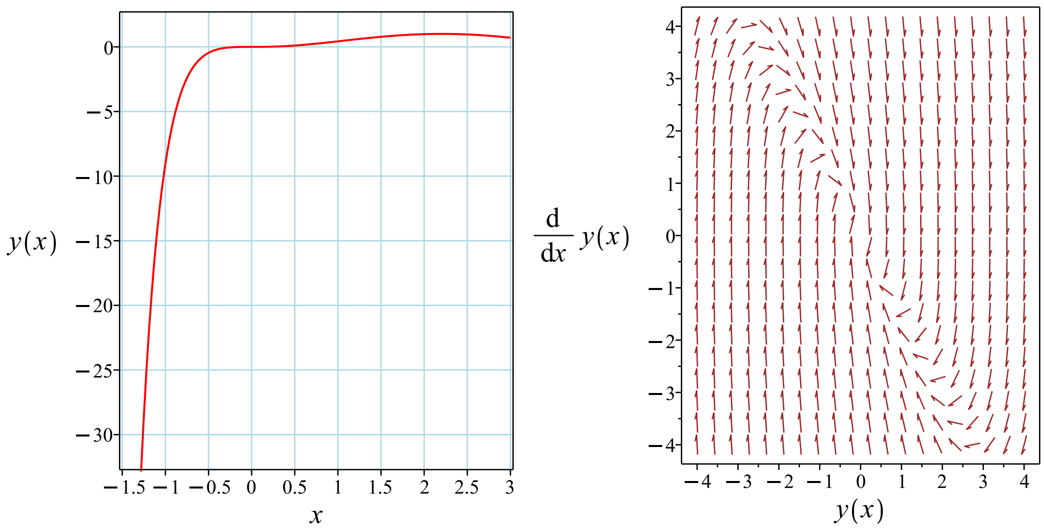
Substituting these values back in above solution results in

$$y = x\,e^{-3x} + \frac{3\,e^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Summary

The solution(s) found are the following

$$y = x\,e^{-3x} + \frac{3\,e^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

Verified OK.

18.7.5 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 9y = 10\sin(x),\, y(0) = 0,\, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 10\sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = 10\,\mathrm{e}^{-3x}\left(-\left(\int \sin(x)\,x\,\mathrm{e}^{3x}dx\right) + x\left(\int \mathrm{e}^{3x}\sin(x)\,dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = -\frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-3x} + c_2 x \mathrm{e}^{-3x} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$

  ◦ Use initial condition $y(0) = 0$

  $0 = -\frac{3}{5} + c_1$

  ◦ Compute derivative of the solution

  $y' = -3c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} - 3c_2 x\,\mathrm{e}^{-3x} + \frac{3\sin(x)}{5} + \frac{4\cos(x)}{5}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = \frac{4}{5} - 3c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = \frac{3}{5}, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$

- Solution to the IVP

  $y = x\,\mathrm{e}^{-3x} + \frac{3\,\mathrm{e}^{-3x}}{5} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=10*sin(x),y(0) = 0, D(y)(0) = 0],y(x), singsol=
```

$$y(x) = \frac{3\,\mathrm{e}^{-3x}}{5} + x\,\mathrm{e}^{-3x} - \frac{3\cos(x)}{5} + \frac{4\sin(x)}{5}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 33

```
DSolve[{y''[x]+6*y'[x]+9*y[x]==10*Sin[x],{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{5}\left(5e^{-3x}x + 3e^{-3x} + 4\sin(x) - 3\cos(x)\right)$$

18.8 problem 597

Internal problem ID [15366]
Internal file name [`OUTPUT/15366_Wednesday_May_08_2024_03_57_07_PM_70106345/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 597.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = 2\cos(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

18.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 2\cos(x) \end{aligned}$$

Hence the ode is

$$y'' + y = 2\cos(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.8.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\,x, \sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\cos(x)\,x + A_2\sin(x)\,x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 2A_2\cos(x) = 2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\sin(x)\,x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\cos(x) + c_2\sin(x) + \sin(x)\,x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + \cos(x)\, x + \sin(x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

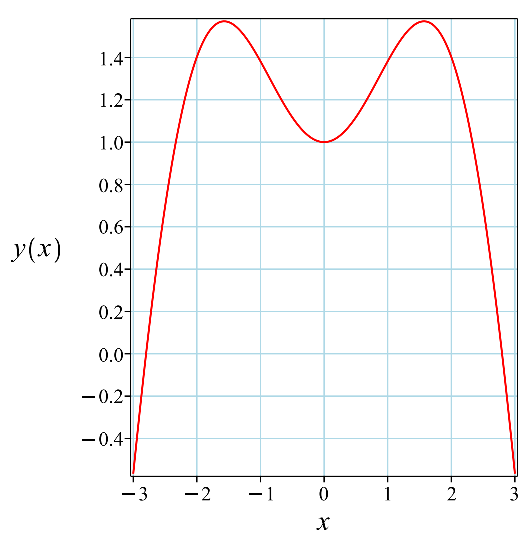
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

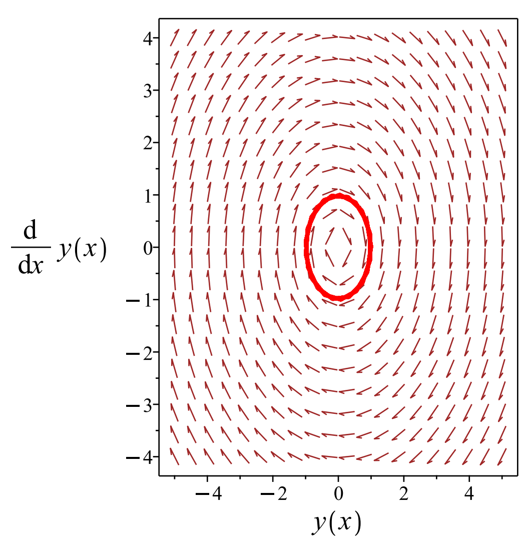
$$y = \sin(x)\, x + \cos(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x)\, x + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)\, x + \cos(x)$$

Verified OK.

18.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 576: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = 2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)\, x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (\sin(x)\, x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + \cos(x) x + \sin(x)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

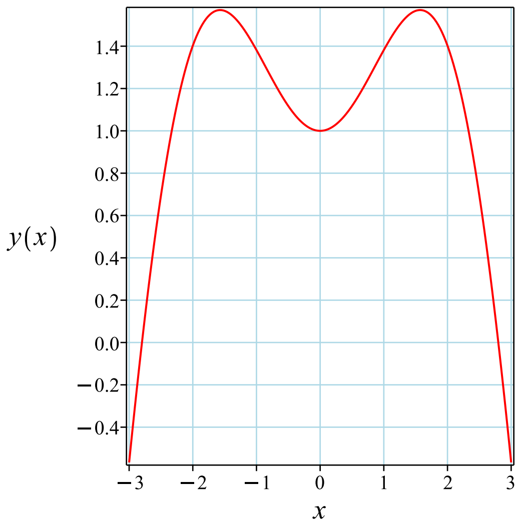
$$\begin{aligned} c_1 &= 1 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

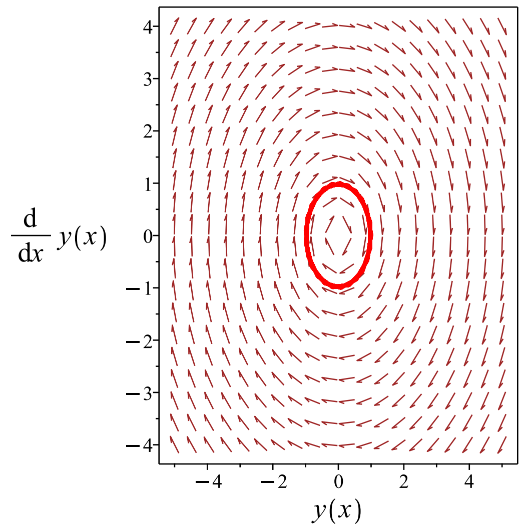
$$y = \sin(x) x + \cos(x)$$

Summary
The solution(s) found are the following

$$y = \sin(x) x + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin(x)\,x + \cos(x)$$

Verified OK.

18.8.4 Maple step by step solution

Let's solve

$\left[y'' + y = 2\cos(x), y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \sin(2x)\, dx\right) + 2\sin(x)\left(\int \cos(x)^2\, dx\right)$

  - Compute integrals

    $y_p(x) = \frac{\cos(x)}{2} + \sin(x)\, x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)}{2} + \sin(x)\, x$

☐ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)}{2} + \sin(x)\, x$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + \frac{1}{2}$

  - Compute derivative of the solution

    $y' = -c_1 \sin(x) + c_2 \cos(x) + \frac{\sin(x)}{2} + \cos(x)\, x$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = \frac{1}{2}, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = \sin(x)\, x + \cos(x)$

- Solution to the IVP

  $y = \sin(x)\, x + \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)+y(x)=2*cos(x),y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \cos(x) + \sin(x)\, x$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 12

```
DSolve[{y''[x]+y[x]==2*Cos[x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\sin(x) + \cos(x)$$

18.9 problem 598

Internal problem ID [15367]
Internal file name [`OUTPUT/15367_Wednesday_May_08_2024_03_57_08_PM_15923070/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 598.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = \sin(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

18.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= \sin(x) \end{aligned}$$

Hence the ode is

$$y'' + 4y = \sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.9.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{\sin(x)}{3}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(x)}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{\cos(x)}{3}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{1}{3} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{1}{3}$$

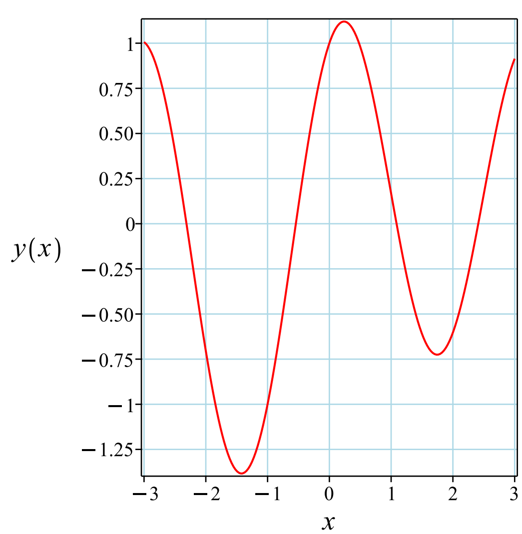
Substituting these values back in above solution results in

$$y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3}$$

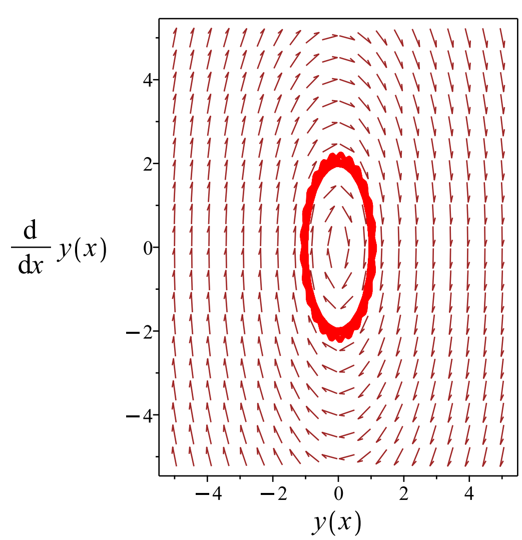
Summary

The solution(s) found are the following

$$y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3}$$

Verified OK.

18.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 578: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\cos(2x)+\frac{c_2\sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2},\cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x)+A_2\sin(x)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\cos(x)+3A_2\sin(x)=\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=0,A_2=\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=\frac{\sin(x)}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{\sin(x)}{3}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{\sin(x)}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \sin(2x) + c_2 \cos(2x) + \frac{\cos(x)}{3}
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = \frac{1}{3} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= \frac{2}{3}
\end{aligned}
$$

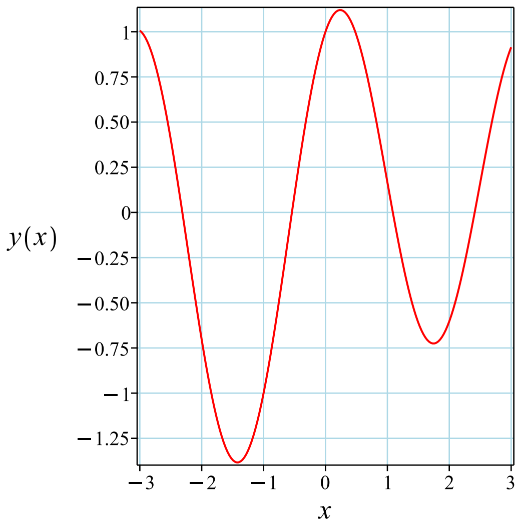
Substituting these values back in above solution results in

$$
y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3}
$$

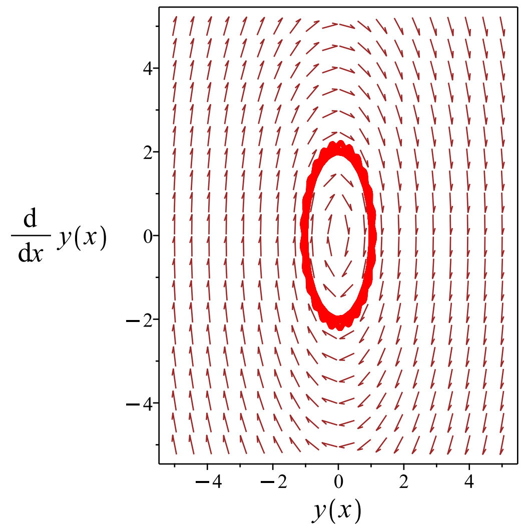
Summary

The solution(s) found are the following

$$
y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3}$$

Verified OK.

18.9.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y = \sin(x), y(0) = 1, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(2x) + c_2\sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(2x)\left(\int \sin(x)^2\cos(x)\,dx\right) + \frac{\sin(2x)\left(\int(\sin(3x)-\sin(x))dx\right)}{4}$

  - Compute integrals

    $y_p(x) = \frac{\sin(x)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2x) + c_2\sin(2x) + \frac{\sin(x)}{3}$

☐ Check validity of solution $y = c_1\cos(2x) + c_2\sin(2x) + \frac{\sin(x)}{3}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1\sin(2x) + 2c_2\cos(2x) + \frac{\cos(x)}{3}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = \frac{1}{3} + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = \frac{1}{3}\}$

- Substitute constant values into general solution and simplify

  $y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3}$

- Solution to the IVP

  $y = \cos(2x) + \frac{\sin(2x)}{3} + \frac{\sin(x)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)+4*y(x)=sin(x),y(0) = 1, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)}{3} + \cos(2x) + \frac{\sin(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 22

```
DSolve[{y''[x]+4*y[x]==Sin[x],{y[0]==1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(\sin(x) + \sin(2x) + 3\cos(2x))$$

18.10 problem 599

Internal problem ID [15368]
Internal file name [`OUTPUT/15368_Wednesday_May_08_2024_03_57_10_PM_35232107/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 599.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = 4\cos(x)\, x}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

18.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 4\cos(x)\, x \end{aligned}$$

Hence the ode is

$$y'' + y = 4\cos(x)\, x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 4\cos(x)\, x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.10.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4\cos(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\, x, \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\cos(x), x^2\sin(x), \cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2\cos(x) + A_2x^2\sin(x) + A_3\cos(x)\, x + A_4\sin(x)\, x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\cos(x) - 4A_1x\sin(x) + 2A_2\sin(x) + 4A_2x\cos(x) - 2A_3\sin(x) + 2A_4\cos(x)\\&= 4\cos(x)\, x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2\sin(x) + \cos(x)\, x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (c_1\cos(x) + c_2\sin(x)) + (x^2\sin(x) + \cos(x)\, x)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x) x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + \sin(x) x + x^2 \cos(x) + \cos(x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^2 \sin(x) + \cos(x) x$$

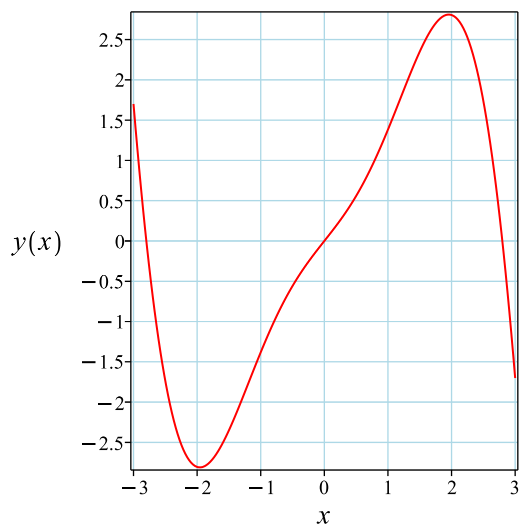
Which simplifies to

$$y = x(\sin(x) x + \cos(x))$$

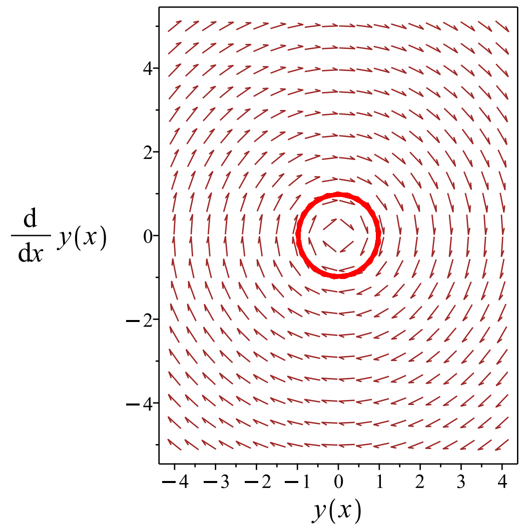
Summary

The solution(s) found are the following

$$y = x(\sin(x) x + \cos(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = x(\sin(x)\, x + \cos(x))$$

Verified OK.

18.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 580: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
4\cos(x)\, x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x)\, x, \sin(x)\, x, \cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\cos(x), \sin(x)\}
$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2\cos(x), x^2\sin(x), \cos(x)\, x, \sin(x)\, x\}]
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Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1x^2\cos(x) + A_2x^2\sin(x) + A_3\cos(x)\, x + A_4\sin(x)\, x
$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\cos(x) - 4A_1x\sin(x) + 2A_2\sin(x) + 4A_2x\cos(x) - 2A_3\sin(x) + 2A_4\cos(x) = 4\cos(x)x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1, A_3 = 1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2\sin(x) + \cos(x)x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (x^2\sin(x) + \cos(x)x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\cos(x) + c_2\sin(x) + x^2\sin(x) + \cos(x)x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1\sin(x) + c_2\cos(x) + \sin(x)x + x^2\cos(x) + \cos(x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^2 \sin(x) + \cos(x) x$$

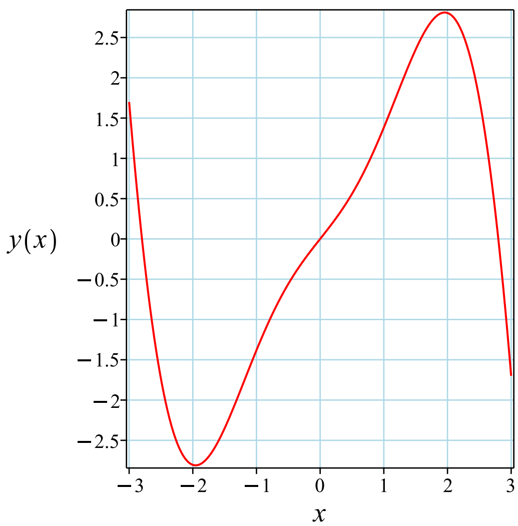
Which simplifies to

$$y = x(\sin(x) x + \cos(x))$$

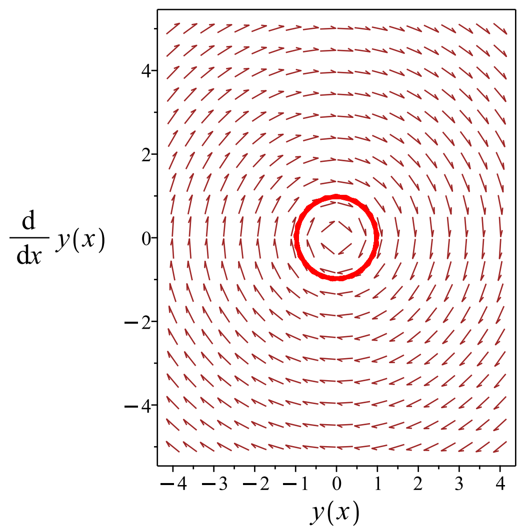
Summary

The solution(s) found are the following

$$y = x(\sin(x) x + \cos(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x(\sin(x) x + \cos(x))$$

Verified OK.

18.10.4 Maple step by step solution

Let's solve

$$\left[y''+y=4\cos\left(x\right)x, y(0)=0, y'\Big|_{\{x=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos\left(x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin\left(x\right)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos\left(x\right)+c_2\sin\left(x\right)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=4\cos\left(x\right)x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x))=\begin{bmatrix}\cos\left(x\right) & \sin\left(x\right)\\ -\sin\left(x\right) & \cos\left(x\right)\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x))=1$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -2\cos(x)\left(\int x\sin(2x)\,dx\right) + 4\sin(x)\left(\int \cos(x)^2 x dx\right)$$

- ◦ Compute integrals

$$y_p(x) = x^2 \sin(x) + \cos(x) x - \sin(x)$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x) x - \sin(x)$$

☐ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) + x^2 \sin(x) + \cos(x) x - \sin(x)$

- ◦ Use initial condition $y(0) = 0$

$$0 = c_1$$

- ◦ Compute derivative of the solution

$$y' = -c_1 \sin(x) + c_2 \cos(x) + \sin(x) x + x^2 \cos(x)$$

- ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

$$1 = c_2$$

- ◦ Solve for c_1 and c_2

$$\{c_1 = 0, c_2 = 1\}$$

- ◦ Substitute constant values into general solution and simplify

$$y = x(\sin(x) x + \cos(x))$$

- Solution to the IVP

$$y = x(\sin(x) x + \cos(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)+y(x)=4*x*cos(x),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = x(\cos(x) + \sin(x)\, x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 14

```
DSolve[{y''[x]+y[x]==4*x*Cos[x],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x \sin(x) + \cos(x))$$

18.11 problem 600

Internal problem ID [15369]
Internal file name [`OUTPUT/15369_Wednesday_May_08_2024_03_57_11_PM_68092082/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 600.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 5y = 2\,\mathrm{e}^{x} x^{2}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 3]$$

18.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 5 \\ F &= 2\,\mathrm{e}^{x} x^{2} \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 5y = 2\,\mathrm{e}^{x}x^{2}$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\,\mathrm{e}^{x}x^{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 5, f(x) = 2\,\mathrm{e}^{x}x^{2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 5\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(5)} \\ &= 2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + i \\ \lambda_2 &= 2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 + i \\ \lambda_2 &= 2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^x x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}x^{2}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(x), e^{2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x} x^{2} + A_3 e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{x} + 2A_1 x\,e^{x} + 2A_2 e^{x} x^{2} - 4A_2 e^{x} x + 2A_2 e^{x} + 2A_3 e^{x} = 2\,e^{x} x^{2}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,e^{x} + e^{x} x^{2} + e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}(c_1\cos(x) + c_2\sin(x))\right) + \left(2x\,e^{x} + e^{x} x^{2} + e^{x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}(c_1\cos(x) + c_2\sin(x)) + 2x\,e^{x} + e^{x} x^{2} + e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,\mathrm{e}^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{2x}(-c_1 \sin(x) + c_2 \cos(x)) + 4x\,\mathrm{e}^{x} + 3\,\mathrm{e}^{x} + \mathrm{e}^{x} x^2$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 2c_1 + c_2 + 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = \mathrm{e}^{x} x^2 + 2x\,\mathrm{e}^{x} + \mathrm{e}^{2x} \cos(x) - 2\,\mathrm{e}^{2x} \sin(x) + \mathrm{e}^{x}$$

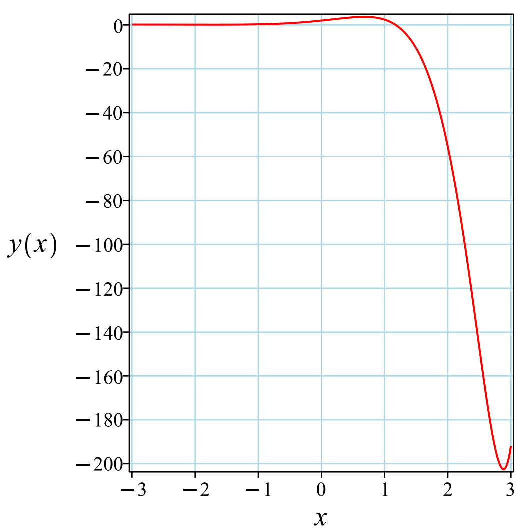
Which simplifies to

$$y = (\cos(x) - 2\sin(x))\,\mathrm{e}^{2x} + \mathrm{e}^{x}(x+1)^2$$

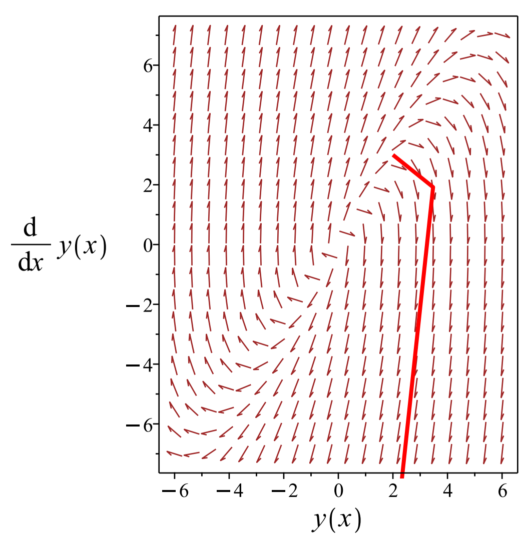
Summary

The solution(s) found are the following

$$y = (\cos(x) - 2\sin(x))\,\mathrm{e}^{2x} + \mathrm{e}^{x}(x+1)^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (\cos(x) - 2\sin(x))\,e^{2x} + e^{x}(x+1)^{2}$$

Verified OK.

18.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 582: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x} \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x} \cos(x)\right) + c_2 \left(\mathrm{e}^{2x} \cos(x)(\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, e^{2x} c_1 + e^{2x} \sin(x)\, c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{x} x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{x}, e^{x} x^2, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} \cos(x)\, , e^{2x} \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\, e^{x} + A_2 e^{x} x^2 + A_3 e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{x} + 2A_1 x\, e^{x} + 2A_2 e^{x} x^2 - 4A_2 e^{x} x + 2A_2 e^{x} + 2A_3 e^{x} = 2\, e^{x} x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 1, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\, e^{x} + e^{x} x^2 + e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x)\, e^{2x} c_1 + e^{2x} \sin(x)\, c_2\right) + \left(2x\, e^{x} + e^{x} x^2 + e^{x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + 2x e^x + e^x x^2 + e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + 2x e^x + e^x x^2 + e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2 e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{2x}(-c_1 \sin(x) + c_2 \cos(x)) + 4x e^x + 3 e^x + e^x x^2$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = 2c_1 + c_2 + 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = e^x x^2 + 2x e^x + e^{2x} \cos(x) - 2 e^{2x} \sin(x) + e^x$$

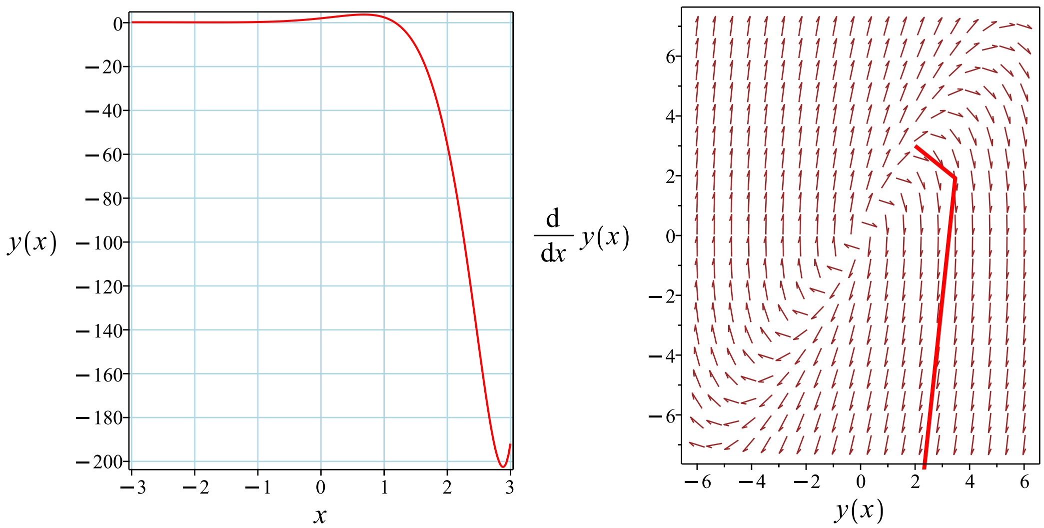
Which simplifies to

$$y = (\cos(x) - 2\sin(x)) e^{2x} + e^x (x+1)^2$$

Summary

The solution(s) found are the following

$$y = (\cos(x) - 2\sin(x)) e^{2x} + e^x (x+1)^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (\cos(x) - 2\sin(x))\,e^{2x} + e^{x}(x+1)^2$$

Verified OK.

18.11.4 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 5y = 2\,e^{x}x^2, y(0) = 2, y'\Big|_{\{x=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - I, 2 + I)$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) e^{2x} c_1 + e^{2x} \sin(x) c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2 e^x x^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} \cos(x) & e^{2x} \sin(x) \\ 2 e^{2x} \cos(x) - e^{2x} \sin(x) & 2 e^{2x} \sin(x) + e^{2x} \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2 e^{2x} \left(\cos(x) \left(\int x^2 \sin(x) e^{-x} dx\right) - \sin(x) \left(\int x^2 e^{-x} \cos(x) dx\right)\right)$

  ◦ Compute integrals

    $y_p(x) = e^x (x+1)^2$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) e^{2x} c_1 + e^{2x} \sin(x) c_2 + e^x (x+1)^2$

☐ Check validity of solution $y = \cos(x) e^{2x} c_1 + e^{2x} \sin(x) c_2 + e^x (x+1)^2$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1 + 1$

  ◦ Compute derivative of the solution

    $y' = -\sin(x) e^{2x} c_1 + 2\cos(x) e^{2x} c_1 + 2 e^{2x} \sin(x) c_2 + e^{2x} \cos(x) c_2 + e^x (x+1)^2 + 2 e^x (x+1)$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 3$

    $3 = 2c_1 + c_2 + 3$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -2\}$

- Substitute constant values into general solution and simplify

  $y = (\cos(x) - 2\sin(x))\,e^{2x} + e^{x}(x+1)^2$

- Solution to the IVP

  $y = (\cos(x) - 2\sin(x))\,e^{2x} + e^{x}(x+1)^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=2*x^2*exp(x),y(0) = 2, D(y)(0) = 3],y(x), sings
```

$$y(x) = (\cos(x) - 2\sin(x))\,e^{2x} + (1+x)^2\,e^{x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 28

```
DSolve[{y''[x]-4*y'[x]+5*y[x]==2*x^2*Exp[x],{y[0]==2,y'[0]==3}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{x}\left((x+1)^2 - 2e^{x}\sin(x) + e^{x}\cos(x)\right)$$

18.12 problem 601

Internal problem ID [15370]
Internal file name [`OUTPUT/15370_Wednesday_May_08_2024_03_57_12_PM_27021038/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 601.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = 16\,\mathrm{e}^{-x} + 9x - 6}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

18.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -6 \\ q(x) &= 9 \\ F &= 16\,e^{-x} + 9x - 6 \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 9y = 16\,e^{-x} + 9x - 6$$

The domain of $p(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 16\,e^{-x} + 9x - 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.12.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 16\,e^{-x} + 9x - 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-6)^2 - (4)\,(1)\,(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 e^{3x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{3x} + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16\, e^{-x} + 9x - 6$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x} x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$16A_1 e^{-x} - 6A_3 + 9A_2 + 9A_3 x = 16\,e^{-x} + 9x - 6$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-x} + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 x\, e^{3x}\right) + \left(e^{-x} + x\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + e^{-x} + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{3x}(c_2 x + c_1) + e^{-x} + x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3\,e^{3x}(c_2x + c_1) + c_2e^{3x} - e^{-x} + 1$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

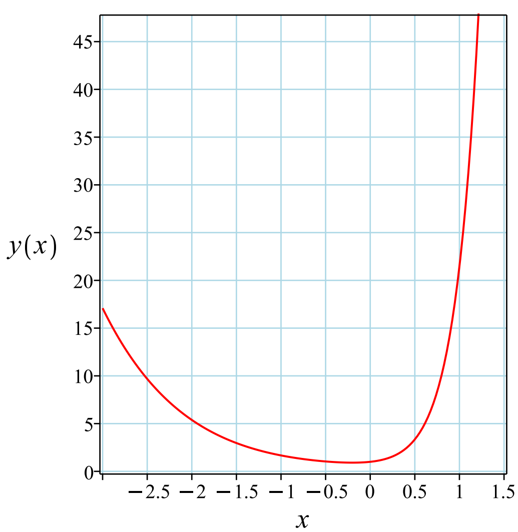
Substituting these values back in above solution results in

$$y = e^{3x}x + e^{-x} + x$$

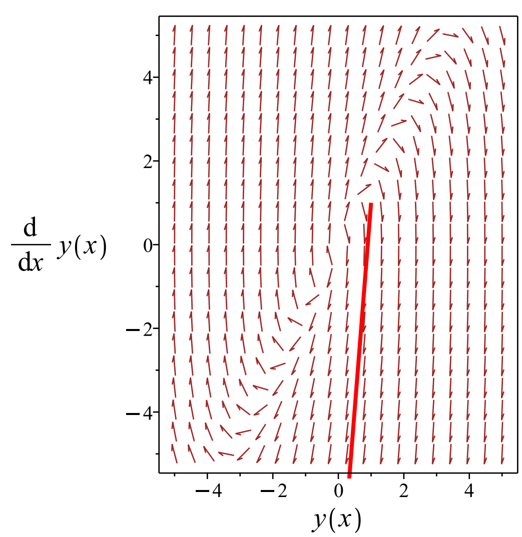
Summary

The solution(s) found are the following

$$y = e^{3x}x + e^{-x} + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{3x}x + e^{-x} + x$$

Verified OK.

18.12.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -6\, dx} \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-3x}\left(16\,\mathrm{e}^{-x} + 9x - 6\right) \\ \left(\mathrm{e}^{-3x} y\right)'' &= \mathrm{e}^{-3x}\left(16\,\mathrm{e}^{-x} + 9x - 6\right) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x} y\right)' = -4\,\mathrm{e}^{-4x} + \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x} y\right) = c_1 x + x\,\mathrm{e}^{-3x} + \mathrm{e}^{-4x} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + x\,\mathrm{e}^{-3x} + \mathrm{e}^{-4x} + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} + x + \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\,\mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} + x + \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = 1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{3x} + 3c_1 x\, e^{3x} + 3c_2 e^{3x} + 1 - e^{-x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

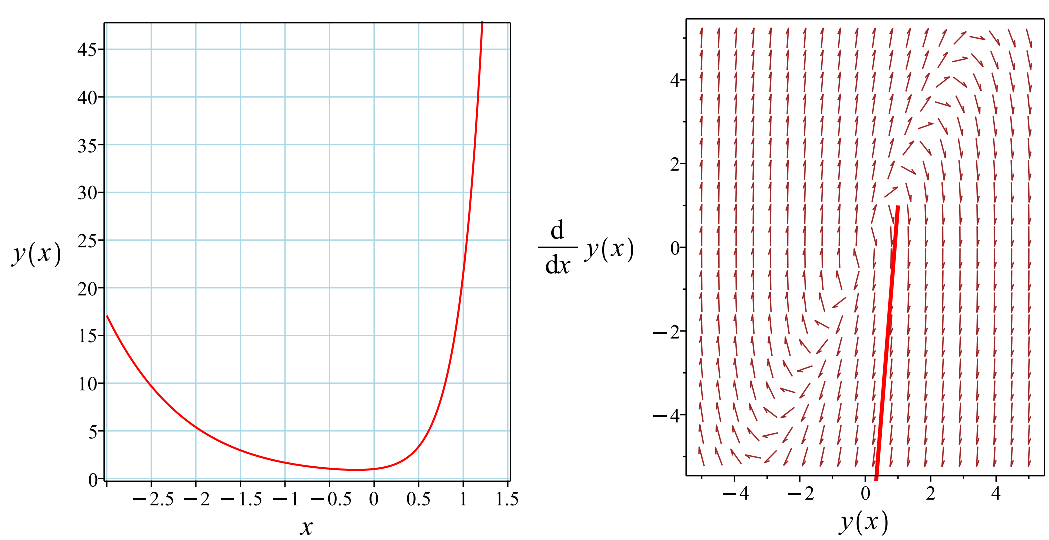
$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = e^{3x} x + e^{-x} + x$$

Summary
The solution(s) found are the following

$$y = e^{3x} x + e^{-x} + x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = e^{3x}x + e^{-x} + x
$$

Verified OK.

18.12.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - 6y' + 9y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= -6 \tag{3} \\
C &= 9
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 0 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 584: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1\left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{3x} + c_2 x \,e^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
16 \,e^{-x} + 9x - 6
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{-x}\}, \{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^{3x} x, e^{3x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{-x} + A_2 + A_3 x
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
16 A_1 e^{-x} - 6 A_3 + 9 A_2 + 9 A_3 x = 16 \,e^{-x} + 9x - 6
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = 1, A_2 = 0, A_3 = 1]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = e^{-x} + x
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{3x} + c_2 x \,e^{3x}\right) + \left(e^{-x} + x\right)
\end{aligned}
$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + e^{-x} + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{3x}(c_2 x + c_1) + e^{-x} + x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3\,e^{3x}(c_2 x + c_1) + c_2 e^{3x} - e^{-x} + 1$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 1$$

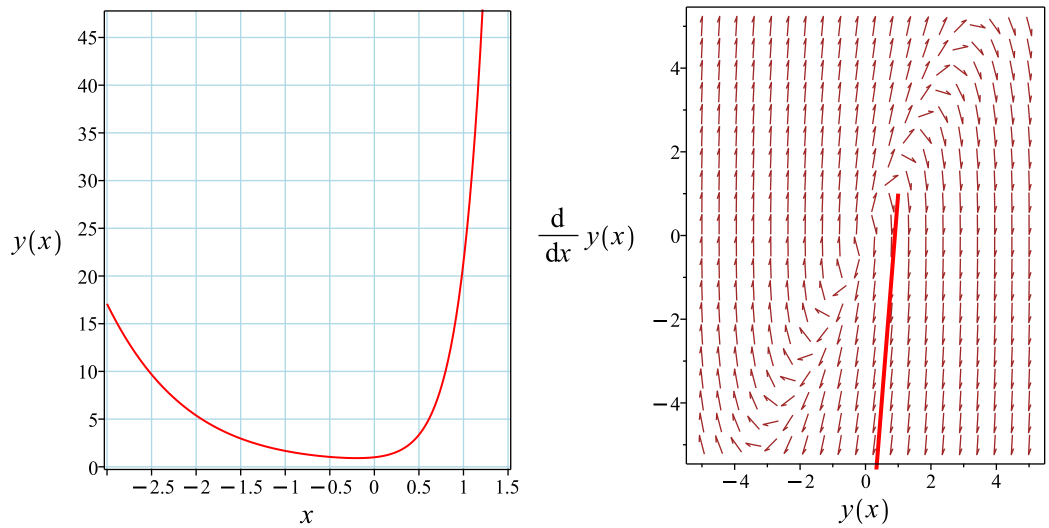
Substituting these values back in above solution results in

$$y = e^{3x}x + e^{-x} + x$$

Summary

The solution(s) found are the following

$$y = e^{3x}x + e^{-x} + x \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{3x}x + e^{-x} + x$$

Verified OK.

18.12.5 Maple step by step solution

Let's solve

$$\left[y'' - 6y' + 9y = 16\,e^{-x} + 9x - 6, y(0) = 1, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{3x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 16\, e^{-x} + 9x - 6\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & e^{3x}x \\ 3\, e^{3x} & 3\, e^{3x}x + e^{3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{6x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{3x}\left(-\left(\int (16\, e^{-x} + 9x - 6)\, x\, e^{-3x}dx\right) + \left(\int e^{-3x}(16\, e^{-x} + 9x - 6)\, dx\right) x\right)$

- ○ Compute integrals

  $y_p(x) = e^{-x} + x$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{3x} + c_2 x\, e^{3x} + e^{-x} + x$

☐ Check validity of solution $y = c_1 e^{3x} + c_2 x e^{3x} + e^{-x} + x$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1 + 1$

- ○ Compute derivative of the solution

  $y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x\, e^{3x} - e^{-x} + 1$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = 3c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$
- Substitute constant values into general solution and simplify

  $y = e^{3x}x + e^{-x} + x$

- Solution to the IVP

  $y = e^{3x}x + e^{-x} + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=16*exp(-x)+9*x-6,y(0) = 1, D(y)(0) = 1],y(x), s
```

$$y(x) = e^{3x}x + x + e^{-x}$$

✓ Solution by Mathematica

Time used: 0.28 (sec). Leaf size: 19

```
DSolve[{y''[x]-6*y'[x]+9*y[x]==16*Exp[-x]+9*x-6,{y[0]==1,y'[0]==1}},y[x],x,IncludeSingularSo
```

$$y(x) \to e^{3x}x + x + e^{-x}$$

18.13 problem 602

Internal problem ID [15371]
Internal file name [`OUTPUT/15371_Wednesday_May_08_2024_03_57_14_PM_64383561/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 602.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - y' = -5\,\mathrm{e}^{-x}(\sin(x) + \cos(x))}$$

With initial conditions

$$[y(0) = -4, y'(0) = 5]$$

18.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= 0 \\ F &= -5\,\mathrm{e}^{-x}(\sin(x) + \cos(x)) \end{aligned}$$

Hence the ode is

$$y'' - y' = -5\,\mathrm{e}^{-x}(\sin(x) + \cos(x))$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = -5\,\mathrm{e}^{-x}(\sin(x) + \cos(x))$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

18.13.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = 0, f(x) = (-5\sin(x) - 5\cos(x))\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(0)}\\ &= \frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(0)x}$$

Or

$$y = \mathrm{e}^{x} c_1 + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x} c_1 + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-5\,e^{-x}(\sin(x)+\cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\cos(x), e^{-x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x}\cos(x) + A_2 e^{-x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&3A_1 e^{-x}\sin(x) - 3A_2 e^{-x}\cos(x) + A_1 e^{-x}\cos(x) + A_2 e^{-x}\sin(x)\\&= (-5\sin(x) - 5\cos(x))\,e^{-x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,e^{-x}\cos(x) + e^{-x}\sin(x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (e^{x}c_1 + c_2) + \left(-2\,e^{-x}\cos(x) + e^{-x}\sin(x)\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x} c_1 + c_2 - 2\,e^{-x} \cos(x) + e^{-x} \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 0$ in the above gives

$$-4 = c_1 + c_2 - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x} c_1 + 3\,e^{-x} \cos(x) + e^{-x} \sin(x)$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = c_1 + 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -4 \end{aligned}$$

Substituting these values back in above solution results in

$$y = e^{-x} \sin(x) - 2\,e^{-x} \cos(x) + 2\,e^{x} - 4$$

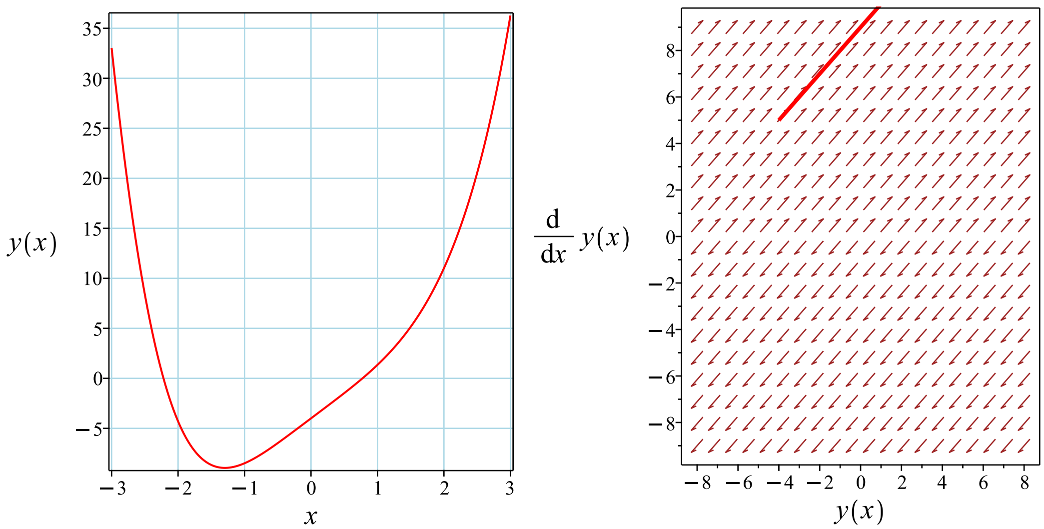
Which simplifies to

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^{x}$$

<ins>Summary</ins>

The solution(s) found are the following

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^{x}$$

Verified OK.

18.13.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\,dx = \int (-5\sin(x) - 5\cos(x))\,e^{-x} dx$$

$$-y + y' = 5\,e^{-x}\cos(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = 5\,e^{-x}\cos(x) + c_1$$

Hence the ode is

$$-y + y' = 5\,e^{-x}\cos(x) + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(5\,e^{-x}\cos(x) + c_1\right)$$
$$\frac{d}{dx}\left(e^{-x} y\right) = \left(e^{-x}\right)\left(5\,e^{-x}\cos(x) + c_1\right)$$
$$d\left(e^{-x} y\right) = \left(\left(5\,e^{-x}\cos(x) + c_1\right)e^{-x}\right)\,dx$$

Integrating gives

$$e^{-x} y = \int \left(5\,e^{-x}\cos(x) + c_1\right)e^{-x}\,dx$$
$$e^{-x} y = -2\,e^{-2x}\cos(x) + e^{-2x}\sin(x) - c_1 e^{-x} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x}\left(-2\,e^{-2x}\cos(x) + e^{-2x}\sin(x) - c_1 e^{-x}\right) + c_2 e^{x}$$

which simplifies to

$$y = (\sin(x) - 2\cos(x))\,e^{-x} - c_1 + c_2 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (\sin(x) - 2\cos(x))\,e^{-x} - c_1 + c_2 e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 0$ in the above gives

$$-4 = c_2 - 2 - c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = (\cos(x) + 2\sin(x))\,e^{-x} - (\sin(x) - 2\cos(x))\,e^{-x} + c_2 e^{x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = c_2 + 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = e^{-x} \sin(x) - 2e^{-x} \cos(x) + 2e^{x} - 4$$

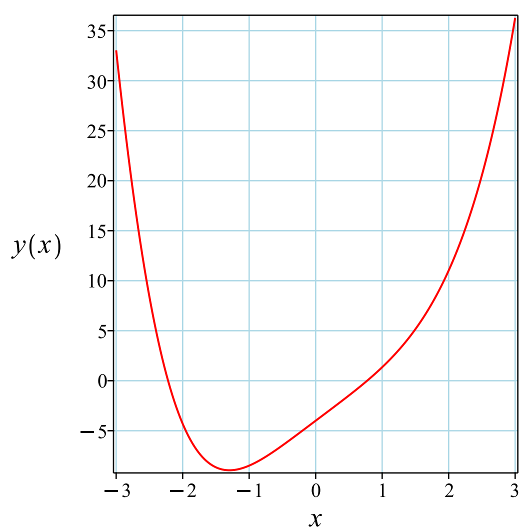
Which simplifies to

$$y = -4 + (\sin(x) - 2\cos(x)) e^{-x} + 2e^{x}$$

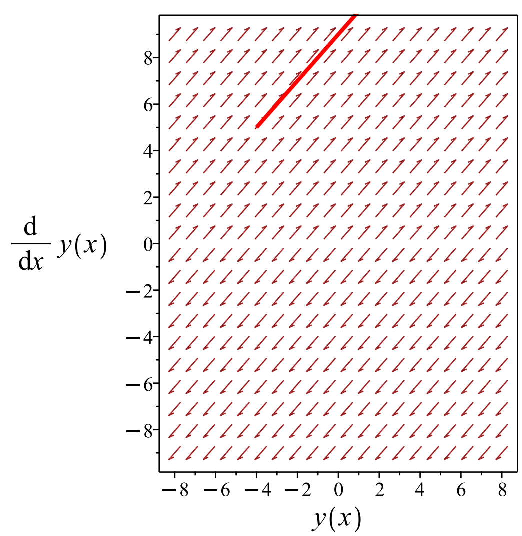
<u>Summary</u>

The solution(s) found are the following

$$y = -4 + (\sin(x) - 2\cos(x)) e^{-x} + 2e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -4 + (\sin(x) - 2\cos(x)) e^{-x} + 2e^{x}$$

Verified OK.

18.13.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - p(x) - (-5\sin(x) - 5\cos(x))\,\mathrm{e}^{-x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-1)dx} \\ &= \mathrm{e}^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(-5\,\mathrm{e}^{-x}(\sin(x) + \cos(x))\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}p\right) &= \left(\mathrm{e}^{-x}\right)\left(-5\,\mathrm{e}^{-x}(\sin(x) + \cos(x))\right) \\ \mathrm{d}\left(\mathrm{e}^{-x}p\right) &= \left(-5(\sin(x) + \cos(x))\,\mathrm{e}^{-2x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-x}p &= \int -5(\sin(x) + \cos(x))\,\mathrm{e}^{-2x}\,\mathrm{d}x \\ \mathrm{e}^{-x}p &= 3\,\mathrm{e}^{-2x}\cos(x) + \mathrm{e}^{-2x}\sin(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$p(x) = \mathrm{e}^{x}\left(3\,\mathrm{e}^{-2x}\cos(x) + \mathrm{e}^{-2x}\sin(x)\right) + \mathrm{e}^{x}c_1$$

which simplifies to

$$p(x) = \mathrm{e}^{-x}\left(\mathrm{e}^{2x}c_1 + \sin(x) + 3\cos(x)\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $p = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = c_1 + 3$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$p(x) = e^{-x}\left(\sin (x) + 3\cos (x) + 2\,e^{2x}\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{-x}\left(\sin (x) + 3\cos (x) + 2\,e^{2x}\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int e^{-x}\left(\sin (x) + 3\cos (x) + 2\,e^{2x}\right) dx \\ &= e^{-x}\sin (x) - 2\,e^{-x}\cos (x) + 2\,e^{x} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = c_2$$

$$c_2 = -4$$

Substituting c_2 found above in the general solution gives

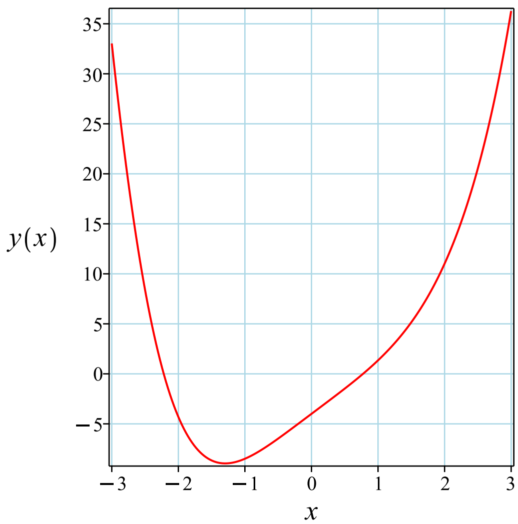
$$y = e^{-x}\sin (x) - 2\,e^{-x}\cos (x) + 2\,e^{x} - 4$$

Initial conditions are used to solve for the constants of integration.

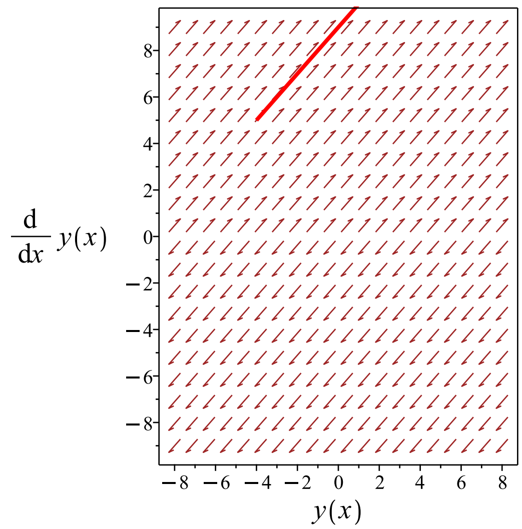
Summary

The solution(s) found are the following

$$y = e^{-x}\sin (x) - 2\,e^{-x}\cos (x) + 2\,e^{x} - 4 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-x} \sin(x) - 2e^{-x} \cos(x) + 2e^{x} - 4$$

Verified OK.

18.13.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - y' = (-5\sin(x) - 5\cos(x))\, e^{-x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - y')\, dx = \int (-5\sin(x) - 5\cos(x))\, e^{-x} dx$$

$$-y + y' = 5e^{-x} \cos(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = 5e^{-x} \cos(x) + c_1$$

Hence the ode is

$$-y+y'=5\,\mathrm{e}^{-x}\cos\left(x\right)+c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int(-1)dx}\\ &= \mathrm{e}^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(5\,\mathrm{e}^{-x}\cos\left(x\right)+c_1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x}y\right) &= \left(\mathrm{e}^{-x}\right)\left(5\,\mathrm{e}^{-x}\cos\left(x\right)+c_1\right)\\ \mathrm{d}\left(\mathrm{e}^{-x}y\right) &= \left(\left(5\,\mathrm{e}^{-x}\cos\left(x\right)+c_1\right)\mathrm{e}^{-x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-x}y &= \int\left(5\,\mathrm{e}^{-x}\cos\left(x\right)+c_1\right)\mathrm{e}^{-x}\,\mathrm{d}x\\ \mathrm{e}^{-x}y &= -2\,\mathrm{e}^{-2x}\cos\left(x\right)+\mathrm{e}^{-2x}\sin\left(x\right)-c_1\mathrm{e}^{-x}+c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\mathrm{e}^{-x}$ results in

$$y=\mathrm{e}^{x}\left(-2\,\mathrm{e}^{-2x}\cos\left(x\right)+\mathrm{e}^{-2x}\sin\left(x\right)-c_1\mathrm{e}^{-x}\right)+c_2\mathrm{e}^{x}$$

which simplifies to

$$y=\left(\sin\left(x\right)-2\cos\left(x\right)\right)\mathrm{e}^{-x}-c_1+c_2\mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=\left(\sin\left(x\right)-2\cos\left(x\right)\right)\mathrm{e}^{-x}-c_1+c_2\mathrm{e}^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=-4$ and $x=0$ in the above gives

$$-4=c_2-2-c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y'=\left(\cos\left(x\right)+2\sin\left(x\right)\right)\mathrm{e}^{-x}-\left(\sin\left(x\right)-2\cos\left(x\right)\right)\mathrm{e}^{-x}+c_2\mathrm{e}^{x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = c_2 + 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = e^{-x} \sin(x) - 2 e^{-x} \cos(x) + 2 e^{x} - 4$$

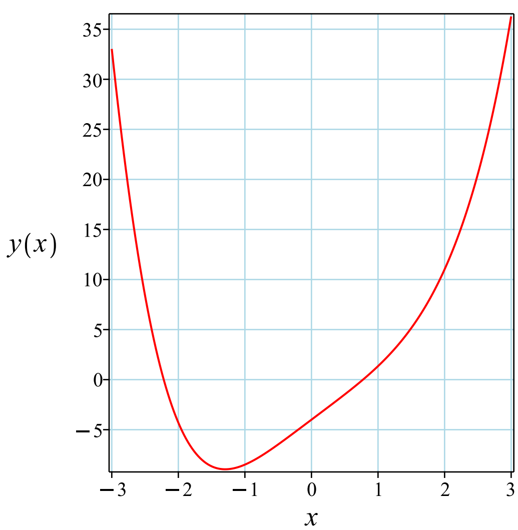
Which simplifies to

$$y = -4 + (\sin(x) - 2\cos(x)) e^{-x} + 2 e^{x}$$

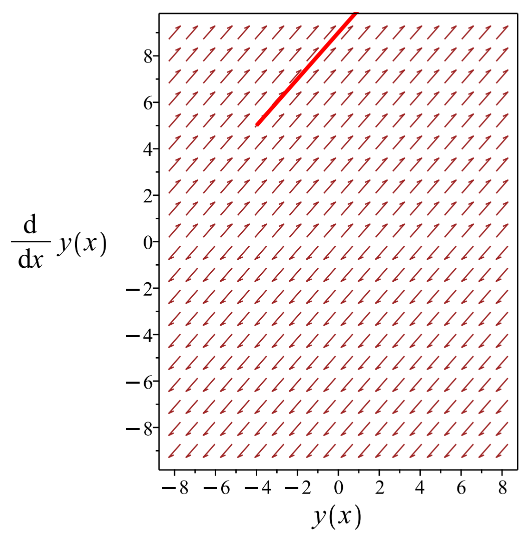
Summary

The solution(s) found are the following

$$y = -4 + (\sin(x) - 2\cos(x)) e^{-x} + 2 e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -4 + (\sin(x) - 2\cos(x)) e^{-x} + 2 e^{x}$$

Verified OK.

18.13.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 586: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{\left(y_1\right)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\mathrm{e}^{x}))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + c_2 \mathrm{e}^{x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-5\,e^{-x}(\sin(x)+\cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\cos(x)\,, e^{-x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x}\cos(x) + A_2 e^{-x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&3A_1 e^{-x}\sin(x) - 3A_2 e^{-x}\cos(x) + A_1 e^{-x}\cos(x) + A_2 e^{-x}\sin(x)\\&= (-5\sin(x) - 5\cos(x))\,e^{-x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,e^{-x}\cos(x) + e^{-x}\sin(x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (c_1 + c_2 e^{x}) + \left(-2\,e^{-x}\cos(x) + e^{-x}\sin(x)\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 e^x - 2\,e^{-x} \cos(x) + e^{-x} \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 0$ in the above gives

$$-4 = c_1 + c_2 - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 e^x + 3\,e^{-x} \cos(x) + e^{-x} \sin(x)$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = c_2 + 3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -4 \\ c_2 &= 2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = e^{-x} \sin(x) - 2\,e^{-x} \cos(x) + 2\,e^x - 4$$

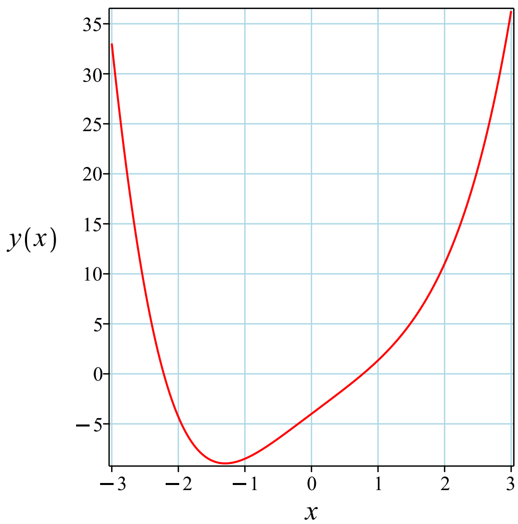
Which simplifies to

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^x$$

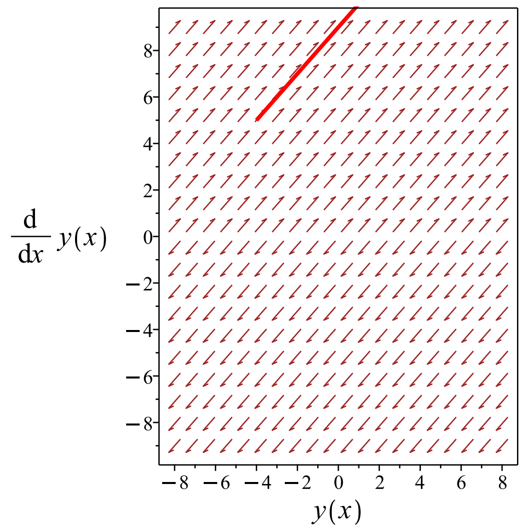
Summary

The solution(s) found are the following

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^x \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^{x}$$

Verified OK.

18.13.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= (-5\sin(x) - 5\cos(x))\,e^{-x} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$-y+y'=\int \left(-5\sin\left(x\right)-5\cos\left(x\right)\right){\mathrm e}^{-x}\,dx$$

We now have a first order ode to solve which is

$$-y+y'=5\,{\mathrm e}^{-x}\cos\left(x\right)+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=-1\\ q(x)&=5\,{\mathrm e}^{-x}\cos\left(x\right)+c_1\end{aligned}$$

Hence the ode is

$$-y+y'=5\,{\mathrm e}^{-x}\cos\left(x\right)+c_1$$

The integrating factor μ is

$$\begin{aligned}\mu&={\mathrm e}^{\int(-1)dx}\\ &={\mathrm e}^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm d}{\mathrm d x}(\mu y)&=(\mu)\left(5\,{\mathrm e}^{-x}\cos\left(x\right)+c_1\right)\\ \frac{\mathrm d}{\mathrm d x}\left({\mathrm e}^{-x}y\right)&=\left({\mathrm e}^{-x}\right)\left(5\,{\mathrm e}^{-x}\cos\left(x\right)+c_1\right)\\ \mathrm d\left({\mathrm e}^{-x}y\right)&=\left(\left(5\,{\mathrm e}^{-x}\cos\left(x\right)+c_1\right){\mathrm e}^{-x}\right)\,\mathrm d x\end{aligned}$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-x} y &= \int \left(5\,\mathrm{e}^{-x} \cos\left(x\right)+c_1\right) \mathrm{e}^{-x}\,\mathrm{d}x \\
\mathrm{e}^{-x} y &= -2\,\mathrm{e}^{-2x} \cos\left(x\right)+\mathrm{e}^{-2x} \sin\left(x\right)-c_1 \mathrm{e}^{-x}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$
y = \mathrm{e}^{x} \left(-2\,\mathrm{e}^{-2x} \cos\left(x\right)+\mathrm{e}^{-2x} \sin\left(x\right)-c_1 \mathrm{e}^{-x}\right)+c_2 \mathrm{e}^{x}
$$

which simplifies to

$$
y = \left(\sin\left(x\right)-2\cos\left(x\right)\right) \mathrm{e}^{-x}-c_1+c_2 \mathrm{e}^{x}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \left(\sin\left(x\right)-2\cos\left(x\right)\right) \mathrm{e}^{-x}-c_1+c_2 \mathrm{e}^{x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 0$ in the above gives

$$
-4 = c_2-2-c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \left(\cos\left(x\right)+2\sin\left(x\right)\right) \mathrm{e}^{-x}-\left(\sin\left(x\right)-2\cos\left(x\right)\right) \mathrm{e}^{-x}+c_2 \mathrm{e}^{x}
$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$
5 = c_2+3 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 4 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{-x} \sin\left(x\right)-2\,\mathrm{e}^{-x} \cos\left(x\right)+2\,\mathrm{e}^{x}-4
$$

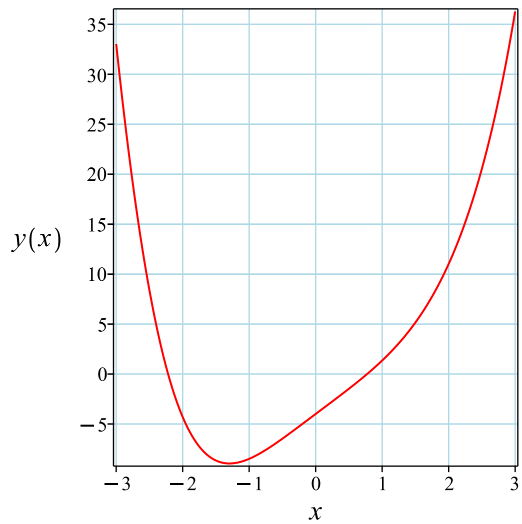
Which simplifies to

$$
y = -4+\left(\sin\left(x\right)-2\cos\left(x\right)\right) \mathrm{e}^{-x}+2\,\mathrm{e}^{x}
$$

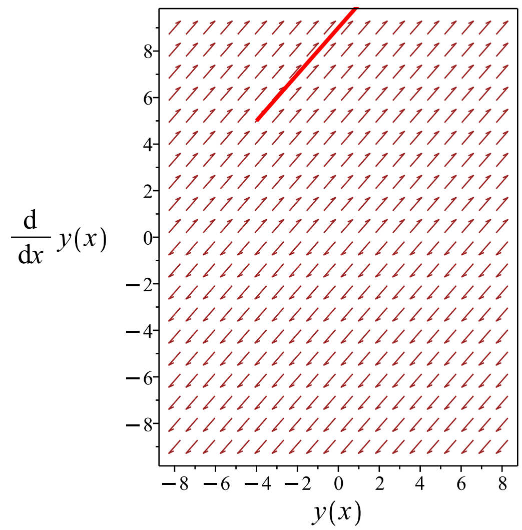
Summary

The solution(s) found are the following

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -4 + (\sin(x) - 2\cos(x))\,e^{-x} + 2\,e^{x}$$

Verified OK.

18.13.8 Maple step by step solution

Let's solve

$$\left[y'' - y' = (-5\sin(x) - 5\cos(x))\,e^{-x}, y(0) = -4, y'\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -5\,e^{-x}\sin(x) - 5\,e^{-x}\cos(x) + y'$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - y' = -5\,e^{-x}(\sin(x) + \cos(x))$$

- Characteristic polynomial of homogeneous ODE

  $r^2 - r = 0$
- Factor the characteristic polynomial

  $r(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (0, 1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = 1$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 \mathrm{e}^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -5\,\mathrm{e}^{-x}(\sin(x) + \cos(x\right.$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & \mathrm{e}^x \\ 0 & \mathrm{e}^x \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^x$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 5\left(\int \mathrm{e}^{-x}(\sin(x) + \cos(x))\, dx\right) - 5\,\mathrm{e}^x\left(\int (\sin(x) + \cos(x))\, \mathrm{e}^{-2x} dx\right)$
  - Compute integrals

    $y_p(x) = (\sin(x) - 2\cos(x))\, \mathrm{e}^{-x}$
- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 \mathrm{e}^x + (\sin(x) - 2\cos(x))\, \mathrm{e}^{-x}$

☐ Check validity of solution $y = c_1 + c_2 \mathrm{e}^x + (\sin(x) - 2\cos(x))\, \mathrm{e}^{-x}$

- Use initial condition $y(0) = -4$

  $-4 = c_1 + c_2 - 2$
- Compute derivative of the solution

  $y' = (\cos(x) + 2\sin(x))\,\mathrm{e}^{-x} - (\sin(x) - 2\cos(x))\,\mathrm{e}^{-x} + c_2\mathrm{e}^{x}$
- Use the initial condition $y'\big|_{\{x=0\}} = 5$

  $5 = c_2 + 3$
- Solve for c_1 and c_2

  $\{c_1 = -4, c_2 = 2\}$
- Substitute constant values into general solution and simplify

  $y = -4 + (\sin(x) - 2\cos(x))\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

- Solution to the IVP

  $y = -4 + (\sin(x) - 2\cos(x))\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -5*exp(-_a)*cos(_a)-5*exp(-_a)*sin(_a)+
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$2)-diff(y(x),x)=-5*exp(-x)*(sin(x)+cos(x)),y(0) = -4, D(y)(0) = 5],y(x),
```

$$y(x) = 2\,\mathrm{e}^{x} - 4 + \mathrm{e}^{-x}(-2\cos(x) + \sin(x))$$

✓ Solution by Mathematica

Time used: 0.173 (sec). Leaf size: 28

```
DSolve[{y''[x]-y'[x]==-5*Exp[-x]*(Sin[x]+Cos[x]),{y[0]==-4,y'[0]==5}},y[x],x,IncludeSingular
```

$$y(x) \rightarrow e^{-x}(2e^{x}(e^{x}-2)+\sin(x)-2\cos(x))$$

18.14 problem 603

Internal problem ID [15372]
Internal file name [`OUTPUT/15372_Wednesday_May_08_2024_03_57_18_PM_76601727/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 603.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = 4\,\mathrm{e}^{x} \cos(x)}$$

With initial conditions

$$[y(\pi) = \pi\,\mathrm{e}^{\pi}, y'(\pi) = \mathrm{e}^{\pi}]$$

18.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= 4\,\mathrm{e}^{x} \cos(x) \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 4\,e^{x} \cos(x)$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is also inside this domain. The domain of $F = 4\,e^{x} \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

18.14.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = 4\,e^{x} \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\,e^{\lambda x} + 2\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^x(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^x(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,e^x \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(x), e^x \sin(x)\}$$

Since $e^x \cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x \cos(x), x e^x \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x \cos(x) + A_2 x e^x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^x \sin(x) + 2A_2 e^x \cos(x) = 4 e^x \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x e^x \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x (c_1 \cos(x) + c_2 \sin(x))) + (2x e^x \sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^x (c_1 \cos(x) + c_2 \sin(x)) + 2x e^x \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \pi\,\mathrm{e}^{\pi}$ and $x = \pi$ in the above gives

$$\pi\,\mathrm{e}^{\pi} = -c_1 \mathrm{e}^{\pi} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{x}(-c_1 \sin(x) + c_2 \cos(x)) + 2\,\mathrm{e}^{x} \sin(x) + 2x\,\mathrm{e}^{x} \sin(x) + 2x\,\mathrm{e}^{x} \cos(x)$$

substituting $y' = \mathrm{e}^{\pi}$ and $x = \pi$ in the above gives

$$\mathrm{e}^{\pi} = (-c_1 - c_2 - 2\pi)\,\mathrm{e}^{\pi} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\pi$$
$$c_2 = -\pi - 1$$

Substituting these values back in above solution results in

$$y = -\mathrm{e}^{x} \cos(x)\,\pi - \mathrm{e}^{x} \sin(x)\,\pi + 2x\,\mathrm{e}^{x} \sin(x) - \mathrm{e}^{x} \sin(x)$$

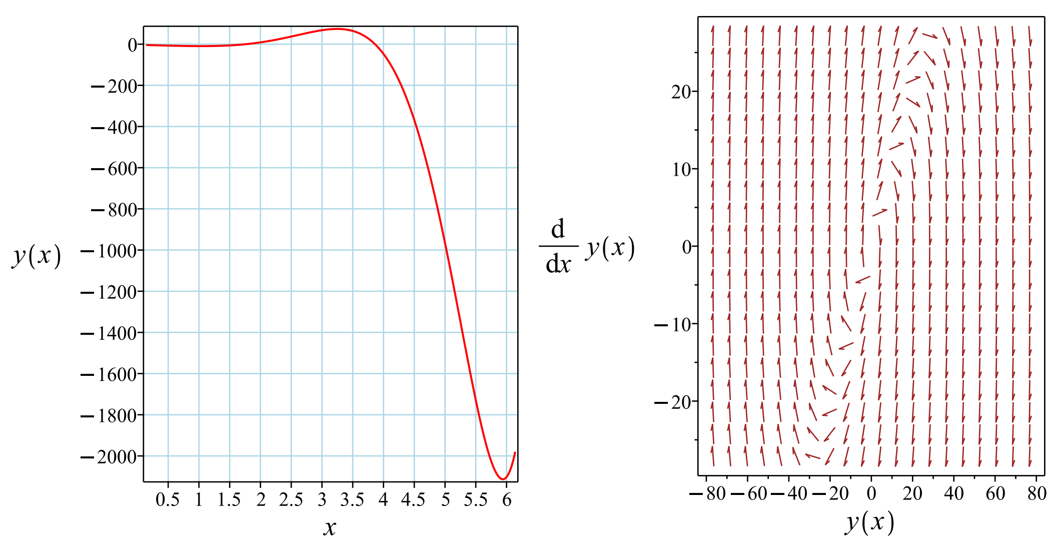
Which simplifies to

$$y = (2x - \pi - 1)\,\mathrm{e}^{x} \sin(x) - \mathrm{e}^{x} \cos(x)\,\pi$$

<u>Summary</u>

The solution(s) found are the following

$$y = (2x - \pi - 1)\,\mathrm{e}^{x} \sin(x) - \mathrm{e}^{x} \cos(x)\,\pi \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (2x - \pi - 1)\,\mathrm{e}^{x} \sin(x) - \mathrm{e}^{x} \cos(x)\,\pi$$

Verified OK.

18.14.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' - 2y' + 2y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= -2 \tag{3} \\ C &= 2 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= -1 \\ t &= 1 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 588: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}\cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}\cos(x)) + c_2(\mathrm{e}^{x}\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x \cos(x) + e^x \sin(x) c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 e^x \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(x), e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(x), e^x \sin(x)\}$$

Since $e^x \cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^x \cos(x), x e^x \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x \cos(x) + A_2 x e^x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^x \sin(x) + 2A_2 e^x \cos(x) = 4 e^x \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x e^x \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x \cos(x) + e^x \sin(x) c_2) + (2x e^x \sin(x)) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + 2x\,e^{x} \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + 2x\,e^{x} \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \pi\,e^{\pi}$ and $x = \pi$ in the above gives

$$\pi\,e^{\pi} = -c_1 e^{\pi} \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-c_1 \sin(x) + c_2 \cos(x)) + 2\,e^{x} \sin(x) + 2x\,e^{x} \sin(x) + 2x\,e^{x} \cos(x)$$

substituting $y' = e^{\pi}$ and $x = \pi$ in the above gives

$$e^{\pi} = (-c_1 - c_2 - 2\pi)\,e^{\pi} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\pi \\ c_2 &= -\pi - 1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -e^{x} \cos(x)\,\pi - e^{x} \sin(x)\,\pi + 2x\,e^{x} \sin(x) - e^{x} \sin(x)$$

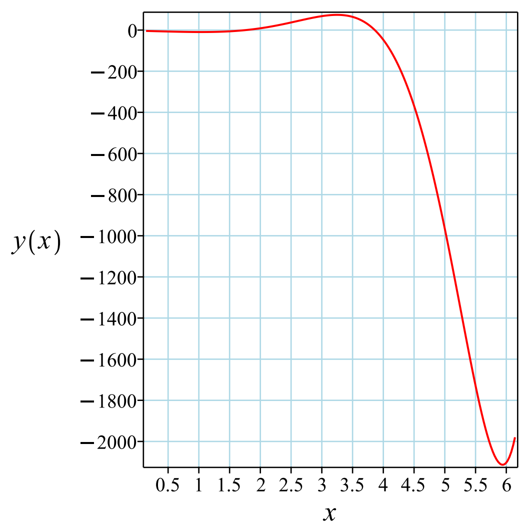
Which simplifies to

$$y = (2x - \pi - 1)\,e^{x} \sin(x) - e^{x} \cos(x)\,\pi$$

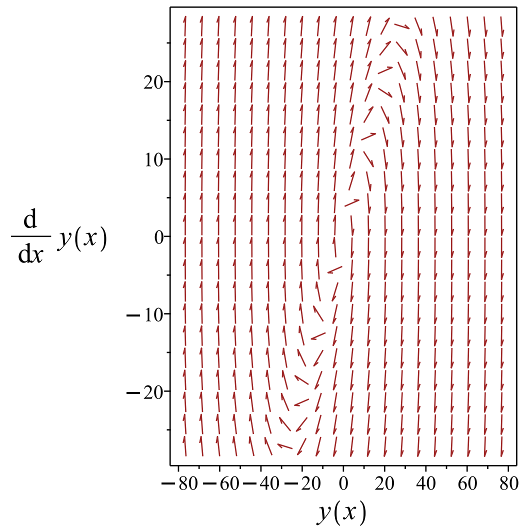
Summary

The solution(s) found are the following

$$y = (2x - \pi - 1)\,e^{x} \sin(x) - e^{x} \cos(x)\,\pi \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (2x - \pi - 1)\, e^{x} \sin(x) - e^{x} \cos(x)\, \pi$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=4*exp(x)*cos(x),y(Pi) = Pi*exp(Pi), D(y)(Pi) =
```

$$y(x) = e^{x}(2x - \pi - 1) \sin(x) - e^{x} \cos(x)\, \pi$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 24

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==4*Exp[x]*Cos[x],{y[Pi]==Pi*Exp[Pi],y'[Pi]==Exp[Pi]}},y[x],x,I
```

$$y(x) \to -e^x((-2x + \pi + 1)\sin(x) + \pi\cos(x))$$

18.15 problem 604

Internal problem ID [15373]
Internal file name [`OUTPUT/15373_Wednesday_May_08_2024_03_57_19_PM_36055126/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 604.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - y' = -2x}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1, y''(0) = 2]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y' = 0$$

The characteristic equation is

$$\lambda^3 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + e^x c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= 1 \\ y_3 &= e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - y' = -2x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2xA_2 - A_1 = -2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^x c_3\right) + \left(x^2\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^x c_3 + x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} + \mathrm{e}^x c_3 + 2x$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 + c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = c_1 \mathrm{e}^{-x} + \mathrm{e}^x c_3 + 2$$

substituting $y'' = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_3 + 2 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_2 = 0$$
$$c_3 = \frac{1}{2}$$

Substituting these values back in above solution results in

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2} + x^2$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2} + x^2 \tag{1}$$

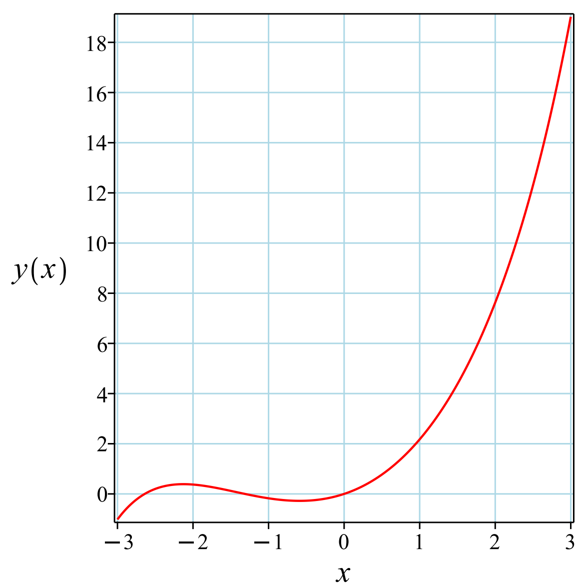


Figure 763: Solution plot

Verification of solutions

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2} + x^2$$

Verified OK.

18.15.1 Maple step by step solution

Let's solve

$$\left[y''' - y' = -2x, y(0) = 0, y'\Big|_{\{x=0\}} = 1, y''\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -2x + y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -2x + y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -2x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -2x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & 1 & e^x \\ -e^{-x} & 0 & e^x \\ e^{-x} & 0 & e^x \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & 1 & e^x \\ -e^{-x} & 0 & e^x \\ e^{-x} & 0 & e^x \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-x}}{2} + \frac{e^x}{2} & \frac{e^{-x}}{2} - 1 + \frac{e^x}{2} \\ 0 & \frac{e^x}{2} + \frac{e^{-x}}{2} & -\frac{e^{-x}}{2} + \frac{e^x}{2} \\ 0 & -\frac{e^{-x}}{2} + \frac{e^x}{2} & \frac{e^x}{2} + \frac{e^{-x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \begin{bmatrix} x^2 - \mathrm{e}^{-x} + 2 - \mathrm{e}^{x} \\ 2x + \mathrm{e}^{-x} - \mathrm{e}^{x} \\ -\mathrm{e}^{-x} + 2 - \mathrm{e}^{x} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} x^2 - \mathrm{e}^{-x} + 2 - \mathrm{e}^{x} \\ 2x + \mathrm{e}^{-x} - \mathrm{e}^{x} \\ -\mathrm{e}^{-x} + 2 - \mathrm{e}^{x} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = (c_1 - 1)\,\mathrm{e}^{-x} + (c_3 - 1)\,\mathrm{e}^{x} + x^2 + c_2 + 2$

- Use the initial condition $y(0) = 0$

  $0 = c_1 + c_2 + c_3$

- Calculate the 1st derivative of the solution

  $y' = -(c_1 - 1)\,e^{-x} + (c_3 - 1)\,e^{x} + 2x$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = -c_1 + c_3$

- Calculate the 2nd derivative of the solution

  $y'' = (c_1 - 1)\,e^{-x} + 2 + (c_3 - 1)\,e^{x}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 2$

  $2 = c_1 + c_3$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{1}{2}, c_2 = -2, c_3 = \frac{3}{2}\right\}$

- Solution to the IVP

  $y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2} + x^2$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _b(_a)-2*_a, _b(_a)`   *** Su
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$3)-diff(y(x),x)=-2*x,y(0) = 0, D(y)(0) = 1, (D@@2)(y)(0) = 2],y(x), sing
```

$$y(x) = -\frac{e^{-x}}{2} + \frac{e^{x}}{2} + x^2$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 25

```
DSolve[{y'''[x]-y'[x]==-2*x,{y[0]==0,y'[0]==1,y''[0]==2}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to x^2 - \frac{e^{-x}}{2} + \frac{e^{x}}{2}$$

18.16 problem 605

Internal problem ID [15374]
Internal file name [`OUTPUT/15374_Wednesday_May_08_2024_03_57_20_PM_99877420/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 605.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' - y = 8\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = -1, y'(0) = 0, y''(0) = 1, y'''(0) = 0]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - y = 8\,e^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{ix}, e^{-x}, e^{-ix}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{x} = 8\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4\right) + \left(2x\,e^x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4 + 2x\,e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + c_2 + c_3 + c_4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + c_2 e^x + i e^{ix} c_3 - i e^{-ix} c_4 + 2x\,e^x + 2\,e^x$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_3 i - c_4 i - c_1 + c_2 + 2 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = c_1 e^{-x} + c_2 e^x - e^{ix} c_3 - e^{-ix} c_4 + 2x\,e^x + 4\,e^x$$

substituting $y'' = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 - c_3 - c_4 + 4 \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = -c_1 e^{-x} + c_2 e^x - i e^{ix} c_3 + i e^{-ix} c_4 + 2x\,e^x + 6\,e^x$$

substituting $y''' = 0$ and $x = 0$ in the above gives

$$0 = -c_3 i + c_4 i - c_1 + c_2 + 6 \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= -3 \\ c_3 &= \frac{1}{2} - i \\ c_4 &= \frac{1}{2} + i \end{aligned}$$

Substituting these values back in above solution results in

$$y = 2x\,e^{x} - 3\,e^{x} + e^{-x} + \cos(x) + 2\sin(x)$$

Which simplifies to

$$y = e^{-x} + (2x - 3)\,e^{x} + \cos(x) + 2\sin(x)$$

Summary

The solution(s) found are the following

$$y = e^{-x} + (2x - 3)\,e^{x} + \cos(x) + 2\sin(x) \tag{1}$$

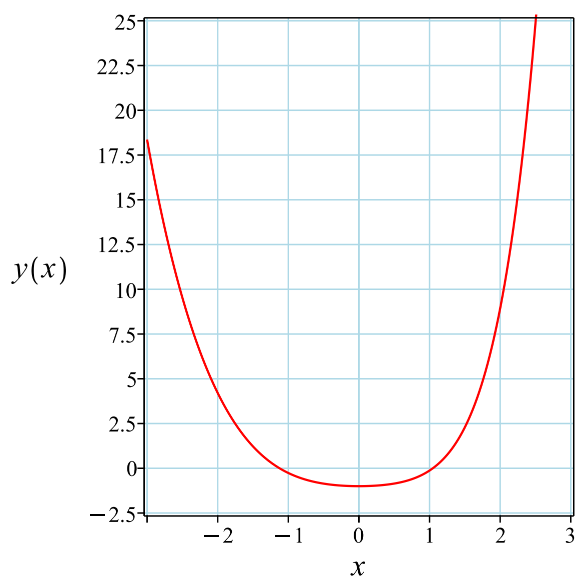


Figure 764: Solution plot

Verification of solutions

$$y = e^{-x} + (2x - 3)\, e^{x} + \cos(x) + 2\sin(x)$$

Verified OK.

18.16.1 Maple step by step solution

Let's solve

$$\left[y'''' - y = 8\, e^{x}, y(0) = -1, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 1, y'''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 8\, e^{x} + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 8\, e^{x} + y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8\,\mathrm{e}^{x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8\,\mathrm{e}^{x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \mathrm{e}^{x} & -\sin(x) & -\cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -\cos(x) & \sin(x) \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & \sin(x) & \cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} \\ \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} e^{-x} + (2x-3)\, e^{x} + 2\cos(x) + 2\sin(x) \\ -e^{-x} + (2x-1)\, e^{x} + 2\cos(x) - 2\sin(x) \\ 2x\, e^{x} + e^{x} + e^{-x} - 2\cos(x) - 2\sin(x) \\ -e^{-x} + (2x+3)\, e^{x} - 2\cos(x) + 2\sin(x) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \begin{bmatrix} e^{-x} + (2x-3)\, e^{x} + 2\cos(x) + 2\sin(x) \\ -e^{-x} + (2x-1)\, e^{x} + 2\cos(x) - 2\sin(x) \\ 2x\, e^{x} + e^{x} + e^{-x} - 2\cos(x) - 2\sin(x) \\ -e^{-x} + (2x+3)\, e^{x} - 2\cos(x) + 2\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = (1-c_1)\, e^{-x} + (c_2 + 2x - 3)\, e^{x} + (-c_4 + 2)\cos(x) - \sin(x)(c_3 - 2)$

- Use the initial condition $y(0) = -1$

$-1 = -c_1 + c_2 - c_4$

- Calculate the 1st derivative of the solution

$y' = -(1-c_1)\, e^{-x} + 2\, e^{x} + (c_2 + 2x - 3)\, e^{x} - (-c_4 + 2)\sin(x) - (c_3 - 2)\cos(x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$0 = c_1 + c_2 - c_3$

- Calculate the 2nd derivative of the solution

$y'' = (1-c_1)\, e^{-x} + 4\, e^{x} + (c_2 + 2x - 3)\, e^{x} - (-c_4 + 2)\cos(x) + \sin(x)(c_3 - 2)$

- Use the initial condition $y''\Big|_{\{x=0\}} = 1$

$1 = -c_1 + c_2 + c_4$

- Calculate the 3rd derivative of the solution

$y''' = -(1-c_1)\, e^{-x} + 6\, e^{x} + (c_2 + 2x - 3)\, e^{x} + (-c_4 + 2)\sin(x) + (c_3 - 2)\cos(x)$

- Use the initial condition $y'''\Big|_{\{x=0\}} = 0$

$0 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $$\{c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 1\}$$

- Solution to the IVP

  $$y = e^{-x} + (2x - 3) e^x + \cos(x) + 2\sin(x)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$4)-y(x)=8*exp(x),y(0) = -1, D(y)(0) = 0, (D@@2)(y)(0) = 1, (D@@3)(y)(0)
```

$$y(x) = e^{-x} + (2x - 3) e^x + \cos(x) + 2\sin(x)$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 28

```
DSolve[{y''''[x]-y[x]==8*Exp[x],{y[0]==-1,y'[0]==0,y''[0]==1,y'''[0]==0}},y[x],x,IncludeSing
```

$$y(x) \to 2e^x x + e^{-x} - 3e^x + 2\sin(x) + \cos(x)$$

18.17 problem 606

Internal problem ID [15375]
Internal file name [`OUTPUT/15375_Wednesday_May_08_2024_03_57_23_PM_34810919/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 606.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - y = 2x}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 2]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y = 0$$

The characteristic equation is

$$\lambda^3 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = e^{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' - y = 2x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x, e^{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x}, e^{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2x - A_1 = 2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^x c_1 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3\right) + (-2x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^x c_1 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 - 2x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^x c_1 + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3 - 2$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -2 + \frac{i(-c_2 + c_3)\sqrt{3}}{2} + c_1 - \frac{c_2}{2} - \frac{c_3}{2} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = e^{x} c_1 + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2 e^{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_2 + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 e^{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_3$$

substituting $y'' = 2$ and $x = 0$ in the above gives

$$2 = \frac{i(c_2 - c_3)\sqrt{3}}{2} + c_1 - \frac{c_2}{2} - \frac{c_3}{2} \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{4}{3} \\ c_2 &= -\frac{2}{3} \\ c_3 &= -\frac{2}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{4\,e^{x}}{3} - \frac{2\,e^{-\frac{(1+i\sqrt{3})x}{2}}}{3} - \frac{2\,e^{\frac{(-1+i\sqrt{3})x}{2}}}{3} - 2x$$

Summary

The solution(s) found are the following

$$y = \frac{4\,e^{x}}{3} - \frac{2\,e^{-\frac{(1+i\sqrt{3})x}{2}}}{3} - \frac{2\,e^{\frac{(-1+i\sqrt{3})x}{2}}}{3} - 2x \tag{1}$$

Verification of solutions

$$y = \frac{4\,e^{x}}{3} - \frac{2\,e^{-\frac{(1+i\sqrt{3})x}{2}}}{3} - \frac{2\,e^{\frac{(-1+i\sqrt{3})x}{2}}}{3} - 2x$$

Verified OK.

18.17.1 Maple step by step solution

Let's solve

$$\left[y''' - y = 2x, y(0) = 0, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 3

y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2x + y_1(x)$

    Convert linear ODE into a system of first order ODEs

    $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2x + y_1(x)]$
- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{x\sqrt{3}}{2}\right) - \mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)-\mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)-\mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) - \mathrm{I}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right], \vec{y}_3(x) = e^{-\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ \frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2} \\ -\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{x\sqrt{3}}{2}\right)}{2}\right) \\ e^{x} & e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{x}}{3} + \frac{2e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} + \frac{2e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} + \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} - \frac{e^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} & \frac{e^{x}}{3} + \frac{2e}{} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{2\,e^{x}}{3} - 2x + \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \frac{2\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{2\,e^{x}}{3} - 2 + \frac{4\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ -\frac{2\,e^{-\frac{x}{2}}\left(-e^{\frac{3x}{2}} + \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3} + \cos\left(\frac{x\sqrt{3}}{2}\right)\right)}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{2\,e^{x}}{3} - 2x + \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{3} - \frac{2\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ \frac{2\,e^{x}}{3} - 2 + \frac{4\,e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{3} \\ -\frac{2\,e^{-\frac{x}{2}}\left(-e^{\frac{3x}{2}} + \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3} + \cos\left(\frac{x\sqrt{3}}{2}\right)\right)}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{2} - \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-c_3\right)e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)}{2} + \frac{(6c_1+4)e^{x}}{6} - 2x$$

- Use the initial condition $y(0) = 0$

$$0 = -\frac{c_3\sqrt{3}}{2} - \frac{c_2}{2} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{4} + \frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{4} + \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-c_3\right)e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)}{4} - \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-\right.}{}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$$0 = \frac{c_3\sqrt{3}}{4} + \frac{c_2}{4} - 1 - \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-c_3\right)\sqrt{3}}{4} + c_1$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)e^{-\frac{x}{2}} \cos\left(\frac{x\sqrt{3}}{2}\right)}{4} - \frac{\left(c_3\sqrt{3}+c_2+\frac{4}{3}\right)e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)\sqrt{3}}{4} + \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-c_3\right)e^{-\frac{x}{2}} \sin\left(\frac{x\sqrt{3}}{2}\right)}{4} + \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-\right.}{}$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 2$

$$2 = \frac{c_3\sqrt{3}}{4} + \frac{c_2}{4} + 1 + \frac{\left(\left(c_2-\frac{4}{3}\right)\sqrt{3}-c_3\right)\sqrt{3}}{4} + c_1$$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{2}{3}, c_2 = \frac{4}{3}, c_3 = 0\right\}$$

- Solution to the IVP

$$y=-\frac{4\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3}-2x+\frac{4\,\mathrm{e}^{x}}{3}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$3)-y(x)=2*x,y(0) = 0, D(y)(0) = 0, (D@@2)(y)(0) = 2],y(x), singsol=all)
```

$$y(x)=-2x+\frac{4\,\mathrm{e}^{x}}{3}-\frac{4\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{x\sqrt{3}}{2}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 38

```
DSolve[{y'''[x]-y[x]==2*x,{y[0]==0,y'[0]==0,y''[0]==2}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x)\to\frac{1}{3}\left(-6x+4e^{x}-4e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)\right)$$

18.18 problem 607

Internal problem ID [15376]
Internal file name [`OUTPUT/15376_Wednesday_May_08_2024_03_57_26_PM_14139834/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 607.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' - y = 8\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 2, y''(0) = 4, y'''(0) = 6]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - y = 8\,e^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{ix}, e^{-x}, e^{-ix}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{x} = 8\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4\right) + (2x\,e^x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 e^x + e^{ix} c_3 + e^{-ix} c_4 + 2x\,e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 + c_3 + c_4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + c_2 e^x + i e^{ix} c_3 - i e^{-ix} c_4 + 2x\,e^x + 2\,e^x$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = c_3 i - c_4 i - c_1 + c_2 + 2 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = c_1 e^{-x} + c_2 e^x - e^{ix} c_3 - e^{-ix} c_4 + 2x\,e^x + 4\,e^x$$

substituting $y'' = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 - c_3 - c_4 + 4 \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = -c_1 e^{-x} + c_2 e^x - i e^{ix} c_3 + i e^{-ix} c_4 + 2x\,e^x + 6\,e^x$$

substituting $y''' = 6$ and $x = 0$ in the above gives

$$6 = -c_3 i + c_4 i - c_1 + c_2 + 6 \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 2x\,e^x$$

Summary

The solution(s) found are the following

$$y = 2x\,e^x \tag{1}$$

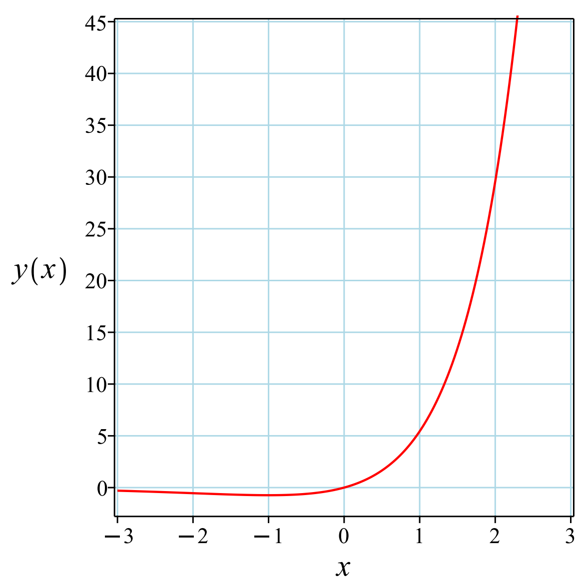


Figure 765: Solution plot

Verification of solutions

$$y = 2x\,e^x$$

Verified OK.

18.18.1 Maple step by step solution

Let's solve

$$\left[y'''' - y = 8\,\mathrm{e}^{x}, y(0) = 0, y'\Big|_{\{x=0\}} = 2, y''\Big|_{\{x=0\}} = 4, y'''\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

☐ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

  $y_1(x) = y$

  ○ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ○ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ○ Define new variable $y_4(x)$

  $y_4(x) = y'''$

  ○ Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 8\,\mathrm{e}^{x} + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_4(x) = y_3'(x)\,, y_4'(x) = 8\,\mathrm{e}^{x} + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}\,'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8\,\mathrm{e}^{x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 8\,\mathrm{e}^{x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \left[\begin{array}{c}1\\1\\1\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \left[\begin{array}{c}-I\\-1\\I\\1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \left[\begin{array}{c}-I\\-1\\I\\1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c}-I\\-1\\I\\1\end{array}\right]$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \mathrm{e}^{x} & -\sin(x) & -\cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -\cos(x) & \sin(x) \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & \sin(x) & \cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\mathrm{e}^{-x} & \mathrm{e}^{x} & -\sin(x) & -\cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -\cos(x) & \sin(x) \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & \sin(x) & \cos(x) \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} \\ \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\cos(x)}{2} & -\frac{e^{-x}}{4}+\frac{e^{x}}{4}-\frac{\sin(x)}{2} & \frac{e^{-x}}{4}+\frac{e^{x}}{4}+\frac{\cos(x)}{2} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

    $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$
  - Take the derivative of the particular solution

    $\vec{y}_p{}'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}\,'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}\,'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}\,'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
  - Cancel like terms

    $\Phi(x)\cdot\vec{v}\,'(x)=\vec{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}\,'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$
  - Integrate to solve for $\vec{v}(x)$

    $\vec{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
  - Plug $\vec{v}(x)$ into the equation for the particular solution

    $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\begin{bmatrix} e^{-x}+(2x-3)\,e^{x}+2\cos(x)+2\sin(x) \\ -e^{-x}+(2x-1)\,e^{x}+2\cos(x)-2\sin(x) \\ 2x\,e^{x}+e^{x}+e^{-x}-2\cos(x)-2\sin(x) \\ -e^{-x}+(2x+3)\,e^{x}-2\cos(x)+2\sin(x) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \begin{bmatrix} e^{-x} + (2x-3)\,e^{x} + 2\cos(x) + 2\sin(x) \\ -e^{-x} + (2x-1)\,e^{x} + 2\cos(x) - 2\sin(x) \\ 2x\,e^{x} + e^{x} + e^{-x} - 2\cos(x) - 2\sin(x) \\ -e^{-x} + (2x+3)\,e^{x} - 2\cos(x) + 2\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = (1-c_1)\,e^{-x} + (c_2 + 2x - 3)\,e^{x} + (-c_4 + 2)\cos(x) - \sin(x)(c_3 - 2)$

- Use the initial condition $y(0) = 0$

  $0 = -c_1 + c_2 - c_4$

- Calculate the 1st derivative of the solution

  $y' = -(1-c_1)\,e^{-x} + 2\,e^{x} + (c_2 + 2x - 3)\,e^{x} - (-c_4 + 2)\sin(x) - (c_3 - 2)\cos(x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 2$

  $2 = c_1 + c_2 - c_3$

- Calculate the 2nd derivative of the solution

  $y'' = (1-c_1)\,e^{-x} + 4\,e^{x} + (c_2 + 2x - 3)\,e^{x} - (-c_4 + 2)\cos(x) + \sin(x)(c_3 - 2)$

- Use the initial condition $y''\Big|_{\{x=0\}} = 4$

  $4 = -c_1 + c_2 + c_4$

- Calculate the 3rd derivative of the solution

  $y''' = -(1-c_1)\,e^{-x} + 6\,e^{x} + (c_2 + 2x - 3)\,e^{x} + (-c_4 + 2)\sin(x) + (c_3 - 2)\cos(x)$

- Use the initial condition $y'''\Big|_{\{x=0\}} = 6$

  $6 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = 3, c_3 = 2, c_4 = 2\}$

- Solution to the IVP

  $y = 2x\,e^{x}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x$4)-y(x)=8*exp(x),y(0) = 0, D(y)(0) = 2, (D@@2)(y)(0) = 4, (D@@3)(y)(0) =
```

$$y(x) = 2\,\mathrm{e}^{x} x$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 11

```
DSolve[{y''''[x]-y[x]==8*Exp[x],{y[0]==0,y'[0]==2,y''[0]==4,y'''[0]==6}},y[x],x,IncludeSingu
```

$$y(x) \to 2e^x x$$

18.19 problem 608

Internal problem ID [15377]
Internal file name [`OUTPUT/15377_Wednesday_May_08_2024_03_57_27_PM_92950036/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 608.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 5y = \sin(x)}$$

18.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 5, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=-4, C=5$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 5\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=-4, C=5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(5)}\\ &= 2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + i\\ \lambda_2 &= 2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 + i\\ \lambda_2 &= 2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(x), e^{2x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\cos(x) + 4A_2\sin(x) + 4A_1\sin(x) - 4A_2\cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}(c_1\cos(x) + c_2\sin(x))\right) + \left(\frac{\cos(x)}{8} + \frac{\sin(x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1\cos(x) + c_2\sin(x)) + \frac{\cos(x)}{8} + \frac{\sin(x)}{8} \tag{1}$$

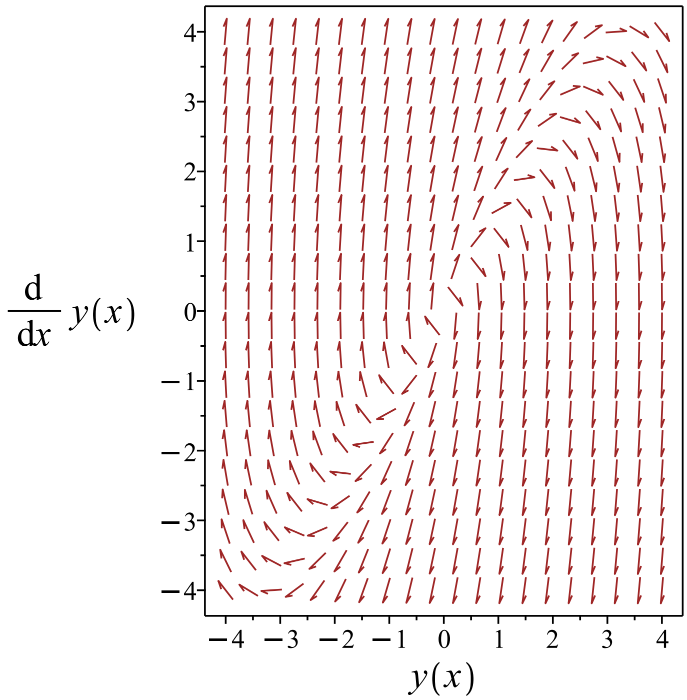


Figure 766: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$$

Verified OK.

18.19.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 593: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx}\\ &= z_1 e^{2x}\\ &= z_1\left(\mathrm{e}^{2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x} \cos\left(x\right)\right) + c_2 \left(\mathrm{e}^{2x} \cos\left(x\right) \left(\tan\left(x\right)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos\left(x\right) \mathrm{e}^{2x} c_1 + \mathrm{e}^{2x} \sin\left(x\right) c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin\left(x\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos\left(x\right), \sin\left(x\right)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{2x} \cos\left(x\right), \mathrm{e}^{2x} \sin\left(x\right)\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos\left(x\right) + A_2 \sin\left(x\right)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_1 \cos\left(x\right) + 4A_2 \sin\left(x\right) + 4A_1 \sin\left(x\right) - 4A_2 \cos\left(x\right) = \sin\left(x\right)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x)\,\mathrm{e}^{2x} c_1 + \mathrm{e}^{2x}\sin(x)\, c_2\right) + \left(\frac{\cos(x)}{8} + \frac{\sin(x)}{8}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{\cos(x)}{8} + \frac{\sin(x)}{8} \tag{1}$$

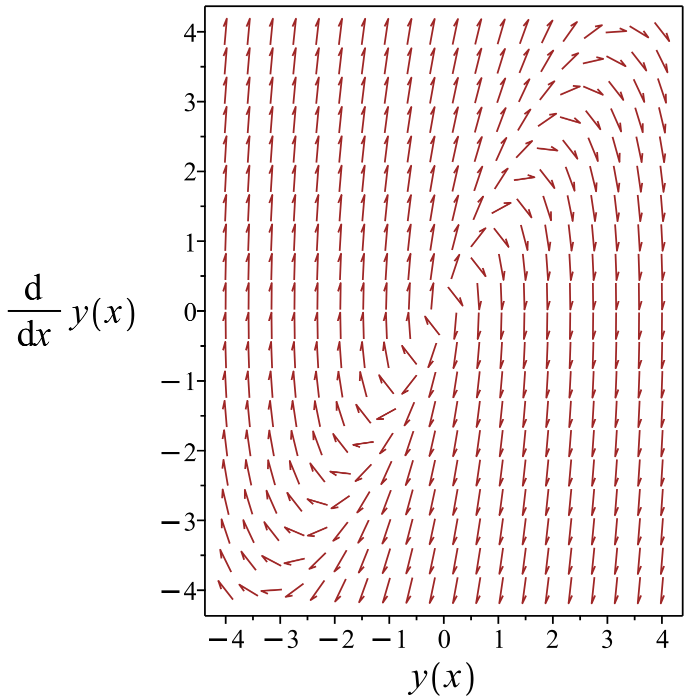


Figure 767: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$$

Verified OK.

18.19.3 Maple step by step solution

Let's solve

$y'' - 4y' + 5y = \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (2-\mathrm{I}, 2+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}\sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\,\mathrm{e}^{2x} c_1 + \mathrm{e}^{2x}\sin(x)\, c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x}\cos(x) & \mathrm{e}^{2x}\sin(x) \\ 2\,\mathrm{e}^{2x}\cos(x) - \mathrm{e}^{2x}\sin(x) & 2\,\mathrm{e}^{2x}\sin(x) + \mathrm{e}^{2x}\cos(x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^{2x}\left(\sin(x)\left(\int \mathrm{e}^{-2x}\sin(2x)dx\right) - 2\cos(x)\left(\int \sin(x)^2 \mathrm{e}^{-2x}dx\right)\right)}{2}$

- ○ Compute integrals

  $y_p(x) = \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{2x}\sin(x)\, c_2 + \cos(x)\,\mathrm{e}^{2x} c_1 + \frac{\sin(x)}{8} + \frac{\cos(x)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=sin(x),y(x), singsol=all)
```

$$y(x) = e^{2x} \sin(x) c_2 + e^{2x} \cos(x) c_1 + \frac{\cos(x)}{8} + \frac{\sin(x)}{8}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 36

```
DSolve[y''[x]-4*y'[x]+5*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(\sin(x) + \cos(x) + 8c_2 e^{2x} \cos(x) + 8c_1 e^{2x} \sin(x)\right)$$

18.20 problem 609

Internal problem ID [15378]
Internal file name [`OUTPUT/15378_Wednesday_May_08_2024_03_57_29_PM_66888193/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 609.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 4\cos(2x) + \sin(2x)}$$

18.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 5, f(x) = 4\cos(2x) + \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(2x) + \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(2x), e^{-x}\sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2x) + A_2\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(2x) + A_2\sin(2x) - 4A_1\sin(2x) + 4A_2\cos(2x) = 4\cos(2x) + \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1\cos(2x) + c_2\sin(2x))\right) + (\sin(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1\cos(2x) + c_2\sin(2x)) + \sin(2x) \tag{1}$$

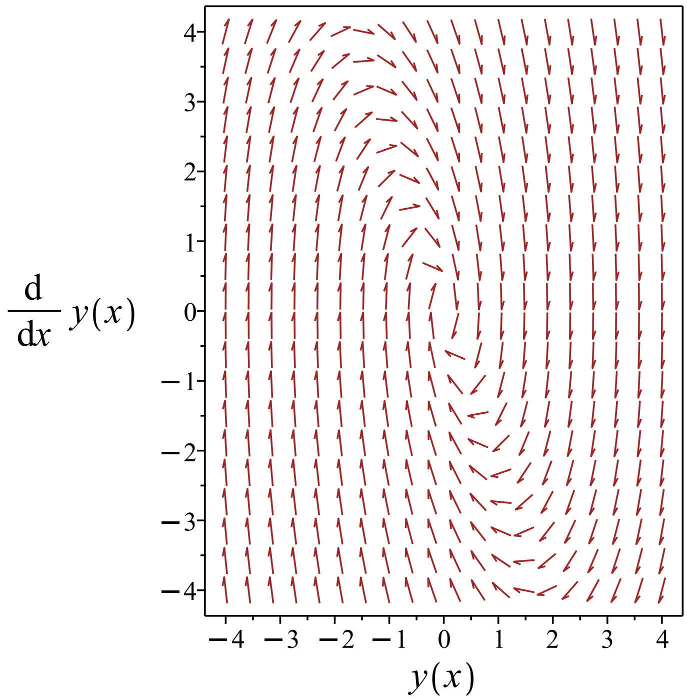


Figure 768: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) + \sin(2x)$$

Verified OK.

18.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 595: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x} \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-x} \cos(2x)\, c_1 + \frac{\mathrm{e}^{-x} \sin(2x)\, c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
4\cos(2x) + \sin(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-x} \cos(2x), \frac{\mathrm{e}^{-x} \sin(2x)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(2x) + A_2 \sin(2x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(2x) + A_2 \sin(2x) - 4A_1 \sin(2x) + 4A_2 \cos(2x) = 4\cos(2x) + \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x} \cos(2x) c_1 + \frac{e^{-x} \sin(2x) c_2}{2}\right) + (\sin(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x} \cos(2x) c_1 + \frac{e^{-x} \sin(2x) c_2}{2} + \sin(2x) \tag{1}$$

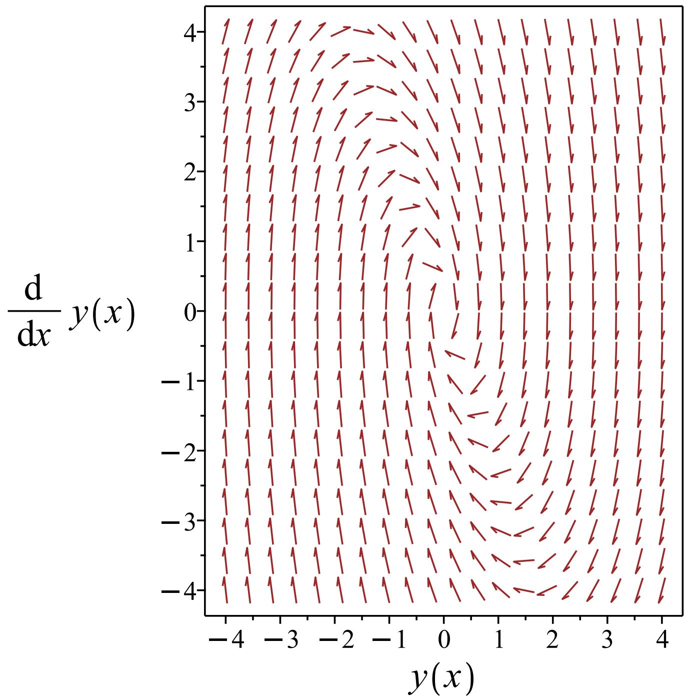


Figure 769: Slope field plot

Verification of solutions

$$y = e^{-x} \cos(2x) c_1 + \frac{e^{-x} \sin(2x) c_2}{2} + \sin(2x)$$

Verified OK.

18.20.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = 4\cos(2x) + \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-1-2\mathrm{I}, -1+2\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}\cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}\sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x}\cos(2x)\, c_1 + \mathrm{e}^{-x}\sin(2x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 4\cos(2x) + \sin(2x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x}\cos(2x) & \mathrm{e}^{-x}\sin(2x) \\ -\mathrm{e}^{-x}\cos(2x) - 2\,\mathrm{e}^{-x}\sin(2x) & -\mathrm{e}^{-x}\sin(2x) + 2\,\mathrm{e}^{-x}\cos(2x) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^{-x}\left(\cos(2x)\left(\int (\cos(4x)-1-4\sin(4x))\mathrm{e}^{x} dx\right)+\sin(2x)\left(\int (\sin(4x)+4\cos(4x)+4)\mathrm{e}^{x} dx\right)\right)}{4}$

- ◦ Compute integrals

  $y_p(x) = \sin(2x)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x}\sin(2x)\, c_2 + \mathrm{e}^{-x}\cos(2x)\, c_1 + \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=4*cos(2*x)+sin(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} \sin(2x) c_2 + \mathrm{e}^{-x} \cos(2x) c_1 + \sin(2x)$$

✓ Solution by Mathematica

Time used: 0.338 (sec). Leaf size: 30

```
DSolve[y''[x]+2*y'[x]+5*y[x]==4*Cos[2*x]+Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2 \cos(2x) + (e^x + c_1)\sin(2x))$$

18.21 problem 610

Internal problem ID [15379]
Internal file name [`OUTPUT/15379_Wednesday_May_08_2024_03_57_30_PM_31675540/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 610.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 1}$$

18.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = e^x c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{-x} c_2\right) + (-1) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^x c_1 + e^{-x} c_2 - 1 \tag{1}$$

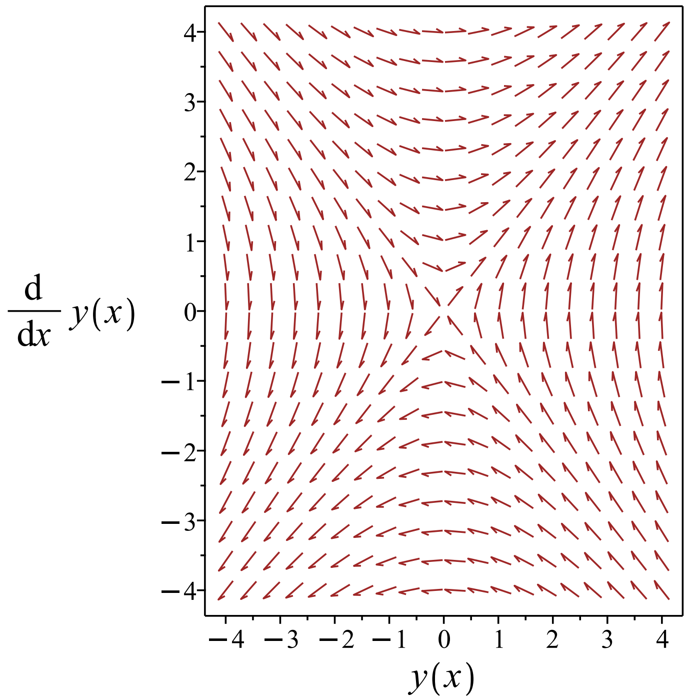


Figure 770: Slope field plot

Verification of solutions

$$y = e^{x}c_1 + e^{-x}c_2 - 1$$

Verified OK.

18.21.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' - y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy' - y')\,dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} - y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{y^2+2c_1+2y}}dy &= \int dx \\
\ln\left(y+1+\sqrt{y^2+2c_1+2y}\right) &= x+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
y+1+\sqrt{y^2+2c_1+2y} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$
y+1+\sqrt{y^2+2c_1+2y} = \mathrm{e}^{x} c_3
$$

Solving equation (2)

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{\sqrt{y^2+2c_1+2y}}dy &= \int dx \\
-\ln\left(y+1+\sqrt{y^2+2c_1+2y}\right) &= x+c_4
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{y+1+\sqrt{y^2+2c_1+2y}} = \mathrm{e}^{x+c_4}
$$

Which simplifies to

$$
\frac{1}{y+1+\sqrt{y^2+2c_1+2y}} = \mathrm{e}^{x} c_5
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(\mathrm{e}^{2x} c_3^2 - 2\,\mathrm{e}^{x} c_3 - 2c_1 + 1\right)\mathrm{e}^{-x}}{2c_3} \tag{1}
$$

$$
y = -\frac{\left(2c_1 \mathrm{e}^{2x} c_5^2 - \mathrm{e}^{2x} c_5^2 + 2\,\mathrm{e}^{x} c_5 - 1\right)\mathrm{e}^{-x}}{2c_5} \tag{2}
$$

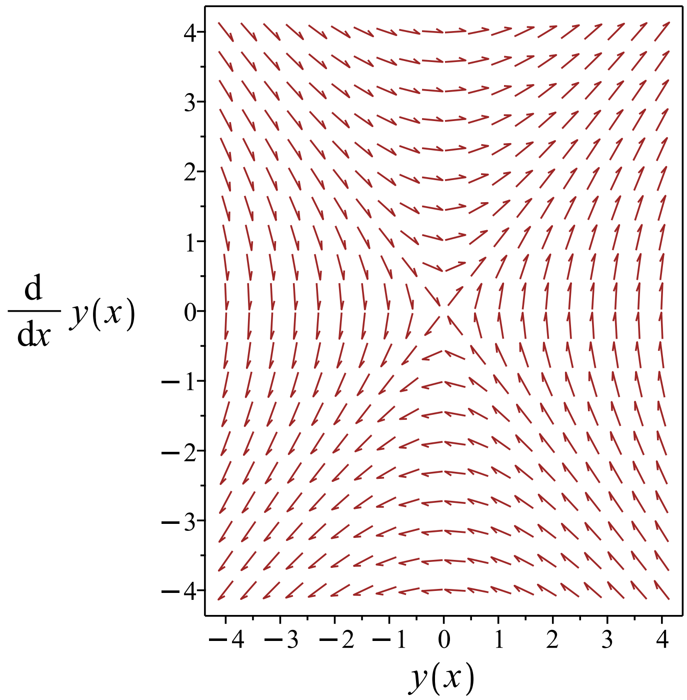


Figure 771: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2\,e^x c_3 - 2c_1 + 1)\,e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1 e^{2x} c_5^2 - e^{2x} c_5^2 + 2\,e^x c_5 - 1)\,e^{-x}}{2c_5}$$

Verified OK.

18.21.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 597: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{x}}{2}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) + (-1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} - 1 \tag{1}$$

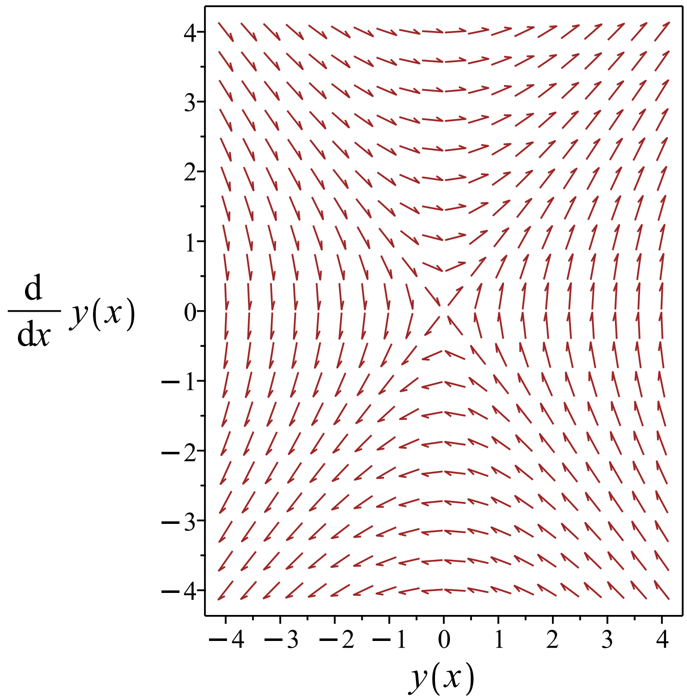


Figure 772: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - 1$$

Verified OK.

18.21.4 Maple step by step solution

Let's solve

$y'' - y = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{-x} dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-y(x)=1,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + c_1 e^{x} - 1$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 21

```
DSolve[y''[x]-y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_2 e^{-x} - 1$$

18.22 problem 611

Internal problem ID [15380]
Internal file name [`OUTPUT/15380_Wednesday_May_08_2024_03_57_31_PM_1715849/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 611.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = -2\cos(x)}$$

18.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = -2\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{x} c_1 + \mathrm{e}^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x} c_1 + \mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \cos(x) - 2A_2 \sin(x) = -2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{x} c_1 + e^{-x} c_2\right) + (\cos(x)) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{x} c_1 + e^{-x} c_2 + \cos(x) \tag{1}$$

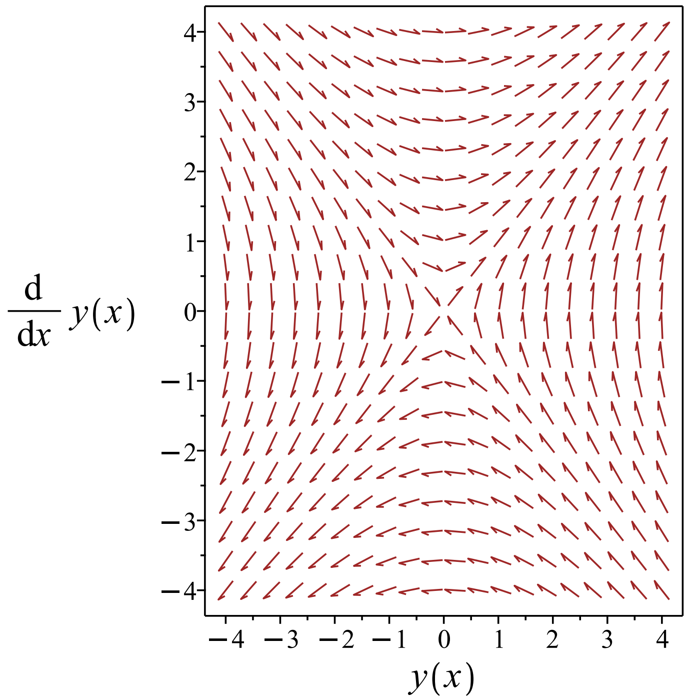


Figure 773: Slope field plot

Verification of solutions

$$y = e^x c_1 + e^{-x} c_2 + \cos(x)$$

Verified OK.

18.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 599: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
-2 \cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \cos(x) - 2A_2 \sin(x) = -2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}\right) + (\cos(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \cos(x) \tag{1}$$

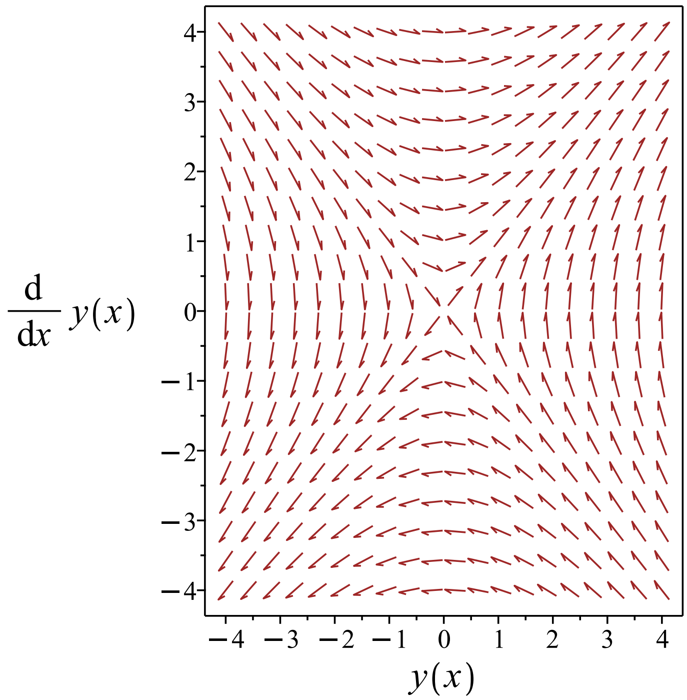


Figure 774: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \cos(x)$$

Verified OK.

18.22.3 Maple step by step solution

Let's solve

$y'' - y = -2\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -2\cos(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-x}\left(\int \mathrm{e}^{x}\cos(x)\,dx\right) - \mathrm{e}^{x}\left(\int \mathrm{e}^{-x}\cos(x)\,dx\right)$
  - Compute integrals

    $y_p(x) = \cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-y(x)=-2*cos(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + c_1 e^{x} + \cos(x)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 22

```
DSolve[y''[x]-y[x]==-2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \cos(x) + c_1 e^{x} + c_2 e^{-x}$$

18.23 problem 612

Internal problem ID [15381]
Internal file name [`OUTPUT/15381_Wednesday_May_08_2024_03_57_32_PM_4419388/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 612.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' - 2y' + y = 4\,\mathrm{e}^{-x}}$$

With initial conditions

$$[y(\infty) = 0]$$

18.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= -2\\ q(x) &= 1\\ F &= 4\,\mathrm{e}^{-x}\end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 4\,\mathrm{e}^{-x}$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = \infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.23.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 4\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x} c_1 + x\,\mathrm{e}^{x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^{-x} = 4\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^x c_1 + x\,\mathrm{e}^x c_2) + \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x (c_2 x + c_1) + \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^x (c_2 x + c_1) + \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \operatorname{signum}(c_2)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.23.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 4\,\mathrm{e}^{-2x} \\ \left(\mathrm{e}^{-x}y\right)'' &= 4\,\mathrm{e}^{-2x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = -2\,\mathrm{e}^{-2x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = c_1x + \mathrm{e}^{-2x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + \mathrm{e}^{-2x} + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \mathrm{signum}\,(c_1)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

18.23.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 601: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + x\,\mathrm{e}^x c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^{-x} = 4\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^{x} c_1 + x\,\mathrm{e}^{x} c_2) + (\mathrm{e}^{-x}) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_2 x + c_1) + \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_2 x + c_1) + e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \text{signum}(c_2)\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.23.5 Maple step by step solution

Let's solve

$[y'' - 2y' + y = 4\,e^{-x}, y(\infty) = 0]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{x} c_1 + x\,e^{x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\,\mathrm{e}^{-x}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & x\,\mathrm{e}^x \\ \mathrm{e}^x & x\,\mathrm{e}^x + \mathrm{e}^x \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -4\,\mathrm{e}^x\left(\int x\,\mathrm{e}^{-2x}dx - \left(\int \mathrm{e}^{-2x}dx\right)x\right)$$

- Compute integrals

  $$y_p(x) = \mathrm{e}^{-x}$$

- Substitute particular solution into general solution to ODE

  $$y = \mathrm{e}^x c_1 + x\,\mathrm{e}^x c_2 + \mathrm{e}^{-x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=4*exp(-x),y(infinity) = 0],y(x), singsol=all)
```

$$y(x) = -\operatorname{signum}(c_1 \mathrm{e}^x)\,\infty$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 10

```
DSolve[{y''[x]-2*y'[x]+y[x]==4*Exp[-x],{y[Infinity]==0}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{-x}$$

18.24 problem 613

Internal problem ID [15382]
Internal file name [`OUTPUT/15382_Wednesday_May_08_2024_03_57_33_PM_81781224/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 613.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + 4y' + 3y = 8\,\mathrm{e}^{x} + 9}$$

With initial conditions

$$[y(-\infty) = 3]$$

18.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 3 \\ F &= 8\,\mathrm{e}^{x} + 9 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = 8\,\mathrm{e}^{x} + 9$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = -\infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.24.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 3, f(x) = 8\,\mathrm{e}^{x} + 9$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(3)} \\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1 \\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{x} + 9$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_2 e^x + 3A_1 = 8\,e^x + 9$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3 + e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-3x}\right) + \left(3 + e^x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 e^{-3x} + 3 + e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = -\infty$ in the above gives

$$3 = \operatorname{signum}(c_2)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.24.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 603: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{\mathrm{e}^{-4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + \frac{\mathrm{e}^{-x} c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{x}+9$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\},\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{-x}}{2},\mathrm{e}^{-3x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_2\mathrm{e}^{x} + 3A_1 = 8\,\mathrm{e}^{x}+9$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3+\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-3x} + \frac{\mathrm{e}^{-x}c_2}{2}\right) + (3+\mathrm{e}^{x}) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + \frac{e^{-x} c_2}{2} + 3 + e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = -\infty$ in the above gives

$$3 = \operatorname{signum}(c_1)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.24.4 Maple step by step solution

Let's solve

$[y'' + 4y' + 3y = 8\,e^{x} + 9, y(-\infty) = 3]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + e^{-x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\,\mathrm{e}^x + 9\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{-x} \\ -3\,\mathrm{e}^{-3x} & -\mathrm{e}^{-x} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-4x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{\mathrm{e}^{-3x}\left(\int (8\,\mathrm{e}^x+9)\mathrm{e}^{3x}dx\right)}{2} + \frac{\mathrm{e}^{-x}\left(\int \left(8(\mathrm{e}^x)^2+9\,\mathrm{e}^x\right)dx\right)}{2}$$

- Compute integrals

  $$y_p(x) = 3 + \mathrm{e}^x$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\mathrm{e}^{-3x} + \mathrm{e}^{-x}c_2 + 3 + \mathrm{e}^x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+3*y(x)=8*exp(x)+9,y(-infinity) = 3],y(x), singsol=all)
```

$$y(x) = -\operatorname{signum}\left(c_1 e^{-x}\right)\infty$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+4*y'[x]+3*y[x]==8*Exp[x]+9,{y[-Infinity]==3}},y[x],x,IncludeSingularSolutions
```

{}

18.25 problem 614

Internal problem ID [15383]
Internal file name [`OUTPUT/15383_Wednesday_May_08_2024_03_57_34_PM_68214933/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 614.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{y'' - y' - 5y = 1}$$

With initial conditions

$$\left[y(\infty) = -\frac{1}{5}\right]$$

18.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -5 \\ F &= 1 \end{aligned}$$

Hence the ode is

$$y'' - y' - 5y = 1$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = \infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.25.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -5, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 5 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-5)} \\ &= \frac{1}{2} \pm \frac{\sqrt{21}}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{21}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{\sqrt{21}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{21}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{\sqrt{21}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}+\frac{\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{1}{2}-\frac{\sqrt{21}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{21}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{21}}{2}\right)x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{21}}{2}\right)x} + c_2 \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{21}}{2}\right)x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(\frac{1}{2}-\frac{\sqrt{21}}{2}\right)x}, e^{\left(\frac{1}{2}+\frac{\sqrt{21}}{2}\right)x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\left(\frac{1}{2}+\frac{\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{1}{2}-\frac{\sqrt{21}}{2}\right)x}\right) + \left(-\frac{1}{5}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{\frac{\left(1+\sqrt{21}\right)x}{2}} + c_2 e^{-\frac{\left(-1+\sqrt{21}\right)x}{2}} - \frac{1}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{\left(1+\sqrt{21}\right)x}{2}} + c_2 \mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}} - \frac{1}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -\frac{1}{5}$ and $x = \infty$ in the above gives

$$-\frac{1}{5} = \mathrm{signum}\left(c_1\right) \infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.25.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{21}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 21 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{21z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 605: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{21}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x\sqrt{21}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(e^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{\left(-1+\sqrt{21}\right)x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{21}\, e^{x\sqrt{21}}}{21}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}} \left(\frac{\sqrt{21}\,\mathrm{e}^{x\sqrt{21}}}{21}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}} + \frac{c_2 \sqrt{21}\,\mathrm{e}^{\frac{\left(1+\sqrt{21}\right)x}{2}}}{21}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\sqrt{21}\,\mathrm{e}^{\frac{\left(1+\sqrt{21}\right)x}{2}}}{21}, \mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}} + \frac{c_2\sqrt{21}\,\mathrm{e}^{\frac{\left(1+\sqrt{21}\right)x}{2}}}{21}\right) + \left(-\frac{1}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{\left(-1+\sqrt{21}\right)x}{2}} + \frac{c_2\sqrt{21}\,\mathrm{e}^{\frac{\left(1+\sqrt{21}\right)x}{2}}}{21} - \frac{1}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -\frac{1}{5}$ and $x = \infty$ in the above gives

$$-\frac{1}{5} = \operatorname{signum}(c_2)\,\infty \tag{1A}$$

Equations $\{1\text{A}\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.25.4 Maple step by step solution

Let's solve

$\left[y'' - y' - 5y = 1, y(\infty) = -\frac{1}{5}\right]$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r - 5 = 0$

- Use quadratic formula to solve for r

$r = \frac{1 \pm \left(\sqrt{21}\right)}{2}$

- Roots of the characteristic polynomial

$r = \left(\frac{1}{2} - \frac{\sqrt{21}}{2}, \frac{1}{2} + \frac{\sqrt{21}}{2}\right)$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{\left(\frac{1}{2} - \frac{\sqrt{21}}{2}\right)x}$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{\left(\frac{1}{2} + \frac{\sqrt{21}}{2}\right)x}$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 e^{\left(\frac{1}{2} - \frac{\sqrt{21}}{2}\right)x} + c_2 e^{\left(\frac{1}{2} + \frac{\sqrt{21}}{2}\right)x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\left(\frac{1}{2} - \frac{\sqrt{21}}{2}\right)x} & e^{\left(\frac{1}{2} + \frac{\sqrt{21}}{2}\right)x} \\ \left(\frac{1}{2} - \frac{\sqrt{21}}{2}\right) e^{\left(\frac{1}{2} - \frac{\sqrt{21}}{2}\right)x} & \left(\frac{1}{2} + \frac{\sqrt{21}}{2}\right) e^{\left(\frac{1}{2} + \frac{\sqrt{21}}{2}\right)x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \sqrt{21}\, e^x$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\sqrt{21}\left(-\mathrm{e}^{-\frac{(-1+\sqrt{21})x}{2}}\left(\int \mathrm{e}^{\frac{(-1+\sqrt{21})x}{2}}dx\right)+\mathrm{e}^{\frac{(1+\sqrt{21})x}{2}}\left(\int \mathrm{e}^{-\frac{(1+\sqrt{21})x}{2}}dx\right)\right)}{21}$$

- Compute integrals

$$y_p(x) = -\frac{1}{5}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{21}}{2}\right)x} + c_2\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{21}}{2}\right)x} - \frac{1}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-5*y(x)=1,y(infinity) = -1/5],y(x), singsol=all)
```

$$y(x) = -\operatorname{signum}\left(c_2\mathrm{e}^{-\frac{(-1+\sqrt{21})x}{2}}\right)\infty$$

✓ Solution by Mathematica

Time used: 0.559 (sec). Leaf size: 26

```
DSolve[{y''[x]-y'[x]-5*y[x]==1,{y[Infinity]==-1/5}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{5} + c_1e^{-\frac{1}{2}(\sqrt{21}-1)x}$$

18.26 problem 615

Internal problem ID [15384]
Internal file name [`OUTPUT/15384_Wednesday_May_08_2024_03_57_36_PM_29793907/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 615.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + 4y' + 4y = 2\,\mathrm{e}^{x}(\sin(x) + 7\cos(x))}$$

With initial conditions

$$[y(-\infty) = 0]$$

18.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 4 \\ F &= 2\,\mathrm{e}^{x}(\sin(x) + 7\cos(x)) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 4y = 2\,\mathrm{e}^{x}(\sin(x) + 7\cos(x))$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = -\infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.26.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = (2\sin(x) + 14\cos(x))\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)}\\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(2\sin(x) + 14\cos(x))\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\cos(x), \mathrm{e}^{x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x}\cos(x) + A_2\mathrm{e}^{x}\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1\mathrm{e}^{x}\sin(x)+6A_2\mathrm{e}^{x}\cos(x)+8A_1\mathrm{e}^{x}\cos(x)+8A_2\mathrm{e}^{x}\sin(x)=(2\sin(x)+14\cos(x))\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x}\sin(x) + \mathrm{e}^{x}\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2x\,\mathrm{e}^{-2x}\right) + \left(\mathrm{e}^{x}\sin(x) + \mathrm{e}^{x}\cos(x)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2x + c_1) + \mathrm{e}^{x}\sin(x) + \mathrm{e}^{x}\cos(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-2x}(c_2x + c_1) + \mathrm{e}^{x}\sin(x) + \mathrm{e}^{x}\cos(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = -\operatorname{signum}(c_2)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.26.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int 4 \, dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x) y)'' &= \mathrm{e}^{2x} (2 \sin(x) + 14 \cos(x))\, \mathrm{e}^{x} \\ \left(\mathrm{e}^{2x} y\right)'' &= \mathrm{e}^{2x} (2 \sin(x) + 14 \cos(x))\, \mathrm{e}^{x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{2x} y\right)' = (2 \sin(x) + 4 \cos(x))\, \mathrm{e}^{3x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{2x} y\right) = (\sin(x) + \cos(x))\, \mathrm{e}^{3x} + c_1 x + c_2$$

Hence the solution is

$$y = \frac{(\sin(x) + \cos(x))\, \mathrm{e}^{3x} + c_1 x + c_2}{\mathrm{e}^{2x}}$$

Or

$$y = \cos(x)\, \mathrm{e}^{x} + \mathrm{e}^{x} \sin(x) + c_1 x\, \mathrm{e}^{-2x} + \mathrm{e}^{-2x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \cos(x)\, \mathrm{e}^{x} + \mathrm{e}^{x} \sin(x) + c_1 x\, \mathrm{e}^{-2x} + \mathrm{e}^{-2x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = -\operatorname{signum}(c_1)\infty \tag{1A}$$

Equations {1A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.26.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 607: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+4y'+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + c_2 x\, e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^x(\sin(x)+7\cos(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\, e^x, e^x \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-2x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\, e^x + A_2 e^x \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(x)\, e^x + 6A_2 e^x \cos(x) + 8A_1 \cos(x)\, e^x + 8A_2 e^x \sin(x) = (2\sin(x)+14\cos(x))\, e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x \sin(x) + \cos(x)\, e^x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}\right) + \left(\mathrm{e}^{x} \sin(x) + \cos(x)\,\mathrm{e}^{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + \mathrm{e}^{x} \sin(x) + \cos(x)\,\mathrm{e}^{x}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^{-2x}(c_2 x + c_1) + \mathrm{e}^{x} \sin(x) + \cos(x)\,\mathrm{e}^{x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$
0 = -\operatorname{signum}(c_2)\,\infty \tag{1A}
$$

Equations $\{1\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.26.5 Maple step by step solution

Let's solve

$[y'' + 4y' + 4y = (2\sin(x) + 14\cos(x))\,\mathrm{e}^{x}, y(-\infty) = 0]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -4y + 2\,\mathrm{e}^{x} \sin(x) + 14\cos(x)\,\mathrm{e}^{x} - 4y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y' + 4y = 2\,\mathrm{e}^{x}(\sin(x) + 7\cos(x))$

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,\mathrm{e}^{x}(\sin(x) + 7\cos(x))\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\,\mathrm{e}^{-2x}\left(-\left(\int x(\sin(x) + 7\cos(x))\,\mathrm{e}^{3x} dx\right) + x\left(\int \mathrm{e}^{3x}(\sin(x) + 7\cos(x))\, dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = \mathrm{e}^{x}(\sin(x) + \cos(x))$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + \mathrm{e}^{x}(\sin(x) + \cos(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=2*exp(x)*(sin(x)+7*cos(x)),y(-infinity) = 0],y(
```

$$y(x) = \operatorname{signum}\left(\mathrm{e}^{-2x}c_1\right)\infty$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+4*y'[x]+4*y[x]==2*Exp[x]*(Sin[x]+7*Cos[x]),{y[-Infinity]==0}},y[x],x,IncludeS
```

Not solved

18.27 problem 616

Internal problem ID [15385]
Internal file name [`OUTPUT/15385_Wednesday_May_08_2024_03_57_37_PM_33875786/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 616.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$y'' - 5y' + 6y = 2\,\mathrm{e}^{-2x}(9\sin(2x) + 4\cos(2x))$$

With initial conditions

$$[y(\infty) = 0]$$

18.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -5 \\ q(x) &= 6 \\ F &= 2\,\mathrm{e}^{-2x}(9\sin(2x) + 4\cos(2x)) \end{aligned}$$

Hence the ode is

$$y'' - 5y' + 6y = 2\,\mathrm{e}^{-2x}(9\sin(2x) + 4\cos(2x))$$

The domain of $p(x) = -5$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = \infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = (18\sin(2x) + 8\cos(2x))\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda\,\mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(6)}\\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{5}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= 2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + \mathrm{e}^{2x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + \mathrm{e}^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$(18 \sin(2x) + 8 \cos(2x))\, \mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x} \cos(2x), \mathrm{e}^{-2x} \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{2x}, \mathrm{e}^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x} \cos(2x) + A_2 e^{-2x} \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$18A_1 e^{-2x} \sin(2x) - 18A_2 e^{-2x} \cos(2x) + 16A_1 e^{-2x} \cos(2x) + 16A_2 e^{-2x} \sin(2x) \\ = (18 \sin(2x) + 8 \cos(2x)) e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{113}{145}, A_2 = \frac{36}{145}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{113 e^{-2x} \cos(2x)}{145} + \frac{36 e^{-2x} \sin(2x)}{145}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{3x} + e^{2x} c_2) + \left(\frac{113 e^{-2x} \cos(2x)}{145} + \frac{36 e^{-2x} \sin(2x)}{145}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + e^{2x} c_2 + \frac{113 e^{-2x} \cos(2x)}{145} + \frac{36 e^{-2x} \sin(2x)}{145} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \operatorname{signum}(c_1) \infty \tag{1A}$$

Equations {1A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.27.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 609: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-2x}(9\sin(2x)+4\cos(2x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\cos(2x)\,,\mathrm{e}^{-2x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{2x},\mathrm{e}^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-2x}\cos(2x) + A_2\mathrm{e}^{-2x}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&18A_1\mathrm{e}^{-2x}\sin(2x) - 18A_2\mathrm{e}^{-2x}\cos(2x) + 16A_1\mathrm{e}^{-2x}\cos(2x) + 16A_2\mathrm{e}^{-2x}\sin(2x)\\ &= (18\sin(2x)+8\cos(2x))\,\mathrm{e}^{-2x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{113}{145}, A_2=\frac{36}{145}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{113\,\mathrm{e}^{-2x}\cos(2x)}{145} + \frac{36\,\mathrm{e}^{-2x}\sin(2x)}{145}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(\mathrm{e}^{2x}c_1 + c_2\mathrm{e}^{3x}\right) + \left(\frac{113\,\mathrm{e}^{-2x}\cos(2x)}{145} + \frac{36\,\mathrm{e}^{-2x}\sin(2x)}{145}\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}c_1 + c_2 e^{3x} + \frac{113\,e^{-2x}\cos(2x)}{145} + \frac{36\,e^{-2x}\sin(2x)}{145} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \text{signum}(c_2)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.27.4 Maple step by step solution

Let's solve

$[y'' - 5y' + 6y = (18\sin(2x) + 8\cos(2x))\,e^{-2x}, y(\infty) = 0]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -6y + 8\,e^{-2x}\cos(2x) + 18\,e^{-2x}\sin(2x) + 5y'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 5y' + 6y = 2\,e^{-2x}(9\sin(2x) + 4\cos(2x))$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,e^{-2x}(9 \sin(2x) + 4\,\mathrm{co}\right.$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{3x} \\ 2\,e^{2x} & 3\,e^{3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{5x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,e^{2x} \left(\int (9 \sin(2x) + 4 \cos(2x))\, e^{-4x} dx\right) + 2\,e^{3x} \left(\int (9 \sin(2x) + 4 \cos(2x))\, e^{-5x} dx\right)$

- ○ Compute integrals

  $y_p(x) = \frac{e^{-2x}(113 \cos(2x) + 36 \sin(2x))}{145}$

- Substitute particular solution into general solution to ODE

  $y = e^{2x} c_1 + c_2 e^{3x} + \frac{e^{-2x}(113 \cos(2x) + 36 \sin(2x))}{145}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 31

```
dsolve([diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=2*exp(-2*x)*(9*sin(2*x)+4*cos(2*x)),y(infinity)
```

$$y(x) = c_2 e^{2x} + \frac{(113 \cos(2x) + 36 \sin(2x)) e^{-2x}}{145}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]-5*y'[x]+6*y[x]==2*Exp[-2*x]*(9*Sin[2*x]+4*Cos[2*x]),{y[Infinity]==0}},y[x],x,
```

Not solved

18.28 problem 617

Internal problem ID [15386]
Internal file name [`OUTPUT/15386_Wednesday_May_08_2024_03_57_38_PM_85770441/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.3 Nonhomogeneous linear equations with constant coefficients. Initial value problem. Exercises page 140
Problem number: 617.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y'' - 4y' + 4y = \mathrm{e}^{-x}\left(9x^2 + 5x - 12\right)}$$

With initial conditions

$$[y(\infty) = 0]$$

18.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 4 \\ F &= \mathrm{e}^{-x}\left(9x^2 + 5x - 12\right) \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = \mathrm{e}^{-x}\left(9x^2 + 5x - 12\right)$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = \infty$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

18.28.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = \mathrm{e}^{-x}(9x^2 + 5x - 12)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x\, e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{2x} c_1 + c_2 x\, e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}\left(9x^2 + 5x - 12\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^{-x}, e^{-x} x^2, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\, e^{-x} + A_2 e^{-x} x^2 + A_3 e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1\mathrm{e}^{-x} + 9A_1x\,\mathrm{e}^{-x} + 9A_2\mathrm{e}^{-x}x^2 - 12A_2\mathrm{e}^{-x}x + 2A_2\mathrm{e}^{-x} + 9A_3\mathrm{e}^{-x} = \mathrm{e}^{-x}\left(9x^2 + 5x - 12\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{17}{9}, A_2 = 1, A_3 = -\frac{8}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{17x\,\mathrm{e}^{-x}}{9} + \mathrm{e}^{-x}x^2 - \frac{8\,\mathrm{e}^{-x}}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{2x}c_1 + c_2x\,\mathrm{e}^{2x}\right) + \left(\frac{17x\,\mathrm{e}^{-x}}{9} + \mathrm{e}^{-x}x^2 - \frac{8\,\mathrm{e}^{-x}}{27}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2x + c_1) + \frac{17x\,\mathrm{e}^{-x}}{9} + \mathrm{e}^{-x}x^2 - \frac{8\,\mathrm{e}^{-x}}{27}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{2x}(c_2x + c_1) + \frac{17x\,\mathrm{e}^{-x}}{9} + \mathrm{e}^{-x}x^2 - \frac{8\,\mathrm{e}^{-x}}{27} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \mathrm{signum}\left(c_2\right)\infty \tag{1A}$$

Equations $\{1\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

18.28.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p \, dx} \\ &= e^{\int -4 \, dx} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} \left(M(x) y\right)'' &= \mathrm{e}^{-2x} \mathrm{e}^{-x} \left(9x^{2} + 5x - 12\right) \\ \left(\mathrm{e}^{-2x} y\right)'' &= \mathrm{e}^{-2x} \mathrm{e}^{-x} \left(9x^{2} + 5x - 12\right) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-2x} y\right)' = -\frac{\mathrm{e}^{-3x} \left(27x^{2} + 33x - 25\right)}{9} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2x} y\right) = \frac{\left(27x^{2} + 51x - 8\right) \mathrm{e}^{-3x}}{27} + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{\left(27x^{2} + 51x - 8\right) \mathrm{e}^{-3x}}{27} + c_1 x + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1 x \,\mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \mathrm{e}^{-x} x^{2} + \frac{17x \,\mathrm{e}^{-x}}{9} - \frac{8 \,\mathrm{e}^{-x}}{27}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x \,\mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \mathrm{e}^{-x} x^{2} + \frac{17x \,\mathrm{e}^{-x}}{9} - \frac{8 \,\mathrm{e}^{-x}}{27} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \text{signum}\left(c_1\right)\infty \tag{1A}$$

Equations {1A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.28.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 611: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(x)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-4y'+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{2x}c_1 + c_2 x\,e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-x}\left(9x^2+5x-12\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, e^{-x}x^2, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1x\,e^{-x} + A_2e^{-x}x^2 + A_3e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1e^{-x}+9A_1x\,e^{-x}+9A_2e^{-x}x^2-12A_2e^{-x}x+2A_2e^{-x}+9A_3e^{-x}=e^{-x}\left(9x^2+5x-12\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{17}{9}, A_2=1, A_3=-\frac{8}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{17x\,e^{-x}}{9}+e^{-x}x^2-\frac{8\,e^{-x}}{27}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{2x} c_1 + c_2 x\, e^{2x}\right) + \left(\frac{17x\, e^{-x}}{9} + e^{-x} x^2 - \frac{8\, e^{-x}}{27}\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{2x}(c_2 x + c_1) + \frac{17x\, e^{-x}}{9} + e^{-x} x^2 - \frac{8\, e^{-x}}{27}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{2x}(c_2 x + c_1) + \frac{17x\, e^{-x}}{9} + e^{-x} x^2 - \frac{8\, e^{-x}}{27} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$
0 = \operatorname{signum}(c_2)\, \infty \tag{1A}
$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

18.28.5 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 4y = e^{-x}(9x^2 + 5x - 12), y(\infty) = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{2x} c_1 + c_2 x\,e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{-x}(9x^2 + 5x - 12)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{2x}\left(-\left(\int x\,e^{-3x}(9x^2 + 5x - 12)\,dx\right) + \left(\int e^{-3x}(9x^2 + 5x - 12)\,dx\right)x\right)$

  - Compute integrals

    $y_p(x) = \frac{e^{-x}(27x^2 + 51x - 8)}{27}$

- Substitute particular solution into general solution to ODE

  $y = e^{2x} c_1 + c_2 x\,e^{2x} + \frac{e^{-x}(27x^2 + 51x - 8)}{27}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.046 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=exp(-x)*(9*x^2+5*x-12),y(infinity) = 0],y(x), s
```

$$y(x) = -\operatorname{signum}\left(c_1 e^{2x}\right) \infty$$

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]-4*y'[x]+4*y[x]==Exp[-x]*(9*x^2+5*x-12),{y[Infinity]==0}},y[x],x,IncludeSingul
```

{}

19 Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143

19.1 problem 618

Internal problem ID [15387]
Internal file name [`OUTPUT/15387_Wednesday_May_08_2024_03_57_39_PM_47562876/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 618.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + xy' - y = 0}$$

19.1.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x$$

Verified OK.

19.1.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +1 \\ \lambda_2 &= -1 \end{aligned}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x}$$

Verified OK.

19.1.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + x y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x} \tag{1}
$$

Verification of solutions

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}
$$

Verified OK.

19.1.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2y'' + xy' - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

19.1.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}
\int (x^2y'' + xy' - y)\,dx &= 0 \\
x^2y' - yx &= c_1
\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\ d\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,dx \\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

19.1.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x) + c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

19.1.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + xy' - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + xy' - y)\, dx = 0$$
$$x^2y' - yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right)\\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)\\
d\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x} &= \int \frac{c_1}{x^3}\,dx\\
\frac{y}{x} &= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

19.1.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + x y' - y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2\\
B &= x \qquad (3)\\
C &= -1
\end{aligned}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 613: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2}$$

Verified OK.

19.1.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= x^2 \\
q(x) &= x \\
r(x) &= -1 \\
s(x) &= 0
\end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2-(1)+(-1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
x^2y'-yx=c_1
$$

We now have a first order ode to solve which is

$$
x^2y'-yx=c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{y}{x}=\frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{c_1}{x^3}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{c_1}{x^3}\, dx$$
$$\frac{y}{x} = -\frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

19.1.10 Maple step by step solution

Let's solve

$x^2 y'' + x y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + xy' - y = 0$
- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - y(t) = 0$
- Simplify

  $\frac{d^2}{dt^2}y(t) - y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{t}$
- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_1}{x} + c_2 x$$

- Simplify

$$y = \frac{c_1}{x} + c_2 x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x} + c_2 x$$

19.2 problem 619

Internal problem ID [15388]
Internal file name [`OUTPUT/15388_Wednesday_May_08_2024_03_57_42_PM_21278193/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 619.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + 3xy' + y = 0}$$

19.2.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -1 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$$

Verified OK.

19.2.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 3xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{x}dx\right)}dx \\ &= \int e^{-3\ln(x)}\,dx \\ &= \int \frac{1}{x^3}dx \\ &= -\frac{1}{2x^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{\frac{1}{x^6}} \\
&= x^4
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + x^4 y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
x^4 = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$
4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
4r(r-1) + 0 + 1 = 0
$$

Or

$$
4r^2 - 4r + 1 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots are equal, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$
y(\tau) = c_1 \sqrt{\tau} + c_2 \sqrt{\tau}\, \ln(\tau)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\sqrt{2}\, \sqrt{-\frac{1}{x^2}}\, \left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{2}\, \sqrt{-\frac{1}{x^2}}\, \left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{\sqrt{2}\, \sqrt{-\frac{1}{x^2}}\, \left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{x^2}\right)\right)}{2}
$$

Verified OK.

19.2.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
x^2 y'' + 3xy' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{3}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 2c \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\, x \ln (x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x}$$

Verified OK.

19.2.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 3xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \frac{c_1 \ln(x) + c_2}{x} \\
&= \frac{c_1 \ln(x) + c_2}{x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + c_2}{x}$$

Verified OK.

19.2.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + 3xy' + y)\, dx = 0$$
$$x^2y' + yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x} dx}$$
$$= x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}(xy) = (x)\left(\frac{c_1}{x^2}\right)$$
$$d(xy) = \left(\frac{c_1}{x}\right) dx$$

Integrating gives

$$xy = \int \frac{c_1}{x}\, dx$$
$$xy = c_1 \ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) + c_2}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + c_2}{x}$$

Verified OK.

19.2.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + 3xy' + y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + 3xy' + y)\,dx = 0$$
$$x^2y' + yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x}dx}$$
$$= x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}(xy) = (x)\left(\frac{c_1}{x^2}\right)$$
$$d(xy) = \left(\frac{c_1}{x}\right) dx$$

Integrating gives

$$xy = \int \frac{c_1}{x} dx$$
$$xy = c_1 \ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{c_1 \ln(x)}{x} + \frac{c_2}{x}$$

which simplifies to

$$y = \frac{c_1 \ln(x) + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + c_2}{x}$$

Verified OK.

19.2.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 3xy' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 615: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=-\frac{1}{4x^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1\left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}
$$

Verified OK.

19.2.8 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= 3x \\ r(x) &= 1 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 3 \end{aligned}$$

Therefore (1) becomes

$$2 - (3) + (1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' + yx = c_1$$

We now have a first order ode to solve which is

$$x^2y' + yx = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{x} = \frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x} dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{d}{dx}(xy) &= (x)\left(\frac{c_1}{x^2}\right) \\
d(xy) &= \left(\frac{c_1}{x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int \frac{c_1}{x} dx \\
xy &= c_1 \ln(x) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{c_1 \ln(x)}{x} + \frac{c_2}{x}
$$

which simplifies to

$$
y = \frac{c_1 \ln(x) + c_2}{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \ln(x) + c_2}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \ln(x) + c_2}{x}
$$

Verified OK.

19.2.9 Maple step by step solution

Let's solve

$x^2y'' + 3xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + 3xy' + y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 3\frac{d}{dt}y(t) + y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$
- Root of the characteristic polynomial

  $r = -1$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$
- Simplify

  $y = \frac{c_1}{x} + \frac{c_2 \ln(x)}{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 \ln(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 17

```
DSolve[x^2*y''[x]+3*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \log(x) + c_1}{x}$$

19.3 problem 620

Internal problem ID [15389]
Internal file name [`OUTPUT/15389_Wednesday_May_08_2024_03_57_44_PM_7101673/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 620.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 2xy' + 6y = 0}$$

19.3.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r + 6 = 0$$

Or

$$r^2 + r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -\frac{1}{2} - \frac{i\sqrt{23}}{2} \\ r_2 &= -\frac{1}{2} + \frac{i\sqrt{23}}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -\frac{1}{2}$ and $\beta = -\frac{\sqrt{23}}{2}$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = -\frac{1}{2}, \beta = -\frac{\sqrt{23}}{2}$, the above becomes

$$y = x^{-\frac{1}{2}}\left(c_1 e^{-\frac{i\sqrt{23}\,\ln(x)}{2}} + c_2 e^{\frac{i\sqrt{23}\,\ln(x)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{\sqrt{x}}\left(c_1 \cos\left(\frac{\sqrt{23}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\frac{\sqrt{23}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y=\frac{c_1\cos\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)+c_2\sin\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

Verified OK.

19.3.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y''+2xy'+6y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{2}{x}\\ q(x)&=\frac{6}{x^2}\end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1=0$. Eq (4) simplifies to

$$\tau''(x)+p(x)\,\tau'(x)=0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\,dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{\frac{1}{x^4}} \\
&= 6x^2 \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 6x^2 y(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$
6x^2 = \frac{6}{\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0
$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 + 6 = 0$$

Or

$$r^2 - r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2} - \frac{i\sqrt{23}}{2} \\ r_2 &= \frac{1}{2} + \frac{i\sqrt{23}}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{23}}{2}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\ &= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\ &= \tau^\alpha\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\ &= \tau^\alpha\left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\ &= \tau^\alpha\left(c_1 e^{i(\beta\ln\tau)} + c_2 e^{-i(\beta\ln\tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{23}}{2}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i\sqrt{23}\,\ln(\tau)}{2}} + c_2 e^{\frac{i\sqrt{23}\,\ln(\tau)}{2}}\right)$$

Using Euler relation, the expression $c_1e^{iA} + c_2e^{-iA}$ is transformed to $c_1\cos A + c_1\sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1\cos\left(\frac{\sqrt{23}\,\ln(\tau)}{2}\right) + c_2\sin\left(\frac{\sqrt{23}\,\ln(\tau)}{2}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y=\sqrt{-\frac{1}{x}}\left(c_1\cos\left(\frac{\sqrt{23}\,\ln\left(-\frac{1}{x}\right)}{2}\right)+c_2\sin\left(\frac{\sqrt{23}\,\ln\left(-\frac{1}{x}\right)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y=\sqrt{-\frac{1}{x}}\left(c_1\cos\left(\frac{\sqrt{23}\,\ln\left(-\frac{1}{x}\right)}{2}\right)+c_2\sin\left(\frac{\sqrt{23}\,\ln\left(-\frac{1}{x}\right)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y=\sqrt{-\frac{1}{x}}\left(c_1\cos\left(\frac{\sqrt{23}\,\ln\left(-\frac{1}{x}\right)}{2}\right)+c_2\sin\left(\frac{\sqrt{23}\,\ln\left(-\frac{1}{x}\right)}{2}\right)\right)$$

Verified OK.

19.3.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y''+2xy'+6y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{2}{x}$$
$$q(x)=\frac{6}{x^2}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{2}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= \frac{c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-\frac{\sqrt{6}\,c\tau}{12}}\left(c_1\cos\left(\frac{c\sqrt{138}\,\tau}{12}\right) + c_2\sin\left(\frac{c\sqrt{138}\,\tau}{12}\right)\right)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1 \cos\left(\frac{\sqrt{23}\, \ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, \ln(x)}{2}\right)}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos\left(\frac{\sqrt{23}\, \ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, \ln(x)}{2}\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(\frac{\sqrt{23}\, \ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, \ln(x)}{2}\right)}{\sqrt{x}}$$

Verified OK.

19.3.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 2xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -\frac{1}{2} + \frac{i\sqrt{23}}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{-1+i\sqrt{23}}{x} + \frac{2}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{\left(i\sqrt{23}+1\right) v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{\left(i\sqrt{23}+1\right) u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\left(-1-i\sqrt{23}\right) u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-i\sqrt{23}}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-i\sqrt{23}}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-i\sqrt{23}}{x}\, dx \\ \ln(u) &= \left(-1-i\sqrt{23}\right)\ln(x) + c_1 \\ u &= e^{\left(-1-i\sqrt{23}\right)\ln(x)+c_1} \\ &= c_1 e^{\left(-1-i\sqrt{23}\right)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-i\sqrt{23}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{i\sqrt{23}\, c_1 x^{-i\sqrt{23}}}{23} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{i\sqrt{23}\, c_1 x^{-i\sqrt{23}}}{23} + c_2\right) x^{-\frac{1}{2}+\frac{i\sqrt{23}}{2}} \\
&= \frac{x^{-\frac{1}{2}-\frac{i\sqrt{23}}{2}} \left(i\sqrt{23}\, c_1 + 23 c_2 x^{i\sqrt{23}}\right)}{23}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i\sqrt{23}\, c_1 x^{-i\sqrt{23}}}{23} + c_2\right) x^{-\frac{1}{2}+\frac{i\sqrt{23}}{2}} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i\sqrt{23}\, c_1 x^{-i\sqrt{23}}}{23} + c_2\right) x^{-\frac{1}{2}+\frac{i\sqrt{23}}{2}}$$

Verified OK.

19.3.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2xy' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 617: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{6}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{23}}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{23}}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{6}{x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{23}}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{23}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + \frac{i\sqrt{23}}{2}$	$\frac{1}{2} - \frac{i\sqrt{23}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + \frac{i\sqrt{23}}{2}$	$\frac{1}{2} - \frac{i\sqrt{23}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - \frac{i\sqrt{23}}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{i\sqrt{23}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{23}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{23}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{23}}{2}}{x} \\
&= \frac{1 - i\sqrt{23}}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{23}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{23}}{2}}{x^2} \right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{23}}{2}}{x} \right)^2 - \left(-\frac{6}{x^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - \frac{i\sqrt{23}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{i\sqrt{23}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-\frac{1}{2}-\frac{i\sqrt{23}}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i x^{i\sqrt{23}}\sqrt{23}}{23}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{-\frac{1}{2}-\frac{i\sqrt{23}}{2}}\right) + c_2\left(x^{-\frac{1}{2}-\frac{i\sqrt{23}}{2}}\left(-\frac{i x^{i\sqrt{23}}\sqrt{23}}{23}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x^{-\frac{1}{2}-\frac{i\sqrt{23}}{2}} - \frac{i c_2\sqrt{23}\, x^{-\frac{1}{2}+\frac{i\sqrt{23}}{2}}}{23} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{-\frac{1}{2}-\frac{i\sqrt{23}}{2}} - \frac{i c_2 \sqrt{23}\, x^{-\frac{1}{2}+\frac{i\sqrt{23}}{2}}}{23}$$

Verified OK.

19.3.6 Maple step by step solution

Let's solve

$x^2 y'' + 2xy' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 2xy' + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+2\frac{d}{dt}y(t)+6y(t)=0$$

- Simplify

$$\frac{d^2}{dt^2}y(t)+\frac{d}{dt}y(t)+6y(t)=0$$

- Characteristic polynomial of ODE

$$r^2+r+6=0$$

- Use quadratic formula to solve for r

$$r=\frac{(-1)\pm\left(\sqrt{-23}\right)}{2}$$

- Roots of the characteristic polynomial

$$r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2},-\frac{1}{2}+\frac{\mathrm{I}\sqrt{23}}{2}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right)$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right)+c_2\mathrm{e}^{-\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)$$

- Change variables back using $t=\ln(x)$

$$y=\frac{c_1\cos\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}}+\frac{c_2\sin\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

- Simplify

$$y=\frac{c_1\cos\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}}+\frac{c_2\sin\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin\left(\frac{\sqrt{23}\,\ln(x)}{2}\right) + c_2 \cos\left(\frac{\sqrt{23}\,\ln(x)}{2}\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 42

```
DSolve[x^2*y''[x]+2*x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \cos\left(\frac{1}{2}\sqrt{23}\log(x)\right) + c_1 \sin\left(\frac{1}{2}\sqrt{23}\log(x)\right)}{\sqrt{x}}$$

19.4 problem 621

Internal problem ID [15390]
Internal file name [`OUTPUT/15390_Wednesday_May_08_2024_03_57_46_PM_92111037/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 621.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' + y' = 0}$$

19.4.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$xy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

19.4.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$xp'(x) + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{p}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{x}\,dx \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{x}\,dx \\
\ln(p) &= -\ln(x) + c_1 \\
p &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \frac{c_1}{x}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{c_1}{x}\,\mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = c_1 \ln(x) + c_2
$$

Verified OK.

19.4.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x \\
B &= 1 \\
C &= 0 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)(0) + (1)(0) + (0)(1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
xv'' + (1) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x}\,\mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (1)\,(c_1 \ln(x) + c_2) \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = c_1 \ln(x) + c_2
$$

Verified OK.

19.4.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$xy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \frac{c_1}{x}\, \mathrm{d}x$$
$$= c_1 \ln (x) + c_2$$

Summary
The solution(s) found are the following

$$y = c_1 \ln (x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln (x) + c_2$$

Verified OK.

19.4.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 619: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{x}\, dx}\\ &= z_1 e^{-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(\ln(x))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 \ln(x) + c_1 \tag{1}$$

Verification of solutions

$$y = c_2 \ln(x) + c_1$$

Verified OK.

19.4.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$xy' = c_1$$

We now have a first order ode to solve which is

$$xy' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

19.4.7 Maple step by step solution

Let's solve

$xy'' + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

  $xy'' + y' = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- ○ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- ○ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

  $\frac{\frac{d^2}{dt^2}y(t)}{x} = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(t) = 1$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_2 t + c_1$

- Change variables back using $t = \ln(x)$

  $y = c_2 \ln(x) + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_2 \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 13

```
DSolve[x*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x) + c_2$$

19.5 problem 622

Internal problem ID [15391]
Internal file name [`OUTPUT/15391_Wednesday_May_08_2024_03_57_46_PM_15776161/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 622.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__x__method__2", "second__order__ode__non__constant__coeff__transformation__on__B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2+x)^2 y'' + 3(2+x) y' - 3y = 0}$$

19.5.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$(2+x)^2 y'' + (3x+6) y' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{2+x}$$
$$q(x) = -\frac{3}{(2+x)^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{3}{2+x}dx\right)}dx \\
&= \int e^{-3\ln(2+x)}\,dx \\
&= \int \frac{1}{(2+x)^3}dx \\
&= -\frac{1}{2\,(2+x)^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{(2+x)^2}}{\frac{1}{(2+x)^6}} \\
&= -3(2+x)^4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) - 3(2+x)^4\,y(\tau) = 0$$

But in terms of τ

$$-3(2+x)^4 = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_1(2+x)^4 + \frac{c_2}{4}\right)\sqrt{2}}{\sqrt{-\frac{1}{(2+x)^2}}\,(2+x)^4}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_1(2+x)^4 + \frac{c_2}{4}\right)\sqrt{2}}{\sqrt{-\frac{1}{(2+x)^2}}\,(2+x)^4} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_1(2+x)^4 + \frac{c_2}{4}\right)\sqrt{2}}{\sqrt{-\frac{1}{(2+x)^2}}\,(2+x)^4}$$

Verified OK.

19.5.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0
\end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= (2 + x)^2 \\
B &= 3x + 6 \\
C &= -3 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left((2 + x)^2\right)(0) + (3x + 6)(3) + (-3)(3x + 6) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$3(2 + x)^3\, v'' + \left(15(2 + x)^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$3(2 + x)^3\, u'(x) + 15(2 + x)^2\, u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{5u}{2 + x}
\end{aligned}$$

Where $f(x) = -\frac{5}{2+x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{5}{2 + x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{5}{2 + x}\, dx \\
\ln(u) &= -5\ln(2 + x) + c_1 \\
u &= \mathrm{e}^{-5\ln(2+x)+c_1} \\
&= \frac{c_1}{(2 + x)^5}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{(2+x)^5} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{(2+x)^5}\, dx \\ &= -\frac{c_1}{4(2+x)^4} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (3x+6)\left(-\frac{c_1}{4(2+x)^4} + c_2\right) \\ &= \frac{3c_2(2+x)^4 - \frac{3c_1}{4}}{(2+x)^3} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{3c_2(2+x)^4 - \frac{3c_1}{4}}{(2+x)^3} \tag{1}$$

Verification of solutions

$$y = \frac{3c_2(2+x)^4 - \frac{3c_1}{4}}{(2+x)^3}$$

Verified OK.

19.5.3 Solving using Kovacic algorithm

Writing the ode as

$$(2+x)^2 y'' + (3x+6) y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (2+x)^2 \\ B &= 3x+6 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4\left(2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4(2+x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4\left(2+x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 621: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(2+x)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{15}{4 (2+x)^2}
$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(2+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4\left(2+x\right)^{2}}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4\left(2+x\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2(2+x)} + (-)(0) \\
&= -\frac{3}{2(2+x)} \\
&= -\frac{3}{2(2+x)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3}{2(2+x)}\right)(0) + \left(\left(\frac{3}{2(2+x)^2}\right) + \left(-\frac{3}{2(2+x)}\right)^2 - \left(\frac{15}{4(2+x)^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3}{2(2+x)} dx} \\
&= \frac{1}{(2+x)^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x+6}{(2+x)^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(2+x)}{2}} \\
&= z_1 \left(\frac{1}{(2+x)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{(2+x)^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x+6}{(2+x)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(2+x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{(2+x)^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{(2+x)^3}\right) + c_2 \left(\frac{1}{(2+x)^3}\left(\frac{(2+x)^4}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{(2+x)^3} + c_2 \left(\frac{1}{2} + \frac{x}{4}\right) \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{(2+x)^3} + c_2\left(\frac{1}{2} + \frac{x}{4}\right)$$

Verified OK.

19.5.4 Maple step by step solution

Let's solve

$(2+x)^2 y'' + (3x+6) y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2+x} + \frac{3y}{(2+x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2+x} - \frac{3y}{(2+x)^2} = 0$

□ Check to see if $x_0 = -2$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{3}{2+x}, P_3(x) = -\frac{3}{(2+x)^2}\right]$

  - $(2+x) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((2+x) \cdot P_2(x)\right)\Big|_{x=-2} = 3$

  - $(2+x)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((2+x)^2 \cdot P_3(x)\right)\Big|_{x=-2} = -3$

  - $x = -2$is a regular singular point

  Check to see if $x_0 = -2$ is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $(2+x)^2 y'' + (3x+6) y' - 3y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $u^2\left(\frac{d^2}{du^2}y(u)\right) + 3u\left(\frac{d}{du}y(u)\right) - 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite DE with series expansions

  - Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} a_k(k+r+3)\,(k+r-1)\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+3)\,(k-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_k = 0$

- Recursion relation for $r = 0$

  $a_k = 0$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$

- Revert the change of variables $u = 2 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k(2+x)^k, a_k = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve((x+2)^2*diff(y(x),x$2)+3*(x+2)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 + c_2(x+2)^4}{(x+2)^3}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 20

```
DSolve[(x+2)^2*y''[x]+3*(x+2)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x+2) + \frac{c_2}{(x+2)^3}$$

19.6 problem 623

Internal problem ID [15392]
Internal file name [`OUTPUT/15392_Wednesday_May_08_2024_03_57_47_PM_17346671/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 623.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__x__method__1", "second__order__ode__non__constant__coeff__transformation__on__B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+2x)^2 y'' - 2(1+2x)\, y' + 4y = 0}$$

19.6.1 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$4\left(x+\frac{1}{2}\right)^2 y'' + (-4x-2)\, y' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{1+2x} \\
q(x) &= \frac{1}{\left(x+\frac{1}{2}\right)^{2}}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^{2}}{d\tau^{2}}y(\tau) + p_{1}\left(\frac{d}{d\tau}y(\tau)\right) + q_{1}y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_{1}(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^{2}} \tag{4}
$$

$$
q_{1}(\tau) = \frac{q(x)}{\tau'(x)^{2}} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^{2}}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^{2}}}\left(x+\frac{1}{2}\right)^{3}}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_{1}(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^{2}} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^{2}}}\left(x+\frac{1}{2}\right)^{3}} - \frac{2}{1+2x}\frac{\sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^{2}}}}{c}}{\left(\frac{\sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^{2}}}}{c}\right)^{2}} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
y(\tau)'' + p_{1}y(\tau)' + q_{1}y(\tau) = 0
$$

$$
\frac{d^{2}}{d\tau^{2}}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^{2}y(\tau) = 0 \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{1}{\left(x+\frac{1}{2}\right)^2}}dx}{c} \\ &= \frac{\sqrt{\frac{1}{(1+2x)^2}}\,(1+2x)\,\sqrt{4}\,\ln\left(1+2x\right)}{2c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = (1+2x)\,c_1$$

Summary
The solution(s) found are the following

$$y = (1+2x)\,c_1 \tag{1}$$

Verification of solutions

$$y = (1+2x)\,c_1$$

Verified OK.

19.6.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+(AB''+BB'+CB)\,v&=0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A&=4\left(x+\frac{1}{2}\right)^2\\
B&=-4x-2\\
C&=4\\
F&=0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB&=\left(4\left(x+\frac{1}{2}\right)^2\right)(0)+(-4x-2)(-4)+(4)(-4x-2)\\
&=0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-2(1+2x)^3\,v''+\left(-16\left(x+\frac{1}{2}\right)^2\right)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
-2(1+2x)^3\,u'(x)-4u(x)\,(1+2x)^2=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(x,u)\\
&=f(x)g(u)\\
&=-\frac{2u}{1+2x}
\end{aligned}
$$

Where $f(x) = -\frac{2}{1+2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{1+2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{1+2x}\,dx \\
\ln(u) &= -\ln(1+2x) + c_1 \\
u &= e^{-\ln(1+2x)+c_1} \\
&= \frac{c_1}{1+2x}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{1+2x}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{1+2x}\,\mathrm{d}x \\
&= \frac{c_1 \ln(1+2x)}{2} + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-4x-2)\left(\frac{c_1 \ln(1+2x)}{2} + c_2\right) \\
&= -2(c_1 \ln(1+2x) + 2c_2)\left(x + \frac{1}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -2(c_1 \ln(1+2x) + 2c_2)\left(x + \frac{1}{2}\right) \tag{1}
$$

Verification of solutions

$$
y = -2(c_1 \ln(1+2x) + 2c_2)\left(x + \frac{1}{2}\right)
$$

Verified OK.

19.6.3 Solving using Kovacic algorithm

Writing the ode as

$$4\left(x+\frac{1}{2}\right)^{2} y''+(-4x-2)\, y'+4y = 0 \tag{1}$$

$$Ay''+By'+Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4\left(x+\frac{1}{2}\right)^{2} \\ B &= -4x-2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^{2}-4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{\left(1+2x\right)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= \left(1+2x\right)^{2} \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{\left(1+2x\right)^{2}}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 623: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4\left(x+\frac{1}{2}\right)^2}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{(1+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{(1+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{1+2x} + (-)\,(0) \\ &= \frac{1}{1+2x} \\ &= \frac{1}{1+2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{1+2x}\right)(0) + \left(\left(-\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(\frac{1}{1+2x}\right)^2 - \left(-\frac{1}{(1+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{1+2x} dx} \\
&= \sqrt{1+2x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x-2}{4\left(x+\frac{1}{2}\right)^2} \, dx} \\
&= z_1 e^{\frac{\ln(1+2x)}{2}} \\
&= z_1 \left(\sqrt{1+2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1 + 2x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x-2}{4\left(x+\frac{1}{2}\right)^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\ln(1+2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (1+2x) + c_2 \left(1 + 2x \left(\frac{\ln(1+2x)}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = (1+2x)\, c_1 + \frac{c_2(1+2x)\ln(1+2x)}{2} \tag{1}$$

Verification of solutions

$$y = (1+2x)\, c_1 + \frac{c_2(1+2x)\ln(1+2x)}{2}$$

Verified OK.

19.6.4 Maple step by step solution

Let's solve

$4\left(x+\frac{1}{2}\right)^2 y'' + (-4x-2)\, y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{1+2x} - \frac{4y}{(1+2x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{1+2x} + \frac{4y}{(1+2x)^2} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{2}{1+2x}, P_3(x) = \frac{4}{(1+2x)^2}\right]$

  ◦ $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$

  ◦ $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 1$

  ◦ $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $(1+2x)^2 y'' + (-4x-2) y' + 4y = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $4u^2 \left(\frac{d^2}{du^2} y(u)\right) - 4u \left(\frac{d}{du} y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

  - Convert $u \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, (k+r-1)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} 4a_k (k+r-1)^2\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $4a_k (k-1)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_k = 0$

- Recursion relation for $r = 0$

  $a_k = 0$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k, a_k = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve((2*x+1)^2*diff(y(x),x$2)-2*(2*x+1)*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(2x+1)\left(-c_2 \ln(2) + c_2 \ln(2x+1) + c_1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 23

```
DSolve[(2*x+1)^2*y''[x]-2*(2*x+1)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (2x+1)(c_2 \log(2x+1) + c_1)$$

19.7 problem 624

Internal problem ID [15393]
Internal file name [OUTPUT/15393_Wednesday_May_08_2024_03_57_48_PM_79081408/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 624.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^2y''' - 3xy'' + 3y' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$x^2v''(x) - 3xv'(x) + 3v(x) = 0$$

This is Euler second order ODE. Let the solution be $v(x) = x^r$, then $v' = rx^{r-1}$ and $v'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$v(x) = c_1v_1 + c_2v_2$$

Where $v_1 = x^{r_1}$ and $v_2 = x^{r_2}$. Hence

$$v(x) = c_2x^3 + c_1x$$

But since $y' = v(x)$ then we now need to solve the ode $y' = c_2x^3 + c_1x$. Integrating both sides gives

$$\begin{aligned} y &= \int c_2x^3 + c_1x \,\mathrm{d}x \\ &= \frac{(c_2x^2 + c_1)^2}{4c_2} + c_3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_2x^2 + c_1)^2}{4c_2} + c_3 \tag{1}$$

Verification of solutions

$$y = \frac{(c_2x^2 + c_1)^2}{4c_2} + c_3$$

Verified OK.

19.7.1 Maple step by step solution

Let's solve

$x^2y''' - 3xy'' + 3y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

$y''' = \frac{3(xy''-y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{x} + \frac{3y'}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y''' - 3xy'' + 3y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - 3x\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{3\left(\frac{d}{dt}y(t)\right)}{x} = 0$

- Simplify

  $\frac{\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 8\frac{d}{dt}y(t)}{x} = 0$

- Isolate 3rd derivative

  $\frac{d^3}{dt^3}y(t) = 6\frac{d^2}{dt^2}y(t) - 8\frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 8\frac{d}{dt}y(t) = 0$

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 6y_3(t) - 8y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 6y_3(t) - 8y_2(t)\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & 6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{4t} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{4t} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y(t) = \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{4t}}{16} + c_1$

- Change variables back using $t = \ln(x)$

  $y = \frac{1}{4} c_2 x^2 + \frac{1}{16} c_3 x^4 + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x$3)-3*x*diff(y(x),x$2)+3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_3 x^4 + c_2 x^2 + c_1$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 26

```
DSolve[x^2*y'''[x]-3*x*y''[x]+3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^4}{4} + \frac{c_1 x^2}{2} + c_3$$

19.8 problem 625

Internal problem ID [15394]
Internal file name [`OUTPUT/15394_Wednesday_May_08_2024_03_57_49_PM_18489764/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 625.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_y]]`

$$\boxed{x^2y''' - 2y' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$x^2v''(x) - 2v(x) = 0$$

This is Euler second order ODE. Let the solution be $v(x) = x^r$, then $v' = rx^{r-1}$ and $v'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$v(x) = c_1 v_1 + c_2 v_2$$

Where $v_1 = x^{r_1}$ and $v_2 = x^{r_2}$. Hence

$$v(x) = \frac{c_1}{x} + c_2 x^2$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \frac{c_1}{x} + c_2 x^2$. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_2 x^3 + c_1}{x} \, \mathrm{d}x \\ &= \frac{c_2 x^3}{3} + c_1 \ln(x) + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x^3}{3} + c_1 \ln(x) + c_3 \tag{1}$$

Verification of solutions

$$y = \frac{c_2 x^3}{3} + c_1 \ln(x) + c_3$$

Verified OK.

19.8.1 Maple step by step solution

Let's solve

$x^2 y''' - 2y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

$y''' = \frac{2y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''' - \frac{2y'}{x^2} = 0$

- Multiply by denominators of the ODE

$x^2 y''' - 2y' = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$
  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - \frac{2\left(\frac{d}{dt}y(t)\right)}{x} = 0$

- Simplify

$\frac{\frac{d^3}{dt^3}y(t) - 3\frac{d^2}{dt^2}y(t)}{x} = 0$

- Isolate 3rd derivative

$\frac{d^3}{dt^3}y(t) = 3\frac{d^2}{dt^2}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^3}{dt^3}y(t) - 3\frac{d^2}{dt^2}y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 3y_3(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 3y_3(t)\right]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

  $$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

  $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$

- Substitute solutions into the general solution

  $$\vec{y} = c_3 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y(t) = \frac{c_3 e^{3t}}{9} + c_1$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_3 x^3}{9} + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$3)=2*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \ln(x) + c_3 x^3$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[x^2*y'''[x]==2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^3}{3} + c_1 \log(x) + c_3$$

19.9 problem 626

Internal problem ID [15395]
Internal file name [`OUTPUT/15395_Wednesday_May_08_2024_03_57_50_PM_74824681/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 626.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{(x+1)^2 y''' - 12y' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$-12v(x) + \left(x^2+2x+1\right) v''(x) = 0$$

Writing the ode as

$$v''(x) (x+1)^2 - 12v(x) = 0 \tag{1}$$
$$Av''(x) + Bv'(x) + Cv(x) = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= 0 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = v(x)\, e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= (x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{(x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $v(x)$ is found using the inverse transformation

$$v(x) = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 627: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{(x+1)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{(x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -3$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -3 - (-3) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{x+1} + (-)(0) \\
&= -\frac{3}{x+1} \\
&= -\frac{3}{x+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{x+1}\right)(0) + \left(\left(\frac{3}{(x+1)^2}\right) + \left(-\frac{3}{x+1}\right)^2 - \left(\frac{12}{(x+1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{3}{x+1} dx} \\ &= \frac{1}{(x+1)^3} \end{aligned}$$

The first solution to the original ode in $v(x)$ is found from

$$v_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} v_1 &= z_1 \\ &= \frac{1}{(x+1)^3} \end{aligned}$$

Which simplifies to

$$v_1 = \frac{1}{(x+1)^3}$$

The second solution v_2 to the original ode is found using reduction of order

$$v_2 = v_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{v_1^2}\, dx$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
v_2 &= v_1 \int \frac{1}{v_1^2}\, dx \\
&= \frac{1}{(x+1)^3} \int \frac{1}{\frac{1}{(x+1)^6}}\, dx \\
&= \frac{1}{(x+1)^3} \left(\frac{(x+1)^7}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
v(x) &= c_1 v_1 + c_2 v_2 \\
&= c_1 \left(\frac{1}{(x+1)^3}\right) + c_2 \left(\frac{1}{(x+1)^3} \left(\frac{(x+1)^7}{7}\right)\right)
\end{aligned}
$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \frac{c_1}{(x+1)^3} + \frac{c_2(x+1)^4}{7}$. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{c_2 x^7 + 7c_2 x^6 + 21c_2 x^5 + 35c_2 x^4 + 35c_2 x^3 + 21c_2 x^2 + 7c_2 x + 7c_1 + c_2}{7\,(x+1)^3}\, dx \\
&= \frac{c_2 x^5}{35} + \frac{c_2 x^4}{7} + \frac{2c_2 x^3}{7} + \frac{2c_2 x^2}{7} + \frac{c_2 x}{7} - \frac{c_1}{2\,(x+1)^2} + c_3
\end{aligned}
$$

This is higher order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$
-12y' + \left(x^2 + 2x + 1\right) y''' = 0
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & \frac{1}{(x+1)^2} & \frac{x(x^4+5x^3+10x^2+10x+5)}{35} \\ 0 & -\frac{2}{(x+1)^3} & \frac{(x+1)^4}{7} \\ 0 & \frac{6}{(x+1)^4} & \frac{4(x+1)^3}{7} \end{bmatrix}$$
$$|W| = -2$$

The determinant simplifies to

$$|W| = -2$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} \frac{1}{(x+1)^2} & \frac{x(x^4+5x^3+10x^2+10x+5)}{35} \\ -\frac{2}{(x+1)^3} & \frac{(x+1)^4}{7} \end{bmatrix}$$
$$= \frac{7x^5 + 35x^4 + 70x^3 + 70x^2 + 35x + 5}{35(x+1)^3}$$

$$W_2(x) = \det \begin{bmatrix} 1 & \frac{x(x^4+5x^3+10x^2+10x+5)}{35} \\ 0 & \frac{(x+1)^4}{7} \end{bmatrix}$$
$$= \frac{(x+1)^4}{7}$$

$$W_3(x) = \det \begin{bmatrix} 1 & \frac{1}{(x+1)^2} \\ 0 & -\frac{2}{(x+1)^3} \end{bmatrix}$$
$$= -\frac{2}{(x+1)^3}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{\left(-(x+1)^2 y''' + (x^2+2x+1)\, y'''\right)\left(\frac{7x^5+35x^4+70x^3+70x^2+35x+5}{35(x+1)^3}\right)}{(x^2+2x+1)(-2)}\, dx \\
&= \int \frac{\frac{\left(-(x+1)^2 y'''+(x^2+2x+1)y'''\right)(7x^5+35x^4+70x^3+70x^2+35x+5)}{35(x+1)^3}}{-2x^2-4x-2}\, dx \\
&= \int (0)\, dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(-(x+1)^2 y''' + (x^2+2x+1)\, y'''\right)\left(\frac{(x+1)^4}{7}\right)}{(x^2+2x+1)(-2)}\, dx \\
&= -\int \frac{\frac{\left(-(x+1)^2 y'''+(x^2+2x+1)y'''\right)(x+1)^4}{7}}{-2x^2-4x-2}\, dx \\
&= -\int (0)\, dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(-(x+1)^2 y''' + (x^2+2x+1)\, y'''\right)\left(-\frac{2}{(x+1)^3}\right)}{(x^2+2x+1)(-2)}\, dx \\
&= \int \frac{-\frac{2\left(-(x+1)^2 y'''+(x^2+2x+1)y'''\right)}{(x+1)^3}}{-2x^2-4x-2}\, dx \\
&= \int (0)\, dx \\
&= 0
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p &= (0) \\
&+ (0)\left(\frac{1}{(x+1)^2}\right) \\
&+ (0)\left(\frac{1}{35}x^5 + \frac{1}{7}x^4 + \frac{2}{7}x^3 + \frac{2}{7}x^2 + \frac{1}{7}x\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = 0$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \Bigg(y \\
&= \frac{c_2x^5}{35} + \frac{c_2x^4}{7} + \frac{2c_2x^3}{7} + \frac{2c_2x^2}{7} + \frac{c_2x}{7} - \frac{c_1}{2(x+1)^2} + c_3\Bigg) + (0)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_2x^5}{35} + \frac{c_2x^4}{7} + \frac{2c_2x^3}{7} + \frac{2c_2x^2}{7} + \frac{c_2x}{7} - \frac{c_1}{2(x+1)^2} + c_3 \tag{1}$$

Verification of solutions

$$y = \frac{c_2x^5}{35} + \frac{c_2x^4}{7} + \frac{2c_2x^3}{7} + \frac{2c_2x^2}{7} + \frac{c_2x}{7} - \frac{c_1}{2(x+1)^2} + c_3$$

Verified OK.

19.9.1 Maple step by step solution

Let's solve

$(x+1)^2 y''' - 12y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Check to see if $x_0 = -1$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = 0, P_3(x) = -\frac{12}{(x+1)^2}, P_4(x) = 0\right]$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right) \bigg|_{x=-1} = 0$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right) \bigg|_{x=-1} = -12$

- $(x+1)^3 \cdot P_4(x)$ is analytic at $x = -1$

  $\left((x+1)^3 \cdot P_4(x)\right) \bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if $x_0 = -1$ is a regular singular point

  $x_0 = -1$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $u^2 \left(\frac{d^3}{du^3} y(u)\right) - 12 \frac{d}{du} y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

- Convert $\frac{d}{du} y(u)$ to series expansion

  $\frac{d}{du} y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $\frac{d}{du} y(u) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, u^{k+r}$

- Convert $u^2 \cdot \left(\frac{d^3}{du^3} y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d^3}{du^3} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)(k+r-2)\, u^{k+r-1}$

- Shift index using $k- > k+1$

  $u^2 \cdot \left(\frac{d^3}{du^3} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r)(k+r-1)\, u^{k+r}$

Rewrite DE with series expansions

$$\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+3+r)\,(k-4+r)\,u^{k+r} = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)\,(k+3)\,(k-4) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = 0$

- Recursion relation for $r = 0$

  $a_{k+1} = 0$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = 0\right]$$

- Revert the change of variables $u = x+1$

  $$\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k\,, a_{k+1} = 0\right]$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((x+1)^2*diff(y(x),x$3)-12*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + \frac{c_2}{(1+x)^2} + c_3(1+x)^5$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 30

```
DSolve[(x+1)^2*y'''[x]-12*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}c_1(x+1)^5 - \frac{c_2}{2(x+1)^2} + c_3$$

19.10 problem 627

Internal problem ID [15396]
Internal file name [`OUTPUT/15396_Wednesday_May_08_2024_03_57_51_PM_19386016/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 627.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_y]]`

$$\boxed{(1+2x)^2 y''' + 2(1+2x) y'' + y' = 0}$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$v(x) + (4x+2) v'(x) + (4x^2+4x+1) v''(x) = 0$$

In normal form the ode

$$4\left(x+\frac{1}{2}\right)^2 v''(x) + (4x+2) v'(x) + v(x) = 0 \tag{1}$$

Becomes

$$v''(x) + p(x) v'(x) + q(x) v(x) = 0 \tag{2}$$

Where

$$p(x) = \frac{4x+2}{4\left(x+\frac{1}{2}\right)^2}$$
$$q(x) = \frac{1}{4\left(x+\frac{1}{2}\right)^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}v(\tau) + p_1\left(\frac{d}{d\tau}v(\tau)\right) + q_1 v(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{4x+2}{4\left(x+\frac{1}{2}\right)^2}dx\right)} dx \\
&= \int e^{-\ln(1+2x)}\,dx \\
&= \int \frac{1}{1+2x} dx \\
&= \frac{\ln\left(1+2x\right)}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'\,(x)^2} \\
&= \frac{\frac{1}{4\left(x+\frac{1}{2}\right)^2}}{\frac{1}{(1+2x)^2}} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}v(\tau) + q_1 v(\tau) = 0 \\
\frac{d^2}{d\tau^2}v(\tau) + v(\tau) = 0
\end{aligned}$$

The above ode is now solved for $v(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(\tau) + Bv'(\tau) + Cv(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$v(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$v(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to $v(x)$ using (6) which results in

$$v(x) = c_1 \cos\left(\frac{\ln(1+2x)}{2}\right) + c_2 \sin\left(\frac{\ln(1+2x)}{2}\right)$$

But since $y' = v(x)$ then we now need to solve the ode $y' = c_1 \cos\left(\frac{\ln(1+2x)}{2}\right) + c_2 \sin\left(\frac{\ln(1+2x)}{2}\right)$. Integrating both sides gives

$$\begin{aligned} y &= \int c_1 \cos\left(\frac{\ln(1+2x)}{2}\right) + c_2 \sin\left(\frac{\ln(1+2x)}{2}\right) dx \\ &= c_1 \left(\frac{2\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5} + \frac{(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right) + c_2 \left(-\frac{\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5} + \frac{2(1+2x)}{}\right. \end{aligned}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y' + (4x+2)\, y'' + (4x^2+4x+1)\, y''' = 0$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & -\frac{(1+2x)\left(\cos\left(\frac{\ln(1+2x)}{2}\right)-2\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{5} & \frac{(1+2x)\left(2\cos\left(\frac{\ln(1+2x)}{2}\right)+\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{5} \\ 0 & \sin\left(\frac{\ln(1+2x)}{2}\right) & \cos\left(\frac{\ln(1+2x)}{2}\right) \\ 0 & \frac{\cos\left(\frac{\ln(1+2x)}{2}\right)}{1+2x} & -\frac{\sin\left(\frac{\ln(1+2x)}{2}\right)}{1+2x} \end{bmatrix}$$

$$|W| = -\frac{\cos\left(\frac{\ln(1+2x)}{2}\right)^2 + \sin\left(\frac{\ln(1+2x)}{2}\right)^2}{1+2x}$$

The determinant simplifies to

$$|W| = -\frac{1}{1+2x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} -\frac{(1+2x)\left(\cos\left(\frac{\ln(1+2x)}{2}\right)-2\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{5} & \frac{(1+2x)\left(2\cos\left(\frac{\ln(1+2x)}{2}\right)+\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{5} \\ \sin\left(\frac{\ln(1+2x)}{2}\right) & \cos\left(\frac{\ln(1+2x)}{2}\right) \end{bmatrix} \\ &= -\frac{1}{5} - \frac{2x}{5} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} 1 & \frac{(1+2x)\left(2\cos\left(\frac{\ln(1+2x)}{2}\right)+\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{5} \\ 0 & \cos\left(\frac{\ln(1+2x)}{2}\right) \end{bmatrix} \\ &= \cos\left(\frac{\ln(1+2x)}{2}\right) \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} 1 & -\frac{(1+2x)\left(\cos\left(\frac{\ln(1+2x)}{2}\right)-2\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{5} \\ 0 & \sin\left(\frac{\ln(1+2x)}{2}\right) \end{bmatrix} \\ &= \sin\left(\frac{\ln(1+2x)}{2}\right) \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x) W_1(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{\left(-(1+2x)^2 y''' - 2(1+2x) y'' + (4x+2) y'' + (4x^2+4x+1) y'''\right) \left(-\frac{1}{5} - \frac{2x}{5}\right)}{(4x^2+4x+1) \left(-\frac{1}{1+2x}\right)} \, dx \\
&= \int \frac{\left(-(1+2x)^2 y''' - 2(1+2x) y'' + (4x+2) y'' + (4x^2+4x+1) y'''\right) \left(-\frac{1}{5} - \frac{2x}{5}\right)}{-\frac{4x^2+4x+1}{1+2x}} \, dx \\
&= \int (0) \, dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{\left(-(1+2x)^2 y''' - 2(1+2x) y'' + (4x+2) y'' + (4x^2+4x+1) y'''\right) \left(\cos\left(\frac{\ln(1+2x)}{2}\right)\right)}{(4x^2+4x+1) \left(-\frac{1}{1+2x}\right)} \, dx \\
&= -\int \frac{\left(-(1+2x)^2 y''' - 2(1+2x) y'' + (4x+2) y'' + (4x^2+4x+1) y'''\right) \cos\left(\frac{\ln(1+2x)}{2}\right)}{-\frac{4x^2+4x+1}{1+2x}} \, dx \\
&= -\int (0) \, dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{\left(-(1+2x)^2 y''' - 2(1+2x) y'' + (4x+2) y'' + (4x^2+4x+1) y'''\right) \left(\sin\left(\frac{\ln(1+2x)}{2}\right)\right)}{(4x^2+4x+1) \left(-\frac{1}{1+2x}\right)} \, dx \\
&= \int \frac{\left(-(1+2x)^2 y''' - 2(1+2x) y'' + (4x+2) y'' + (4x^2+4x+1) y'''\right) \sin\left(\frac{\ln(1+2x)}{2}\right)}{-\frac{4x^2+4x+1}{1+2x}} \, dx \\
&= \int (0) \, dx \\
&= 0
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p &= (0)\\
&+ (0)\left(-\frac{\cos\left(\frac{\ln(1+2x)}{2}\right)}{5}-\frac{2\cos\left(\frac{\ln(1+2x)}{2}\right)x}{5}+\frac{2\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}+\frac{4\sin\left(\frac{\ln(1+2x)}{2}\right)x}{5}\right)\\
&+ (0)\left(\frac{2\cos\left(\frac{\ln(1+2x)}{2}\right)}{5}+\frac{4\cos\left(\frac{\ln(1+2x)}{2}\right)x}{5}+\frac{\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}+\frac{2\sin\left(\frac{\ln(1+2x)}{2}\right)x}{5}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = 0
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= \Bigg(y\\
&= c_1\left(\frac{2\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5}+\frac{(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right)\\
&\quad + c_2\left(-\frac{\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5}+\frac{2(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right)+c_3\Bigg)+(0)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1\left(\frac{2\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5}+\frac{(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right)\\
+ c_2\left(-\frac{\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5}+\frac{2(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right)+c_3
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(\frac{2\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5}+\frac{(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right) + c_2\left(-\frac{\cos\left(\frac{\ln(1+2x)}{2}\right)(1+2x)}{5}+\frac{2(1+2x)\sin\left(\frac{\ln(1+2x)}{2}\right)}{5}\right)+c_3$$

Verified OK.

19.10.1 Maple step by step solution

Let's solve

$(1+2x)^2 y''' + 2(1+2x)y'' + y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{2}{1+2x}, P_3(x) = \frac{1}{(1+2x)^2}, P_4(x) = 0\right]$
  - $\left(x+\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)\cdot P_2(x)\right)\Big|_{x=-\frac{1}{2}} = 1$
  - $\left(x+\frac{1}{2}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^2\cdot P_3(x)\right)\Big|_{x=-\frac{1}{2}} = \frac{1}{4}$
  - $\left(x+\frac{1}{2}\right)^3\cdot P_4(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x+\frac{1}{2}\right)^3\cdot P_4(x)\right)\Big|_{x=-\frac{1}{2}} = 0$
  - $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $y' + (4x+2)y'' + (1+2x)^2 y''' = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $4u^2 \left(\frac{d^3}{du^3} y(u)\right) + 4u \left(\frac{d^2}{du^2} y(u)\right) + \frac{d}{du} y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

  - Convert $\frac{d}{du} y(u)$ to series expansion

    $\frac{d}{du} y(u) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1}$
  - Shift index using $k- > k+1$

    $\frac{d}{du} y(u) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, u^{k+r}$
  - Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-1}$
  - Shift index using $k- > k+1$

    $u \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, u^{k+r}$
  - Convert $u^2 \cdot \left(\frac{d^3}{du^3} y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^3}{du^3} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,(k+r-2)\, u^{k+r-1}$
  - Shift index using $k- > k+1$

    $u^2 \cdot \left(\frac{d^3}{du^3} y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\,(k+r-1)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=-1}^{\infty} a_{k+1} (k+1+r) \left(4(k+1)^2 + 8(k+1)\, r + 4r^2 - 8k - 3 - 8r\right) u^{k+r} = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1} (k+1)\,(4k^2+1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = 0$

- Recursion relation for $r = 0$

  $a_{k+1} = 0$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = 0\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{1}{2}\right)^k, a_{k+1} = 0\right]$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve((2*x+1)^2*diff(y(x),x$3)+2*(2*x+1)*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + \frac{c_2(2x+1)\sin\left(-\frac{\ln(2)}{2} + \frac{\ln(2x+1)}{2}\right)}{2} + \frac{c_3(2x+1)\cos\left(-\frac{\ln(2)}{2} + \frac{\ln(2x+1)}{2}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 58

```
DSolve[(2*x+1)^2*y'''[x]+2*(2*x+1)*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}(2x+1)\left((2c_1 - c_2)\cos\left(\frac{1}{2}\log(2x+1)\right) + (c_1 + 2c_2)\sin\left(\frac{1}{2}\log(2x+1)\right)\right) + c_3$$

19.11 problem 628

Internal problem ID [15397]
Internal file name [`OUTPUT/15397_Wednesday_May_08_2024_03_57_52_PM_10192106/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 628.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + y = x(6 - \ln(x))}$$

19.11.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = -x\ln(x) + 6x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' + xy' + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 1 = 0$$

Or

$$r^2 + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -i \\ r_2 &= i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \\ &= c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta} \\ &= x^\alpha\left(c_1x^{i\beta} + c_2x^{-i\beta}\right) \\ &= x^\alpha\left(c_1e^{\ln\left(x^{i\beta}\right)} + c_2e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha\left(c_1e^{i(\beta\ln x)} + c_2e^{-i(\beta\ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -1$, the above becomes

$$y = x^0\left(c_1e^{-i\ln(x)} + c_2e^{i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Next, we find the particular solution to the ODE

$$x^2 y'' + x y' + y = -x \ln(x) + 6x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(\ln(x)) \\ y_2 &= -\sin(\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & -\sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(-\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & -\sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & -\frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(-\frac{\cos(\ln(x))}{x}\right) - (-\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = -\frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\sin(\ln(x))(-x\ln(x) + 6x)}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int -\sin(\ln(x))(\ln(x) - 6)\,dx$$

Hence

$$u_1 = \left(-\frac{\ln(x)}{2} + \frac{7}{2}\right)x\cos(\ln(x)) + \left(\frac{\ln(x)}{2} - 3\right)x\sin(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))(-x\ln(x) + 6x)}{-x}\,dx$$

Which simplifies to

$$u_2 = \int \cos(\ln(x))(\ln(x) - 6)\,dx$$

Hence

$$u_2 = \left(\frac{\ln(x)}{2} - 3\right)x\cos(\ln(x)) - \left(-\frac{\ln(x)}{2} + \frac{7}{2}\right)x\sin(\ln(x))$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -\frac{x((-7+\ln(x))\cos(\ln(x)) - \sin(\ln(x))(\ln(x)-6))}{2} \\
u_2 &= \frac{x(\cos(\ln(x))(\ln(x)-6) + \sin(\ln(x))(-7+\ln(x)))}{2}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = &-\frac{x((-7+\ln(x))\cos(\ln(x)) - \sin(\ln(x))(\ln(x)-6))\cos(\ln(x))}{2} \\
&- \frac{x(\cos(\ln(x))(\ln(x)-6) + \sin(\ln(x))(-7+\ln(x)))\sin(\ln(x))}{2}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -\frac{x(-7+\ln(x))}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= -\frac{x\ln(x)}{2} + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + \frac{7x}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x\ln(x)}{2} + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + \frac{7x}{2} \tag{1}
$$

Verification of solutions

$$
y = -\frac{x\ln(x)}{2} + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + \frac{7x}{2}
$$

Verified OK.

19.11.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' + y = 0$$

In normal form the ode

$$x^2y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln (x) \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= 1 \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda \tau} + \mathrm{e}^{\lambda \tau} = 0 \qquad (1)
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$
\lambda^2 + 1 = 0 \qquad (2)
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(1)}$$
$$= \pm i$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(\ln(x))$$
$$y_2 = \sin(\ln(x))$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(\frac{\cos(\ln(x))}{x}\right) - (\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(\ln(x))(-x\ln(x)+6x)}{x}\,dx$$

Which simplifies to

$$u_1 = -\int -\sin(\ln(x))(\ln(x)-6)\,dx$$

Hence

$$u_1 = \left(-\frac{\ln(x)}{2}+\frac{7}{2}\right)x\cos(\ln(x)) + \left(\frac{\ln(x)}{2}-3\right)x\sin(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))(-x\ln(x)+6x)}{x}\,dx$$

Which simplifies to

$$u_2 = \int -\cos(\ln(x))(\ln(x)-6)\,dx$$

Hence

$$u_2 = -\left(\frac{\ln(x)}{2}-3\right)x\cos(\ln(x)) + \left(-\frac{\ln(x)}{2}+\frac{7}{2}\right)x\sin(\ln(x))$$

Which simplifies to

$$u_1 = -\frac{x((-7+\ln(x))\cos(\ln(x)) - \sin(\ln(x))(\ln(x)-6))}{2}$$
$$u_2 = -\frac{x(\cos(\ln(x))(\ln(x)-6) + \sin(\ln(x))(-7+\ln(x)))}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x((-7+\ln(x))\cos(\ln(x)) - \sin(\ln(x))(\ln(x)-6))\cos(\ln(x))}{2} - \frac{x(\cos(\ln(x))(\ln(x)-6) + \sin(\ln(x))(-7+\ln(x)))\sin(\ln(x))}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x(-7+\ln(x))}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))) + \left(-\frac{x(-7+\ln(x))}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x(-7+\ln(x))}{2} + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}$$

Verification of solutions

$$y = -\frac{x(-7+\ln(x))}{2} + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Verified OK.

19.11.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = -x\ln(x) + 6x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' + y = 0$$

In normal form the ode

$$x^2y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' + y = -x\ln(x) + 6x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \cos(\ln(x)) \\ y_2 &= \sin(\ln(x))\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(\frac{\cos(\ln(x))}{x}\right) - (\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(\ln(x))(-x\ln(x) + 6x)}{x}\,dx$$

Which simplifies to

$$u_1 = -\int -\sin(\ln(x))(\ln(x) - 6)\,dx$$

Hence

$$u_1 = \left(-\frac{\ln(x)}{2} + \frac{7}{2}\right) x\cos(\ln(x)) + \left(\frac{\ln(x)}{2} - 3\right) x\sin(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))(-x\ln(x) + 6x)}{x}\,dx$$

Which simplifies to

$$u_2 = \int -\cos(\ln(x))(\ln(x) - 6)\,dx$$

Hence

$$u_2 = -\left(\frac{\ln(x)}{2} - 3\right) x\cos(\ln(x)) + \left(-\frac{\ln(x)}{2} + \frac{7}{2}\right) x\sin(\ln(x))$$

Which simplifies to

$$u_1 = -\frac{x((-7 + \ln(x))\cos(\ln(x)) - \sin(\ln(x))(\ln(x) - 6))}{2}$$
$$u_2 = -\frac{x(\cos(\ln(x))(\ln(x) - 6) + \sin(\ln(x))(-7 + \ln(x)))}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x((-7 + \ln(x))\cos(\ln(x)) - \sin(\ln(x))(\ln(x) - 6))\cos(\ln(x))}{2} - \frac{x(\cos(\ln(x))(\ln(x) - 6) + \sin(\ln(x))(-7 + \ln(x)))\sin(\ln(x))}{2}$$

Which simplifies to

$$y_p(x) = -\frac{x(-7 + \ln(x))}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(\ln(x)) + c_2\sin(\ln(x))) + \left(-\frac{x(-7 + \ln(x))}{2}\right) \\ &= -\frac{x(-7 + \ln(x))}{2} + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) \end{aligned}$$

Which simplifies to

$$y = -\frac{x\ln(x)}{2} + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) + \frac{7x}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{x \ln(x)}{2} + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) + \frac{7x}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x \ln(x)}{2} + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) + \frac{7x}{2}$$

Verified OK.

19.11.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = -x \ln(x) + 6x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' + y = 0$$

In normal form the ode

$$x^2y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = i \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2i}{x} + \frac{1}{x}\right) v'(x) = 0$$

$$v''(x) + \frac{(1+2i)\, v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+2i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{(-1-2i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-2i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-2i}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-2i}{x}\, dx \\ \ln(u) &= (-1-2i)\ln(x) + c_1 \\ u &= \mathrm{e}^{(-1-2i)\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{(-1-2i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{i c_1 x^{-2i}}{2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i \\
&= x^i c_2 + \frac{i x^{-i} c_1}{2}
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' + x y' + y = -x \ln(x) + 6x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x^i \\
y_2 &= x^{-i}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^i & x^{-i} \\ \frac{d}{dx}(x^i) & \frac{d}{dx}(x^{-i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^i & x^{-i} \\ \frac{i x^i}{x} & -\frac{i x^{-i}}{x} \end{vmatrix}$$

Therefore

$$W = (x^i)\left(-\frac{i x^{-i}}{x}\right) - (x^{-i})\left(\frac{i x^i}{x}\right)$$

Which simplifies to

$$W = -\frac{2i}{x}$$

Which simplifies to

$$W = -\frac{2i}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{-i}(-x\ln(x) + 6x)}{-2ix}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{i(\ln(x) - 6)\,x^{-i}}{2} dx$$

Hence

$$u_1 = \frac{(7-6i+(-1+i)\ln(x))\,x^{1-i}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^i(-x\ln(x)+6x)}{-2ix}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{ix^i(\ln(x)-6)}{2}dx$$

Hence

$$u_2 = -\frac{x^{1+i}(-7-6i+(1+i)\ln(x))}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(7-6i+(-1+i)\ln(x))\,x^{1-i}x^i}{4} - \frac{x^{1+i}(-7-6i+(1+i)\ln(x))\,x^{-i}}{4}$$

Which simplifies to

$$y_p(x) = -\frac{x(-7+\ln(x))}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{ic_1x^{-2i}}{2}+c_2\right)x^i\right) + \left(-\frac{x(-7+\ln(x))}{2}\right) \\
&= -\frac{x(-7+\ln(x))}{2} + \left(\frac{ic_1x^{-2i}}{2}+c_2\right)x^i
\end{aligned}$$

Which simplifies to

$$y = \frac{ix^{-i}c_1}{2} + x^ic_2 - \frac{x(-7+\ln(x))}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{i x^{-i} c_1}{2} + x^i c_2 - \frac{x(-7 + \ln(x))}{2} \tag{1}$$

Verification of solutions

$$y = \frac{i x^{-i} c_1}{2} + x^i c_2 - \frac{x(-7 + \ln(x))}{2}$$

Verified OK.

19.11.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + x y' + y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -5 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 630: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-}=\frac{1}{2}-i$ then

$$\begin{aligned}
d&=\alpha_\infty^{-}-\left(\alpha_{c_1}^{-}\right)\\
&=\frac{1}{2}-i-\left(\frac{1}{2}-i\right)\\
&=0
\end{aligned}$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega&=\left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^{-}}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&=\frac{\frac{1}{2}-i}{x}+(-)\,(0)\\
&=\frac{\frac{1}{2}-i}{x}\\
&=\frac{\frac{1}{2}-i}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-i}{x}\right)(0)+\left(\left(\frac{-\frac{1}{2}+i}{x^2}\right)+\left(\frac{\frac{1}{2}-i}{x}\right)^2-\left(-\frac{5}{4x^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2}-i}{x}dx} \\
&= x^{\frac{1}{2}-i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i x^{2i}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-i}\right) + c_2 \left(x^{-i} \left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + x y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = x^{-i} c_1 - \frac{i c_2 x^i}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^{-i} \\
y_2 &= -\frac{i x^i}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-i} & -\frac{i x^{i}}{2} \\ \frac{d}{dx}(x^{-i}) & \frac{d}{dx}\left(-\frac{i x^{i}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-i} & -\frac{i x^{i}}{2} \\ -\frac{i x^{-i}}{x} & \frac{x^{i}}{2x} \end{vmatrix}$$

Therefore

$$W = (x^{-i})\left(\frac{x^{i}}{2x}\right) - \left(-\frac{i x^{i}}{2}\right)\left(-\frac{i x^{-i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{-i} x^{i}}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i x^{i}(-x\ln(x)+6x)}{2}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{i x^{i}(\ln(x) - 6)}{2} dx$$

Hence

$$u_1 = -\frac{x^{1+i}(-7 - 6i + (1+i)\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{-i}(-x\ln(x) + 6x)}{x}\, dx$$

Which simplifies to

$$u_2 = \int (6 - \ln(x))\, x^{-i} dx$$

Hence

$$u_2 = -\frac{(-6 - 7i + (1 + i)\ln(x))\, x^{1-i}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{1+i}(-7 - 6i + (1 + i)\ln(x))\, x^{-i}}{4} + \frac{i(-6 - 7i + (1 + i)\ln(x))\, x^{1-i} x^{i}}{4}$$

Which simplifies to

$$y_p(x) = -\frac{x(-7 + \ln(x))}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{-i} c_1 - \frac{i c_2 x^{i}}{2}\right) + \left(-\frac{x(-7 + \ln(x))}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{-i} c_1 - \frac{i c_2 x^{i}}{2} - \frac{x(-7 + \ln(x))}{2} \tag{1}$$

Verification of solutions

$$y = x^{-i} c_1 - \frac{i c_2 x^{i}}{2} - \frac{x(-7 + \ln(x))}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=x*(6-ln(x)),y(x), singsol=all)
```

$$y(x) = \sin(\ln(x))\, c_2 + \cos(\ln(x))\, c_1 - \frac{x(\ln(x) - 7)}{2}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]+x*y'[x]+y[x]==x*(6-Log[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}x(\log(x) - 7) + c_1 \cos(\log(x)) + c_2 \sin(\log(x))$$

19.12 problem 629

Internal problem ID [15398]
Internal file name [`OUTPUT/15398_Wednesday_May_08_2024_03_57_54_PM_80775769/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 629.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 2y = \sin(\ln(x))}$$

19.12.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 0, C = -2, f(x) = \sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 2y = \sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(x^2\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(2x) - \left(x^2\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 3$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 \sin(\ln(x))}{3x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(\ln(x))}{3} dx$$

Hence

$$u_1 = \frac{x\cos(\ln(x))}{6} - \frac{x\sin(\ln(x))}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sin(\ln(x))}{x}}{3x^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sin(\ln(x))}{3x^3}dx$$

Hence

$$u_2 = \frac{-\frac{1}{15} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{15} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{15}}{x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)}$$

Which simplifies to

$$u_1 = -\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{6}$$

$$u_2 = \frac{-\cos(\ln(x)) - 2\sin(\ln(x))}{15x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10} + \frac{c_1}{x} + c_2x^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10} + \frac{c_1}{x} + c_2x^2 \tag{1}$$

Verification of solutions

$$y = \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10} + \frac{c_1}{x} + c_2x^2$$

Verified OK.

19.12.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - 2y)\,dx = \int \sin(\ln(x))\,dx$$

$$x^2y' - 2yx = -\frac{x\cos(\ln(x))}{2} + \frac{x\sin(\ln(x))}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}\right)$$
$$d\left(\frac{y}{x^2}\right) = \left(\frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^4}\right)dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{x \sin(\ln(x)) - x \cos(\ln(x)) + 2c_1}{2x^4} \mathrm{d}x$$

$$\frac{y}{x^2} = -\frac{c_1}{3x^3} - \frac{-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{3x^3} - \frac{-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)}\right) + c_2x^2$$

which simplifies to

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2x^3}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2x^3}{x} \tag{1}$$

Verification of solutions

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2x^3}{x}$$

Verified OK.

19.12.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' - 2y = \sin(\ln(x))$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - 2y)\, dx = \int \sin(\ln(x))\, dx$$

$$x^2y' - 2yx = -\frac{x\cos(\ln(x))}{2} + \frac{x\sin(\ln(x))}{2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^2}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^4}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{x\sin(\ln(x)) - x\cos(\ln(x)) + 2c_1}{2x^4}\,\mathrm{d}x \\ \frac{y}{x^2} &= -\frac{c_1}{3x^3} - \frac{-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{3x^3} - \frac{-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)}\right) + c_2x^2$$

which simplifies to

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2 x^3}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2 x^3}{x} \tag{1}$$

Verification of solutions

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2 x^3}{x}$$

Verified OK.

19.12.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 631: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x} + \frac{c_2 x^2}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= \frac{x^2}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ -\frac{1}{x^2} & \frac{2x}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{2x}{3}\right) - \left(\frac{x^2}{3}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2 \sin(\ln(x))}{3}}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(\ln(x))}{3} dx$$

Hence

$$u_1 = \frac{x\cos(\ln(x))}{6} - \frac{x\sin(\ln(x))}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sin(\ln(x))}{x}}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sin(\ln(x))}{x^3} dx$$

Hence

$$u_2 = \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)}$$

Which simplifies to

$$u_1 = -\frac{x(-\cos(\ln(x)) + \sin(\ln(x)))}{6}$$
$$u_2 = \frac{-\cos(\ln(x)) - 2\sin(\ln(x))}{5x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x^2}{3}\right) + \left(\frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x^2}{3} + \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x^2}{3} + \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10}$$

Verified OK.

19.12.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= 0 \\ r(x) &= -2 \\ s(x) &= \sin(\ln(x)) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$2 - (0) + (-2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' - 2yx = \int \sin(\ln(x))\;dx$$

We now have a first order ode to solve which is

$$x^2y' - 2yx = -\frac{x\cos(\ln(x))}{2} + \frac{x\sin(\ln(x))}{2} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{c_1 + \frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}}{x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{c_1 + \frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1 + \frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{c_1 + \frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}}{x^2}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{c_1 + \frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}}{x^4}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{c_1 + \frac{x(-\cos(\ln(x))+\sin(\ln(x)))}{2}}{x^4} dx \\ \frac{y}{x^2} &= -\frac{c_1}{3x^3} - \frac{-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{c_1}{3x^3} - \frac{-\frac{2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)^2}{5} + \frac{2\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)} + \frac{-\frac{1}{5} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{5} - \frac{4\tan\left(\frac{\ln(x)}{2}\right)}{5}}{2x^2\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right)}\right) + c_2x^2$$

which simplifies to

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2 x^3}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2 x^3}{x} \tag{1}$$

Verification of solutions

$$y = \frac{-\frac{(3\sin(\ln(x))-\cos(\ln(x)))x}{10} - \frac{c_1}{3} + c_2 x^3}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(x^2*diff(y(x),x$2)-2*y(x)=sin(ln(x)),y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + c_2 x^2 + \frac{\cos(\ln(x))}{10} - \frac{3\sin(\ln(x))}{10}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 31

```
DSolve[x^2*y''[x]-2*y[x]==Sin[Log[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x^2 + \frac{c_1}{x} + \frac{1}{10}(\cos(\log(x)) - 3\sin(\log(x)))$$

19.13 problem 630

Internal problem ID [15399]
Internal file name [`OUTPUT/15399_Wednesday_May_08_2024_03_57_56_PM_900528/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 630.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$x^2y'' - xy' - 3y = -\frac{16\ln(x)}{x}$$

19.13.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = -3, f(x) = -\frac{16\ln(x)}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' - 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} - 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r - 3 = 0$$

Or

$$r^2 - 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x^3$$

Next, we find the particular solution to the ODE

$$x^2y'' - xy' - 3y = -\frac{16\ln(x)}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ -\frac{1}{x^2} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(3x^2\right) - \left(x^3\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-16x^2 \ln(x)}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{4\ln(x)}{x} dx$$

Hence

$$u_1 = 2\ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{16\ln(x)}{x^2}}{4x^3}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{4\ln(x)}{x^5} dx$$

Hence

$$u_2 = \frac{\ln(x)}{x^4} + \frac{1}{4x^4}$$

Which simplifies to

$$u_1 = 2\ln(x)^2$$
$$u_2 = \frac{\frac{1}{4} + \ln(x)}{x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\ln(x)^2}{x} + \frac{\frac{1}{4} + \ln(x)}{x}$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) + 4c_1 + 1}{4x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) + 4c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) + 4c_1 + 1}{4x}$$

Verified OK.

19.13.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' - 3y = 0$$

In normal form the ode

$$x^2y'' - xy' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{3}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\, \tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x} dx\right)} dx \\
&= \int e^{\ln(x)}\, dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{x^2}}{x^2} \\
&= -\frac{3}{x^4}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{x^4} = 0$$

But in terms of τ

$$-\frac{3}{x^4} = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,(c_2x^4 + 4c_1)}{4x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ -\frac{1}{x^2} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(3x^2) - (x^3)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-16x^2 \ln(x)}{4x^3} dx$$

Which simplifies to

$$u_1 = -\int -\frac{4\ln(x)}{x} dx$$

Hence

$$u_1 = 2\ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{16\ln(x)}{x^2}}{4x^3} dx$$

Which simplifies to

$$u_2 = \int -\frac{4\ln(x)}{x^5} dx$$

Hence

$$u_2 = \frac{\ln(x)}{x^4} + \frac{1}{4x^4}$$

Which simplifies to

$$u_1 = 2\ln(x)^2$$

$$u_2 = \frac{\frac{1}{4}+\ln(x)}{x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\ln(x)^2}{x} + \frac{\frac{1}{4}+\ln(x)}{x}$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)^2+\frac{1}{4}+\ln(x)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,(c_2x^4+4c_1)}{4x}\right) + \left(\frac{2\ln(x)^2+\frac{1}{4}+\ln(x)}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,(c_2x^4+4c_1)}{4x} + \frac{2\ln(x)^2+\frac{1}{4}+\ln(x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,(c_2x^4+4c_1)}{4x} + \frac{2\ln(x)^2+\frac{1}{4}+\ln(x)}{x}$$

Verified OK.

19.13.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = -3, f(x) = -\frac{16\ln(x)}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' - 3y = 0$$

In normal form the ode

$$x^2y'' - xy' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -\frac{3}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} - \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{5v'(x)}{x} = 0$$
$$v''(x) + \frac{5v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\ \ln(u) &= -5\ln(x) + c_1 \\ u &= \mathrm{e}^{-5\ln(x)+c_1} \\ &= \frac{c_1}{x^5} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{4x^4} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^3 \\
&= \frac{4c_2x^4 - c_1}{4x}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2y'' - xy' - 3y = -\frac{16 \ln(x)}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x} \\
y_2 &= x^3
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^3) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^3 \\ -\frac{1}{x^2} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(3x^2) - (x^3)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 4x$$

Which simplifies to

$$W = 4x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-16x^2 \ln(x)}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{4\ln(x)}{x} dx$$

Hence

$$u_1 = 2\ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{16\ln(x)}{x^2}}{4x^3}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{4\ln(x)}{x^5} dx$$

Hence

$$u_2 = \frac{\ln(x)}{x^4} + \frac{1}{4x^4}$$

Which simplifies to

$$u_1 = 2\ln(x)^2$$

$$u_2 = \frac{\frac{1}{4} + \ln(x)}{x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\ln(x)^2}{x} + \frac{\frac{1}{4} + \ln(x)}{x}$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right)x^3\right) + \left(\frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x}\right) \\ &= \frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x} + \left(-\frac{c_1}{4x^4} + c_2\right)x^3 \end{aligned}$$

Which simplifies to

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Verified OK.

19.13.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - xy' - 3y)\, dx = \int -\frac{16\ln(x)}{x} dx$$
$$x^2y' - 3yx = -8\ln(x)^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{-8\ln(x)^2 + c_1}{x^2}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{-8\ln(x)^2 + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x}dx}$$
$$= \frac{1}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{-8\ln(x)^2 + c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \left(\frac{1}{x^3}\right)\left(\frac{-8\ln(x)^2 + c_1}{x^2}\right)$$
$$d\left(\frac{y}{x^3}\right) = \left(\frac{-8\ln(x)^2 + c_1}{x^5}\right) dx$$

Integrating gives

$$\frac{y}{x^3} = \int \frac{-8\ln(x)^2 + c_1}{x^5} dx$$
$$\frac{y}{x^3} = \frac{2\ln(x)^2}{x^4} + \frac{\ln(x)}{x^4} + \frac{1}{4x^4} - \frac{c_1}{4x^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3 \left(\frac{2\ln(x)^2}{x^4} + \frac{\ln(x)}{x^4} + \frac{1}{4x^4} - \frac{c_1}{4x^4}\right) + c_2 x^3$$

which simplifies to

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Verified OK.

19.13.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' - xy' - 3y = -\frac{16\ln(x)}{x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - xy' - 3y)\, dx = \int -\frac{16\ln(x)}{x} dx$$
$$x^2y' - 3yx = -8\ln(x)^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{-8\ln(x)^2 + c_1}{x^2}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{-8 \ln(x)^2 + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{-8\ln(x)^2 + c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(\frac{-8\ln(x)^2 + c_1}{x^2}\right) \\ d\left(\frac{y}{x^3}\right) &= \left(\frac{-8\ln(x)^2 + c_1}{x^5}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^3} &= \int \frac{-8\ln(x)^2 + c_1}{x^5} dx \\ \frac{y}{x^3} &= \frac{2\ln(x)^2}{x^4} + \frac{\ln(x)}{x^4} + \frac{1}{4x^4} - \frac{c_1}{4x^4} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3\left(\frac{2\ln(x)^2}{x^4} + \frac{\ln(x)}{x^4} + \frac{1}{4x^4} - \frac{c_1}{4x^4}\right) + c_2 x^3$$

which simplifies to

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Verified OK.

19.13.6 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x^2y'' - xy' - 3y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x^2\\
B &= -x \tag{3}\\
C &= -3
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{15}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 15\\
t &= 4x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 632: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{2x} + (-)\,(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - xy' - 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 x^3}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x^3}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^3}{4} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^3}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^3}{4} \\ -\frac{1}{x^2} & \frac{3x^2}{4} \end{vmatrix}$$

Therefore

$$W=\left(\frac{1}{x}\right)\left(\frac{3x^2}{4}\right)-\left(\frac{x^3}{4}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W=x$$

Which simplifies to

$$W=x$$

Therefore Eq. (2) becomes

$$u_1=-\int\frac{-4x^2\ln(x)}{x^3}\,dx$$

Which simplifies to

$$u_1=-\int-\frac{4\ln(x)}{x}dx$$

Hence

$$u_1=2\ln(x)^2$$

And Eq. (3) becomes

$$u_2=\int\frac{-\frac{16\ln(x)}{x^2}}{x^3}\,dx$$

Which simplifies to

$$u_2=\int-\frac{16\ln(x)}{x^5}dx$$

Hence

$$u_2=\frac{4\ln(x)}{x^4}+\frac{1}{x^4}$$

Which simplifies to

$$u_1=2\ln(x)^2$$

$$u_2=\frac{1+4\ln(x)}{x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\ln(x)^2}{x} + \frac{1+4\ln(x)}{4x}$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x^3}{4}\right) + \left(\frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x^3}{4} + \frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x^3}{4} + \frac{2\ln(x)^2 + \frac{1}{4} + \ln(x)}{x}$$

Verified OK.

19.13.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= -x \\ r(x) &= -3 \\ s(x) &= -\frac{16\ln(x)}{x} \end{aligned}$$

Hence

$$p''(x) = 2$$
$$q'(x) = -1$$

Therefore (1) becomes

$$2 - (-1) + (-3) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x^2 y' - 3yx = \int -\frac{16\ln(x)}{x}\, dx$$

We now have a first order ode to solve which is

$$x^2 y' - 3yx = -8\ln(x)^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{-8\ln(x)^2 + c_1}{x^2}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{-8\ln(x)^2 + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x} dx}$$
$$= \frac{1}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{-8\ln(x)^2 + c_1}{x^2}\right)$$

$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \left(\frac{1}{x^3}\right)\left(\frac{-8\ln(x)^2 + c_1}{x^2}\right)$$

$$d\left(\frac{y}{x^3}\right) = \left(\frac{-8\ln(x)^2 + c_1}{x^5}\right) dx$$

Integrating gives

$$\frac{y}{x^3} = \int \frac{-8\ln(x)^2 + c_1}{x^5} dx$$

$$\frac{y}{x^3} = \frac{2\ln(x)^2}{x^4} + \frac{\ln(x)}{x^4} + \frac{1}{4x^4} - \frac{c_1}{4x^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3\left(\frac{2\ln(x)^2}{x^4} + \frac{\ln(x)}{x^4} + \frac{1}{4x^4} - \frac{c_1}{4x^4}\right) + c_2 x^3$$

which simplifies to

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) - c_1 + 1}{4x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=-16*ln(x)/x,y(x), singsol=all)
```

$$y(x) = \frac{4c_2x^4 + 8\ln(x)^2 + 4\ln(x) + 4c_1 + 1}{4x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 35

```
DSolve[x^2*y''[x]-x*y'[x]-3*y[x]==-16*Log[x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_2x^4 + 8\log^2(x) + 4\log(x) + 1 + 4c_1}{4x}$$

19.14 problem 631

Internal problem ID [15400]
Internal file name [`OUTPUT/15400_Wednesday_May_08_2024_03_57_59_PM_52426067/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 631.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' - 2y = x^2 - 2x + 2}$$

19.14.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = -2, f(x) = x^2 - 2x + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' - 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r - 2 = 0$$

Or

$$r^2 - 3r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{3}{2} - \frac{\sqrt{17}}{2}$$
$$r_2 = \frac{3}{2} + \frac{\sqrt{17}}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^{\frac{3}{2} - \frac{\sqrt{17}}{2}} + c_2 x^{\frac{3}{2} + \frac{\sqrt{17}}{2}}$$

Next, we find the particular solution to the ODE

$$x^2 y'' - 2xy' - 2y = x^2 - 2x + 2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{\frac{3}{2} - \frac{\sqrt{17}}{2}}$$

$$y_2 = x^{\frac{3}{2} + \frac{\sqrt{17}}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} & x^{\frac{3}{2}+\frac{\sqrt{17}}{2}} \\ \frac{d}{dx}\left(x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\right) & \frac{d}{dx}\left(x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} & x^{\frac{3}{2}+\frac{\sqrt{17}}{2}} \\ \frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)}{x} & \frac{x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)}{x} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\right)\left(\frac{x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)}{x}\right) - \left(x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\right)\left(\frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)}{x}\right)$$

Which simplifies to

$$W = \frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} x^{\frac{3}{2}+\frac{\sqrt{17}}{2}} \sqrt{17}}{x}$$

Which simplifies to

$$W = x^2\sqrt{17}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}(x^2-2x+2)}{x^4\sqrt{17}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{-\frac{5}{2}+\frac{\sqrt{17}}{2}} \sqrt{17}\,(x^2-2x+2)}{17} dx$$

Hence

$$u_1 = -\frac{x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}} \sqrt{17}\left(x^2\sqrt{17}-2\sqrt{17}\,x-x^2+4\sqrt{17}-2x+12\right)}{136}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}(x^2-2x+2)}{x^4\sqrt{17}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^{-\frac{5}{2}-\frac{\sqrt{17}}{2}} \sqrt{17}\,(x^2-2x+2)}{17} dx$$

Hence

$$u_2 = -\frac{x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}} \sqrt{17}\left(x^2\sqrt{17}-2\sqrt{17}\,x+x^2+4\sqrt{17}+2x-12\right)}{136}$$

Which simplifies to

$$u_1 = -\frac{\left((x^2-2x+4)\sqrt{17}-x^2-2x+12\right)\sqrt{17}\,x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}}{136}$$

$$u_2 = -\frac{\left((x^2-2x+4)\sqrt{17}+x^2+2x-12\right)\sqrt{17}\,x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}}{136}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left((x^2-2x+4)\sqrt{17}-x^2-2x+12\right)\sqrt{17}\,x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}\,x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}}{136} - \frac{\left((x^2-2x+4)\sqrt{17}+x^2+2x-12\right)\sqrt{17}\,x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}\,x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{136}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x^2+\frac{1}{2}x-1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= -\frac{x^2}{4} + \frac{x}{2} - 1 + c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + c_2 x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -\frac{x^2}{4} + \frac{x}{2} - 1 + c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + c_2 x^{\frac{3}{2}+\frac{\sqrt{17}}{2}} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = -\frac{x^2}{4} + \frac{x}{2} - 1 + c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + c_2 x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}
$$

Verified OK.

19.14.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 2xy' - 2y = 0
$$

In normal form the ode

$$
x^2y'' - 2xy' - 2y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= -\frac{2}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{2}{x}dx\right)} dx \\
&= \int e^{2\ln(x)}\, dx \\
&= \int x^2 dx \\
&= \frac{x^3}{3}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{x^2}}{x^4} \\
&= -\frac{2}{x^6}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{x^6} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{2}{x^6} = -\frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 - 2 = 0$$

Or

$$9r^2 - 9r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{\sqrt{17}}{6}$$
$$r_2 = \frac{1}{2} + \frac{\sqrt{17}}{6}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{2}-\frac{\sqrt{17}}{6}} + c_2\tau^{\frac{1}{2}+\frac{\sqrt{17}}{6}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{3}\,\sqrt{x^3}\left(c_1 3^{\frac{\sqrt{17}}{6}} \left(x^3\right)^{-\frac{\sqrt{17}}{6}} + c_2 3^{-\frac{\sqrt{17}}{6}} \left(x^3\right)^{\frac{\sqrt{17}}{6}}\right)}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{3}\,\sqrt{x^3}\left(c_1 3^{\frac{\sqrt{17}}{6}} \left(x^3\right)^{-\frac{\sqrt{17}}{6}} + c_2 3^{-\frac{\sqrt{17}}{6}} \left(x^3\right)^{\frac{\sqrt{17}}{6}}\right)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}}$$

$$y_2 = \sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}} & \sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}} \\ \frac{d}{dx}\left(\sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}}\right) & \frac{d}{dx}\left(\sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}} & \sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}} \\ \frac{3(x^3)^{-\frac{\sqrt{17}}{6}} x^2}{2\sqrt{x^3}} - \frac{\sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}}\sqrt{17}}{2x} & \frac{3(x^3)^{\frac{\sqrt{17}}{6}} x^2}{2\sqrt{x^3}} + \frac{\sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}}\sqrt{17}}{2x} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}}\right)\left(\frac{3(x^3)^{\frac{\sqrt{17}}{6}} x^2}{2\sqrt{x^3}} + \frac{\sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}}\sqrt{17}}{2x}\right) - \left(\sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}}\right)\left(\frac{3(x^3)^{-\frac{\sqrt{17}}{6}} x^2}{2\sqrt{x^3}} - \frac{\sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}}\sqrt{17}}{2x}\right)$$

Which simplifies to

$$W = x^2\left(x^3\right)^{-\frac{\sqrt{17}}{6}}\left(x^3\right)^{\frac{\sqrt{17}}{6}}\sqrt{17}$$

Which simplifies to

$$W = x^2\sqrt{17}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sqrt{x^3}\left(x^3\right)^{\frac{\sqrt{17}}{6}}\left(x^2 - 2x + 2\right)}{x^4\sqrt{17}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{17}\left(x^3\right)^{\frac{1}{2}+\frac{\sqrt{17}}{6}}\left(x^2 - 2x + 2\right)}{17x^4} dx$$

Hence

$$u_1 = -\frac{\left(x^2\sqrt{17} - 2\sqrt{17}\,x - x^2 + 4\sqrt{17} - 2x + 12\right)\sqrt{17}\left(x^3\right)^{\frac{1}{2}+\frac{\sqrt{17}}{6}}}{136x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^3}\left(x^3\right)^{-\frac{\sqrt{17}}{6}}\left(x^2 - 2x + 2\right)}{x^4\sqrt{17}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{17}\,(x^3)^{\frac{1}{2}-\frac{\sqrt{17}}{6}}\,(x^2-2x+2)}{17x^4}dx$$

Hence

$$u_2 = -\frac{\left(x^2\sqrt{17}-2\sqrt{17}\,x+x^2+4\sqrt{17}+2x-12\right)\sqrt{17}\,(x^3)^{\frac{1}{2}-\frac{\sqrt{17}}{6}}}{136x^3}$$

Which simplifies to

$$u_1 = -\frac{\left((x^2-2x+4)\sqrt{17}-x^2-2x+12\right)\sqrt{17}\,(x^3)^{\frac{1}{2}+\frac{\sqrt{17}}{6}}}{136x^3}$$

$$u_2 = -\frac{\left((x^2-2x+4)\sqrt{17}+x^2+2x-12\right)\sqrt{17}\,(x^3)^{\frac{1}{2}-\frac{\sqrt{17}}{6}}}{136x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left((x^2-2x+4)\sqrt{17}-x^2-2x+12\right)\sqrt{17}\,(x^3)^{\frac{1}{2}+\frac{\sqrt{17}}{6}}\sqrt{x^3}\,(x^3)^{-\frac{\sqrt{17}}{6}}}{136x^3} - \frac{\left((x^2-2x+4)\sqrt{17}+x^2+2x-12\right)\sqrt{17}\,(x^3)^{\frac{1}{2}-\frac{\sqrt{17}}{6}}\sqrt{x^3}\,(x^3)^{\frac{\sqrt{17}}{6}}}{136x^3}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x^2+\frac{1}{2}x-1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{3}\,\sqrt{x^3}\left(c_1 3^{\frac{\sqrt{17}}{6}}(x^3)^{-\frac{\sqrt{17}}{6}}+c_2 3^{-\frac{\sqrt{17}}{6}}(x^3)^{\frac{\sqrt{17}}{6}}\right)}{3}\right)+\left(-\frac{1}{4}x^2+\frac{1}{2}x-1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\sqrt{x^3}\left(c_1 3^{\frac{\sqrt{17}}{6}}(x^3)^{-\frac{\sqrt{17}}{6}}+c_2 3^{-\frac{\sqrt{17}}{6}}(x^3)^{\frac{\sqrt{17}}{6}}\right)}{3}-\frac{x^2}{4}+\frac{x}{2}-1 \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{3}\,\sqrt{x^3}\left(c_1 3^{\frac{\sqrt{17}}{6}} (x^3)^{-\frac{\sqrt{17}}{6}} + c_2 3^{-\frac{\sqrt{17}}{6}} (x^3)^{\frac{\sqrt{17}}{6}}\right)}{3} - \frac{x^2}{4} + \frac{x}{2} - 1$$

Verified OK.

19.14.3 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = -2, f(x) = x^2 - 2x + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' - 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -\frac{2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{2n}{x^2}-\frac{2}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=\frac{3}{2}+\frac{\sqrt{17}}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{3+\sqrt{17}}{x}-\frac{2}{x}\right)v'(x)&=0\\ v''(x)+\frac{\left(1+\sqrt{17}\right)v'(x)}{x}&=0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{\left(1+\sqrt{17}\right)u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u'&=F(x,u)\\ &=f(x)g(u)\\ &=\frac{\left(-1-\sqrt{17}\right)u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-\sqrt{17}}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-\sqrt{17}}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-\sqrt{17}}{x}\,dx \\
\ln(u) &= \left(-1-\sqrt{17}\right)\ln(x) + c_1 \\
u &= \mathrm{e}^{\left(-1-\sqrt{17}\right)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{\left(-1-\sqrt{17}\right)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-\sqrt{17}}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2\right) x^{\frac{3}{2}+\frac{\sqrt{17}}{2}} \\
&= -\frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(-17c_2 x^{\sqrt{17}} + \sqrt{17}\,c_1\right)}{17}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 2xy' - 2y = x^2 - 2x + 2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}}$$

$$y_2 = x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} & x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}} \\ \frac{d}{dx}\left(x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}}\right) & \frac{d}{dx}\left(x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} & x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}} \\ \frac{3\sqrt{x}\, x^{\frac{\sqrt{17}}{2}}}{2} + \frac{\sqrt{x}\, x^{\frac{\sqrt{17}}{2}} \sqrt{17}}{2} & \frac{3\sqrt{x}\, x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}}}{2} - \frac{\sqrt{x}\, x^{\frac{\sqrt{17}}{2}} \sqrt{17}\, x^{-\sqrt{17}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}}\right) \left(\frac{3\sqrt{x}\, x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}}}{2} - \frac{\sqrt{x}\, x^{\frac{\sqrt{17}}{2}} \sqrt{17}\, x^{-\sqrt{17}}}{2}\right) - \left(x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}}\right) \left(\frac{3\sqrt{x}\, x^{\frac{\sqrt{17}}{2}}}{2} + \frac{\sqrt{x}\, x^{\frac{\sqrt{17}}{2}} \sqrt{17}}{2}\right)$$

Which simplifies to

$$W = -x^2 x^{\sqrt{17}} \sqrt{17}\, x^{-\sqrt{17}}$$

Which simplifies to

$$W = -x^2 \sqrt{17}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} x^{-\sqrt{17}} (x^2 - 2x + 2)}{-x^4 \sqrt{17}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{-\frac{5}{2}-\frac{\sqrt{17}}{2}} \sqrt{17}\, (x^2 - 2x + 2)}{17} dx$$

Hence

$$u_1 = -\frac{x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}} \sqrt{17}\, \left(x^2\sqrt{17} - 2\sqrt{17}\, x + x^2 + 4\sqrt{17} + 2x - 12\right)}{136}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{3}{2}} x^{\frac{\sqrt{17}}{2}} (x^2 - 2x + 2)}{-x^4 \sqrt{17}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{x^{-\frac{5}{2}+\frac{\sqrt{17}}{2}} \sqrt{17}\, (x^2 - 2x + 2)}{17} dx$$

Hence

$$u_2 = -\frac{x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}} \sqrt{17}\, \left(x^2\sqrt{17} - 2\sqrt{17}\, x - x^2 + 4\sqrt{17} - 2x + 12\right)}{136}$$

Which simplifies to

$$u_1 = -\frac{\left((x^2 - 2x + 4)\sqrt{17} + x^2 + 2x - 12\right)\sqrt{17}\, x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}}{136}$$

$$u_2 = -\frac{\left((x^2 - 2x + 4)\sqrt{17} - x^2 - 2x + 12\right)\sqrt{17}\, x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}}{136}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\left(x^2-2x+4\right)\sqrt{17}+x^2+2x-12\right)\sqrt{17}\,x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}x^{\frac{3}{2}}x^{\frac{\sqrt{17}}{2}}}{136} - \frac{\left(\left(x^2-2x+4\right)\sqrt{17}-x^2-2x+12\right)\sqrt{17}\,x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}x^{\frac{3}{2}}x^{-\frac{\sqrt{17}}{2}}x^{-\sqrt{17}}}{136}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x^2 + \frac{1}{2}x - 1$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2\right)x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\right) + \left(-\frac{1}{4}x^2 + \frac{1}{2}x - 1\right) \\
&= -\frac{x^2}{4} + \frac{x}{2} - 1 + \left(-\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2\right)x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}
\end{aligned}$$

Which simplifies to

$$y = -\frac{x^2}{4} + \frac{x}{2} - 1 + \left(-\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2\right)x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{4} + \frac{x}{2} - 1 + \left(-\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2\right)x^{\frac{3}{2}+\frac{\sqrt{17}}{2}} \tag{1}$$

Verification of solutions

$$y = -\frac{x^2}{4} + \frac{x}{2} - 1 + \left(-\frac{\sqrt{17}\,c_1 x^{-\sqrt{17}}}{17} + c_2\right)x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}$$

Verified OK.

19.14.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2xy' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 633: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{4}{x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 4$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{17}}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{17}}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{4}{x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 4$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{17}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{17}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{\sqrt{17}}{2}$	$\frac{1}{2} - \frac{\sqrt{17}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{\sqrt{17}}{2}$	$\frac{1}{2} - \frac{\sqrt{17}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - \frac{\sqrt{17}}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \frac{\sqrt{17}}{2} - \left(\frac{1}{2} - \frac{\sqrt{17}}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{x} + (-)\,(0) \\ &= \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{x} \\ &= -\frac{\sqrt{17} - 1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{x}\right)^2 - \left(\frac{4}{x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - \frac{\sqrt{17}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{\sqrt{17}}{2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^{\sqrt{17}}\sqrt{17}}{17}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\right) + c_2\left(x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(\frac{x^{\sqrt{17}}\sqrt{17}}{17}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''-2xy'-2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + \frac{c_2\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}$$

$$y_2 = \frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} & \frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17} \\ \frac{d}{dx}\left(x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\right) & \frac{d}{dx}\left(\frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} & \frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17} \\ \frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)}{x} & \frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)}{17x} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\right)\left(\frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}\left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)}{17x}\right) - \left(\frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17}\right)\left(\frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(\frac{3}{2}-\frac{\sqrt{17}}{2}\right)}{x}\right)$$

Which simplifies to

$$W = \frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{x}$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{17}\, x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}(x^2-2x+2)}{17}}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{-\frac{5}{2}+\frac{\sqrt{17}}{2}}\sqrt{17}\,(x^2-2x+2)}{17} dx$$

Hence

$$u_1 = -\frac{x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}\sqrt{17}\,\left(x^2\sqrt{17} - 2\sqrt{17}\, x - x^2 + 4\sqrt{17} - 2x + 12\right)}{136}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}(x^2-2x+2)}{x^4}\, dx$$

Which simplifies to

$$u_2 = \int x^{-\frac{5}{2}-\frac{\sqrt{17}}{2}}\left(x^2-2x+2\right) dx$$

Hence

$$u_2 = -\frac{x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}\left(x^2\sqrt{17} - 2\sqrt{17}\,x + x^2 + 4\sqrt{17} + 2x - 12\right)}{8}$$

Which simplifies to

$$u_1 = -\frac{\left(\left(x^2-2x+4\right)\sqrt{17} - x^2 - 2x + 12\right)\sqrt{17}\,x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}}{136}$$

$$u_2 = -\frac{\left(\left(x^2-2x+4\right)\sqrt{17} + x^2 + 2x - 12\right)x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\left(x^2-2x+4\right)\sqrt{17} - x^2 - 2x + 12\right)\sqrt{17}\,x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}}\,x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}}{136} - \frac{\left(\left(x^2-2x+4\right)\sqrt{17} + x^2 + 2x - 12\right)\sqrt{17}\,x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}\,x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{136}$$

Which simplifies to

$$y_p(x) = -\frac{1}{4}x^2 + \frac{1}{2}x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + \frac{c_2\sqrt{17}\,x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17}\right) + \left(-\frac{1}{4}x^2 + \frac{1}{2}x - 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + \frac{c_2\sqrt{17}\,x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17} - \frac{x^2}{4} + \frac{x}{2} - 1 \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}-\frac{\sqrt{17}}{2}} + \frac{c_2\sqrt{17}\,x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}}{17} - \frac{x^2}{4} + \frac{x}{2} - 1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)-2*y(x)=x^2-2*x+2,y(x), singsol=all)
```

$$y(x)=x^{\frac{3}{2}+\frac{\sqrt{17}}{2}}c_2+x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}c_1-\frac{x^2}{4}+\frac{x}{2}-1$$

✓ Solution by Mathematica

Time used: 0.398 (sec). Leaf size: 53

```
DSolve[x^2*y''[x]-2*x*y'[x]-2*y[x]==x^2-2*x+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_2x^{\frac{1}{2}\left(3+\sqrt{17}\right)}+c_1x^{\frac{3}{2}-\frac{\sqrt{17}}{2}}-\frac{x^2}{4}+\frac{x}{2}-1$$

19.15 problem 632

Internal problem ID [15401]
Internal file name [`OUTPUT/15401_Wednesday_May_08_2024_03_58_01_PM_6244923/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 632.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + xy' - y = x^m}$$

19.15.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^m$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x$$

Next, we find the particular solution to the ODE

$$x^2y'' + xy' - y = x^m$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2}{x}$$

Which simplifies to

$$W = \frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\, x^m}{2x} dx$$

Which simplifies to

$$u_1 = -\int \frac{x^m}{2} dx$$

Hence

$$u_1 = -\frac{x^{m+1}}{2\,(m+1)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^m}{x}}{2x} dx$$

Which simplifies to

$$u_2 = \int \frac{x^{m-2}}{2} dx$$

Hence

$$u_2 = \frac{x^{m-1}}{2m-2}$$

Which simplifies to

$$u_1 = -\frac{x^{m+1}}{2m+2}$$
$$u_2 = \frac{x^{m-1}}{2m-2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{m+1}}{(2m+2)\,x} + \frac{x^{m-1}x}{2m-2}$$

Which simplifies to

$$y_p(x) = \frac{x^m}{m^2-1}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \frac{x^m}{m^2-1} + \frac{c_1}{x} + c_2 x\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^m}{m^2-1} + \frac{c_1}{x} + c_2 x \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{x^m}{m^2-1} + \frac{c_1}{x} + c_2 x$$

Verified OK.

19.15.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= -\frac{1}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= -1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda \tau} - e^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^2 + c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^m}{x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{m-2}}{2}dx$$

Hence

$$u_1 = \frac{x^{m-1}}{2m-2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,x^m}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^m}{2}dx$$

Hence

$$u_2 = -\frac{x^{m+1}}{2\,(m+1)}$$

Which simplifies to

$$u_1 = \frac{x^{m-1}}{2m-2}$$
$$u_2 = -\frac{x^{m+1}}{2m+2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{m+1}}{(2m+2)\,x} + \frac{x^{m-1}x}{2m-2}$$

Which simplifies to

$$y_p(x) = \frac{x^m}{m^2-1}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^2 + c_2}{x}\right) + \left(\frac{x^m}{m^2-1}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} + \frac{x^m}{m^2-1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x} + \frac{x^m}{m^2-1}$$

Verified OK.

19.15.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^m$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = \frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1)\,x^2 - ic_2 + c_1}{2x}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + xy' - y = x^m
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^m}{x}}{-2x} dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{m-2}}{2} dx$$

Hence

$$u_1 = \frac{x^{m-1}}{2m-2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,x^m}{-2x} dx$$

Which simplifies to

$$u_2 = \int -\frac{x^m}{2} dx$$

Hence

$$u_2 = -\frac{x^{m+1}}{2(m+1)}$$

Which simplifies to

$$u_1 = \frac{x^{m-1}}{2m-2}$$
$$u_2 = -\frac{x^{m+1}}{2m+2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{m+1}}{(2m+2)\,x} + \frac{x^{m-1}x}{2m-2}$$

Which simplifies to

$$y_p(x) = \frac{x^m}{m^2 - 1}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}\right) + \left(\frac{x^m}{m^2 - 1}\right) \\ &= \frac{x^m}{m^2 - 1} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \end{aligned}$$

Which simplifies to

$$y = \frac{x^m}{m^2 - 1} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^m}{m^2 - 1} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{x^m}{m^2 - 1} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Verified OK.

19.15.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x^m$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' - y = x^m$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^m}{x}}{-2x} dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{m-2}}{2} dx$$

Hence

$$u_1 = \frac{x^{m-1}}{2m-2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\, x^m}{-2x} dx$$

Which simplifies to

$$u_2 = \int -\frac{x^m}{2} dx$$

Hence

$$u_2 = -\frac{x^{m+1}}{2\,(m+1)}$$

Which simplifies to

$$u_1 = \frac{x^{m-1}}{2m-2}$$
$$u_2 = -\frac{x^{m+1}}{2m+2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{m+1}}{(2m+2)\,x} + \frac{x^{m-1}x}{2m-2}$$

Which simplifies to

$$y_p(x) = \frac{x^m}{m^2 - 1}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{x^m}{m^2 - 1}\right) \\ &= \frac{x^m}{m^2 - 1} + \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = \frac{x^m}{m^2 - 1} + \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Summary
The solution(s) found are the following

$$y = \frac{x^m}{m^2 - 1} + \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \frac{x^m}{m^2 - 1} + \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

19.15.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int \left(x^2 y'' + x y' - y\right) dx &= \int x^m dx \\ x^2 y' - y x &= \frac{x^{m+1}}{m+1} + c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x) y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{y}{x} = \frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}\right) \\
\mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1(m+1) + x^{m+1}}{x^3\,(m+1)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{c_1(m+1) + x^{m+1}}{x^3\,(m+1)}\,\mathrm{d}x \\
\frac{y}{x} &= \frac{\frac{\mathrm{e}^{(m+1)\ln(x)}}{m^2-1} - \frac{c_1}{2}}{x^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = \frac{\frac{\mathrm{e}^{(m+1)\ln(x)}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x
$$

which simplifies to

$$
y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x
$$

Summary

The solution(s) found are the following

$$
y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x \tag{1}
$$

Verification of solutions

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

Verified OK.

19.15.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= x^m \end{aligned}$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
x^3 v'' + \left(3x^2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
x^2 (u'(x) x + 3u(x)) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x) g(u) \\
&= -\frac{3u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x) + c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2}+c_2\right) \\ &= \left(-\frac{c_1}{2x^2}+c_2\right)x\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\ y_2 &= \frac{1}{x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^m}{x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x^{m-2}}{2}dx$$

Hence

$$u_1 = \frac{x^{m-1}}{2m-2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,x^m}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^m}{2}dx$$

Hence

$$u_2 = -\frac{x^{m+1}}{2\,(m+1)}$$

Which simplifies to

$$u_1 = \frac{x^{m-1}}{2m-2}$$

$$u_2 = -\frac{x^{m+1}}{2m+2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{m+1}}{(2m+2)\,x} + \frac{x^{m-1}x}{2m-2}$$

Which simplifies to

$$y_p(x) = \frac{x^m}{m^2-1}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{2x^2} + c_2\right)x\right) + \left(\frac{x^m}{m^2-1}\right) \\
&= \frac{x^m}{m^2-1} - \frac{-2c_2x^2 + c_1}{2x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^m}{m^2-1} - \frac{-2c_2x^2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{x^m}{m^2-1} - \frac{-2c_2x^2 + c_1}{2x}$$

Verified OK.

19.15.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + xy' - y = x^m$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}
\int (x^2y'' + xy' - y)\,dx &= \int x^m dx \\
x^2y' - yx &= \frac{x^{m+1}}{m+1} + c_1
\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1(m+1) + x^{m+1}}{x^2\,(m+1)}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1(m+1) + x^{m+1}}{x^3\,(m+1)}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{c_1(m+1) + x^{m+1}}{x^3\,(m+1)}\,\mathrm{d}x \\ \frac{y}{x} &= \frac{\frac{\mathrm{e}^{(m+1)\ln(x)}}{m^2-1} - \frac{c_1}{2}}{x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = \frac{\frac{\mathrm{e}^{(m+1)\ln(x)}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

which simplifies to

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

Verified OK.

19.15.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + x y' - y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 634: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=\frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1\int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1\int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{x} \\ y_2 &= \frac{x}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,x^m}{2}}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^m}{2}dx$$

Hence

$$u_1 = -\frac{x^{m+1}}{2\,(m+1)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x^m}{x}}{x}\,dx$$

Which simplifies to

$$u_2 = \int x^{m-2} dx$$

Hence

$$u_2 = \frac{x^{m-1}}{m-1}$$

Which simplifies to

$$u_1 = -\frac{x^{m+1}}{2m+2}$$
$$u_2 = \frac{x^{m-1}}{m-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{m+1}}{(2m+2)\,x} + \frac{x^{m-1}x}{2m-2}$$

Which simplifies to

$$y_p(x) = \frac{x^m}{m^2-1}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right) + \left(\frac{x^m}{m^2-1}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{x^m}{m^2-1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{x^m}{m^2-1}$$

Verified OK.

19.15.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= x^2 \\
q(x) &= x \\
r(x) &= -1 \\
s(x) &= x^m
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' - yx = \int x^m\,dx$$

We now have a first order ode to solve which is

$$x^2y' - yx = \frac{x^{m+1}}{m+1} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{c_1(m+1) + x^{m+1}}{x^2 (m+1)} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1(m+1) + x^{m+1}}{x^2 (m+1)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{c_1(m+1) + x^{m+1}}{x^2 (m+1)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right) \left(\frac{c_1(m+1) + x^{m+1}}{x^2 (m+1)}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1(m+1) + x^{m+1}}{x^3 (m+1)}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{c_1(m+1) + x^{m+1}}{x^3 (m+1)} \,\mathrm{d}x \\ \frac{y}{x} &= \frac{\frac{\mathrm{e}^{(m+1)\ln(x)}}{m^2-1} - \frac{c_1}{2}}{x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = \frac{\frac{\mathrm{e}^{(m+1)\ln(x)}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

which simplifies to

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{\frac{x^{m+1}}{m^2-1} - \frac{c_1}{2}}{x} + c_2 x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=x^m,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x} + c_2 x + \frac{x^m}{(m-1)(m+1)}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 27

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==x^m,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^m}{m^2-1} + c_2 x + \frac{c_1}{x}$$

19.16 problem 633

Internal problem ID [15402]
Internal file name [`OUTPUT/15402_Wednesday_May_08_2024_03_58_04_PM_40084235/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 633.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

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[[_2nd_order, _exact, _linear, _nonhomogeneous]]
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$$x^2y'' + 4xy' + 2y = 2\ln(x)^2 + 12x$$

19.16.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 2\ln(x)^2 + 12x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 4r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + \frac{c_2}{x}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 4xy' + 2y = 2\ln(x)^2 + 12x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\ln(x)^2+12x}{x}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(2x\ln(x)^2 + 12x^2\right) dx$$

Hence

$$u_1 = -x^2\ln(x)^2 + x^2\ln(x) - \frac{x^2}{2} - 4x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\ln(x)^2+12x}{x^2}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \left(2\ln(x)^2 + 12x\right) dx$$

Hence

$$u_2 = 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2$$

Which simplifies to

$$u_1 = -\frac{x^2\left(2\ln(x)^2 - 2\ln(x) + 8x + 1\right)}{2}$$
$$u_2 = 2x\left(\ln(x)^2 - 2\ln(x) + 3x + 2\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \ln (x)^2 - 3\ln (x) + 2x + \frac{7}{2} + \frac{c_1}{x^2} + \frac{c_2}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \ln (x)^2 - 3\ln (x) + 2x + \frac{7}{2} + \frac{c_1}{x^2} + \frac{c_2}{x} \tag{1}$$

Verification of solutions

$$y = \ln (x)^2 - 3\ln (x) + 2x + \frac{7}{2} + \frac{c_1}{x^2} + \frac{c_2}{x}$$

Verified OK.

19.16.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int \frac{4}{x}\, dx} \\ &= x^2 \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 2\ln (x)^2 + 12x \\ \left(x^2 y\right)'' &= 2\ln (x)^2 + 12x \end{aligned}$$

Integrating once gives

$$\left(x^2 y\right)' = 2x\left(\ln (x)^2 - 2\ln (x) + 3x + 2\right) + c_1$$

Integrating again gives

$$\left(x^2 y\right) = x^2 \ln (x)^2 - 3x^2 \ln (x) + \frac{x(4x^2 + 2c_1 + 7x)}{2} + c_2$$

Hence the solution is

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + \frac{x(4x^2+2c_1+7x)}{2} + c_2}{x^2}$$

Or

$$y = \ln(x)^2 + 2x - 3\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{7}{2}$$

Summary
The solution(s) found are the following

$$y = \ln(x)^2 + 2x - 3\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{7}{2} \tag{1}$$

Verification of solutions

$$y = \ln(x)^2 + 2x - 3\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{7}{2}$$

Verified OK.

19.16.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{4}{x}dx\right)}dx \\
&= \int e^{-4\ln(x)}\,dx \\
&= \int \frac{1}{x^4}dx \\
&= -\frac{1}{3x^3}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{\frac{1}{x^8}} \\
&= 2x^6
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2x^6 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$2x^6 = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$
$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

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$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(2\ln(x)^2 + 12x\right)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int 2\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(\ln(x)^2 + 6x\right) x^2 dx$$

Hence

$$u_1 = -2x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)^2 - 6\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4 + 4x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x) - 4x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \left(2\ln(x)^2 + 12x\right)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \left(\ln(x)^2 + 6x\right) x^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x)^2 - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4 + \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3}{2}$$

Which simplifies to

$$u_1 = -2x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(\ln(x)^2 - 2\ln(x) + 3x + 2\right)$$

$$u_2 = \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3\left(2\ln(x)^2 - 2\ln(x) + 8x + 1\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}\right) + \left(\ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} + \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} + \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Verified OK.

19.16.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 2\ln(x)^2 + 12x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{4}{x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}
$$

Now the particular solution to this ODE is found

$$
x^2y'' + 4xy' + 2y = 2\ln(x)^2 + 12x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(2\ln\left(x\right)^2 + 12x\right)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int 2\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(\ln\left(x\right)^2 + 6x\right) x^2 dx$$

Hence

$$u_1 = -2x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln\left(x\right)^2 - 6\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4 + 4x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln\left(x\right) - 4x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \left(2\ln\left(x\right)^2 + 12x\right)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \left(\ln\left(x\right)^2 + 6x\right) x^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln\left(x\right)^2 - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln\left(x\right) + 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4 + \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3}{2}$$

Which simplifies to

$$u_1 = -2x^3 \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(\ln(x)^2 - 2\ln(x) + 3x + 2\right)$$

$$u_2 = \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \left(2\ln(x)^2 - 2\ln(x) + 8x + 1\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right) + \left(\ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}\right) \\ &= \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Which simplifies to

$$y = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Verified OK.

19.16.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{4}{x}\right)'}{2} - \frac{\left(\frac{4}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(-\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\ &= e^{-\int \frac{\frac{4}{x}}{2}} \\ &= \frac{1}{x^2} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = 2\ln(x)^2 + 12x$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1$$

Integrating again gives

$$v(x) = x^2\ln(x)^2 - 3x^2\ln(x) + \frac{7x^2}{2} + 2x^3 + c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\,z(x) \\ &= \left(x^2\ln(x)^2 - 3x^2\ln(x) + 2x^3 + c_1x + \frac{7x^2}{2} + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2}$$

Hence (7) becomes

$$y = \frac{x^2\ln(x)^2 - 3x^2\ln(x) + 2x^3 + c_1x + \frac{7x^2}{2} + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{x^2\ln(x)^2 - 3x^2\ln(x) + 2x^3 + c_1x + \frac{7x^2}{2} + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(2\ln(x)^2 + 12x\right)}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int 2\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(\ln(x)^2 + 6x\right) x^2 dx$$

Hence

$$u_1 = -2x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)^2 - 6\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4 + 4x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x) - 4x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \left(2\ln(x)^2 + 12x\right)}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_2 = \int 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \left(\ln(x)^2 + 6x\right) x^2 dx$$

Hence

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x)^2 - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4 + \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3}{2}$$

Which simplifies to

$$u_1 = -2x^3\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \left(\ln(x)^2 - 2\ln(x) + 3x + 2\right)$$

$$u_2 = \frac{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \left(2\ln(x)^2 - 2\ln(x) + 8x + 1\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{x^2\ln(x)^2 - 3x^2\ln(x) + 2x^3 + c_1x + \frac{7x^2}{2} + c_2}{x^2}\right) + \left(\ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2\ln(x)^2 - 3x^2\ln(x) + 2x^3 + c_1x + \frac{7x^2}{2} + c_2}{x^2} + \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2\ln(x)^2 - 3x^2\ln(x) + 2x^3 + c_1x + \frac{7x^2}{2} + c_2}{x^2} + \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Verified OK.

19.16.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 2\ln(x)^2 + 12x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4xy' + 2y = 0$$

In normal form the ode

$$x^2y'' + 4xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{4n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{2u}{x}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= -\frac{2}{x}\,dx\\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx\\ \ln(u) &= -2\ln(x) + c_1\\ u &= \mathrm{e}^{-2\ln(x)+c_1}\\ &= \frac{c_1}{x^2}\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\,dx + c_2\\ &= -\frac{c_1}{x} + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n\\ &= \frac{-\frac{c_1}{x} + c_2}{x}\\ &= \frac{c_2 x - c_1}{x^2}\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' + 4xy' + 2y = 2\ln(x)^2 + 12x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\ln(x)^2+12x}{x}}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int \left(2x\ln(x)^2 + 12x^2\right) dx$$

Hence

$$u_1 = -x^2\ln(x)^2 + x^2\ln(x) - \frac{x^2}{2} - 4x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\ln(x)^2+12x}{x^2}}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int \left(2\ln(x)^2 + 12x\right) dx$$

Hence

$$u_2 = 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2$$

Which simplifies to

$$u_1 = -\frac{x^2\left(2\ln(x)^2 - 2\ln(x) + 8x + 1\right)}{2}$$
$$u_2 = 2x\left(\ln(x)^2 - 2\ln(x) + 3x + 2\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{-\frac{c_1}{x} + c_2}{x}\right) + \left(\ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2}\right) \\
&= \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{-\frac{c_1}{x} + c_2}{x}
\end{aligned}$$

Which simplifies to

$$y = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{-\frac{c_1}{x} + c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{-\frac{c_1}{x} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \ln(x)^2 - 3\ln(x) + 2x + \frac{7}{2} + \frac{-\frac{c_1}{x} + c_2}{x}$$

Verified OK.

19.16.7 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + 4xy' + 2y)\,dx = \int (2\ln(x)^2 + 12x)\,dx$$
$$x^2y' + 2yx = 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x}dx}$$
$$= x^2$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x\ln(x)^2-4x\ln(x)+4x+6x^2+c_1}{x^2}\right)\\
\frac{d}{dx}(x^2 y) &= (x^2)\left(\frac{2x\ln(x)^2-4x\ln(x)+4x+6x^2+c_1}{x^2}\right)\\
d(x^2 y) &= \left(2x\ln(x)^2-4x\ln(x)+4x+6x^2+c_1\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 y &= \int 2x\ln(x)^2-4x\ln(x)+4x+6x^2+c_1\, dx\\
x^2 y &= c_1 x+x^2\ln(x)^2-3x^2\ln(x)+\frac{7x^2}{2}+2x^3+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{c_1 x+x^2\ln(x)^2-3x^2\ln(x)+\frac{7x^2}{2}+2x^3}{x^2}+\frac{c_2}{x^2}
$$

which simplifies to

$$
y = \frac{x^2\ln(x)^2-3x^2\ln(x)+2x^3+c_1 x+\frac{7x^2}{2}+c_2}{x^2}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x^2\ln(x)^2-3x^2\ln(x)+2x^3+c_1 x+\frac{7x^2}{2}+c_2}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{x^2\ln(x)^2-3x^2\ln(x)+2x^3+c_1 x+\frac{7x^2}{2}+c_2}{x^2}
$$

Verified OK.

19.16.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
x^2 y''+4xy'+2y = 2\ln(x)^2+12x
$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2y'' + 4xy' + 2y\right) dx = \int \left(2\ln(x)^2 + 12x\right) dx$$

$$x^2y' + 2yx = 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x}dx}$$
$$= x^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(x^2y\right) = \left(x^2\right)\left(\frac{2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1}{x^2}\right)$$
$$d\left(x^2y\right) = \left(2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1\right) dx$$

Integrating gives

$$x^2y = \int 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1\, dx$$
$$x^2y = c_1x + x^2\ln(x)^2 - 3x^2\ln(x) + \frac{7x^2}{2} + 2x^3 + c_2$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1 x + x^2 \ln(x)^2 - 3x^2 \ln(x) + \frac{7x^2}{2} + 2x^3}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + 2x^3 + c_1 x + \frac{7x^2}{2} + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + 2x^3 + c_1 x + \frac{7x^2}{2} + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + 2x^3 + c_1 x + \frac{7x^2}{2} + c_2}{x^2}$$

Verified OK.

19.16.9 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + 4xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 635: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \\ &= z_1 e^{-2\ln(x)} \\ &= z_1\left(\frac{1}{x^2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + 4xy' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x^2} + \frac{c_2}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\ln(x)^2+12x}{x}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(2x\ln(x)^2 + 12x^2\right) dx$$

Hence

$$u_1 = -x^2\ln(x)^2 + x^2\ln(x) - \frac{x^2}{2} - 4x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2\ln(x)^2+12x}{x^2}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int \left(2 \ln (x)^2 + 12x\right) dx$$

Hence

$$u_2 = 2x \ln (x)^2 - 4x \ln (x) + 4x + 6x^2$$

Which simplifies to

$$u_1 = -\frac{x^2 \left(2 \ln (x)^2 - 2 \ln (x) + 8x + 1\right)}{2}$$
$$u_2 = 2x \left(\ln (x)^2 - 2 \ln (x) + 3x + 2\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln (x)^2 - 3 \ln (x) + 2x + \frac{7}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2}{x}\right) + \left(\ln (x)^2 - 3 \ln (x) + 2x + \frac{7}{2}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 x + c_1}{x^2} + \ln (x)^2 - 3 \ln (x) + 2x + \frac{7}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x + c_1}{x^2} + \ln (x)^2 - 3 \ln (x) + 2x + \frac{7}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 x + c_1}{x^2} + \ln (x)^2 - 3 \ln (x) + 2x + \frac{7}{2}$$

Verified OK.

19.16.10 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= 4x \\ r(x) &= 2 \\ s(x) &= 2\ln(x)^2 + 12x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 4 \end{aligned}$$

Therefore (1) becomes

$$2 - (4) + (2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\ dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' + 2yx = \int 2\ln(x)^2 + 12x\,dx$$

We now have a first order ode to solve which is

$$x^2y' + 2yx = 2x\ln(x)^2 - 4x\ln(x) + 4x + 6x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= \frac{2x \ln(x)^2 - 4x \ln(x) + 4x + 6x^2 + c_1}{x^2}
\end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{2x \ln(x)^2 - 4x \ln(x) + 4x + 6x^2 + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu) \left(\frac{2x \ln(x)^2 - 4x \ln(x) + 4x + 6x^2 + c_1}{x^2}\right) \\
\frac{d}{dx}(x^2 y) &= (x^2) \left(\frac{2x \ln(x)^2 - 4x \ln(x) + 4x + 6x^2 + c_1}{x^2}\right) \\
d(x^2 y) &= \left(2x \ln(x)^2 - 4x \ln(x) + 4x + 6x^2 + c_1\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
x^2 y &= \int 2x \ln(x)^2 - 4x \ln(x) + 4x + 6x^2 + c_1 \, dx \\
x^2 y &= c_1 x + x^2 \ln(x)^2 - 3x^2 \ln(x) + \frac{7x^2}{2} + 2x^3 + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1 x + x^2 \ln(x)^2 - 3x^2 \ln(x) + \frac{7x^2}{2} + 2x^3}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + 2x^3 + c_1 x + \frac{7x^2}{2} + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + 2x^3 + c_1 x + \frac{7x^2}{2} + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2 \ln(x)^2 - 3x^2 \ln(x) + 2x^3 + c_1 x + \frac{7x^2}{2} + c_2}{x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=2*(ln(x))^2+12*x,y(x), singsol=all)
```

$$y(x) = \frac{c_2}{x^2} + 2x + \frac{7}{2} + \frac{c_1}{x} - 3\ln(x) + \ln(x)^2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]+4*x*y'[x]+2*y[x]==2*(Log[x])^2+12*x,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{c_1}{x^2} + 2x + \log^2(x) - 3\log(x) + \frac{c_2}{x} + \frac{7}{2}$$

19.17 problem 634

Internal problem ID [15403]
Internal file name [`OUTPUT/15403_Wednesday_May_08_2024_03_58_07_PM_97886931/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 634.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(x+1)^3 y'' + 3(x+1)^2 y' + (x+1) y = 6\ln(x+1)}$$

19.17.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1) y = 0$$

In normal form the ode

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1) y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x+1} \\ q(x) &= \frac{1}{(x+1)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x) dx\right)} dx \\ &= \int e^{-\left(\int \frac{3}{x+1} dx\right)} dx \\ &= \int e^{-3\ln(x+1)}\, dx \\ &= \int \frac{1}{(x+1)^3} dx \\ &= -\frac{1}{2(x+1)^2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{(x+1)^2}}{\frac{1}{(x+1)^6}} \\ &= (x+1)^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + (x+1)^4\, y(\tau) = 0$$

But in terms of τ

$$(x+1)^4 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{(x+1)^2}}$$

$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\,\ln\left(-\frac{1}{(x+1)^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{(x+1)^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{(x+1)^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)\ln(x+1)}{\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int 3\sqrt{-\frac{1}{(x+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(x+1)^2}\right)\right)\ln(x+1)\,dx$$

Hence

$$u_1 = -\left(\int_0^x 3\sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{6\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)}{\sqrt{2}}\,dx$$

Which simplifies to

$$u_2 = \int 3\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)\sqrt{2}dx$$

Hence

$$u_2 = \frac{3\sqrt{2}\,(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)^2}{2}$$

Which simplifies to

$$u_1 = -3\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)$$

$$u_2 = \frac{3\sqrt{2}\,(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= -3\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)\sqrt{-\frac{1}{(x+1)^2}} \\ &\quad + \frac{3\sqrt{2}\,(x+1)\sqrt{-\frac{1}{(x+1)^2}}\,\ln(x+1)^2\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2}+\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)}{2} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{-6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\left(\ln(2)-\mathrm{l}\right.}{2x+2}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2)+c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2}\right) + \left(\frac{-6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\left(\ln(\right.}{2x+2}\right.$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2)+c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2} + \frac{-6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\left(\ln(2)-\right.}{2x+2} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\left(c_1 - c_2\ln(2)+c_2\ln\left(-\frac{1}{(x+1)^2}\right)\right)}{2} + \frac{-6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2)+\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\left(\ln(2)-\right.}{2x+2}$$

Verified OK.

19.17.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = (x+1)^3, B = 3(x+1)^2, C = x+1, f(x) = 6\ln(x+1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1)y = 0$$

In normal form the ode

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1)y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x+1} \\ q(x) &= \frac{1}{(x+1)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{(x+1)^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{(x+1)^2}}\,(x+1)^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{3}{x+1}\frac{\sqrt{\frac{1}{(x+1)^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{(x+1)^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-c\tau}c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{(x+1)^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{(x+1)^2}}\,(x+1)\ln(x+1)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1}{x+1}
$$

Now the particular solution to this ODE is found

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1) y = 6\ln(x+1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{(x+1)^2}}$$

$$y_2 = -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{d}{dx}\left(\sqrt{-\frac{1}{(x+1)^2}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{(x+1)^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{(x+1)^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{(x+1)^2}\right)}{2\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}}{x+1}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{(x+1)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{(x+1)^2}}\ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)\ln(x+1)}{\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int 3\sqrt{-\frac{1}{(x+1)^2}}\left(-\ln(2) + \ln\left(-\frac{1}{(x+1)^2}\right)\right)\ln(x+1)\,dx$$

Hence

$$u_1 = -\left(\int_0^x 3\sqrt{-\frac{1}{(\alpha+1)^2}}\left(-\ln(2) + \ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{6\sqrt{-\frac{1}{(x+1)^2}}\, \ln(x+1)}{\sqrt{2}} dx$$

Which simplifies to

$$u_2 = \int 3\sqrt{-\frac{1}{(x+1)^2}}\, \ln(x+1)\, \sqrt{2}dx$$

Hence

$$u_2 = \frac{3\sqrt{2}\,(x+1)\,\sqrt{-\frac{1}{(x+1)^2}}\, \ln(x+1)^2}{2}$$

Which simplifies to

$$u_1 = 3\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\, \left(\ln(2) - \ln\left(-\frac{1}{(\alpha+1)^2}\right)\right) \ln(\alpha+1)\, d\alpha\right)$$

$$u_2 = \frac{3\sqrt{2}\,(x+1)\,\sqrt{-\frac{1}{(x+1)^2}}\, \ln(x+1)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\, \left(\ln(2) - \ln\left(-\frac{1}{(\alpha+1)^2}\right)\right) \ln(\alpha+1)\, d\alpha\right) \sqrt{-\frac{1}{(x+1)^2}} + \frac{3\sqrt{2}\,(x+1)\,\sqrt{-\frac{1}{(x+1)^2}}\, \ln(x+1)^2 \left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\, \ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{(x+1)^2}}\, \ln\left(-\frac{1}{(x+1)^2}\right)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\, \left(\ln(2) - \ln\left(-\frac{1}{(\alpha+1)^2}\right)\right) \ln(\alpha+1)\, d\alpha\right) + 3\ln(x+1)^2 \left(\ln(2) - \ln\left(-\right.\right.}{2x+2}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{c_1}{x+1}\right)$$

$$+ \left(\frac{6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(\ln(2)-\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\left(\ln(2)-\right.}{2x+2}\right.$$

$$= \frac{6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(\ln(2)-\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\left(\ln(2)-\ln\left(\right.\right.}{2x+2}$$

$$+ \frac{c_1}{x+1}$$

Which simplifies to

$$y = \frac{6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(\ln(2)-\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\ln(2)-3\ln(x}{2x+2}$$

Summary

The solution(s) found are the following

$$y = \frac{6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(\ln(2)-\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\ln(2)-3\ln(x}{2x+2} \tag{1}$$

Verification of solutions

$$y = \frac{6\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\left(\int_0^x \sqrt{-\frac{1}{(\alpha+1)^2}}\left(\ln(2)-\ln\left(-\frac{1}{(\alpha+1)^2}\right)\right)\ln(\alpha+1)\,d\alpha\right)+3\ln(x+1)^2\ln(2)-3\ln(x}{2x+2}$$

Verified OK.

19.17.3 Solving using Kovacic algorithm

Writing the ode as

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^3 \\ B &= 3(x+1)^2 \\ C &= x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4(x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4(x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 636: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x+1)^2}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=-\frac{1}{4\,(x+1)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=-\frac{1}{4\,(x+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x + 2} + (-)(0) \\ &= \frac{1}{2x + 2} \\ &= \frac{1}{2x + 2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x + 2}\right)(0) + \left(\left(-\frac{1}{2(x+1)^2}\right) + \left(\frac{1}{2x + 2}\right)^2 - \left(-\frac{1}{4(x+1)^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x+2} dx} \\ &= \sqrt{x + 1} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3(x+1)^2}{(x+1)^3}\,dx} \\
&= z_1 e^{-\frac{3\ln(x+1)}{2}} \\
&= z_1\left(\frac{1}{(x+1)^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x+1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3(x+1)^2}{(x+1)^3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x+1))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x+1}\right) + c_2\left(\frac{1}{x+1}(\ln(x+1))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x+1)^3 y'' + 3(x+1)^2 y' + (x+1)\, y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x+1} + \frac{c_2 \ln(x+1)}{x+1}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x+1}$$
$$y_2 = \frac{\ln(x+1)}{x+1}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{\ln(x+1)}{x+1} \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{\ln(x+1)}{x+1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{\ln(x+1)}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{\ln(x+1)}{(x+1)^2} + \frac{1}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)\left(-\frac{\ln(x+1)}{(x+1)^2} + \frac{1}{(x+1)^2}\right) - \left(\frac{\ln(x+1)}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{1}{(x+1)^3}$$

Which simplifies to

$$W = \frac{1}{(x+1)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{6\ln(x+1)^2}{x+1}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{6\ln(x+1)^2}{x+1}dx$$

Hence

$$u_1 = -2\ln(x+1)^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{6\ln(x+1)}{x+1}}{1}\,dx$$

Which simplifies to

$$u_2 = \int \frac{6\ln(x+1)}{x+1}dx$$

Hence

$$u_2 = 3\ln(x+1)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x+1)^3}{x+1}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x+1} + \frac{c_2 \ln(x+1)}{x+1}\right) + \left(\frac{\ln(x+1)^3}{x+1}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 \ln(x+1) + c_1}{x+1} + \frac{\ln(x+1)^3}{x+1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 \ln(x+1) + c_1}{x+1} + \frac{\ln(x+1)^3}{x+1} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 \ln(x+1) + c_1}{x+1} + \frac{\ln(x+1)^3}{x+1}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve((x+1)^3*diff(y(x),x$2)+3*(x+1)^2*diff(y(x),x)+(x+1)*y(x)=6*ln(x+1),y(x), singsol=all)
```

$$y(x) = \frac{c_1 \ln(1+x) + \ln(1+x)^3 + c_2}{1+x}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 27

```
DSolve[(x+1)^3*y''[x]+3*(x+1)^2*y'[x]+(x+1)*y[x]==6*Log[x+1],y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{\log^3(x+1) + c_2 \log(x+1) + c_1}{x+1}$$

19.18 problem 635

Internal problem ID [15404]
Internal file name [`OUTPUT/15404_Wednesday_May_08_2024_03_58_10_PM_56908608/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.4 Nonhomogeneous linear equations with constant coefficients. The Euler equations. Exercises page 143
Problem number: 635.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x-2)^2 y''-3(x-2) y'+4y=x}$$

19.18.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(x-2)^2 y''+(-3x+6) y'+4y=0$$

In normal form the ode

$$(x-2)^2 y''+(-3x+6) y'+4y=0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{-x+2} \\ q(x) &= \frac{4}{(x-2)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{-x+2} dx\right)} dx \\ &= \int e^{3\ln(-x+2)}\, dx \\ &= \int (-x+2)^3\, dx \\ &= -\frac{(-x+2)^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{(x-2)^2}}{(-x+2)^6} \\ &= \frac{4}{(x-2)^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{(x-2)^8} &= 0\end{aligned}$$

But in terms of τ

$$\frac{4}{(x-2)^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1 &= \frac{1}{2}\\ r_2 &= \frac{1}{2}\end{aligned}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_2 \ln\left(-(x-2)^4\right) - 2c_2 \ln(2) + c_1\right)\sqrt{-(x-2)^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\left(c_2 \ln\left(-(x-2)^4\right) - 2c_2 \ln(2) + c_1\right)\sqrt{-(x-2)^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-x^4 + 8x^3 - 24x^2 + 32x - 16}$$
$$y_2 = \frac{\sqrt{-x^4 + 8x^3 - 24x^2 + 32x - 16}\,\ln\left(-(x-2)^4\right)}{2} - \sqrt{-x^4 + 8x^3 - 24x^2 + 32x - 16}\,\ln(2)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-x^4+8x^3-24x^2+32x-16} & \frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(-(x-2)^4\right)}{2} - \sqrt{-x^4+8x^3-24x^2+3 \\ \frac{d}{dx}\left(\sqrt{-x^4+8x^3-24x^2+32x-16}\right) & \frac{d}{dx}\left(\frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(-(x-2)^4\right)}{2} - \sqrt{-x^4+8x^3-24x^2+ \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-x^4+8x^3-24x^2+32x-16} & \frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(-(x-2)^4\right)}{2} - \sqrt{-x^4+8x^3-24x^2+32x- \\ \frac{-4x^3+24x^2-48x+32}{2\sqrt{-x^4+8x^3-24x^2+32x-16}} & \frac{\ln\left(-(x-2)^4\right)\left(-4x^3+24x^2-48x+32\right)}{4\sqrt{-x^4+8x^3-24x^2+32x-16}} + \frac{2\sqrt{-x^4+8x^3-24x^2+32x-16}}{x-2} - \frac{\ln(2)\left(-4x^3+\right.}{2\sqrt{-x^4+8x^3} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-x^4+8x^3-24x^2+32x-16}\right)\left(\frac{\ln\left(-(x-2)^4\right)\left(-4x^3+24x^2-48x+32\right)}{4\sqrt{-x^4+8x^3-24x^2+32x-16}} + \frac{2\sqrt{-x^4+8x^3-24x^2+32x-16}}{x-2} - \frac{\ln(2)\left(-4x^3+24x^2-48x+32\right)}{2\sqrt{-x^4+8x^3-24x^2+32x-16}}\right) - \left(\frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(-(x-2)^4\right)}{2} - \sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln(2)\right)\left(\frac{-4x^3+24x^2-48x+32}{2\sqrt{-x^4+8x^3-24x^2+32x-16}}\right)$$

Which simplifies to

$$W = -2(x-2)\left(x^2-4x+4\right)$$

Which simplifies to

$$W = -2(x-2)^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(-(x-2)^4\right)}{2} - \sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln(2)\right)x}{-2\,(x-2)^5}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{-\left(x-2\right)^4}\,\left(-2\ln\left(2\right)+\ln\left(-\left(x-2\right)^4\right)\right)x}{4\left(x-2\right)^5}dx$$

Hence

$$u_1 = -\frac{\sqrt{-\left(x-2\right)^4}\,\left(x-1\right)\ln\left(-\left(x-2\right)^4\right)}{4\left(x-2\right)^4}+\frac{\sqrt{-\left(x-2\right)^4}\,\left(\ln\left(2\right)x-\ln\left(2\right)-2x+3\right)}{2\left(x-2\right)^4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,x}{-2\left(x-2\right)^5}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\sqrt{-\left(x-2\right)^4}\,x}{2\left(x-2\right)^5}dx$$

Hence

$$u_2 = \frac{\left(x-1\right)\sqrt{-\left(x-2\right)^4}}{2\left(x-2\right)^4}$$

Which simplifies to

$$u_1 = \frac{\left(\left(1-x\right)\ln\left(-\left(x-2\right)^4\right)+\left(2x-2\right)\ln\left(2\right)-4x+6\right)\sqrt{-\left(x-2\right)^4}}{4\left(x-2\right)^4}$$

$$u_2 = \frac{\left(x-1\right)\sqrt{-\left(x-2\right)^4}}{2\left(x-2\right)^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\left(1-x\right)\ln\left(-\left(x-2\right)^4\right)+\left(2x-2\right)\ln\left(2\right)-4x+6\right)\sqrt{-\left(x-2\right)^4}\,\sqrt{-x^4+8x^3-24x^2+32x-16}}{4\left(x-2\right)^4} + \frac{\left(x-1\right)\sqrt{-\left(x-2\right)^4}\,\left(\frac{\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(-\left(x-2\right)^4\right)}{2}-\sqrt{-x^4+8x^3-24x^2+32x-16}\,\ln\left(2\right)\right)}{2\left(x-2\right)^4}$$

Which simplifies to

$$y_p(x) = x - \frac{3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\left(c_2 \ln\left(-(x-2)^4\right) - 2c_2 \ln(2) + c_1\right)\sqrt{-\left(x-2\right)^4}}{2}\right) + \left(x - \frac{3}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_2 \ln\left(-(x-2)^4\right) - 2c_2 \ln(2) + c_1\right)\sqrt{-\left(x-2\right)^4}}{2} + x - \frac{3}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_2 \ln\left(-(x-2)^4\right) - 2c_2 \ln(2) + c_1\right)\sqrt{-\left(x-2\right)^4}}{2} + x - \frac{3}{2}$$

Verified OK.

19.18.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(x-2\right)^2 y'' + \left(-3x+6\right) y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \left(x-2\right)^2 \\ B &= -3x+6 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4(x-2)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4(x-2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4(x-2)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 637: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-2)^2$. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-2)^2}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4(x-2)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4(x-2)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
2	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x-4} + (-)(0) \\ &= \frac{1}{2x-4} \\ &= \frac{1}{2x-4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x-4}\right)(0) + \left(\left(-\frac{1}{2\left(x-2\right)^2}\right) + \left(\frac{1}{2x-4}\right)^2 - \left(-\frac{1}{4\left(x-2\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x-4} dx} \\
&= \sqrt{x-2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x+6}{(x-2)^2}\, dx} \\
&= z_1 e^{\frac{3\ln(x-2)}{2}} \\
&= z_1\left(\left(x-2\right)^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \left(x-2\right)^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x+6}{(x-2)^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3\ln(x-2)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x-2))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x-2)^2\right) + c_2\left((x-2)^2 (\ln(x-2))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
(x-2)^2 y'' + (-3x+6)\, y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(x-2)^2 + c_2(x-2)^2 \ln(x-2)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (x-2)^2 \\
y_2 &= (x-2)^2 \ln(x-2)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x-2)^2 & (x-2)^2 \ln(x-2) \\ \frac{d}{dx}\left((x-2)^2\right) & \frac{d}{dx}\left((x-2)^2 \ln(x-2)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x-2)^2 & (x-2)^2 \ln(x-2) \\ 2x-4 & 2(x-2)\ln(x-2) + x - 2 \end{vmatrix}$$

Therefore

$$W = \left((x-2)^2\right)\left(2(x-2)\ln(x-2) + x - 2\right) - \left((x-2)^2 \ln(x-2)\right)(2x-4)$$

Which simplifies to

$$W = x^3 - 6x^2 + 12x - 8$$

Which simplifies to

$$W = (x-2)^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x-2)^2 \ln(x-2)\, x}{(x-2)^5} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x-2)\, x}{(x-2)^3} dx$$

Hence

$$u_1 = \frac{\ln(x-2)}{(x-2)^2} + \frac{1}{2(x-2)^2} + \frac{\ln(x-2)}{x-2} + \frac{1}{x-2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x-2)^2 x}{(x-2)^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{x}{(x-2)^3} dx$$

Hence

$$u_2 = -\frac{1}{(x-2)^2} - \frac{1}{x-2}$$

Which simplifies to

$$u_1 = \frac{(2x-2)\ln(x-2) + 2x - 3}{2(x-2)^2}$$
$$u_2 = \frac{1-x}{(x-2)^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(2x-2)\ln(x-2)}{2} + x - \frac{3}{2} + (1-x)\ln(x-2)$$

Which simplifies to

$$y_p(x) = x - \frac{3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1(x-2)^2 + c_2(x-2)^2 \ln(x-2)\right) + \left(x - \frac{3}{2}\right) \end{aligned}$$

Which simplifies to

$$y = (x-2)^2 (c_1 + c_2 \ln(x-2)) + x - \frac{3}{2}$$

Summary

The solution(s) found are the following

$$y = (x-2)^2 (c_1 + c_2 \ln(x-2)) + x - \frac{3}{2} \tag{1}$$

Verification of solutions

$$y = (x-2)^2 (c_1 + c_2 \ln(x-2)) + x - \frac{3}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve((x-2)^2*diff(y(x),x$2)-3*(x-2)*diff(y(x),x)+4*y(x)=x,y(x), singsol=all)
```

$$y(x) = (x-2)^2 c_2 + (x-2)^2 \ln(x-2) c_1 + x - \frac{3}{2}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 31

```
DSolve[(x-2)^2*y''[x]-3*(x-2)*y'[x]+4*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1(x-2)^2 + 2c_2(x-2)^2 \log(x-2) - \frac{3}{2}$$

20 Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148

20.1 problem 636

Internal problem ID [15405]
Internal file name [`OUTPUT/15405_Wednesday_May_08_2024_03_58_11_PM_64581847/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 636.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+2x)\,y''+(4x-2)\,y'-8y=0}$$

20.1.1 Solving using Kovacic algorithm

Writing the ode as

$$(1+2x)\,y''+(4x-2)\,y'-8y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1+2x \\ B &= 4x-2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 + 12x + 13}{(1 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^2 + 12x + 13 \\ t &= (1 + 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 + 12x + 13}{(1 + 2x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 638: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1+2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{2}{\left(x+\frac{1}{2}\right)^2} + \frac{2}{x+\frac{1}{2}}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 + \frac{1}{x} - \frac{3}{4x^3} + \frac{11}{8x^4} - \frac{29}{16x^5} + \frac{29}{16x^6} - \frac{61}{64x^7} - \frac{5}{4x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 + 12x + 13}{4x^2 + 4x + 1} \\ &= Q + \frac{R}{4x^2 + 4x + 1} \\ &= (1) + \left(\frac{8x + 12}{4x^2 + 4x + 1}\right) \\ &= 1 + \frac{8x + 12}{4x^2 + 4x + 1} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is 8. Dividing this by leading coefficient in t which is 4 gives 2. Now b can be found.

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{2}{1} - 0\right) &= 1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{2}{1} - 0\right) &= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^2 + 12x + 13}{(1 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	1	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x+\frac{1}{2}} + (-)(1) \\
&= -\frac{1}{x+\frac{1}{2}} - 1 \\
&= \frac{-2x-3}{1+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x+\frac{1}{2}} - 1\right)(0) + \left(\left(\frac{1}{\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{x+\frac{1}{2}} - 1\right)^2 - \left(\frac{4x^2+12x+13}{(1+2x)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{x+\frac{1}{2}} - 1\right) dx} \\
&= \frac{\mathrm{e}^{-x}}{1+2x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x-2}{1+2x} \, dx} \\
&= z_1 e^{-x+\ln(1+2x)} \\
&= z_1\left((1+2x)\,\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x-2}{1+2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x+2\ln(1+2x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{(4x^2+1)\,\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{(4x^2+1)\,\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-2x} + c_2\left(2x^2 + \frac{1}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{-2x} + c_2\left(2x^2 + \frac{1}{2}\right)$$

Verified OK.

20.1.2 Maple step by step solution

Let's solve

$(1 + 2x)\, y'' + (4x - 2)\, y' - 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{8y}{1+2x} - \frac{2(2x-1)y'}{1+2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2(2x-1)y'}{1+2x} - \frac{8y}{1+2x} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2(2x-1)}{1+2x}, P_3(x) = -\frac{8}{1+2x}\right]$

  - $\left(x + \frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x + \frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -2$

  - $\left(x + \frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

    $\left(\left(x + \frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

  - $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $(1 + 2x)\, y'' + (4x - 2)\, y' - 8y = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2} y(u)\right) + (4u - 4)\left(\frac{d}{du} y(u)\right) - 8y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m} \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\, u^{k+r-1+m}$
- Shift index using $k- >k+1-m$

  $u^{m} \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
- Convert $u \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, u^{k+r-1}$
- Shift index using $k- >k+1$

  $u \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $2a_{0}r(-3+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(k+r-2) + 4a_{k}(k+r-2)\right) u^{k+r}\right) = 0$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $2r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term in the series must be 0, giving the recursion relation

  $2\left(a_{k+1}(k+1+r) + 2a_{k}\right)(k+r-2) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{2a_{k}}{k+1+r}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{2a_{k}}{k+1}$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_{k} u^{k}, a_{k+1} = -\frac{2a_{k}}{k+1}\right]$
- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_{k} \left(x+\frac{1}{2}\right)^{k}, a_{k+1} = -\frac{2a_{k}}{k+1}\right]$

- Recursion relation for $r = 3$

  $a_{k+1} = -\frac{2a_k}{k+4}$

- Solution for $r = 3$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+1} = -\frac{2a_k}{k+4}\right]$

- Revert the change of variables $u = x + \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k+3}, a_{k+1} = -\frac{2a_k}{k+4}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k \left(x+\frac{1}{2}\right)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k \left(x+\frac{1}{2}\right)^{k+3}\right), a_{k+1} = -\frac{2a_k}{k+1}, b_{k+1} = -\frac{2b_k}{k+4}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((2*x+1)*diff(y(x),x$2)+(4*x-2)*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = 4c_1x^2 + c_2\mathrm{e}^{-2x} + c_1$$

✓ Solution by Mathematica

Time used: 0.192 (sec). Leaf size: 27

```
DSolve[(2*x+1)*y''[x]+(4*x-2)*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}c_2(4x^2+1) + c_1e^{-2x}$$

20.2 problem 637

Internal problem ID [15406]
Internal file name [`OUTPUT/15406_Wednesday_May_08_2024_03_58_12_PM_21344098/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 637.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{(x^2 - x)\, y'' + (2x - 3)\, y' - 2y = 0}$$

20.2.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x^2 - x)\, y'' + (2x - 3)\, y' - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2x - 3}{x\,(x - 1)}$$
$$q(x) = -\frac{2}{x\,(x - 1)}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(2x-3)}{x^2\,(x-1)} - \frac{2}{x\,(x-1)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(-\frac{4}{x} + \frac{2x-3}{x\,(x-1)}\right) v'(x) &= 0 \\ v''(x) + \frac{(-2x+1)\,v'(x)}{x\,(x-1)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-2x+1)\,u(x)}{x\,(x-1)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(2x-1)}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{2x-1}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{2x-1}{x(x-1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2x-1}{x(x-1)}\,dx \\
\ln(u) &= \ln(x(x-1)) + c_1 \\
u &= e^{\ln(x(x-1))+c_1} \\
&= c_1 x(x-1)
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \frac{c_1\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) + c_2}{x^2} \\
&= \frac{2c_1x^3 - 3c_1x^2 + 6c_2}{6x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) + c_2}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right) + c_2}{x^2}
$$

Verified OK.

20.2.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 - x \\ B &= 2x - 3 \\ C &= -2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^2 - x)\,(0) + (2x - 3)\,(2) + (-2)\,(2x - 3) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$2x^3 - 5x^2 + 3xv'' + (8x^2 - 16x + 9)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$(2x^3 - 5x^2 + 3x)\, u'(x) + 8\left(x^2 - 2x + \frac{9}{8}\right) u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(8x^2 - 16x + 9)}{x\,(2x^2 - 5x + 3)}
\end{aligned}$$

Where $f(x) = -\frac{8x^2-16x+9}{x(2x^2-5x+3)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{8x^2 - 16x + 9}{x\,(2x^2 - 5x + 3)}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{8x^2 - 16x + 9}{x\,(2x^2 - 5x + 3)}\, dx \\
\ln(u) &= -2\ln(2x - 3) + \ln(x - 1) - 3\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(2x-3)+\ln(x-1)-3\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{-2\ln(2x-3)+\ln(x-1)-3\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{1}{(2x-3)^2 x^2} - \frac{1}{(2x-3)^2 x^3}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1\left(\frac{1}{(2x-3)^2 x^2} - \frac{1}{(2x-3)^2 x^3}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1(x-1)}{(2x-3)^2 x^3}\, \mathrm{d}x \\
&= c_1\left(-\frac{2}{27\,(2x-3)} + \frac{1}{18x^2} + \frac{1}{27x}\right) + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (2x-3)\left(c_1\left(-\frac{2}{27(2x-3)}+\frac{1}{18x^2}+\frac{1}{27x}\right)+c_2\right) \\ &= \frac{12c_2x^3-18c_2x^2-c_1}{6x^2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{12c_2x^3-18c_2x^2-c_1}{6x^2} \tag{1}$$

Verification of solutions

$$y = \frac{12c_2x^3-18c_2x^2-c_1}{6x^2}$$

Verified OK.

20.2.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2-x)\,y''+(2x-3)\,y'-2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2-x \\ B &= 2x-3 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{8x^2 - 8x + 3}{4\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 8x^2 - 8x + 3 \\ t &= 4\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{8x^2 - 8x + 3}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 640: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-1)^2} + \frac{3}{4x^2} - \frac{1}{2x} + \frac{1}{2x-2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{8x^2 - 8x + 3}{4(x^2 - x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{8x^2 - 8x + 3}{4\left(x^2 - x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} - \frac{1}{2\left(x - 1\right)} + (-)\left(0\right) \\ &= -\frac{1}{2x} - \frac{1}{2\left(x - 1\right)} \\ &= -\frac{2x - 1}{2x\left(x - 1\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(x-1)}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2(x-1)}\right)^2 - \left(\frac{8x^2 - 8x + 3}{4(x^2 - x)^2}\right)\right) = 0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= e^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-1)}\right) dx} \\
&= \frac{1}{\sqrt{x}\,\sqrt{x-1}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x-3}{x^2-x}\, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{2} - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x-1}}{x^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\sqrt{x-1}}{\sqrt{x(x-1)}\, x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x-3}{x^2-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x-1)-3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\sqrt{x-1}}{\sqrt{x\,(x-1)}\, x^{\frac{3}{2}}}\right) + c_2 \left(\frac{\sqrt{x-1}}{\sqrt{x\,(x-1)}\, x^{\frac{3}{2}}} \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{x-1}}{\sqrt{x\,(x-1)}\, x^{\frac{3}{2}}} + \frac{c_2\sqrt{x-1}\,(2x-3)\,\sqrt{x}}{6\sqrt{x\,(x-1)}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1\sqrt{x-1}}{\sqrt{x\,(x-1)}\, x^{\frac{3}{2}}} + \frac{c_2\sqrt{x-1}\,(2x-3)\,\sqrt{x}}{6\sqrt{x\,(x-1)}}
$$

Verified OK.

20.2.4 Maple step by step solution

Let's solve

$(x^2-x)\, y'' + (2x-3)\, y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x(x-1)} - \frac{(2x-3)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(2x-3)y'}{x(x-1)} - \frac{2y}{x(x-1)} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = \frac{2x-3}{x(x-1)}, P_3(x) = -\frac{2}{x(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 3$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(x-1) + (2x-3)\, y' - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$-a_0 r(2+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(k+3+r) + a_k(k+r+2)\,(k+r-1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term in the series must be 0, giving the recursion relation

  $-a_{k+1}(k+1+r)\,(k+3+r) + a_k(k+r+2)\,(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r+2)(k+r-1)}{(k+1+r)(k+3+r)}$

- Recursion relation for $r = -2$; series terminates at $k = 3$

  $a_{k+1} = \frac{a_k k(k-3)}{(k-1)(k+1)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 1$

  $a_{k+1} = \frac{a_k k(k-3)}{(k-1)(k+1)}$

- Recursion relation for $r = 0$; series terminates at $k = 1$

  $a_{k+1} = \frac{a_k(k+2)(k-1)}{(k+1)(k+3)}$

- Apply recursion relation for $k = 0$

  $a_1 = -\frac{2a_0}{3}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y = a_0 \cdot \left(1 - \frac{2x}{3}\right)$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x^2-x)*diff(y(x),x$2)+(2*x-3)*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^2} + c_2\left(x - \frac{3}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 23

```
DSolve[(x^2-x)*y''[x]+(2*x-3)*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2} + \frac{1}{6}c_2(3 - 2x)$$

20.3 problem 638

Internal problem ID [15407]
Internal file name [`OUTPUT/15407_Wednesday_May_08_2024_03_58_13_PM_34672586/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 638.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(2x^2+3x)\,y''-6y'(x+1)+6y=6}$$

20.3.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=2x^2+3x, B=-6x-6, C=6, f(x)=6$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(2x^2+3x)\,y''+(-6x-6)\,y'+6y=0$$

In normal form the ode

$$\left(2x^2+3x\right) y''+(-6x-6)\, y'+6y = 0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-6x-6}{x\,(2x+3)} \\ q(x) &= \frac{6}{(2x+3)\, x} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(-6x-6)}{x^2\,(2x+3)}+\frac{6}{(2x+3)\, x} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{6}{x}+\frac{-6x-6}{x\,(2x+3)}\right) v'(x) &= 0 \\ v''(x)+\frac{(6x+12)\, v'(x)}{x\,(2x+3)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(6x+12)\,u(x)}{x\,(2x+3)}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{6u(2+x)}{x\,(2x+3)}
\end{aligned}$$

Where $f(x)=-\frac{6(2+x)}{x(2x+3)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{6(2+x)}{x\,(2x+3)}\,dx\\
\int\frac{1}{u}\,du &= \int -\frac{6(2+x)}{x\,(2x+3)}\,dx\\
\ln(u) &= \ln(2x+3)-4\ln(x)+c_1\\
u &= \mathrm{e}^{\ln(2x+3)-4\ln(x)+c_1}\\
&= c_1\mathrm{e}^{\ln(2x+3)-4\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{2}{x^3}+\frac{3}{x^4}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_2\\
&= c_1\left(-\frac{1}{x^3}-\frac{1}{x^2}\right)+c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n\\
&= \left(c_1\left(-\frac{1}{x^3}-\frac{1}{x^2}\right)+c_2\right)x^3\\
&= (-x-1)\,c_1+c_2x^3
\end{aligned}$$

Now the particular solution to this ODE is found

$$(2x^2 + 3x)\, y'' + (-6x - 6)\, y' + 6y = 6$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^3 \\ y_2 &= -x - 1 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^3 & -x - 1 \\ \frac{d}{dx}(x^3) & \frac{d}{dx}(-x - 1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^3 & -x - 1 \\ 3x^2 & -1 \end{vmatrix}$$

Therefore

$$W = (x^3)(-1) - (-x - 1)(3x^2)$$

Which simplifies to

$$W = 2x^3 + 3x^2$$

Which simplifies to

$$W = 2x^3 + 3x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-6x - 6}{(2x^2 + 3x)(2x^3 + 3x^2)}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-6x - 6}{x^3 (2x + 3)^2} dx$$

Hence

$$u_1 = -\frac{4}{9(2x + 3)} - \frac{1}{3x^2} + \frac{2}{9x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{6x^3}{(2x^2 + 3x)(2x^3 + 3x^2)}\, dx$$

Which simplifies to

$$u_2 = \int \frac{6}{(2x + 3)^2} dx$$

Hence

$$u_2 = -\frac{3}{2x + 3}$$

Which simplifies to

$$u_1 = -\frac{1}{x^2 (2x + 3)}$$

$$u_2 = -\frac{3}{2x + 3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{2x+3} - \frac{3(-x-1)}{2x+3}$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\left(-\frac{1}{x^3} - \frac{1}{x^2}\right) + c_2\right) x^3\right) + (1) \\ &= 1 + \left(c_1\left(-\frac{1}{x^3} - \frac{1}{x^2}\right) + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = c_2 x^3 - c_1 x - c_1 + 1$$

Summary

The solution(s) found are the following

$$y = c_2 x^3 - c_1 x - c_1 + 1 \tag{1}$$

Verification of solutions

$$y = c_2 x^3 - c_1 x - c_1 + 1$$

Verified OK.

20.3.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv = 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v = 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 2x^2 + 3x \\
B &= -6x - 6 \\
C &= 6 \\
F &= 6
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(2x^2 + 3x\right)(0) + (-6x - 6)(-6) + (6)(-6x - 6) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-12x^3 - 30x^2 - 18xv'' + \left(12x^2 + 36x + 36\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(-12x^3 - 30x^2 - 18x\right) u'(x) + 12\left(x^2 + 3x + 3\right) u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{2u(x^2+3x+3)}{x\left(2x^2+5x+3\right)}
\end{aligned}$$

Where $f(x) = \frac{2x^2+6x+6}{x(2x^2+5x+3)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{2x^2+6x+6}{x\left(2x^2+5x+3\right)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2x^2+6x+6}{x\left(2x^2+5x+3\right)}\,dx \\
\ln\left(u\right) &= -2\ln\left(x+1\right)+\ln\left(2x+3\right)+2\ln\left(x\right)+c_1 \\
u &= \mathrm{e}^{-2\ln(x+1)+\ln(2x+3)+2\ln(x)+c_1} \\
&= c_1\mathrm{e}^{-2\ln(x+1)+\ln(2x+3)+2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{2x^3}{\left(x+1\right)^2}+\frac{3x^2}{\left(x+1\right)^2}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1\left(\frac{2x^3}{\left(x+1\right)^2}+\frac{3x^2}{\left(x+1\right)^2}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1x^2(2x+3)}{\left(x+1\right)^2}\,\mathrm{d}x \\
&= c_1\left(x^2-x-\frac{1}{x+1}\right)+c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-6x-6)\left(c_1\left(x^2-x-\frac{1}{x+1}\right)+c_2\right) \\
&= \left(-6x^3+6x+6\right)c_1-6c_2(x+1)
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -6x - 6$$
$$y_2 = -6x^3 + 6x + 6$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -6x - 6 & -6x^3 + 6x + 6 \\ \frac{d}{dx}(-6x - 6) & \frac{d}{dx}(-6x^3 + 6x + 6) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -6x - 6 & -6x^3 + 6x + 6 \\ -6 & -18x^2 + 6 \end{vmatrix}$$

Therefore

$$W = (-6x - 6)\left(-18x^2 + 6\right) - \left(-6x^3 + 6x + 6\right)(-6)$$

Which simplifies to

$$W = 72x^3 + 108x^2$$

Which simplifies to

$$W = 72x^3 + 108x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-36x^3 + 36x + 36}{(2x^2 + 3x)(72x^3 + 108x^2)} dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^3 + x + 1}{x^3 (2x + 3)^2} dx$$

Hence

$$u_1 = -\frac{23}{54 (2x + 3)} + \frac{1}{18x^2} - \frac{1}{27x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-36x - 36}{(2x^2 + 3x)(72x^3 + 108x^2)} dx$$

Which simplifies to

$$u_2 = \int \frac{-x - 1}{x^3 (2x + 3)^2} dx$$

Hence

$$u_2 = \frac{2}{27 (2x + 3)} + \frac{1}{18x^2} - \frac{1}{27x}$$

Which simplifies to

$$u_1 = \frac{-3x^2 + 1}{12x^3 + 18x^2}$$
$$u_2 = \frac{1}{12x^3 + 18x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(-3x^2 + 1)(-6x - 6)}{12x^3 + 18x^2} + \frac{-6x^3 + 6x + 6}{12x^3 + 18x^2}$$

Which simplifies to

$$y_p(x) = 1$$

Hence the complete solution is

$$\begin{aligned}y(x) &= y_h + y_p \\ &= \left(\left(-6x^3 + 6x + 6\right) c_1 - 6c_2(x+1)\right) + (1) \\ &= \left(-6x^3 + 6x + 6\right) c_1 - 6c_2 x - 6c_2 + 1\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-6x^3 + 6x + 6\right) c_1 - 6c_2 x - 6c_2 + 1 \tag{1}$$

Verification of solutions

$$y = \left(-6x^3 + 6x + 6\right) c_1 - 6c_2 x - 6c_2 + 1$$

Verified OK.

20.3.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(2x^2 + 3x\right) y'' + (-6x - 6) y' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 2x^2 + 3x \\ B &= -6x - 6 \\ C &= 6\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3x^2 + 12x + 18}{(2x^2 + 3x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2 + 12x + 18 \\ t &= (2x^2 + 3x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3x^2 + 12x + 18}{(2x^2 + 3x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 642: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x^2 + 3x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -\frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x + \frac{3}{2}\right)^2} + \frac{4}{3\left(x + \frac{3}{2}\right)} + \frac{2}{x^2} - \frac{4}{3x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

For the pole at $x = -\frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3x^2 + 12x + 18}{(2x^2 + 3x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{3}{2}\\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3x^2+12x+18}{(2x^2+3x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1
$-\frac{3}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+=\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right)\\
&= \frac{3}{2} - \left(\frac{3}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\ge 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2}\right) + (+)[\sqrt{r}]_{\infty} \\
&= \frac{2}{x} - \frac{1}{2\left(x+\frac{3}{2}\right)} + (0) \\
&= \frac{2}{x} - \frac{1}{2\left(x+\frac{3}{2}\right)} \\
&= \frac{3x+6}{x(2x+3)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{2}{x} - \frac{1}{2\left(x+\frac{3}{2}\right)}\right)(0) + \left(\left(-\frac{2}{x^2} + \frac{1}{2\left(x+\frac{3}{2}\right)^2}\right) + \left(\frac{2}{x} - \frac{1}{2\left(x+\frac{3}{2}\right)}\right)^2 - \left(\frac{3x^2+12x+18}{\left(2x^2+3x\right)^2}\right)\right) = 0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{2}{x} - \frac{1}{2\left(x+\frac{3}{2}\right)}\right) dx} \\
&= \frac{x^2}{\sqrt{2x+3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x-6}{2x^2+3x} \, dx} \\
&= z_1 e^{\frac{\ln(2x+3)}{2} + \ln(x)} \\
&= z_1 \left(\sqrt{2x+3}\, x\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6x-6}{2x^2+3x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\ln(2x+3)+2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{-x-1}{x^3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^3) + c_2 \left(x^3 \left(\frac{-x-1}{x^3} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\left(2x^2 + 3x\right) y'' + (-6x - 6)\, y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^3 + c_2(-x - 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^3 \\ y_2 &= -x - 1 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^3 & -x - 1 \\ \frac{d}{dx}(x^3) & \frac{d}{dx}(-x - 1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^3 & -x - 1 \\ 3x^2 & -1 \end{vmatrix}$$

Therefore

$$W = (x^3)(-1) - (-x - 1)(3x^2)$$

Which simplifies to

$$W = 2x^3 + 3x^2$$

Which simplifies to

$$W = 2x^3 + 3x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-6x-6}{(2x^2+3x)(2x^3+3x^2)}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-6x-6}{x^3 (2x+3)^2} dx$$

Hence

$$u_1 = -\frac{4}{9(2x+3)} - \frac{1}{3x^2} + \frac{2}{9x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{6x^3}{(2x^2+3x)(2x^3+3x^2)}\, dx$$

Which simplifies to

$$u_2 = \int \frac{6}{(2x+3)^2} dx$$

Hence

$$u_2 = -\frac{3}{2x+3}$$

Which simplifies to

$$u_1 = -\frac{1}{x^2(2x+3)}$$

$$u_2 = -\frac{3}{2x+3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{2x+3} - \frac{3(-x-1)}{2x+3}$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x^3 + c_2(-x-1)\right) + (1)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x^3 + c_2(-x-1) + 1 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3 + c_2(-x-1) + 1
$$

Verified OK.
Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 15

```
dsolve((3*x+2*x^2)*diff(y(x),x$2)-6*(1+x)*diff(y(x),x)+6*y(x)=6,y(x), singsol=all)
```

$$
y(x) = c_2 x^3 + c_1 x + c_1 + 1
$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 20

```
DSolve[(3*x+2*x^2)*y''[x]-6*(1+x)*y'[x]+6*y[x]==6,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^3 - c_2(x+1) + 1$$

20.4 problem 639

Internal problem ID [15408]
Internal file name [OUTPUT/15408_Wednesday_May_08_2024_03_58_15_PM_82899032/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 639.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(\ln(x)-1)\,y''-xy'+y=0}$$

Given that one solution of the ode is

$$y_1=x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int\frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x)=-\frac{1}{x\,(\ln(x)-1)}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -\frac{1}{x(\ln(x)-1)}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{\ln(x)-1}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{\ln(x)-1}{x^2}dx\right)$$

$$y_2(x) = -\ln(x)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x - c_2\ln(x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x - c_2\ln(x) \tag{1}$$

Verification of solutions

$$y = c_1x - c_2\ln(x)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
   Change of variables used:
      [x = exp(t)]
   Linear ODE actually solved:
      u(t)-t*diff(u(t),t)+(t-1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 12

```
dsolve([x^2*(ln(x)-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,x],singsol=all)
```

$$y(x) = c_1 x + c_2 \ln(x)$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 16

```
DSolve[x^2*(Log[x]-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x - c_2 \log(x)$$

20.5 problem 640

Internal problem ID [15409]
Internal file name [OUTPUT/15409_Wednesday_May_08_2024_03_58_15_PM_32792851/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 640.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + (\tan(x) - 2\cot(x))\, y' + 2\cot(x)^2\, y = 0}$$

Given that one solution of the ode is

$$y_1 = \sin(x)$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \tan(x) - 2\cot(x)$$

Therefore

$$y_2(x) = \sin(x)\left(\int \frac{e^{-\left(\int (\tan(x)-2\cot(x))dx\right)}}{\sin(x)^2} dx\right)$$

$$y_2(x) = \sin(x)\int \frac{e^{2\ln(\sin(x))+\ln(\cos(x))}}{\sin(x)^2}, dx$$

$$y_2(x) = \sin(x)\left(\int \csc(x)^2 \sin(x)^2 \cos(x)\, dx\right)$$

$$y_2(x) = \sin(x)^2$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sin(x)\, c_1 + c_2 \sin(x)^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sin(x)\, c_1 + c_2 \sin(x)^2 \tag{1}$$

Verification of solutions

$$y = \sin(x)\, c_1 + c_2 \sin(x)^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      -2*u(t)+2*t*diff(u(t),t)-t^2*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)+(tan(x)-2*cot(x))*diff(y(x),x)+2*cot(x)^2*y(x)=0,sin(x)],singsol=all)
```

$$y(x) = \sin(x)(\sin(x) c_2 + c_1)$$

✓ Solution by Mathematica

Time used: 2.22 (sec). Leaf size: 27

```
DSolve[y''[x]+(Tan[x]-2*Cot[x])*y'[x]+2*Cot[x]^2*y[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_1 \sqrt{-\sin^2(x)} - c_2 \sin^2(x)$$

20.6 problem 641

Internal problem ID [15410]
Internal file name [`OUTPUT/15410_Wednesday_May_08_2024_03_58_16_PM_27062736/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 641.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' \tan(x) + \cos(x)^2 y = 0}$$

Given that one solution of the ode is

$$y_1 = \cos(\sin(x))$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \tan(x)$$

Therefore

$$y_2(x) = \cos(\sin(x)) \left(\int \frac{e^{-\left(\int \tan(x)dx\right)}}{\cos(\sin(x))^2} dx\right)$$

$$y_2(x) = \cos(\sin(x)) \int \frac{\cos(x)}{\cos(\sin(x))^2}, dx$$

$$y_2(x) = \cos(\sin(x)) \left(\int \sec(\sin(x))^2 \cos(x)\, dx\right)$$

$$y_2(x) = \cos(\sin(x)) \tan(\sin(x))$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \cos(\sin(x))\, c_1 + c_2 \cos(\sin(x)) \tan(\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(\sin(x))\, c_1 + c_2 \cos(\sin(x)) \tan(\sin(x)) \tag{1}$$

Verification of solutions

$$y = \cos(\sin(x))\, c_1 + c_2 \cos(\sin(x)) \tan(\sin(x))$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   Change of variables used:
      [x = arcsin(t)]
   Linear ODE actually solved:
      (-2*t^2+2)*u(t)+(-2*t^2+2)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)+tan(x)*diff(y(x),x)+cos(x)^2*y(x)=0,cos(sin(x))],singsol=all)
```

$$y(x) = c_1 \sin(\sin(x)) + c_2 \cos(\sin(x))$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 18

```
DSolve[y''[x]+Tan[x]*y'[x]+Cos[x]^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \sin(\sin(x)) + c_1 \cos(\sin(x))$$

20.7 problem 642

Internal problem ID [15411]
Internal file name [`OUTPUT/15411_Wednesday_May_08_2024_03_58_17_PM_35724180/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 642.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(x^2+1)\,y''+xy'-y=1}$$

Given that one solution of the ode is

$$y_1 = x$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2+1, B=x, C=-1, f(x)=1$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(x^2+1)\,y''+xy'-y=0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{x}{x^2 + 1}$$

Therefore

$$y_2(x) = x \left(\int \frac{e^{-\left(\int \frac{x}{x^2+1} dx\right)}}{x^2} dx \right)$$

$$y_2(x) = x \int \frac{\frac{1}{\sqrt{x^2+1}}}{x^2}, dx$$

$$y_2(x) = x \left(\int \frac{1}{\sqrt{x^2+1}\, x^2} dx \right)$$

$$y_2(x) = -\sqrt{x^2+1}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x - c_2 \sqrt{x^2+1} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x - c_2 \sqrt{x^2+1}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = -\sqrt{x^2+1}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -\sqrt{x^2+1} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(-\sqrt{x^2+1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -\sqrt{x^2+1} \\ 1 & -\frac{x}{\sqrt{x^2+1}} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{x}{\sqrt{x^2+1}}\right) - \left(-\sqrt{x^2+1}\right)(1)$$

Which simplifies to

$$W = \frac{1}{\sqrt{x^2+1}}$$

Which simplifies to

$$W = \frac{1}{\sqrt{x^2+1}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\sqrt{x^2+1}}{\sqrt{x^2+1}}\,dx$$

Which simplifies to

$$u_1 = -\int (-1)\,dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x}{\sqrt{x^2+1}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x}{\sqrt{x^2+1}}dx$$

Hence

$$u_2 = \sqrt{x^2+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x - c_2\sqrt{x^2+1}\right) + (-1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x - c_2\sqrt{x^2+1} - 1 \tag{1}$$

Verification of solutions

$$y = c_1 x - c_2\sqrt{x^2+1} - 1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([(1+x^2)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=1,x],singsol=all)
```

$$y(x) = \sqrt{x^2+1}\, c_2 + c_1 x - 1$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 80

```
DSolve[(1+x^2)*y''[x]+x*y'[x]-y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\sqrt{x^2+1} + (c_1 - ic_2)x^2 + x\left(c_1\left(-\sqrt{x^2+1}\right) + ic_2\sqrt{x^2+1} + 1\right) + c_1}{\sqrt{x^2+1} - x}$$

20.8 problem 643

Internal problem ID [15412]
Internal file name [`OUTPUT/15412_Wednesday_May_08_2024_03_58_18_PM_47095460/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 643.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - xy' - 3y = 5x^4}$$

Given that one solution of the ode is

$$y_1 = \frac{1}{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = -3, f(x) = 5x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' - 3y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{1}{x}$$

Therefore

$$y_2(x) = \frac{\int e^{-(\int -\frac{1}{x} dx)} x^2 dx}{x}$$

$$y_2(x) = \frac{1}{x} \int \frac{x}{\frac{1}{x^2}}, dx$$

$$y_2(x) = \frac{\int x^3 dx}{x}$$

$$y_2(x) = \frac{x^3}{4}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1}{x} + \frac{c_2 x^3}{4} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1}{x} + \frac{c_2 x^3}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = \frac{x^3}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^3}{4} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^3}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^3}{4} \\ -\frac{1}{x^2} & \frac{3x^2}{4} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{3x^2}{4}\right) - \left(\frac{x^3}{4}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{5x^7}{4}}{x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{5x^4}{4}dx$$

Hence

$$u_1 = -\frac{x^5}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{5x^3}{x^3}\,dx$$

Which simplifies to

$$u_2 = \int 5dx$$

Hence

$$u_2 = 5x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x^3}{4}\right) + (x^4) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x^3}{4} + x^4 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x^3}{4} + x^4$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([x^2*diff(y(x),x$2)-x*diff(y(x),x)-3*y(x)=5*x^4,1/x],singsol=all)
```

$$y(x) = \frac{c_2 x^4 + x^5 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]-x*y'[x]-3*y[x]==5*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^5 + c_2 x^4 + c_1}{x}$$

20.9 problem 644

Internal problem ID [15413]
Internal file name [`OUTPUT/15413_Wednesday_May_08_2024_03_58_18_PM_36343905/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 644.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=(x-1)^2\,\mathrm{e}^{x}}$$

Given that one solution of the ode is

$$y_1=\mathrm{e}^{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x-1, B=-x, C=1, f(x)=(x-1)^2\,\mathrm{e}^{x}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(x-1)\,y''-xy'+y=0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{x}{x-1}$$

Therefore

$$y_2(x) = e^x\left(\int e^{-\left(\int -\frac{x}{x-1}dx\right)}e^{-2x}dx\right)$$

$$y_2(x) = e^x\int \frac{e^{x+\ln(x-1)}}{e^{2x}}, dx$$

$$y_2(x) = e^x\left(\int e^{-x}(x-1)\,dx\right)$$

$$y_2(x) = -e^x x\,e^{-x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= e^x c_1 - c_2 e^x x\,e^{-x} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 - c_2 e^x x\,e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = -e^x x\, e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & -e^x x\, e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(-e^x x\, e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & -e^x x\, e^{-x} \\ e^x & -e^{-x} e^x \end{vmatrix}$$

Therefore

$$W = (e^x)\left(-e^{-x} e^x\right) - \left(-e^x x\, e^{-x}\right)(e^x)$$

Which simplifies to

$$W = -e^{-x} e^{2x} + e^{2x} x\, e^{-x}$$

Which simplifies to

$$W = (x-1)\, e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-e^{2x} x\, e^{-x} (x-1)^2}{(x-1)^2 e^x}\, dx$$

Which simplifies to

$$u_1 = -\int -x dx$$

Hence

$$u_1 = \frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x-1)^2 \, e^{2x}}{(x-1)^2 \, e^{x}} \, dx$$

Which simplifies to

$$u_2 = \int e^{x} dx$$

Hence

$$u_2 = e^{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{x} x^2}{2} - e^{2x} x \, e^{-x}$$

Which simplifies to

$$y_p(x) = \frac{e^{x} x(x-2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{x} c_1 - c_2 e^{x} x \, e^{-x}\right) + \left(\frac{e^{x} x(x-2)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{x} c_1 - c_2 x + \frac{e^{x} x(x-2)}{2}$$

Summary
The solution(s) found are the following

$$y = e^{x} c_1 - c_2 x + \frac{e^{x} x(x-2)}{2} \tag{1}$$

Verification of solutions

$$y = e^x c_1 - c_2 x + \frac{e^x x(x-2)}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([(x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=(x-1)^2*exp(x),exp(x)],singsol=all)
```

$$y(x) = \frac{(x^2 + 2c_1 - 2x)\, e^x}{2} + c_2 x$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 28

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==(x-1)^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(\frac{x^2}{2} - x + c_1 \right) - c_2 x$$

20.10 problem 645

Internal problem ID [15414]
Internal file name [`OUTPUT/15414_Wednesday_May_08_2024_03_58_19_PM_57410697/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 645.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode_form_A", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + \mathrm{e}^{-2x}y = \mathrm{e}^{-3x}}$$

Given that one solution of the ode is

$$y_1 = \cos\left(\mathrm{e}^{-x}\right)$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = \mathrm{e}^{-2x}, f(x) = \mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + \mathrm{e}^{-2x}y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = 1$$

Therefore

$$y_2(x) = \cos\left(e^{-x}\right) \left(\int \frac{e^{-\left(\int 1 dx\right)}}{\cos\left(e^{-x}\right)^2} dx \right)$$

$$y_2(x) = \cos\left(e^{-x}\right) \int \frac{e^{-x}}{\cos\left(e^{-x}\right)^2}, dx$$

$$y_2(x) = \cos\left(e^{-x}\right) \left(\int \sec\left(e^{-x}\right)^2 e^{-x} dx \right)$$

$$y_2(x) = -\cos\left(e^{-x}\right) \tan\left(e^{-x}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \cos\left(e^{-x}\right) c_1 - c_2 \cos\left(e^{-x}\right) \tan\left(e^{-x}\right) \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \cos\left(e^{-x}\right) c_1 - c_2 \cos\left(e^{-x}\right) \tan\left(e^{-x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos\left(\mathrm{e}^{-x}\right) \\
y_2 &= -\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \cos\left(\mathrm{e}^{-x}\right) & -\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right) \\ \frac{d}{dx}\left(\cos\left(\mathrm{e}^{-x}\right)\right) & \frac{d}{dx}\left(-\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right)\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \cos\left(\mathrm{e}^{-x}\right) & -\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right) \\ \mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right) & -\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right) + \cos\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-x}\left(1+\tan\left(\mathrm{e}^{-x}\right)^2\right) \end{vmatrix}
$$

Therefore

$$
\begin{aligned}
W = {} & \left(\cos\left(\mathrm{e}^{-x}\right)\right)\left(-\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right) + \cos\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-x}\left(1+\tan\left(\mathrm{e}^{-x}\right)^2\right)\right) \\
& - \left(-\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right)\right)\left(\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

Which simplifies to

$$
W = \cos\left(\mathrm{e}^{-x}\right)^2\tan\left(\mathrm{e}^{-x}\right)^2\mathrm{e}^{-x} + \cos\left(\mathrm{e}^{-x}\right)^2\mathrm{e}^{-x}
$$

Which simplifies to

$$
W = \mathrm{e}^{-x}
$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-3x}}{\mathrm{e}^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int -\sin\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-2x} dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{-x}\tan\left(\frac{\mathrm{e}^{-x}}{2}\right)^2 - \mathrm{e}^{-x} + 2\tan\left(\frac{\mathrm{e}^{-x}}{2}\right)}{1+\tan\left(\frac{\mathrm{e}^{-x}}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-3x}}{\mathrm{e}^{-x}}\, dx$$

Which simplifies to

$$u_2 = \int \cos\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-2x} dx$$

Hence

$$u_2 = \frac{-2\,\mathrm{e}^{-x}\tan\left(\frac{\mathrm{e}^{-x}}{2}\right) - 2}{1+\tan\left(\frac{\mathrm{e}^{-x}}{2}\right)^2}$$

Which simplifies to

$$u_1 = \mathrm{e}^{-x}\cos\left(\mathrm{e}^{-x}\right) - \sin\left(\mathrm{e}^{-x}\right)$$
$$u_2 = -\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right) - \cos\left(\mathrm{e}^{-x}\right) - 1$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(\mathrm{e}^{-x}\cos\left(\mathrm{e}^{-x}\right) - \sin\left(\mathrm{e}^{-x}\right)\right)\cos\left(\mathrm{e}^{-x}\right) \\ & - \left(-\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right) - \cos\left(\mathrm{e}^{-x}\right) - 1\right)\cos\left(\mathrm{e}^{-x}\right)\tan\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-x} + \sin\left(\mathrm{e}^{-x}\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\cos\left(\mathrm{e}^{-x}\right) c_1 - c_2 \cos\left(\mathrm{e}^{-x}\right) \tan\left(\mathrm{e}^{-x}\right)\right) + \left(\mathrm{e}^{-x} + \sin\left(\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \cos\left(\mathrm{e}^{-x}\right) c_1 - c_2 \sin\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{-x} + \sin\left(\mathrm{e}^{-x}\right)
$$

Summary

The solution(s) found are the following

$$
y = \cos\left(\mathrm{e}^{-x}\right) c_1 - c_2 \sin\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{-x} + \sin\left(\mathrm{e}^{-x}\right) \tag{1}
$$

Verification of solutions

$$
y = \cos\left(\mathrm{e}^{-x}\right) c_1 - c_2 \sin\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{-x} + \sin\left(\mathrm{e}^{-x}\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(y(x),x$2)+diff(y(x),x)+exp(-2*x)*y(x)=exp(-3*x),cos(exp(-x))],singsol=all)
```

$$
y(x) = \sin\left(\mathrm{e}^{-x}\right) c_2 + \cos\left(\mathrm{e}^{-x}\right) c_1 + \sin\left(\mathrm{e}^{-x}\right) + \mathrm{e}^{-x}
$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 30

```
DSolve[y''[x]+y'[x]+Exp[-2*x]*y[x]==Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} + c_1 \cos\left(e^{-x}\right) - c_2 \sin\left(e^{-x}\right)$$

20.11 problem 646

Internal problem ID [15415]
Internal file name [`OUTPUT/15415_Wednesday_May_08_2024_03_58_19_PM_49778489/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 646.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{\left(x^{4}-x^{3}\right) y''+\left(2x^{3}-2x^{2}-x\right) y'-y=\frac{\left(x-1\right)^{2}}{x}}$$

Given that one solution of the ode is

$$y_1=\frac{1}{x}$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^4-x^3, B=2x^3-2x^2-x, C=-1, f(x)=\frac{(x-1)^2}{x}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$\left(x^{4}-x^{3}\right) y''+\left(2x^{3}-2x^{2}-x\right) y'-y=0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2x^3 - 2x^2 - x}{x^4 - x^3}$$

Therefore

$$y_2(x) = \frac{\int e^{-\left(\int \frac{2x^3-2x^2-x}{x^4-x^3} dx\right)} x^2 dx}{x}$$

$$y_2(x) = \frac{1}{x} \int \frac{e^{\ln(x-1)+\frac{1}{x}-3\ln(x)}}{\frac{1}{x^2}}, dx$$

$$y_2(x) = \frac{\int \frac{(x-1)e^{\frac{1}{x}}}{x} dx}{x}$$

$$y_2(x) = e^{\frac{1}{x}}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1}{x} + c_2 e^{\frac{1}{x}} \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1}{x} + c_2 e^{\frac{1}{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$

$$y_2 = e^{\frac{1}{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & e^{\frac{1}{x}} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(e^{\frac{1}{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & e^{\frac{1}{x}} \\ -\frac{1}{x^2} & -\frac{e^{\frac{1}{x}}}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(-\frac{e^{\frac{1}{x}}}{x^2}\right) - \left(e^{\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{(x-1)\,e^{\frac{1}{x}}}{x^3}$$

Which simplifies to

$$W = \frac{(x-1)\,e^{\frac{1}{x}}}{x^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{\frac{1}{x}}(x-1)^2}{x}}{\frac{(x^4-x^3)(x-1)e^{\frac{1}{x}}}{x^3}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x}dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{(x-1)^2}{x^2}}{\frac{(x^4-x^3)(x-1)e^{\frac{1}{x}}}{x^3}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-\frac{1}{x}}}{x^2}dx$$

Hence

$$u_2 = e^{-\frac{1}{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{x} + e^{-\frac{1}{x}}e^{\frac{1}{x}}$$

Which simplifies to

$$y_p(x) = \frac{-\ln(x) + x}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + c_2 e^{\frac{1}{x}}\right) + \left(\frac{-\ln(x) + x}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 e^{\frac{1}{x}} + \frac{-\ln(x) + x}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 e^{\frac{1}{x}} + \frac{-\ln(x) + x}{x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve([(x^4-x^3)*diff(y(x),x$2)+(2*x^3-2*x^2-x)*diff(y(x),x)-y(x)=(x-1)^2/x,1/x],singsol=al
```

$$y(x) = \frac{e^{\frac{1}{x}} c_1 x - \ln(x) + c_2 + x}{x}$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 27

```
DSolve[(x^4-x^3)*y''[x]+(2*x^3-2*x^2-x)*y'[x]-y[x]==(x-1)^2/x,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{x - \log(x) + c_2\left(-e^{\frac{1}{x}}\right)x + c_1}{x}$$

20.12 problem 647

Internal problem ID [15416]
Internal file name [`OUTPUT/15416_Wednesday_May_08_2024_03_58_20_PM_14284348/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 647.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_bessel_ode_form_A", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' + e^{2x} y = x\, e^{2x} - 1}$$

Given that one solution of the ode is

$$y_1 = \sin(e^x)$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = e^{2x}, f(x) = x\, e^{2x} - 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' + e^{2x} y = 0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -1$$

Therefore

$$y_2(x) = \sin(e^x) \left(\int \frac{e^{-(\int (-1) dx)}}{\sin(e^x)^2} dx \right)$$

$$y_2(x) = \sin(e^x) \int \frac{e^x}{\sin(e^x)^2}, dx$$

$$y_2(x) = \sin(e^x) \left(\int \csc(e^x)^2 e^x dx \right)$$

$$y_2(x) = -\sin(e^x) \cot(e^x)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= \sin(e^x) c_1 - c_2 \sin(e^x) \cot(e^x) \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = \sin(e^x) c_1 - c_2 \sin(e^x) \cot(e^x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sin(e^x)$$
$$y_2 = -\sin(e^x)\cot(e^x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sin(e^x) & -\sin(e^x)\cot(e^x) \\ \frac{d}{dx}(\sin(e^x)) & \frac{d}{dx}(-\sin(e^x)\cot(e^x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sin(e^x) & -\sin(e^x)\cot(e^x) \\ e^x\cos(e^x) & -e^x\cos(e^x)\cot(e^x) - \sin(e^x)e^x(-1-\cot(e^x)^2) \end{vmatrix}$$

Therefore

$$W = (\sin(e^x))\left(-e^x\cos(e^x)\cot(e^x) - \sin(e^x)e^x(-1-\cot(e^x)^2)\right) - (-\sin(e^x)\cot(e^x))(e^x\cos(e^x))$$

Which simplifies to

$$W = \cot(e^x)^2\sin(e^x)^2 e^x + \sin(e^x)^2 e^x$$

Which simplifies to

$$W = e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\sin\left(e^{x}\right) \cot\left(e^{x}\right) \left(x\,e^{2x}-1\right)}{e^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\cos\left(e^{x}\right) \left(x\,e^{x}-e^{-x}\right) dx$$

Hence

$$u_1 = -\frac{\left(-1+\tan\left(\frac{e^{x}}{2}\right)^{2}-2\,e^{x} x \tan\left(\frac{e^{x}}{2}\right)\right) e^{-x}}{1+\tan\left(\frac{e^{x}}{2}\right)^{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin\left(e^{x}\right) \left(x\,e^{2x}-1\right)}{e^{x}}\,dx$$

Which simplifies to

$$u_2 = \int \sin\left(e^{x}\right) \left(x\,e^{x}-e^{-x}\right) dx$$

Hence

$$u_2 = \frac{\left(e^{x} x \tan\left(\frac{e^{x}}{2}\right)^{2}-x\,e^{x}+2\tan\left(\frac{e^{x}}{2}\right)\right) e^{-x}}{1+\tan\left(\frac{e^{x}}{2}\right)^{2}}$$

Which simplifies to

$$u_1 = x\sin\left(e^{x}\right)+\cos\left(e^{x}\right) e^{-x}$$
$$u_2 = -x\cos\left(e^{x}\right)+\sin\left(e^{x}\right) e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x\sin\left(e^{x}\right)+\cos\left(e^{x}\right) e^{-x}\right) \sin\left(e^{x}\right)-\left(-x\cos\left(e^{x}\right)+\sin\left(e^{x}\right) e^{-x}\right) \sin\left(e^{x}\right) \cot\left(e^{x}\right)$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\sin(e^x) c_1 - c_2 \sin(e^x) \cot(e^x)) + (x)
\end{aligned}
$$

Which simplifies to

$$y = \sin(e^x) c_1 - c_2 \cos(e^x) + x$$

Summary

The solution(s) found are the following

$$y = \sin(e^x) c_1 - c_2 \cos(e^x) + x \tag{1}$$

Verification of solutions

$$y = \sin(e^x) c_1 - c_2 \cos(e^x) + x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)-diff(y(x),x)+exp(2*x)*y(x)=x*exp(2*x)-1,sin(exp(x))],singsol=all)
```

$$y(x) = \sin(e^x) c_2 + \cos(e^x) c_1 + x$$

✓ Solution by Mathematica

Time used: 0.341 (sec). Leaf size: 21

```
DSolve[y''[x]-y'[x]+Exp[2*x]*y[x]==x*Exp[2*x]-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 \cos(e^x) + c_2 \sin(e^x)$$

20.13 problem 648

Internal problem ID [15417]
Internal file name [OUTPUT/15417_Wednesday_May_08_2024_03_58_21_PM_48278293/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 648.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x-1)\, y''-(2x-1)\, y'+2y=(2x-3)\, x^{2}}$$

Given that one solution of the ode is

$$y_1 = x^2$$

This is second order nonhomogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2 - x, B = -2x + 1, C = 2, f(x) = 2x^3 - 3x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the inhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$\left(x^{2}-x\right) y''+\left(-2x+1\right) y'+2y=0$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2x + 1}{x^2 - x}$$

Therefore

$$y_2(x) = x^2 \left(\int \frac{e^{-\left(\int \frac{-2x+1}{x^2-x} dx\right)}}{x^4} dx \right)$$

$$y_2(x) = x^2 \int \frac{x(x-1)}{x^4}, dx$$

$$y_2(x) = x^2 \left(\int \frac{x-1}{x^3} dx \right)$$

$$y_2(x) = x^2 \left(\frac{1}{2x^2} - \frac{1}{x} \right)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 + c_2 x^2 \left(\frac{1}{2x^2} - \frac{1}{x} \right) \end{aligned}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x^2 + c_2 x^2 \left(\frac{1}{2x^2} - \frac{1}{x} \right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^2\left(\frac{1}{2x^2} - \frac{1}{x}\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^2\left(\frac{1}{2x^2} - \frac{1}{x}\right) \\ \frac{d}{dx}(x^2) & \frac{d}{dx}\left(x^2\left(\frac{1}{2x^2} - \frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^2\left(\frac{1}{2x^2} - \frac{1}{x}\right) \\ 2x & 2x\left(\frac{1}{2x^2} - \frac{1}{x}\right) + x^2\left(-\frac{1}{x^3} + \frac{1}{x^2}\right) \end{vmatrix}$$

Therefore

$$W = (x^2)\left(2x\left(\frac{1}{2x^2} - \frac{1}{x}\right) + x^2\left(-\frac{1}{x^3} + \frac{1}{x^2}\right)\right) - \left(x^2\left(\frac{1}{2x^2} - \frac{1}{x}\right)\right)(2x)$$

Which simplifies to

$$W = x(x - 1)$$

Which simplifies to

$$W = x(x-1)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 \left(\frac{1}{2x^2} - \frac{1}{x}\right)(2x^3 - 3x^2)}{(x^2 - x)\, x\,(x-1)}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-4x^2 + 8x - 3}{2\,(x-1)^2} dx$$

Hence

$$u_1 = 2x + \frac{1}{2x-2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(2x^3 - 3x^2)}{(x^2 - x)\, x\,(x-1)}\, dx$$

Which simplifies to

$$u_2 = \int \frac{x^2(2x-3)}{(x-1)^2} dx$$

Hence

$$u_2 = x^2 + x + \frac{1}{x-1}$$

Which simplifies to

$$u_1 = 2x + \frac{1}{2x-2}$$
$$u_2 = x^2 + x + \frac{1}{x-1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(2x + \frac{1}{2x-2}\right) x^2 + \left(x^2 + x + \frac{1}{x-1}\right) x^2 \left(\frac{1}{2x^2} - \frac{1}{x}\right)$$

Which simplifies to

$$y_p(x) = x^3 - \frac{1}{2}x^2 + x - \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 x^2 \left(\frac{1}{2x^2} - \frac{1}{x}\right)\right) + \left(x^3 - \frac{1}{2}x^2 + x - \frac{1}{2}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 x^2 - c_2 x + \frac{1}{2}c_2 + x^3 - \frac{1}{2}x^2 + x - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 - c_2 x + \frac{1}{2}c_2 + x^3 - \frac{1}{2}x^2 + x - \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 - c_2 x + \frac{1}{2}c_2 + x^3 - \frac{1}{2}x^2 + x - \frac{1}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve([x*(x-1)*diff(y(x),x$2)-(2*x-1)*diff(y(x),x)+2*y(x)=x^2*(2*x-3),x^2],singsol=all)
```

$$y(x) = c_2x^2 + x^3 - 2c_1x + c_1$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 40

```
DSolve[x*(x-1)*y''[x]-(2*x-1)*y'[x]+2*y[x]==x^2*(2*x-3),y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to x^3 + \left(-\frac{1}{2} + c_1\right)x^2 + (1 - 2c_1 + c_2)x - \frac{1}{2} + c_1 - \frac{c_2}{2}$$

20.14 problem 653

Internal problem ID [15418]
Internal file name [`OUTPUT/15418_Wednesday_May_08_2024_03_58_22_PM_10368088/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 653.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \frac{1}{\sin(x)}}$$

20.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cot(x)\,dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\,x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (-\cos(x)\, x + \ln(\sin(x)) \sin(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - \cos(x)\, x + \ln(\sin(x)) \sin(x) \tag{1}
$$

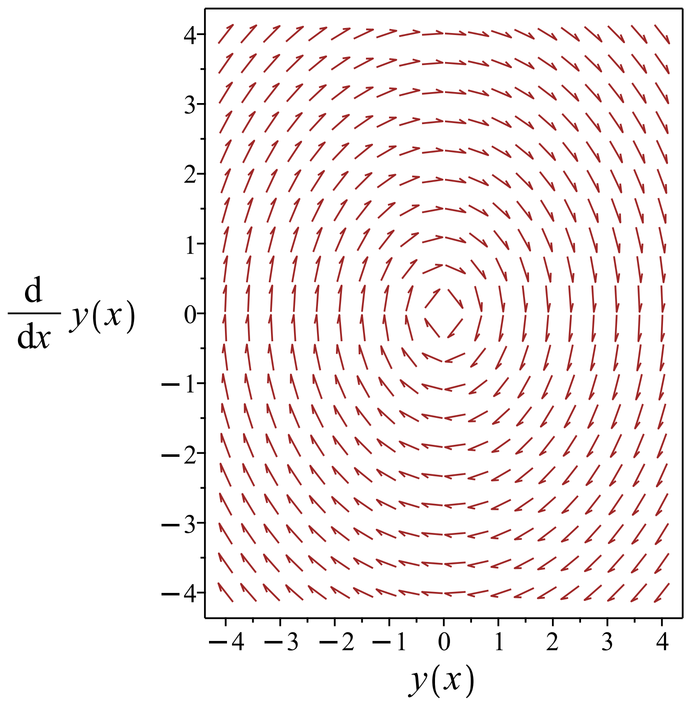

Figure 775: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - \cos(x)\, x + \ln(\sin(x)) \sin(x)
$$

Verified OK.

20.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 643: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cot(x)\,dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\,x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (-\cos(x)\,x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) - \cos(x)\,x + \ln(\sin(x))\sin(x) \tag{1}$$

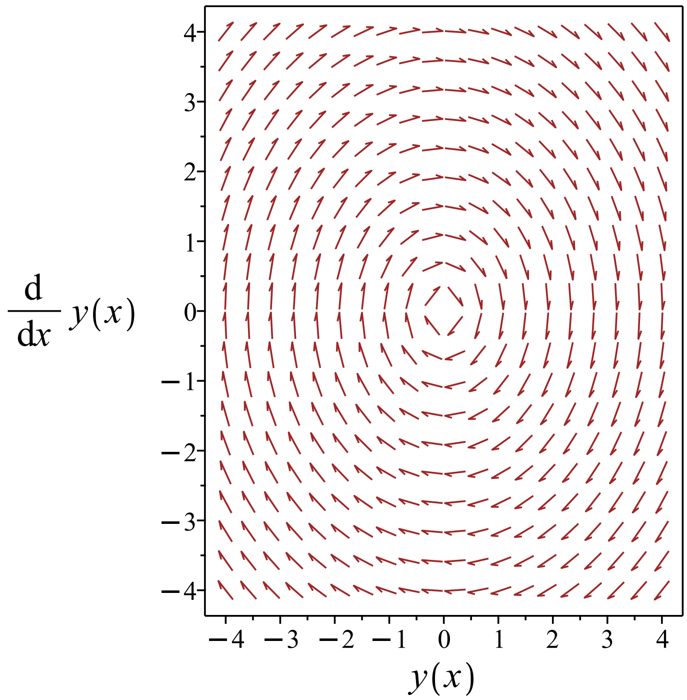

Figure 776: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x)\, x + \ln(\sin(x)) \sin(x)$$

Verified OK.

20.14.3 Maple step by step solution

Let's solve

$y'' + y = \csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(x)\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int 1 dx\right) + \sin(x)\left(\int \cot(x) dx\right)$
- Compute integrals

  $y_p(x) = -\cos(x) x + \ln(\sin(x)) \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=1/sin(x),y(x), singsol=all)
```

$$y(x) = \ln(\sin(x))\sin(x) + (c_1 - x)\cos(x) + \sin(x)\,c_2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 24

```
DSolve[y''[x]+y[x]==1/Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + c_1)\cos(x) + \sin(x)(\log(\sin(x)) + c_2)$$

20.15 problem 654

Internal problem ID [15419]
Internal file name [`OUTPUT/15419_Wednesday_May_08_2024_03_58_22_PM_40415851/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 654.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = \frac{1}{1 + e^{x}}}$$

20.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = \frac{1}{1+e^x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + \mathrm{e}^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + \mathrm{e}^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & e^{-x} \\ \frac{d}{dx}(1) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (1)\left(-e^{-x}\right) - \left(e^{-x}\right)(0)$$

Which simplifies to

$$W = -e^{-x}$$

Which simplifies to

$$W = -e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-x}}{1+e^x}}{-e^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{1+e^x}dx$$

Hence

$$u_1 = -\ln(1+e^x) + \ln(e^x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{1+e^x}}{-e^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{e^x}{1+e^x}dx$$

Hence

$$u_2 = -\ln(1 + e^x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(1 + e^x) + \ln(e^x) - \ln(1 + e^x)\, e^{-x}$$

Which simplifies to

$$y_p(x) = (-e^{-x} - 1) \ln(1 + e^x) + \ln(e^x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + e^{-x} c_2) + ((-e^{-x} - 1) \ln(1 + e^x) + \ln(e^x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-x} c_2 + (-e^{-x} - 1) \ln(1 + e^x) + \ln(e^x) \tag{1}$$

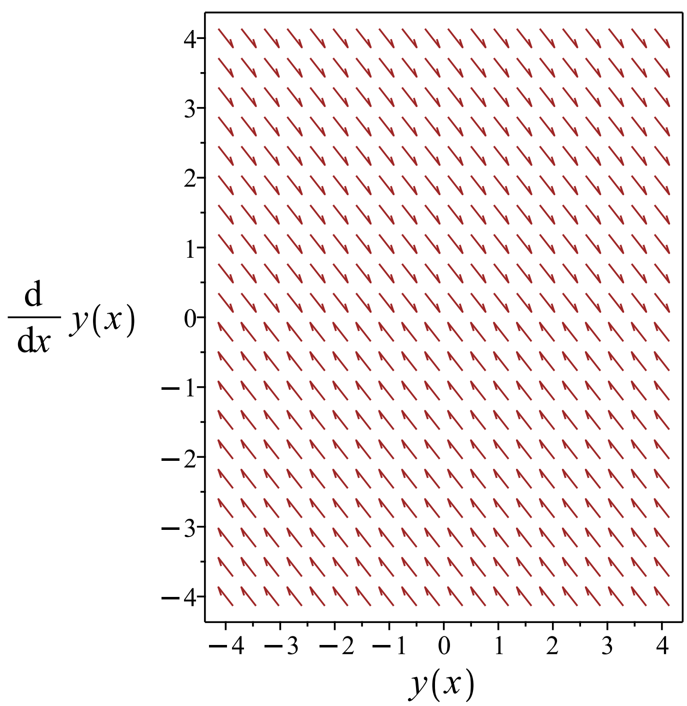

Figure 777: Slope field plot

Verification of solutions

$$y = c_1 + e^{-x}c_2 + \left(-e^{-x} - 1\right)\ln\left(1 + e^x\right) + \ln\left(e^x\right)$$

Verified OK.

20.15.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'' + y'\right) dx = \int \frac{1}{1 + e^x} dx$$
$$y' + y = -\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = -\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1$$

Hence the ode is

$$y' + y = -\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^x$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1\right)$$
$$\frac{d}{dx}\left(e^x y\right) = \left(e^x\right)\left(-\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1\right)$$
$$d\left(e^x y\right) = \left(\left(-\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1\right)e^x\right) dx$$

Integrating gives

$$e^x y = \int \left(-\ln\left(1 + e^x\right) + \ln\left(e^x\right) + c_1\right)e^x \, dx$$
$$e^x y = e^x c_1 - \left(1 + e^x\right)\ln\left(1 + e^x\right) + 1 + e^x\ln\left(e^x\right) + c_2$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(e^x c_1 - (1 + e^x)\ln(1 + e^x) + 1 + e^x \ln(e^x)\right) + e^{-x} c_2$$

which simplifies to

$$y = \left(-e^{-x} - 1\right)\ln(1 + e^x) + (1 + c_2)\, e^{-x} + c_1 + \ln(e^x)$$

Summary

The solution(s) found are the following

$$y = \left(-e^{-x} - 1\right)\ln(1 + e^x) + (1 + c_2)\, e^{-x} + c_1 + \ln(e^x) \tag{1}$$

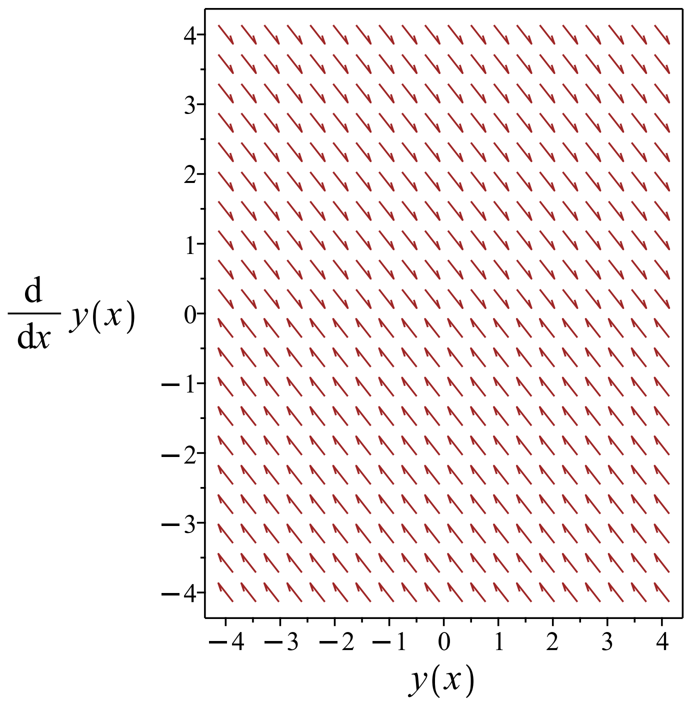

Figure 778: Slope field plot

Verification of solutions

$$y = \left(-e^{-x} - 1\right)\ln(1 + e^x) + (1 + c_2)\, e^{-x} + c_1 + \ln(e^x)$$

Verified OK.

20.15.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - \frac{1}{1 + e^x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \frac{1}{1 + e^x} \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = \frac{1}{1 + e^x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu)\left(\frac{1}{1 + e^x}\right) \\ \frac{d}{dx}(p\,e^x) &= (e^x)\left(\frac{1}{1 + e^x}\right) \\ d(p\,e^x) &= \left(\frac{e^x}{1 + e^x}\right) dx \end{aligned}$$

Integrating gives

$$
\begin{aligned}
p\,\mathrm{e}^{x} &= \int \frac{\mathrm{e}^{x}}{1+\mathrm{e}^{x}}\,\mathrm{d}x \\
p\,\mathrm{e}^{x} &= \ln\left(1+\mathrm{e}^{x}\right)+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
p(x) = \ln\left(1+\mathrm{e}^{x}\right)\mathrm{e}^{-x}+c_1\mathrm{e}^{-x}
$$

which simplifies to

$$
p(x) = \mathrm{e}^{-x}\left(\ln\left(1+\mathrm{e}^{x}\right)+c_1\right)
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \mathrm{e}^{-x}\left(\ln\left(1+\mathrm{e}^{x}\right)+c_1\right)
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \mathrm{e}^{-x}\left(\ln\left(1+\mathrm{e}^{x}\right)+c_1\right)\,\mathrm{d}x \\
&= -c_1\mathrm{e}^{-x}+\ln\left(\mathrm{e}^{x}\right)-\ln\left(1+\mathrm{e}^{x}\right)\left(1+\mathrm{e}^{x}\right)\mathrm{e}^{-x}+c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -c_1\mathrm{e}^{-x}+\ln\left(\mathrm{e}^{x}\right)-\ln\left(1+\mathrm{e}^{x}\right)\left(1+\mathrm{e}^{x}\right)\mathrm{e}^{-x}+c_2 \tag{1}
$$

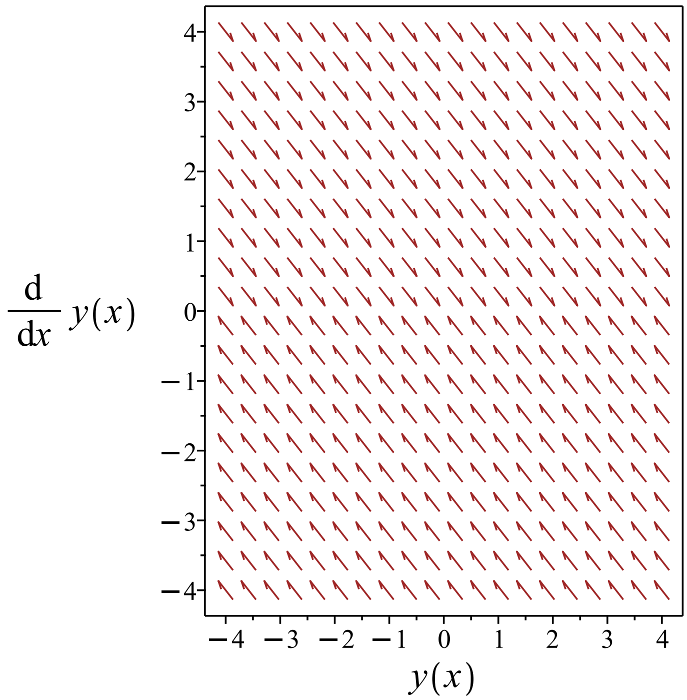

Figure 779: Slope field plot

Verification of solutions

$$y = -c_1 e^{-x} + \ln(e^x) - \ln(1 + e^x)(1 + e^x) e^{-x} + c_2$$

Verified OK.

20.15.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = \frac{1}{1 + e^x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = \int \frac{1}{1 + e^x} dx$$

$$y' + y = -\ln(1 + e^x) + \ln(e^x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -\ln(1+\mathrm{e}^{x}) + \ln(\mathrm{e}^{x}) + c_1
\end{aligned}
$$

Hence the ode is

$$
y' + y = -\ln(1+\mathrm{e}^{x}) + \ln(\mathrm{e}^{x}) + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\ln(1+\mathrm{e}^{x}) + \ln(\mathrm{e}^{x}) + c_1\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\left(-\ln(1+\mathrm{e}^{x}) + \ln(\mathrm{e}^{x}) + c_1\right) \\
\mathrm{d}(\mathrm{e}^{x} y) &= \left(\left(-\ln(1+\mathrm{e}^{x}) + \ln(\mathrm{e}^{x}) + c_1\right)\mathrm{e}^{x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int \left(-\ln(1+\mathrm{e}^{x}) + \ln(\mathrm{e}^{x}) + c_1\right)\mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{x} y &= \mathrm{e}^{x} c_1 - (1+\mathrm{e}^{x})\ln(1+\mathrm{e}^{x}) + 1 + \mathrm{e}^{x}\ln(\mathrm{e}^{x}) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
y = \mathrm{e}^{-x}\left(\mathrm{e}^{x} c_1 - (1+\mathrm{e}^{x})\ln(1+\mathrm{e}^{x}) + 1 + \mathrm{e}^{x}\ln(\mathrm{e}^{x})\right) + \mathrm{e}^{-x} c_2
$$

which simplifies to

$$
y = \left(-\mathrm{e}^{-x} - 1\right)\ln(1+\mathrm{e}^{x}) + (1+c_2)\,\mathrm{e}^{-x} + c_1 + \ln(\mathrm{e}^{x})
$$

Summary

The solution(s) found are the following

$$
y = \left(-\mathrm{e}^{-x} - 1\right)\ln(1+\mathrm{e}^{x}) + (1+c_2)\,\mathrm{e}^{-x} + c_1 + \ln(\mathrm{e}^{x}) \tag{1}
$$

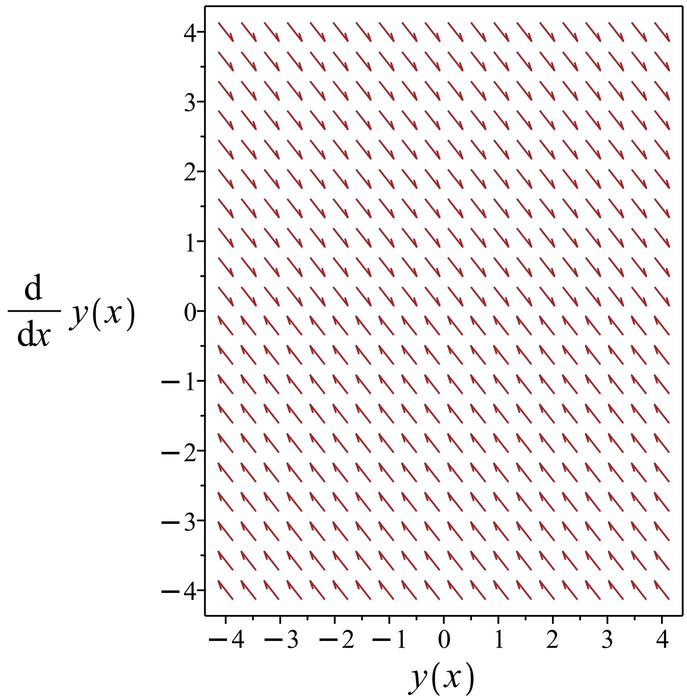

Figure 780: Slope field plot

Verification of solutions

$$y = \left(-\mathrm{e}^{-x} - 1\right) \ln\left(1 + \mathrm{e}^{x}\right) + (1 + c_2)\,\mathrm{e}^{-x} + c_1 + \ln\left(\mathrm{e}^{x}\right)$$

Verified OK.

20.15.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 645: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & 1 \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}(1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)(0) - (1)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{1+e^{x}}}{e^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{x}}{1+e^{x}} dx$$

Hence

$$u_1 = -\ln\left(1+e^{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-x}}{1+e^{x}}}{e^{-x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{1+e^{x}} dx$$

Hence

$$u_2 = -\ln(1+e^x) + \ln(e^x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(1+e^x) + \ln(e^x) - \ln(1+e^x)\,e^{-x}$$

Which simplifies to

$$y_p(x) = \left(-e^{-x} - 1\right)\ln(1+e^x) + \ln(e^x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2\right) + \left(\left(-e^{-x} - 1\right)\ln(1+e^x) + \ln(e^x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + \left(-e^{-x} - 1\right)\ln(1+e^x) + \ln(e^x) \tag{1}$$

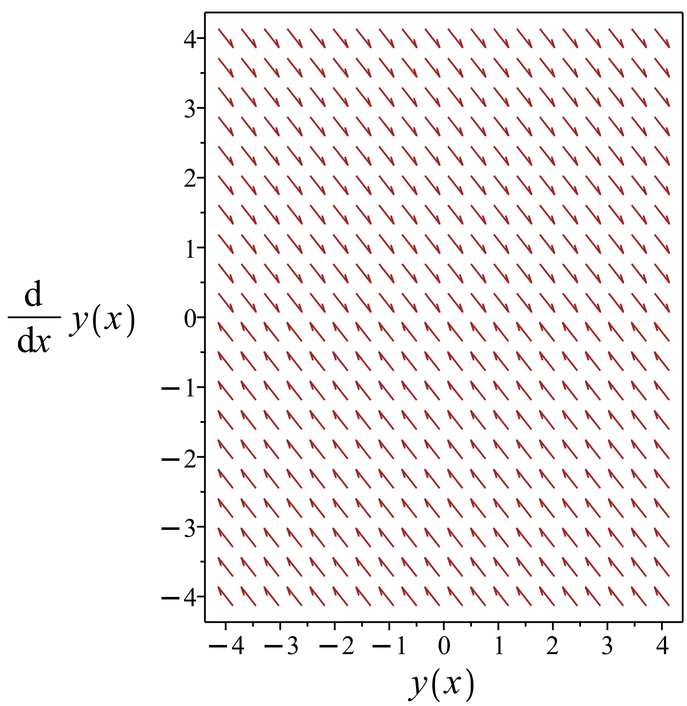

Figure 781: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 + \left(-e^{-x} - 1\right) \ln\left(1 + e^{x}\right) + \ln\left(e^{x}\right)$$

Verified OK.

20.15.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= \frac{1}{1 + e^{x}} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\, y' + \left(q(x) - p'(x)\right) y\right)' = s(x)$$

Integrating gives

$$p(x)\, y' + \left(q(x) - p'(x)\right) y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int \frac{1}{1 + e^{x}}\, dx$$

We now have a first order ode to solve which is

$$y' + y = -\ln(1 + e^x) + \ln(e^x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -\ln(1 + e^x) + \ln(e^x) + c_1 \end{aligned}$$

Hence the ode is

$$y' + y = -\ln(1 + e^x) + \ln(e^x) + c_1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)(-\ln(1 + e^x) + \ln(e^x) + c_1) \\ \frac{d}{dx}(e^x y) &= (e^x)(-\ln(1 + e^x) + \ln(e^x) + c_1) \\ d(e^x y) &= ((-\ln(1 + e^x) + \ln(e^x) + c_1) e^x)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x y &= \int (-\ln(1 + e^x) + \ln(e^x) + c_1) e^x\, dx \\ e^x y &= e^x c_1 - (1 + e^x)\ln(1 + e^x) + 1 + e^x \ln(e^x) + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}(e^x c_1 - (1 + e^x)\ln(1 + e^x) + 1 + e^x \ln(e^x)) + e^{-x} c_2$$

which simplifies to

$$y = (-e^{-x} - 1)\ln(1 + e^x) + (1 + c_2) e^{-x} + c_1 + \ln(e^x)$$

Summary

The solution(s) found are the following

$$y = \left(-e^{-x} - 1\right) \ln\left(1 + e^{x}\right) + \left(1 + c_2\right) e^{-x} + c_1 + \ln\left(e^{x}\right) \tag{1}$$

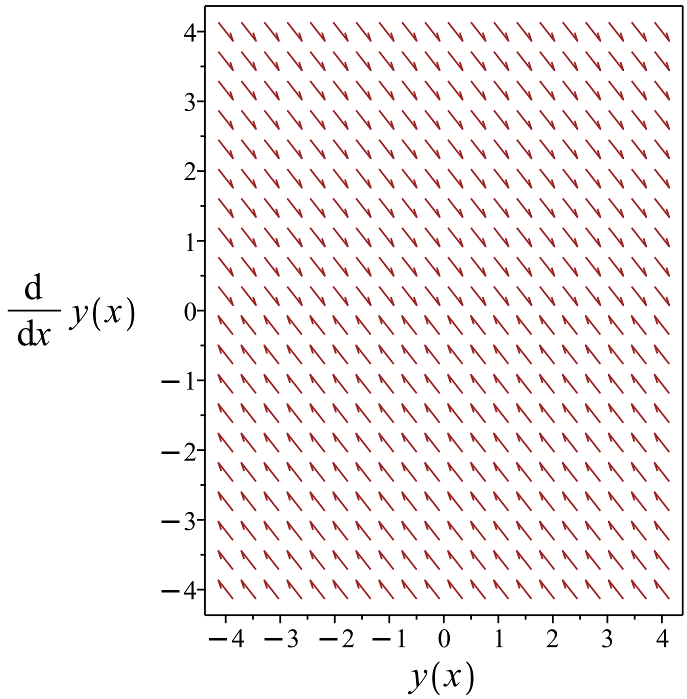

Figure 782: Slope field plot

Verification of solutions

$$y = \left(-e^{-x} - 1\right) \ln\left(1 + e^{x}\right) + \left(1 + c_2\right) e^{-x} + c_1 + \ln\left(e^{x}\right)$$

Verified OK.

20.15.7 Maple step by step solution

Let's solve

$y'' + y' = \frac{1}{1+e^{x}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1)=0$
- Roots of the characteristic polynomial

  $r=(-1,0)$
- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x)=1$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-x}+c_2+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\frac{1}{1+\mathrm{e}^x}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\left[\begin{array}{cc} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{array}\right]$
  - Compute Wronskian

    $W(y_1(x), y_2(x))=\mathrm{e}^{-x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^x}{1+\mathrm{e}^x}dx\right)+\int \frac{1}{1+\mathrm{e}^x}dx$
  - Compute integrals

    $y_p(x)=(-\mathrm{e}^{-x}-1)\ln(1+\mathrm{e}^x)+\ln(\mathrm{e}^x)$
- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-x}+c_2+(-\mathrm{e}^{-x}-1)\ln(1+\mathrm{e}^x)+\ln(\mathrm{e}^x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)*exp(_a)+_b(_a)-1)/(1+exp(_a)),
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=1/(1+exp(x)),y(x), singsol=all)
```

$$y(x) = \left(-\mathrm{e}^{-x} - 1\right) \ln \left(1 + \mathrm{e}^{x}\right) - c_1 \mathrm{e}^{-x} + c_2 + \ln \left(\mathrm{e}^{x}\right)$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 33

```
DSolve[y''[x]+y'[x]==1/(1+Exp[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \log \left(e^{x} + 1\right) - e^{-x} (\log \left(e^{x} + 1\right) + c_1) + c_2$$

20.16 problem 655

Internal problem ID [15420]
Internal file name [`OUTPUT/15420_Wednesday_May_08_2024_03_58_24_PM_67258881/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 655.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''+y=\frac{1}{\cos\left(x\right)^{3}}}$$

20.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1, B=0, C=1, f(x)=\sec\left(x\right)^{3}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)^3}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sec(x)^2 dx$$

Hence

$$u_1 = -\frac{\sec(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)^3}{1} dx$$

Which simplifies to

$$u_2 = \int \sec(x)^2 dx$$

Hence

$$u_2 = \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\sec(x)^2\cos(x)}{2} + \tan(x)\sin(x)$$

Which simplifies to

$$y_p(x) = -\cos(x) + \frac{\sec(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\cos(x) + \frac{\sec(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2} \tag{1}$$

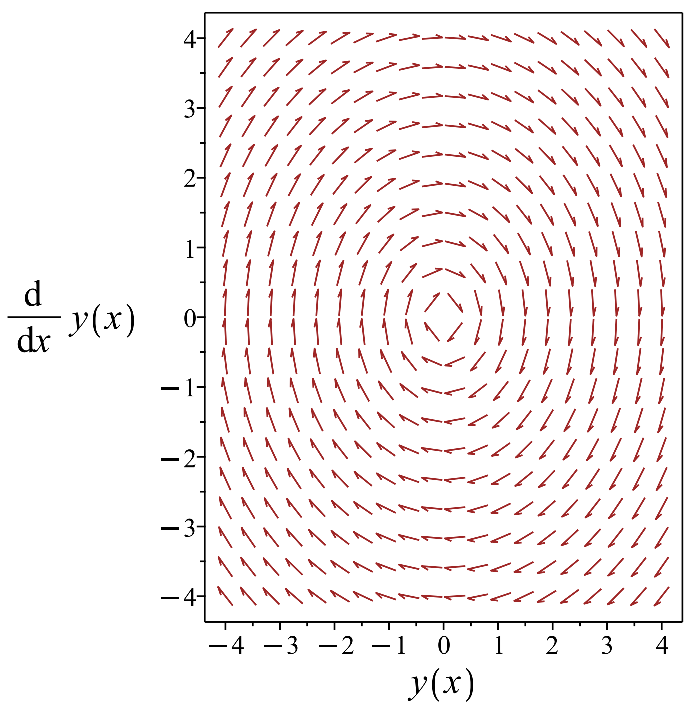

Figure 783: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2}$$

Verified OK.

20.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 647: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)^3}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sec(x)^2 dx$$

Hence

$$u_1 = -\frac{\sec(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)^3}{1} dx$$

Which simplifies to

$$u_2 = \int \sec(x)^2 dx$$

Hence

$$u_2 = \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\sec(x)^2\cos(x)}{2} + \tan(x)\sin(x)$$

Which simplifies to

$$y_p(x) = -\cos(x) + \frac{\sec(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(-\cos(x) + \frac{\sec(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2} \tag{1}$$

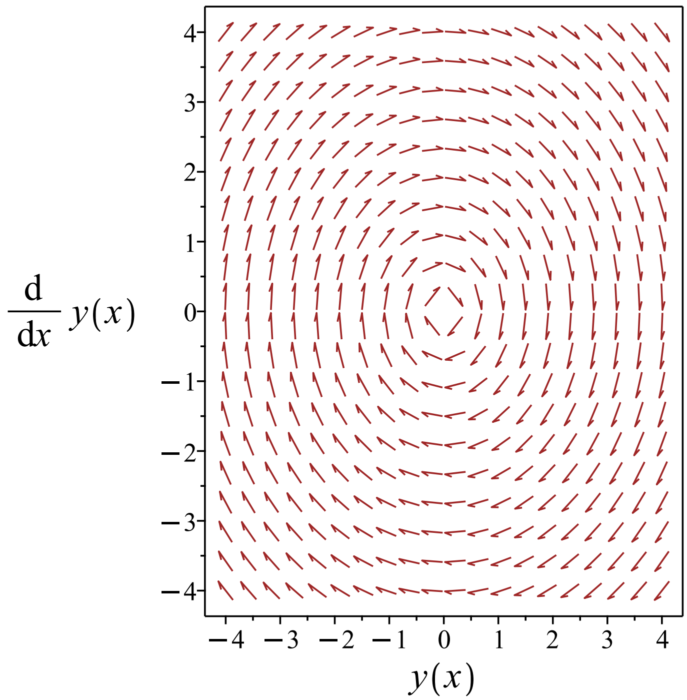

Figure 784: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2}$$

Verified OK.

20.16.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)^3\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x) \left(\int \tan(x) \sec(x)^2 dx\right) + \sin(x) \left(\int \sec(x)^2 dx\right)$

  ◦ Compute integrals

  $y_p(x) = -\cos(x) + \frac{\sec(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+y(x)=1/cos(x)^3,y(x), singsol=all)
```

$$y(x) = (-1 + c_1)\cos(x) + \sin(x)\, c_2 + \frac{\sec(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 25

```
DSolve[y''[x]+y[x]==1/Cos[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sec(x)}{2} + c_1 \cos(x) + \sin(x)(\tan(x) + c_2)$$

20.17 problem 656

Internal problem ID [15421]
Internal file name [`OUTPUT/15421_Wednesday_May_08_2024_03_58_25_PM_47235270/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 656.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \frac{1}{\sqrt{\sin(x)^5 \cos(x)}}}$$

20.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \frac{1}{\sqrt{\sin(x)^5 \cos(x)}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)}{\sqrt{\sin(x)^5 \cos(x)}}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx$$

Hence

$$u_1 = \frac{\cos(x)\sin(x)^2 \sqrt{2}\, \sqrt{32}}{4\sqrt{\sin(x)^5 \cos(x)}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{\sqrt{\sin(x)^5 \cos(x)}}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx$$

Hence

$$u_2 = -\frac{\sqrt{2}\, \cos(x)^2 \sin(x)\, \sqrt{32}}{12\sqrt{\sin(x)^5 \cos(x)}}$$

Which simplifies to

$$u_1 = \frac{2\cos(x)\sin(x)^2}{\sqrt{\sin(x)^5\cos(x)}}$$

$$u_2 = -\frac{2\cos(x)^2\sin(x)}{3\sqrt{\sin(x)^5\cos(x)}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4\cos(x)^2\sin(x)^2}{3\sqrt{\sin(x)^5\cos(x)}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{4\cos(x)^2\sin(x)^2}{3\sqrt{\sin(x)^5\cos(x)}}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \frac{4\cos(x)^2\sin(x)^2}{3\sqrt{\sin(x)^5\cos(x)}} \tag{1}$$

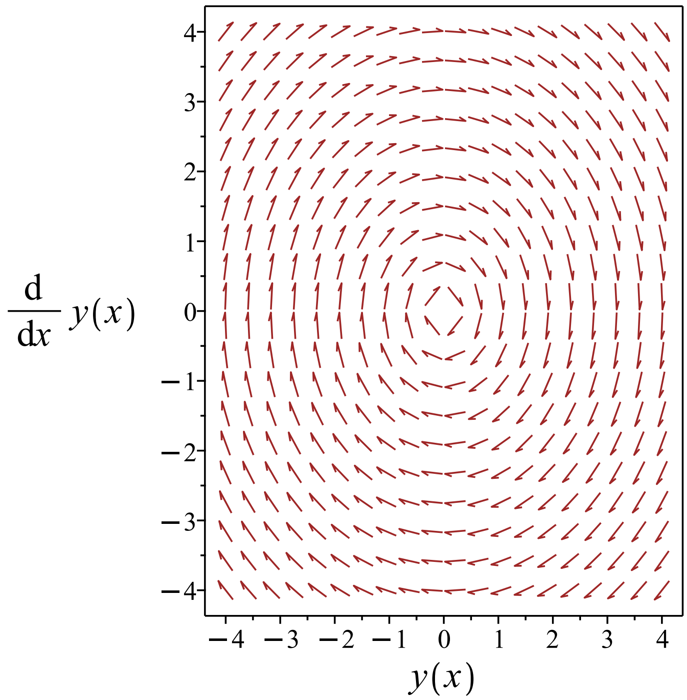

Figure 785: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{4 \cos(x)^2 \sin(x)^2}{3 \sqrt{\sin(x)^5 \cos(x)}}$$

Verified OK.

20.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 649: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} \, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos(x) \\
y_2 &= \sin(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)}{\sqrt{\sin(x)^5 \cos(x)}}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx$$

Hence

$$u_1 = \frac{\cos(x)\sin(x)^2 \sqrt{2}\sqrt{32}}{4\sqrt{\sin(x)^5 \cos(x)}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{\sqrt{\sin(x)^5 \cos(x)}}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx$$

Hence

$$u_2 = -\frac{\sqrt{2}\,\cos(x)^2 \sin(x)\,\sqrt{32}}{12\sqrt{\sin(x)^5 \cos(x)}}$$

Which simplifies to

$$u_1 = \frac{2\cos(x)\sin(x)^2}{\sqrt{\sin(x)^5 \cos(x)}}$$

$$u_2 = -\frac{2\cos(x)^2 \sin(x)}{3\sqrt{\sin(x)^5 \cos(x)}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4\cos(x)^2 \sin(x)^2}{3\sqrt{\sin(x)^5 \cos(x)}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{4\cos(x)^2 \sin(x)^2}{3\sqrt{\sin(x)^5 \cos(x)}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{4\cos(x)^2 \sin(x)^2}{3\sqrt{\sin(x)^5 \cos(x)}} \tag{1}$$

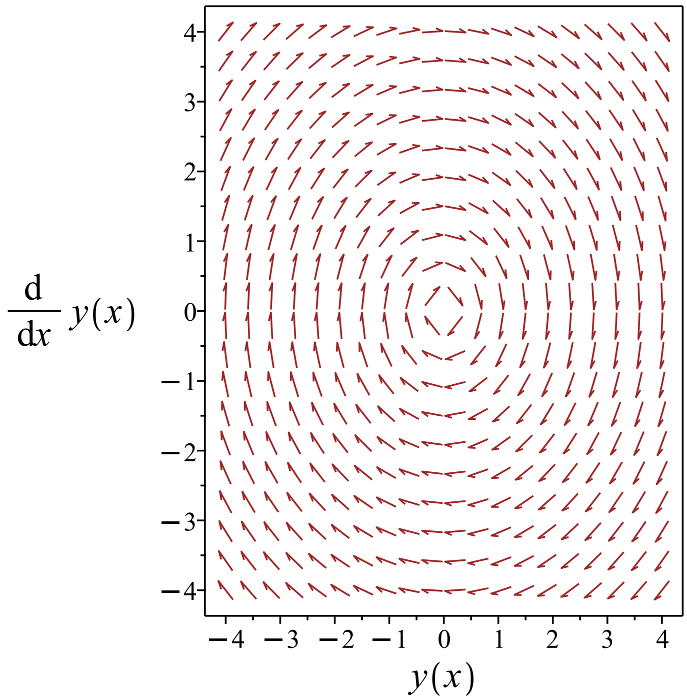

Figure 786: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{4 \cos(x)^2 \sin(x)^2}{3\sqrt{\sin(x)^5 \cos(x)}}$$

Verified OK.

20.17.3 Maple step by step solution

Let's solve

$y'' + y = \frac{1}{\sqrt{\sin(x)^5 \cos(x)}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{\sqrt{\sin(x)^5 \cos(x)}}\right]$$

  ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ○ Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\cos(x)\left(\int \frac{\sin(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx\right) + \sin(x)\left(\int \frac{\cos(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx\right)$$

  ○ Compute integrals

  $$y_p(x) = \frac{4\cos(x)^2 \sin(x)^2}{3\sqrt{\sin(x)^5 \cos(x)}}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \cos(x) + c_2 \sin(x) + \frac{4\cos(x)^2 \sin(x)^2}{3\sqrt{\sin(x)^5 \cos(x)}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$2)+y(x)=1/sqrt(sin(x)^5*cos(x)),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 + \left(\int \frac{\cos(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx\right) \sin(x) - \left(\int \frac{\sin(x)}{\sqrt{\sin(x)^5 \cos(x)}} dx\right) \cos(x)$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 35

```
DSolve[y''[x]+y[x]==1/Sqrt[Sin[x]^5*Cos[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x) + \frac{4}{3} \csc^8(x) \left(\sin^5(x) \cos(x)\right)^{3/2}$$

20.18 problem 657

Internal problem ID [15422]
Internal file name [`OUTPUT/15422_Wednesday_May_08_2024_03_58_28_PM_97596783/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 657.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 2y' + y = \frac{e^x}{x^2 + 1}}$$

20.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = \frac{e^x}{x^2+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\
&= 1
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + x e^x c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= e^x \\
y_2 &= x e^x
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2x}}{x^2+1}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x^2+1} dx$$

Hence

$$u_1 = -\frac{\ln(x^2+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{x^2+1}}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2+1} dx$$

Hence

$$u_2 = \arctan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x^2+1)\,e^x}{2} + \arctan(x)\,x\,e^x$$

Which simplifies to

$$y_p(x) = e^x \left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + x\,e^x c_2) + \left(e^x \left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2 x + c_1) + e^x \left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)$$

Summary

The solution(s) found are the following

$$y = e^x(c_2 x + c_1) + e^x \left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right) \tag{1}$$

Verification of solutions

$$y = e^x(c_2 x + c_1) + e^x \left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)$$

Verified OK.

20.18.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int -2\, dx} \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= \frac{\mathrm{e}^{-x} \mathrm{e}^{x}}{x^{2}+1} \\
\left(\mathrm{e}^{-x} y\right)'' &= \frac{\mathrm{e}^{-x} \mathrm{e}^{x}}{x^{2}+1}
\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x} y\right)' = \arctan(x) + c_{1}$$

Integrating again gives

$$\left(\mathrm{e}^{-x} y\right) = c_{1} x + x \arctan(x) - \frac{\ln(x^{2}+1)}{2} + c_{2}$$

Hence the solution is

$$y = \frac{c_{1} x + x \arctan(x) - \frac{\ln(x^{2}+1)}{2} + c_{2}}{\mathrm{e}^{-x}}$$

Or

$$y = c_{1} x\, \mathrm{e}^{x} + x \arctan(x)\, \mathrm{e}^{x} + c_{2} \mathrm{e}^{x} - \frac{\mathrm{e}^{x} \ln(x^{2}+1)}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_{1} x\, \mathrm{e}^{x} + x \arctan(x)\, \mathrm{e}^{x} + c_{2} \mathrm{e}^{x} - \frac{\mathrm{e}^{x} \ln(x^{2}+1)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x\, e^{x} + x \arctan(x)\, e^{x} + c_2 e^{x} - \frac{e^{x} \ln(x^2+1)}{2}$$

Verified OK.

20.18.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 651: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^x c_1 + x\,e^x c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^x \\ y_2 &= x\,e^x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\,e^{2x}}{x^2+1}}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{x^2+1}dx$$

Hence

$$u_1 = -\frac{\ln(x^2+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{2x}}{x^2+1}}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2+1}dx$$

Hence

$$u_2 = \arctan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^x \ln(x^2+1)}{2} + x\arctan(x)\,e^x$$

Which simplifies to

$$y_p(x) = e^x\left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\mathrm{e}^{x} c_1 + x\,\mathrm{e}^{x} c_2) + \left(\mathrm{e}^{x}\left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{x}(c_2 x + c_1) + \mathrm{e}^{x}\left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{x}(c_2 x + c_1) + \mathrm{e}^{x}\left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right) \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{x}(c_2 x + c_1) + \mathrm{e}^{x}\left(-\frac{\ln(x^2+1)}{2} + x\arctan(x)\right)
$$

Verified OK.

20.18.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = \frac{\mathrm{e}^{x}}{x^2+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

$y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + x\,e^x c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{e^x}{x^2+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^x\left(-\left(\int \frac{x}{x^2+1} dx\right) + \left(\int \frac{1}{x^2+1} dx\right) x\right)$

  - Compute integrals

    $y_p(x) = -\frac{e^x\left(-2x\arctan(x) + \ln(x^2+1)\right)}{2}$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + x\,e^x c_2 - \frac{e^x\left(-2x\arctan(x) + \ln(x^2+1)\right)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(x)/(x^2+1),y(x), singsol=all)
```

$$y(x) = e^{x}\left(c_2 + c_1 x - \frac{\ln(x^2+1)}{2} + x\arctan(x)\right)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 35

```
DSolve[y''[x]-2*y'[x]+y[x]==Exp[x]/(1+x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{x}\left(2x\arctan(x) - \log\left(x^2+1\right) + 2(c_2 x + c_1)\right)$$

20.19 problem 658

Internal problem ID [15423]
Internal file name [`OUTPUT/15423_Wednesday_May_08_2024_03_58_29_PM_97041349/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 658.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = \frac{e^{-x}}{\sin(x)}}$$

20.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = \csc(x) e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)}\\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i\\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x} \cos(x)$$
$$y_2 = e^{-x} \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} \cos(x) & e^{-x} \sin(x) \\ \frac{d}{dx}(e^{-x} \cos(x)) & \frac{d}{dx}(e^{-x} \sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} \cos(x) & e^{-x} \sin(x) \\ -e^{-x} \cos(x) - e^{-x} \sin(x) & -e^{-x} \sin(x) + e^{-x} \cos(x) \end{vmatrix}$$

Therefore

$$W = (e^{-x} \cos(x)) (-e^{-x} \sin(x) + e^{-x} \cos(x)) - (e^{-x} \sin(x)) (-e^{-x} \cos(x) - e^{-x} \sin(x))$$

Which simplifies to

$$W = \sin(x)^2 e^{-2x} + \cos(x)^2 e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x} \sin(x) \csc(x)}{e^{-2x}} dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x} \cos(x) \csc(x)}{e^{-2x}} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x \cos(x) e^{-x} + \ln(\sin(x)) e^{-x} \sin(x)$$

Which simplifies to

$$y_p(x) = e^{-x}(-\cos(x) x + \ln(\sin(x)) \sin(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x))\right) + \left(\mathrm{e}^{-x}(-\cos(x)\, x + \ln(\sin(x)) \sin(x))\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{-x}(-\cos(x)\, x + \ln(\sin(x)) \sin(x)) \tag{1}$$

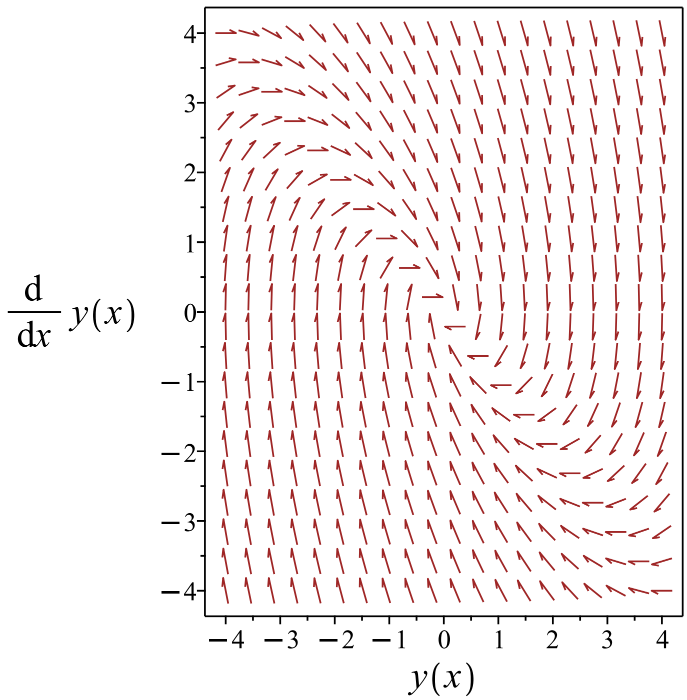

Figure 787: Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x)) + \mathrm{e}^{-x}(-\cos(x)\, x + \ln(\sin(x)) \sin(x))$$

Verified OK.

20.19.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 653: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x} \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(x) (\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-x} \cos(x) c_1 + \mathrm{e}^{-x} \sin(x) c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x} \cos(x)$$
$$y_2 = e^{-x} \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x}\cos(x) & e^{-x}\sin(x) \\ \frac{d}{dx}(e^{-x}\cos(x)) & \frac{d}{dx}(e^{-x}\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x}\cos(x) & e^{-x}\sin(x) \\ -e^{-x}\cos(x) - e^{-x}\sin(x) & -e^{-x}\sin(x) + e^{-x}\cos(x) \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\cos(x)\right)\left(-e^{-x}\sin(x) + e^{-x}\cos(x)\right) - \left(e^{-x}\sin(x)\right)\left(-e^{-x}\cos(x) - e^{-x}\sin(x)\right)$$

Which simplifies to

$$W = \sin(x)^2 e^{-2x} + \cos(x)^2 e^{-2x}$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-2x}\sin(x)\csc(x)}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-2x}\cos(x)\csc(x)}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \cot(x)\,dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x\cos(x)\,\mathrm{e}^{-x} + \ln(\sin(x))\,\mathrm{e}^{-x}\sin(x)$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-x}(-\cos(x)\,x + \ln(\sin(x))\sin(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x}\cos(x)\,c_1 + \mathrm{e}^{-x}\sin(x)\,c_2\right) + \left(\mathrm{e}^{-x}(-\cos(x)\,x + \ln(\sin(x))\sin(x))\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-x}(-\cos(x) x + \ln(\sin(x)) \sin(x))$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-x}(-\cos(x) x + \ln(\sin(x)) \sin(x)) \tag{1}$$

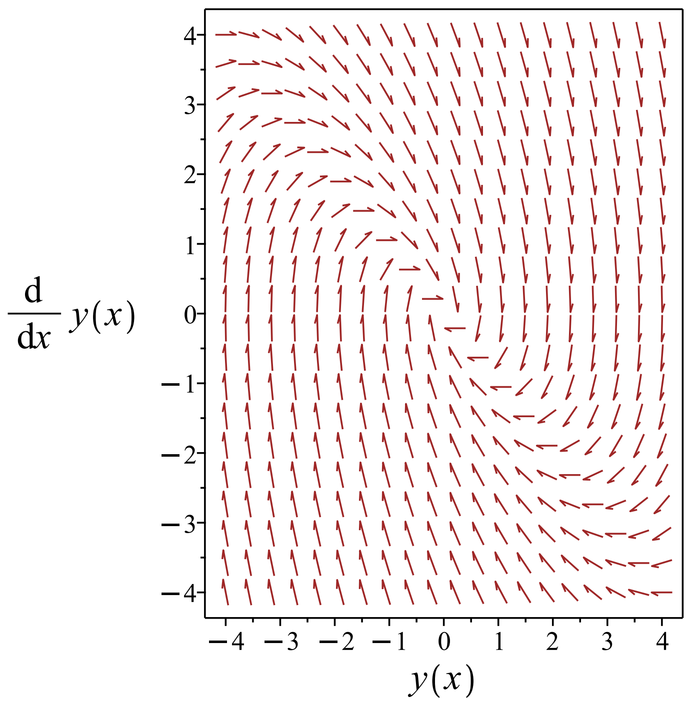

Figure 788: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-x}(-\cos(x) x + \ln(\sin(x)) \sin(x))$$

Verified OK.

20.19.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = \csc(x)\, e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - I, -1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}\sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-x}\cos(x)\, c_1 + e^{-x}\sin(x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \csc(x)\, e^{-x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x}\cos(x) & e^{-x}\sin(x) \\ -e^{-x}\cos(x) - e^{-x}\sin(x) & -e^{-x}\sin(x) + e^{-x}\cos(x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-2x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \mathrm{e}^{-x}\left(-\cos(x)\left(\int 1dx\right) + \sin(x)\left(\int \cot(x)\,dx\right)\right)$$

- Compute integrals

$$y_p(x) = \mathrm{e}^{-x}(-\cos(x)\,x + \ln(\sin(x))\sin(x))$$

- Substitute particular solution into general solution to ODE

$$y = \mathrm{e}^{-x}\cos(x)\,c_1 + \mathrm{e}^{-x}\sin(x)\,c_2 + \mathrm{e}^{-x}(-\cos(x)\,x + \ln(\sin(x))\sin(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=1/(exp(x)*sin(x)),y(x), singsol=all)
```

$$y(x) = -(-\ln(\sin(x))\sin(x) + (x - c_1)\cos(x) - \sin(x)\,c_2)\,\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 30

```
DSolve[y''[x]+2*y'[x]+2*y[x]==1/(Exp[x]*Sin[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}((-x + c_2)\cos(x) + \sin(x)(\log(\sin(x)) + c_1))$$

20.20 problem 659

Internal problem ID [15424]
Internal file name [`OUTPUT/15424_Wednesday_May_08_2024_03_58_30_PM_81546042/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 659.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + y = \frac{2}{\sin(x)^3}$$

20.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\csc(x)^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(x)\csc(x)^3}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 2\csc(x)^2\,dx$$

Hence

$$u_1 = 2\cot(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(x)\csc(x)^3}{1}\,dx$$

Which simplifies to

$$u_2 = \int 2\csc(x)^2\cot(x)\,dx$$

Hence

$$u_2 = -\cot(x)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2\cot(x)\cos(x) - \cot(x)^2\sin(x)$$

Which simplifies to

$$y_p(x) = \cot(x)\cos(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (\cot(x)\cos(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \cot(x)\cos(x) \tag{1}
$$

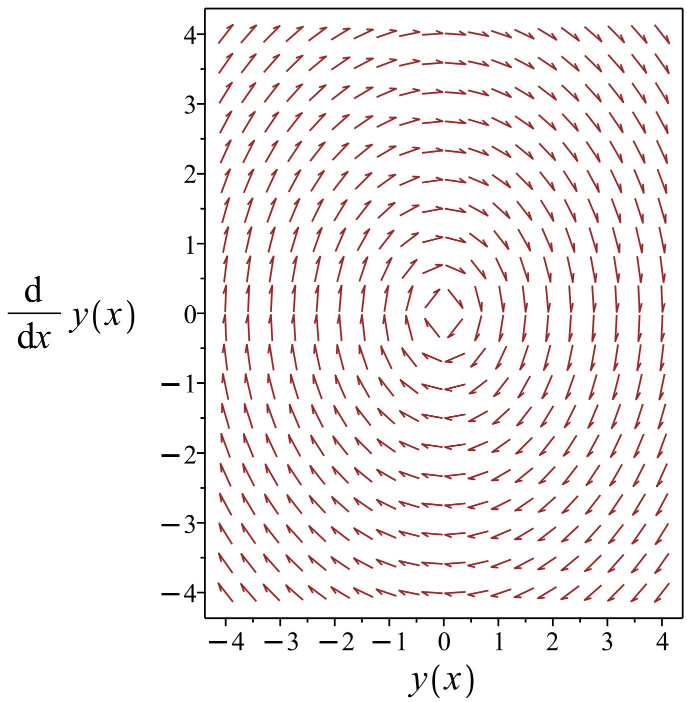

Figure 789: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \cot(x)\cos(x)
$$

Verified OK.

20.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 655: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x) (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(x)\csc(x)^3}{1} dx$$

Which simplifies to

$$u_1 = -\int 2\csc(x)^2 dx$$

Hence

$$u_1 = 2\cot(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(x)\csc(x)^3}{1} dx$$

Which simplifies to

$$u_2 = \int 2\csc(x)^2 \cot(x) dx$$

Hence

$$u_2 = -\cot(x)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2\cot(x)\cos(x) - \cot(x)^2 \sin(x)$$

Which simplifies to

$$y_p(x) = \cot(x)\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (\cot(x)\cos(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \cot(x)\cos(x) \tag{1}$$

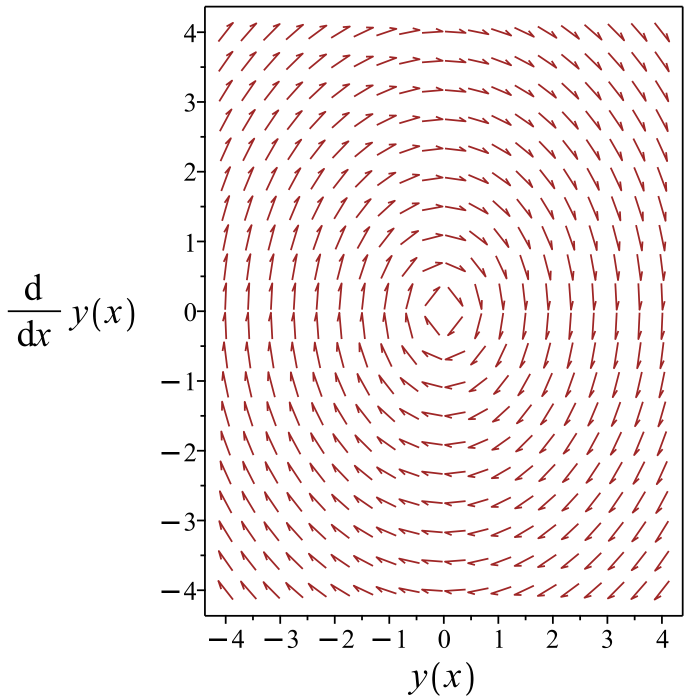

Figure 790: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \cot(x)\cos(x)$$

Verified OK.

20.20.3 Maple step by step solution

Let's solve

$y'' + y = 2\csc(x)^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\csc(x)^3\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\cos(x)\left(\int \csc(x)^2 dx\right) + 2\sin(x)\left(\int \csc(x)^2 \cot(x) dx\right)$

  - Compute integrals

    $y_p(x) = \cot(x)\cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \cot(x)\cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+y(x)=2/sin(x)^3,y(x), singsol=all)
```

$$y(x) = (c_1 + 2\cot(x))\cos(x) + \sin(x)\, c_2 - \csc(x)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 25

```
DSolve[y''[x]+y[x]==2/Sin[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\csc(x) + c_2 \sin(x) + \cos(x)(2\cot(x) + c_1)$$

20.21 problem 660

Internal problem ID [15425]
Internal file name [`OUTPUT/15425_Wednesday_May_08_2024_03_58_30_PM_67343751/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 660.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' = e^{2x} \cos\left(e^{x}\right)}$$

20.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 0, f(x) = e^{2x} \cos\left(e^{x}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=1, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+\lambda \mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=1, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{-1}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{1^2-(4)(1)(0)}\\&=-\frac{1}{2}\pm\frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=-\frac{1}{2}+\frac{1}{2}\\\lambda_2&=-\frac{1}{2}-\frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=0\\\lambda_2&=-1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y&=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}\\y&=c_1e^{(0)x}+c_2e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 + e^{-x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + e^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & e^{-x} \\ \frac{d}{dx}(1) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (1)\left(-\mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x}\right)(0)$$

Which simplifies to

$$W = -\mathrm{e}^{-x}$$

Which simplifies to

$$W = -\mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{\mathrm{e}^{-x}\mathrm{e}^{2x}\cos\left(\mathrm{e}^{x}\right)}{-\mathrm{e}^{-x}}dx\right)$$

Which simplifies to

$$u_1 = -\left(\int -\mathrm{e}^{2x}\cos\left(\mathrm{e}^{x}\right)dx\right)$$

Hence

$$u_1 = -\frac{-2\,\mathrm{e}^{x}\tan\left(\frac{\mathrm{e}^{x}}{2}\right) - 2}{1 + \tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{2x}\cos\left(\mathrm{e}^{x}\right)}{-\mathrm{e}^{-x}}dx$$

Which simplifies to

$$u_2 = \int -\cos\left(\mathrm{e}^{x}\right)\mathrm{e}^{3x}dx$$

Hence

$$u_2 = \frac{2\,\mathrm{e}^{x}\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2} - 2\,\mathrm{e}^{2x}\tan\left(\frac{\mathrm{e}^{x}}{2}\right) - 2\,\mathrm{e}^{x} + 4\tan\left(\frac{\mathrm{e}^{x}}{2}\right)}{1 + \tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}}$$

Which simplifies to

$$u_1 = 1 + \mathrm{e}^{x}\sin\left(\mathrm{e}^{x}\right) + \cos\left(\mathrm{e}^{x}\right)$$
$$u_2 = -\mathrm{e}^{2x}\sin\left(\mathrm{e}^{x}\right) - 2\,\mathrm{e}^{x}\cos\left(\mathrm{e}^{x}\right) + 2\sin\left(\mathrm{e}^{x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 + e^x \sin(e^x) + \cos(e^x) + \left(-e^{2x}\sin(e^x) - 2\,e^x\cos(e^x) + 2\sin(e^x)\right)e^{-x}$$

Which simplifies to

$$y_p(x) = -\cos(e^x) + 2\sin(e^x)\,e^{-x} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{-x}c_2\right) + \left(-\cos(e^x) + 2\sin(e^x)\,e^{-x} + 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + e^{-x}c_2 - \cos(e^x) + 2\sin(e^x)\,e^{-x} + 1 \tag{1}$$

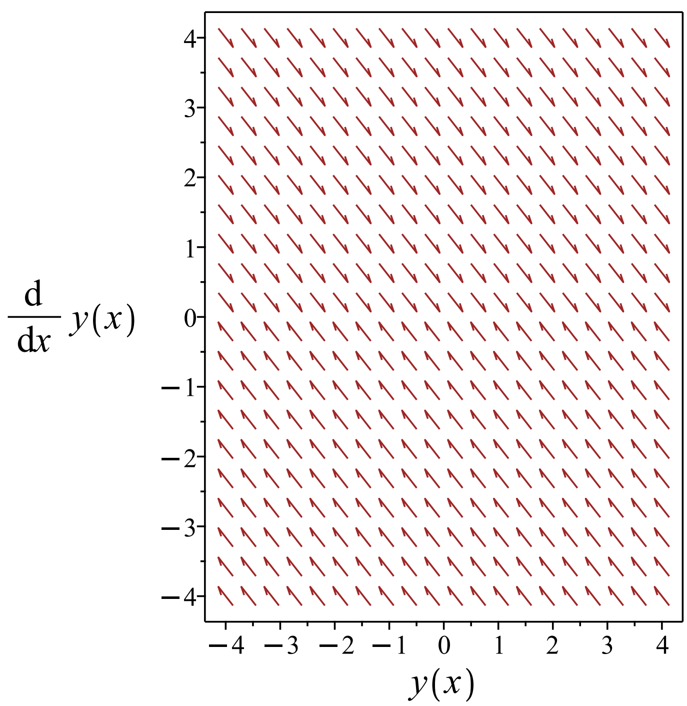

Figure 791: Slope field plot

Verification of solutions

$$y = c_1 + e^{-x}c_2 - \cos(e^x) + 2\sin(e^x)\,e^{-x} + 1$$

Verified OK.

20.21.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y''+y')\,dx = \int \mathrm{e}^{2x}\cos(\mathrm{e}^{x})\,dx$$

$$y'+y = \frac{2\,\mathrm{e}^{x}\tan\left(\frac{\mathrm{e}^{x}}{2}\right)+2}{1+\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 1\\ q(x) &= \mathrm{e}^{x}\sin(\mathrm{e}^{x})+\cos(\mathrm{e}^{x})+c_1+1\end{aligned}$$

Hence the ode is

$$y'+y = \mathrm{e}^{x}\sin(\mathrm{e}^{x})+\cos(\mathrm{e}^{x})+c_1+1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 1dx}\\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\mathrm{e}^{x}\sin(\mathrm{e}^{x})+\cos(\mathrm{e}^{x})+c_1+1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x}y) &= (\mathrm{e}^{x})\left(\mathrm{e}^{x}\sin(\mathrm{e}^{x})+\cos(\mathrm{e}^{x})+c_1+1\right)\\ \mathrm{d}(\mathrm{e}^{x}y) &= \left(\left(\mathrm{e}^{x}\sin(\mathrm{e}^{x})+\cos(\mathrm{e}^{x})+c_1+1\right)\mathrm{e}^{x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{x}y &= \int \left(\mathrm{e}^{x}\sin(\mathrm{e}^{x})+\cos(\mathrm{e}^{x})+c_1+1\right)\mathrm{e}^{x}\,\mathrm{d}x\\ \mathrm{e}^{x}y &= \mathrm{e}^{x}c_1+\mathrm{e}^{x}+2\sin(\mathrm{e}^{x})-\mathrm{e}^{x}\cos(\mathrm{e}^{x})+c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\mathrm{e}^{x}$ results in

$$y = \mathrm{e}^{-x}\left(\mathrm{e}^{x}c_1+\mathrm{e}^{x}+2\sin(\mathrm{e}^{x})-\mathrm{e}^{x}\cos(\mathrm{e}^{x})\right)+\mathrm{e}^{-x}c_2$$

which simplifies to

$$y = (c_2 + 2\sin(e^x))e^{-x} + c_1 - \cos(e^x) + 1$$

Summary

The solution(s) found are the following

$$y = (c_2 + 2\sin(e^x))e^{-x} + c_1 - \cos(e^x) + 1 \tag{1}$$

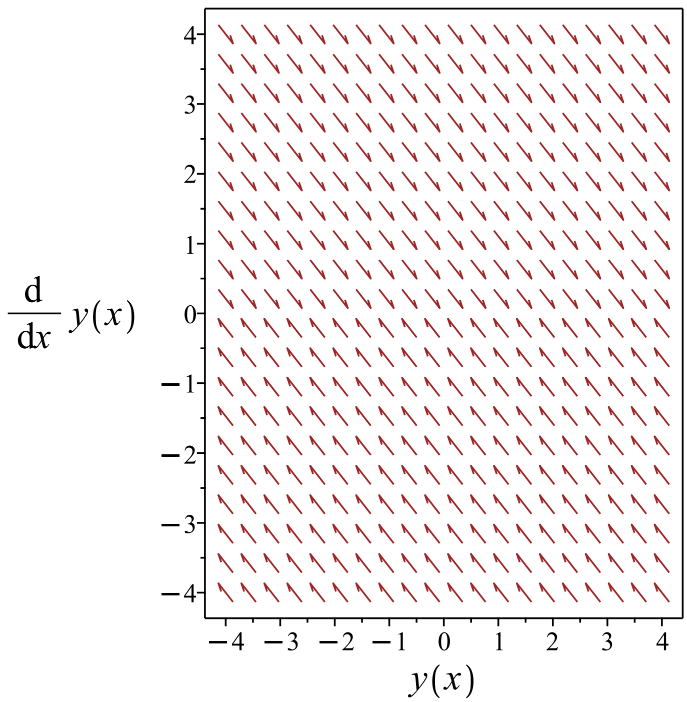

Figure 792: Slope field plot

Verification of solutions

$$y = (c_2 + 2\sin(e^x))e^{-x} + c_1 - \cos(e^x) + 1$$

Verified OK.

20.21.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) - e^{2x} \cos(e^x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= e^{2x} \cos(e^x) \end{aligned}$$

Hence the ode is

$$p'(x) + p(x) = e^{2x} \cos(e^x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu) \left(e^{2x} \cos(e^x)\right) \\ \frac{d}{dx}(e^x p) &= (e^x) \left(e^{2x} \cos(e^x)\right) \\ d(e^x p) &= \left(\cos(e^x) e^{3x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x p &= \int \cos(e^x) e^{3x} dx \\ e^x p &= \frac{-2 e^x \tan\left(\frac{e^x}{2}\right)^2 + 2 e^{2x} \tan\left(\frac{e^x}{2}\right) + 2 e^x - 4 \tan\left(\frac{e^x}{2}\right)}{1 + \tan\left(\frac{e^x}{2}\right)^2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$p(x) = \frac{\mathrm{e}^{-x}\left(-2\,\mathrm{e}^{x} \tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}+2\,\mathrm{e}^{2x} \tan\left(\frac{\mathrm{e}^{x}}{2}\right)+2\,\mathrm{e}^{x}-4\tan\left(\frac{\mathrm{e}^{x}}{2}\right)\right)}{1+\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}}+c_1\mathrm{e}^{-x}$$

which simplifies to

$$p(x) = (c_1 - 2\sin(\mathrm{e}^{x}))\,\mathrm{e}^{-x}+\mathrm{e}^{x}\sin(\mathrm{e}^{x})+2\cos(\mathrm{e}^{x})$$

Since $p = y'$ then the new first order ode to solve is

$$y' = (c_1 - 2\sin(\mathrm{e}^{x}))\,\mathrm{e}^{-x}+\mathrm{e}^{x}\sin(\mathrm{e}^{x})+2\cos(\mathrm{e}^{x})$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \mathrm{e}^{x}\sin(\mathrm{e}^{x})+2\cos(\mathrm{e}^{x})+c_1\mathrm{e}^{-x}-2\sin(\mathrm{e}^{x})\,\mathrm{e}^{-x}\,\mathrm{d}x \\
&= \frac{\left(-2\,\mathrm{e}^{x}-c_1-\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2} c_1+4\tan\left(\frac{\mathrm{e}^{x}}{2}\right)\right)\mathrm{e}^{-x}}{1+\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}}+c_2
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-2\,\mathrm{e}^{x}-c_1-\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2} c_1+4\tan\left(\frac{\mathrm{e}^{x}}{2}\right)\right)\mathrm{e}^{-x}}{1+\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^{2}}+c_2 \tag{1}$$

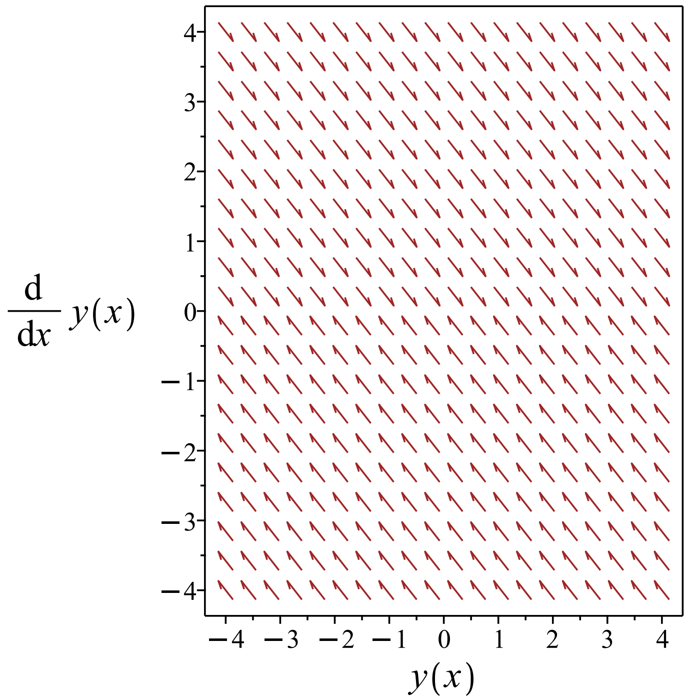

Figure 793: Slope field plot

Verification of solutions

$$y = \frac{\left(-2\,\mathrm{e}^{x} - c_1 - \tan\left(\frac{\mathrm{e}^{x}}{2}\right)^2 c_1 + 4\tan\left(\frac{\mathrm{e}^{x}}{2}\right)\right)\mathrm{e}^{-x}}{1+\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^2} + c_2$$

Verified OK.

20.21.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = \mathrm{e}^{2x}\cos\left(\mathrm{e}^{x}\right)$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'' + y'\right) dx = \int \mathrm{e}^{2x}\cos\left(\mathrm{e}^{x}\right) dx$$

$$y' + y = \frac{2\,\mathrm{e}^{x}\tan\left(\frac{\mathrm{e}^{x}}{2}\right) + 2}{1+\tan\left(\frac{\mathrm{e}^{x}}{2}\right)^2} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= e^x \sin(e^x) + \cos(e^x) + c_1 + 1
\end{aligned}$$

Hence the ode is

$$y' + y = e^x \sin(e^x) + \cos(e^x) + c_1 + 1$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int 1dx} \\
&= e^x
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^x \sin(e^x) + \cos(e^x) + c_1 + 1\right) \\
\frac{d}{dx}(e^x y) &= (e^x)\left(e^x \sin(e^x) + \cos(e^x) + c_1 + 1\right) \\
d(e^x y) &= \left(\left(e^x \sin(e^x) + \cos(e^x) + c_1 + 1\right) e^x\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^x y &= \int \left(e^x \sin(e^x) + \cos(e^x) + c_1 + 1\right) e^x \, dx \\
e^x y &= e^x c_1 + e^x + 2\sin(e^x) - e^x \cos(e^x) + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(e^x c_1 + e^x + 2\sin(e^x) - e^x \cos(e^x)\right) + e^{-x} c_2$$

which simplifies to

$$y = (c_2 + 2\sin(e^x))\, e^{-x} + c_1 - \cos(e^x) + 1$$

Summary

The solution(s) found are the following

$$y = (c_2 + 2\sin(e^x))\, e^{-x} + c_1 - \cos(e^x) + 1 \tag{1}$$

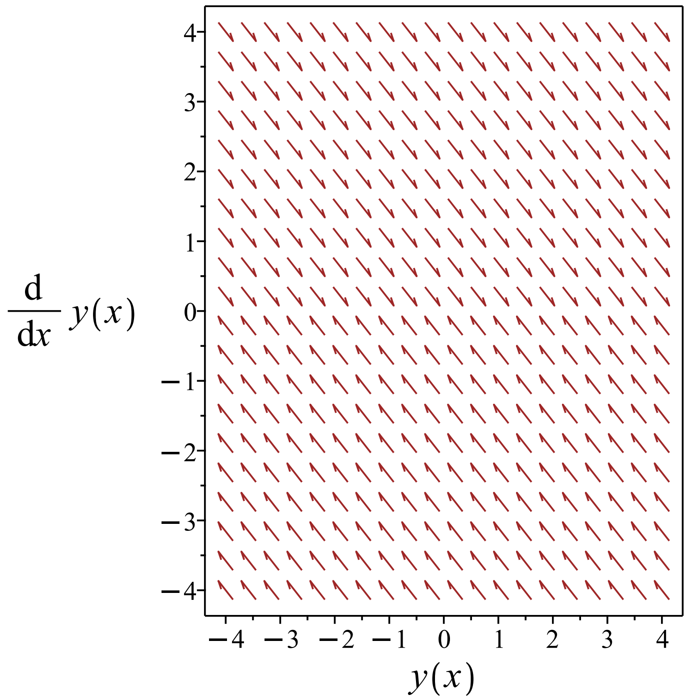

Figure 794: Slope field plot

<u>Verification of solutions</u>

$$y = (c_2 + 2\sin(e^x))\,e^{-x} + c_1 - \cos(e^x) + 1$$

Verified OK.

20.21.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = 1 \tag{3}$$
$$C = 0$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 657: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & 1 \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}(1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)(0) - (1)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{2x} \cos\left(e^x\right)}{e^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int \cos\left(e^x\right) e^{3x} dx$$

Hence

$$u_1 = -\frac{-2\, e^x \tan\left(\frac{e^x}{2}\right)^2 + 2\, e^{2x} \tan\left(\frac{e^x}{2}\right) + 2\, e^x - 4 \tan\left(\frac{e^x}{2}\right)}{1 + \tan\left(\frac{e^x}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} e^{2x} \cos\left(e^x\right)}{e^{-x}}\, dx$$

Which simplifies to

$$u_2 = \int e^{2x} \cos(e^x) \, dx$$

Hence

$$u_2 = \frac{2\,e^x \tan\left(\frac{e^x}{2}\right) + 2}{1 + \tan\left(\frac{e^x}{2}\right)^2}$$

Which simplifies to

$$\begin{aligned} u_1 &= -e^{2x} \sin(e^x) - 2\,e^x \cos(e^x) + 2\sin(e^x) \\ u_2 &= 1 + e^x \sin(e^x) + \cos(e^x) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 + e^x \sin(e^x) + \cos(e^x) + \left(-e^{2x} \sin(e^x) - 2\,e^x \cos(e^x) + 2\sin(e^x)\right) e^{-x}$$

Which simplifies to

$$y_p(x) = -\cos(e^x) + 2\sin(e^x)\,e^{-x} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2\right) + \left(-\cos(e^x) + 2\sin(e^x)\,e^{-x} + 1\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 - \cos(e^x) + 2\sin(e^x)\,e^{-x} + 1 \tag{1}$$

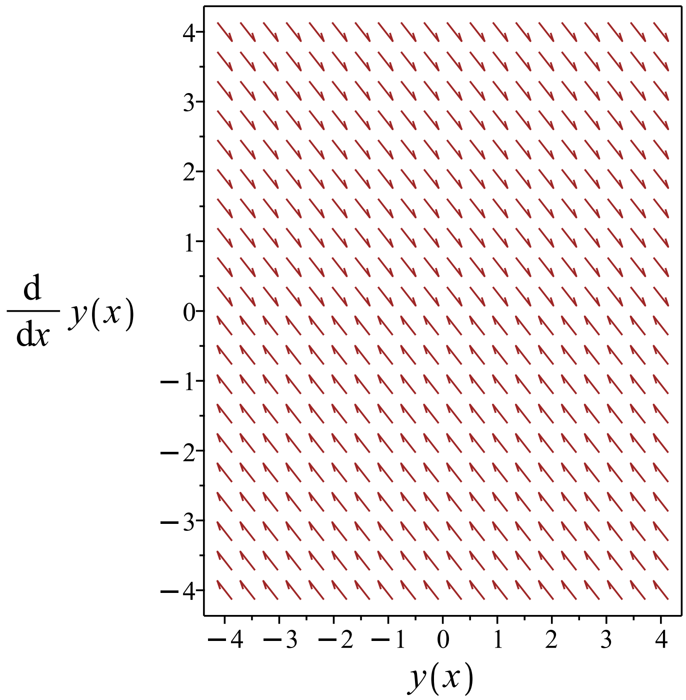

Figure 795: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 - \cos(e^x) + 2\sin(e^x)\,e^{-x} + 1$$

Verified OK.

20.21.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= e^{2x}\cos(e^x) \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$y' + y = \int e^{2x} \cos(e^{x})\, dx$$

We now have a first order ode to solve which is

$$y' + y = \frac{2\, e^{x} \tan\left(\frac{e^{x}}{2}\right) + 2}{1 + \tan\left(\frac{e^{x}}{2}\right)^{2}} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = e^{x} \sin(e^{x}) + \cos(e^{x}) + c_1 + 1$$

Hence the ode is

$$y' + y = e^{x} \sin(e^{x}) + \cos(e^{x}) + c_1 + 1$$

The integrating factor μ is

$$\mu = e^{\int 1 dx}$$
$$= e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(e^{x} \sin\left(e^{x}\right)+\cos\left(e^{x}\right)+c_1+1\right)$$
$$\frac{d}{dx}\left(e^{x} y\right) = \left(e^{x}\right)\left(e^{x} \sin\left(e^{x}\right)+\cos\left(e^{x}\right)+c_1+1\right)$$
$$d\left(e^{x} y\right) = \left(\left(e^{x} \sin\left(e^{x}\right)+\cos\left(e^{x}\right)+c_1+1\right) e^{x}\right)\,dx$$

Integrating gives

$$e^{x} y = \int \left(e^{x} \sin\left(e^{x}\right)+\cos\left(e^{x}\right)+c_1+1\right) e^{x}\,dx$$
$$e^{x} y = e^{x} c_1 + e^{x} + 2\sin\left(e^{x}\right) - e^{x}\cos\left(e^{x}\right) + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}\left(e^{x} c_1 + e^{x} + 2\sin\left(e^{x}\right) - e^{x}\cos\left(e^{x}\right)\right) + e^{-x} c_2$$

which simplifies to

$$y = \left(c_2 + 2\sin\left(e^{x}\right)\right) e^{-x} + c_1 - \cos\left(e^{x}\right) + 1$$

Summary

The solution(s) found are the following

$$y = \left(c_2 + 2\sin\left(e^{x}\right)\right) e^{-x} + c_1 - \cos\left(e^{x}\right) + 1 \tag{1}$$

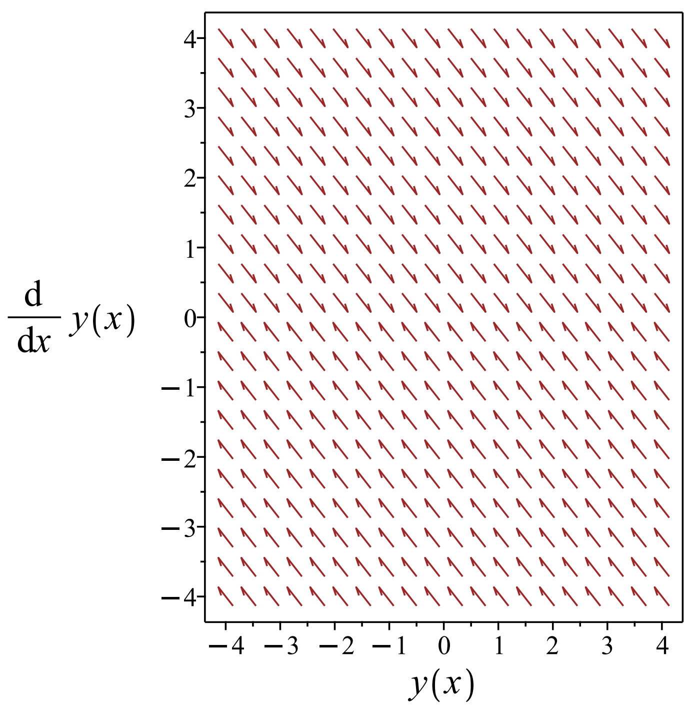

Figure 796: Slope field plot

Verification of solutions

$$y = (c_2 + 2\sin(e^x))\,e^{-x} + c_1 - \cos(e^x) + 1$$

Verified OK.

20.21.7 Maple step by step solution

Let's solve

$y'' + y' = e^{2x}\cos(e^x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{2x}\cos(e^x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-x}\left(\int \cos(e^x) e^{3x} dx\right) + \int e^{2x} \cos(e^x) dx$

- Compute integrals

  $y_p(x) = -\cos(e^x) + 2\sin(e^x) e^{-x} + 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 - \cos(e^x) + 2\sin(e^x) e^{-x} + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = exp(2*_a)*cos(exp(_a))-_b(_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+diff(y(x),x)=exp(2*x)*cos(exp(x)),y(x), singsol=all)
```

$$y(x) = (-c_1 + 2\sin(e^x)) e^{-x} + c_2 - \cos(e^x) - 1$$

✓ Solution by Mathematica

Time used: 0.258 (sec). Leaf size: 32

```
DSolve[y''[x]+y'[x]==Exp[2*x]*Cos[Exp[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - e^{-x}(-2\sin(e^x) + e^x \cos(e^x) + c_1)$$

20.22 problem 661

Internal problem ID [15426]
Internal file name [`OUTPUT/15426_Wednesday_May_08_2024_03_58_33_PM_17635758/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 661.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + y'' = \frac{x - 1}{x^3}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= 1 \\ y_3 &= x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + y'' = \frac{x-1}{x^3}$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & 1 & x \\ -e^{-x} & 0 & 1 \\ e^{-x} & 0 & 0 \end{bmatrix}$$

$$|W| = e^{-x}$$

The determinant simplifies to

$$|W| = e^{-x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\ &= 1 \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{-x} & x \\ -e^{-x} & 1 \end{bmatrix} \\ &= e^{-x}(x+1) \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{-x} & 1 \\ -e^{-x} & 0 \end{bmatrix} \\ &= e^{-x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x) W_1(x)}{a W(x)} \, dx \\ &= (-1)^2 \int \frac{\left(\frac{x-1}{x^3}\right)(1)}{(1)\left(e^{-x}\right)} \, dx \\ &= \int \frac{\frac{x-1}{x^3}}{e^{-x}} \, dx \\ &= \int \left(\frac{(x-1)\, e^x}{x^3}\right) dx \\ &= -\frac{e^x}{2x} - \frac{\text{expIntegral}_1(-x)}{2} + \frac{e^x}{2x^2} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(\frac{x-1}{x^3}\right)\left(\mathrm{e}^{-x}(x+1)\right)}{(1)\left(\mathrm{e}^{-x}\right)}\,dx \\
&= -\int \frac{\frac{(x-1)\mathrm{e}^{-x}(x+1)}{x^3}}{\mathrm{e}^{-x}}\,dx \\
&= -\int \left(\frac{x^2-1}{x^3}\right)dx \\
&= -\frac{1}{2x^2} - \ln(x)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{\left(\frac{x-1}{x^3}\right)\left(\mathrm{e}^{-x}\right)}{(1)\left(\mathrm{e}^{-x}\right)}\,dx \\
&= \int \frac{\frac{(x-1)\mathrm{e}^{-x}}{x^3}}{\mathrm{e}^{-x}}\,dx \\
&= \int \left(\frac{x-1}{x^3}\right)dx \\
&= \frac{1}{2x^2} - \frac{1}{x}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{\mathrm{e}^{x}}{2x} - \frac{\text{expIntegral}_1(-x)}{2} + \frac{\mathrm{e}^{x}}{2x^2}\right)\left(\mathrm{e}^{-x}\right) \\
&+ \left(-\frac{1}{2x^2} - \ln(x)\right) \\
&+ \left(\frac{1}{2x^2} - \frac{1}{x}\right)(x)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{\text{expIntegral}_1(-x)\,\mathrm{e}^{-x}}{2} - \ln(x) - 1
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + c_2 + c_3 x\right) + \left(-\frac{\text{expIntegral}_1(-x)\,\mathrm{e}^{-x}}{2} - \ln(x) - 1\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x - \frac{\mathrm{expIntegral}_1(-x)\,\mathrm{e}^{-x}}{2} - \ln(x) - 1 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + c_3 x - \frac{\mathrm{expIntegral}_1(-x)\,\mathrm{e}^{-x}}{2} - \ln(x) - 1$$

Verified OK.

20.22.1 Maple step by step solution

Let's solve

$y''' + y'' = \frac{x-1}{x^3}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{y_3(x)x^3 - x + 1}{x^3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{y_3(x)x^3 - x + 1}{x^3}\right]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + c_2$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)*_a^3-_a+1)/_a^3, _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)=(x-1)/x^3,y(x), singsol=all)
```

$$y(x) = -\frac{\left(\int\int\frac{\mathrm{e}^{-x}\,\mathrm{expIntegral}_1(-x)x^2-2\,\mathrm{e}^{-x}c_1x^2+x-1}{x^2}dxdx\right)}{2}+c_2x+c_3$$

✓ Solution by Mathematica

Time used: 0.238 (sec). Leaf size: 35

```
DSolve[y'''[x]+y''[x]==(x-1)/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}\,\mathrm{ExpIntegralEi}(x)}{2} - \log(x) + c_1e^{-x} + c_3x + c_2$$

20.23 problem 662

Internal problem ID [15427]
Internal file name [`OUTPUT/15427_Wednesday_May_08_2024_03_58_34_PM_61185690/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 662.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_y]]`

$$\boxed{xy'' - (2x^2+1)\, y' = 4x^3 \mathrm{e}^{x^2}}$$

20.23.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$xp'(x) + (-2x^2-1)\, p(x) - 4x^3 \mathrm{e}^{x^2} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{2x^2+1}{x}\\ q(x) &= 4x^2 e^{x^2}\end{aligned}$$

Hence the ode is

$$p'(x) - \frac{(2x^2+1)\, p(x)}{x} = 4x^2 e^{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2x^2+1}{x}dx}\\ &= e^{-x^2-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{e^{-x^2}}{x}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(4x^2 e^{x^2}\right)\\ \frac{d}{dx}\left(\frac{e^{-x^2} p}{x}\right) &= \left(\frac{e^{-x^2}}{x}\right)\left(4x^2 e^{x^2}\right)\\ d\left(\frac{e^{-x^2} p}{x}\right) &= (4x)\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{e^{-x^2} p}{x} &= \int 4x\, dx\\ \frac{e^{-x^2} p}{x} &= 2x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-x^2}}{x}$ results in

$$p(x) = 2x^3 e^{x^2} + c_1 x\, e^{x^2}$$

which simplifies to

$$p(x) = e^{x^2} x(2x^2 + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = e^{x^2} x(2x^2 + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int e^{x^2} x(2x^2 + c_1) \, dx \\ &= \frac{(2x^2 + c_1 - 2) e^{x^2}}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(2x^2 + c_1 - 2) e^{x^2}}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{(2x^2 + c_1 - 2) e^{x^2}}{2} + c_2$$

Verified OK.

20.23.2 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-2x^2 - 1) y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x^2 - 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 660: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{8x^3} - \frac{9}{128x^7} + \frac{27}{1024x^{11}} - \frac{405}{32768x^{15}} + \frac{1701}{262144x^{19}} - \frac{15309}{4194304x^{23}} + \frac{72171}{33554432x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (x^2) + \left(\frac{3}{4x^2}\right) \\ &= x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) &&= \frac{1}{2}\left(\frac{0}{1}-1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) &&= \frac{1}{2}\left(-\frac{0}{1}-1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)(x) \\
&= -\frac{1}{2x} - x \\
&= -\frac{1}{2x} - x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - x\right)(0) + \left(\left(\frac{1}{2x^2} - 1\right) + \left(-\frac{1}{2x} - x\right)^2 - \left(\frac{4x^4 + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - x\right) dx} \\
&= \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 - 1}{x} \, dx} \\
&= z_1 e^{\frac{x^2}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x^2+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{x^2}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2 \left(1\left(\frac{e^{x^2}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' + \left(-2x^2 - 1\right) y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 e^{x^2}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$

$$y_2 = \frac{e^{x^2}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{e^{x^2}}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{e^{x^2}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{e^{x^2}}{2} \\ 0 & x\,e^{x^2} \end{vmatrix}$$

Therefore

$$W = (1)\left(x\,e^{x^2}\right) - \left(\frac{e^{x^2}}{2}\right)(0)$$

Which simplifies to

$$W = x\,e^{x^2}$$

Which simplifies to

$$W = x\,e^{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,e^{2x^2} x^3}{x^2 e^{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int 2x\,e^{x^2}\,dx$$

Hence

$$u_1 = -e^{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x^3 e^{x^2}}{x^2 e^{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -e^{x^2} + x^2 e^{x^2}$$

Which simplifies to

$$y_p(x) = e^{x^2}\left(x^2 - 1\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \frac{c_2 \mathrm{e}^{x^2}}{2}\right) + \left(\mathrm{e}^{x^2} \left(x^2 - 1\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \frac{c_2 \mathrm{e}^{x^2}}{2} + \mathrm{e}^{x^2} \left(x^2 - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 + \frac{c_2 \mathrm{e}^{x^2}}{2} + \mathrm{e}^{x^2} \left(x^2 - 1\right)
$$

Verified OK.

20.23.3 Maple step by step solution

Let's solve

$xy'' + (-2x^2 - 1)\, y' = 4x^3 \mathrm{e}^{x^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $xu'(x) + (-2x^2 - 1)\, u(x) = 4x^3 \mathrm{e}^{x^2}$

- Isolate the derivative

  $u'(x) = \frac{(2x^2+1)u(x)}{x} + 4x^2 \mathrm{e}^{x^2}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{(2x^2+1)u(x)}{x} = 4x^2 \mathrm{e}^{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(u'(x) - \frac{(2x^2+1)u(x)}{x}\right) = 4\mu(x)\, x^2 \mathrm{e}^{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

  $\mu(x) \left(u'(x) - \frac{(2x^2+1)u(x)}{x}\right) = \mu'(x)\, u(x) + \mu(x)\, u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)(2x^2+1)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{\mathrm{e}^{-x^2}}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) u(x))\right) dx = \int 4\mu(x) x^2 \mathrm{e}^{x^2} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x) u(x) = \int 4\mu(x) x^2 \mathrm{e}^{x^2} dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int 4\mu(x)x^2\mathrm{e}^{x^2} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{\mathrm{e}^{-x^2}}{x}$

  $u(x) = \frac{x\left(\int 4x\,\mathrm{e}^{x^2}\mathrm{e}^{-x^2} dx + c_1\right)}{\mathrm{e}^{-x^2}}$
- Evaluate the integrals on the rhs

  $u(x) = \frac{x(2x^2+c_1)}{\mathrm{e}^{-x^2}}$
- Simplify

  $u(x) = \mathrm{e}^{x^2} x(2x^2 + c_1)$
- Solve 1st ODE for $u(x)$

  $u(x) = \mathrm{e}^{x^2} x(2x^2 + c_1)$
- Make substitution $u = y'$

  $y' = \mathrm{e}^{x^2} x(2x^2 + c_1)$
- Integrate both sides to solve for y

  $\int y' dx = \int \mathrm{e}^{x^2} x(2x^2 + c_1)\, dx + c_2$
- Compute integrals

  $y = \frac{(2x^2+c_1-2)\mathrm{e}^{x^2}}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (4*_a^3*exp(_a^2)+2*_b(_a)*_a^2+_b(_a))
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x*diff(y(x),x$2)-(1+2*x^2)*diff(y(x),x)=4*x^3*exp(x^2),y(x), singsol=all)
```

$$y(x) = \frac{(2x^2 + c_1 - 2)\,e^{x^2}}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 25

```
DSolve[x*y''[x]-(1+2*x^2)*y'[x]==4*x^3*Exp[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}\left(x^2 - 1 + \frac{c_1}{2}\right) + c_2$$

20.24 problem 663

Internal problem ID [15428]
Internal file name [`OUTPUT/15428_Wednesday_May_08_2024_03_58_35_PM_39720521/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 663.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 2y' \tan(x) = 1}$$

20.24.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 2p(x) \tan(x) - 1 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2\tan(x) \\ q(x) &= 1 \end{aligned}$$

Hence the ode is

$$p'(x) - 2p(x)\tan(x) = 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -2\tan(x)dx} \\ &= \cos(x)^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)^2 p\right) &= \cos(x)^2 \\ \mathrm{d}\left(\cos(x)^2 p\right) &= \cos(x)^2 \,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \cos(x)^2 p &= \int \cos(x)^2 \,\mathrm{d}x \\ \cos(x)^2 p &= \frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)^2$ results in

$$p(x) = \sec(x)^2 \left(\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right) + c_1 \sec(x)^2$$

which simplifies to

$$p(x) = \frac{(2c_1 + x)\sec(x)^2}{2} + \frac{\tan(x)}{2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{(2c_1 + x)\sec(x)^2}{2} + \frac{\tan(x)}{2}$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{\tan(x)}{2} + c_1 \sec(x)^2 + \frac{\sec(x)^2 x}{2} \, \mathrm{d}x \\
&= \frac{\tan(x) x}{2} + c_1 \tan(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\tan(x) x}{2} + c_1 \tan(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = \frac{\tan(x) x}{2} + c_1 \tan(x) + c_2
$$

Verified OK.

20.24.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + (2AB' + B^2) v' + (AB'' + BB' + CB) v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + (2AB' + B^2) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$AB u' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= 1 \\
B &= -2\tan(x) \\
C &= 0 \\
F &= 1
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (1)\left(-4\tan(x)\left(1+\tan(x)^2\right)\right) + (-2\tan(x))\left(-2-2\tan(x)^2\right) + (0)(-2\tan(x)) \\
&= -4\tan(x)\left(1+\tan(x)^2\right) - 2\tan(x)\left(-2-2\tan(x)^2\right) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2\tan(x)\, v'' + (-4)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2\tan(x)\, u'(x) - 4u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u}{\tan(x)}
\end{aligned}$$

Where $f(x) = -\frac{2}{\tan(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{\tan(x)}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{\tan(x)}\, dx \\
\ln(u) &= -2\ln(\sin(x)) + c_1 \\
u &= e^{-2\ln(\sin(x)) + c_1} \\
&= \frac{c_1}{\sin(x)^2}
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{\sin(x)^2}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{\sin(x)^2}\, \mathrm{d}x \\
&= -c_1 \cot(x) + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2\tan(x))\,(-c_1 \cot(x) + c_2) \\
&= 2c_1 - 2\tan(x)\, c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 2 \\
y_2 &= \tan(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 2 & \tan(x) \\ \frac{d}{dx}(2) & \frac{d}{dx}(\tan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 2 & \tan(x) \\ 0 & 1 + \tan(x)^2 \end{vmatrix}$$

Therefore

$$W = (2)\left(1 + \tan(x)^2\right) - (\tan(x))(0)$$

Which simplifies to

$$W = 2 + 2\tan(x)^2$$

Which simplifies to

$$W = 2\sec(x)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan(x)}{2\sec(x)^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)}{4} dx$$

Hence

$$u_1 = \frac{\cos(2x)}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2}{2\sec(x)^2}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)^2 \, dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)}{8}$$

$$u_2 = \frac{\sin(2x)}{4} + \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)}{4} + \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right)\tan(x)$$

Which simplifies to

$$y_p(x) = \frac{1}{4} + \frac{\tan(x)\, x}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (2c_1 - 2\tan(x)\, c_2) + \left(\frac{1}{4} + \frac{\tan(x)\, x}{2}\right) \\ &= \frac{(-8c_2 + 2x)\tan(x)}{4} + 2c_1 + \frac{1}{4} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(-8c_2 + 2x)\tan(x)}{4} + 2c_1 + \frac{1}{4} \tag{1}$$

Verification of solutions

$$y = \frac{(-8c_2 + 2x)\tan(x)}{4} + 2c_1 + \frac{1}{4}$$

Verified OK.

20.24.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' \tan(x) = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2\tan(x) \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 662: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2} \frac{-2\tan(x)}{1}\, dx} \\
&= z_1 e^{-\ln(\cos(x))} \\
&= z_1(\sec(x))
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2\tan(x)}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(\cos(x))}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y'\tan(x) = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \tan(x)\, c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \tan(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \tan(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\tan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \tan(x) \\ 0 & 1 + \tan(x)^2 \end{vmatrix}$$

Therefore

$$W = (1)\left(1 + \tan(x)^2\right) - (\tan(x))(0)$$

Which simplifies to

$$W = 1 + \tan(x)^2$$

Which simplifies to

$$W = \sec(x)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\tan(x)}{\sec(x)^2} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x)}{2} dx$$

Hence

$$u_1 = \frac{\cos(2x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{1}{\sec(x)^2} dx$$

Which simplifies to

$$u_2 = \int \cos(x)^2 dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Which simplifies to

$$u_1 = \frac{\cos(2x)}{4}$$

$$u_2 = \frac{\sin(2x)}{4} + \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)}{4} + \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right)\tan(x)$$

Which simplifies to

$$y_p(x) = \frac{1}{4} + \frac{\tan(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + \tan(x)\, c_2) + \left(\frac{1}{4} + \frac{\tan(x)\, x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \tan(x)\, c_2 + \frac{1}{4} + \frac{\tan(x)\, x}{2} \tag{1}$$

Verification of solutions

$$y = c_1 + \tan(x)\, c_2 + \frac{1}{4} + \frac{\tan(x)\, x}{2}$$

Verified OK.

20.24.4 Maple step by step solution

Let's solve

$y'' - 2y' \tan(x) = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) - 2u(x) \tan(x) = 1$

- Isolate the derivative

  $u'(x) = 2u(x) \tan(x) + 1$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - 2u(x) \tan(x) = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) (u'(x) - 2u(x) \tan(x)) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

$$\mu(x)\left(u'(x)-2u(x)\tan(x)\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$$

- Isolate $\mu'(x)$

$$\mu'(x)=-2\mu(x)\tan(x)$$

- Solve to find the integrating factor

$$\mu(x)=\cos(x)^2$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\mu(x)\,dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,u(x)=\int\mu(x)\,dx+c_1$$

- Solve for $u(x)$

$$u(x)=\frac{\int\mu(x)dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\cos(x)^2$

$$u(x)=\frac{\int\cos(x)^2dx+c_1}{\cos(x)^2}$$

- Evaluate the integrals on the rhs

$$u(x)=\frac{\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1}{\cos(x)^2}$$

- Simplify

$$u(x)=\frac{(2c_1+x)\sec(x)^2}{2}+\frac{\tan(x)}{2}$$

- Solve 1st ODE for $u(x)$

$$u(x)=\frac{(2c_1+x)\sec(x)^2}{2}+\frac{\tan(x)}{2}$$

- Make substitution $u=y'$

$$y'=\frac{(2c_1+x)\sec(x)^2}{2}+\frac{\tan(x)}{2}$$

- Integrate both sides to solve for y

$$\int y'dx=\int\left(\frac{(2c_1+x)\sec(x)^2}{2}+\frac{\tan(x)}{2}\right)dx+c_2$$

- Compute integrals

$$y=\frac{\tan(x)x}{2}+c_1\tan(x)+c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*_b(_a)*tan(_a)+1, _b(_a)`   *** Suble
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-2*tan(x)*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = -\frac{\ln(1+\cos(2x))}{4} + \frac{\ln(\cos(x))}{2} + \frac{(4c_1+2x)\tan(x)}{4} + c_2$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 19

```
DSolve[y''[x]-2*Tan[x]*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{x}{2} + c_1\right)\tan(x) + c_2$$

20.25 problem 664

Internal problem ID [15429]
Internal file name [`OUTPUT/15429_Wednesday_May_08_2024_03_58_36_PM_34969694/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 664.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x \ln(x) y'' - y' = \ln(x)^2}$$

20.25.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x \ln(x) p'(x) - p(x) - \ln(x)^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x \ln(x)} \\ q(x) &= \frac{\ln(x)}{x} \end{aligned}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x \ln(x)} = \frac{\ln(x)}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x \ln(x)} dx} \\ &= \frac{1}{\ln(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu p) &= (\mu) \left(\frac{\ln(x)}{x}\right) \\ \frac{d}{dx}\left(\frac{p}{\ln(x)}\right) &= \left(\frac{1}{\ln(x)}\right) \left(\frac{\ln(x)}{x}\right) \\ d\left(\frac{p}{\ln(x)}\right) &= \frac{1}{x} dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{p}{\ln(x)} &= \int \frac{1}{x} dx \\ \frac{p}{\ln(x)} &= \ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\ln(x)}$ results in

$$p(x) = \ln(x)^2 + c_1 \ln(x)$$

which simplifies to

$$p(x) = \ln(x) (\ln(x) + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \ln(x)(\ln(x) + c_1)$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \ln(x)(\ln(x) + c_1) \, \mathrm{d}x \\
&= x \ln(x)^2 - 2x \ln(x) + 2x + c_1(x \ln(x) - x) + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x \ln(x)^2 - 2x \ln(x) + 2x + c_1(x \ln(x) - x) + c_2 \tag{1}$$

Verification of solutions

$$y = x \ln(x)^2 - 2x \ln(x) + 2x + c_1(x \ln(x) - x) + c_2$$

Verified OK.

20.25.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + (2AB' + B^2) v' + (AB'' + BB' + CB) v &= 0
\end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$AB u' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x \ln(x) \\
B &= -1 \\
C &= 0 \\
F &= \ln(x)^2
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x \ln(x))(0) + (-1)(0) + (0)(-1) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x \ln(x) v'' + (1) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x \ln(x) u'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= \frac{u}{x \ln(x)}
\end{aligned}$$

Where $f(x) = \frac{1}{x \ln(x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u} du &= \frac{1}{x \ln(x)} dx \\
\int \frac{1}{u} du &= \int \frac{1}{x \ln(x)} dx \\
\ln(u) &= \ln(\ln(x)) + c_1 \\
u &= e^{\ln(\ln(x)) + c_1} \\
&= c_1 \ln(x)
\end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 \ln(x)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 \ln(x) \,\mathrm{d}x \\
&= c_1(x \ln(x) - x) + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-1)\,(c_1(x \ln(x) - x) + c_2) \\
&= -x \ln(x)\, c_1 + c_1 x - c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= -1 \\
y_2 &= -x \ln(x) + x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} -1 & -x \ln(x) + x \\ \frac{d}{dx}(-1) & \frac{d}{dx}(-x \ln(x) + x) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} -1 & -x\ln(x) + x \\ 0 & -\ln(x) \end{vmatrix}$$

Therefore

$$W = (-1)(-\ln(x)) - (-x\ln(x) + x)(0)$$

Which simplifies to

$$W = \ln(x)$$

Which simplifies to

$$W = \ln(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-x\ln(x) + x)\ln(x)^2}{x\ln(x)^2}\,dx$$

Which simplifies to

$$u_1 = -\int (-\ln(x) + 1)\,dx$$

Hence

$$u_1 = -2x + x\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\ln(x)^2}{x\ln(x)^2}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{x}dx$$

Hence

$$u_2 = -\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2x - x\ln(x) - \ln(x)(-x\ln(x) + x)$$

Which simplifies to

$$y_p(x) = x\left(\ln(x)^2 - 2\ln(x) + 2\right)$$

Hence the complete solution is

$$\begin{aligned}y(x) &= y_h + y_p \\ &= (-x\ln(x)\,c_1 + c_1 x - c_2) + \left(x\left(\ln(x)^2 - 2\ln(x) + 2\right)\right) \\ &= x\ln(x)^2 - x(2+c_1)\ln(x) + x(2+c_1) - c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\ln(x)^2 - x(2+c_1)\ln(x) + x(2+c_1) - c_2 \tag{1}$$

Verification of solutions

$$y = x\ln(x)^2 - x(2+c_1)\ln(x) + x(2+c_1) - c_2$$

Verified OK.

20.25.3 Maple step by step solution

Let's solve
$x\ln(x)\,y'' - y' = \ln(x)^2$

- Highest derivative means the order of the ODE is 2
  y''
- Make substitution $u = y'$ to reduce order of ODE
  $x\ln(x)\,u'(x) - u(x) = \ln(x)^2$
- Isolate the derivative
  $u'(x) = \frac{u(x)}{x\ln(x)} + \frac{\ln(x)}{x}$
- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $u'(x) - \frac{u(x)}{x\ln(x)} = \frac{\ln(x)}{x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(u'(x) - \frac{u(x)}{x\ln(x)}\right) = \frac{\mu(x)\ln(x)}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

$\mu(x)\left(u'(x)-\frac{u(x)}{x\ln(x)}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x\ln(x)}$

- Solve to find the integrating factor

  $\mu(x)=\frac{1}{\ln(x)}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\frac{\mu(x)\ln(x)}{x}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\frac{\mu(x)\ln(x)}{x}dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int\frac{\mu(x)\ln(x)}{x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{\ln(x)}$

  $u(x)=\ln(x)\left(\int\frac{1}{x}dx+c_1\right)$

- Evaluate the integrals on the rhs

  $u(x)=\ln(x)\,(\ln(x)+c_1)$

- Solve 1st ODE for $u(x)$

  $u(x)=\ln(x)\,(\ln(x)+c_1)$

- Make substitution $u=y'$

  $y'=\ln(x)\,(\ln(x)+c_1)$

- Integrate both sides to solve for y

  $\int y'dx=\int\ln(x)\,(\ln(x)+c_1)\,dx+c_2$

- Compute integrals

  $y=x\ln(x)^2-2x\ln(x)+2x+c_1(x\ln(x)-x)+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (ln(_a)^2+_b(_a))/(ln(_a)*_a), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(x*ln(x)*diff(y(x),x$2)-diff(y(x),x)=ln(x)^2,y(x), singsol=all)
```

$$y(x) = \ln(x)^2 x + x(c_1 - 2)\ln(x) + (-c_1 + 2)x + c_2$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 29

```
DSolve[x*Log[x]*y''[x]-y'[x]==Log[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\log^2(x) - (-2 + c_1)x + (-2 + c_1)x\log(x) + c_2$$

20.26 problem 665

Internal problem ID [15430]
Internal file name [`OUTPUT/15430_Wednesday_May_08_2024_03_58_38_PM_87810356/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 665.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' + (2x - 1)\, y' = -4x^2}$$

20.26.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$xp'(x) + (2x - 1)\, p(x) + 4x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{-2x+1}{x}\\ q(x) &= -4x\end{aligned}$$

Hence the ode is

$$p'(x) - \frac{(-2x+1)\,p(x)}{x} = -4x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{-2x+1}{x}dx}\\ &= \mathrm{e}^{2x-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{2x}}{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\,(-4x)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{2x}p}{x}\right) &= \left(\frac{\mathrm{e}^{2x}}{x}\right)(-4x)\\ \mathrm{d}\left(\frac{\mathrm{e}^{2x}p}{x}\right) &= \left(-4\,\mathrm{e}^{2x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\mathrm{e}^{2x}p}{x} &= \int -4\,\mathrm{e}^{2x}\,\mathrm{d}x\\ \frac{\mathrm{e}^{2x}p}{x} &= -2\,\mathrm{e}^{2x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{2x}}{x}$ results in

$$p(x) = -2x\,\mathrm{e}^{-2x}\mathrm{e}^{2x} + c_1 x\,\mathrm{e}^{-2x}$$

which simplifies to

$$p(x) = x\left(-2 + c_1\mathrm{e}^{-2x}\right)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x\left(-2 + c_1 \mathrm{e}^{-2x}\right)$$

Integrating both sides gives

$$\begin{aligned} y &= \int x\left(-2 + c_1 \mathrm{e}^{-2x}\right) \mathrm{d}x \\ &= \frac{c_1\left(-2x\,\mathrm{e}^{-2x} - \mathrm{e}^{-2x}\right)}{4} - x^2 + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\left(-2x\,\mathrm{e}^{-2x} - \mathrm{e}^{-2x}\right)}{4} - x^2 + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1\left(-2x\,\mathrm{e}^{-2x} - \mathrm{e}^{-2x}\right)}{4} - x^2 + c_2$$

Verified OK.

20.26.2 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (2x - 1)\,y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2x - 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^2 - 4x + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4x^2 - 4x + 3$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^2 - 4x + 3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 665: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = 1 + \frac{3}{4x^2} - \frac{1}{x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx 1 - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{32x^4} - \frac{1}{64x^5} - \frac{3}{128x^6} - \frac{3}{256x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^2 - 4x + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (1) + \left(\frac{-4x + 3}{4x^2}\right) \\ &= 1 + \frac{-4x + 3}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-1}{1} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-1}{1} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4x^2 - 4x + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (1) \\
&= 1 - \frac{1}{2x} \\
&= 1 - \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(1 - \frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(1 - \frac{1}{2x}\right)^2 - \left(\frac{4x^2 - 4x + 3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(1 - \frac{1}{2x}\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x-1}{x} \, dx} \\
&= z_1 e^{-x + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\, \mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x-1}{x}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x+\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(-\frac{(1+2x)\,e^{-2x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2\left(1\left(-\frac{(1+2x)\,e^{-2x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' + (2x-1)\,y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 - \frac{c_2(1+2x)\,e^{-2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= -\frac{(1+2x)\,\mathrm{e}^{-2x}}{4} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & -\frac{(1+2x)\mathrm{e}^{-2x}}{4} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(-\frac{(1+2x)\mathrm{e}^{-2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & -\frac{(1+2x)\mathrm{e}^{-2x}}{4} \\ 0 & -\frac{\mathrm{e}^{-2x}}{2} + \frac{(1+2x)\mathrm{e}^{-2x}}{2} \end{vmatrix}$$

Therefore

$$W = (1)\left(-\frac{\mathrm{e}^{-2x}}{2} + \frac{(1+2x)\,\mathrm{e}^{-2x}}{2}\right) - \left(-\frac{(1+2x)\,\mathrm{e}^{-2x}}{4}\right)(0)$$

Which simplifies to

$$W = x\,\mathrm{e}^{-2x}$$

Which simplifies to

$$W = x\,\mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(1+2x)\,\mathrm{e}^{-2x}x^2}{\mathrm{e}^{-2x}x^2}\,dx$$

Which simplifies to

$$u_1 = -\int (1+2x)\,dx$$

Hence

$$u_1 = -x^2 - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{-4x^2}{\mathrm{e}^{-2x}x^2}\,dx$$

Which simplifies to

$$u_2 = \int -4\,\mathrm{e}^{2x}dx$$

Hence

$$u_2 = -2\,\mathrm{e}^{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^2 - x + \frac{\mathrm{e}^{2x}(1+2x)\,\mathrm{e}^{-2x}}{2}$$

Which simplifies to

$$y_p(x) = -x^2 + \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 - \frac{c_2(1+2x)\,\mathrm{e}^{-2x}}{4}\right) + \left(-x^2 + \frac{1}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 - \frac{c_2(1+2x)\,\mathrm{e}^{-2x}}{4} - x^2 + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = c_1 - \frac{c_2(1+2x)\,\mathrm{e}^{-2x}}{4} - x^2 + \frac{1}{2}$$

Verified OK.

20.26.3 Maple step by step solution

Let's solve

$xy'' + (2x-1)\,y' = -4x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $xu'(x) + (2x-1)\,u(x) = -4x^2$

- Isolate the derivative

  $u'(x) = -\frac{(2x-1)u(x)}{x} - 4x$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{(2x-1)u(x)}{x} = -4x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x) + \frac{(2x-1)u(x)}{x}\right) = -4\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x) + \frac{(2x-1)u(x)}{x}\right) = \mu'(x)\,u(x) + \mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(2x-1)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{\mathrm{e}^{2x}}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,u(x))\right) dx = \int -4\mu(x)\,x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x) = \int -4\mu(x)\,x dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int -4\mu(x)x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{e^{2x}}{x}$

  $u(x) = \frac{x\left(\int -4\,e^{2x} dx + c_1\right)}{e^{2x}}$
- Evaluate the integrals on the rhs

  $u(x) = \frac{x(-2\,e^{2x} + c_1)}{e^{2x}}$
- Simplify

  $u(x) = x(-2 + c_1 e^{-2x})$
- Solve 1st ODE for $u(x)$

  $u(x) = x(-2 + c_1 e^{-2x})$
- Make substitution $u = y'$

  $y' = x(-2 + c_1 e^{-2x})$
- Integrate both sides to solve for y

  $\int y' dx = \int x(-2 + c_1 e^{-2x})\, dx + c_2$
- Compute integrals

  $y = \frac{c_1(-2x\,e^{-2x} - e^{-2x})}{4} - x^2 + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(2*_b(_a)*_a+4*_a^2-_b(_a))/_a, _b(_a)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)+(2*x-1)*diff(y(x),x)=-4*x^2,y(x), singsol=all)
```

$$y(x) = \frac{(-2x-1)\,c_1\mathrm{e}^{-2x}}{4} - x^2 + c_2$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 35

```
DSolve[x*y''[x]+(2*x-1)*y'[x]==-4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{4}e^{-2x}\left(4e^{2x}x^2 + 2c_1x + c_1\right)$$

20.27 problem 666

Internal problem ID [15431]
Internal file name [`OUTPUT/15431_Wednesday_May_08_2024_03_58_38_PM_85480017/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 666.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + y' \tan(x) = \cot(x)\cos(x)}$$

20.27.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)\tan(x) - \cot(x)\cos(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \tan(x) \\
q(x) &= \cot(x)\cos(x)
\end{aligned}
$$

Hence the ode is

$$
p'(x) + p(x)\tan(x) = \cot(x)\cos(x)
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \tan(x)dx} \\
&= \frac{1}{\cos(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \sec(x)
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu p) &= (\mu)(\cot(x)\cos(x)) \\
\frac{d}{dx}(\sec(x)\, p) &= (\sec(x))(\cot(x)\cos(x)) \\
d(\sec(x)\, p) &= \cot(x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sec(x)\, p &= \int \cot(x)\, dx \\
\sec(x)\, p &= \ln(\sin(x)) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$
p(x) = \cos(x)\ln(\sin(x)) + c_1\cos(x)
$$

which simplifies to

$$
p(x) = \cos(x)(\ln(\sin(x)) + c_1)
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \cos(x)(\ln(\sin(x)) + c_1)
$$

Integrating both sides gives

$$\begin{aligned} y &= \int \cos(x)\left(\ln(\sin(x)) + c_1\right) \mathrm{d}x \\ &= c_1 \sin(x) + \ln(\sin(x)) \sin(x) - \sin(x) + c_2 \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \sin(x) + \ln(\sin(x)) \sin(x) - \sin(x) + c_2 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \sin(x) + \ln(\sin(x)) \sin(x) - \sin(x) + c_2$$

Verified OK.

20.27.2 Maple step by step solution

Let's solve

$y'' + y' \tan(x) = \cot(x)\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''
- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)\tan(x) = \cot(x)\cos(x)$
- Isolate the derivative

  $u'(x) = -u(x)\tan(x) + \cot(x)\cos(x)$
- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + u(x)\tan(x) = \cot(x)\cos(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x) + u(x)\tan(x)\right) = \mu(x)\cot(x)\cos(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x) + u(x)\tan(x)\right) = \mu'(x)\,u(x) + \mu(x)\,u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\tan(x)$
- Solve to find the integrating factor

$\mu(x) = \frac{1}{\cos(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int \mu(x) \cot(x) \cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, u(x) = \int \mu(x) \cot(x) \cos(x)\, dx + c_1$

- Solve for $u(x)$

  $u(x) = \frac{\int \mu(x) \cot(x) \cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)}$

  $u(x) = \cos(x) \left(\int \cot(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $u(x) = \cos(x) (\ln(\sin(x)) + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \cos(x) (\ln(\sin(x)) + c_1)$

- Make substitution $u = y'$

  $y' = \cos(x) (\ln(\sin(x)) + c_1)$

- Integrate both sides to solve for y

  $\int y' dx = \int \cos(x) (\ln(\sin(x)) + c_1)\, dx + c_2$

- Compute integrals

  $y = c_1 \sin(x) + \ln(\sin(x)) \sin(x) - \sin(x) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)*tan(_a)+cot(_a)*cos(_a), _b(_a)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+tan(x)*diff(y(x),x)=cos(x)*cot(x),y(x), singsol=all)
```

$$y(x) = c_2 + \sin(x)(-1 + \ln(\sin(x)) + c_1)$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 39

```
DSolve[y''[x]+Tan[x]*y'[x]==Cos[x]*Cot[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{\sin^2(x)}\log\left(\sin^2(x)\right) - (1 + c_2)\sqrt{\sin^2(x)} + c_1$$

20.28 problem 667

Internal problem ID [15432]
Internal file name [`OUTPUT/15432_Wednesday_May_08_2024_03_58_39_PM_82807764/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 667.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{4xy'' + 2y' + y = 1}$$

With initial conditions

$$[y(\infty) = 1]$$

20.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{1}{4x} \\ F &= \frac{1}{4x} \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{2x} + \frac{y}{4x} = \frac{1}{4x}$$

The domain of $p(x) = \frac{1}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = \frac{1}{4x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is also inside this domain. The domain of $F = \frac{1}{4x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

20.28.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$4xy'' + 2y' + y = 0$$

In normal form the ode

$$4xy'' + 2y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{1}{4x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)} dx \\ &= \int e^{-\left(\int \frac{1}{2x} dx\right)} dx \\ &= \int e^{-\frac{\ln(x)}{2}} dx \\ &= \int \frac{1}{\sqrt{x}} dx \\ &= 2\sqrt{x} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{4x}}{\frac{1}{x}} \\
&= \frac{1}{4}
\end{aligned}
\tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4} &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = \frac{1}{4}$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda \tau} + \frac{\mathrm{e}^{\lambda \tau}}{4} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$
\lambda^2 + \frac{1}{4} = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = \frac{1}{4}$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)\left(\frac{1}{4}\right)} \\
&= \pm \frac{i}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +\frac{i}{2} \\
\lambda_2 &= -\frac{i}{2}
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = \frac{i}{2}$$
$$\lambda_2 = -\frac{i}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0\left(c_1 \cos\left(\frac{\tau}{2}\right) + c_2 \sin\left(\frac{\tau}{2}\right)\right)$$

Or

$$y(\tau) = c_1 \cos\left(\frac{\tau}{2}\right) + c_2 \sin\left(\frac{\tau}{2}\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\sqrt{x}\right)$$
$$y_2 = \sin\left(\sqrt{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\sqrt{x}\right) & \sin\left(\sqrt{x}\right) \\ \frac{d}{dx}\left(\cos\left(\sqrt{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\sqrt{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\sqrt{x}\right) & \sin\left(\sqrt{x}\right) \\ -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}} & \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\sqrt{x}\right)\right)\left(\frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}}\right) - \left(\sin\left(\sqrt{x}\right)\right)\left(-\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{\cos\left(\sqrt{x}\right)^2 + \sin\left(\sqrt{x}\right)^2}{2\sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{2\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}} dx$$

Hence

$$u_1 = \cos\left(\sqrt{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}} dx$$

Hence

$$u_2 = \sin\left(\sqrt{x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos\left(\sqrt{x}\right)^2 + \sin\left(\sqrt{x}\right)^2$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)\right) + (1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right) + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \infty$ in the above gives

$$1 = -|c_1| - |c_2| + 1..|c_1| + |c_2| + 1 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.28.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x, B = 2, C = 1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$4xy'' + 2y' + y = 0$$

In normal form the ode

$$4xy'' + 2y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{1}{4x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x}}}{2c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{1}{4c\sqrt{\frac{1}{x}}\, x^2}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{4c\sqrt{\frac{1}{x}}\,x^2} + \frac{1}{2x}\frac{\sqrt{\frac{1}{x}}}{2c}}{\left(\frac{\sqrt{\frac{1}{x}}}{2c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{\sqrt{\frac{1}{x}}}{2} dx}{c} \\
&= \frac{x\sqrt{\frac{1}{x}}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)
$$

Now the particular solution to this ODE is found

$$
4xy'' + 2y' + y = 1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(x\sqrt{\frac{1}{x}}\right)$$

$$y_2 = \sin\left(x\sqrt{\frac{1}{x}}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(x\sqrt{\frac{1}{x}}\right) & \sin\left(x\sqrt{\frac{1}{x}}\right) \\ \frac{d}{dx}\left(\cos\left(x\sqrt{\frac{1}{x}}\right)\right) & \frac{d}{dx}\left(\sin\left(x\sqrt{\frac{1}{x}}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(x\sqrt{\frac{1}{x}}\right) & \sin\left(x\sqrt{\frac{1}{x}}\right) \\ -\left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\sin\left(x\sqrt{\frac{1}{x}}\right) & \left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\cos\left(x\sqrt{\frac{1}{x}}\right) \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(x\sqrt{\frac{1}{x}}\right)\right)\left(\left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\cos\left(x\sqrt{\frac{1}{x}}\right)\right) - \left(\sin\left(x\sqrt{\frac{1}{x}}\right)\right)\left(-\left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\sin\left(x\sqrt{\frac{1}{x}}\right)\right)$$

Which simplifies to

$$W = \frac{\cos\left(x\sqrt{\frac{1}{x}}\right)^2 + \sin\left(x\sqrt{\frac{1}{x}}\right)^2}{2x\sqrt{\frac{1}{x}}}$$

Which simplifies to

$$W = \frac{1}{2x\sqrt{\frac{1}{x}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(x\sqrt{\frac{1}{x}}\right)}{\frac{2}{\sqrt{\frac{1}{x}}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(x\sqrt{\frac{1}{x}}\right)\sqrt{\frac{1}{x}}}{2}dx$$

Hence

$$u_1 = \cos\left(x\sqrt{\frac{1}{x}}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(x\sqrt{\frac{1}{x}}\right)}{\frac{2}{\sqrt{\frac{1}{x}}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(x\sqrt{\frac{1}{x}}\right)\sqrt{\frac{1}{x}}}{2}dx$$

Hence

$$u_2 = \sin\left(x\sqrt{\frac{1}{x}}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos\left(x\sqrt{\frac{1}{x}}\right)^2 + \sin\left(x\sqrt{\frac{1}{x}}\right)^2$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(\sqrt{x}\right) + c_2\sin\left(\sqrt{x}\right)\right) + (1) \\ &= c_1\cos\left(\sqrt{x}\right) + c_2\sin\left(\sqrt{x}\right) + 1 \end{aligned}$$

Which simplifies to

$$y = c_1\cos\left(\sqrt{x}\right) + c_2\sin\left(\sqrt{x}\right) + 1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\cos\left(\sqrt{x}\right) + c_2\sin\left(\sqrt{x}\right) + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \infty$ in the above gives

$$1 = -|c_1| - |c_2| + 1..|c_1| + |c_2| + 1 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.28.4 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{xy'}{2} + \frac{yx}{4} = \frac{x}{4} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{4}$$
$$\beta = 1$$
$$n = \frac{1}{2}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

$$y_2 = -\frac{\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\sqrt{2}\sin(\sqrt{x})}{\sqrt{\pi}} & -\frac{\sqrt{2}\cos(\sqrt{x})}{\sqrt{\pi}} \\ \frac{d}{dx}\left(\frac{\sqrt{2}\sin(\sqrt{x})}{\sqrt{\pi}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\cos(\sqrt{x})}{\sqrt{\pi}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\sqrt{2}\sin(\sqrt{x})}{\sqrt{\pi}} & -\frac{\sqrt{2}\cos(\sqrt{x})}{\sqrt{\pi}} \\ \frac{\sqrt{2}\cos(\sqrt{x})}{2\sqrt{\pi}\,\sqrt{x}} & \frac{\sqrt{2}\sin(\sqrt{x})}{2\sqrt{\pi}\,\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}}\right)\left(\frac{\sqrt{2}\,\sin\left(\sqrt{x}\right)}{2\sqrt{\pi}\,\sqrt{x}}\right) - \left(-\frac{\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}\right)\left(\frac{\sqrt{2}\,\cos\left(\sqrt{x}\right)}{2\sqrt{\pi}\,\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{\cos\left(\sqrt{x}\right)^2 + \sin\left(\sqrt{x}\right)^2}{\pi \sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{\pi \sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\sqrt{2}\, \cos(\sqrt{x})x}{4\sqrt{\pi}}}{\frac{x^{\frac{3}{2}}}{\pi}} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{2}\,\sqrt{\pi}\, \cos\left(\sqrt{x}\right)}{4\sqrt{x}} dx$$

Hence

$$u_1 = \frac{\sqrt{2}\,\sqrt{\pi}\, \sin\left(\sqrt{x}\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{2}\, \sin(\sqrt{x})x}{4\sqrt{\pi}}}{\frac{x^{\frac{3}{2}}}{\pi}} dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\,\sqrt{\pi}\, \sin\left(\sqrt{x}\right)}{4\sqrt{x}} dx$$

Hence

$$u_2 = -\frac{\sqrt{2}\,\sqrt{\pi}\, \cos\left(\sqrt{x}\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \cos\left(\sqrt{x}\right)^2 + \sin\left(\sqrt{x}\right)^2$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}\right) + (1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}} + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \infty$ in the above gives

$$1 = -\frac{-\sqrt{\pi} + (|c_1| + |c_2|)\sqrt{2}}{\sqrt{\pi}} .. \frac{\sqrt{\pi} + (|c_1| + |c_2|)\sqrt{2}}{\sqrt{\pi}} \tag{1A}$$

Equations $\{1\text{A}\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.28.5 Solving using Kovacic algorithm

Writing the ode as

$$4xy'' + 2y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 668: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned}
d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right)\\
&= \frac{1}{2}(1 - (1))\\
&= 0
\end{aligned}$$

We now form the following rational function

$$\begin{aligned}
\theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x - c}\\
&= \frac{1}{2}\left(\frac{1}{(x - (0))}\right)\\
&= \frac{1}{2x}
\end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{4x+1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{4x} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{4x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\sqrt{-x}}\right) + c_2\left(\mathrm{e}^{\sqrt{-x}}\left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4xy'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}} + \mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\sqrt{-x}}$$
$$y_2 = \frac{\sqrt{-x}\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{\sqrt{x}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\sqrt{-x}} & \frac{\sqrt{-x}\left(-e^{\sqrt{-x}}+e^{-\sqrt{-x}}\right)}{\sqrt{x}} \\ \frac{d}{dx}\left(e^{\sqrt{-x}}\right) & \frac{d}{dx}\left(\frac{\sqrt{-x}\left(-e^{\sqrt{-x}}+e^{-\sqrt{-x}}\right)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\sqrt{-x}} & \frac{\sqrt{-x}\left(-e^{\sqrt{-x}}+e^{-\sqrt{-x}}\right)}{\sqrt{x}} \\ -\frac{e^{\sqrt{-x}}}{2\sqrt{-x}} & -\frac{-e^{\sqrt{-x}}+e^{-\sqrt{-x}}}{2\sqrt{-x}\sqrt{x}} + \frac{\sqrt{-x}\left(\frac{e^{\sqrt{-x}}}{2\sqrt{-x}}+\frac{e^{-\sqrt{-x}}}{2\sqrt{-x}}\right)}{\sqrt{x}} - \frac{\sqrt{-x}\left(-e^{\sqrt{-x}}+e^{-\sqrt{-x}}\right)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(e^{\sqrt{-x}}\right)\left(-\frac{-e^{\sqrt{-x}} + e^{-\sqrt{-x}}}{2\sqrt{-x}\sqrt{x}} + \frac{\sqrt{-x}\left(\frac{e^{\sqrt{-x}}}{2\sqrt{-x}} + \frac{e^{-\sqrt{-x}}}{2\sqrt{-x}}\right)}{\sqrt{x}} - \frac{\sqrt{-x}\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{2x^{\frac{3}{2}}}\right) - \left(\frac{\sqrt{-x}\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{\sqrt{x}}\right)\left(-\frac{e^{\sqrt{-x}}}{2\sqrt{-x}}\right)$$

Which simplifies to

$$W = \frac{e^{\sqrt{-x}} e^{-\sqrt{-x}}}{\sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-x}\left(-e^{\sqrt{-x}}+e^{-\sqrt{-x}}\right)}{\sqrt{x}}}{4\sqrt{x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{\sqrt{-x}} - e^{-\sqrt{-x}}}{4\sqrt{-x}} dx$$

Hence

$$u_1 = \frac{e^{\sqrt{-x}}}{2} + \frac{e^{-\sqrt{-x}}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\sqrt{-x}}}{4\sqrt{x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{\sqrt{-x}}}{4\sqrt{x}} dx$$

Hence

$$u_2 = \frac{\sqrt{-x}\left(1 - e^{\sqrt{-x}}\right)}{2\sqrt{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{e^{\sqrt{-x}}}{2} + \frac{e^{-\sqrt{-x}}}{2}\right) e^{\sqrt{-x}} - \frac{\left(1 - e^{\sqrt{-x}}\right)\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{2}$$

Which simplifies to

$$y_p(x) = 1 + \frac{e^{\sqrt{-x}}}{2} - \frac{e^{-\sqrt{-x}}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-x}} + \frac{c_2 \sqrt{-x}\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{\sqrt{x}}\right) + \left(1 + \frac{e^{\sqrt{-x}}}{2} - \frac{e^{-\sqrt{-x}}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\sqrt{-x}} + \frac{c_2 \sqrt{-x}\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{\sqrt{x}} + 1 + \frac{e^{\sqrt{-x}}}{2} - \frac{e^{-\sqrt{-x}}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \infty$ in the above gives

$$1 = -|\Re((1+i)\, c_1)\,| - i|\Im((1+i)\, c_1)\,| + 2\min\left(-\Re((1-i)\, c_2)\,, -\Re((1+i)\, c_2)\,, \Re((1-i)\, c_2)\,, \Re((1+i)\, c_2 \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✗ Solution by Maple

```
dsolve([4*x*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=1,y(infinity) = 1],y(x), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 25

```
DSolve[{4*x*y''[x]+2*y'[x]+y[x]==1,{y[Infinity]==1}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right) + 1$$

20.29 problem 668

Internal problem ID [15433]
Internal file name [`OUTPUT/15433_Wednesday_May_08_2024_03_58_42_PM_47599306/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 668.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{4xy'' + 2y' + y = \frac{6 + x}{x^2}}$$

With initial conditions

$$[y(\infty) = 0]$$

20.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{1}{4x} \\ F &= \frac{6+x}{4x^3} \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{2x} + \frac{y}{4x} = \frac{6+x}{4x^3}$$

The domain of $p(x) = \frac{1}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = \frac{1}{4x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is also inside this domain. The domain of $F = \frac{6+x}{4x^3}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

20.29.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4xy'' + 2y' + y = 0$$

In normal form the ode

$$4xy'' + 2y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{1}{4x} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{2x}dx\right)}dx \\ &= \int e^{-\frac{\ln(x)}{2}}\,dx \\ &= \int \frac{1}{\sqrt{x}}dx \\ &= 2\sqrt{x} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{4x}}{\frac{1}{x}} \\ &= \frac{1}{4} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4} &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = \frac{1}{4}$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + \frac{e^{\lambda\tau}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \frac{1}{4}$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)\left(\frac{1}{4}\right)} \\ &= \pm\frac{i}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +\frac{i}{2} \\ \lambda_2 &= -\frac{i}{2} \end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{i}{2}$$
$$\lambda_2 = -\frac{i}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0\left(c_1 \cos\left(\frac{\tau}{2}\right) + c_2 \sin\left(\frac{\tau}{2}\right)\right)$$

Or

$$y(\tau) = c_1 \cos\left(\frac{\tau}{2}\right) + c_2 \sin\left(\frac{\tau}{2}\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\sqrt{x}\right)$$
$$y_2 = \sin\left(\sqrt{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\sqrt{x}) & \sin(\sqrt{x}) \\ \frac{d}{dx}(\cos(\sqrt{x})) & \frac{d}{dx}(\sin(\sqrt{x})) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\sqrt{x}) & \sin(\sqrt{x}) \\ -\frac{\sin(\sqrt{x})}{2\sqrt{x}} & \frac{\cos(\sqrt{x})}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (\cos(\sqrt{x}))\left(\frac{\cos(\sqrt{x})}{2\sqrt{x}}\right) - (\sin(\sqrt{x}))\left(-\frac{\sin(\sqrt{x})}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{\cos(\sqrt{x})^2 + \sin(\sqrt{x})^2}{2\sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{2\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(\sqrt{x})(6+x)}{x^2}}{\frac{1}{2\sqrt{x}}} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(\sqrt{x}\right)(6+x)}{2x^{\frac{5}{2}}} dx$$

Hence

$$u_1 = \frac{2\sin\left(\sqrt{x}\right)}{x^{\frac{3}{2}}} + \frac{\cos\left(\sqrt{x}\right)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\sqrt{x}\right)(6+x)}{x^2}}{2\sqrt{x}} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(\sqrt{x}\right)(6+x)}{2x^{\frac{5}{2}}} dx$$

Hence

$$u_2 = -\frac{2\cos\left(\sqrt{x}\right)}{x^{\frac{3}{2}}} + \frac{\sin\left(\sqrt{x}\right)}{x}$$

Which simplifies to

$$u_1 = \frac{\cos\left(\sqrt{x}\right)\sqrt{x} + 2\sin\left(\sqrt{x}\right)}{x^{\frac{3}{2}}}$$

$$u_2 = \frac{\sin\left(\sqrt{x}\right)\sqrt{x} - 2\cos\left(\sqrt{x}\right)}{x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(\cos\left(\sqrt{x}\right)\sqrt{x} + 2\sin\left(\sqrt{x}\right)\right)\cos\left(\sqrt{x}\right)}{x^{\frac{3}{2}}} + \frac{\left(\sin\left(\sqrt{x}\right)\sqrt{x} - 2\cos\left(\sqrt{x}\right)\right)\sin\left(\sqrt{x}\right)}{x^{\frac{3}{2}}}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)\right) + \left(\frac{1}{x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right) + \frac{1}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = -|c_1| - |c_2|..|c_1| + |c_2| \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.29.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x, B = 2, C = 1, f(x) = \frac{6+x}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$4xy'' + 2y' + y = 0$$

In normal form the ode

$$4xy'' + 2y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{4x}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x}}}{2c} \\ \tau'' &= -\frac{1}{4c\sqrt{\frac{1}{x}}\,x^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{4c\sqrt{\frac{1}{x}}\,x^2} + \frac{1}{2x}\frac{\sqrt{\frac{1}{x}}}{2c}}{\left(\frac{\sqrt{\frac{1}{x}}}{2c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau) + c_2\sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{\sqrt{\frac{1}{x}}}{2}dx}{c} \\
&= \frac{x\sqrt{\frac{1}{x}}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)
$$

Now the particular solution to this ODE is found

$$
4xy'' + 2y' + y = \frac{6+x}{x^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos\left(x\sqrt{\frac{1}{x}}\right) \\
y_2 &= \sin\left(x\sqrt{\frac{1}{x}}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(x\sqrt{\frac{1}{x}}\right) & \sin\left(x\sqrt{\frac{1}{x}}\right) \\ \frac{d}{dx}\left(\cos\left(x\sqrt{\frac{1}{x}}\right)\right) & \frac{d}{dx}\left(\sin\left(x\sqrt{\frac{1}{x}}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(x\sqrt{\frac{1}{x}}\right) & \sin\left(x\sqrt{\frac{1}{x}}\right) \\ -\left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\sin\left(x\sqrt{\frac{1}{x}}\right) & \left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\cos\left(x\sqrt{\frac{1}{x}}\right) \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(x\sqrt{\frac{1}{x}}\right)\right)\left(\left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\cos\left(x\sqrt{\frac{1}{x}}\right)\right) - \left(\sin\left(x\sqrt{\frac{1}{x}}\right)\right)\left(-\left(\sqrt{\frac{1}{x}} - \frac{1}{2x\sqrt{\frac{1}{x}}}\right)\sin\left(x\sqrt{\frac{1}{x}}\right)\right)$$

Which simplifies to

$$W = \frac{\cos\left(x\sqrt{\frac{1}{x}}\right)^2 + \sin\left(x\sqrt{\frac{1}{x}}\right)^2}{2x\sqrt{\frac{1}{x}}}$$

Which simplifies to

$$W = \frac{1}{2x\sqrt{\frac{1}{x}}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(x\sqrt{\frac{1}{x}}\right)(6+x)}{x^2}}{\frac{2}{\sqrt{\frac{1}{x}}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin\left(x\sqrt{\frac{1}{x}}\right)(6+x)\sqrt{\frac{1}{x}}}{2x^2}dx$$

Hence

$$u_1 = -\frac{3\sqrt{\pi}\left(-\frac{8}{\sqrt{\pi}\,x} - \frac{4\left(2\gamma - \frac{11}{3} + \ln(x) + 2\ln\left(\sqrt{\frac{1}{x}}\,\sqrt{x}\right)\right)}{3\sqrt{\pi}} + \frac{-\frac{44x}{9}+8}{\sqrt{\pi}\,x} + \frac{8\gamma}{3\sqrt{\pi}} + \frac{8\ln(2)}{3\sqrt{\pi}} + \frac{8\ln\left(\frac{\sqrt{x}}{2}\right)}{3\sqrt{\pi}} - \frac{8\cos\left(\sqrt{x}\right)}{3\sqrt{\pi}\,x} - \frac{16\left(-\frac{5x}{2}+5\right)\sin}{15\sqrt{\pi}\,x^{\frac{3}{2}}}}{8}$$

$$- \frac{\sqrt{\pi}\left(\frac{4\gamma - 4 + 2\ln(x) + 4\ln\left(\sqrt{\frac{1}{x}}\,\sqrt{x}\right)}{\sqrt{\pi}} + \frac{4}{\sqrt{\pi}} - \frac{4\gamma}{\sqrt{\pi}} - \frac{4\ln(2)}{\sqrt{\pi}} - \frac{4\ln\left(\frac{\sqrt{x}}{2}\right)}{\sqrt{\pi}} - \frac{4\sin\left(\sqrt{x}\right)}{\sqrt{\pi}\,\sqrt{x}} + \frac{4\,\operatorname{Ci}\left(\sqrt{x}\right)}{\sqrt{\pi}}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(x\sqrt{\frac{1}{x}}\right)(6+x)}{x^2}}{\frac{2}{\sqrt{\frac{1}{x}}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos\left(x\sqrt{\frac{1}{x}}\right)(6+x)\sqrt{\frac{1}{x}}}{2x^2}dx$$

Hence

$$u_2 = \frac{3\sqrt{\pi}\,\sqrt{\frac{1}{x}}\,\sqrt{x}\left(-\frac{8(-x+2)\cos\left(\sqrt{x}\right)}{3x^{\frac{3}{2}}\sqrt{\pi}} + \frac{8\sin\left(\sqrt{x}\right)}{3x\sqrt{\pi}} + \frac{8\,\operatorname{Si}\left(\sqrt{x}\right)}{3\sqrt{\pi}}\right)}{8} + \frac{\sqrt{\pi}\,\sqrt{\frac{1}{x}}\,\sqrt{x}\left(-\frac{4\cos\left(\sqrt{x}\right)}{\sqrt{x}\,\sqrt{\pi}} - \frac{4\,\operatorname{Si}\left(\sqrt{x}\right)}{\sqrt{\pi}}\right)}{4}$$

Which simplifies to

$$u_1 = \frac{\cos\left(\sqrt{x}\right)\sqrt{x} + 2\sin\left(\sqrt{x}\right)}{x^{\frac{3}{2}}}$$

$$u_2 = \frac{\sqrt{\frac{1}{x}}\left(\sin\left(\sqrt{x}\right)\sqrt{x} - 2\cos\left(\sqrt{x}\right)\right)}{x}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \frac{\left(\cos \left(\sqrt{x}\right) \sqrt{x}+2 \sin \left(\sqrt{x}\right)\right) \cos \left(x \sqrt{\frac{1}{x}}\right)}{x^{\frac{3}{2}}} \\
& + \frac{\sqrt{\frac{1}{x}}\, \left(\sin \left(\sqrt{x}\right) \sqrt{x}-2 \cos \left(\sqrt{x}\right)\right) \sin \left(x \sqrt{\frac{1}{x}}\right)}{x}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
& y_p(x) \\
& = \frac{\left(\cos \left(\sqrt{x}\right) \sqrt{x}+2 \sin \left(\sqrt{x}\right)\right) \cos \left(x \sqrt{\frac{1}{x}}\right)+\sqrt{\frac{1}{x}}\, \sin \left(x \sqrt{\frac{1}{x}}\right) \left(\sin \left(\sqrt{x}\right) x-2 \cos \left(\sqrt{x}\right) \sqrt{x}\right)}{x^{\frac{3}{2}}}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos \left(\sqrt{x}\right)+c_2 \sin \left(\sqrt{x}\right)\right) \\
&\quad + \left(\frac{\left(\cos \left(\sqrt{x}\right) \sqrt{x}+2 \sin \left(\sqrt{x}\right)\right) \cos \left(x \sqrt{\frac{1}{x}}\right)+\sqrt{\frac{1}{x}}\, \sin \left(x \sqrt{\frac{1}{x}}\right) \left(\sin \left(\sqrt{x}\right) x-2 \cos \left(\sqrt{x}\right) \sqrt{x}\right)}{x^{\frac{3}{2}}}\right) \\
&= \frac{\left(\cos \left(\sqrt{x}\right) \sqrt{x}+2 \sin \left(\sqrt{x}\right)\right) \cos \left(x \sqrt{\frac{1}{x}}\right)+\sqrt{\frac{1}{x}}\, \sin \left(x \sqrt{\frac{1}{x}}\right) \left(\sin \left(\sqrt{x}\right) x-2 \cos \left(\sqrt{x}\right) \sqrt{x}\right)}{x^{\frac{3}{2}}} \\
&\quad + c_1 \cos \left(\sqrt{x}\right)+c_2 \sin \left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
& y \\
& = \frac{\left(\cos \left(\sqrt{x}\right) \sqrt{x}+2 \sin \left(\sqrt{x}\right)\right) \cos \left(x \sqrt{\frac{1}{x}}\right)+\sqrt{\frac{1}{x}}\, \sin \left(x \sqrt{\frac{1}{x}}\right) \left(\sin \left(\sqrt{x}\right) x-2 \cos \left(\sqrt{x}\right) \sqrt{x}\right)}{x^{\frac{3}{2}}} \\
& \quad + c_1 \cos \left(\sqrt{x}\right)+c_2 \sin \left(\sqrt{x}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{\left(\cos \left(\sqrt{x}\right) \sqrt{x}+2 \sin \left(\sqrt{x}\right)\right) \cos \left(x \sqrt{\frac{1}{x}}\right)+\sqrt{\frac{1}{x}}\, \sin \left(x \sqrt{\frac{1}{x}}\right) \left(\sin \left(\sqrt{x}\right) x-2 \cos \left(\sqrt{x}\right) \sqrt{x}\right)}{x^{\frac{3}{2}}} + c_1 \cos \left(\sqrt{\right. \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = -|c_1| - |c_2|..|c_1| + |c_2| \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.29.4 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + \frac{xy'}{2} + \frac{yx}{4} = \frac{6+x}{4x} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \,\text{BesselJ}\,(n, \beta\, x^\gamma) + c_2 \,\text{BesselY}\,(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{4}$$
$$\beta = 1$$
$$n = \frac{1}{2}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{2}\, \sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2 \sqrt{2}\, \cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \sqrt{2}\, \sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2 \sqrt{2}\, \cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\sqrt{2}\, \sin\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

$$y_2 = -\frac{\sqrt{2}\, \cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\sqrt{2}\, \sin(\sqrt{x})}{\sqrt{\pi}} & -\frac{\sqrt{2}\, \cos(\sqrt{x})}{\sqrt{\pi}} \\ \frac{d}{dx}\left(\frac{\sqrt{2}\, \sin(\sqrt{x})}{\sqrt{\pi}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\, \cos(\sqrt{x})}{\sqrt{\pi}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\sqrt{2}\,\sin(\sqrt{x})}{\sqrt{\pi}} & -\frac{\sqrt{2}\,\cos(\sqrt{x})}{\sqrt{\pi}} \\ \frac{\sqrt{2}\,\cos(\sqrt{x})}{2\sqrt{\pi}\,\sqrt{x}} & \frac{\sqrt{2}\,\sin(\sqrt{x})}{2\sqrt{\pi}\,\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}}\right)\left(\frac{\sqrt{2}\,\sin\left(\sqrt{x}\right)}{2\sqrt{\pi}\,\sqrt{x}}\right) - \left(-\frac{\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}\right)\left(\frac{\sqrt{2}\,\cos\left(\sqrt{x}\right)}{2\sqrt{\pi}\,\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{\cos\left(\sqrt{x}\right)^2 + \sin\left(\sqrt{x}\right)^2}{\pi\sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{\pi\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\sqrt{2}\,\cos(\sqrt{x})(6+x)}{4\sqrt{\pi}\,x}}{\frac{x^{\frac{3}{2}}}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{2}\,\sqrt{\pi}\,\cos\left(\sqrt{x}\right)(6+x)}{4x^{\frac{5}{2}}}dx$$

Hence

$$u_1 = \frac{\sqrt{2}\,\sqrt{\pi}\left(-\frac{4\cos(\sqrt{x})}{x^{\frac{3}{2}}} + \frac{2\sin(\sqrt{x})}{x}\right)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\sqrt{2}\,\sin(\sqrt{x})(6+x)}{4\sqrt{\pi}\,x}}{\frac{x^{\frac{3}{2}}}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}\,\sqrt{\pi}\,\sin\left(\sqrt{x}\right)(6+x)}{4x^{\frac{5}{2}}}dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\sqrt{\pi}\left(-\frac{4\sin\left(\sqrt{x}\right)}{x^{\frac{3}{2}}} - \frac{2\cos\left(\sqrt{x}\right)}{x}\right)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(-\frac{4\cos\left(\sqrt{x}\right)}{x^{\frac{3}{2}}} + \frac{2\sin\left(\sqrt{x}\right)}{x}\right)\sin\left(\sqrt{x}\right)}{2} - \frac{\left(-\frac{4\sin\left(\sqrt{x}\right)}{x^{\frac{3}{2}}} - \frac{2\cos\left(\sqrt{x}\right)}{x}\right)\cos\left(\sqrt{x}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}\right) + \left(\frac{1}{x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}} + \frac{1}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \frac{(-|c_1| - |c_2|)\sqrt{2}}{\sqrt{\pi}} .. \frac{\sqrt{2}\,(|c_1| + |c_2|)}{\sqrt{\pi}} \tag{1A}$$

Equations {1A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.29.5 Solving using Kovacic algorithm

Writing the ode as

$$4xy'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 669: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned}
E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\
&= \{1, 2, 3\}
\end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1 - (1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d = 0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right) p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) p = 0 \tag{1A}$$

Since $d = 0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned}\phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x}\end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{4x+1}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1 + 2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{4x} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{4x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-x}}\right) + c_2 \left(\mathrm{e}^{\sqrt{-x}} \left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}} \right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4xy'' + 2y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}} + \mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{\sqrt{-x}} \\ y_2 &= \frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}} + \mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-x}} & \frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}} \\ \frac{d}{dx}\left(\mathrm{e}^{\sqrt{-x}}\right) & \frac{d}{dx}\left(\frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\sqrt{-x}} & \frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}} \\ -\frac{\mathrm{e}^{\sqrt{-x}}}{2\sqrt{-x}} & -\frac{-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}}{2\sqrt{-x}\,\sqrt{x}} + \frac{\sqrt{-x}\left(\frac{\mathrm{e}^{\sqrt{-x}}}{2\sqrt{-x}}+\frac{\mathrm{e}^{-\sqrt{-x}}}{2\sqrt{-x}}\right)}{\sqrt{x}} - \frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{2x^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\sqrt{-x}}\right)\left(-\frac{-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}}{2\sqrt{-x}\,\sqrt{x}}+\frac{\sqrt{-x}\left(\frac{\mathrm{e}^{\sqrt{-x}}}{2\sqrt{-x}}+\frac{\mathrm{e}^{-\sqrt{-x}}}{2\sqrt{-x}}\right)}{\sqrt{x}}-\frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{2x^{\frac{3}{2}}}\right)
- \left(\frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}}\right)\left(-\frac{\mathrm{e}^{\sqrt{-x}}}{2\sqrt{-x}}\right)$$

Which simplifies to

$$W = \frac{\mathrm{e}^{\sqrt{-x}}\mathrm{e}^{-\sqrt{-x}}}{\sqrt{x}}$$

Which simplifies to

$$W = \frac{1}{\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}}+\mathrm{e}^{-\sqrt{-x}}\right)(6+x)}{x^{\frac{5}{2}}}}{4\sqrt{x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(6+x)\left(\mathrm{e}^{\sqrt{-x}}-\mathrm{e}^{-\sqrt{-x}}\right)}{4\,(-x)^{\frac{5}{2}}}dx$$

Hence

$$u_1 = \frac{\mathrm{e}^{-\sqrt{-x}}}{(-x)^{\frac{3}{2}}}+\frac{\mathrm{e}^{-\sqrt{-x}}}{2x}-\frac{\mathrm{e}^{\sqrt{-x}}}{(-x)^{\frac{3}{2}}}+\frac{\mathrm{e}^{\sqrt{-x}}}{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{\sqrt{-x}}(6+x)}{x^2}}{4\sqrt{x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{\sqrt{-x}}(6+x)}{4x^{\frac{5}{2}}}dx$$

Hence

$$u_2 = \frac{3(-x)^{\frac{5}{2}} \left(\frac{1}{3(-x)^{\frac{3}{2}}} - \frac{1}{2x} + \frac{1}{2\sqrt{-x}} + \frac{11}{36} - \frac{\ln(x)}{12} - \frac{\ln\left(-\frac{\sqrt{-x}}{\sqrt{x}}\right)}{6} - \frac{\sqrt{-x}\left(22(-x)^{\frac{3}{2}} - 36x + 36\sqrt{-x} + 24\right)}{72x^2} + \frac{\sqrt{-x}\,(-4x+4\sqrt{-x}+8)\mathrm{e}}{24x^2}}{x^{\frac{5}{2}}} + \frac{(-x)^{\frac{3}{2}} \left(\frac{1}{\sqrt{-x}} + 1 - \frac{\ln(x)}{2} - \ln\left(-\frac{\sqrt{-x}}{\sqrt{x}}\right) + \frac{\sqrt{-x}\,(2\sqrt{-x}+2)}{2x} - \frac{\sqrt{-x}\,\mathrm{e}^{\sqrt{-x}}}{x} + \ln\left(-\sqrt{-x}\right) + \mathrm{expIntegral}_1\left(-\sqrt{-}\right.\right.}{2x^{\frac{3}{2}}}$$

Which simplifies to

$$u_1 = -\frac{\left(\sqrt{-x}-2\right)\mathrm{e}^{-\sqrt{-x}} + \mathrm{e}^{\sqrt{-x}}\left(\sqrt{-x}+2\right)}{2\,(-x)^{\frac{3}{2}}}$$

$$u_2 = -\frac{\mathrm{e}^{\sqrt{-x}}\left(\sqrt{-x}+2\right)}{2x^{\frac{3}{2}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(\left(\sqrt{-x}-2\right)\mathrm{e}^{-\sqrt{-x}} + \mathrm{e}^{\sqrt{-x}}\left(\sqrt{-x}+2\right)\right)\mathrm{e}^{\sqrt{-x}}}{2\,(-x)^{\frac{3}{2}}} - \frac{\mathrm{e}^{\sqrt{-x}}\left(\sqrt{-x}+2\right)\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}} + \mathrm{e}^{-\sqrt{-x}}\right)}{2x^2}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{\sqrt{-x}} + \frac{c_2\sqrt{-x}\left(-\mathrm{e}^{\sqrt{-x}} + \mathrm{e}^{-\sqrt{-x}}\right)}{\sqrt{x}}\right) + \left(\frac{1}{x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\sqrt{-x}} + \frac{c_2 \sqrt{-x}\left(-e^{\sqrt{-x}} + e^{-\sqrt{-x}}\right)}{\sqrt{x}} + \frac{1}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = -|\Re((1+i)\,c_1)| - i|\Im((1+i)\,c_1)| + 2\min\left(-\Re((1-i)\,c_2), -\Re((1+i)\,c_2), \Re((1-i)\,c_2), \Re((1+i)\,c_2 \right. \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.75 (sec). Leaf size: 41

```
dsolve([4*x*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=(6+x)/x^2,y(infinity) = 0],y(x), singsol=all)
```

$$y(x) = \textit{undefined}$$

✓ Solution by Mathematica

Time used: 0.192 (sec). Leaf size: 27

```
DSolve[{4*x*y''[x]+2*y'[x]+y[x]==(6+x)/x^2,{y[Infinity]==0}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{x} + c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

20.30 problem 669

Internal problem ID [15434]
Internal file name [`OUTPUT/15434_Wednesday_May_08_2024_03_58_44_PM_33700203/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 669.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{\left(x^2+1\right) y''+2xy'=\frac{1}{x^2+1}}$$

With initial conditions

$$\left[y(\infty)=\frac{\pi^2}{8}, y'(0)=0\right]$$

20.30.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^2+1\right) y''+2xy'\right) dx = \int \frac{1}{x^2+1} dx$$
$$\left(x^2+1\right) y' = \arctan\left(x\right)+c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\arctan\left(x\right)+c_1}{x^2+1}\,\mathrm{d}x \\ &= \frac{\arctan\left(x\right)^2}{2}+\arctan\left(x\right) c_1+c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\arctan\left(x\right)^2}{2}+\arctan\left(x\right) c_1+c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=\frac{\pi^2}{8}$ and $x=\infty$ in the above gives

$$\frac{\pi^2}{8} = \frac{1}{8}\pi^2+\frac{1}{2}\pi c_1+c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\arctan\left(x\right)}{x^2+1}+\frac{c_1}{x^2+1}$$

substituting $y'=0$ and $x=0$ in the above gives

$$0 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{\arctan\left(x\right)^2}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\arctan(x)^2}{2} \tag{1}$$

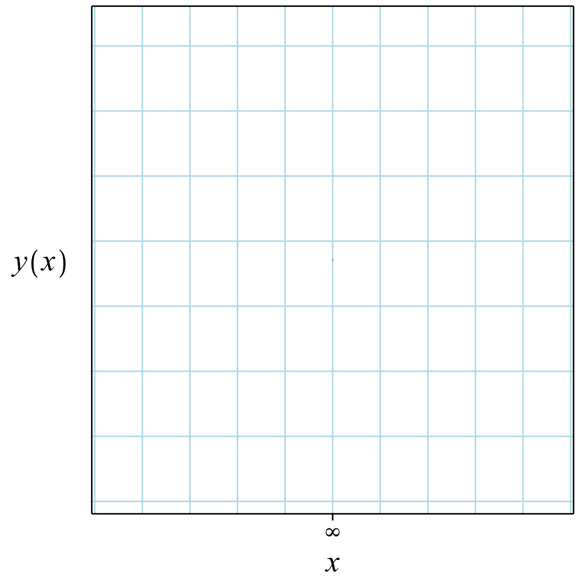

Figure 797: Solution plot

Verification of solutions

$$y = \frac{\arctan(x)^2}{2}$$

Verified OK.

20.30.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(x^2+1)\, p'(x) + 2p(x)\, x - \frac{1}{x^2+1} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2x}{x^2+1}dx}\\ &= x^2+1\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(\frac{1}{(x^2+1)^2}\right)\\ \frac{d}{dx}\left((x^2+1)\,p\right) &= (x^2+1)\left(\frac{1}{(x^2+1)^2}\right)\\ d\left((x^2+1)\,p\right) &= \frac{1}{x^2+1}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(x^2+1)\,p &= \int \frac{1}{x^2+1}\,dx\\ (x^2+1)\,p &= \arctan(x)+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2+1$ results in

$$p(x) = \frac{\arctan(x)}{x^2+1}+\frac{c_1}{x^2+1}$$

which simplifies to

$$p(x) = \frac{\arctan(x)+c_1}{x^2+1}$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $p=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(x) = \frac{\arctan(x)}{x^2+1}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\arctan(x)}{x^2+1}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\arctan(x)}{x^2+1}\, \mathrm{d}x \\ &= \frac{\arctan(x)^2}{2} + c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = \infty$ and $y = \frac{\pi^2}{8}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi^2}{8} = \frac{\pi^2}{8} + c_2$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$y = \frac{\arctan(x)^2}{2}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{\arctan(x)^2}{2} \tag{1}$$

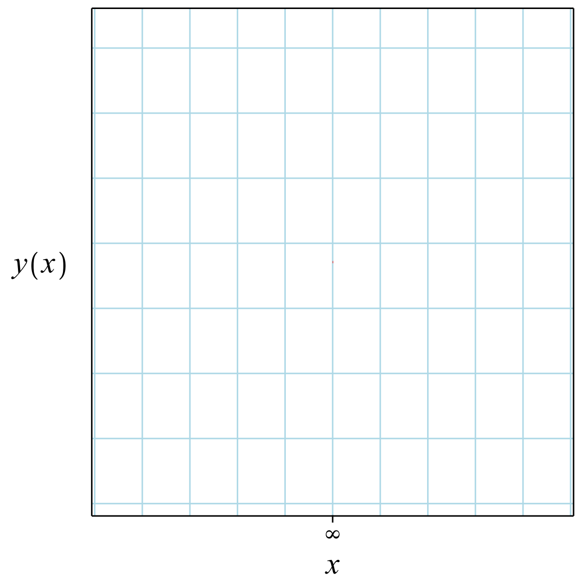

Figure 798: Solution plot

Verification of solutions

$$y = \frac{\arctan(x)^2}{2}$$

Verified OK.

20.30.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2+1)\,y'' + 2xy' = \frac{1}{x^2+1}$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x^2+1)\,y'' + 2xy'\right) dx = \int \frac{1}{x^2+1} dx$$
$$(x^2+1)\,y' = \arctan(x) + c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\arctan(x) + c_1}{x^2+1}\, \mathrm{d}x \\ &= \frac{\arctan(x)^2}{2} + \arctan(x)\, c_1 + c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\arctan(x)^2}{2} + \arctan(x) c_1 + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi^2}{8}$ and $x = \infty$ in the above gives

$$\frac{\pi^2}{8} = \frac{1}{8}\pi^2 + \frac{1}{2}\pi c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\arctan(x)}{x^2+1} + \frac{c_1}{x^2+1}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{\arctan(x)^2}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\arctan(x)^2}{2} \tag{1}$$

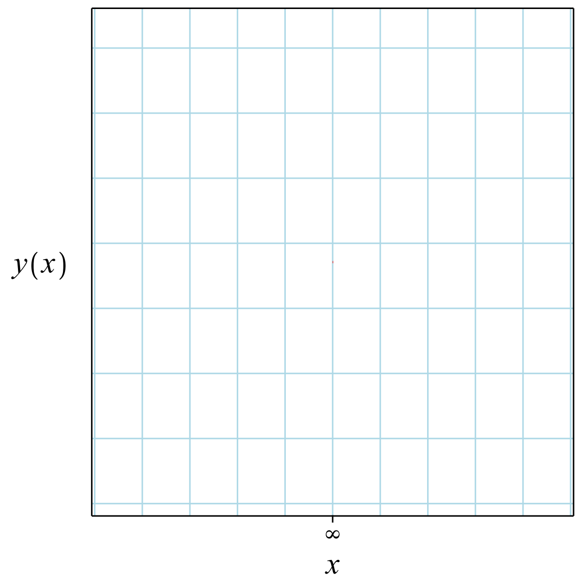

Figure 799: Solution plot

Verification of solutions

$$y = \frac{\arctan(x)^2}{2}$$

Verified OK.

20.30.4 Solving using Kovacic algorithm

Writing the ode as

$$(x^2+1)\, y'' + 2xy' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= 2x \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = \left(x^2+1\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 670: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{i}{4(x-i)} + \frac{i}{4x+4i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x-2i} + \frac{1}{2x+2i} + (-)(0) \\ &= \frac{1}{2x-2i} + \frac{1}{2x+2i} \\ &= \frac{x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)(0) + \left(\left(-\frac{1}{2\left(x - i\right)^2} - \frac{1}{2\left(x + i\right)^2}\right) + \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right)^2 - \left(\frac{1}{\left(x^2 + 1\right)^2}\right.\right.$$
$$0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x - 2i} + \frac{1}{2x + 2i}\right) dx} \\
&= \sqrt{x^2 + 1}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x}{x^2 + 1} \, dx} \\
&= z_1 e^{-\frac{\ln\left(x^2 + 1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2 + 1}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1(\arctan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\arctan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x^2+1)\, y'' + 2xy' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + c_2 \arctan(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \arctan(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \arctan(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\arctan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \arctan(x) \\ 0 & \frac{1}{x^2+1} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x^2+1}\right) - (\arctan(x))(0)$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\arctan(x)}{x^2+1}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\arctan(x)}{x^2+1}dx$$

Hence

$$u_1 = -\frac{\arctan(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2+1}}{1}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2+1} dx$$

Hence

$$u_2 = \arctan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\arctan(x)^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \arctan(x)) + \left(\frac{\arctan(x)^2}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 \arctan(x) + \frac{\arctan(x)^2}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi^2}{8}$ and $x = \infty$ in the above gives

$$\frac{\pi^2}{8} = c_1 + \frac{1}{2}c_2\pi + \frac{1}{8}\pi^2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_2}{x^2+1} + \frac{\arctan(x)}{x^2+1}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{\arctan(x)^2}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\arctan(x)^2}{2} \tag{1}$$

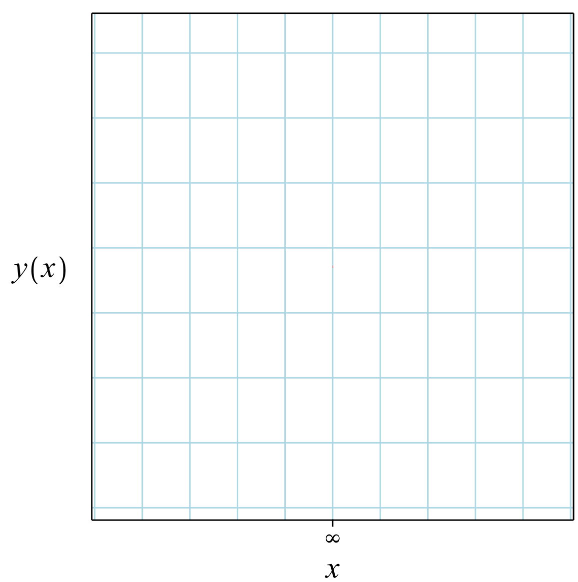

Figure 800: Solution plot

Verification of solutions

$$y = \frac{\arctan(x)^2}{2}$$

Verified OK.

20.30.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=x^2+1\\q(x)&=2x\\r(x)&=0\\s(x)&=\frac{1}{x^2+1}\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=2\\q'(x)&=2\end{aligned}$$

Therefore (1) becomes

$$2-(2)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\,dx$$

Substituting the above values for p,q,r,s gives

$$\left(x^2+1\right)y'=\int\frac{1}{x^2+1}\,dx$$

We now have a first order ode to solve which is

$$\left(x^2+1\right)y'=\arctan\left(x\right)+c_1$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{\arctan(x) + c_1}{x^2 + 1}\, \mathrm{d}x \\
&= \frac{\arctan(x)^2}{2} + \arctan(x)\, c_1 + c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{\arctan(x)^2}{2} + \arctan(x)\, c_1 + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{\pi^2}{8}$ and $x = \infty$ in the above gives

$$
\frac{\pi^2}{8} = \frac{1}{8}\pi^2 + \frac{1}{2}\pi c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{\arctan(x)}{x^2 + 1} + \frac{c_1}{x^2 + 1}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{\arctan(x)^2}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\arctan(x)^2}{2} \tag{1}
$$

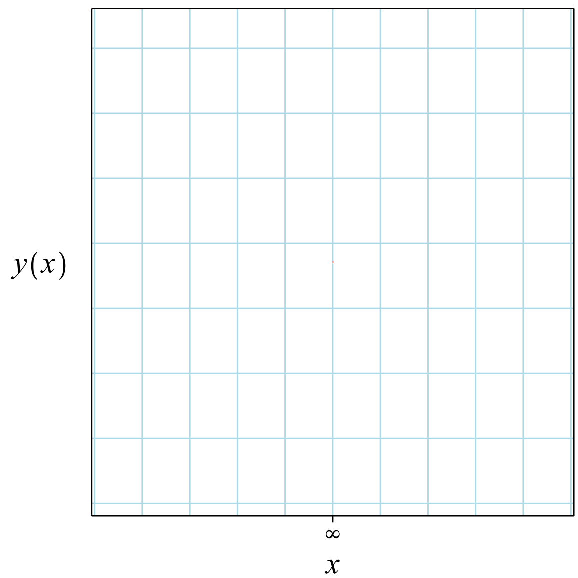

Figure 801: Solution plot

Verification of solutions

$$y = \frac{\arctan(x)^2}{2}$$

Verified OK.

20.30.6 Maple step by step solution

Let's solve

$$\left[(x^2+1)\,y'' + 2xy' = \frac{1}{x^2+1}, y(\infty) = \frac{\pi^2}{8}, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $(x^2+1)\,u'(x) + 2xu(x) = \frac{1}{x^2+1}$

- Isolate the derivative

  $u'(x) = -\frac{2xu(x)}{x^2+1} + \frac{1}{(x^2+1)^2}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{2xu(x)}{x^2+1} = \frac{1}{(x^2+1)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{2xu(x)}{x^2+1}\right)=\frac{\mu(x)}{(x^2+1)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{2xu(x)}{x^2+1}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)x}{x^2+1}$

- Solve to find the integrating factor

  $\mu(x)=x^2+1$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\frac{\mu(x)}{(x^2+1)^2}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\frac{\mu(x)}{(x^2+1)^2}dx+c_1$

- Solve for $u(x)$

  $u(x)=\frac{\int\frac{\mu(x)}{\left(x^2+1\right)^2}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x^2+1$

  $u(x)=\frac{\int\frac{1}{x^2+1}dx+c_1}{x^2+1}$

- Evaluate the integrals on the rhs

  $u(x)=\frac{\arctan(x)+c_1}{x^2+1}$

- Solve 1st ODE for $u(x)$

  $u(x)=\frac{\arctan(x)+c_1}{x^2+1}$

- Make substitution $u=y'$

  $y'=\frac{\arctan(x)+c_1}{x^2+1}$

- Integrate both sides to solve for y

  $\int y'dx=\int\frac{\arctan(x)+c_1}{x^2+1}dx+c_2$

- Compute integrals

  $y=\frac{\arctan(x)^2}{2}+\arctan(x)\,c_1+c_2$

☐ Check validity of solution $y=\frac{\arctan(x)^2}{2}+\arctan(x)\,c_1+c_2$

- Use initial condition $y(\infty) = \frac{\pi^2}{8}$

  $\frac{\pi^2}{8} = \frac{1}{8}\pi^2 + \frac{1}{2}\pi c_1 + c_2$

- Compute derivative of the solution

  $y' = \frac{\arctan(x)}{x^2+1} + \frac{c_1}{x^2+1}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = c_1$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\arctan(x)^2}{2}$

- Solution to the IVP

  $y = \frac{\arctan(x)^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(2*_b(_a)*_a^3+2*_b(_a)*_a-1)/(_a^2+1)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 10

```
dsolve([(1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)=1/(1+x^2),y(infinity) = 1/8*Pi^2, D(y)(0) =
```

$$y(x) = \frac{\arctan(x)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 13

```
DSolve[{(1+x^2)*y''[x]+2*x*y'[x]==1/(1+x^2),{y[Infinity]==Pi^2/8,y'[0]==0}},y[x],x,IncludeSi
```

$$y(x) \to \frac{\arctan(x)^2}{2}$$

20.31 problem 670

Internal problem ID [15435]
Internal file name [`OUTPUT/15435_Wednesday_May_08_2024_03_58_46_PM_11576560/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 670.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
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Unable to solve or complete the solution.

$$\boxed{(1-x)\,y''+xy'-y=(x-1)^2\,\mathrm{e}^x}$$

With initial conditions

$$[y(-\infty)=0, y'(0)=1]$$

20.31.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1 - x, B = x, C = -1, f(x) = (x-1)^2 e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$(1-x)\,y'' + xy' - y = 0$$

In normal form the ode

$$(1-x)\,y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{x}{x-1} \\ q(x) &= \frac{1}{x-1} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x-1} + \frac{1}{x-1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 2x + 2)}{x(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{x^2 - 2x + 2}{x(x-1)}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{x^2 - 2x + 2}{x(x-1)}\, dx \\ \ln(u) &= x + \ln(x-1) - 2\ln(x) + c_1 \\ u &= e^{x + \ln(x-1) - 2\ln(x) + c_1} \\ &= c_1 e^{x + \ln(x-1) - 2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{e^x c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{e^x c_1}{x} + c_2\right) x \\
&= e^x c_1 + c_2 x
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
(1-x)\, y'' + xy' - y = (x-1)^2\, e^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= e^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & e^x \\ \frac{d}{dx}(x) & \frac{d}{dx}(e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}$$

Therefore

$$W = (x)(e^x) - (e^x)(1)$$

Which simplifies to

$$W = x\,e^x - e^x$$

Which simplifies to

$$W = (x - 1)\,e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x-1)^2\,e^{2x}}{(1-x)(x-1)\,e^x}\,dx$$

Which simplifies to

$$u_1 = -\int -e^x dx$$

Hence

$$u_1 = e^x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x-1)^2\,e^x}{(1-x)(x-1)\,e^x}\,dx$$

Which simplifies to

$$u_2 = \int -x dx$$

Hence

$$u_2 = -\frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x\,\mathrm{e}^{x} - \frac{\mathrm{e}^{x} x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\mathrm{e}^{x} x(x-2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{\mathrm{e}^{x} c_1}{x} + c_2\right) x\right) + \left(-\frac{\mathrm{e}^{x} x(x-2)}{2}\right) \\ &= -\frac{\mathrm{e}^{x} x(x-2)}{2} + \left(\frac{\mathrm{e}^{x} c_1}{x} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = \frac{(-x^2 + 2c_1 + 2x)\,\mathrm{e}^{x}}{2} + c_2 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(-x^2 + 2c_1 + 2x)\,\mathrm{e}^{x}}{2} + c_2 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = -\operatorname{signum}(c_2)\,\infty \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{(2-2x)\,\mathrm{e}^{x}}{2} + \frac{(-x^{2}+2c_1+2x)\,\mathrm{e}^{x}}{2} + c_2$$

substituting $y'=1$ and $x=0$ in the above gives

$$1 = 1 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

20.31.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^{2}\right)v' + (AB'' + BB' + CB)\,v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^{2}\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^{2}\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 1-x \\
B &= x \\
C &= -1 \\
F &= (x-1)^2 \, \mathrm{e}^x
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (1-x)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-x(x-1) \, v'' + (x^2 - 2x + 2) \, v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
(-x^2 + x) \, u'(x) + (x^2 - 2x + 2) \, u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(x^2 - 2x + 2)}{x\,(x-1)}
\end{aligned}
$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u} \, du &= \frac{x^2 - 2x + 2}{x\,(x-1)} \, dx \\
\int \frac{1}{u} \, du &= \int \frac{x^2 - 2x + 2}{x\,(x-1)} \, dx \\
\ln(u) &= x + \ln(x-1) - 2\ln(x) + c_1 \\
u &= \mathrm{e}^{x + \ln(x-1) - 2\ln(x) + c_1} \\
&= c_1 \mathrm{e}^{x + \ln(x-1) - 2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1 \left(\frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^2} \right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 \left(\frac{e^x}{x} - \frac{e^x}{x^2} \right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1 e^x (x-1)}{x^2} \, dx \\
&= \frac{e^x c_1}{x} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (x) \left(\frac{e^x c_1}{x} + c_2 \right) \\
&= e^x c_1 + c_2 x
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= e^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & e^x \\ \frac{d}{dx}(x) & \frac{d}{dx}(e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}$$

Therefore

$$W = (x)(e^x) - (e^x)(1)$$

Which simplifies to

$$W = x\,e^x - e^x$$

Which simplifies to

$$W = (x-1)\,e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x-1)^2 e^{2x}}{(1-x)(x-1)e^x}\,dx$$

Which simplifies to

$$u_1 = -\int -e^x dx$$

Hence

$$u_1 = e^x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x-1)^2 e^x}{(1-x)(x-1)e^x}\,dx$$

Which simplifies to

$$u_2 = \int -x dx$$

Hence

$$u_2 = -\frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x\,\mathrm{e}^x - \frac{\mathrm{e}^x x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\mathrm{e}^x x(x-2)}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (\mathrm{e}^x c_1 + c_2 x) + \left(-\frac{\mathrm{e}^x x(x-2)}{2}\right) \\ &= \frac{(-x^2 + 2c_1 + 2x)\,\mathrm{e}^x}{2} + c_2 x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(-x^2 + 2c_1 + 2x)\,\mathrm{e}^x}{2} + c_2 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = -\operatorname{signum}(c_2)\,\infty \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{(2-2x)\,\mathrm{e}^x}{2} + \frac{(-x^2 + 2c_1 + 2x)\,\mathrm{e}^x}{2} + c_2$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 1 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

20.31.3 Solving using Kovacic algorithm

Writing the ode as

$$(1 - x) y'' + xy' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 - x \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4 (x - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 672: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4\left(x-1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x-1\right)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2\left(x-1\right)} + \frac{1}{2} \\ &= \frac{x-2}{2x-2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1-x}\, dx} \\ &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\ &= z_1 \left(\sqrt{x-1}\, e^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1-x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-x\, \mathrm{e}^{-x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{x}\right) + c_2 \left(\mathrm{e}^{x} \left(-x\, \mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(1-x)\, y'' + xy' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{x} c_1 - c_2 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= -x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & -x \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(-x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & -x \\ \mathrm{e}^{x} & -1 \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x})(-1) - (-x)(\mathrm{e}^{x})$$

Which simplifies to

$$W = x\,\mathrm{e}^{x} - \mathrm{e}^{x}$$

Which simplifies to

$$W = (x-1)\,\mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-x(x-1)^2\,\mathrm{e}^{x}}{(1-x)(x-1)\,\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int x\,dx$$

Hence

$$u_1 = -\frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x-1)^2 \,\mathrm{e}^{2x}}{(1-x)\,(x-1)\,\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_2 = \int -\mathrm{e}^{x} dx$$

Hence

$$u_2 = -\mathrm{e}^{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x\,\mathrm{e}^{x} - \frac{\mathrm{e}^{x} x^2}{2}$$

Which simplifies to

$$y_p(x) = -\frac{\mathrm{e}^{x} x(x-2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^{x} c_1 - c_2 x) + \left(-\frac{\mathrm{e}^{x} x(x-2)}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x} c_1 - c_2 x - \frac{\mathrm{e}^{x} x(x-2)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = \operatorname{signum}(c_2)\,\infty \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{x} c_1 - c_2 - \frac{\mathrm{e}^{x} x(x-2)}{2} - \frac{(x-2)\,\mathrm{e}^{x}}{2} - \frac{x\,\mathrm{e}^{x}}{2}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 - c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 16

```
dsolve([(1-x)*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=(x-1)^2*exp(x),y(-infinity) = 0, D(y)(0) =
```

$$y(x) = -\frac{x(x-2)\,\mathrm{e}^{x}}{2}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 16

```
DSolve[{(1-x)*y''[x]+x*y'[x]-y[x]==(x-1)^2*Exp[x],{y[-Infinity]==0,y'[0]==1}},y[x],x,Include
```

$$y(x) \to -\frac{1}{2}e^x(x-2)x$$

20.32 problem 671

Internal problem ID [15436]
Internal file name [`OUTPUT/15436_Wednesday_May_08_2024_03_58_47_PM_46832233/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 671.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{2x^2(2-\ln(x))\,y''+x(4-\ln(x))\,y'-y=\frac{(2-\ln(x))^2}{\sqrt{x}}}$$

With initial conditions

$$[y(\infty)=0]$$

20.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-x \ln(x) + 4x}{2x^2 (\ln(x) - 2)} \\
q(x) &= \frac{1}{2x^2 (\ln(x) - 2)} \\
F &= -\frac{\ln(x) - 2}{2x^{\frac{5}{2}}}
\end{aligned}
$$

Hence the ode is

$$
y'' - \frac{(-x \ln(x) + 4x) y'}{2x^2 (\ln(x) - 2)} + \frac{y}{2x^2 (\ln(x) - 2)} = -\frac{\ln(x) - 2}{2x^{\frac{5}{2}}}
$$

The domain of $p(x) = -\frac{-x \ln(x)+4x}{2x^2(\ln(x)-2)}$ is

$$
\{0 < x < e^2, e^2 < x \leq \infty\}
$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = \frac{1}{2x^2(\ln(x)-2)}$ is

$$
\{0 < x < e^2, e^2 < x \leq \infty\}
$$

And the point $x_0 = \infty$ is also inside this domain. The domain of $F = -\frac{\ln(x)-2}{2x^{\frac{5}{2}}}$ is

$$
\{0 < x\}
$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

20.32.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = -2x^2(\ln(x) - 2), B = -x \ln(x) + 4x, C = -1, f(x) = \frac{(\ln(x)-2)^2}{\sqrt{x}}$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$
-2y''x^2(\ln(x) - 2) + (-x \ln(x) + 4x) y' - y = 0
$$

In normal form the ode

$$-2y''x^2(\ln(x)-2)+(-x\ln(x)+4x)\,y'-y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{4-\ln(x)}{x(-2\ln(x)+4)}\\q(x)&=\frac{1}{2x^2(\ln(x)-2)}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(4-\ln(x))}{x^2(-2\ln(x)+4)}+\frac{1}{2x^2(\ln(x)-2)}=0 \tag{5}$$

Solving (5) for n gives

$$n=\frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\left(\frac{1}{x}+\frac{4-\ln(x)}{x(-2\ln(x)+4)}\right)v'(x)&=0\\v''(x)+\frac{(3\ln(x)-8)\,v'(x)}{2x(\ln(x)-2)}&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(3\ln(x) - 8)\, u(x)}{2x\,(\ln(x) - 2)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{(3\ln(x) - 8)\, u}{2x\,(\ln(x) - 2)} \end{aligned}$$

Where $f(x) = -\frac{3\ln(x)-8}{2x(\ln(x)-2)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3\ln(x) - 8}{2x\,(\ln(x) - 2)}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3\ln(x) - 8}{2x\,(\ln(x) - 2)}\, dx \\ \ln(u) &= -\frac{3\ln(x)}{2} + \ln(\ln(x) - 2) + c_1 \\ u &= \mathrm{e}^{-\frac{3\ln(x)}{2} + \ln(\ln(x)-2) + c_1} \\ &= c_1 \mathrm{e}^{-\frac{3\ln(x)}{2} + \ln(\ln(x)-2)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{\ln(x)}{x^{\frac{3}{2}}} - \frac{2}{x^{\frac{3}{2}}}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{2c_1 \ln(x)}{\sqrt{x}} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{2c_1 \ln(x)}{\sqrt{x}} + c_2\right) \sqrt{x} \\ &= c_2 \sqrt{x} - 2c_1 \ln(x) \end{aligned}$$

Now the particular solution to this ODE is found

$$-2y''x^2(\ln(x)-2)+(-x\ln(x)+4x)\,y'-y=\frac{(\ln(x)-2)^2}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\sqrt{x}$$
$$y_2=\ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \sqrt{x} & \ln(x) \\ \frac{d}{dx}(\sqrt{x}) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} \sqrt{x} & \ln(x) \\ \frac{1}{2\sqrt{x}} & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (\sqrt{x})\left(\frac{1}{x}\right) - (\ln(x))\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = -\frac{\ln(x) - 2}{2\sqrt{x}}$$

Which simplifies to

$$W = -\frac{\ln(x) - 2}{2\sqrt{x}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\ln(x)(\ln(x)-2)^2}{\sqrt{x}}}{x^{\frac{3}{2}}(\ln(x) - 2)^2} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(\ln(x) - 2)^2}{x^{\frac{3}{2}}(\ln(x) - 2)^2} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^{\frac{3}{2}}} dx$$

Hence

$$u_2 = -\frac{2}{\sqrt{x}}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{\ln (x) + 1}{x} \\
u_2 &= -\frac{2}{\sqrt{x}}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \frac{\ln (x) + 1}{\sqrt{x}} - \frac{2 \ln (x)}{\sqrt{x}}
$$

Which simplifies to

$$
y_p(x) = -\frac{\ln (x) - 1}{\sqrt{x}}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{2c_1 \ln (x)}{\sqrt{x}} + c_2\right) \sqrt{x}\right) + \left(-\frac{\ln (x) - 1}{\sqrt{x}}\right) \\
&= -\frac{\ln (x) - 1}{\sqrt{x}} + \left(-\frac{2c_1 \ln (x)}{\sqrt{x}} + c_2\right) \sqrt{x}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_2 x - 2\sqrt{x}\, \ln (x)\, c_1 - \ln (x) + 1}{\sqrt{x}}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_2 x - 2\sqrt{x}\, \ln (x)\, c_1 - \ln (x) + 1}{\sqrt{x}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$
0 = \operatorname{signum}(c_2)\, \infty \tag{1A}
$$

Equations {1A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
      Change of variables used:
         [x = exp(t)]
      Linear ODE actually solved:
         u(t)-t*diff(u(t),t)+(2*t-4)*diff(diff(u(t),t),t) = 0
   <- change of variables successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 21

```
dsolve([2*x^2*(2-ln(x))*diff(y(x),x$2)+x*(4-ln(x))*diff(y(x),x)-y(x)=(2-ln(x))^2/sqrt(x),y(i
```

$$y(x) = \frac{\sqrt{x}\,\ln(x)\,c_2 - \ln(x) + 1}{\sqrt{x}}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{2*x^2*(2-Log[x])*y''[x]+x*(4-Log[x])*y'[x]-y[x]==(2-Log[x])^2/Sqrt[x],{y[Infinity]==
```

Not solved

20.33 problem 672

Internal problem ID [15437]
Internal file name [`OUTPUT/15437_Wednesday_May_08_2024_03_58_49_PM_98827332/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 672.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \frac{2y'}{x} - y = 4\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(-\infty) = 0, y'(-1) = -\mathrm{e}^{-1}]$$

20.33.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + \frac{2y'}{x} - y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= -1 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\ &= -1 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= -1 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= -1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{\frac{2}{x}}{2}} \\ &= \frac{1}{x} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$-v(x) + v''(x) = 4x\,\mathrm{e}^{x}$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 4x\,\mathrm{e}^{x}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$-v(x) + v''(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$v(x) = e^x c_1 + e^{-x} c_2$$

Therefore the homogeneous solution v_h is

$$v_h = e^x c_1 + e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x\,e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x, e^x x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 x\,e^x + A_2 e^x x^2$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + 4A_2 e^x x + 2A_2 e^x = 4x\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -x\,e^x + e^x x^2$$

Therefore the general solution is

$$\begin{aligned}v &= v_h + v_p\\ &= \left(e^x c_1 + e^{-x} c_2\right) + \left(-x\,e^x + e^x x^2\right)\end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\,z(x)\\ &= \left(e^x c_1 + e^{-x} c_2 - x\,e^x + e^x x^2\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{e^x c_1 + e^{-x} c_2 - x\,e^x + e^x x^2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{e^x c_1 + e^{-x} c_2 - x\,e^x + e^x x^2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^x}{x}$$

$$y_2 = \frac{e^{-x}}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ \frac{d}{dx}\left(\frac{e^x}{x}\right) & \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ \frac{e^x}{x} - \frac{e^x}{x^2} & -\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^x}{x}\right)\left(-\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}\right) - \left(\frac{e^{-x}}{x}\right)\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)$$

Which simplifies to

$$W = -\frac{2\,e^x e^{-x}}{x^2}$$

Which simplifies to

$$W = -\frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\,e^{-x}e^{x}}{x}}{-\frac{2}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int -2x dx$$

Hence

$$u_1 = x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{2x}}{x}}{-\frac{2}{x^2}} dx$$

Which simplifies to

$$u_2 = \int -2x\,e^{2x} dx$$

Hence

$$u_2 = -\frac{(2x-1)\,e^{2x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x\,e^{x} - \frac{(2x-1)\,e^{2x} e^{-x}}{2x}$$

Which simplifies to

$$y_p(x) = \frac{e^{x}(2x^2 - 2x + 1)}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{e^{x} c_1 + e^{-x} c_2 - x\,e^{x} + e^{x} x^2}{x}\right) + \left(\frac{e^{x}(2x^2 - 2x + 1)}{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{\mathrm{e}^{-x} c_2 + \mathrm{e}^{x}(x^2 + c_1 - x)}{x} + \frac{\mathrm{e}^{x}(2x^2 - 2x + 1)}{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\mathrm{e}^{-x} c_2 + \mathrm{e}^{x}(x^2 + c_1 - x)}{x} + \frac{\mathrm{e}^{x}(2x^2 - 2x + 1)}{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = -\operatorname{signum}(c_2)\,\infty \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-\mathrm{e}^{-x} c_2 + \mathrm{e}^{x}(x^2 + c_1 - x) + (2x - 1)\,\mathrm{e}^{x}}{x} - \frac{\mathrm{e}^{-x} c_2 + \mathrm{e}^{x}(x^2 + c_1 - x)}{x^2} + \frac{\mathrm{e}^{x}(2x^2 - 2x + 1)}{2x} + \frac{\mathrm{e}^{x}(4x - 2)}{2x}$$

substituting $y' = -\mathrm{e}^{-1}$ and $x = -1$ in the above gives

$$-\mathrm{e}^{-1} = \mathrm{e}^{-1}(-3 - 2c_1) \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

20.33.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2xy' - yx^2 = 4\,\mathrm{e}^{x} x^2 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2)\,y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2}\\
\beta &= i\\
n &= \frac{1}{2}\\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} - \frac{c_2\sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} - \frac{c_2\sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{i\sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\\
y_2 &= -\frac{\sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{i\sqrt{2}\,\sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} & -\frac{\sqrt{2}\,\cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} \\ \frac{d}{dx}\left(\frac{i\sqrt{2}\,\sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right) & \frac{d}{dx}\left(-\frac{\sqrt{2}\,\cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{i\sqrt{2}\,\sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} & -\frac{\sqrt{2}\,\cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} \\ -\frac{i\sqrt{2}\,\sinh(x)}{2x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} + \frac{\sqrt{2}\,\sinh(x)}{2\sqrt{x}\,\sqrt{\pi}\,(ix)^{\frac{3}{2}}} + \frac{i\sqrt{2}\,\cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} & \frac{\sqrt{2}\,\cosh(x)}{2x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} + \frac{i\sqrt{2}\,\cosh(x)}{2\sqrt{x}\,\sqrt{\pi}\,(ix)^{\frac{3}{2}}} - \frac{\sqrt{2}\,\sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\frac{i\sqrt{2}\,\sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right)\left(\frac{\sqrt{2}\,\cosh(x)}{2x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} + \frac{i\sqrt{2}\,\cosh(x)}{2\sqrt{x}\,\sqrt{\pi}\,(ix)^{\frac{3}{2}}} - \frac{\sqrt{2}\,\sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right) \\ & - \left(-\frac{\sqrt{2}\,\cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right)\left(-\frac{i\sqrt{2}\,\sinh(x)}{2x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{ix}} + \frac{\sqrt{2}\,\sinh(x)}{2\sqrt{x}\,\sqrt{\pi}\,(ix)^{\frac{3}{2}}} + \frac{i\sqrt{2}\,\cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{ix}}\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{2\left(\sinh(x)^2 - \cosh(x)^2\right)}{x^2\pi}$$

Which simplifies to

$$W = \frac{2}{x^2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{4x^{\frac{3}{2}}\sqrt{2}\,\cosh(x)\mathrm{e}^{x}}{\sqrt{\pi}\,\sqrt{ix}}}{\frac{2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2x^{\frac{3}{2}}\sqrt{2}\,\sqrt{\pi}\,\cosh(x)\,\mathrm{e}^{x}}{\sqrt{ix}}dx$$

Hence

$$u_1 = -\left(\int_0^x -\frac{2\alpha^{\frac{3}{2}}\sqrt{2}\,\sqrt{\pi}\,\cosh(\alpha)\,\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4ix^{\frac{3}{2}}\sqrt{2}\,\sinh(x)\mathrm{e}^{x}}{\sqrt{\pi}\,\sqrt{ix}}}{\frac{2}{\pi}}dx$$

Which simplifies to

$$u_2 = \int \frac{2ix^{\frac{3}{2}}\sqrt{2}\,\sqrt{\pi}\,\sinh(x)\,\mathrm{e}^{x}}{\sqrt{ix}}dx$$

Hence

$$u_2 = \int_0^x \frac{2i\alpha^{\frac{3}{2}}\sqrt{2}\,\sqrt{\pi}\,\sinh(\alpha)\,\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha$$

Which simplifies to

$$u_1 = 2\sqrt{2}\,\sqrt{\pi}\left(\int_0^x \frac{\alpha^{\frac{3}{2}}\cosh(\alpha)\,\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)$$

$$u_2 = 2i\sqrt{2}\,\sqrt{\pi}\left(\int_0^x \frac{\alpha^{\frac{3}{2}}\sinh(\alpha)\,\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{4i\left(\int_0^x \frac{\alpha^{\frac{3}{2}}\cosh(\alpha)\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)\sinh(x)}{\sqrt{x}\,\sqrt{ix}} - \frac{4i\left(\int_0^x \frac{\alpha^{\frac{3}{2}}\sinh(\alpha)\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)\cosh(x)}{\sqrt{x}\,\sqrt{ix}}$$

Which simplifies to

$$y_p(x) = \frac{4i\left(\left(\int_0^x \frac{\alpha^{\frac{3}{2}}\cosh(\alpha)\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)\sinh(x) - \left(\int_0^x \frac{\alpha^{\frac{3}{2}}\sinh(\alpha)\mathrm{e}^{\alpha}}{\sqrt{i\alpha}}d\alpha\right)\cosh(x)\right)}{\sqrt{x}\,\sqrt{ix}}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{i x}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{i x}}\right) \\
&\quad + \left(\frac{4 i \left(\left(\int_0^x \frac{\alpha^{\frac{3}{2}} \cosh(\alpha) \mathrm{e}^{\alpha}}{\sqrt{i \alpha}} d\alpha\right) \sinh(x) - \left(\int_0^x \frac{\alpha^{\frac{3}{2}} \sinh(\alpha) \mathrm{e}^{\alpha}}{\sqrt{i \alpha}} d\alpha\right) \cosh(x)\right)}{\sqrt{x}\,\sqrt{i x}}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{i c_1 \sqrt{2}\, \sinh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{i x}} - \frac{c_2 \sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{i x}} + \frac{4 i \left(\left(\int_0^x \frac{\alpha^{\frac{3}{2}} \cosh(\alpha) \mathrm{e}^{\alpha}}{\sqrt{i \alpha}} d\alpha\right) \sinh(x) - \left(\int_0^x \frac{\alpha^{\frac{3}{2}} \sinh(\alpha) \mathrm{e}^{\alpha}}{\sqrt{i \alpha}} d\alpha\right) \cosh(x)\right)}{\sqrt{x}\,\sqrt{i x}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$
0 = \frac{\lim_{x \to -\infty} \left(-\frac{4\left(-i \sinh(x) \left(\int_0^x \frac{\alpha^{\frac{3}{2}} \cosh(\alpha) \mathrm{e}^{\alpha}}{\sqrt{i \alpha}} d\alpha\right) \sqrt{\pi} + i \cosh(x) \left(\int_0^x \frac{\alpha^{\frac{3}{2}} \sinh(\alpha) \mathrm{e}^{\alpha}}{\sqrt{i \alpha}} d\alpha\right) \sqrt{\pi} - \frac{(i \sinh(x) c_1 - \cosh(x) c_2)\sqrt{2}}{4}\right)}{\sqrt{x}\,\sqrt{i x}}\right)}{\sqrt{\pi}} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{i c_1 \sqrt{2}\, \sinh(x)}{2 x^{\frac{3}{2}} \sqrt{\pi}\,\sqrt{i x}} + \frac{c_1 \sqrt{2}\, \sinh(x)}{2\sqrt{x}\,\sqrt{\pi}\,(i x)^{\frac{3}{2}}} + \frac{i c_1 \sqrt{2}\, \cosh(x)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{i x}} + \frac{c_2 \sqrt{2}\, \cosh(x)}{2 x^{\frac{3}{2}} \sqrt{\pi}\,\sqrt{i x}} + \frac{i c_2 \sqrt{2}\, \cosh(x)}{2\sqrt{x}\,\sqrt{\pi}\,(i x)^{\frac{3}{2}}} - \frac{c_2 \sqrt{2}\, \mathrm{sin}}{\sqrt{x}\,\sqrt{\pi}}
$$

substituting $y' = -\mathrm{e}^{-1}$ and $x = -1$ in the above gives

$$
-\mathrm{e}^{-1} = \frac{(1+i)\, \mathrm{e}^{-1} \left((-1+i)\sqrt{\pi} + i c_2 + c_1\right)}{\sqrt{\pi}} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.33.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \frac{2y'}{x} - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \frac{2}{x} \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 673: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{2}{x}}{1}\, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^{-x}}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{2}{x}}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2 \left(\frac{\mathrm{e}^{-x}}{x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + \frac{2y'}{x} - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 e^{-x}}{x} + \frac{c_2 e^{x}}{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{e^{-x}}{x} \\ y_2 &= \frac{e^{x}}{2x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^{-x}}{x} & \frac{e^{x}}{2x} \\ \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) & \frac{d}{dx}\left(\frac{e^{x}}{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^{-x}}{x} & \frac{e^{x}}{2x} \\ -\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} & \frac{e^{x}}{2x} - \frac{e^{x}}{2x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^{-x}}{x}\right)\left(\frac{e^{x}}{2x} - \frac{e^{x}}{2x^{2}}\right) - \left(\frac{e^{x}}{2x}\right)\left(-\frac{e^{-x}}{x} - \frac{e^{-x}}{x^{2}}\right)$$

Which simplifies to

$$W = \frac{e^{x} e^{-x}}{x^{2}}$$

Which simplifies to

$$W = \frac{1}{x^{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\,e^{2x}}{x}}{\frac{1}{x^{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int 2x\,e^{2x} dx$$

Hence

$$u_1 = -\frac{(2x-1)\,e^{2x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\,e^{-x} e^{x}}{x}}{\frac{1}{x^{2}}}\, dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x\,e^{x} - \frac{(2x-1)\,e^{2x} e^{-x}}{2x}$$

Which simplifies to

$$y_p(x) = \frac{e^x(2x^2 - 2x + 1)}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 e^{-x}}{x} + \frac{c_2 e^x}{2x}\right) + \left(\frac{e^x(2x^2 - 2x + 1)}{2x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{2c_1 e^{-x} + c_2 e^x}{2x} + \frac{e^x(2x^2 - 2x + 1)}{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{2c_1 e^{-x} + c_2 e^x}{2x} + \frac{e^x(2x^2 - 2x + 1)}{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = -\infty$ in the above gives

$$0 = -\operatorname{signum}(c_1)\infty \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-2c_1 e^{-x} + c_2 e^x}{2x} - \frac{2c_1 e^{-x} + c_2 e^x}{2x^2} + \frac{e^x(2x^2 - 2x + 1)}{2x} + \frac{e^x(4x - 2)}{2x} - \frac{e^x(2x^2 - 2x + 1)}{2x^2}$$

substituting $y' = -e^{-1}$ and $x = -1$ in the above gives

$$-e^{-1} = e^{-1}(-2 - c_2) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.141 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)+2/x*diff(y(x),x)-y(x)=4*exp(x),y(-infinity) = 0, D(y)(-1) = -1/exp(1)
```

$$y(x) = (x-1)\,\mathrm{e}^{x}$$

✓ Solution by Mathematica
Time used: 0.079 (sec). Leaf size: 12

```
DSolve[{y''[x]+2/x*y'[x]-y[x]==4*Exp[x],{y[-Infinity]==0,y'[-1]==-1/Exp[1]}},y[x],x,IncludeS
```

$$y(x) \to e^{x}(x-1)$$

20.34 problem 673

Internal problem ID [15438]
Internal file name [`OUTPUT/15438_Wednesday_May_08_2024_03_58_53_PM_72127052/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 673.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^3(\ln(x)-1)\,y''-x^2y'+yx=2\ln(x)}$$

With initial conditions

$$[y(\infty)=0]$$

20.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$p(x)=-\frac{1}{x\,(\ln(x)-1)}$$
$$q(x)=\frac{1}{x^2\,(\ln(x)-1)}$$
$$F=\frac{2\ln(x)}{x^3\,(\ln(x)-1)}$$

Hence the ode is

$$y'' - \frac{y'}{x\left(\ln\left(x\right)-1\right)} + \frac{y}{x^{2}\left(\ln\left(x\right)-1\right)} = \frac{2\ln\left(x\right)}{x^{3}\left(\ln\left(x\right)-1\right)}$$

The domain of $p(x) = -\frac{1}{x(\ln(x)-1)}$ is

$$\{0 < x < \mathrm{e}, \mathrm{e} < x \leq \infty\}$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = \frac{1}{x^2(\ln(x)-1)}$ is

$$\{0 < x < \mathrm{e}, \mathrm{e} < x \leq \infty\}$$

And the point $x_0 = \infty$ is also inside this domain. The domain of $F = \frac{2\ln(x)}{x^3(\ln(x)-1)}$ is

$$\{0 < x < \mathrm{e}, \mathrm{e} < x \leq \infty\}$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

20.34.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^3(\ln(x) - 1), B = -x^2, C = x, f(x) = 2\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^3(\ln(x) - 1)\, y'' - x^2 y' + yx = 0$$

In normal form the ode

$$x^3(\ln(x) - 1)\, y'' - x^2 y' + yx = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x\left(\ln\left(x\right)-1\right)} \\
q(x) &= \frac{1}{x^{2}\left(\ln\left(x\right)-1\right)}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} - \frac{n}{x^{2}\left(\ln\left(x\right)-1\right)} + \frac{1}{x^{2}\left(\ln\left(x\right)-1\right)} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = 1 \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2}{x} - \frac{1}{x\left(\ln\left(x\right)-1\right)}\right) v'(x) &= 0 \\
v''(x) + \left(\frac{2}{x} - \frac{1}{x\left(\ln\left(x\right)-1\right)}\right) v'(x) &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \left(\frac{2}{x} - \frac{1}{x\left(\ln\left(x\right)-1\right)}\right) u(x) = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(2\ln(x)-3)}{x(\ln(x)-1)} \end{aligned}$$

Where $f(x) = -\frac{2\ln(x)-3}{x(\ln(x)-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2\ln(x)-3}{x(\ln(x)-1)}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2\ln(x)-3}{x(\ln(x)-1)}\,dx \\ \ln(u) &= -2\ln(x)+\ln(\ln(x)-1)+c_1 \\ u &= \mathrm{e}^{-2\ln(x)+\ln(\ln(x)-1)+c_1} \\ &= c_1\mathrm{e}^{-2\ln(x)+\ln(\ln(x)-1)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(\frac{\ln(x)}{x^2}-\frac{1}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1\ln(x)}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1\ln(x)}{x}+c_2\right)x \\ &= -c_1\ln(x)+c_2x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^3(\ln(x)-1)\,y'' - x^2y' + yx = 2\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x) \\ 1 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{1}{x}\right) - (\ln(x))(1)$$

Which simplifies to

$$W = -\ln(x) + 1$$

Which simplifies to

$$W = -\ln(x) + 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\ln(x)^2}{x^3(\ln(x)-1)(-\ln(x)+1)}dx$$

Which simplifies to

$$u_1 = -\int -\frac{2\ln(x)^2}{x^3(\ln(x)-1)^2}dx$$

Hence

$$u_1 = -\frac{\ln(x)+1}{(\ln(x)-1)x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x\ln(x)}{x^3(\ln(x)-1)(-\ln(x)+1)}dx$$

Which simplifies to

$$u_2 = \int -\frac{2\ln(x)}{x^2(\ln(x)-1)^2}dx$$

Hence

$$u_2 = \frac{2}{x(\ln(x)-1)}$$

Which simplifies to

$$u_1 = \frac{-\ln(x)-1}{x^2(\ln(x)-1)}$$

$$u_2 = \frac{2}{x(\ln(x)-1)}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\ln(x)-1}{x(\ln(x)-1)} + \frac{2\ln(x)}{x(\ln(x)-1)}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1 \ln (x)}{x} + c_2\right) x\right) + \left(\frac{1}{x}\right) \\ &= \frac{1}{x} + \left(-\frac{c_1 \ln (x)}{x} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = \frac{1}{x} + \left(-\frac{c_1 \ln (x)}{x} + c_2\right) x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{x} + \left(-\frac{c_1 \ln (x)}{x} + c_2\right) x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = \infty$ in the above gives

$$0 = \operatorname{signum}(c_2) \infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
Try integration with the canonical coordinates of the symmetry [0, x]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(2*_a^3*_b(_a)*ln(_a)-3*_b(_a)*_a^3-2*
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- differential order: 2; canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 16

```
dsolve([x^3*(ln(x)-1)*diff(y(x),x$2)-x^2*diff(y(x),x)+x*y(x)=2*ln(x),y(infinity) = 0],y(x),
```

$$y(x) = \frac{-c_1 \ln(x) x + 1}{x}$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 8

```
DSolve[{x^3*(Log[x]-1)*y''[x]-x^2*y'[x]+x*y[x]==2*Log[x],{y[Infinity]==0}},y[x],x,IncludeSin
```

$$y(x) \to \frac{1}{x}$$

20.35 problem 674

Internal problem ID [15439]
Internal file name [`OUTPUT/15439_Wednesday_May_08_2024_03_58_54_PM_84795186/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 15.5 Linear equations with variable coefficients. The Lagrange method. Exercises page 148
Problem number: 674.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__change__of__variable__on__y__method__2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{(x^2-2x)\,y''+(-x^2+2)\,y'-2(1-x)\,y=2x-2}$$

With initial conditions

$$[y(\infty)=1]$$

20.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$
\begin{aligned}
p(x) &= \frac{-x^2+2}{x^2-2x} \\
q(x) &= \frac{2x-2}{x^2-2x} \\
F &= \frac{2x-2}{x^2-2x}
\end{aligned}
$$

Hence the ode is

$$
y'' + \frac{(-x^2+2)\, y'}{x^2-2x} + \frac{(2x-2)\, y}{x^2-2x} = \frac{2x-2}{x^2-2x}
$$

The domain of $p(x) = \frac{-x^2+2}{x^2-2x}$ is

$$
\{-\infty \le x < 0, 0 < x < 2, 2 < x \le \infty\}
$$

And the point $x_0 = \infty$ is inside this domain. The domain of $q(x) = \frac{2x-2}{x^2-2x}$ is

$$
\{-\infty \le x < 0, 0 < x < 2, 2 < x \le \infty\}
$$

And the point $x_0 = \infty$ is also inside this domain. The domain of $F = \frac{2x-2}{x^2-2x}$ is

$$
\{-\infty \le x < 0, 0 < x < 2, 2 < x \le \infty\}
$$

And the point $x_0 = \infty$ is also inside this domain. Hence solution exists and is unique.

20.35.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2 - 2x, B = -x^2 + 2, C = 2x - 2, f(x) = 2x - 2$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
\left(x^2-2x\right) y'' + \left(-x^2+2\right) y' + (2x-2)\, y = 0
$$

In normal form the ode

$$\left(x^2-2x\right) y''+\left(-x^2+2\right) y'+\left(2x-2\right) y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-x^2+2}{x\left(x-2\right)}\\ q(x) &= \frac{2x-2}{x\left(x-2\right)} \end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(-x^2+2)}{x^2\left(x-2\right)}+\frac{2x-2}{x\left(x-2\right)}=0 \tag{5}$$

Solving (5) for n gives

$$n=2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{4}{x}+\frac{-x^2+2}{x\left(x-2\right)}\right) v'(x) &= 0\\ v''(x)+\frac{\left(-x^2+4x-6\right) v'(x)}{x\left(x-2\right)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(-x^2+4x-6)\,u(x)}{x\,(x-2)}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u(x^2-4x+6)}{x\,(x-2)}\end{aligned}$$

Where $f(x)=\frac{x^2-4x+6}{x(x-2)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{x^2-4x+6}{x\,(x-2)}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{x^2-4x+6}{x\,(x-2)}\,dx\\ \ln(u) &= x+\ln(x-2)-3\ln(x)+c_1\\ u &= \mathrm{e}^{x+\ln(x-2)-3\ln(x)+c_1}\\ &= c_1\mathrm{e}^{x+\ln(x-2)-3\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{\mathrm{e}^x}{x^2}-\frac{2\,\mathrm{e}^x}{x^3}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\,dx+c_2\\ &= \frac{\mathrm{e}^x c_1}{x^2}+c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n\\ &= \left(\frac{\mathrm{e}^x c_1}{x^2}+c_2\right)x^2\\ &= c_2x^2+\mathrm{e}^x c_1\end{aligned}$$

Now the particular solution to this ODE is found

$$(x^2-2x)\,y''+(-x^2+2)\,y'+(2x-2)\,y=2x-2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=x^2$$
$$y_2=\mathrm{e}^x$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} x^2 & \mathrm{e}^x \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\mathrm{e}^x) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} x^2 & \mathrm{e}^x \\ 2x & \mathrm{e}^x \end{vmatrix}$$

Therefore

$$W=(x^2)(\mathrm{e}^x)-(\mathrm{e}^x)(2x)$$

Which simplifies to

$$W = e^{x}x^{2} - 2x\,e^{x}$$

Which simplifies to

$$W = e^{x}x(x-2)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{x}(2x-2)}{(x^{2}-2x)\,e^{x}x\,(x-2)}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2x-2}{x^{2}\,(x-2)^{2}}dx$$

Hence

$$u_1 = \frac{1}{2x-4} - \frac{1}{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{2}(2x-2)}{(x^{2}-2x)\,e^{x}x\,(x-2)}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2(x-1)\,e^{-x}}{(x-2)^{2}}dx$$

Hence

$$u_2 = -\frac{2\,e^{-x}}{x-2}$$

Which simplifies to

$$u_1 = \frac{1}{x\,(x-2)}$$

$$u_2 = -\frac{2\,e^{-x}}{x-2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x}{x-2} - \frac{2\,\mathrm{e}^{-x}\mathrm{e}^{x}}{x-2}$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{\mathrm{e}^{x} c_1}{x^2} + c_2\right) x^2\right) + (1) \\ &= 1 + \left(\frac{\mathrm{e}^{x} c_1}{x^2} + c_2\right) x^2 \end{aligned}$$

Which simplifies to

$$y = c_2 x^2 + \mathrm{e}^{x} c_1 + 1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^2 + \mathrm{e}^{x} c_1 + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \infty$ in the above gives

$$1 = \operatorname{signum}(c_1)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

20.35.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 - 2x) y'' + (-x^2 + 2) y' + (2x - 2) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 2x \\ B &= -x^2 + 2 \\ C &= 2x - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 8x^3 + 24x^2 - 24x + 12 \\ t &= 4\left(x^2 - 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 674: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-2)^2} - \frac{1}{4(x-2)} + \frac{3}{4x^2} - \frac{3}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} - \frac{1}{2(x-2)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{2x - 4} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-2)+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1\left(-\mathrm{e}^{-x} x^2\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^{x} \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\mathrm{e}^{x} \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}}\left(-\mathrm{e}^{-x} x^2\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
\left(x^2-2x\right) y'' + \left(-x^2+2\right) y' + (2x-2)\, y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 \mathrm{e}^{x} \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

$$y_2 = -\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}} & -\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x(x-2)}} \\ \frac{d}{dx}\left(\frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}}\right) & \frac{d}{dx}\left(-\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x(x-2)}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}} & -\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x(x-2)}} \\ \frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}} - \frac{e^{x}\sqrt{x}\,\sqrt{x-2}\,(2x-2)}{2(x(x-2))^{\frac{3}{2}}} + \frac{e^{x}\sqrt{x-2}}{2\sqrt{x(x-2)}\,\sqrt{x}} + \frac{e^{x}\sqrt{x}}{2\sqrt{x(x-2)}\,\sqrt{x-2}} & -\frac{5x^{\frac{3}{2}}\sqrt{x-2}}{2\sqrt{x(x-2)}} - \frac{x^{\frac{5}{2}}}{2\sqrt{x-2}\,\sqrt{x(x-2)}} + \frac{x^{\frac{5}{2}}\sqrt{x-2}\,(2x-2)}{2(x(x-2))^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right)\left(-\frac{5x^{\frac{3}{2}}\sqrt{x-2}}{2\sqrt{x\,(x-2)}} - \frac{x^{\frac{5}{2}}}{2\sqrt{x-2}\,\sqrt{x\,(x-2)}} + \frac{x^{\frac{5}{2}}\sqrt{x-2}\,(2x-2)}{2\,(x\,(x-2))^{\frac{3}{2}}}\right) - \left(-\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right)\left(\frac{e^{x}\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{e^{x}\sqrt{x}\,\sqrt{x-2}\,(2x-2)}{2\,(x\,(x-2))^{\frac{3}{2}}} + \frac{e^{x}\sqrt{x-2}}{2\sqrt{x\,(x-2)}\,\sqrt{x}} + \frac{e^{x}\sqrt{x}}{2\sqrt{x\,(x-2)}\,\sqrt{x-2}}\right)$$

Which simplifies to

$$W = e^{x} x(x-2)$$

Which simplifies to

$$W = e^{x} x(x-2)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{x^{\frac{5}{2}} \sqrt{x-2}\,(2x-2)}{\sqrt{x(x-2)}}}{(x^2-2x)\,e^{x} x\,(x-2)}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{2\sqrt{x}\,(x-1)\,e^{-x}}{(x-2)^{\frac{3}{2}} \sqrt{x\,(x-2)}} dx$$

Hence

$$u_1 = -\frac{2\sqrt{x}\,e^{-x}}{\sqrt{x-2}\,\sqrt{x\,(x-2)}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{x} \sqrt{x}\,\sqrt{x-2}\,(2x-2)}{\sqrt{x(x-2)}}}{(x^2-2x)\,e^{x} x\,(x-2)}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2x-2}{\sqrt{x\,(x-2)}\,x^{\frac{3}{2}}\,(x-2)^{\frac{3}{2}}} dx$$

Hence

$$u_2 = -\frac{1}{\sqrt{x}\,\sqrt{x-2}\,\sqrt{x\,(x-2)}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x}{x-2} - \frac{2\,e^{-x} e^{x}}{x-2}$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \mathrm{e}^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}\right) + (1) \end{aligned}$$

Which simplifies to

$$y = \frac{\sqrt{x}\,\sqrt{x-2}\,(-c_2 x^2 + \mathrm{e}^x c_1)}{\sqrt{x\,(x-2)}} + 1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\sqrt{x}\,\sqrt{x-2}\,(-c_2 x^2 + \mathrm{e}^x c_1)}{\sqrt{x\,(x-2)}} + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = \infty$ in the above gives

$$1 = \operatorname{signum}(c_1)\,\infty \tag{1A}$$

Equations $\{1A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([(x^2-2*x)*diff(y(x),x$2)+(2-x^2)*diff(y(x),x)-2*(1-x)*y(x)=2*(x-1),y(infinity) = 1],
```

$$y(x) = -\operatorname{signum}\left(c_1x^2\right)\infty$$

✓ Solution by Mathematica

Time used: 0.314 (sec). Leaf size: 6

```
DSolve[{(x^2-2*x)*y''[x]+(2-x^2)*y'[x]-2*(1-x)*y[x]==2*(x-1),{y[Infinity]==1}},y[x],x,Includ
```

$$y(x) \to 1$$

21 Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158

21.1 problem 696

Internal problem ID [15440]
Internal file name [`OUTPUT/15440_Wednesday_May_08_2024_03_58_55_PM_31210361/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 696.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + x' + x = 0}$$

21.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + \lambda \, \mathrm{e}^{\lambda t} + \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\lambda_{1,2} = \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}$$
$$= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right) \tag{1}$$

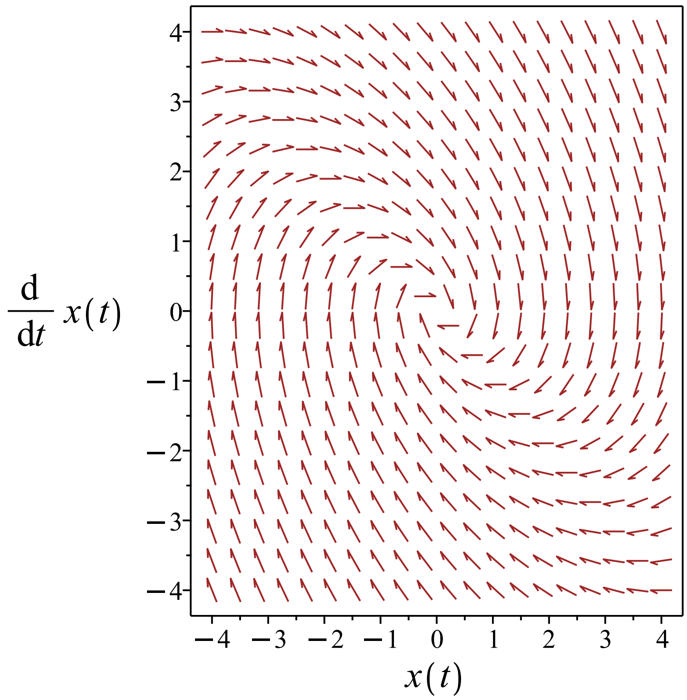


Figure 802: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

Verified OK.

21.1.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 675: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dt} \\ &= z_1 e^{-\frac{t}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{1}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
x = c_1\mathrm{e}^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{2c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\mathrm{e}^{-\frac{t}{2}}\sqrt{3}}{3} \tag{1}
$$

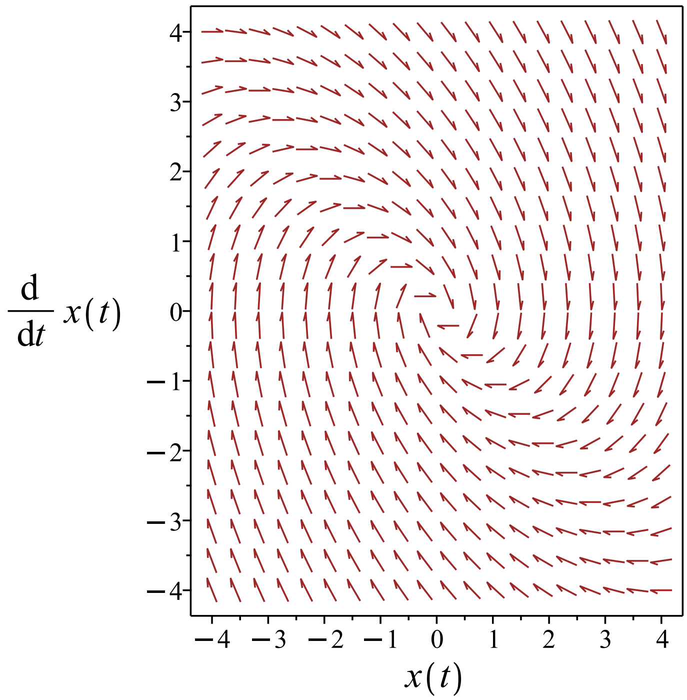


Figure 803: Slope field plot

Verification of solutions

$$x = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) + \frac{2c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right) e^{-\frac{t}{2}} \sqrt{3}}{3}$$

Verified OK.

21.1.3 Maple step by step solution

Let's solve

$x'' + x' + x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(diff(x(t),t$2)+diff(x(t),t)+x(t)=0,x(t), singsol=all)
```

$$x(t) = \mathrm{e}^{-\frac{t}{2}} \left(c_1 \sin\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 42

```
DSolve[x''[t]+x'[t]+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t/2}\left(c_2 \cos\left(\frac{\sqrt{3}t}{2}\right) + c_1 \sin\left(\frac{\sqrt{3}t}{2}\right)\right)$$

21.2 problem 697

Internal problem ID [15441]
Internal file name [`OUTPUT/15441_Wednesday_May_08_2024_03_58_56_PM_57460650/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 697.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 2x' + 6x = 0}$$

21.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 6$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + 6\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(6)}\\ &= -1 \pm i\sqrt{5}\end{aligned}$$

Hence

$$\lambda_1 = -1 + i\sqrt{5}$$
$$\lambda_2 = -1 - i\sqrt{5}$$

Which simplifies to

$$\lambda_1 = -1 + i\sqrt{5}$$
$$\lambda_2 = -1 - i\sqrt{5}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = \sqrt{5}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}\left(c_1 \cos\left(\sqrt{5}\, t\right) + c_2 \sin\left(\sqrt{5}\, t\right)\right)$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-t}\left(c_1 \cos\left(\sqrt{5}\, t\right) + c_2 \sin\left(\sqrt{5}\, t\right)\right) \tag{1}$$

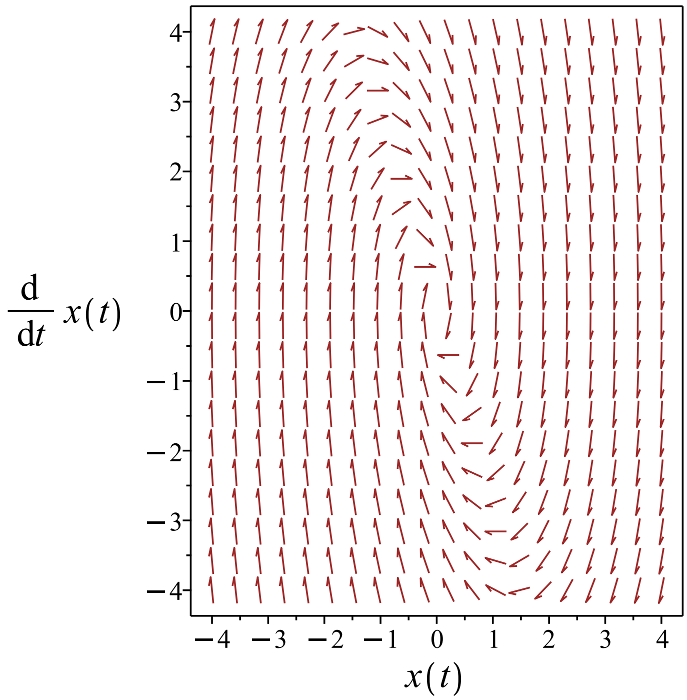


Figure 804: Slope field plot

Verification of solutions

$$x = e^{-t}\left(c_1 \cos\left(\sqrt{5}\, t\right) + c_2 \sin\left(\sqrt{5}\, t\right)\right)$$

Verified OK.

21.2.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + 6x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -5z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 677: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -5$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\sqrt{5}\, t\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2}{1}\, dt} \\
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{\left(x_1\right)^2}\, dt \\
&= x_1 \int \frac{e^{-2t}}{\left(x_1\right)^2}\, dt \\
&= x_1 \left(\frac{\sqrt{5}\, \tan\left(\sqrt{5}\, t\right)}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right)\right) + c_2\left(\mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right)\left(\frac{\sqrt{5}\, \tan\left(\sqrt{5}\, t\right)}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right) + \frac{c_2 \sin\left(\sqrt{5}\, t\right) \mathrm{e}^{-t} \sqrt{5}}{5} \tag{1}
$$

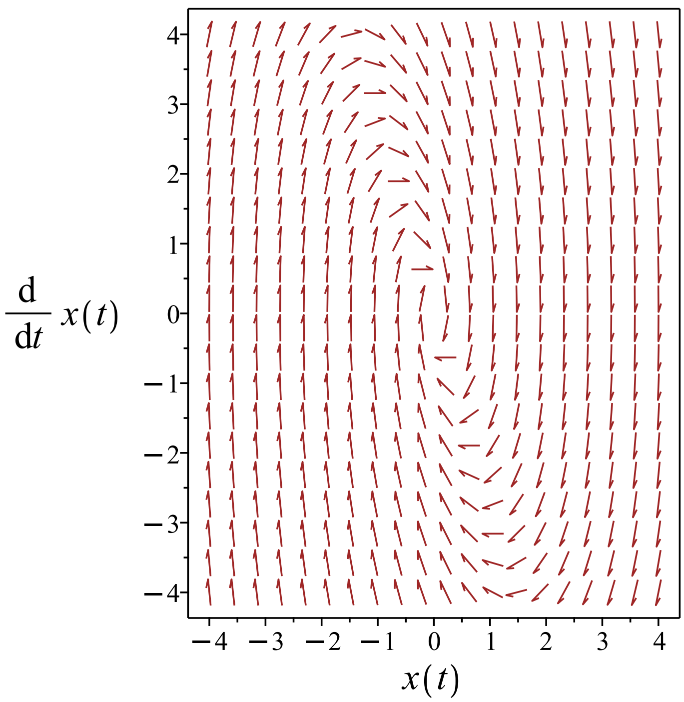


Figure 805: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right) + \frac{c_2 \sin\left(\sqrt{5}\, t\right) \mathrm{e}^{-t} \sqrt{5}}{5}
$$

Verified OK.

21.2.3 Maple step by step solution

Let's solve

$x'' + 2x' + 6x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 2r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-20})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-1 - \mathrm{I}\sqrt{5}, -1 + \mathrm{I}\sqrt{5}\right)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-t} \sin\left(\sqrt{5}\, t\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{-t} \cos\left(\sqrt{5}\, t\right) + c_2 \mathrm{e}^{-t} \sin\left(\sqrt{5}\, t\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(x(t),t$2)+2*diff(x(t),t)+6*x(t)=0,x(t), singsol=all)
```

$$x(t) = e^{-t}\left(c_1 \sin\left(\sqrt{5}\, t\right) + c_2 \cos\left(\sqrt{5}\, t\right)\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 34

```
DSolve[x''[t]+2*x'[t]+6*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t}\left(c_2 \cos\left(\sqrt{5}t\right) + c_1 \sin\left(\sqrt{5}t\right)\right)$$

21.3 problem 698

Internal problem ID [15442]
Internal file name [`OUTPUT/15442_Wednesday_May_08_2024_03_58_57_PM_87463915/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 698.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 2x' + x = 0}$$

21.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$x = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Summary

The solution(s) found are the following

$$x = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

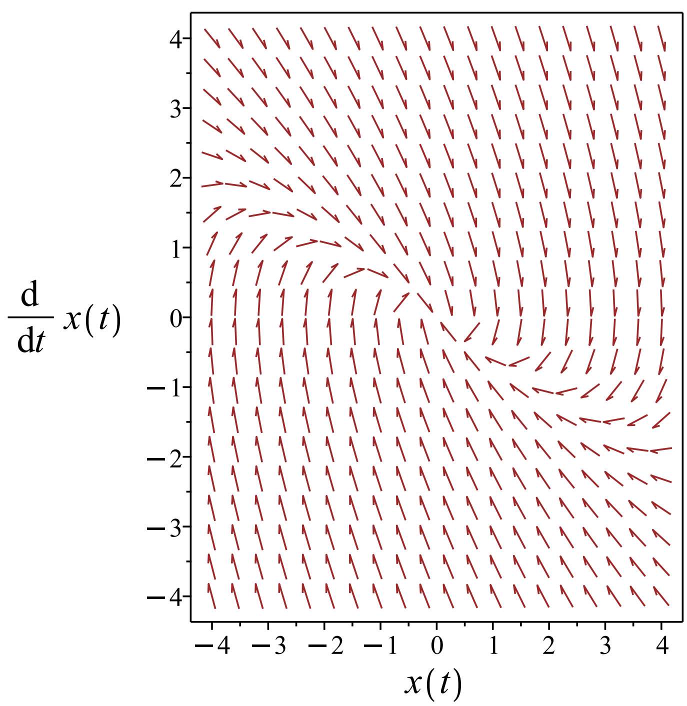


Figure 806: Slope field plot

Verification of solutions

$$x = c_1 e^{-t} + c_2 t\, e^{-t}$$

Verified OK.

21.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\, x' + \frac{\left(p(t)^2 + p'(t)\right) x}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int 2 \, dx} \\ &= \mathrm{e}^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\mathrm{e}^{t} x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{t} x\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{t} x\right) = c_1 t + c_2$$

Hence the solution is

$$x = \frac{c_1 t + c_2}{\mathrm{e}^{t}}$$

Or

$$x = c_1 t\, \mathrm{e}^{-t} + \mathrm{e}^{-t} c_2$$

Summary
The solution(s) found are the following

$$x = c_1 t\, \mathrm{e}^{-t} + \mathrm{e}^{-t} c_2 \tag{1}$$

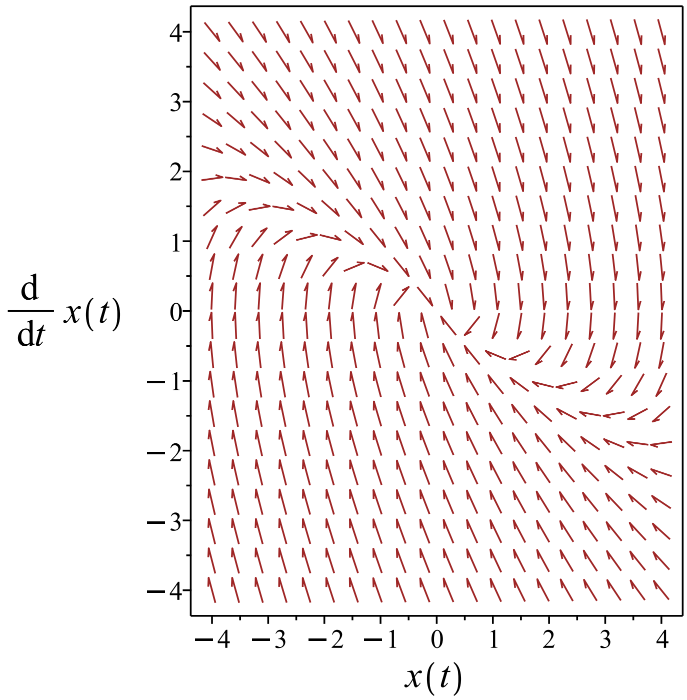


Figure 807: Slope field plot

Verification of solutions

$$x = c_1 t\, e^{-t} + e^{-t} c_2$$

Verified OK.

21.3.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 679: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\ &= z_1 e^{-t} \\ &= z_1 \left(\mathrm{e}^{-t}\right)\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-2t}}{(x_1)^2}\,dt \\ &= x_1(t)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t} \tag{1}
$$

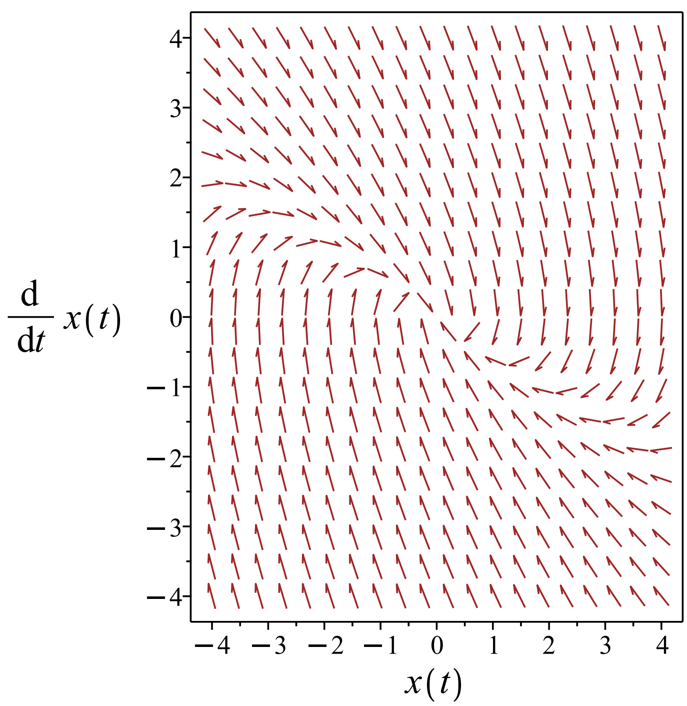


Figure 808: Slope field plot

Verification of solutions

$$
x = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}
$$

Verified OK.

21.3.4 Maple step by step solution

Let's solve

$x'' + 2x' + x = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $x_1(t) = e^{-t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t\,e^{-t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-t} + c_2 t\,e^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve(diff(x(t),t$2)+2*diff(x(t),t)+x(t)=0,x(t), singsol=all)
```

$$x(t) = e^{-t}(c_2 t + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[x''[t]+2*x'[t]+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t}(c_2 t + c_1)$$

21.4 problem 699

Internal problem ID [15443]
Internal file name [`OUTPUT/15443_Wednesday_May_08_2024_03_58_57_PM_93223807/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 699.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + x'^2 + x = 0}$$

21.4.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + p(x)^2 + x = 0$$

Which is now solved as first order ode for $p(x)$. Writing the ode as

$$\frac{d}{dx}p(x) = -\frac{p^2 + x}{p}$$
$$\frac{d}{dx}p(x) = \omega(x, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_p - \xi_x) - \omega^2\xi_p - \omega_x\xi - \omega_p\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 681: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,p) &= 0\\ \eta(x,p) &= \frac{e^{-2x}}{p}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial p}\right) S(x,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\frac{e^{-2x}}{p}} dy\end{aligned}$$

Which results in

$$S = \frac{p^2 e^{2x}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,p)S_p}{R_x + \omega(x,p)R_p} \tag{2}$$

Where in the above R_x, R_p, S_x, S_p are all partial derivatives and $\omega(x,p)$ is the right hand side of the original ode given by

$$\omega(x,p) = -\frac{p^2 + x}{p}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_p &= 0 \\
S_x &= p^2 \mathrm{e}^{2x} \\
S_p &= p\,\mathrm{e}^{2x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -x\,\mathrm{e}^{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -R\,\mathrm{e}^{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{(2R-1)\,\mathrm{e}^{2R}}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, p coordinates. This results in

$$
\frac{p(x)^2\,\mathrm{e}^{2x}}{2} = -\frac{(2x-1)\,\mathrm{e}^{2x}}{4} + c_1
$$

Which simplifies to

$$
\frac{\left(2p(x)^2 + 2x - 1\right)\mathrm{e}^{2x}}{4} - c_1 = 0
$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$
\frac{\left(2x'^2 + 2x - 1\right)\mathrm{e}^{2x}}{4} - c_1 = 0
$$

Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
x' = \frac{\mathrm{e}^{-2x}\sqrt{-2\,\mathrm{e}^{2x}\left(2\,\mathrm{e}^{2x}x - \mathrm{e}^{2x} - 4c_1\right)}}{2} \tag{1}
$$

$$
x' = -\frac{\mathrm{e}^{-2x}\sqrt{-2\,\mathrm{e}^{2x}\left(2\,\mathrm{e}^{2x}x - \mathrm{e}^{2x} - 4c_1\right)}}{2} \tag{2}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2\,e^{2x}}{\sqrt{-2\,e^{2x}\left(2x\,e^{2x}-e^{2x}-4c_1\right)}}dx = \int dt$$

$$2\left(\int^{x}\frac{e^{2\_a}}{\sqrt{-2\,e^{2\_a}\left(2\_a\,e^{2\_a}-e^{2\_a}-4c_1\right)}}d\_a\right) = t+c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2\,e^{2x}}{\sqrt{-2\,e^{2x}\left(2x\,e^{2x}-e^{2x}-4c_1\right)}}dx = \int dt$$

$$-2\left(\int^{x}\frac{e^{2\_a}}{\sqrt{-2\,e^{2\_a}\left(2\_a\,e^{2\_a}-e^{2\_a}-4c_1\right)}}d\_a\right) = t+c_3$$

Summary

The solution(s) found are the following

$$2\left(\int^{x}\frac{e^{2\_a}}{\sqrt{-2\,e^{2\_a}\left(2\_a\,e^{2\_a}-e^{2\_a}-4c_1\right)}}d\_a\right) = t+c_2 \tag{1}$$

$$-2\left(\int^{x}\frac{e^{2\_a}}{\sqrt{-2\,e^{2\_a}\left(2\_a\,e^{2\_a}-e^{2\_a}-4c_1\right)}}d\_a\right) = t+c_3 \tag{2}$$

Verification of solutions

$$2\left(\int^{x}\frac{e^{2\_a}}{\sqrt{-2\,e^{2\_a}\left(2\_a\,e^{2\_a}-e^{2\_a}-4c_1\right)}}d\_a\right) = t+c_2$$

Verified OK.

$$-2\left(\int^{x}\frac{e^{2\_a}}{\sqrt{-2\,e^{2\_a}\left(2\_a\,e^{2\_a}-e^{2\_a}-4c_1\right)}}d\_a\right) = t+c_3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2+_a = 0, _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 61

```
dsolve(diff(x(t),t$2)+diff(x(t),t)^2+x(t)=0,x(t), singsol=all)
```

$$-2\left(\int^{x(t)}\frac{1}{\sqrt{2+4\,\mathrm{e}^{-2\_a}c_1-4\_a}}d\_a\right)-t-c_2=0$$

$$2\left(\int^{x(t)}\frac{1}{\sqrt{2+4\,\mathrm{e}^{-2\_a}c_1-4\_a}}d\_a\right)-t-c_2=0$$

✓ Solution by Mathematica

Time used: 0.81 (sec). Leaf size: 272

```
DSolve[x''[t]+x'[t]^2+x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}c_1 - 2K[1] + 1}} dK[1]\&\right][t + c_2]$$
$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}c_1 - 2K[2] + 1}} dK[2]\&\right][t + c_2]$$
$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}(-c_1) - 2K[1] + 1}} dK[1]\&\right][t + c_2]$$
$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} -\frac{\sqrt{2}}{\sqrt{2e^{-2K[1]}c_1 - 2K[1] + 1}} dK[1]\&\right][t + c_2]$$
$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}(-c_1) - 2K[2] + 1}} dK[2]\&\right][t + c_2]$$
$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{\sqrt{2}}{\sqrt{2e^{-2K[2]}c_1 - 2K[2] + 1}} dK[2]\&\right][t + c_2]$$

21.5 problem 700

Internal problem ID [15444]
Internal file name [OUTPUT/15444_Wednesday_May_08_2024_03_58_59_PM_52636469/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 700.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{x'' - 2x'^2 + x' - 2x = 0}$$

21.5.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + (-2p(x) + 1)\,p(x) - 2x = 0$$

Which is now solved as first order ode for $p(x)$. Unable to determine ODE type.

Unable to solve. Terminating

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-2*_b(_a)^2+_b(_a)-2*_a = 0, _b(_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 2
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
   -> trying a symmetry pattern of the form [F(x),G(x)]
   -> trying a symmetry pattern of the form [F(y),G(y)]
   -> trying a symmetry pattern of the form [F(x)+G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(diff(x(t),t$2)-2*diff(x(t),t)^2+diff(x(t),t)-2*x(t)=0,x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x''[t]-2*x'[t]^2+x'[t]-2*x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

Not solved

21.6 problem 701

Internal problem ID [15445]
Internal file name [`OUTPUT/15445_Wednesday_May_08_2024_03_58_59_PM_64750972/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 701.
ODE order: 2.
ODE degree: 0.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{x'' - x\,\mathrm{e}^{x'} = 0}$$

21.6.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) - x\,\mathrm{e}^{p(x)} = 0$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{x\,\mathrm{e}^{p}}{p} \end{aligned}$$

Where $f(x) = x$ and $g(p) = \frac{\mathrm{e}^{p}}{p}$. Integrating both sides gives

$$\frac{1}{\frac{\mathrm{e}^{p}}{p}}\, dp = x\, dx$$

$$\int \frac{1}{\frac{\mathrm{e}^{p}}{p}}\, dp = \int x\, dx$$

$$-(p+1)\,\mathrm{e}^{-p} = \frac{x^2}{2} + c_1$$

The solution is

$$-(p(x)+1)\,\mathrm{e}^{-p(x)} - \frac{x^2}{2} - c_1 = 0$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$-(x'+1)\,\mathrm{e}^{-x'} - \frac{x^2}{2} - c_1 = 0$$

Integrating both sides gives

$$\int \frac{1}{-\mathrm{LambertW}\left(\frac{(x^2+2c_1)\mathrm{e}^{-1}}{2}\right) - 1} dx = \int dt$$

$$\int^{x} \frac{1}{-\mathrm{LambertW}\left(\frac{(\_a^2+2c_1)\mathrm{e}^{-1}}{2}\right) - 1} d\_a = t + c_2$$

Summary

The solution(s) found are the following

$$\int^{x} \frac{1}{-\mathrm{LambertW}\left(\frac{(\_a^2+2c_1)\mathrm{e}^{-1}}{2}\right) - 1} d\_a = t + c_2 \tag{1}$$

Verification of solutions

$$\int^{x} \frac{1}{-\operatorname{LambertW}\left(\frac{(\_a^{2}+2c_{1})\,\mathrm{e}^{-1}}{2}\right)-1}d\_a = t+c_{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_a*exp(_b(_a)) = 0, _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   <- separable successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(diff(x(t),t$2)-x(t)*exp(diff(x(t),t))=0,x(t), singsol=all)
```

$$-\left(\int^{x(t)} \frac{1}{\operatorname{LambertW}\left(\frac{(\_a^{2}+2c_{1})\,\mathrm{e}^{-1}}{2}\right)+1}d\_a\right)-t-c_{2} = 0$$

✓ Solution by Mathematica

Time used: 0.389 (sec). Leaf size: 126

```
DSolve[x''[t]-x[t]*Exp[x'[t]]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{-W\left(\frac{K[1]^2+2c_1}{2e}\right)-1}dK[1]\&\right][t+c_2]$$

$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{-W\left(\frac{K[1]^2+2(-1)c_1}{2e}\right)-1}dK[1]\&\right][t+c_2]$$

$$x(t) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{-W\left(\frac{K[1]^2+2c_1}{2e}\right)-1}dK[1]\&\right][t+c_2]$$

21.7 problem 702

Internal problem ID [15446]
Internal file name [`OUTPUT/15446_Wednesday_May_08_2024_03_59_00_PM_39363274/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 702.
ODE order: 2.
ODE degree: 0.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{x'' + \mathrm{e}^{-x'} - x = 0}$$

21.7.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) - x = -\mathrm{e}^{-p(x)}$$

Which is now solved as first order ode for $p(x)$. Unable to determine ODE type.

Unable to solve. Terminating

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+exp(-_b(_a))-_a = 0, _b(_a)`   *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 5
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
   `, `-> Computing symmetries using: way = HINT
      -> Calling odsolve with the ODE`, diff(y(x), x), y(x)`         *** Sublevel 3 ***
         Methods for first order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         trying 1st order linear
         <- 1st order linear successful
```

✗ Solution by Maple

```
dsolve(diff(x(t),t$2)+exp(-diff(x(t),t))-x(t)=0,x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x''[t]+Exp[-x'[t]]-x[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

Not solved

21.8 problem 703

Internal problem ID [15447]
Internal file name [`OUTPUT/15447_Wednesday_May_08_2024_03_59_01_PM_47068444/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 703.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{x'' + x{x'}^2 = 0}$$

21.8.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right)+xp(x)^2=0$$

Which is now solved as first order ode for $p(x)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(x,p)\\ &= f(x)g(p)\\ &= -px\end{aligned}$$

Where $f(x)=-x$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}\frac{1}{p}\,dp &= -x\,dx\\ \int\frac{1}{p}\,dp &= \int -x\,dx\\ \ln(p) &= -\frac{x^2}{2}+c_1\\ p &= \mathrm{e}^{-\frac{x^2}{2}+c_1}\\ &= c_1\mathrm{e}^{-\frac{x^2}{2}}\end{aligned}$$

For solution (1) found earlier, since $p=x'$ then we now have a new first order ode to solve which is

$$x'=c_1\mathrm{e}^{-\frac{x^2}{2}}$$

Integrating both sides gives

$$\begin{aligned}\int\frac{\mathrm{e}^{\frac{x^2}{2}}}{c_1}dx &= \int dt\\ \int^x\frac{\mathrm{e}^{\frac{\_a^2}{2}}}{c_1}d\_a &= t+c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$\int^x\frac{\mathrm{e}^{\frac{\_a^2}{2}}}{c_1}d\_a=t+c_2 \tag{1}$$

Verification of solutions

$$\int^x\frac{\mathrm{e}^{\frac{\_a^2}{2}}}{c_1}d\_a=t+c_2$$

Verified OK.

21.8.2 Maple step by step solution

Let's solve

$x'' + x x'^2 = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Define new dependent variable u

  $u(t) = x'$

- Compute x''

  $u'(t) = x''$

- Use chain rule on the lhs

  $x' \left(\frac{d}{dx} u(x)\right) = x''$

- Substitute in the definition of u

  $u(x) \left(\frac{d}{dx} u(x)\right) = x''$

- Make substitutions $x' = u(x)\,, x'' = u(x) \left(\frac{d}{dx} u(x)\right)$ to reduce order of ODE

  $u(x) \left(\frac{d}{dx} u(x)\right) + x u(x)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dx} u(x)}{u(x)} = -x$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx} u(x)}{u(x)} dx = \int -x dx + c_1$

- Evaluate integral

  $\ln\left(u(x)\right) = -\frac{x^2}{2} + c_1$

- Solve for $u(x)$

  $u(x) = \mathrm{e}^{-\frac{x^2}{2} + c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = \mathrm{e}^{-\frac{x^2}{2} + c_1}$

- Revert to original variables with substitution $u(x) = x', x = x$

  $x' = \mathrm{e}^{-\frac{x^2}{2} + c_1}$

- Separate variables

$$\frac{x'}{e^{-\frac{x^2}{2}+c_1}} = 1$$

- Integrate both sides with respect to t

$$\int \frac{x'}{e^{-\frac{x^2}{2}+c_1}} dt = \int 1 dt + c_2$$

- Evaluate integral

$$\frac{-\frac{I}{2}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{I}{2}\sqrt{2}\,x\right)}{e^{c_1}} = t + c_2$$

- Solve for x

$$\left\{-I\,\mathit{RootOf}\left(I\,e^{c_1}\sqrt{2}\,c_2 + I\,e^{c_1}\sqrt{2}\,t - \mathrm{erf}(\_Z)\,\sqrt{\pi}\right)\sqrt{2}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(x(t),t$2)+x(t)*diff(x(t),t)^2=0,x(t), singsol=all)
```

$$x(t) = -i\,\mathrm{RootOf}\left(i\sqrt{2}\,c_1 t + i\sqrt{2}\,c_2 - \mathrm{erf}\left(\_Z\right)\sqrt{\pi}\right)\sqrt{2}$$

✓ Solution by Mathematica

Time used: 1.757 (sec). Leaf size: 34

```
DSolve[x''[t]+x[t]*x'[t]^2==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -i\sqrt{2}\mathrm{erf}^{-1}\left(i\sqrt{\frac{2}{\pi}}c_1(t+c_2)\right)$$

21.9 problem 704

Internal problem ID [15448]
Internal file name [`OUTPUT/15448_Wednesday_May_08_2024_03_59_02_PM_16519201/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 704.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{x'' + (x+2)\,x' = 0}$$

21.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t t gives

$$\int (x'' + (x+2)\,x')\,dt = 0$$

$$2x + \frac{x^2}{2} + x' = c_1$$

Which is now solved for x. Integrating both sides gives

$$\int \frac{1}{-2x - \frac{1}{2}x^2 + c_1} dx = t + c_2$$
$$\frac{2 \operatorname{arctanh}\left(\frac{2x+4}{2\sqrt{2c_1+4}}\right)}{\sqrt{2c_1 + 4}} = t + c_2$$

Solving for x gives these solutions

$$x_1 = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2$$

Summary

The solution(s) found are the following

$$x = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2 \tag{1}$$

Verification of solutions

$$x = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2$$

Verified OK.

21.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) + (2 + x)\, p(x) = 0$$

Which is now solved as first order ode for $p(x)$. Integrating both sides gives

$$\begin{aligned} p(x) &= \int -x - 2 \,\mathrm{d}x \\ &= -2x - \frac{1}{2}x^2 + c_1 \end{aligned}$$

For solution (1) found earlier, since $p = x'$ then we now have a new first order ode to solve which is

$$x' = -2x - \frac{x^2}{2} + c_1$$

Integrating both sides gives

$$\begin{aligned} \int \frac{1}{-2x - \frac{1}{2}x^2 + c_1} dx &= t + c_2 \\ \frac{2\,\operatorname{arctanh}\left(\frac{2x+4}{2\sqrt{2c_1+4}}\right)}{\sqrt{2c_1 + 4}} &= t + c_2 \end{aligned}$$

Solving for x gives these solutions

$$x_1 = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2$$

Summary

The solution(s) found are the following

$$x = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2 \tag{1}$$

Verification of solutions

$$x = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2$$

Verified OK.

21.9.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x'' + (x + 2)\,x' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\begin{aligned} \int (x'' + (x + 2)\,x')\,dt &= 0 \\ 2x + \frac{x^2}{2} + x' &= c_1 \end{aligned}$$

Which is now solved for x. Integrating both sides gives

$$\int \frac{1}{-2x-\frac{1}{2}x^2+c_1}dx = t+c_2$$
$$\frac{2\,\mathrm{arctanh}\left(\frac{2x+4}{2\sqrt{2c_1+4}}\right)}{\sqrt{2c_1+4}} = t+c_2$$

Solving for x gives these solutions

$$x_1 = \tanh\left(\frac{c_2\sqrt{2c_1+4}}{2}+\frac{t\sqrt{2c_1+4}}{2}\right)\sqrt{2c_1+4}-2$$

Summary

The solution(s) found are the following

$$x = \tanh\left(\frac{c_2\sqrt{2c_1+4}}{2}+\frac{t\sqrt{2c_1+4}}{2}\right)\sqrt{2c_1+4}-2 \tag{1}$$

Verification of solutions

$$x = \tanh\left(\frac{c_2\sqrt{2c_1+4}}{2}+\frac{t\sqrt{2c_1+4}}{2}\right)\sqrt{2c_1+4}-2$$

Verified OK.

21.9.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(t,x,x')\,x''+a_1(t,x,x')\,x'+a_0(t,x,x')=0$$

Where the following conditions are satisfied

$$\frac{\partial a_2}{\partial x} = \frac{\partial a_1}{\partial x'}$$
$$\frac{\partial a_2}{\partial t} = \frac{\partial a_0}{\partial x'}$$
$$\frac{\partial a_1}{\partial t} = \frac{\partial a_0}{\partial x}$$

Looking at the the ode given we see that

$$a_2 = 1$$
$$a_1 = x+2$$
$$a_0 = 0$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2 \, dx' + \int a_1 \, dx + \int a_0 \, dt = c_1$$
$$\int 1 \, dx' + \int x + 2 \, dx + \int 0 \, dt = c_1$$

Which results in

$$2x + \frac{x^2}{2} + x' = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{-2x - \frac{1}{2}x^2 + c_1} dx = t + c_2$$
$$\frac{2 \operatorname{arctanh}\left(\frac{2x+4}{2\sqrt{2c_1+4}}\right)}{\sqrt{2c_1 + 4}} = t + c_2$$

Solving for x gives these solutions

$$x_1 = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2$$

Summary
The solution(s) found are the following

$$x = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2 \tag{1}$$

Verification of solutions

$$x = \tanh\left(\frac{c_2\sqrt{2c_1 + 4}}{2} + \frac{t\sqrt{2c_1 + 4}}{2}\right)\sqrt{2c_1 + 4} - 2$$

Verified OK.

21.9.5 Maple step by step solution

Let's solve

$x'' + (x + 2)\, x' = 0$

- Highest derivative means the order of the ODE is 2

  x''

- Define new dependent variable u

  $u(t) = x'$
- Compute x''

  $u'(t) = x''$
- Use chain rule on the lhs

  $x'\left(\frac{d}{dx}u(x)\right) = x''$
- Substitute in the definition of u

  $u(x)\left(\frac{d}{dx}u(x)\right) = x''$
- Make substitutions $x' = u(x)\,, x'' = u(x)\left(\frac{d}{dx}u(x)\right)$ to reduce order of ODE

  $u(x)\left(\frac{d}{dx}u(x)\right) + (2+x)\,u(x) = 0$
- Separate variables

  $\frac{d}{dx}u(x) = -x-2$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}u(x)\right) dx = \int (-x-2)\, dx + c_1$
- Evaluate integral

  $u(x) = -2x - \frac{1}{2}x^2 + c_1$
- Solve for $u(x)$

  $u(x) = -2x - \frac{1}{2}x^2 + c_1$
- Solve 1st ODE for $u(x)$

  $u(x) = -2x - \frac{1}{2}x^2 + c_1$
- Revert to original variables with substitution $u(x) = x', x = x$

  $x' = -2x - \frac{x^2}{2} + c_1$
- Separate variables

  $\frac{x'}{-2x-\frac{x^2}{2}+c_1} = 1$
- Integrate both sides with respect to t

  $\int \frac{x'}{-2x-\frac{x^2}{2}+c_1} dt = \int 1 dt + c_2$
- Evaluate integral

  $\frac{2\,\mathrm{arctanh}\left(\frac{2x+4}{2\sqrt{2c_1+4}}\right)}{\sqrt{2c_1+4}} = t + c_2$

- Solve for x

  $x = \tanh\left(\frac{c_2\sqrt{2c_1+4}}{2} + \frac{t\sqrt{2c_1+4}}{2}\right)\sqrt{2c_1+4} - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(_a+2)*_b(_a) = 0, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[-_a-2, -2*_b]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve(diff(x(t),t$2)+(x(t)+2)*diff(x(t),t)=0,x(t), singsol=all)
```

$$x(t) = -\frac{\left(\sqrt{2}\, c_1 - \tanh\left(\frac{(t+c_2)\sqrt{2}}{2c_1}\right)\right)\sqrt{2}}{c_1}$$

✓ Solution by Mathematica

Time used: 60.064 (sec). Leaf size: 40

```
DSolve[x''[t]+(x[t]+2)*x'[t]==0,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -2 + \sqrt{2}\sqrt{2+c_1}\tanh\left(\frac{\sqrt{2+c_1}(t+c_2)}{\sqrt{2}}\right)$$

21.10 problem 705

Internal problem ID [15449]
Internal file name [`OUTPUT/15449_Wednesday_May_08_2024_03_59_02_PM_5954560/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 16. The method of isoclines for differential equations of the second order. Exercises page 158
Problem number: 705.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{x'' - x' + x - x^2 = 0}$$

21.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable x an independent variable. Using

$$x' = p(x)$$

Then

$$\begin{aligned} x'' &= \frac{dp}{dt} \\ &= \frac{dx}{dt}\frac{dp}{dx} \\ &= p\frac{dp}{dx} \end{aligned}$$

Hence the ode becomes

$$p(x)\left(\frac{d}{dx}p(x)\right) - p(x) + (1 - x)\, x = 0$$

Which is now solved as first order ode for $p(x)$. Unable to determine ODE type.

Unable to solve. Terminating

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-_b(_a)+_a-_a^2 = 0, _b(_a)`   **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 2
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
   -> trying a symmetry pattern of the form [F(x),G(x)]
   -> trying a symmetry pattern of the form [F(y),G(y)]
   -> trying a symmetry pattern of the form [F(x)+G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)+G(y)]
```

✗ Solution by Maple

```
dsolve(diff(x(t),t$2)-diff(x(t),t)+x(t)-x(t)^2=0,x(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x''[t]-x'[t]+x[t]-x[t]^2==0,x[t],t,IncludeSingularSolutions -> True]
```

Not solved

22 Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163

22.1 problem 706 (a)

Internal problem ID [15450]
Internal file name [`OUTPUT/15450_Wednesday_May_08_2024_03_59_03_PM_92668551/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 706 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \lambda y = 0}$$

With initial conditions

$$[y'(0) = 0, y'(\pi) = 0]$$

22.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \lambda$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \lambda$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\lambda)}\\ &= \pm\sqrt{-\lambda}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\lambda}\\ \lambda_2 &= -\sqrt{-\lambda}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\lambda}\\ \lambda_2 &= -\sqrt{-\lambda}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(\sqrt{-\lambda})x} + c_2 e^{(-\sqrt{-\lambda})x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}\, x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}\, x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda}\,\mathrm{e}^{\sqrt{-\lambda}\, x} - c_2\sqrt{-\lambda}\,\mathrm{e}^{-\sqrt{-\lambda}\, x}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \left(c_1 e^{\pi\sqrt{-\lambda}} - c_2 e^{-\pi\sqrt{-\lambda}}\right)\sqrt{-\lambda} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\, x} - c_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\, x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = (c_1 - c_2)\sqrt{-\lambda} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

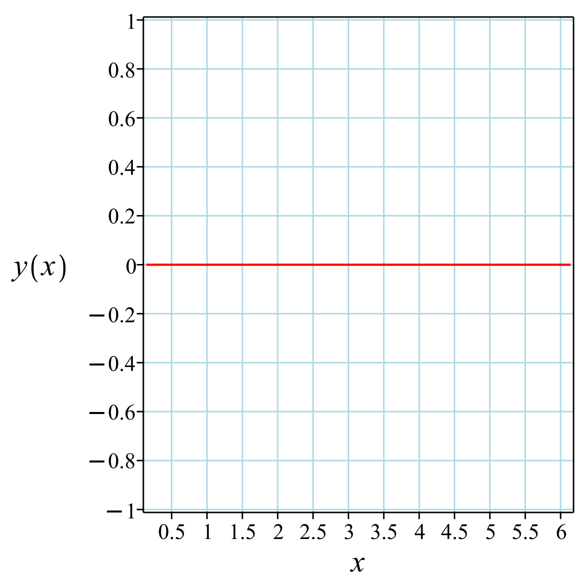


Figure 809: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

22.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \lambda y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + \lambda y'y)\, dx = 0$$
$$\frac{y'^2}{2} + \frac{\lambda y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\lambda y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\lambda y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\lambda\, y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^2+2c_1}}\right)}{\sqrt{\lambda}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\lambda\, y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^2+2c_1}}\right)}{\sqrt{\lambda}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^{2}+2 c_{1}}}\right)}{\sqrt{\lambda}} = x + c_{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = \frac{\sqrt{2}\,\sqrt{\lambda}\,\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2} c_{1}}{\sqrt{\lambda\left(\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}+1\right) c_{1}}} - \frac{2\sqrt{2}\,\sqrt{\lambda\left(\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}+1\right) c_{1}}\,\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}}{\sqrt{\lambda}\left(\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}+1\right)}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \frac{\sqrt{\lambda}\, c_{1}\sqrt{2}}{\sqrt{\sec\left(\left(\pi + c_{2}\right)\sqrt{\lambda}\right)^{2} c_{1}\lambda}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\sqrt{2}\,\sqrt{\lambda}\,\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2} c_{1}}{\sqrt{\lambda\left(\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}+1\right) c_{1}}} - \frac{2\sqrt{2}\,\sqrt{\lambda\left(\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}+1\right) c_{1}}\,\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}}{\sqrt{\lambda}\left(\tan\left(c_{2}\sqrt{\lambda}+x\sqrt{\lambda}\right)^{2}+1\right)}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{\sqrt{\lambda}\, c_{1}\sqrt{2}}{\sqrt{\sec\left(c_{2}\sqrt{\lambda}\right)^{2} c_{1}\lambda}} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^{2}+2 c_{1}}}\right)}{\sqrt{\lambda}} = x + c_{3} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = -\frac{\sqrt{2}\,\sqrt{\lambda}\,\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2 c_1}{\sqrt{\lambda\left(\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2+1\right)c_1}} + \frac{2\sqrt{2}\,\sqrt{\lambda\left(\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2+1\right)c_1}\,\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2}{\sqrt{\lambda}\left(\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2+1\right)}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = -\frac{\sqrt{\lambda}\,c_1\sqrt{2}}{\sqrt{\sec\left((\pi+c_3)\sqrt{\lambda}\right)^2 c_1\lambda}} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\sqrt{2}\,\sqrt{\lambda}\,\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2 c_1}{\sqrt{\lambda\left(\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2+1\right)c_1}} + \frac{2\sqrt{2}\,\sqrt{\lambda\left(\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2+1\right)c_1}\,\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2}{\sqrt{\lambda}\left(\tan\left(c_3\sqrt{\lambda}+x\sqrt{\lambda}\right)^2+1\right)}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{\sqrt{\lambda}\,c_1\sqrt{2}}{\sqrt{\sec\left(c_3\sqrt{\lambda}\right)^2 c_1\lambda}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

22.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \lambda y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-\lambda)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 685: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\lambda$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{-\lambda}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\sqrt{-\lambda}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-\lambda}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-\lambda}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda}\, x} \left(-\frac{\mathrm{e}^{-2\sqrt{-\lambda}\, x}}{2\sqrt{-\lambda}} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{-\lambda}\, x} \right) + c_2 \left(\mathrm{e}^{\sqrt{-\lambda}\, x} \left(-\frac{\mathrm{e}^{-2\sqrt{-\lambda}\, x}}{2\sqrt{-\lambda}} \right) \right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}\, x} - \frac{c_2 \mathrm{e}^{-\sqrt{-\lambda}\, x}}{2\sqrt{-\lambda}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\lambda}\, \mathrm{e}^{\sqrt{-\lambda}\, x} + \frac{c_2 \mathrm{e}^{-\sqrt{-\lambda}\, x}}{2}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \sqrt{-\lambda}\, \mathrm{e}^{\pi\sqrt{-\lambda}} c_1 + \frac{c_2 \mathrm{e}^{-\pi\sqrt{-\lambda}}}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda}\,\mathrm{e}^{\sqrt{-\lambda}\,x} + \frac{c_2 \mathrm{e}^{-\sqrt{-\lambda}\,x}}{2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1\sqrt{-\lambda} + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

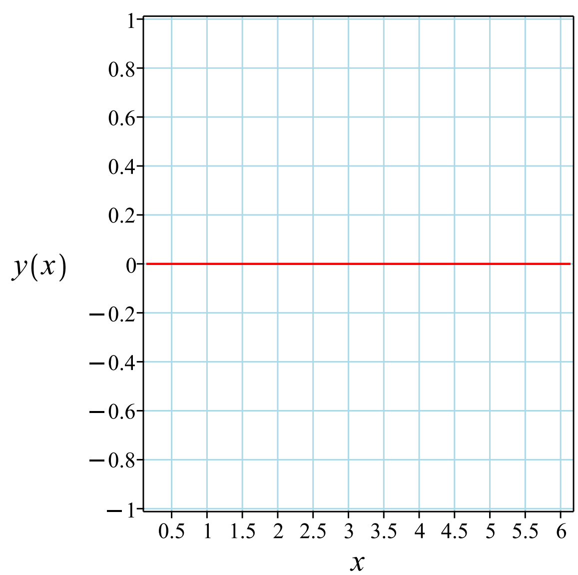


Figure 810: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$2)+lambda*y(x)=0,D(y)(0) = 0, D(y)(Pi) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 32

```
DSolve[{y''[x]+\[Lambda]*y[x]==0,{y'[0]==0,y'[Pi]==0}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \begin{cases} c_1 \cos\left(x\sqrt{\lambda}\right) & n \in \mathbb{Z} \land n \geq 0 \land \lambda = n^2 \\ 0 & \text{True} \end{cases}$$

22.2 problem 707

Internal problem ID [15451]
Internal file name [OUTPUT/15451_Wednesday_May_08_2024_03_59_06_PM_65752335/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 707.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \lambda y = 0}$$

With initial conditions

$$[y(0) = 0, y(1) = 0]$$

22.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \lambda$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \lambda$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(\lambda)}\\ &= \pm\sqrt{-\lambda}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\lambda}\\ \lambda_2 &= -\sqrt{-\lambda}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\lambda}\\ \lambda_2 &= -\sqrt{-\lambda}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(\sqrt{-\lambda})x} + c_2 e^{(-\sqrt{-\lambda})x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}\,x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}\,x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}\,x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}\,x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 \mathrm{e}^{\sqrt{-\lambda}} + c_2 \mathrm{e}^{-\sqrt{-\lambda}} \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

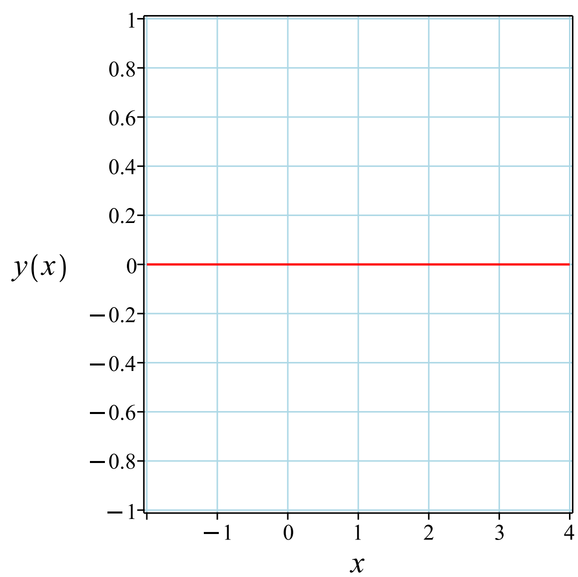


Figure 811: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

22.2.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \lambda y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + \lambda y'y)\, dx = 0$$
$$\frac{y'^2}{2} + \frac{\lambda y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\lambda y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\lambda y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\lambda\, y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^2+2c_1}}\right)}{\sqrt{\lambda}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\lambda\, y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^2+2c_1}}\right)}{\sqrt{\lambda}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^2+2c_1}}\right)}{\sqrt{\lambda}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = 1 + c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\arctan\left(\frac{\sqrt{\lambda}\, y}{\sqrt{-\lambda y^2+2c_1}}\right)}{\sqrt{\lambda}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = 1 + c_3 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

22.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \lambda y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \lambda \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-\lambda) \, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 686: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\lambda$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{-\lambda}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{-\lambda}\, x}
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-\lambda}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-\lambda}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda}\, x} \left(-\frac{\mathrm{e}^{-2\sqrt{-\lambda}\, x}}{2\sqrt{-\lambda}}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{-\lambda}\, x}\right) + c_2 \left(\mathrm{e}^{\sqrt{-\lambda}\, x} \left(-\frac{\mathrm{e}^{-2\sqrt{-\lambda}\, x}}{2\sqrt{-\lambda}}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda}\, x} - \frac{c_2 \mathrm{e}^{-\sqrt{-\lambda}\, x}}{2\sqrt{-\lambda}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 \mathrm{e}^{\sqrt{-\lambda}} - \frac{c_2 \mathrm{e}^{-\sqrt{-\lambda}}}{2\sqrt{-\lambda}} \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{2c_1\sqrt{-\lambda} - c_2}{2\sqrt{-\lambda}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

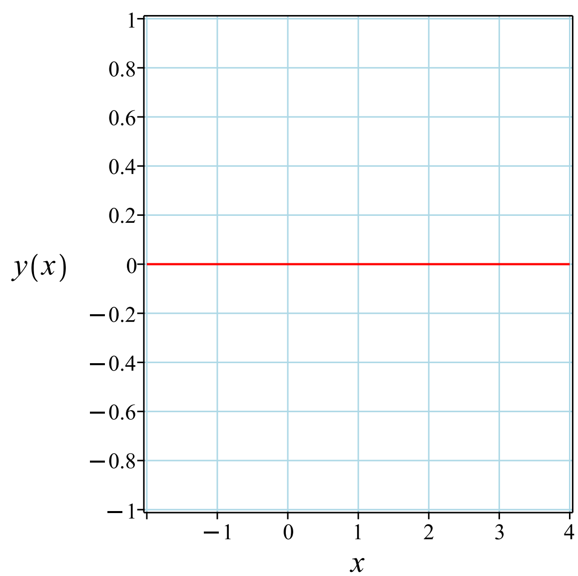


Figure 812: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

22.2.4 Maple step by step solution

Let's solve

$[y'' + \lambda y = 0, y(0) = 0, y(1) = 0]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + \lambda = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4\lambda})}{2}$

- Roots of the characteristic polynomial

  $r = (\sqrt{-\lambda}, -\sqrt{-\lambda})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\sqrt{-\lambda}\, x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\sqrt{-\lambda}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\sqrt{-\lambda}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}\, x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$2)+lambda*y(x)=0,y(0) = 0, y(1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
DSolve[{y''[x]+\[Lambda]*y[x]==0,{y[0]==0,y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \begin{cases} c_1 \sin\left(x\sqrt{\lambda}\right) & n \in \mathbb{Z} \land n \geq 1 \land \lambda = n^2\pi^2 \\ 0 & \text{True} \end{cases}$$

22.3 problem 708 (a)

Internal problem ID [15452]
Internal file name [`OUTPUT/15452_Wednesday_May_08_2024_03_59_06_PM_90510706/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 708 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With initial conditions

$$[y(0) = 0, y(2\pi) = 1]$$

22.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = \mathrm{e}^{x} c_1 + \mathrm{e}^{-x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x} c_1 + \mathrm{e}^{-x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$1 = \mathrm{e}^{2\pi} c_1 + \mathrm{e}^{-2\pi} c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{e^{2\pi}}{e^{4\pi} - 1} \\ c_2 &= -\frac{e^{2\pi}}{e^{4\pi} - 1} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{e^{2\pi - x} - e^{6\pi - x} + e^{x + 6\pi} - e^{x + 2\pi}}{e^{8\pi} - 2\,e^{4\pi} + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{2\pi - x} - e^{6\pi - x} + e^{x + 6\pi} - e^{x + 2\pi}}{e^{8\pi} - 2\,e^{4\pi} + 1} \tag{1}$$

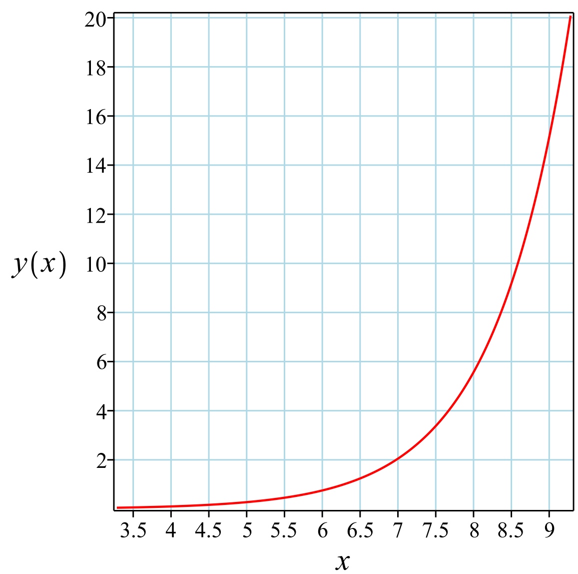


Figure 813: Solution plot

Verification of solutions

$$y = \frac{e^{2\pi - x} - e^{6\pi - x} + e^{x + 6\pi} - e^{x + 2\pi}}{e^{8\pi} - 2\,e^{4\pi} + 1}$$

Verified OK.

22.3.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y + \sqrt{y^2 + 2c_1} = e^{x + c_2}$$

Which simplifies to

$$y + \sqrt{y^2 + 2c_1} = e^x c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$-\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x} c_5$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(\mathrm{e}^{2x} c_3^2 - 2c_1\right)\mathrm{e}^{-x}}{2c_3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$1 = \frac{c_3^2 \mathrm{e}^{2\pi} - 2\,\mathrm{e}^{-2\pi} c_1}{2c_3} \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_3^2 - 2c_1}{2c_3} \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = \frac{2\,\mathrm{e}^{4\pi}}{\left(\mathrm{e}^{4\pi}-1\right)^2}$$
$$c_3 = \frac{2\,\mathrm{e}^{2\pi}}{\mathrm{e}^{4\pi}-1}$$

Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{-2\pi-x}\mathrm{e}^{4\pi} - 2\,\mathrm{e}^{-2\pi-x}\mathrm{e}^{8\pi} + 2\,\mathrm{e}^{-2\pi-x}\mathrm{e}^{8\pi+2x} - \mathrm{e}^{-2\pi-x}\mathrm{e}^{4\pi+2x} - \mathrm{e}^{-2\pi-x}\mathrm{e}^{12\pi+2x} + \mathrm{e}^{-2\pi-x}\mathrm{e}^{12\pi}}{3\,\mathrm{e}^{8\pi} - \mathrm{e}^{8\pi}\mathrm{e}^{4\pi} - 3\,\mathrm{e}^{4\pi} + 1}$$

Which simplifies to

$$y = \frac{-\mathrm{e}^{2\pi-x} + 2\,\mathrm{e}^{6\pi-x} - 2\,\mathrm{e}^{x+6\pi} + \mathrm{e}^{x+2\pi} + \mathrm{e}^{10\pi+x} - \mathrm{e}^{10\pi-x}}{-3\,\mathrm{e}^{8\pi} + \mathrm{e}^{12\pi} + 3\,\mathrm{e}^{4\pi} - 1}$$

Looking at the Second solution

$$y = -\frac{\left(2c_1 \mathrm{e}^{2x} c_5^2 - 1\right)\mathrm{e}^{-x}}{2c_5} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$1 = \frac{-2c_1 c_5^2 \mathrm{e}^{2\pi} + \mathrm{e}^{-2\pi}}{2c_5} \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{-2c_1 c_5^2 + 1}{2c_5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{2\,\mathrm{e}^{4\pi}}{\left(\mathrm{e}^{4\pi} - 1\right)^2} \\ c_5 &= -\frac{\mathrm{e}^{2\pi}}{2} + \frac{\mathrm{e}^{-2\pi}}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{2\,\mathrm{e}^{-x}\mathrm{e}^{4\pi} - \mathrm{e}^{-x}\mathrm{e}^{8\pi} + \mathrm{e}^{-x}\mathrm{e}^{8\pi+2x} - 2\,\mathrm{e}^{-x}\mathrm{e}^{4\pi+2x} + \mathrm{e}^{-x}\mathrm{e}^{2x} - \mathrm{e}^{-x}}{-3\,\mathrm{e}^{6\pi} + \mathrm{e}^{10\pi} + 3\,\mathrm{e}^{2\pi} - \mathrm{e}^{-2\pi}}$$

Which simplifies to

$$y = \frac{\mathrm{e}^{2\pi-x}\left(\mathrm{e}^{8\pi+2x} - 2\,\mathrm{e}^{4\pi+2x} + \mathrm{e}^{2x} + 2\,\mathrm{e}^{4\pi} - \mathrm{e}^{8\pi} - 1\right)}{-3\,\mathrm{e}^{8\pi} + \mathrm{e}^{12\pi} + 3\,\mathrm{e}^{4\pi} - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-\mathrm{e}^{2\pi-x} + 2\,\mathrm{e}^{6\pi-x} - 2\,\mathrm{e}^{x+6\pi} + \mathrm{e}^{x+2\pi} + \mathrm{e}^{10\pi+x} - \mathrm{e}^{10\pi-x}}{-3\,\mathrm{e}^{8\pi} + \mathrm{e}^{12\pi} + 3\,\mathrm{e}^{4\pi} - 1} \tag{1}$$

$$y = \frac{\mathrm{e}^{2\pi-x}\left(\mathrm{e}^{8\pi+2x} - 2\,\mathrm{e}^{4\pi+2x} + \mathrm{e}^{2x} + 2\,\mathrm{e}^{4\pi} - \mathrm{e}^{8\pi} - 1\right)}{-3\,\mathrm{e}^{8\pi} + \mathrm{e}^{12\pi} + 3\,\mathrm{e}^{4\pi} - 1} \tag{2}$$

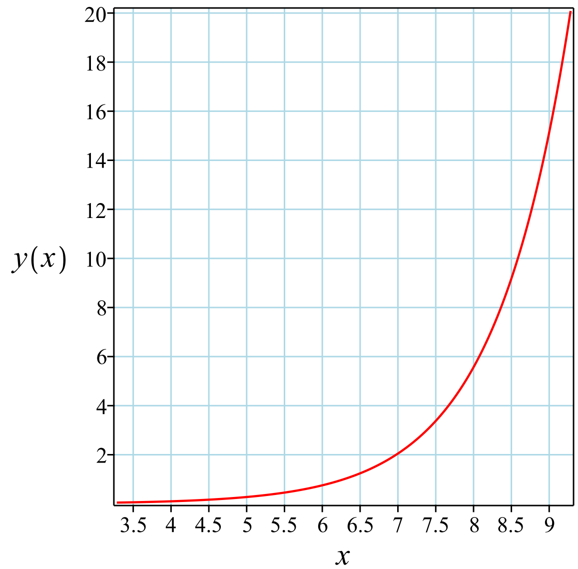


Figure 814: Solution plot

Verification of solutions

$$y = \frac{-e^{2\pi - x} + 2\,e^{6\pi - x} - 2\,e^{x + 6\pi} + e^{x + 2\pi} + e^{10\pi + x} - e^{10\pi - x}}{-3\,e^{8\pi} + e^{12\pi} + 3\,e^{4\pi} - 1}$$

Verified OK.

$$y = \frac{e^{2\pi - x}\left(e^{8\pi + 2x} - 2\,e^{4\pi + 2x} + e^{2x} + 2\,e^{4\pi} - e^{8\pi} - 1\right)}{-3\,e^{8\pi} + e^{12\pi} + 3\,e^{4\pi} - 1}$$

Verified OK.

22.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 688: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\
&= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$
1 = \mathrm{e}^{-2\pi} c_1 + \frac{c_2 \mathrm{e}^{2\pi}}{2} \tag{1A}
$$

substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{\mathrm{e}^{2\pi}}{\mathrm{e}^{4\pi} - 1} \\
c_2 &= \frac{2\,\mathrm{e}^{2\pi}}{\mathrm{e}^{4\pi} - 1}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{\mathrm{e}^{2\pi - x} - \mathrm{e}^{6\pi - x} + \mathrm{e}^{x + 6\pi} - \mathrm{e}^{x + 2\pi}}{\mathrm{e}^{8\pi} - 2\,\mathrm{e}^{4\pi} + 1}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{2\pi - x} - \mathrm{e}^{6\pi - x} + \mathrm{e}^{x + 6\pi} - \mathrm{e}^{x + 2\pi}}{\mathrm{e}^{8\pi} - 2\,\mathrm{e}^{4\pi} + 1} \tag{1}
$$

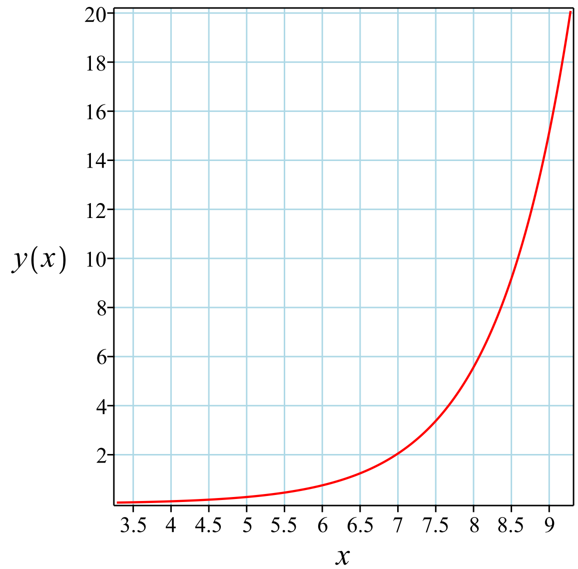


Figure 815: Solution plot

Verification of solutions

$$y = \frac{e^{2\pi - x} - e^{6\pi - x} + e^{x+6\pi} - e^{x+2\pi}}{e^{8\pi} - 2\,e^{4\pi} + 1}$$

Verified OK.

22.3.4 Maple step by step solution

Let's solve

$[y'' - y = 0, y(0) = 0, y(2\pi) = 1]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2 e^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 27

```
dsolve([diff(y(x),x$2)-y(x)=0,y(0) = 0, y(2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e^{-x+2\pi}(e^{2x} - 1)}{e^{4\pi} - 1}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 31

```
DSolve[{y''[x]-y[x]==0,{y[0]==0,y[2*Pi]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2\pi - x}(e^{2x} - 1)}{e^{4\pi} - 1}$$

22.4 problem 708 (b)

Internal problem ID [15453]
Internal file name [`OUTPUT/15453_Wednesday_May_08_2024_03_59_08_PM_64885066/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 708 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + y = 0}$$

With initial conditions

$$[y(0) = 0, y(2\pi) = 1]$$

22.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$1 = c_1 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

22.4.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\,dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1}}dy = \int dx$$
$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$\arctan\left(\frac{1}{\sqrt{-1+2c_1}}\right) = 2\pi + c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Looking at the Second solution

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$-\arctan\left(\frac{1}{\sqrt{-1+2c_1}}\right) = 2\pi + c_3 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

22.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 690: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 2\pi$ in the above gives

$$1 = c_1 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

22.4.4 Maple step by step solution

Let's solve

$[y'' + y = 0, y(0) = 0, y(2\pi) = 1]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x$2)+y(x)=0,y(0) = 0, y(2*Pi) = 1],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+y[x]==0,{y[0]==0,y[2*Pi]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$\{\}$$

22.5 problem 710

Internal problem ID [15454]
Internal file name [`OUTPUT/15454_Wednesday_May_08_2024_03_59_09_PM_89884284/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 710.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{yy'' + y'^2 = -1}$$

With initial conditions

$$[y(0) = 1, y(1) = 2]$$

22.5.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = \int (-1)\, dx$$
$$yy' = -x + c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{-x + c_1}{y} \end{aligned}$$

Where $f(x) = -x + c_1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = -x + c_1\, dx$$
$$\int \frac{1}{\frac{1}{y}}\, dy = \int -x + c_1\, dx$$
$$\frac{y^2}{2} = -\frac{1}{2}x^2 + c_1 x + c_2$$

The solution is

$$\frac{y^2}{2} + \frac{x^2}{2} - c_1 x - c_2 = 0$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\frac{y^2}{2} + \frac{x^2}{2} - c_1 x - c_2 = 0 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$\frac{5}{2} - c_1 - c_2 = 0 \tag{1A}$$

substituting $y = 1$ and $x = 0$ in the above gives

$$\frac{1}{2} - c_2 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= \frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$\frac{y^2}{2} + \frac{x^2}{2} - 2x - \frac{1}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} + \frac{x^2}{2} - 2x - \frac{1}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} + \frac{x^2}{2} - 2x - \frac{1}{2} = 0$$

Verified OK.

22.5.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p^2+1}{yp}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = \frac{p^2+1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{p^2+1}{p}}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{\frac{p^2+1}{p}}\,dp &= \int -\frac{1}{y}\,dy \\
\frac{\ln(p^2+1)}{2} &= -\ln(y) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{p^2+1} = \mathrm{e}^{-\ln(y)+c_1}
$$

Which simplifies to

$$
\sqrt{p^2+1} = \frac{c_2}{y}
$$

Which simplifies to

$$
\sqrt{p(y)^2+1} = \frac{c_2\mathrm{e}^{c_1}}{y}
$$

The solution is

$$
\sqrt{p(y)^2+1} = \frac{c_2\mathrm{e}^{c_1}}{y}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\sqrt{y'^2+1} = \frac{c_2\mathrm{e}^{c_1}}{y}
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{c_2^2 e^{2c_1} - y^2}}{y} \tag{1}$$

$$y' = -\frac{\sqrt{c_2^2 e^{2c_1} - y^2}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{\sqrt{c_2^2 e^{2c_1} - y^2}} dy = \int dx$$

$$-\sqrt{c_2^2 e^{2c_1} - y^2} = x + c_3$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$-\sqrt{c_2^2 e^{2c_1} - 4} = 1 + c_3$$

$$c_1 = \frac{\ln\left(\frac{c_3^2 + 2c_3 + 5}{c_2^2}\right)}{2}$$

Substituting c_1 found above in the general solution gives

$$-\sqrt{c_3^2 - y^2 + 2c_3 + 5} = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{c_2^2 e^{2c_1} - y^2}} dy = \int dx$$

$$\sqrt{c_2^2 e^{2c_1} - y^2} = x + c_4$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{c_2^2 e^{2c_1} - 4} = 1 + c_4$$

$$c_1 = \frac{\ln\left(\frac{c_4^2+2c_4+5}{c_2^2}\right)}{2}$$

Substituting c_1 found above in the general solution gives

$$\sqrt{c_4^2 - y^2 + 2c_4 + 5} = x + c_4$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$-\sqrt{c_3^2 - y^2 + 2c_3 + 5} = x + c_3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$-\operatorname{csgn}(1 + c_3)(1 + c_3) = 1 + c_3 \tag{1A}$$

substituting $y = 1$ and $x = 0$ in the above gives

$$-\sqrt{c_3^2 + 2c_3 + 4} = c_3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_3\}$. Solving for the constants gives

$$c_3 = -2$$

Substituting these values back in above solution results in

$$-\sqrt{5 - y^2} = x - 2$$

Looking at the Second solution

$$\sqrt{c_4^2 - y^2 + 2c_4 + 5} = x + c_4 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$\operatorname{csgn}(1 + c_4)(1 + c_4) = 1 + c_4 \tag{1A}$$

substituting $y = 1$ and $x = 0$ in the above gives

$$\sqrt{c_4^2 + 2c_4 + 4} = c_4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_4\}$. There is no solution for the constants of integrations. This solution is removed.

Summary

The solution(s) found are the following

$$
-\sqrt{5-y^2} = x-2 \tag{1}
$$

Verification of solutions

$$
-\sqrt{5-y^2} = x-2
$$

Verified OK.

22.5.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
yy'' + y'^2 = -1
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(yy'' + y'^2\right) dx &= \int (-1)\, dx \\
yy' &= -x + c_1
\end{aligned}
$$

Which is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{-x+c_1}{y}
\end{aligned}
$$

Where $f(x) = -x + c_1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\, dy &= -x + c_1\, dx \\
\int \frac{1}{\frac{1}{y}}\, dy &= \int -x + c_1\, dx \\
\frac{y^2}{2} &= -\frac{1}{2}x^2 + c_1 x + c_2
\end{aligned}
$$

The solution is

$$\frac{y^2}{2}+\frac{x^2}{2}-c_1x-c_2=0$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\frac{y^2}{2}+\frac{x^2}{2}-c_1x-c_2=0 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=2$ and $x=1$ in the above gives

$$\frac{5}{2}-c_1-c_2=0 \tag{1A}$$

substituting $y=1$ and $x=0$ in the above gives

$$\frac{1}{2}-c_2=0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1,c_2\}$. Solving for the constants gives

$$c_1=2$$
$$c_2=\frac{1}{2}$$

Substituting these values back in above solution results in

$$\frac{y^2}{2}+\frac{x^2}{2}-2x-\frac{1}{2}=0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2}+\frac{x^2}{2}-2x-\frac{1}{2}=0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2}+\frac{x^2}{2}-2x-\frac{1}{2}=0$$

Verified OK.

22.5.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= y \\ a_1 &= y' \\ a_0 &= 1\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\ \int y\, dy' + \int y'\, dy + \int 1\, dx &= c_1\end{aligned}$$

Which results in

$$2yy' + x = c_1$$

Which is now solved In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{-\frac{x}{2} + \frac{c_1}{2}}{y}\end{aligned}$$

Where $f(x) = -\frac{x}{2} + \frac{c_1}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\, dy &= -\frac{x}{2} + \frac{c_1}{2}\, dx \\ \int \frac{1}{\frac{1}{y}}\, dy &= \int -\frac{x}{2} + \frac{c_1}{2}\, dx \\ \frac{y^2}{2} &= -\frac{1}{4}x^2 + \frac{1}{2}c_1 x + c_2\end{aligned}$$

The solution is

$$\frac{y^2}{2}+\frac{x^2}{4}-\frac{c_1 x}{2}-c_2=0$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\frac{y^2}{2}+\frac{x^2}{4}-\frac{c_1 x}{2}-c_2=0 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=2$ and $x=1$ in the above gives

$$\frac{9}{4}-\frac{c_1}{2}-c_2=0 \tag{1A}$$

substituting $y=1$ and $x=0$ in the above gives

$$\frac{1}{2}-c_2=0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1=\frac{7}{2}$$
$$c_2=\frac{1}{2}$$

Substituting these values back in above solution results in

$$\frac{y^2}{2}+\frac{x^2}{4}-\frac{7x}{4}-\frac{1}{2}=0$$

Summary
The solution(s) found are the following

$$\frac{y^2}{2}+\frac{x^2}{4}-\frac{7x}{4}-\frac{1}{2}=0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2}+\frac{x^2}{4}-\frac{7x}{4}-\frac{1}{2}=0$$

Warning, solution could not be verified

22.5.5 Maple step by step solution

Let's solve

$[yy'' + y'^2 = -1, y(0) = 1, y(1) = 2]$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\, , y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + u(y)^2 = -1$

- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2-1} = \frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2-1}dy = \int \frac{1}{y}dy + c_1$

- Evaluate integral

  $-\frac{\ln\left(u(y)^2+1\right)}{2} = \ln(y) + c_1$

- Solve for $u(y)$

  $\left\{u(y) = \frac{\sqrt{1-y^2(\mathrm{e}^{c_1})^2}}{\mathrm{e}^{c_1}y}, u(y) = -\frac{\sqrt{1-y^2(\mathrm{e}^{c_1})^2}}{\mathrm{e}^{c_1}y}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{\sqrt{1-y^2(\mathrm{e}^{c_1})^2}}{\mathrm{e}^{c_1}y}$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}}{\mathrm{e}^{c_1} y}$$

- Separate variables

$$\frac{y' y}{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}} = \frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y}{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}} dx = \int \frac{1}{\mathrm{e}^{c_1}} dx + c_2$$

- Evaluate integral

$$-\frac{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}}{(\mathrm{e}^{c_1})^2} = \frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = \frac{\sqrt{1-(\mathrm{e}^{c_1})^4 c_2^2 - 2(\mathrm{e}^{c_1})^3 c_2 x - (\mathrm{e}^{c_1})^2 x^2}}{\mathrm{e}^{c_1}}, y = -\frac{\sqrt{1-(\mathrm{e}^{c_1})^4 c_2^2 - 2(\mathrm{e}^{c_1})^3 c_2 x - (\mathrm{e}^{c_1})^2 x^2}}{\mathrm{e}^{c_1}}\right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}}{\mathrm{e}^{c_1} y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}}{\mathrm{e}^{c_1} y}$$

- Separate variables

$$\frac{y' y}{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}} = -\frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y}{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}} dx = \int -\frac{1}{\mathrm{e}^{c_1}} dx + c_2$$

- Evaluate integral

$$-\frac{\sqrt{1-y^2 (\mathrm{e}^{c_1})^2}}{(\mathrm{e}^{c_1})^2} = -\frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = \frac{\sqrt{1-(\mathrm{e}^{c_1})^4 c_2^2 + 2(\mathrm{e}^{c_1})^3 c_2 x - (\mathrm{e}^{c_1})^2 x^2}}{\mathrm{e}^{c_1}}, y = -\frac{\sqrt{1-(\mathrm{e}^{c_1})^4 c_2^2 + 2(\mathrm{e}^{c_1})^3 c_2 x - (\mathrm{e}^{c_1})^2 x^2}}{\mathrm{e}^{c_1}}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.781 (sec). Leaf size: 16

```
dsolve([y(x)*diff(y(x),x$2)+diff(y(x),x)^2+1=0,y(0) = 1, y(1) = 2],y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2 + 4x + 1}$$

✓ Solution by Mathematica

Time used: 12.271 (sec). Leaf size: 19

```
DSolve[{y[x]*y''[x]+y'[x]^2+1==0,{y[0]==1,y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{-x^2 + 4x + 1}$$

22.6 problem 711

Internal problem ID [15455]
Internal file name [`OUTPUT/15455_Wednesday_May_08_2024_03_59_14_PM_49463104/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 711.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With initial conditions

$$\left[y(0) = 0, y\left(\frac{\pi}{2}\right) = \alpha\right]$$

22.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \alpha$ and $x = \frac{\pi}{2}$ in the above gives

$$\alpha = c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \alpha \end{aligned}$$

Substituting these values back in above solution results in

$$y = \alpha \sin(x)$$

Summary

The solution(s) found are the following

$$y = \alpha \sin(x) \tag{1}$$

Verification of solutions

$$y = \alpha \sin(x)$$

Verified OK.

22.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\,dx = 0$$

$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2+2c_1} \tag{1}$$

$$y' = -\sqrt{-y^2+2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \alpha$ and $x = \frac{\pi}{2}$ in the above gives

$$\arctan\left(\frac{\alpha}{\sqrt{-\alpha^2+2c_1}}\right) = \frac{\pi}{2} + c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Warning, unable to solve for constants of integrations.

Looking at the Second solution

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \alpha$ and $x = \frac{\pi}{2}$ in the above gives

$$-\arctan\left(\frac{\alpha}{\sqrt{-\alpha^2+2c_1}}\right) = \frac{\pi}{2} + c_3 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

22.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 693: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x) + c_2 \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \alpha$ and $x = \frac{\pi}{2}$ in the above gives

$$
\alpha = c_2 \tag{1A}
$$

substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= \alpha
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \alpha \sin(x)
$$

Summary

The solution(s) found are the following

$$
y = \alpha \sin(x) \tag{1}
$$

Verification of solutions

$$
y = \alpha \sin(x)
$$

Verified OK.

22.6.4 Maple step by step solution

Let's solve

$\left[y''+y=0, y(0)=0, y\left(\frac{\pi}{2}\right)=\alpha\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x)=\cos(x)$

- 2nd solution of the ODE

  $y_2(x)=\sin(x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\cos(x)+c_2\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve([diff(y(x),x$2)+y(x)=0,y(0) = 0, y(1/2*Pi) = alpha],y(x), singsol=all)
```

$$y(x) = \sin(x)\,\alpha$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 9

```
DSolve[{y''[x]+y[x]==0,{y[0]==0,y[Pi/2]==\[Alpha]}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \alpha \sin(x)$$

22.7 problem 712

Internal problem ID [15456]
Internal file name [`OUTPUT/15456_Wednesday_May_08_2024_03_59_15_PM_94501898/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 712.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With initial conditions

$$[y(0) = 0, y'(1) = 1]$$

22.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{x} c_1 + \mathrm{e}^{-x} c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x} c_1 + \mathrm{e}^{-x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x} c_1 - e^{-x} c_2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = e c_1 - c_2 e^{-1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{e}{e^2 + 1}$$
$$c_2 = -\frac{e}{e^2 + 1}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2 e^2 + 1}$$

Which simplifies to

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2 e^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2 e^2 + 1} \tag{1}$$

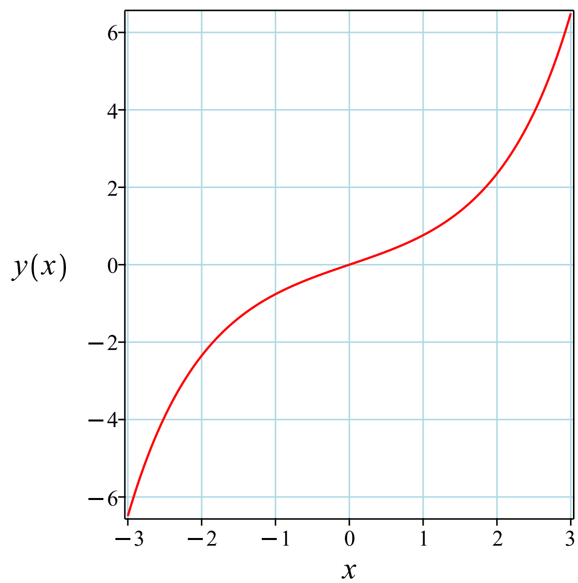


Figure 816: Solution plot

Verification of solutions

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2e^2 + 1}$$

Verified OK.

22.7.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y + \sqrt{y^2 + 2c_1} = e^{x + c_2}$$

Which simplifies to

$$y + \sqrt{y^2 + 2c_1} = e^x c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = e^{x}c_5$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(e^{2x}c_3^2 - 2c_1\right)e^{-x}}{2c_3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_3^2 - 2c_1}{2c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}c_3 - \frac{\left(e^{2x}c_3^2 - 2c_1\right)e^{-x}}{2c_3}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{ec_3^2 + 2\,e^{-1}c_1}{2c_3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = \frac{2\,e^2}{\left(e^2+1\right)^2}$$
$$c_3 = \frac{2\,e}{e^2+1}$$

Substituting these values back in above solution results in

$$y = \frac{-\mathrm{e}^{-x-1}\mathrm{e}^{2} + \mathrm{e}^{-x-1}\mathrm{e}^{6+2x} - \mathrm{e}^{-x-1}\mathrm{e}^{6} + 2\,\mathrm{e}^{-x-1}\mathrm{e}^{2x+4} + \mathrm{e}^{-x-1}\mathrm{e}^{2x+2} - 2\,\mathrm{e}^{-x-1}\mathrm{e}^{4}}{3\,\mathrm{e}^{4} + \mathrm{e}^{4}\mathrm{e}^{2} + 3\,\mathrm{e}^{2} + 1}$$

Which simplifies to

$$y = \frac{-\mathrm{e}^{1-x} + \mathrm{e}^{5+x} - \mathrm{e}^{5-x} + 2\,\mathrm{e}^{x+3} + \mathrm{e}^{x+1} - 2\,\mathrm{e}^{3-x}}{3\,\mathrm{e}^{4} + \mathrm{e}^{6} + 3\,\mathrm{e}^{2} + 1}$$

Looking at the Second solution

$$y = -\frac{(2c_1\mathrm{e}^{2x}c_5^2 - 1)\,\mathrm{e}^{-x}}{2c_5} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{-2c_1c_5^2 + 1}{2c_5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1\mathrm{e}^{x}c_5 + \frac{(2c_1\mathrm{e}^{2x}c_5^2 - 1)\,\mathrm{e}^{-x}}{2c_5}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{-2\,\mathrm{e}c_1c_5^2 - \mathrm{e}^{-1}}{2c_5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = \frac{2\,\mathrm{e}^{2}}{(\mathrm{e}^{2} + 1)^2}$$
$$c_5 = -\frac{\mathrm{e}}{2} - \frac{\mathrm{e}^{-1}}{2}$$

Substituting these values back in above solution results in

$$y = \frac{\mathrm{e}^{-x}\mathrm{e}^{2x} - 2\,\mathrm{e}^{-x}\mathrm{e}^{2} + \mathrm{e}^{-x}\mathrm{e}^{2x+4} + 2\,\mathrm{e}^{-x}\mathrm{e}^{2x+2} - \mathrm{e}^{-x}\mathrm{e}^{4} - \mathrm{e}^{-x}}{3\,\mathrm{e}^{3} + \mathrm{e}^{5} + 3\,\mathrm{e} + \mathrm{e}^{-1}}$$

Which simplifies to

$$y = \frac{\mathrm{e}^{1-x}(\mathrm{e}^{2x+4} + 2\,\mathrm{e}^{2x+2} + \mathrm{e}^{2x} - 2\,\mathrm{e}^{2} - \mathrm{e}^{4} - 1)}{3\,\mathrm{e}^{4} + \mathrm{e}^{6} + 3\,\mathrm{e}^{2} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{1-x} + e^{5+x} - e^{5-x} + 2\,e^{x+3} + e^{x+1} - 2\,e^{3-x}}{3\,e^{4} + e^{6} + 3\,e^{2} + 1} \tag{1}$$

$$y = \frac{e^{1-x}\left(e^{2x+4} + 2\,e^{2x+2} + e^{2x} - 2\,e^{2} - e^{4} - 1\right)}{3\,e^{4} + e^{6} + 3\,e^{2} + 1} \tag{2}$$

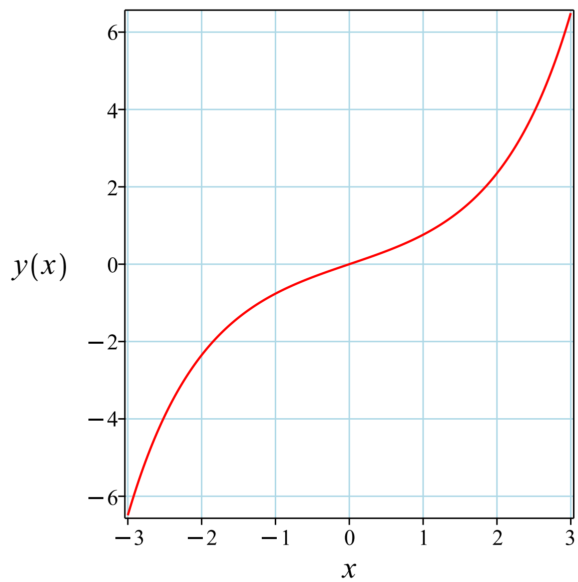


Figure 817: Solution plot

Verification of solutions

$$y = \frac{-e^{1-x} + e^{5+x} - e^{5-x} + 2\,e^{x+3} + e^{x+1} - 2\,e^{3-x}}{3\,e^{4} + e^{6} + 3\,e^{2} + 1}$$

Verified OK.

$$y = \frac{e^{1-x}\left(e^{2x+4} + 2\,e^{2x+2} + e^{2x} - 2\,e^{2} - e^{4} - 1\right)}{3\,e^{4} + e^{6} + 3\,e^{2} + 1}$$

Verified OK.

22.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 695: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=0$ and $x=0$ in the above gives

$$
0 = c_1 + \frac{c_2}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{c_2 e^x}{2}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -e^{-1} c_1 + \frac{c_2 e}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{e}{e^2 + 1}$$
$$c_2 = \frac{2\,e}{e^2 + 1}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2\,e^2 + 1}$$

Which simplifies to

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2\,e^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2\,e^2 + 1} \tag{1}$$

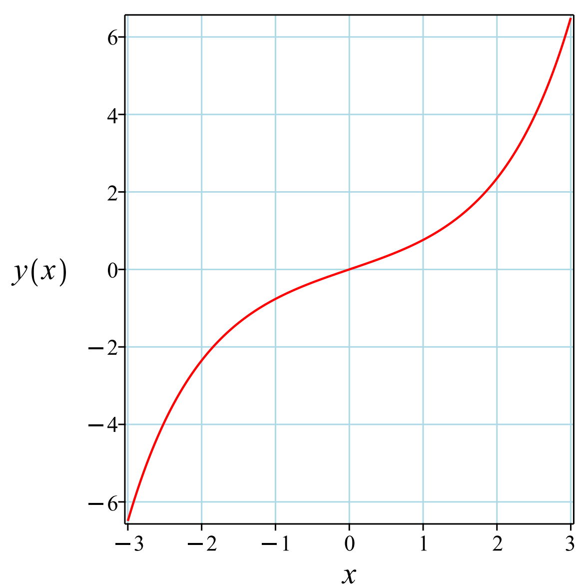


Figure 818: Solution plot

Verification of solutions

$$y = \frac{-e^{3-x} - e^{1-x} + e^{x+3} + e^{x+1}}{e^4 + 2e^2 + 1}$$

Verified OK.

22.7.4 Maple step by step solution

Let's solve

$\left[y'' - y = 0, y(0) = 0, y'\Big|_{\{x=1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the ODE

  $y_2(x) = e^x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^x$

□ Check validity of solution $y = c_1 e^{-x} + c_2 e^x$

- ◦ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$
- ◦ Compute derivative of the solution

  $y' = -c_1 e^{-x} + c_2 e^x$

- Use the initial condition $y'\Big|_{\{x=1\}} = 1$

  $1 = -\mathrm{e}^{-1} c_1 + c_2 \mathrm{e}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{\mathrm{e}+\mathrm{e}^{-1}}, c_2 = \frac{1}{\mathrm{e}+\mathrm{e}^{-1}}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\mathrm{e}^{1-x}(\mathrm{e}^{2x}-1)}{\mathrm{e}^2+1}$

- Solution to the IVP

  $y = \frac{\mathrm{e}^{1-x}(\mathrm{e}^{2x}-1)}{\mathrm{e}^2+1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)-y(x)=0,y(0) = 0, D(y)(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{1-x}(\mathrm{e}^{2x} - 1)}{\mathrm{e}^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 27

```
DSolve[{y''[x]-y[x]==0,{y[0]==0,y'[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{1-x}(e^{2x} - 1)}{1 + e^2}$$

22.8 problem 713

Internal problem ID [15457]
Internal file name [OUTPUT/15457_Wednesday_May_08_2024_03_59_17_PM_2608336/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 713.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y''-2y'+2y=0}$$

With initial conditions

$$[y(0)=0, y'(\pi)=\mathrm{e}^{\pi}]$$

22.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=-2, C=2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2\mathrm{e}^{\lambda x}-2\lambda\,\mathrm{e}^{\lambda x}+2\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2-2\lambda+2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^x(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^x(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^x (c_1 \cos(x) + c_2 \sin(x)) + e^x (-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = e^\pi$ and $x = \pi$ in the above gives

$$e^\pi = (-c_1 - c_2) e^\pi \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -e^x \sin(x)$$

Summary

The solution(s) found are the following

$$y = -e^x \sin(x) \tag{1}$$

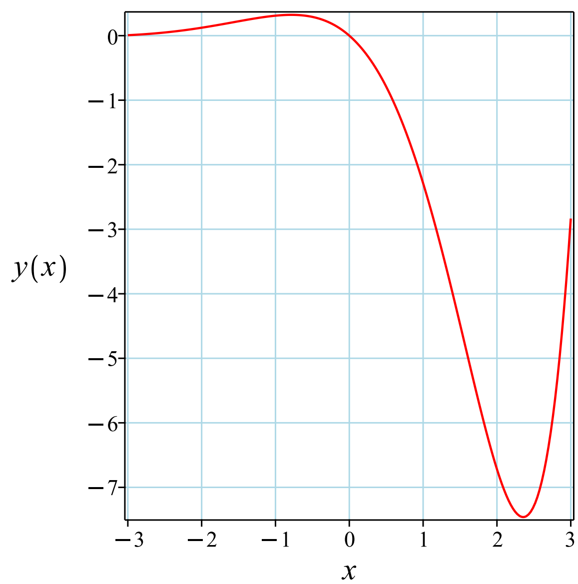


Figure 819: Solution plot

Verification of solutions

$$y = -e^x \sin(x)$$

Verified OK.

22.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 697: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)\,\mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)\,\mathrm{e}^x) + c_2(\cos(x)\,\mathrm{e}^x(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x)\,\mathrm{e}^x + \mathrm{e}^x \sin(x)\,c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \sin(x)\,\mathrm{e}^x + c_1 \cos(x)\,\mathrm{e}^x + \mathrm{e}^x \sin(x)\,c_2 + \mathrm{e}^x \cos(x)\,c_2
$$

substituting $y' = \mathrm{e}^{\pi}$ and $x = \pi$ in the above gives

$$\mathrm{e}^{\pi} = (-c_1 - c_2)\,\mathrm{e}^{\pi} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -\mathrm{e}^{x} \sin(x)$$

Summary

The solution(s) found are the following

$$y = -\mathrm{e}^{x} \sin(x) \tag{1}$$

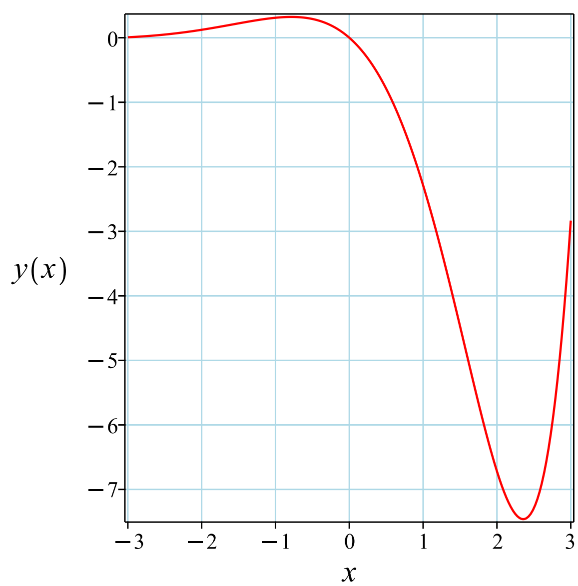


Figure 820: Solution plot

Verification of solutions

$$y = -\mathrm{e}^{x} \sin(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=0,y(0) = 0, D(y)(Pi) = exp(Pi)],y(x), singsol=a
```

$$y(x) = -e^x \sin(x)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 12

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==0,{y[0]==0,y'[Pi]==Exp[Pi]}},y[x],x,IncludeSingularSolutions
```

$$y(x) \rightarrow -e^x \sin(x)$$

22.9 problem 714

Internal problem ID [15458]
Internal file name [`OUTPUT/15458_Wednesday_May_08_2024_03_59_18_PM_96601685/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 714.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \alpha y' = 0}$$

With initial conditions

$$[y(0) = e^{\alpha}, y'(1) = 0]$$

22.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = \alpha, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \alpha\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\alpha\lambda + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = \alpha, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-\alpha}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{\alpha^2 - (4)(1)(0)}\\ &= -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2}}{2}\\ \lambda_2 &= -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{\alpha}{2} + \frac{\sqrt{\alpha^2}}{2}\\ \lambda_2 &= -\frac{\alpha}{2} - \frac{\sqrt{\alpha^2}}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2}}{2}\right)x} + c_2 e^{\left(-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2}}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 e^{\left(-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2}}{2}\right)x} + c_2 e^{\left(-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2}}{2}\right)x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\left(-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2}}{2}\right)x} + c_2 e^{\left(-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2}}{2}\right)x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = e^{\alpha}$ and $x = 0$ in the above gives

$$e^{\alpha} = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \left(-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2}}{2}\right) e^{\left(-\frac{\alpha}{2}+\frac{\sqrt{\alpha^2}}{2}\right)x} + c_2 \left(-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2}}{2}\right) e^{\left(-\frac{\alpha}{2}-\frac{\sqrt{\alpha^2}}{2}\right)x}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{\left(c_1(-1+\operatorname{csgn}(\alpha))\, e^{\frac{\alpha(-1+\operatorname{csgn}(\alpha))}{2}} - c_2 e^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}}(1+\operatorname{csgn}(\alpha))\right)\alpha}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{(1+\operatorname{csgn}(\alpha))\, e^{-\frac{\alpha(-1+\operatorname{csgn}(\alpha))}{2}}}{\operatorname{csgn}(\alpha)\, e^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}} + \operatorname{csgn}(\alpha)\, e^{\frac{\alpha(-1+\operatorname{csgn}(\alpha))}{2}} + e^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}} - e^{\frac{\alpha(-1+\operatorname{csgn}(\alpha))}{2}}}$$

$$c_2 = \frac{(-1+\operatorname{csgn}(\alpha))\, e^{\alpha(1+\operatorname{csgn}(\alpha))}}{\operatorname{csgn}(\alpha)\, e^{\operatorname{csgn}(\alpha)\alpha} - e^{\operatorname{csgn}(\alpha)\alpha} + \operatorname{csgn}(\alpha) + 1}$$

Substituting these values back in above solution results in

$$y = \frac{2\, e^{-\frac{\alpha(x\ \operatorname{csgn}(\alpha)-3\ \operatorname{csgn}(\alpha)+x-1)}{2}} + 2\, e^{\frac{\alpha(x-1)}{}}}{\operatorname{csgn}(\alpha)^2 e^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}} e^{\operatorname{csgn}(\alpha)\alpha} + \operatorname{csgn}(\alpha)^2 e^{\frac{\alpha(-1+\operatorname{csgn}(\alpha))}{2}} e^{\operatorname{csgn}(\alpha)\alpha} + \operatorname{csgn}(\alpha)^2 e^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}} + \operatorname{csgn}(\alpha)^2 e^{\frac{\alpha(}{}}}$$

Which simplifies to

$$y = \frac{(1-\operatorname{csgn}(\alpha))\, e^{-\frac{((-3+x)\operatorname{csgn}(\alpha)+x-1)\alpha}{2}} + (1+\operatorname{csgn}(\alpha))\, e^{\frac{\alpha(x-1)(-1+\operatorname{csgn}(\alpha))}{2}}}{(1-\operatorname{csgn}(\alpha))\, e^{\frac{\alpha(3\ \operatorname{csgn}(\alpha)-1)}{2}} + e^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}}(1+\operatorname{csgn}(\alpha))}$$

Summary

The solution(s) found are the following

$$y = \frac{(1-\operatorname{csgn}(\alpha))\,\mathrm{e}^{-\frac{((-3+x)\operatorname{csgn}(\alpha)+x-1)\alpha}{2}} + (1+\operatorname{csgn}(\alpha))\,\mathrm{e}^{\frac{\alpha(x-1)(-1+\operatorname{csgn}(\alpha))}{2}}}{(1-\operatorname{csgn}(\alpha))\,\mathrm{e}^{\frac{\alpha(3\operatorname{csgn}(\alpha)-1)}{2}} + \mathrm{e}^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}}(1+\operatorname{csgn}(\alpha))} \tag{1}$$

Verification of solutions

$$y = \frac{(1-\operatorname{csgn}(\alpha))\,\mathrm{e}^{-\frac{((-3+x)\operatorname{csgn}(\alpha)+x-1)\alpha}{2}} + (1+\operatorname{csgn}(\alpha))\,\mathrm{e}^{\frac{\alpha(x-1)(-1+\operatorname{csgn}(\alpha))}{2}}}{(1-\operatorname{csgn}(\alpha))\,\mathrm{e}^{\frac{\alpha(3\operatorname{csgn}(\alpha)-1)}{2}} + \mathrm{e}^{-\frac{\alpha(1+\operatorname{csgn}(\alpha))}{2}}(1+\operatorname{csgn}(\alpha))}$$

Verified OK.

22.9.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + \alpha y')\, dx = 0$$
$$\alpha y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\alpha y + c_1} dy = \int dx$$
$$-\frac{\ln(-\alpha y + c_1)}{\alpha} = x + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln(-\alpha y + c_1)}{\alpha}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$(-\alpha y + c_1)^{-\frac{1}{\alpha}} = \mathrm{e}^x c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{(\mathrm{e}^x c_3)^{-\alpha} - c_1}{\alpha} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \mathrm{e}^{\alpha}$ and $x = 0$ in the above gives

$$\mathrm{e}^{\alpha} = \frac{-c_3^{-\alpha} + c_1}{\alpha} \tag{1A}$$

Taking derivative of the solution gives

$$y' = (\mathrm{e}^{x} c_3)^{-\alpha}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = c_3^{-\alpha} \mathrm{e}^{-\alpha} \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \mathrm{e}^{\alpha}\alpha \\ c_3 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \lim_{c_3 \to 0} \frac{-(\mathrm{e}^{x} c_3)^{-\alpha} + \mathrm{e}^{\alpha}\alpha}{\alpha}$$

Summary
The solution(s) found are the following

$$y = \lim_{c_3 \to 0} \frac{-(\mathrm{e}^{x} c_3)^{-\alpha} + \mathrm{e}^{\alpha}\alpha}{\alpha} \tag{1}$$

Verification of solutions

$$y = \lim_{c_3 \to 0} \frac{-(\mathrm{e}^{x} c_3)^{-\alpha} + \mathrm{e}^{\alpha}\alpha}{\alpha}$$

Verified OK.

22.9.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + \alpha p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{\alpha p} dp = \int dx$$
$$-\frac{\ln(p)}{\alpha} = x + c_1$$

Raising both side to exponential gives

$$e^{-\frac{\ln(p)}{\alpha}} = e^{x+c_1}$$

Which simplifies to

$$p^{-\frac{1}{\alpha}} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2^{-\alpha} e^{-\alpha}$$

$$c_2 = 0$$

Substituting c_2 found above in the general solution gives

$$p(x) = \lim_{c_2 \to 0} (c_2 e^x)^{-\alpha}$$

But this does not satisfy the initial conditions. Hence no solution can be found.

22.9.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + \alpha y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + \alpha y')\, dx = 0$$

$$\alpha y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\alpha y + c_1} dy = \int dx$$
$$-\frac{\ln(-\alpha y + c_1)}{\alpha} = x + c_2$$

Raising both side to exponential gives

$$e^{-\frac{\ln(-\alpha y+c_1)}{\alpha}} = e^{x+c_2}$$

Which simplifies to

$$(-\alpha y + c_1)^{-\frac{1}{\alpha}} = e^x c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{(e^x c_3)^{-\alpha} - c_1}{\alpha} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = e^\alpha$ and $x = 0$ in the above gives

$$e^\alpha = \frac{-c_3^{-\alpha} + c_1}{\alpha} \tag{1A}$$

Taking derivative of the solution gives

$$y' = (e^x c_3)^{-\alpha}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = c_3^{-\alpha} e^{-\alpha} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = e^\alpha \alpha$$
$$c_3 = 0$$

Substituting these values back in above solution results in

$$y = \lim_{c_3 \to 0} \frac{-(e^x c_3)^{-\alpha} + e^\alpha \alpha}{\alpha}$$

Summary
The solution(s) found are the following

$$y = \lim_{c_3 \to 0} \frac{-(e^x c_3)^{-\alpha} + e^\alpha \alpha}{\alpha} \tag{1}$$

Verification of solutions

$$y = \lim_{c_3 \to 0} \frac{-(e^x c_3)^{-\alpha} + e^\alpha \alpha}{\alpha}$$

Verified OK.

22.9.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \alpha y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= \alpha \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{\alpha^2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= \alpha^2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{\alpha^2}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 698: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{\alpha^2}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\frac{x\sqrt{\alpha^2}}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2}\frac{\alpha}{1}\, dx} \\
&= z_1 e^{-\frac{x\alpha}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x\alpha}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x\alpha(-1+\operatorname{csgn}(\alpha))}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{\alpha}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x\alpha}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\operatorname{csgn}(\alpha)\, \mathrm{e}^{-x \operatorname{csgn}(\alpha)\alpha}}{\alpha}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x\alpha(-1+\operatorname{csgn}(\alpha))}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{x\alpha(-1+\operatorname{csgn}(\alpha))}{2}} \left(-\frac{\operatorname{csgn}(\alpha)\, \mathrm{e}^{-x \operatorname{csgn}(\alpha)\alpha}}{\alpha}\right)\right)
\end{aligned}
$$

Simplifying the solution $y = c_1 \mathrm{e}^{\frac{x\alpha(-1+\operatorname{csgn}(\alpha))}{2}} - \frac{c_2 \operatorname{csgn}(\alpha) \mathrm{e}^{-\frac{x\alpha(1+\operatorname{csgn}(\alpha))}{2}}}{\alpha}$ to $y = c_1 - \frac{c_2 \mathrm{e}^{-x\alpha}}{\alpha}$ Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 - \frac{c_2 \mathrm{e}^{-x\alpha}}{\alpha} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \mathrm{e}^{\alpha}$ and $x = 0$ in the above gives

$$
\mathrm{e}^{\alpha} = \frac{\alpha c_1 - c_2}{\alpha} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = c_2 e^{-x\alpha}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = c_2 e^{-\alpha} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= e^{\alpha} \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

$$y = e^{\alpha}$$

Summary
The solution(s) found are the following

$$y = e^{\alpha} \tag{1}$$

Verification of solutions

$$y = e^{\alpha}$$

Verified OK.

22.9.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \alpha \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$\alpha y + y' = c_1$$

We now have a first order ode to solve which is

$$\alpha y + y' = c_1$$

Integrating both sides gives

$$\int \frac{1}{-\alpha y + c_1} dy = \int dx$$
$$-\frac{\ln (-\alpha y + c_1)}{\alpha} = x + c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln(-\alpha y + c_1)}{\alpha}} = \mathrm{e}^{x + c_2}$$

Which simplifies to

$$(-\alpha y + c_1)^{-\frac{1}{\alpha}} = \mathrm{e}^{x} c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{(\mathrm{e}^{x} c_3)^{-\alpha} - c_1}{\alpha} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = e^{\alpha}$ and $x = 0$ in the above gives

$$e^{\alpha} = \frac{-c_3^{-\alpha} + c_1}{\alpha} \tag{1A}$$

Taking derivative of the solution gives

$$y' = (e^{x} c_3)^{-\alpha}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = c_3^{-\alpha} e^{-\alpha} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = e^{\alpha}\alpha$$
$$c_3 = 0$$

Substituting these values back in above solution results in

$$y = \lim_{c_3 \to 0} \frac{-(e^{x} c_3)^{-\alpha} + e^{\alpha}\alpha}{\alpha}$$

Summary

The solution(s) found are the following

$$y = \lim_{c_3 \to 0} \frac{-(e^{x} c_3)^{-\alpha} + e^{\alpha}\alpha}{\alpha} \tag{1}$$

Verification of solutions

$$y = \lim_{c_3 \to 0} \frac{-(e^{x} c_3)^{-\alpha} + e^{\alpha}\alpha}{\alpha}$$

Verified OK.

22.9.7 Maple step by step solution

Let's solve

$$\left[y'' + \alpha y' = 0, y(0) = e^{\alpha}, y'\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of ODE

  $\alpha r + r^2 = 0$
- Factor the characteristic polynomial

  $r(\alpha + r) = 0$
- Roots of the characteristic polynomial

  $r = (0, -\alpha)$
- 1st solution of the ODE

  $y_1(x) = 1$
- 2nd solution of the ODE

  $y_2(x) = e^{-x\alpha}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 + c_2 e^{-x\alpha}$

□ Check validity of solution $y = c_1 + c_2 e^{-x\alpha}$

  - Use initial condition $y(0) = e^{\alpha}$

    $e^{\alpha} = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -c_2 \alpha e^{-x\alpha}$
  - Use the initial condition $y'\Big|_{\{x=1\}} = 0$

    $0 = -c_2 \alpha e^{-\alpha}$
  - Solve for c_1 and c_2

    $\{c_1 = e^{\alpha}, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $y = e^{\alpha}$

- Solution to the IVP

  $y = e^{\alpha}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 6

```
dsolve([diff(y(x),x$2)+alpha*diff(y(x),x)=0,y(0) = exp(alpha), D(y)(1) = 0],y(x), singsol=al
```

$$y(x) = e^{\alpha}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 8

```
DSolve[{y''[x]+\[Alpha]*y'[x]==0,{y[0]==Exp[\[Alpha]],y'[1]==0}},y[x],x,IncludeSingularSolut
```

$$y(x) \to e^{\alpha}$$

22.10 problem 715

Internal problem ID [15459]
Internal file name [`OUTPUT/15459_Wednesday_May_08_2024_03_59_20_PM_28052777/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 715.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \alpha^2 y = 1}$$

With initial conditions

$$[y'(0) = \alpha, y'(\pi) = 0]$$

22.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = \alpha^2, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + \alpha^2 y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \alpha^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \alpha^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\alpha^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \alpha^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\alpha^2)}\\ &= \pm\sqrt{-\alpha^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\alpha^2}\\ \lambda_2 &= -\sqrt{-\alpha^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\alpha^2}\\ \lambda_2 &= -\sqrt{-\alpha^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(\sqrt{-\alpha^2}\right)x} + c_2 e^{\left(-\sqrt{-\alpha^2}\right)x}\end{aligned}$$

Or

$$y = c_1 e^{\sqrt{-\alpha^2}\, x} + c_2 e^{-\sqrt{-\alpha^2}\, x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{\sqrt{-\alpha^2}\, x} + c_2 e^{-\sqrt{-\alpha^2}\, x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\sqrt{-\alpha^2}\, x}, e^{-\sqrt{-\alpha^2}\, x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\alpha^2 A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{\alpha^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{\alpha^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-\alpha^2}\, x} + c_2 e^{-\sqrt{-\alpha^2}\, x}\right) + \left(\frac{1}{\alpha^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\sqrt{-\alpha^2}\, x} + c_2 e^{-\sqrt{-\alpha^2}\, x} + \frac{1}{\alpha^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\alpha^2}\, e^{\sqrt{-\alpha^2}\, x} - c_2 \sqrt{-\alpha^2}\, e^{-\sqrt{-\alpha^2}\, x}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \left(c_1 e^{\pi\sqrt{-\alpha^2}} - e^{-\pi\sqrt{-\alpha^2}} c_2\right) \sqrt{-\alpha^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\alpha^2}\, e^{\sqrt{-\alpha^2}\, x} - c_2 \sqrt{-\alpha^2}\, e^{-\sqrt{-\alpha^2}\, x}$$

substituting $y' = \alpha$ and $x = 0$ in the above gives

$$\alpha = (c_1 - c_2) \sqrt{-\alpha^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\alpha}{\left(-e^{2\pi\sqrt{-\alpha^2}} + 1\right)\sqrt{-\alpha^2}}$$

$$c_2 = \frac{e^{2\pi\sqrt{-\alpha^2}}\alpha}{\left(-e^{2\pi\sqrt{-\alpha^2}} + 1\right)\sqrt{-\alpha^2}}$$

Substituting these values back in above solution results in

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\,x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}} \cdots}{\sqrt{-\alpha^2}\,\alpha^2 e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\,\alpha^2 e^{2\pi\sqrt{-\alpha^2}} + \sqrt{-\alpha^2}\,\alpha^2}$$

Which simplifies to

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\,x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}} \cdots}{\sqrt{-\alpha^2}\,\alpha^2 \left(e^{4\pi\sqrt{-\alpha^2}} - 2\, e^{2\pi\sqrt{-\alpha^2}} + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\,x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}} \cdots}{\sqrt{-\alpha^2}\,\alpha^2 \left(e^{4\pi\sqrt{-\alpha^2}} - 2\, e^{2\pi\sqrt{-\alpha^2}} + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\,x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}}}{\sqrt{-\alpha^2}\,\alpha^2 \left(e^{4\pi\sqrt{-\alpha^2}} - 2\, e^{2\pi\sqrt{-\alpha^2}} + 1\right)}$$

Verified OK.

22.10.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \alpha^2 y'y - y' = 0$$

Integrating the above w.r.t x gives

$$\int \left(y'y'' + \alpha^2 y'y - y'\right) dx = 0$$
$$\frac{y'^2}{2} + \frac{\alpha^2 y^2}{2} - y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\alpha^2 y^2 + 2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\alpha^2 y^2 + 2y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\alpha^2 y^2 + 2c_1 + 2y}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\alpha^2}\left(y - \frac{1}{\alpha^2}\right)}{\sqrt{-\alpha^2 y^2 + 2y + 2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\alpha^2 y^2 + 2c_1 + 2y}} dy = \int dx$$

$$-\frac{\arctan\left(\frac{\sqrt{\alpha^2}\left(y-\frac{1}{\alpha^2}\right)}{\sqrt{-\alpha^2 y^2+2y+2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{\sqrt{\alpha^2}\left(y-\frac{1}{\alpha^2}\right)}{\sqrt{-\alpha^2 y^2+2y+2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = \frac{2\tan\left(c_2\sqrt{\alpha^2} + x\sqrt{\alpha^2}\right)\sqrt{\alpha^2}\left(\tan\left(c_2\sqrt{\alpha^2} + x\sqrt{\alpha^2}\right)^2 + 1\right) + \frac{8\tan\left(c_2\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)^3 c_1\alpha^2\sqrt{\alpha^2}\left(\tan\left(c_2\sqrt{\alpha^2}+x\sqrt{\alpha\right.\right.}}{}}{}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \frac{\left(2\cos\left(\alpha(\pi + c_2)\right)^2 - \cos\left(\operatorname{csgn}\left(\alpha\right)\alpha(\pi + c_2)\right)^2\right)\sqrt{\sec\left(\operatorname{csgn}\left(\alpha\right)\alpha\left(\pi + c_2\right)\right)^4\sin\left(\alpha\left(\pi + c_2\right)\right)^2\left(2\alpha^2 c_1 + \right.}}{\alpha} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\tan\left(c_2\sqrt{\alpha^2} + x\sqrt{\alpha^2}\right)\sqrt{\alpha^2}\left(\tan\left(c_2\sqrt{\alpha^2} + x\sqrt{\alpha^2}\right)^2 + 1\right) + \frac{8\tan\left(c_2\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)^3 c_1\alpha^2\sqrt{\alpha^2}\left(\tan\left(c_2\sqrt{\alpha^2}+x\sqrt{\alpha\right.\right.}}{}}{}$$

substituting $y' = \alpha$ and $x = 0$ in the above gives

$$\alpha = \frac{\csc\left(\alpha c_2\right)\sqrt{\sec\left(c_2\operatorname{csgn}\left(\alpha\right)\alpha\right)^4\sin\left(\alpha c_2\right)^2\left(2\alpha^2 c_1 + 1\right)}\cos\left(c_2\operatorname{csgn}\left(\alpha\right)\alpha\right)\left(2\cos\left(\alpha c_2\right)^2 - \cos\left(c_2\operatorname{csgn}\left(\alpha\right.\right.\right.}{\alpha} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\arctan\left(\frac{\sqrt{\alpha^2}\left(y-\frac{1}{\alpha^2}\right)}{\sqrt{-\alpha^2 y^2+2y+2c_1}}\right)}{\sqrt{\alpha^2}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = \frac{2\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)\sqrt{\alpha^2}\left(\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)^2+1\right)+\frac{8\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)^3 c_1\alpha^2\sqrt{\alpha^2}\left(\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha\right.\right.}{}}{}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \frac{\left(2\cos\left(\alpha(\pi+c_3)\right)^2-\cos\left(\operatorname{csgn}\left(\alpha\right)\alpha(\pi+c_3)\right)^2\right)\sqrt{\sec\left(\operatorname{csgn}\left(\alpha\right)\alpha\left(\pi+c_3\right)\right)^4\left(2\alpha^2c_1+1\right)\sin\left(\alpha\left(\pi+c_3\right)\right.}}{\alpha} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)\sqrt{\alpha^2}\left(\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)^2+1\right)+\frac{8\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha^2}\right)^3 c_1\alpha^2\sqrt{\alpha^2}\left(\tan\left(c_3\sqrt{\alpha^2}+x\sqrt{\alpha\right.\right.}{}}{}$$

substituting $y' = \alpha$ and $x = 0$ in the above gives

$$\alpha = \frac{\csc\left(\alpha c_3\right)\sqrt{\sec\left(c_3\operatorname{csgn}\left(\alpha\right)\alpha\right)^4\sin\left(\alpha c_3\right)^2\left(2\alpha^2c_1+1\right)}\cos\left(c_3\operatorname{csgn}\left(\alpha\right)\alpha\right)\left(2\cos\left(\alpha c_3\right)^2-\cos\left(c_3\operatorname{csgn}\left(\alpha\right.\right.\right.}{\alpha} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

22.10.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + \alpha^2 y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3} \\
C &= \alpha^2
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-\alpha^2}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -\alpha^2 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\alpha^2\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 700: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\alpha^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{-\alpha^2}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{\sqrt{-\alpha^2}\, x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\sqrt{-\alpha^2}\, x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{\sqrt{-\alpha^2}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-\alpha^2}\, x}}\, dx \\
&= \mathrm{e}^{\sqrt{-\alpha^2}\, x} \left(\frac{\sqrt{-\alpha^2}\, \mathrm{e}^{-2\sqrt{-\alpha^2}\, x}}{2\alpha^2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-\alpha^2}\, x} \right) + c_2 \left(\mathrm{e}^{\sqrt{-\alpha^2}\, x} \left(\frac{\sqrt{-\alpha^2}\, \mathrm{e}^{-2\sqrt{-\alpha^2}\, x}}{2\alpha^2} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + \alpha^2 y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\sqrt{-\alpha^2}\, x} + \frac{c_2 \sqrt{-\alpha^2}\, e^{-\sqrt{-\alpha^2}\, x}}{2\alpha^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{-\alpha^2}\, e^{-\sqrt{-\alpha^2}\, x}}{2\alpha^2}, e^{\sqrt{-\alpha^2}\, x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\alpha^2 A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{\alpha^2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{\alpha^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\sqrt{-\alpha^2}\, x} + \frac{c_2 \sqrt{-\alpha^2}\, e^{-\sqrt{-\alpha^2}\, x}}{2\alpha^2}\right) + \left(\frac{1}{\alpha^2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\sqrt{-\alpha^2}\, x} + \frac{c_2 \sqrt{-\alpha^2}\, e^{-\sqrt{-\alpha^2}\, x}}{2\alpha^2} + \frac{1}{\alpha^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\alpha^2}\, e^{\sqrt{-\alpha^2}\, x} + \frac{c_2 e^{-\sqrt{-\alpha^2}\, x}}{2}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = e^{\pi\sqrt{-\alpha^2}} \sqrt{-\alpha^2}\, c_1 + \frac{e^{-\pi\sqrt{-\alpha^2}} c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\alpha^2}\, e^{\sqrt{-\alpha^2}\, x} + \frac{c_2 e^{-\sqrt{-\alpha^2}\, x}}{2}$$

substituting $y' = \alpha$ and $x = 0$ in the above gives

$$\alpha = c_1 \sqrt{-\alpha^2} + \frac{c_2}{2} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\alpha}{\left(-e^{2\pi\sqrt{-\alpha^2}} + 1\right)\sqrt{-\alpha^2}}$$

$$c_2 = \frac{2\alpha\, e^{2\pi\sqrt{-\alpha^2}}}{e^{2\pi\sqrt{-\alpha^2}} - 1}$$

Substituting these values back in above solution results in

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\, x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}}}{\sqrt{-\alpha^2}\,\alpha^2 e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\,\alpha^2 e^{2\pi\sqrt{-\alpha^2}} + \sqrt{-\alpha^2}\,\alpha^2}$$

Which simplifies to

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\, x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}}}{\sqrt{-\alpha^2}\,\alpha^2\left(e^{4\pi\sqrt{-\alpha^2}} - 2\, e^{2\pi\sqrt{-\alpha^2}} + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\,x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}}}{\sqrt{-\alpha^2}\,\alpha^2 \left(e^{4\pi\sqrt{-\alpha^2}} - 2\, e^{2\pi\sqrt{-\alpha^2}} + 1\right)} \tag{1}$$

Verification of solutions

$$y = \frac{-\alpha^3 e^{\sqrt{-\alpha^2}\,(x+2\pi)} + \alpha^3 e^{-\sqrt{-\alpha^2}\,(-2\pi+x)} - \alpha^3 e^{-\sqrt{-\alpha^2}\,(x-4\pi)} + e^{\sqrt{-\alpha^2}\,x}\alpha^3 + \sqrt{-\alpha^2}\, e^{4\pi\sqrt{-\alpha^2}} - 2\sqrt{-\alpha^2}\, e^{2\pi\sqrt{-\alpha^2}}}{\sqrt{-\alpha^2}\,\alpha^2 \left(e^{4\pi\sqrt{-\alpha^2}} - 2\, e^{2\pi\sqrt{-\alpha^2}} + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+alpha^2*y(x)=1,D(y)(0) = alpha, D(y)(Pi) = 0],y(x), singsol=all)
```

$$y(x) = \sin(\alpha x) + \cos(\alpha x)\cot(\alpha \pi) + \frac{1}{\alpha^2}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+\[Alpha]^2*y'[x]==1,{y'[0]==\[Alpha],y'[Pi]==0}},y[x],x,IncludeSingularSoluti
```

{}

22.11 problem 716

Internal problem ID [15460]
Internal file name [`OUTPUT/15460_Wednesday_May_08_2024_04_00_38_PM_97445917/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 716.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 1}$$

With initial conditions

$$[y(0) = 0, y'(\pi) = 0]$$

22.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x)$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -\cos(x) + 1$$

Summary

The solution(s) found are the following

$$y = -\cos(x) + 1 \tag{1}$$

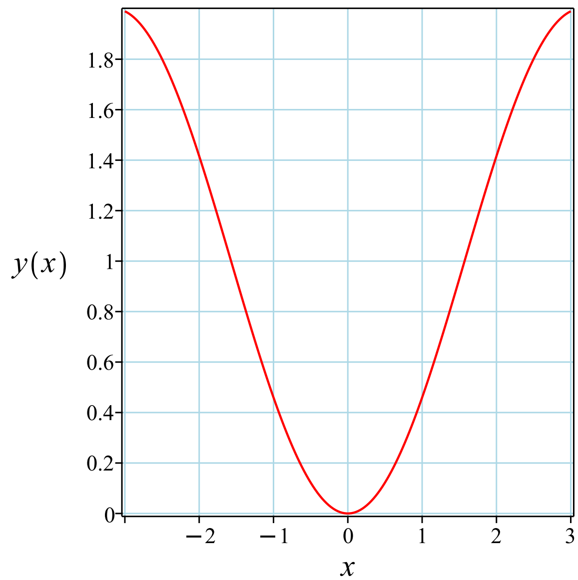


Figure 821: Solution plot

Verification of solutions

$$y = -\cos(x) + 1$$

Verified OK.

22.11.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' - y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy' - y')\, dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} - y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+2c_1+2y}}dy = \int dx$$
$$\arctan\left(\frac{y-1}{\sqrt{-y^2+2y+2c_1}}\right) = x+c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1+2y}}dy = \int dx$$
$$-\arctan\left(\frac{y-1}{\sqrt{-y^2+2y+2c_1}}\right) = x+c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y-1}{\sqrt{-y^2+2y+2c_1}}\right) = x+c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=0$ and $x=0$ in the above gives

$$-\arctan\left(\frac{\sqrt{2}}{2\sqrt{c_1}}\right) = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \left(\tan(x+c_2)^2+1\right)\sqrt{\frac{1+2c_1}{\tan(x+c_2)^2+1}} - \frac{\tan(x+c_2)^2(1+2c_1)}{\sqrt{\frac{1+2c_1}{\tan(x+c_2)^2+1}}\left(\tan(x+c_2)^2+1\right)}$$

substituting $y'=0$ and $x=\pi$ in the above gives

$$0 = \sqrt{\cos(c_2)^2(1+2c_1)} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$\arctan\left(\frac{y-1}{\sqrt{-y^2+2y}}\right) = x + c_2$$

Which can be written as

$$\arctan\left(\frac{y-1}{\sqrt{-y(y-2)}}\right) = x + c_2$$

Looking at the Second solution

$$-\arctan\left(\frac{y-1}{\sqrt{-y^2+2y+2c_1}}\right) = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$\arctan\left(\frac{\sqrt{2}}{2\sqrt{c_1}}\right) = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\left(\tan(x+c_3)^2 + 1\right)\sqrt{\frac{1+2c_1}{\tan(x+c_3)^2+1}} + \frac{\tan(x+c_3)^2(1+2c_1)}{\sqrt{\frac{1+2c_1}{\tan(x+c_3)^2+1}}\left(\tan(x+c_3)^2+1\right)}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \frac{\cos(c_3)^2(-2c_1-1)}{\sqrt{\cos(c_3)^2(1+2c_1)}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y-1}{\sqrt{-y(y-2)}}\right) = x + c_2 \tag{1}$$

Verification of solutions

$$\arctan\left(\frac{y-1}{\sqrt{-y\,(y-2)}}\right) = x + c_2$$

Verified OK.

22.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 701: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x)$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -\cos(x) + 1$$

Summary

The solution(s) found are the following

$$y = -\cos(x) + 1 \tag{1}$$

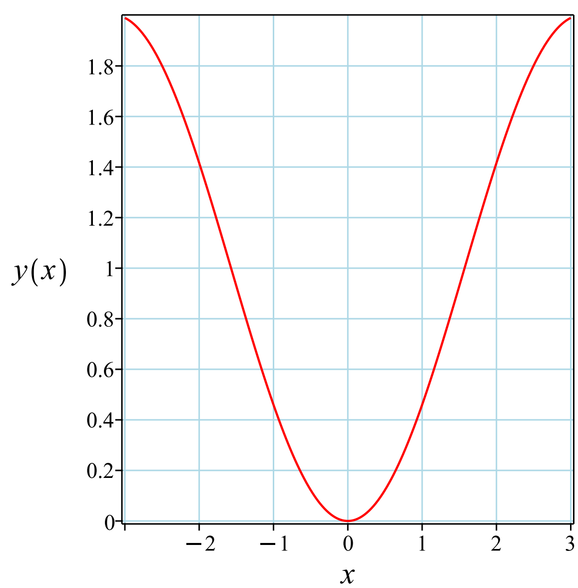


Figure 822: Solution plot

Verification of solutions

$$y = -\cos(x) + 1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)+y(x)=1,y(0) = 0, D(y)(Pi) = 0],y(x), singsol=all)
```

$$y(x) = 1 - \cos(x)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 11

```
DSolve[{y''[x]+y[x]==1,{y[0]==0,y'[Pi]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - \cos(x)$$

22.12 problem 717

Internal problem ID [15461]
Internal file name [`OUTPUT/15461_Wednesday_May_08_2024_04_00_41_PM_14134547/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 717.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \lambda^2 y = 0}$$

With initial conditions

$$[y'(0) = 0, y'(\pi) = 0]$$

22.12.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \lambda^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \lambda^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(\lambda^2)} \\ &= \pm\sqrt{-\lambda^2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{-\lambda^2} \\ \lambda_2 &= -\sqrt{-\lambda^2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{-\lambda^2} \\ \lambda_2 &= -\sqrt{-\lambda^2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{-\lambda^2}\right)x} + c_2 e^{\left(-\sqrt{-\lambda^2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda^2}\, x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda^2}\, x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda^2}\, \mathrm{e}^{\sqrt{-\lambda^2}\, x} - c_2\sqrt{-\lambda^2}\, \mathrm{e}^{-\sqrt{-\lambda^2}\, x}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \left(c_1 e^{\pi\sqrt{-\lambda^2}} - c_2 e^{-\pi\sqrt{-\lambda^2}}\right)\sqrt{-\lambda^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda^2}\, e^{\sqrt{-\lambda^2}\, x} - c_2\sqrt{-\lambda^2}\, e^{-\sqrt{-\lambda^2}\, x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = (c_1 - c_2)\sqrt{-\lambda^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

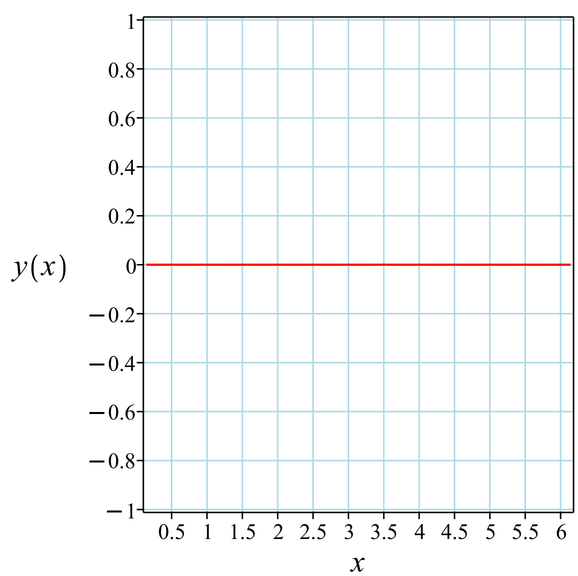


Figure 823: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

22.12.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \lambda^2 y' y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + \lambda^2 y' y)\, dx = 0$$
$$\frac{y'^2}{2} + \frac{\lambda^2 y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\lambda^2 y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\lambda^2 y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\lambda^2 y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2 + 2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\lambda^2 y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2 + 2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2+2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = -\frac{2\sqrt{2}\sqrt{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)c_1}\,\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2\sqrt{\lambda^2}}{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)\lambda} + \frac{\sqrt{2}\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2}{\sqrt{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2 \right.}}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = -\left(\cos\left(\lambda(\pi+c_2)\right)^2 - 2\cos\left(\operatorname{csgn}(\lambda)\,\lambda(\pi+c_2)\right)^2\right)\sqrt{\sec\left(\operatorname{csgn}(\lambda)\,\lambda\,(\pi+c_2)\right)^2 c_1}\,\sqrt{2}\,\operatorname{csgn}(\lambda) \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\sqrt{2}\sqrt{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)c_1}\,\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2\sqrt{\lambda^2}}{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)\lambda} + \frac{\sqrt{2}\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2}{\sqrt{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2 \right.}}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \left(-\cos\left(\lambda c_2\right)^2 + 2\cos\left(c_2\operatorname{csgn}(\lambda)\,\lambda\right)^2\right)\sqrt{\sec\left(c_2\operatorname{csgn}(\lambda)\,\lambda\right)^2 c_1}\,\sqrt{2}\,\operatorname{csgn}(\lambda) \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x + c_2$$

Looking at the Second solution

$$-\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2+2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = \frac{2\sqrt{2}\,\sqrt{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)c_1}\,\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2\sqrt{\lambda^2}}{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)\lambda} - \frac{\sqrt{2}\,\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2 \cdots}{\sqrt{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+\cdots}}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \left(\cos\left(\lambda(\pi+c_3)\right)^2 - 2\cos\left(\operatorname{csgn}\left(\lambda\right)\lambda(\pi+c_3)\right)^2\right)\sqrt{\sec\left(\operatorname{csgn}\left(\lambda\right)\lambda\left(\pi+c_3\right)\right)^2 c_1}\,\sqrt{2}\operatorname{csgn}\left(\lambda\right) \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\sqrt{2}\,\sqrt{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)c_1}\,\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2\sqrt{\lambda^2}}{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)\lambda} - \frac{\sqrt{2}\,\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2 \cdots}{\sqrt{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+\cdots}}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \left(\cos\left(\lambda c_3\right)^2 - 2\cos\left(c_3\operatorname{csgn}\left(\lambda\right)\lambda\right)^2\right)\sqrt{\sec\left(c_3\operatorname{csgn}\left(\lambda\right)\lambda\right)^2 c_1}\,\sqrt{2}\operatorname{csgn}\left(\lambda\right) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 0$$

Substituting these values back in above solution results in

$$-\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x + c_3$$

Summary

The solution(s) found are the following

$$\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x + c_2 \tag{1}$$

$$-\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x + c_3 \tag{2}$$

Verification of solutions

$$\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x + c_2$$

Warning, solution could not be verified

$$-\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x + c_3$$

Warning, solution could not be verified

22.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \lambda^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \lambda^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -\lambda^2 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\lambda^2\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 702: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\lambda^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{-\lambda^2}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-\lambda^2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-\lambda^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda^2}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-\lambda^2}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda^2}\, x} \left(\frac{\sqrt{-\lambda^2}\, \mathrm{e}^{-2\sqrt{-\lambda^2}\, x}}{2\lambda^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{-\lambda^2}\, x}\right) + c_2 \left(\mathrm{e}^{\sqrt{-\lambda^2}\, x} \left(\frac{\sqrt{-\lambda^2}\, \mathrm{e}^{-2\sqrt{-\lambda^2}\, x}}{2\lambda^2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\sqrt{-\lambda^2}\, x} + \frac{c_2 \sqrt{-\lambda^2}\, e^{-\sqrt{-\lambda^2}\, x}}{2 \lambda^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\lambda^2}\, e^{\sqrt{-\lambda^2}\, x} + \frac{c_2 e^{-\sqrt{-\lambda^2}\, x}}{2}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \sqrt{-\lambda^2}\, e^{\pi \sqrt{-\lambda^2}} c_1 + \frac{c_2 e^{-\pi \sqrt{-\lambda^2}}}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \sqrt{-\lambda^2}\, e^{\sqrt{-\lambda^2}\, x} + \frac{c_2 e^{-\sqrt{-\lambda^2}\, x}}{2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 \sqrt{-\lambda^2} + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

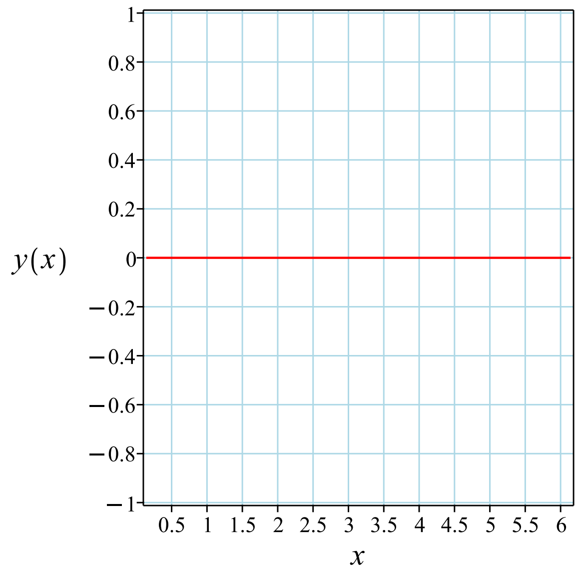


Figure 824: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$2)+lambda^2*y(x)=0,D(y)(0) = 0, D(y)(Pi) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
DSolve[{y''[x]+\[Lambda]^2*y[x]==0,{y'[0]==0,y'[Pi]==0}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \begin{cases} c_1 \cos\left(x\sqrt{\lambda^2}\right) & n \in \mathbb{Z} \land n \geq 0 \land \lambda^2 = n^2 \\ 0 & \text{True} \end{cases}$$

22.13 problem 718

Internal problem ID [15462]
Internal file name [`OUTPUT/15462_Wednesday_May_08_2024_04_00_51_PM_30253002/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 718.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + \lambda^2 y = 0}$$

With initial conditions

$$[y(0) = 0, y'(\pi) = 0]$$

22.13.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \lambda^2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda^2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \lambda^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\lambda^2)} \\ &= \pm\sqrt{-\lambda^2}\end{aligned}$$

Hence

$$\lambda_1 = +\sqrt{-\lambda^2}$$
$$\lambda_2 = -\sqrt{-\lambda^2}$$

Which simplifies to

$$\lambda_1 = \sqrt{-\lambda^2}$$
$$\lambda_2 = -\sqrt{-\lambda^2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\sqrt{-\lambda^2}\right)x} + c_2 e^{\left(-\sqrt{-\lambda^2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda^2}\, x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda^2}\, x} + c_2 \mathrm{e}^{-\sqrt{-\lambda^2}\, x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda^2}\,\mathrm{e}^{\sqrt{-\lambda^2}\,x} - c_2\sqrt{-\lambda^2}\,\mathrm{e}^{-\sqrt{-\lambda^2}\,x}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \left(c_1\mathrm{e}^{\pi\sqrt{-\lambda^2}} - c_2\mathrm{e}^{-\pi\sqrt{-\lambda^2}}\right)\sqrt{-\lambda^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

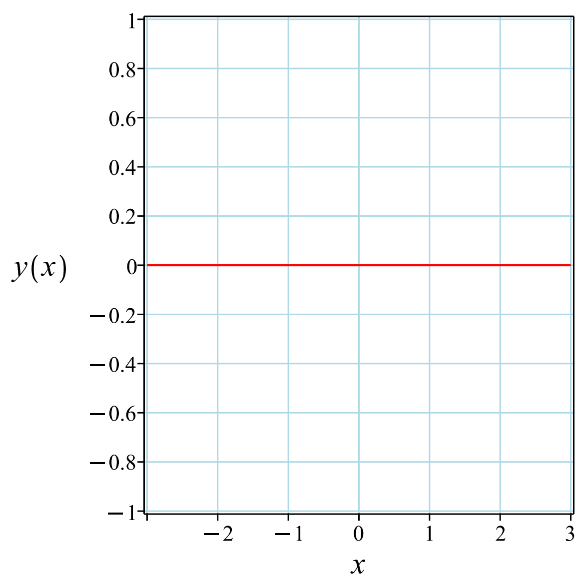


Figure 825: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

22.13.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \lambda^2 y' y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + \lambda^2 y' y)\, dx = 0$$

$$\frac{y'^2}{2} + \frac{\lambda^2 y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\lambda^2 y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-\lambda^2 y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\lambda^2 y^2 + 2c_1}} dy = \int dx$$

$$\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2 + 2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\lambda^2 y^2 + 2c_1}} dy = \int dx$$

$$-\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2 + 2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2+2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2\sqrt{2}\,\sqrt{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)c_1}\,\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2\sqrt{\lambda^2}}{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)\lambda} + \frac{\sqrt{2}\,\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2}{\sqrt{\left(\tan\left(c_2\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2 +}}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = -\left(\cos\left(\lambda(\pi+c_2)\right)^2 - 2\cos\left(\operatorname{csgn}(\lambda)\,\lambda(\pi+c_2)\right)^2\right)\sqrt{\sec\left(\operatorname{csgn}(\lambda)\,\lambda\,(\pi+c_2)\right)^2 c_1}\,\sqrt{2}\,\operatorname{csgn}(\lambda) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x$$

Looking at the Second solution

$$-\frac{\arctan\left(\frac{\sqrt{\lambda^2}\, y}{\sqrt{-\lambda^2 y^2+2c_1}}\right)}{\sqrt{\lambda^2}} = x + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2\sqrt{2}\,\sqrt{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)c_1}\,\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2\sqrt{\lambda^2}}{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+1\right)\lambda} - \frac{\sqrt{2}\,\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2 \sqrt{}}{\sqrt{\left(\tan\left(c_3\sqrt{\lambda^2}+x\sqrt{\lambda^2}\right)^2+\right.}}$$

substituting $y'=0$ and $x=\pi$ in the above gives

$$0 = \left(\cos\left(\lambda(\pi+c_3)\right)^2 - 2\cos\left(\operatorname{csgn}(\lambda)\,\lambda(\pi+c_3)\right)^2\right)\sqrt{\sec\left(\operatorname{csgn}(\lambda)\,\lambda\,(\pi+c_3)\right)^2 c_1}\,\sqrt{2}\,\operatorname{csgn}(\lambda) \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_3 = 0$$

Substituting these values back in above solution results in

$$-\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x$$

Summary

The solution(s) found are the following

$$\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x \tag{1}$$

$$-\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x \tag{2}$$

Verification of solutions

$$\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x$$

Warning, solution could not be verified

$$-\operatorname{signum}\left(\frac{y}{\sqrt{-\lambda^2 y^2}}\right)\infty = x$$

Warning, solution could not be verified

22.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + \lambda^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \lambda^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\lambda^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\lambda^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\lambda^2\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 703: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\lambda^2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{\sqrt{-\lambda^2}\, x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{-\lambda^2}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-\lambda^2}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda^2}\, x} \int \frac{1}{\mathrm{e}^{2\sqrt{-\lambda^2}\, x}}\, dx \\ &= \mathrm{e}^{\sqrt{-\lambda^2}\, x} \left(\frac{\sqrt{-\lambda^2}\, \mathrm{e}^{-2\sqrt{-\lambda^2}\, x}}{2\lambda^2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{\sqrt{-\lambda^2}\, x}\right) + c_2 \left(\mathrm{e}^{\sqrt{-\lambda^2}\, x} \left(\frac{\sqrt{-\lambda^2}\, \mathrm{e}^{-2\sqrt{-\lambda^2}\, x}}{2\lambda^2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\sqrt{-\lambda^2}\, x} + \frac{c_2 \sqrt{-\lambda^2}\, \mathrm{e}^{-\sqrt{-\lambda^2}\, x}}{2\lambda^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{2c_1\lambda^2 + c_2\sqrt{-\lambda^2}}{2\lambda^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\sqrt{-\lambda^2}\,\mathrm{e}^{\sqrt{-\lambda^2}\,x} + \frac{c_2\mathrm{e}^{-\sqrt{-\lambda^2}\,x}}{2}$$

substituting $y' = 0$ and $x = \pi$ in the above gives

$$0 = \sqrt{-\lambda^2}\,\mathrm{e}^{\pi\sqrt{-\lambda^2}}c_1 + \frac{c_2\mathrm{e}^{-\pi\sqrt{-\lambda^2}}}{2} \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

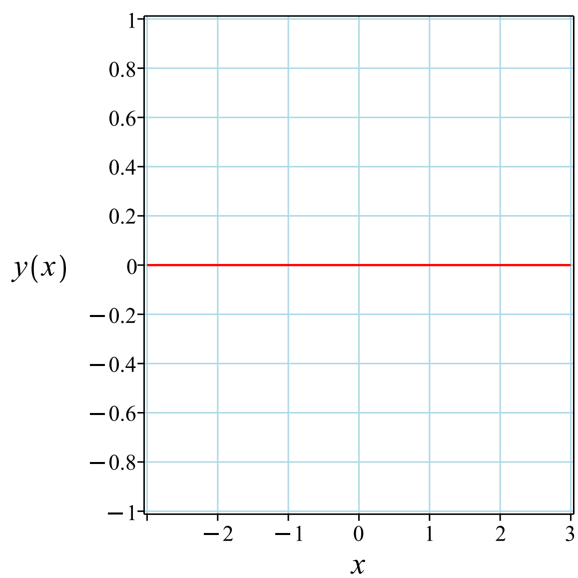


Figure 826: Solution plot

Verification of solutions

$$y = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$2)+lambda^2*y(x)=0,y(0) = 0, D(y)(Pi) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 40

```
DSolve[{y''[x]+\[Lambda]^2*y[x]==0,{y[0]==0,y'[Pi]==0}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \begin{cases} c_1 \sin\left(x\sqrt{\lambda^2}\right) & n \in \mathbb{Z} \land n \geq 1 \land \lambda^2 = \left(n - \frac{1}{2}\right)^2 \\ 0 & \text{True} \end{cases}$$

22.14 problem 719

Internal problem ID [15463]
Internal file name [`OUTPUT/15463_Wednesday_May_08_2024_04_00_53_PM_48923003/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 719.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_x]]`

Unable to solve or complete the solution.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$3)+diff(y(x),x$2)-diff(y(x),x)-y(x)=0,y(0) = -1, y(1) = 0, D(y)(0) = 2],
```

$$y(x) = e^{-x}(x-1)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 14

```
DSolve[{y'''[x]+y''[x]-y'[x]-y[x]==0,{y[0]==-1,y[1]==0,y'[0]==2}},y[x],x,IncludeSingularSolu
```

$$y(x) \to e^{-x}(x-1)$$

22.15 problem 720

Internal problem ID [15464]
Internal file name [`OUTPUT/15464_Wednesday_May_08_2024_04_00_53_PM_73637771/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 720.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

Unable to solve or complete the solution.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$4)-lambda^4*y(x)=0,y(0) = 0, (D@@2)(y)(0) = 0, y(Pi) = 0, (D@@2)(y)(Pi)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 6

```
DSolve[{y''''[x]-\[Lambda]^4*y[x]==0,{y[0]==0,y''[0]==0,y[Pi]==0,y''[Pi]==0}},y[x],x,Include
```

$$y(x) \to 0$$

22.16 problem 721

Internal problem ID [15465]
Internal file name [`OUTPUT/15465_Wednesday_May_08_2024_04_00_53_PM_5979204/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 721.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' + y' = 0}$$

22.16.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$xy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

22.16.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$xp'(x) + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{p}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{x}\,dx \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{x}\,dx \\
\ln(p) &= -\ln(x) + c_1 \\
p &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1}{x}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int \frac{c_1}{x}\,\mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

22.16.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x \\
B &= 1 \\
C &= 0 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x)(0) + (1)(0) + (0)(1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
xv'' + (1) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
xu'(x) + u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (1)\,(c_1 \ln(x) + c_2) \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = c_1 \ln(x) + c_2
$$

Verified OK.

22.16.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' + y')\, dx = 0$$
$$xy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \frac{c_1}{x}\, \mathrm{d}x$$
$$= c_1 \ln(x) + c_2$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

22.16.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 704: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{x} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 + c_2 \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 + c_2 \ln(x)
$$

Verified OK.

22.16.6 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x \\
q(x) &= 1 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$xy' = c_1$$

We now have a first order ode to solve which is

$$xy' = c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK.

22.16.7 Maple step by step solution

Let's solve

$xy'' + y' = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

$xy'' + y' = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

$\frac{\frac{d^2}{dt^2}y(t)}{x} = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(t) = 1$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_2 t + c_1$

- Change variables back using $t = \ln(x)$

  $y = c_1 + c_2 \ln(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_2 \ln (x) + c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 13

```
DSolve[x*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x) + c_2$$

22.17 problem 722

Internal problem ID [15466]
Internal file name [`OUTPUT/15466_Wednesday_May_08_2024_04_00_55_PM_99259692/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 722.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{x^2y'''' + 4xy''' + 2y'' = 0}$$

With initial conditions

$$[y(1) = 0, y'(1) = 0]$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$x^2v'''(x) + 4xv''(x) + 2v'(x) = 0$$

Since $v(x)$ is missing from the ode then we can use the substitution $v'(x) = w(x)$ to reduce the order by one. The ODE becomes

$$x^2w''(x) + 4xw'(x) + 2w(x) = 0$$

This is Euler second order ODE. Let the solution be $w(x) = x^r$, then $w' = rx^{r-1}$ and $w'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 4r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= -1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$w(x) = c_1 w_1 + c_2 w_2$$

Where $w_1 = x^{r_1}$ and $w_2 = x^{r_2}$. Hence

$$w(x) = \frac{c_1}{x^2} + \frac{c_2}{x}$$

But since $v'(x) = w(x)$ then we now need to solve the ode $v'(x) = \frac{c_1}{x^2} + \frac{c_2}{x}$. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_2 x + c_1}{x^2}\, dx \\ &= -\frac{c_1}{x} + c_2 \ln(x) + c_3 \end{aligned}$$

But since $y' = v(x)$ then we now need to solve the ode $y' = -\frac{c_1}{x} + c_2 \ln(x) + c_3$. Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_2 \ln(x)\, x + c_3 x - c_1}{x}\, dx \\ &= c_2(x \ln(x) - x) + c_3 x - c_1 \ln(x) + c_4 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2(x \ln(x) - x) + c_3 x - c_1 \ln(x) + c_4 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -c_2 + c_3 + c_4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x} + c_2 \ln(x) + c_3$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = -c_1 + c_3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$c_1 = c_3$$
$$c_2 = c_3 + c_4$$

Substituting these values back in above solution results in

$$y = x\ln(x) c_3 + x\ln(x) c_4 - xc_4 - c_3\ln(x) + c_4$$

Which simplifies to

$$y = (xc_4 + c_3(x-1))\ln(x) - c_4(x-1)$$

Summary
The solution(s) found are the following

$$y = (xc_4 + c_3(x-1))\ln(x) - c_4(x-1) \tag{1}$$

Verification of solutions

$$y = (xc_4 + c_3(x-1))\ln(x) - c_4(x-1)$$

Verified OK.

22.17.1 Maple step by step solution

Let's solve

$$\left[x^2y'''' + 4xy''' + 2y'' = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -\frac{2(2xy'''+y'')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + \frac{4y'''}{x} + \frac{2y''}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'''' + 4xy''' + 2y'' = 0$

- Make a change of variables

  $t = \ln{(x)}$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  - Calculate the 4th derivative of y with respect to x , using the chain rule

    $y'''' = \left(\frac{d^4}{dt^4}y(t)\right)t'(x)^4 + 3t'(x)^2t''(x)\left(\frac{d^3}{dt^3}y(t)\right) + 3t''(x)^2\left(\frac{d^2}{dt^2}y(t)\right) + 3\left(t'''(x)\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d^3}{dt^3}y(\right.\right.$

  - Compute derivative

    $y'''' = \frac{\frac{d^4}{dt^4}y(t)}{x^4} - \frac{3\left(\frac{d^3}{dt^3}y(t)\right)}{x^4} + \frac{5\left(\frac{d^2}{dt^2}y(t)\right)}{x^4} + \frac{3\left(\frac{2\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} - \frac{\frac{d^3}{dt^3}y(t)}{x^3}\right)}{x} - \frac{6\left(\frac{d}{dt}y(t)\right)}{x^4}$

Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^4}{dt^4}y(t)}{x^4}-\frac{3\left(\frac{d^3}{dt^3}y(t)\right)}{x^4}+\frac{5\left(\frac{d^2}{dt^2}y(t)\right)}{x^4}+\frac{3\left(\frac{2\left(\frac{d^2}{dt^2}y(t)\right)}{x^3}-\frac{\frac{d^3}{dt^3}y(t)}{x^3}\right)}{x}-\frac{6\left(\frac{d}{dt}y(t)\right)}{x^4}\right)+4x\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3}-\frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3}+\right.$$

- Simplify

$$\frac{\frac{d^4}{dt^4}y(t)-2\frac{d^3}{dt^3}y(t)+\frac{d^2}{dt^2}y(t)}{x^2}=0$$

- Isolate 4th derivative

$$\frac{d^4}{dt^4}y(t)=2\frac{d^3}{dt^3}y(t)-\frac{d^2}{dt^2}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^4}{dt^4}y(t)-2\frac{d^3}{dt^3}y(t)+\frac{d^2}{dt^2}y(t)=0$$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(t)$

$$y_1(t)=y(t)$$

  ◦ Define new variable $y_2(t)$

$$y_2(t)=\frac{d}{dt}y(t)$$

  ◦ Define new variable $y_3(t)$

$$y_3(t)=\frac{d^2}{dt^2}y(t)$$

  ◦ Define new variable $y_4(t)$

$$y_4(t)=\frac{d^3}{dt^3}y(t)$$

  ◦ Isolate for $\frac{d}{dt}y_4(t)$ using original ODE

$$\frac{d}{dt}y_4(t)=2y_4(t)-y_3(t)$$

Convert linear ODE into a system of first order ODEs

$$\left[y_2(t)=\frac{d}{dt}y_1(t),y_3(t)=\frac{d}{dt}y_2(t),y_4(t)=\frac{d}{dt}y_3(t),\frac{d}{dt}y_4(t)=2y_4(t)-y_3(t)\right]$$

- Define vector

$$\vec{y}(t)=\begin{bmatrix}y_1(t)\\y_2(t)\\y_3(t)\\y_4(t)\end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- First solution from eigenvalue 1

$$\vec{y}_3(t) = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_4(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\vec{y}_4(t)$ into the homogeneous system

$$\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_4(t)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right] - 1 \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 1

$$\overrightarrow{y}_4(t) = \mathrm{e}^{t} \cdot \left(t \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3(t) + c_4 \overrightarrow{y}_4(t)$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_3 \mathrm{e}^{t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + c_4 \mathrm{e}^{t} \cdot \left(t \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right) + \left[\begin{array}{c} c_1 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$y(t) = ((t-1)\, c_4 + c_3)\, \mathrm{e}^{t} + c_1$

- Change variables back using $t = \ln(x)$

$$y = ((\ln(x) - 1) c_4 + c_3) x + c_1$$

- Simplify

$$y = x \ln(x) c_4 + (c_3 - c_4) x + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve([x^2*diff(y(x),x$4)+4*x*diff(y(x),x$3)+2*diff(y(x),x$2)=0,y(1) = 0, D(y)(1) = 0],y(x)
```

$$y(x) = (-c_3 + (x - 1) c_4) \ln(x) + c_3(x - 1)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 29

```
DSolve[{x^2*y''''[x]+4*x*y'''[x]+2*y''[x]==0,{y[1]==0,y'[1]==0}},y[x],x,IncludeSingularSolut
```

$$y(x) \to (c_1 - c_2)(x - 1) + (c_2 x - c_1) \log(x)$$

22.18 problem 723

Internal problem ID [15467]
Internal file name [`OUTPUT/15467_Wednesday_May_08_2024_04_00_56_PM_9770541/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 17. Boundary value problems. Exercises page 163
Problem number: 723.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{x^3y'''' + 6x^2y''' + 6xy'' = 0}$$

With initial conditions

$$[y(1) = 0, y'(1) = 0]$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$x^2v'''(x) + 6xv''(x) + 6v'(x) = 0$$

Since $v(x)$ is missing from the ode then we can use the substitution $v'(x) = w(x)$ to reduce the order by one. The ODE becomes

$$x^2w''(x) + 6xw'(x) + 6w(x) = 0$$

This is Euler second order ODE. Let the solution be $w(x) = x^r$, then $w' = rx^{r-1}$ and $w'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 6xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r+6r\,x^r+6x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$r(r-1)+6r+6=0$$

Or

$$r^2+5r+6=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1&=-3\\ r_2&=-2\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$w(x)=c_1w_1+c_2w_2$$

Where $w_1=x^{r_1}$ and $w_2=x^{r_2}$. Hence

$$w(x)=\frac{c_1}{x^3}+\frac{c_2}{x^2}$$

But since $v'(x)=w(x)$ then we now need to solve the ode $v'(x)=\frac{c_1}{x^3}+\frac{c_2}{x^2}$. Integrating both sides gives

$$\begin{aligned}v(x)&=\int\frac{c_2x+c_1}{x^3}\,\mathrm{d}x\\ &=-\frac{c_1}{2x^2}-\frac{c_2}{x}+c_3\end{aligned}$$

But since $y'=v(x)$ then we now need to solve the ode $y'=-\frac{c_1}{2x^2}-\frac{c_2}{x}+c_3$. Integrating both sides gives

$$\begin{aligned}y&=\int-\frac{-2c_3x^2+2c_2x+c_1}{2x^2}\,\mathrm{d}x\\ &=c_3x+\frac{c_1}{2x}-c_2\ln(x)+c_4\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=c_3x+\frac{c_1}{2x}-c_2\ln(x)+c_4 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = \frac{c_1}{2} + c_3 + c_4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{2x^2} - \frac{c_2}{x} + c_3$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = -\frac{c_1}{2} - c_2 + c_3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2c_3 - 2c_4 \\ c_2 &= 2c_3 + c_4 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{-2x \ln(x) c_3 - x \ln(x) c_4 + c_3 x^2 + x c_4 - c_3 - c_4}{x}$$

Which simplifies to

$$y = \frac{-2x\left(c_3 + \frac{c_4}{2}\right) \ln(x) + (x-1)(c_3 x + c_3 + c_4)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-2x\left(c_3 + \frac{c_4}{2}\right) \ln(x) + (x-1)(c_3 x + c_3 + c_4)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{-2x\left(c_3 + \frac{c_4}{2}\right) \ln(x) + (x-1)(c_3 x + c_3 + c_4)}{x}$$

Verified OK.

22.18.1 Maple step by step solution

Let's solve

$$\left[x^2y'''' + 6xy''' + 6y'' = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = -\frac{6(xy'''+y'')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' + \frac{6y'''}{x} + \frac{6y''}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'''' + 6xy''' + 6y'' = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  ◦ Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  ◦ Calculate the 4th derivative of y with respect to x , using the chain rule

$$y'''' = \left(\frac{d^4}{dt^4}y(t)\right)t'(x)^4 + 3t'(x)^2 t''(x)\left(\frac{d^3}{dt^3}y(t)\right) + 3t''(x)^2\left(\frac{d^2}{dt^2}y(t)\right) + 3\left(t'''(x)\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d^3}{dt^3}y(\right.\right.$$

- Compute derivative

$$y'''' = \frac{\frac{d^4}{dt^4}y(t)}{x^4} - \frac{3\left(\frac{d^3}{dt^3}y(t)\right)}{x^4} + \frac{5\left(\frac{d^2}{dt^2}y(t)\right)}{x^4} + \frac{3\left(\frac{2\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} - \frac{\frac{d^3}{dt^3}y(t)}{x^3}\right)}{x} - \frac{6\left(\frac{d}{dt}y(t)\right)}{x^4}$$

Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^4}{dt^4}y(t)}{x^4} - \frac{3\left(\frac{d^3}{dt^3}y(t)\right)}{x^4} + \frac{5\left(\frac{d^2}{dt^2}y(t)\right)}{x^4} + \frac{3\left(\frac{2\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} - \frac{\frac{d^3}{dt^3}y(t)}{x^3}\right)}{x} - \frac{6\left(\frac{d}{dt}y(t)\right)}{x^4}\right) + 6x\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \right.$$

- Simplify

$$\frac{\frac{d^4}{dt^4}y(t) - \frac{d^2}{dt^2}y(t)}{x^2} = 0$$

- Isolate 4th derivative

$$\frac{d^4}{dt^4}y(t) = \frac{d^2}{dt^2}y(t)$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^4}{dt^4}y(t) - \frac{d^2}{dt^2}y(t) = 0$$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y(t)$
  - Define new variable $y_2(t)$

    $y_2(t) = \frac{d}{dt}y(t)$
  - Define new variable $y_3(t)$

    $y_3(t) = \frac{d^2}{dt^2}y(t)$
  - Define new variable $y_4(t)$

    $y_4(t) = \frac{d^3}{dt^3}y(t)$
  - Isolate for $\frac{d}{dt}y_4(t)$ using original ODE

    $\frac{d}{dt}y_4(t) = y_3(t)$

    Convert linear ODE into a system of first order ODEs

    $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), y_4(t) = \frac{d}{dt}y_3(t), \frac{d}{dt}y_4(t) = y_3(t)\right]$
- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-t} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_4 e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = -c_1 e^{-t} + c_4 e^t + c_2$$

- Change variables back using $t = \ln(x)$

$$y = -\frac{c_1}{x} + x c_4 + c_2$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve([x^3*diff(y(x),x$4)+6*x^2*diff(y(x),x$3)+6*x*diff(y(x),x$2)=0,y(1) = 0, D(y)(1) = 0],
```

$$y(x) = -c_3 - c_4 + (c_3 - c_4)\ln(x) + \frac{c_3}{x} + c_4 x$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 34

```
DSolve[{x^3*y''''[x]+6*x^2*y'''[x]+6*x*y''[x]==0,{y[1]==0,y'[1]==0}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{(x-1)(c_1(x-1) + 2c_2 x)}{2x} - c_2 \log(x)$$

23 Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171

23.1 problem 724

Internal problem ID [15468]
Internal file name [`OUTPUT/15468_Wednesday_May_08_2024_04_00_57_PM_85384528/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 724.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + yx = 1}$$

With initial conditions

$$[y(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

23.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x$$
$$q(x) = 1$$

Hence the ode is

$$y' + yx = 1$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

23.1.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f & (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f & (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f & (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned} \tag{5}$$

For example, for $n=1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0,y_0} \tag{6}$$

Hence

$$
\begin{aligned}
F_0 &= 1 - yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= yx^2 - y - x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \left(-x^3 + 3x\right) y + x^2 - 2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \left(x^4 - 6x^2 + 3\right) y - x^3 + 5x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \left(-x^5 + 10x^3 - 15x\right) y + x^4 - 9x^2 + 8
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 0 \\
F_2 &= -2 \\
F_3 &= 0 \\
F_4 &= 8
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = x - \frac{x^3}{3} + \frac{x^5}{15} + O\left(x^6\right)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$
y(0) = y(0)
$$

Therefore the solution becomes

$$y = x - \frac{1}{3}x^3 + \frac{1}{15}x^5$$

Hence the solution can be written as

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

which simplifies to

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' + yx &= 1 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= x \\ p(x) &= 1 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) x = 1 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 1 \tag{3}$$

$n = 0$ gives

$$(a_1) 1 = 1$$
$$a_1 = 1$$

Or

$$a_1 = 1$$

For $1 \leq n$, the recurrence equation is

$$((1+n) a_{1+n} + a_{n-1}) x^n = 1 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$(2a_2 + a_0) x = 0$$
$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(3a_3 + a_1)\, x^2 &= 0\\ 3a_3 + a_1 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{1}{3}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(4a_4 + a_2)\, x^3 &= 0\\ 4a_4 + a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(5a_5 + a_3)\, x^4 &= 0\\ 5a_5 + a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{15}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(6a_6 + a_4)\, x^5 &= 0\\ 6a_6 + a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + x - \frac{1}{2} a_0 x^2 - \frac{1}{3} x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4\right) a_0 + x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4\right) y(0) + x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 0$$

Therefore the solution becomes

$$y = x - \frac{1}{3} x^3 + \frac{1}{15} x^5$$

Hence the solution can be written as

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

which simplifies to

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6) \tag{1}$$
$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

Verified OK.

$$y = x - \frac{x^3}{3} + \frac{x^5}{15} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x)=1-x*y(x),y(0) = 0],y(x),type='series',x=0);
```

$$y(x) = x - \frac{1}{3}x^3 + \frac{1}{15}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y'[x]==1-x*y[x],{y[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{15} - \frac{x^3}{3} + x$$

23.2 problem 725

Internal problem ID [15469]
Internal file name [`OUTPUT/15469_Wednesday_May_08_2024_04_00_59_PM_27445916/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 725.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y - x}{y + x} = 0}$$

With initial conditions

$$[y(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

23.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y-x}{y+x} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x<-1\vee -1<x\}$$

And the point $x_0=0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y-x}{y+x}\right) \\ &= \frac{1}{y+x}-\frac{y-x}{(y+x)^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=1$ is

$$\{x<-1\vee -1<x\}$$

And the point $x_0=0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=1$ is inside this domain. Therefore solution exists and is unique.

23.2.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f \qquad (1)\\
\frac{d^2f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f \qquad (2)\\
\frac{d^3f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2f}{dx^2}\right)f \qquad (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0=f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) \qquad (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)F_0 \qquad (5)
\end{aligned}
$$

For example, for $n=1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right)F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0, y_0} \tag{6}
$$

Hence

$$
\begin{aligned}
F_0 &= \frac{y-x}{y+x} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{-2x^2-2y^2}{(y+x)^3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\frac{4(x-2y)\,(x^2+y^2)}{(y+x)^5} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= -\frac{20(x^2+y^2)\,(x^2-2yx+3y^2)}{(y+x)^7} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \frac{40(16y^3-13y^2x+10yx^2-3x^3)\,(x^2+y^2)}{(y+x)^9}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0)=0$ and $y(0)=1$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -2 \\
F_2 &= 8 \\
F_3 &= -60 \\
F_4 &= 640
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = -x^2 + x + 1 + \frac{4x^3}{3} - \frac{5x^4}{2} + \frac{16x^5}{3} + O(x^6)
$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = -x^2 + x + 1 + \frac{4}{3}x^3 - \frac{5}{2}x^4 + \frac{16}{3}x^5$$

Hence the solution can be written as

$$y = -x^2 + x + 1 + \frac{4x^3}{3} - \frac{5x^4}{2} + \frac{16x^5}{3} + O(x^6)$$

which simplifies to

$$y = -x^2 + x + 1 + \frac{4x^3}{3} - \frac{5x^4}{2} + \frac{16x^5}{3} + O(x^6)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = -x^2 + x + 1 + \frac{4x^3}{3} - \frac{5x^4}{2} + \frac{16x^5}{3} + O(x^6) \tag{1}$$

Verification of solutions

$$y = -x^2 + x + 1 + \frac{4x^3}{3} - \frac{5x^4}{2} + \frac{16x^5}{3} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x)=(y(x)-x)/(y(x)+x),y(0) = 1],y(x),type='series',x=0);
```

$$y(x) = 1 + x - x^2 + \frac{4}{3}x^3 - \frac{5}{2}x^4 + \frac{16}{3}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 32

```
AsymptoticDSolveValue[{y'[x]==(y[x]-x)/(y[x]+x),{y[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{16x^5}{3} - \frac{5x^4}{2} + \frac{4x^3}{3} - x^2 + x + 1$$

23.3 problem 726

Internal problem ID [15470]
Internal file name [`OUTPUT/15470_Wednesday_May_08_2024_04_01_00_PM_28126000/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 726.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \sin(x)\, y = 0}$$

With initial conditions

$$[y(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

23.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\sin(x)\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' - \sin(x)\, y = 0$$

The domain of $p(x) = -\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

23.3.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots\\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots\\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned} \tag{2}$$

$$\begin{aligned}\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \end{aligned} \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$\begin{aligned}F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \end{aligned} \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned}F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \sin(x) y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= y(\cos(x) + \sin(x)^2) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\sin(x)\cos(x)(\cos(x) - 3) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= (\cos(x)^4 - 6\cos(x)^3 + 5\cos(x)^2 + 5\cos(x) - 3) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \sin(x) y(\cos(x)^4 - 10\cos(x)^3 + 23\cos(x)^2 - 5\cos(x) - 8)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= 0 \\
F_3 &= 2 \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = 1 + \frac{1}{2}x^2 + \frac{1}{12}x^4$$

Hence the solution can be written as

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

which simplifies to

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \sin(x)\, y &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\sin(x) \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \sin\left(x\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding $-\sin\left(x\right)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -\sin\left(x\right) &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \dots \\ &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(-x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7\right) \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + -x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) + \frac{x^7}{5040} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6}\right) \\ &+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+7} a_n}{5040}\right) = 0 \end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} (-a_{n-1} x^n)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6} = \sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) = \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+7} a_n}{5040} = \sum_{n=7}^{\infty} \frac{a_{n-7} x^n}{5040}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \sum_{n=1}^{\infty} (-a_{n-1} x^n) + \left(\sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}\right) + \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right) + \left(\sum_{n=7}^{\infty} \frac{a_{n-7} x^n}{5040}\right) = 0 \tag{3}$$

$n = 1$ gives

$$2a_2 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_2 = \frac{a_0}{2}$$

$n = 2$ gives

$$3a_3 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$3a_3 = 0$$

Or

$$a_3 = 0$$

$n = 3$ gives

$$4a_4 - a_2 + \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{12}$$

$n = 4$ gives

$$5a_5 - a_3 + \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$5a_5 = 0$$

Or

$$a_5 = 0$$

$n = 5$ gives

$$6a_6 - a_4 + \frac{a_2}{6} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{720}$$

For $7 \leq n$, the recurrence equation is

$$(1+n)\, a_{1+n} - a_{n-1} + \frac{a_{n-3}}{6} - \frac{a_{n-5}}{120} + \frac{a_{n-7}}{5040} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = -\frac{-5040a_{n-1} + 840a_{n-3} - 42a_{n-5} + a_{n-7}}{5040\,(1+n)} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{2}a_0x^2 + \frac{1}{12}a_0x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{12}x^4\right)a_0 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{12}x^4\right)y(0) + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 1$$

Therefore the solution becomes

$$y = 1 + \frac{1}{2}x^2 + \frac{1}{12}x^4$$

Hence the solution can be written as

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

which simplifies to

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6) \tag{1}$$
$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

Verified OK.

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{12} + O(x^6)$$

Verified OK.

23.3.3 Maple step by step solution

Let's solve

$[y' - \sin(x)\, y = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \sin(x)\, dx + c_1$

- Evaluate integral

  $\ln(y) = -\cos(x) + c_1$

- Solve for y

  $y = e^{-\cos(x)+c_1}$

- Use initial condition $y(0) = 1$

  $1 = e^{c_1-1}$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = e^{-\cos(x)+1}$

- Solution to the IVP

  $y = e^{-\cos(x)+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x)=sin(x)*y(x),y(0) = 1],y(x),type='series',x=0);
```

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{12}x^4 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y'[x]==Sin[x]*y[x],{y[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{12} + \frac{x^2}{2} + 1$$

23.4 problem 727

Internal problem ID [15471]
Internal file name [`OUTPUT/15471_Wednesday_May_08_2024_04_01_02_PM_56188045/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 727.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

23.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= x \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + yx = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1209}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1210}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -xy' - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^2 - 2y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x - \frac{x^4}{12} + O(x^6)
$$

$$
y = x - \frac{x^4}{12} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

$$y = x - \frac{x^4}{12} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = x - \frac{x^4}{12} + O(x^6) \tag{1}$$
$$y = x - \frac{x^4}{12} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x - \frac{x^4}{12} + O(x^6)$$

Verified OK.

$$y = x - \frac{x^4}{12} + O(x^6)$$

Verified OK.

23.4.2 Maple step by step solution

Let's solve

$$\left[y'' = -yx, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + yx = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$$

  ◦ Shift index using $k- >k - 1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  ◦ Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2 + 3k + 2\right) a_{k+2} + a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2+3k+5\right) a_{k+3}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
Order:=6;
dsolve([diff(y(x),x$2)+x*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x)=x-\frac{1}{12}x^4+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 12

```
AsymptoticDSolveValue[{y''[x]+x*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x)\to x-\frac{x^4}{12}$$

23.5 problem 728

Internal problem ID [15472]
Internal file name [`OUTPUT/15472_Wednesday_May_08_2024_04_01_03_PM_71973850/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 728.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - y' \sin(x) = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

23.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\sin(x) \\ q(x) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' \sin(x) = 0$$

The domain of $p(x) = -\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1212)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1213)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y' \sin(x) \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \left(\sin(x)^2 + \cos(x)\right) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -\sin(x)\cos(x)\left(\cos(x) - 3\right) y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\cos(x)^4 - 6\cos(x)^3 + 5\cos(x)^2 + 5\cos(x) - 3\right) y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y' \sin(x)\left(\cos(x)^4 - 10\cos(x)^3 + 23\cos(x)^2 - 5\cos(x) - 8\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= 0 \\
F_3 &= 2 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O\left(x^6\right)
$$

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \sin(x) \tag{1}$$

Expanding $-\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -\sin(x) &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \dots \\ &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(-x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0$$

Expanding the second term in (1) gives

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + -x \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^3}{6} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \\ &- \frac{x^5}{120} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \frac{x^7}{5040} \cdot \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) = 0 \end{aligned}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=1}^{\infty} \frac{n\, x^{n+2} a_n}{6}\right) \\ &+ \sum_{n=1}^{\infty} \left(-\frac{n\, x^{n+4} a_n}{120}\right) + \left(\sum_{n=1}^{\infty} \frac{n\, x^{n+6} a_n}{5040}\right) = 0 \end{aligned} \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n \\
\sum_{n=1}^{\infty} \frac{n x^{n+2} a_n}{6} &= \sum_{n=3}^{\infty} \frac{(n-2) a_{n-2} x^n}{6} \\
\sum_{n=1}^{\infty} \left(-\frac{n x^{n+4} a_n}{120}\right) &= \sum_{n=5}^{\infty} \left(-\frac{(n-4) a_{n-4} x^n}{120}\right) \\
\sum_{n=1}^{\infty} \frac{n x^{n+6} a_n}{5040} &= \sum_{n=7}^{\infty} \frac{(n-6) a_{n-6} x^n}{5040}
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-n a_n x^n) + \left(\sum_{n=3}^{\infty} \frac{(n-2) a_{n-2} x^n}{6}\right) \\
&+ \sum_{n=5}^{\infty} \left(-\frac{(n-4) a_{n-4} x^n}{120}\right) + \left(\sum_{n=7}^{\infty} \frac{(n-6) a_{n-6} x^n}{5040}\right) = 0
\end{aligned} \tag{3}
$$

$n = 1$ gives

$$6a_3 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{6}$$

$n = 2$ gives

$$12a_4 - 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$12a_4 = 0$$

Or

$$a_4 = 0$$

$n = 3$ gives

$$20a_5 - 3a_3 + \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_1}{60}$$

$n = 4$ gives

$$30a_6 - 4a_4 + \frac{a_2}{3} = 0$$

Which after substituting earlier equations, simplifies to

$$30a_6 = 0$$

Or

$$a_6 = 0$$

$n = 5$ gives

$$42a_7 - 5a_5 + \frac{a_3}{2} - \frac{a_1}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_1}{5040}$$

For $7 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n + \frac{(n-2)\, a_{n-2}}{6} - \frac{(n-4)\, a_{n-4}}{120} + \frac{(n-6)\, a_{n-6}}{5040} = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{5040 n a_n - n a_{n-6} + 42 n a_{n-4} - 840 n a_{n-2} + 6 a_{n-6} - 168 a_{n-4} + 1680 a_{n-2}}{5040\,(n+2)\,(n+1)} \quad (5)$$

$$= \frac{n a_n}{(n+2)\,(n+1)} + \frac{(-n+6)\, a_{n-6}}{5040\,(n+2)\,(n+1)} + \frac{(42n-168)\, a_{n-4}}{5040\,(n+2)\,(n+1)} + \frac{(-840n+1680)\, a_{n-2}}{5040\,(n+2)\,(n+1)}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \frac{1}{6} a_1 x^3 + \frac{1}{60} a_1 x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = a_0 + \left(x + \frac{1}{6} x^3 + \frac{1}{60} x^5\right) a_1 + O(x^6) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = c_1 + \left(x + \frac{1}{6} x^3 + \frac{1}{60} x^5\right) c_2 + O(x^6)
$$

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O(x^6)
$$

Summary

The solution(s) found are the following

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O(x^6) \tag{1}
$$

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O(x^6) \tag{2}
$$

Verification of solutions

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O(x^6)
$$

Verified OK.

$$
y = x + \frac{x^3}{6} + \frac{x^5}{60} + O(x^6)
$$

Verified OK.

23.5.2 Maple step by step solution

Let's solve

$\left[y'' = y' \sin(x), y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) = u(x)\sin(x)$

- Separate variables

  $\frac{u'(x)}{u(x)} = \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{u(x)}dx = \int \sin(x)\, dx + c_1$

- Evaluate integral

  $\ln(u(x)) = -\cos(x) + c_1$

- Solve for $u(x)$

  $u(x) = \mathrm{e}^{-\cos(x)+c_1}$

- Solve 1st ODE for $u(x)$

  $u(x) = \mathrm{e}^{-\cos(x)+c_1}$

- Make substitution $u = y'$

  $y' = \mathrm{e}^{-\cos(x)+c_1}$

- Integrate both sides to solve for y

  $\int y' dx = \int \mathrm{e}^{-\cos(x)+c_1} dx + c_2$

- Compute integrals

  $y = \int \mathrm{e}^{-\cos(x)+c_1} dx + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-diff(y(x),x)*sin(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0);
```

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+Sin[x]*y'[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{30} - \frac{x^3}{6} + x$$

23.6 problem 729

Internal problem ID [15473]
Internal file name [`OUTPUT/15473_Wednesday_May_08_2024_04_01_05_PM_36847634/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 729.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' + \sin(x)\, y = x}$$

With initial conditions

$$[y(\pi) = 1, y'(\pi) = 0]$$

With the expansion point for the power series method at $x = \pi$.

23.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{\sin(x)}{x}$$
$$F = 1$$

Hence the ode is

$$y'' + \frac{\sin(x)\,y}{x} = 1$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = \frac{\sin(x)}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \pi$ is also inside this domain. The domain of $F = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - \pi$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+\pi)\left(\frac{d^2}{dt^2}y(t)\right) - \sin(t)\,y(t) = t + \pi$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = 1$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1216) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1217) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{\sin(t)\,y(t) + t + \pi}{t + \pi} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= \frac{\sin(t)\,(t+\pi)\left(\frac{d}{dt}y(t)\right) + (-\sin(t) + \cos(t)\,(t+\pi))\,y(t)}{(t+\pi)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= \frac{2(t+\pi)\,(-\sin(t) + \cos(t)\,(t+\pi))\left(\frac{d}{dt}y(t)\right) + y(t)\,(t+\pi)\sin(t)^2 + \left((-\pi^2 - 2\pi t - t^2 + 2)\,y(t) + (t\right.}{(t+\pi)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{-3(t+\pi)\left(\frac{(-t-\pi)\sin(t)^2}{3} + (\pi^2 + 2\pi t + t^2 - 2)\sin(t) + 2\cos(t)\,(t+\pi)\right)\left(\frac{d}{dt}y(t)\right) - 4y(t)\,(t+\pi)\sin(}{} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= \frac{-4\left(\frac{3(t+\pi)\sin(t)^2}{2} + 3\left(2 - \frac{(t+\pi)^2\cos(t)}{2} - t^2 - 2\pi t - \pi^2\right)\sin(t) + \cos(t)\,(t+\pi)\,(\pi^2 + 2\pi t + t^2 - 6)\right)(t}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = 1$ and

$y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= \frac{1}{\pi} \\
F_2 &= -\frac{2}{\pi^2} \\
F_3 &= \frac{2\pi^2 + 6}{\pi^3} \\
F_4 &= \frac{-4\pi^2 - 24}{\pi^4}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = 1 + \frac{t^2}{2} + \frac{t^3}{6\pi} - \frac{t^4}{12\pi^2} + \frac{t^5}{60\pi} + \frac{t^5}{20\pi^3} - \frac{t^6}{180\pi^2} - \frac{t^6}{30\pi^4} + O(t^6)
$$

$$
y(t) = 1 + \frac{t^2}{2} + \frac{t^3}{6\pi} - \frac{t^4}{12\pi^2} + \frac{t^5}{60\pi} + \frac{t^5}{20\pi^3} - \frac{t^6}{180\pi^2} - \frac{t^6}{30\pi^4} + O(t^6)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(t + \pi)\left(\frac{d^2}{dt^2}y(t)\right) - \sin(t)\, y(t) = t + \pi
$$

Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(t + \pi)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) - \sin(t)\left(\sum_{n=0}^{\infty} a_n t^n\right) = t + \pi \tag{1}
$$

Expanding $-\sin(t)$ as Taylor series around $t=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\sin(t) &= -t+\frac{1}{6}t^3-\frac{1}{120}t^5+\frac{1}{5040}t^7+\dots\\
&= -t+\frac{1}{6}t^3-\frac{1}{120}t^5+\frac{1}{5040}t^7
\end{aligned}
$$

Expanding $t+\pi$ as Taylor series around $t=0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
t+\pi &= t+\pi+\dots\\
&= t+\pi
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
(t+\pi)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\left(-t+\frac{1}{6}t^3-\frac{1}{120}t^5+\frac{1}{5040}t^7\right)\left(\sum_{n=0}^{\infty} a_n t^n\right)=t+\pi
$$

Expanding the second term in (1) gives

$$
\begin{aligned}
&(t+\pi)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+-t\cdot\left(\sum_{n=0}^{\infty} a_n t^n\right)+\frac{t^3}{6}\\
&\cdot\left(\sum_{n=0}^{\infty} a_n t^n\right)-\frac{t^5}{120}\cdot\left(\sum_{n=0}^{\infty} a_n t^n\right)+\frac{t^7}{5040}\cdot\left(\sum_{n=0}^{\infty} a_n t^n\right)=t+\pi
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right)+\left(\sum_{n=2}^{\infty} n\pi\, t^{n-2} a_n (n-1)\right)+\sum_{n=0}^{\infty}\left(-t^{1+n} a_n\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{t^{n+3} a_n}{6}\right)+\sum_{n=0}^{\infty}\left(-\frac{t^{n+5} a_n}{120}\right)+\left(\sum_{n=0}^{\infty}\frac{t^{n+7} a_n}{5040}\right)=t+\pi
\end{aligned}
\tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^{n}$$

$$\sum_{n=2}^{\infty} n\pi\, t^{n-2} a_n (n-1) = \sum_{n=0}^{\infty} \pi (n+2)\, a_{n+2} (1+n)\, t^{n}$$

$$\sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} t^{n}\right)$$

$$\sum_{n=0}^{\infty} \frac{t^{n+3} a_n}{6} = \sum_{n=3}^{\infty} \frac{a_{n-3} t^{n}}{6}$$

$$\sum_{n=0}^{\infty} \left(-\frac{t^{n+5} a_n}{120}\right) = \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} t^{n}}{120}\right)$$

$$\sum_{n=0}^{\infty} \frac{t^{n+7} a_n}{5040} = \sum_{n=7}^{\infty} \frac{a_{n-7} t^{n}}{5040}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, t^{n}\right) + \left(\sum_{n=0}^{\infty} \pi (n+2)\, a_{n+2} (1+n)\, t^{n}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} t^{n}\right) + \left(\sum_{n=3}^{\infty} \frac{a_{n-3} t^{n}}{6}\right) + \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} t^{n}}{120}\right) + \left(\sum_{n=7}^{\infty} \frac{a_{n-7} t^{n}}{5040}\right) = t + \pi \tag{3}$$

$n = 0$ gives

$$(2\pi a_2)\, 1 = \pi$$
$$2\pi a_2 = \pi$$

Or

$$a_2 = \frac{1}{2}$$

$n = 1$ gives

$$(6\pi a_3 - a_0 + 2a_2)\, t = t$$
$$6\pi a_3 - a_0 + 2a_2 = 1$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6\pi}$$

$n = 2$ gives

$$12\pi a_4 - a_1 + 6a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{\pi a_1 - a_0}{12\pi^2}$$

$n = 3$ gives

$$20\pi a_5 - a_2 + 12a_4 + \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = -\frac{\pi^2 a_0 - 3\pi^2 + 6\pi a_1 - 6a_0}{120\pi^3}$$

$n = 4$ gives

$$30\pi a_6 - a_3 + 20a_5 + \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{\pi^3 a_1 - 2\pi^2 a_0 + 3\pi^2 - 6\pi a_1 + 6a_0}{180\pi^4}$$

$n = 5$ gives

$$42\pi a_7 - a_4 + 30a_6 + \frac{a_2}{6} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{\pi^4 a_0 - 10\pi^4 + 30\pi^3 a_1 - 50\pi^2 a_0 + 60\pi^2 - 120\pi a_1 + 120a_0}{5040\pi^5}$$

For $7 \leq n$, the recurrence equation is

$$\left((1+n)\, a_{1+n} n + \pi(n+2)\, a_{n+2}(1+n) - a_{n-1} + \frac{a_{n-3}}{6} - \frac{a_{n-5}}{120} + \frac{a_{n-7}}{5040}\right) t^n = t + \pi \quad (4)$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t + \frac{t^2}{2} + \frac{a_0 t^3}{6\pi} + \frac{(\pi a_1 - a_0)\, t^4}{12\pi^2} - \frac{(\pi^2 a_0 - 3\pi^2 + 6\pi a_1 - 6a_0)\, t^5}{120\pi^3} + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1+\frac{t^3}{6\pi}-\frac{t^4}{12\pi^2}-\frac{(\pi^2-6)\, t^5}{120\pi^3}\right) a_0+\left(t+\frac{t^4}{12\pi}-\frac{t^5}{20\pi^2}\right) a_1+\frac{t^2}{2}+\frac{t^5}{40\pi}+O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1+\frac{t^3}{6\pi}-\frac{t^4}{12\pi^2}-\frac{(\pi^2-6)\, t^5}{120\pi^3}\right) c_1+\left(t+\frac{t^4}{12\pi}-\frac{t^5}{20\pi^2}\right) c_2+\frac{t^2}{2}+\frac{t^5}{40\pi}+O(t^6)
$$

$$
y(t) = 1 + \frac{t^3}{6\pi} - \frac{t^4}{12\pi^2} + \frac{t^5}{60\pi} + \frac{t^5}{20\pi^3} + \frac{t^2}{2} + O(t^6)
$$

Replacing t in the above with the original independent variable $x$susing $t = x - \pi$ results in

$$
\begin{aligned}
y = 1 &+ \frac{(x-\pi)^2}{2} + \frac{(x-\pi)^3}{6\pi} - \frac{(x-\pi)^4}{12\pi^2} + \frac{(x-\pi)^5}{60\pi} \\
&+ \frac{(x-\pi)^5}{20\pi^3} - \frac{(x-\pi)^6}{180\pi^2} - \frac{(x-\pi)^6}{30\pi^4} + O\left((x-\pi)^6\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = 1 &+ \frac{(x-\pi)^2}{2} + \frac{(x-\pi)^3}{6\pi} - \frac{(x-\pi)^4}{12\pi^2} + \frac{(x-\pi)^5}{60\pi} \\
&+ \frac{(x-\pi)^5}{20\pi^3} - \frac{(x-\pi)^6}{180\pi^2} - \frac{(x-\pi)^6}{30\pi^4} + O\left((x-\pi)^6\right)
\end{aligned} \quad (1)
$$

Verification of solutions

$$
y = 1 + \frac{(x-\pi)^2}{2} + \frac{(x-\pi)^3}{6\pi} - \frac{(x-\pi)^4}{12\pi^2} + \frac{(x-\pi)^5}{60\pi} + \frac{(x-\pi)^5}{20\pi^3} - \frac{(x-\pi)^6}{180\pi^2} - \frac{(x-\pi)^6}{30\pi^4} + O\left((x-\pi)^6\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         -> trying with_periodic_functions in the coefficients
            --- Trying Lie symmetry methods, 2nd order ---
            `, `-> Computing symmetries using: way = 5
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
         trying Riccati sub-methods:
            trying Riccati_symmetries
            -> trying a symmetry pattern of the form [F(x)*G(y), 0]
            -> trying a symmetry pattern of the form [0, F(x)*G(y)]
            -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, y]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve([x*diff(y(x),x$2)+y(x)*sin(x)=x,y(Pi) = 1, D(y)(Pi) = 0],y(x),type='series',x=Pi);
```

$$y(x) = 1 + \frac{1}{2}(-\pi + x)^2 + \frac{1}{6}\frac{1}{\pi}(-\pi + x)^3 - \frac{1}{12}\frac{1}{\pi^2}(-\pi + x)^4 + \frac{1}{60}\frac{\pi^2 + 3}{\pi^3}(-\pi + x)^5 + \mathrm{O}\left((-\pi + x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 75

```
AsymptoticDSolveValue[{x*y''[x]+Sin[x]*y[x]==x,{y[Pi]==1,y'[Pi]==0}},y[x],{x,Pi,5}]
```

$$y(x) \to \frac{1}{60}\left(\frac{3}{2\pi} - \frac{\pi^2 - 6}{2\pi^3}\right)(x - \pi)^5 - \frac{(x - \pi)^4}{12\pi^2} + \frac{(x - \pi)^3}{6\pi} + \frac{1}{2}(x - \pi)^2 + 1$$

23.7 problem 730

Internal problem ID [15474]
Internal file name [`OUTPUT/15474_Wednesday_May_08_2024_04_01_09_PM_89370389/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 730.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\ln(x)\, y'' - y\sin(x) = 0}$$

With initial conditions

$$[y(\mathrm{e}) = \mathrm{e}^{-1}, y'(\mathrm{e}) = 0]$$

With the expansion point for the power series method at $x = \mathrm{e}$.

23.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{\sin(x)}{\ln(x)} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y \sin(x)}{\ln(x)} = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \mathrm{e}$ is inside this domain. The domain of $q(x) = -\frac{\sin(x)}{\ln(x)}$ is

$$\{0 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = \mathrm{e}$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - \mathrm{e}$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\ln(t + \mathrm{e}) \left(\frac{d^2}{dt^2} y(t)\right) - y(t) \sin(t + \mathrm{e}) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$y(0) = \mathrm{e}^{-1}$$
$$y'(0) = 0$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1219)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1220)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\ \ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y(t)\sin(t+\mathrm{e})}{\ln(t+\mathrm{e})}\\
F_1 &= \frac{dF_0}{dt}\\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0\\
&= \frac{\left(y(t)\cos(t+\mathrm{e})+\left(\frac{d}{dt}y(t)\right)\sin(t+\mathrm{e})\right)(t+\mathrm{e})\ln(t+\mathrm{e})-y(t)\sin(t+\mathrm{e})}{\ln(t+\mathrm{e})^2(t+\mathrm{e})}\\
F_2 &= \frac{dF_1}{dt}\\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1\\
&= \frac{-(t^2+2\,\mathrm{e}t+\mathrm{e}^2)\left(y(t)\sin(t+\mathrm{e})-2\left(\frac{d}{dt}y(t)\right)\cos(t+\mathrm{e})\right)\ln(t+\mathrm{e})^2+\left(y(t)(t^2+2\,\mathrm{e}t+\mathrm{e}^2)\sin(t+\mathrm{e}\right.}{\ln(t+\mathrm{e})^3}\\
F_3 &= \frac{dF_2}{dt}\\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2\\
&= \frac{-(t^3+3t^2\mathrm{e}+3\,\mathrm{e}^2t+\mathrm{e}^3)\left(y(t)\cos(t+\mathrm{e})+3\left(\frac{d}{dt}y(t)\right)\sin(t+\mathrm{e})\right)\ln(t+\mathrm{e})^3+\left(\left(\frac{d}{dt}y(t)\right)(t^3+3t^2\mathrm{e}+\right.}{}\\
F_4 &= \frac{dF_3}{dt}\\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3\\
&= \frac{(t^4+4t^3\mathrm{e}+6t^2\mathrm{e}^2+4t\,\mathrm{e}^3+\mathrm{e}^4)\left(y(t)\sin(t+\mathrm{e})-4\left(\frac{d}{dt}y(t)\right)\cos(t+\mathrm{e})\right)\ln(t+\mathrm{e})^4+\left(\left(6\left(\frac{d}{dt}y(t)\right)(t^4+\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=\mathrm{e}^{-1}$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= \mathrm{e}^{-1}\sin(\mathrm{e})\\
F_1 &= -\mathrm{e}^{-2}\sin(\mathrm{e})+\cos(\mathrm{e})\,\mathrm{e}^{-1}\\
F_2 &= \sin(\mathrm{e})^2\,\mathrm{e}^{-1}+3\sin(\mathrm{e})\,\mathrm{e}^{-3}-\mathrm{e}^{-1}\sin(\mathrm{e})-2\,\mathrm{e}^{-2}\cos(\mathrm{e})\\
F_3 &= -4\sin(\mathrm{e})^2\,\mathrm{e}^{-2}+9\cos(\mathrm{e})\,\mathrm{e}^{-3}-\cos(\mathrm{e})\,\mathrm{e}^{-1}+2\sin(2\,\mathrm{e})\,\mathrm{e}^{-1}-14\sin(\mathrm{e})\,\mathrm{e}^{-4}+3\,\mathrm{e}^{-2}\sin(\mathrm{e})\\
F_4 &= 88\sin(\mathrm{e})\,\mathrm{e}^{-5}-56\,\mathrm{e}^{-4}\cos(\mathrm{e})+25\sin(\mathrm{e})^2\,\mathrm{e}^{-3}-18\sin(\mathrm{e})\,\mathrm{e}^{-3}+4\,\mathrm{e}^{-2}\cos(\mathrm{e})-11\sin(2\,\mathrm{e})\,\mathrm{e}^{-2}-\frac{\mathrm{e}^{-1}\mathrm{s}}{}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = {} & \frac{t^{5} \mathrm{e}^{-1} \sin(\mathrm{e}) \cos(\mathrm{e})}{30} - \frac{t^{6} \sin(\mathrm{e}) \cos(\mathrm{e})^{2} \mathrm{e}^{-1}}{720} - \frac{11 t^{6} \mathrm{e}^{-2} \sin(\mathrm{e}) \cos(\mathrm{e})}{360} \\
& + \mathrm{e}^{-1} + \frac{\mathrm{e}^{-1} \sin(\mathrm{e}) t^{2}}{2} + O(t^{6}) - \frac{t^{3} \mathrm{e}^{-2} \sin(\mathrm{e})}{6} + \frac{t^{3} \cos(\mathrm{e}) \mathrm{e}^{-1}}{6} \\
& + \frac{t^{4} \sin(\mathrm{e})^{2} \mathrm{e}^{-1}}{24} + \frac{t^{4} \sin(\mathrm{e}) \mathrm{e}^{-3}}{8} - \frac{t^{4} \mathrm{e}^{-1} \sin(\mathrm{e})}{24} - \frac{t^{4} \mathrm{e}^{-2} \cos(\mathrm{e})}{12} \\
& - \frac{t^{5} \sin(\mathrm{e})^{2} \mathrm{e}^{-2}}{30} + \frac{3 t^{5} \cos(\mathrm{e}) \mathrm{e}^{-3}}{40} - \frac{t^{5} \cos(\mathrm{e}) \mathrm{e}^{-1}}{120} - \frac{7 t^{5} \sin(\mathrm{e}) \mathrm{e}^{-4}}{60} \\
& + \frac{t^{5} \mathrm{e}^{-2} \sin(\mathrm{e})}{40} + \frac{5 t^{6} \sin(\mathrm{e})^{2} \mathrm{e}^{-3}}{144} + \frac{11 t^{6} \mathrm{e}^{-1} \cos(\mathrm{e})^{2}}{720} + \frac{11 t^{6} \sin(\mathrm{e}) \mathrm{e}^{-5}}{90} \\
& - \frac{7 t^{6} \mathrm{e}^{-4} \cos(\mathrm{e})}{90} - \frac{t^{6} \sin(\mathrm{e}) \mathrm{e}^{-3}}{40} + \frac{t^{6} \mathrm{e}^{-2} \cos(\mathrm{e})}{180} + \frac{t^{6} \mathrm{e}^{-1} \sin(\mathrm{e})}{360} - \frac{7 t^{6} \mathrm{e}^{-1}}{720}
\end{aligned}
$$

$$
\begin{aligned}
y(t) = {} & \frac{t^{5} \mathrm{e}^{-1} \sin(\mathrm{e}) \cos(\mathrm{e})}{30} - \frac{t^{6} \sin(\mathrm{e}) \cos(\mathrm{e})^{2} \mathrm{e}^{-1}}{720} - \frac{11 t^{6} \mathrm{e}^{-2} \sin(\mathrm{e}) \cos(\mathrm{e})}{360} \\
& + \mathrm{e}^{-1} + \frac{\mathrm{e}^{-1} \sin(\mathrm{e}) t^{2}}{2} + O(t^{6}) - \frac{t^{3} \mathrm{e}^{-2} \sin(\mathrm{e})}{6} + \frac{t^{3} \cos(\mathrm{e}) \mathrm{e}^{-1}}{6} \\
& + \frac{t^{4} \sin(\mathrm{e})^{2} \mathrm{e}^{-1}}{24} + \frac{t^{4} \sin(\mathrm{e}) \mathrm{e}^{-3}}{8} - \frac{t^{4} \mathrm{e}^{-1} \sin(\mathrm{e})}{24} - \frac{t^{4} \mathrm{e}^{-2} \cos(\mathrm{e})}{12} \\
& - \frac{t^{5} \sin(\mathrm{e})^{2} \mathrm{e}^{-2}}{30} + \frac{3 t^{5} \cos(\mathrm{e}) \mathrm{e}^{-3}}{40} - \frac{t^{5} \cos(\mathrm{e}) \mathrm{e}^{-1}}{120} - \frac{7 t^{5} \sin(\mathrm{e}) \mathrm{e}^{-4}}{60} \\
& + \frac{t^{5} \mathrm{e}^{-2} \sin(\mathrm{e})}{40} + \frac{5 t^{6} \sin(\mathrm{e})^{2} \mathrm{e}^{-3}}{144} + \frac{11 t^{6} \mathrm{e}^{-1} \cos(\mathrm{e})^{2}}{720} + \frac{11 t^{6} \sin(\mathrm{e}) \mathrm{e}^{-5}}{90} \\
& - \frac{7 t^{6} \mathrm{e}^{-4} \cos(\mathrm{e})}{90} - \frac{t^{6} \sin(\mathrm{e}) \mathrm{e}^{-3}}{40} + \frac{t^{6} \mathrm{e}^{-2} \cos(\mathrm{e})}{180} + \frac{t^{6} \mathrm{e}^{-1} \sin(\mathrm{e})}{360} - \frac{7 t^{6} \mathrm{e}^{-1}}{720}
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt} y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2} y(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \frac{\left(\sum_{n=0}^{\infty} a_n t^n\right) \sin(t+\mathrm{e})}{\ln(t+\mathrm{e})} \tag{1}$$

Expanding $\ln(t+\mathrm{e})$ as Taylor series around $t=0$ and keeping only the first 6 terms gives

$$\begin{aligned}
\ln(t+\mathrm{e}) &= 1+\mathrm{e}^{-1}t-\frac{\mathrm{e}^{-2}t^2}{2}+\frac{\mathrm{e}^{-3}t^3}{3}-\frac{\mathrm{e}^{-4}t^4}{4}+\frac{\mathrm{e}^{-5}t^5}{5}-\frac{\mathrm{e}^{-6}t^6}{6}+\dots \\
&= 1+\mathrm{e}^{-1}t-\frac{\mathrm{e}^{-2}t^2}{2}+\frac{\mathrm{e}^{-3}t^3}{3}-\frac{\mathrm{e}^{-4}t^4}{4}+\frac{\mathrm{e}^{-5}t^5}{5}-\frac{\mathrm{e}^{-6}t^6}{6}
\end{aligned}$$

Expanding $-\sin(t+\mathrm{e})$ as Taylor series around $t=0$ and keeping only the first 6 terms gives

$$\begin{aligned}
-\sin(t+\mathrm{e}) &= -\sin(\mathrm{e})-\cos(\mathrm{e})\,t+\frac{\sin(\mathrm{e})\,t^2}{2}+\frac{\cos(\mathrm{e})\,t^3}{6}-\frac{\sin(\mathrm{e})\,t^4}{24}-\frac{\cos(\mathrm{e})\,t^5}{120}+\frac{\sin(\mathrm{e})\,t^6}{720}+\dots \\
&= -\sin(\mathrm{e})-\cos(\mathrm{e})\,t+\frac{\sin(\mathrm{e})\,t^2}{2}+\frac{\cos(\mathrm{e})\,t^3}{6}-\frac{\sin(\mathrm{e})\,t^4}{24}-\frac{\cos(\mathrm{e})\,t^5}{120}+\frac{\sin(\mathrm{e})\,t^6}{720}
\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned}
&\left(1+\mathrm{e}^{-1}t-\frac{\mathrm{e}^{-2}t^2}{2}+\frac{\mathrm{e}^{-3}t^3}{3}-\frac{\mathrm{e}^{-4}t^4}{4}+\frac{\mathrm{e}^{-5}t^5}{5}-\frac{\mathrm{e}^{-6}t^6}{6}\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) \\
&+\left(-\sin(\mathrm{e})-\cos(\mathrm{e})\,t+\frac{\sin(\mathrm{e})\,t^2}{2}+\frac{\cos(\mathrm{e})\,t^3}{6}\right. \\
&\left.-\frac{\sin(\mathrm{e})\,t^4}{24}-\frac{\cos(\mathrm{e})\,t^5}{120}+\frac{\sin(\mathrm{e})\,t^6}{720}\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0
\end{aligned}$$

Expanding the first term in (1) gives

$$\begin{aligned}
&1\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\frac{t}{\mathrm{e}}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)-\frac{t^2}{2\left(\mathrm{e}\right)^2} \\
&\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\frac{t^3}{3\left(\mathrm{e}\right)^3}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)-\frac{t^4}{4\left(\mathrm{e}\right)^4} \\
&\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\frac{t^5}{5\left(\mathrm{e}\right)^5}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)-\frac{t^6}{6\left(\mathrm{e}\right)^6} \\
&\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\left(-\sin(\mathrm{e})-\cos(\mathrm{e})\,t+\frac{\sin(\mathrm{e})\,t^2}{2}\right. \\
&\left.+\frac{\cos(\mathrm{e})\,t^3}{6}-\frac{\sin(\mathrm{e})\,t^4}{24}-\frac{\cos(\mathrm{e})\,t^5}{120}+\frac{\sin(\mathrm{e})\,t^6}{720}\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0
\end{aligned}$$

Expanding the second term in (1) gives

Expression too large to display

Which simplifies to

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \left(-\frac{n\,t^{n+4} a_n (n-1)\,\mathrm{e}^{-6}}{6}\right) + \left(\sum_{n=2}^{\infty} \frac{n\,t^{n+3} a_n (n-1)\,\mathrm{e}^{-5}}{5}\right) \\
&+ \sum_{n=2}^{\infty} \left(-\frac{n\,t^{n+2} a_n (n-1)\,\mathrm{e}^{-4}}{4}\right) + \left(\sum_{n=2}^{\infty} \frac{n\,t^{1+n} a_n (n-1)\,\mathrm{e}^{-3}}{3}\right) \\
&+ \sum_{n=2}^{\infty} \left(-\frac{n a_n t^n (n-1)\,\mathrm{e}^{-2}}{2}\right) + \left(\sum_{n=2}^{\infty} n\,t^{n-1} a_n (n-1)\,\mathrm{e}^{-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2}\right) + \sum_{n=0}^{\infty} \left(-a_n t^n \sin\left(\mathrm{e}\right)\right) \\
&+ \sum_{n=0}^{\infty} \left(-t^{1+n} a_n \cos\left(\mathrm{e}\right)\right) + \left(\sum_{n=0}^{\infty} \frac{t^{n+2} a_n \sin\left(\mathrm{e}\right)}{2}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{t^{n+3} a_n \cos\left(\mathrm{e}\right)}{6}\right) + \sum_{n=0}^{\infty} \left(-\frac{t^{n+4} a_n \sin\left(\mathrm{e}\right)}{24}\right) \\
&+ \sum_{n=0}^{\infty} \left(-\frac{t^{n+5} a_n \cos\left(\mathrm{e}\right)}{120}\right) + \left(\sum_{n=0}^{\infty} \frac{t^{n+6} a_n \sin\left(\mathrm{e}\right)}{720}\right) = 0
\end{aligned}
\tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} \left(-\frac{n\,t^{n+4} a_n (n-1)\,\mathrm{e}^{-6}}{6}\right) = \sum_{n=6}^{\infty} \left(-\frac{(n-4)\,a_{n-4} (n-5)\,\mathrm{e}^{-6} t^n}{6}\right)
$$

$$
\sum_{n=2}^{\infty} \frac{n\,t^{n+3} a_n (n-1)\,\mathrm{e}^{-5}}{5} = \sum_{n=5}^{\infty} \frac{(n-3)\,a_{n-3} (n-4)\,\mathrm{e}^{-5} t^n}{5}
$$

$$
\sum_{n=2}^{\infty} \left(-\frac{n\,t^{n+2} a_n (n-1)\,\mathrm{e}^{-4}}{4}\right) = \sum_{n=4}^{\infty} \left(-\frac{(n-2)\,a_{n-2} (n-3)\,\mathrm{e}^{-4} t^n}{4}\right)
$$

$$
\sum_{n=2}^{\infty} \frac{n\,t^{1+n} a_n (n-1)\,\mathrm{e}^{-3}}{3} = \sum_{n=3}^{\infty} \frac{(n-1)\,a_{n-1} (n-2)\,\mathrm{e}^{-3} t^n}{3}
$$

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\, \mathrm{e}^{-1} = \sum_{n=1}^{\infty} (1+n)\, a_{1+n} n\, \mathrm{e}^{-1} t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, t^n$$

$$\sum_{n=0}^{\infty} \left(-t^{1+n} a_n \cos(\mathrm{e})\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} \cos(\mathrm{e})\, t^n\right)$$

$$\sum_{n=0}^{\infty} \frac{t^{n+2} a_n \sin(\mathrm{e})}{2} = \sum_{n=2}^{\infty} \frac{a_{n-2} \sin(\mathrm{e})\, t^n}{2}$$

$$\sum_{n=0}^{\infty} \frac{t^{n+3} a_n \cos(\mathrm{e})}{6} = \sum_{n=3}^{\infty} \frac{a_{n-3} \cos(\mathrm{e})\, t^n}{6}$$

$$\sum_{n=0}^{\infty} \left(-\frac{t^{n+4} a_n \sin(\mathrm{e})}{24}\right) = \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} \sin(\mathrm{e})\, t^n}{24}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{t^{n+5} a_n \cos(\mathrm{e})}{120}\right) = \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} \cos(\mathrm{e})\, t^n}{120}\right)$$

$$\sum_{n=0}^{\infty} \frac{t^{n+6} a_n \sin(\mathrm{e})}{720} = \sum_{n=6}^{\infty} \frac{a_{n-6} \sin(\mathrm{e})\, t^n}{720}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\sum_{n=6}^{\infty}\left(-\frac{(n-4)\, a_{n-4}(n-5)\, \mathrm{e}^{-6} t^{n}}{6}\right)+\left(\sum_{n=5}^{\infty}\frac{(n-3)\, a_{n-3}(n-4)\, \mathrm{e}^{-5} t^{n}}{5}\right)$$
$$+\sum_{n=4}^{\infty}\left(-\frac{(n-2)\, a_{n-2}(n-3)\, \mathrm{e}^{-4} t^{n}}{4}\right)$$
$$+\left(\sum_{n=3}^{\infty}\frac{(n-1)\, a_{n-1}(n-2)\, \mathrm{e}^{-3} t^{n}}{3}\right)+\sum_{n=2}^{\infty}\left(-\frac{n a_{n} t^{n}(n-1)\, \mathrm{e}^{-2}}{2}\right)$$
$$+\left(\sum_{n=1}^{\infty}(1+n)\, a_{1+n} n\, \mathrm{e}^{-1} t^{n}\right)+\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, t^{n}\right) \tag{3}$$
$$+\sum_{n=0}^{\infty}\left(-a_{n} t^{n} \sin(\mathrm{e})\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}\cos(\mathrm{e})\, t^{n}\right)+\left(\sum_{n=2}^{\infty}\frac{a_{n-2}\sin(\mathrm{e})\, t^{n}}{2}\right)$$
$$+\left(\sum_{n=3}^{\infty}\frac{a_{n-3}\cos(\mathrm{e})\, t^{n}}{6}\right)+\sum_{n=4}^{\infty}\left(-\frac{a_{n-4}\sin(\mathrm{e})\, t^{n}}{24}\right)$$
$$+\sum_{n=5}^{\infty}\left(-\frac{a_{n-5}\cos(\mathrm{e})\, t^{n}}{120}\right)+\left(\sum_{n=6}^{\infty}\frac{a_{n-6}\sin(\mathrm{e})\, t^{n}}{720}\right)=0$$

$n=0$ gives

$$2a_2 - a_0\sin(\mathrm{e}) = 0$$

$$a_2 = \frac{a_0\sin(\mathrm{e})}{2}$$

$n=1$ gives

$$2a_2\mathrm{e}^{-1} + 6a_3 - a_1\sin(\mathrm{e}) - a_0\cos(\mathrm{e}) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0\sin(\mathrm{e})\,\mathrm{e}^{-1}}{6} + \frac{a_0\cos(\mathrm{e})}{6} + \frac{a_1\sin(\mathrm{e})}{6}$$

$n=2$ gives

$$-a_2\mathrm{e}^{-2} + 6a_3\mathrm{e}^{-1} + 12a_4 - a_2\sin(\mathrm{e}) - a_1\cos(\mathrm{e}) + \frac{a_0\sin(\mathrm{e})}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0 \sin(e) e^{-2}}{8} - \frac{e^{-1} a_0 \cos(e)}{12} + \frac{a_0 \sin(e)^2}{24} - \frac{a_0 \sin(e)}{24} - \frac{e^{-1} a_1 \sin(e)}{12} + \frac{a_1 \cos(e)}{12}$$

$n = 3$ gives

$$\frac{2a_2 e^{-3}}{3} - 3a_3 e^{-2} + 12a_4 e^{-1} + 20a_5 - a_3 \sin(e) - a_2 \cos(e) + \frac{a_1 \sin(e)}{2} + \frac{a_0 \cos(e)}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$\begin{aligned} a_5 = &-\frac{a_0 \sin(e) e^{-3}}{15} - \frac{e^{-2} a_0 \sin(e) e^{-1}}{20} + \frac{3 e^{-2} a_0 \cos(e)}{40} \\ &- \frac{e^{-1} a_0 \sin(e)^2}{30} + \frac{a_0 \sin(e) e^{-1}}{40} + \frac{a_0 \sin(e) \cos(e)}{30} - \frac{a_0 \cos(e)}{120} \\ &+ \frac{3 e^{-2} a_1 \sin(e)}{40} - \frac{e^{-1} a_1 \cos(e)}{20} + \frac{a_1 \sin(e)^2}{120} - \frac{a_1 \sin(e)}{40} \end{aligned}$$

$n = 4$ gives

$$\begin{aligned} &-\frac{a_2 e^{-4}}{2} + 2a_3 e^{-3} - 6a_4 e^{-2} + 20a_5 e^{-1} + 30a_6 - a_4 \sin(e) \\ &- a_3 \cos(e) + \frac{a_2 \sin(e)}{2} + \frac{a_1 \cos(e)}{6} - \frac{a_0 \sin(e)}{24} = 0 \end{aligned}$$

Which after substituting earlier equations, simplifies to

$$\begin{aligned} a_6 = &\frac{a_0 \sin(e)^3}{720} + \frac{a_0 \cos(e)^2}{180} + \frac{e^{-1} a_0 \cos(e)}{180} + \frac{e^{-1} a_1 \sin(e)}{60} \\ &+ \frac{a_0 \sin(e) e^{-4}}{20} - \frac{2 e^{-3} a_0 \cos(e)}{45} - \frac{2 e^{-3} a_1 \sin(e)}{45} + \frac{a_0 \sin(e) (e^{-2})^2}{20} \\ &+ \frac{5 e^{-2} a_0 \sin(e)^2}{144} + \frac{e^{-2} a_1 \cos(e)}{20} - \frac{e^{-1} a_1 \sin(e)^2}{120} + \frac{\sin(e) a_1 \cos(e)}{120} \\ &- \frac{a_0 \sin(e) e^{-2}}{40} - \frac{7 a_0 \sin(e)^2}{720} - \frac{e^{-2} e^{-1} a_0 \cos(e)}{30} - \frac{e^{-2} e^{-1} a_1 \sin(e)}{30} \\ &- \frac{11 e^{-1} a_0 \sin(e) \cos(e)}{360} + \frac{e^{-3} a_0 \sin(e) e^{-1}}{45} + \frac{a_0 \sin(e)}{720} - \frac{a_1 \cos(e)}{180} \end{aligned}$$

$n = 5$ gives

$$\begin{aligned} &\frac{2a_2 e^{-5}}{5} - \frac{3a_3 e^{-4}}{2} + 4a_4 e^{-3} - 10a_5 e^{-2} + 30a_6 e^{-1} + 42a_7 - a_5 \sin(e) \\ &- a_4 \cos(e) + \frac{a_3 \sin(e)}{2} + \frac{a_2 \cos(e)}{6} - \frac{a_1 \sin(e)}{24} - \frac{a_0 \cos(e)}{120} = 0 \end{aligned}$$

Which after substituting earlier equations, simplifies to

$$
\begin{aligned}
a_7 = &-\frac{a_0 \sin(\mathrm{e})\,\mathrm{e}^{-5}}{35} + \frac{\mathrm{e}^{-4} a_0 \cos(\mathrm{e})}{28} + \frac{\mathrm{e}^{-4} a_1 \sin(\mathrm{e})}{28} - \frac{59\,\mathrm{e}^{-3} a_0 \sin(\mathrm{e})^2}{2520} \\
&- \frac{2\,\mathrm{e}^{-3} a_1 \cos(\mathrm{e})}{63} + \frac{\left(\mathrm{e}^{-2}\right)^2 a_0 \cos(\mathrm{e})}{28} + \frac{\left(\mathrm{e}^{-2}\right)^2 a_1 \sin(\mathrm{e})}{28} + \frac{7\,\mathrm{e}^{-2} a_1 \sin(\mathrm{e})^2}{720} \\
&- \frac{\mathrm{e}^{-1} a_0 \sin(\mathrm{e})^3}{560} - \frac{\mathrm{e}^{-1} a_0 \cos(\mathrm{e})^2}{168} + \frac{a_0 \sin(\mathrm{e})^2 \cos(\mathrm{e})}{560} - \frac{13 a_1 \sin(\mathrm{e})^2}{5040} \\
&+ \frac{a_1 \sin(\mathrm{e})^3}{5040} + \frac{a_1 \cos(\mathrm{e})^2}{504} - \frac{a_0 \sin(\mathrm{e})\,\mathrm{e}^{-1}}{1008} - \frac{5\,\mathrm{e}^{-4} a_0 \sin(\mathrm{e})\,\mathrm{e}^{-1}}{168} \\
&- \frac{5\,\mathrm{e}^{-3} a_0 \sin(\mathrm{e})\,\mathrm{e}^{-2}}{63} + \frac{\mathrm{e}^{-3}\mathrm{e}^{-1} a_0 \cos(\mathrm{e})}{63} + \frac{\mathrm{e}^{-3}\mathrm{e}^{-1} a_1 \sin(\mathrm{e})}{63} - \frac{9\,\mathrm{e}^{-2}\mathrm{e}^{-1} a_0 \sin(\mathrm{e})^2}{560} \\
&+ \frac{29\,\mathrm{e}^{-2} a_0 \sin(\mathrm{e}) \cos(\mathrm{e})}{840} - \frac{\mathrm{e}^{-2}\mathrm{e}^{-1} a_1 \cos(\mathrm{e})}{42} - \frac{23\,\mathrm{e}^{-1} \sin(\mathrm{e})\, a_1 \cos(\mathrm{e})}{2520} \\
&- \frac{\mathrm{e}^{-2} a_0 \cos(\mathrm{e})}{168} - \frac{\mathrm{e}^{-2} a_1 \sin(\mathrm{e})}{56} + \frac{\mathrm{e}^{-1} a_0 \sin(\mathrm{e})^2}{105} + \frac{\mathrm{e}^{-1} a_1 \cos(\mathrm{e})}{252} + \frac{a_0 \sin(\mathrm{e})\,\mathrm{e}^{-3}}{63} \\
&- \frac{13 a_0 \sin(\mathrm{e}) \cos(\mathrm{e})}{2520} + \frac{\mathrm{e}^{-2} a_0 \sin(\mathrm{e})\,\mathrm{e}^{-1}}{84} + \frac{a_0 \cos(\mathrm{e})}{5040} + \frac{a_1 \sin(\mathrm{e})}{1008}
\end{aligned}
$$

For $6 \leq n$, the recurrence equation is

$$
\begin{aligned}
&-\frac{(n-4)\, a_{n-4}(n-5)\,\mathrm{e}^{-6}}{6} + \frac{(n-3)\, a_{n-3}(n-4)\,\mathrm{e}^{-5}}{5} \\
&- \frac{(n-2)\, a_{n-2}(n-3)\,\mathrm{e}^{-4}}{4} + \frac{(n-1)\, a_{n-1}(n-2)\,\mathrm{e}^{-3}}{3} - \frac{n a_n (n-1)\,\mathrm{e}^{-2}}{2} \\
&+ (1+n)\, a_{1+n} n\,\mathrm{e}^{-1} + (n+2)\, a_{n+2}(1+n) - a_n \sin(\mathrm{e}) - a_{n-1} \cos(\mathrm{e}) \\
&+ \frac{a_{n-2} \sin(\mathrm{e})}{2} + \frac{a_{n-3} \cos(\mathrm{e})}{6} - \frac{a_{n-4} \sin(\mathrm{e})}{24} - \frac{a_{n-5} \cos(\mathrm{e})}{120} + \frac{a_{n-6} \sin(\mathrm{e})}{720} = 0
\end{aligned} \tag{4}
$$

Solving for a_{n+2}, gives

$$
a_{n+2} = \tag{5}
$$

$$
-\frac{144\,\mathrm{e}^{-5} n^2 a_{n-3} - 120\,\mathrm{e}^{-6} n^2 a_{n-4} + 720 a_{1+n} n^2 \mathrm{e}^{-1} + 240\,\mathrm{e}^{-3} n^2 a_{n-1} - 360 n^2 a_n \mathrm{e}^{-2} - 180\,\mathrm{e}^{-4} n^2 a_{n-2} - 1}{}
$$

$$
\begin{aligned}
&= -\frac{\left(-360\,\mathrm{e}^{-2} n^{2}+360\,\mathrm{e}^{-2} n -720 \sin \left(\mathrm{e}\right)\right) a_{n}}{720 \left(n +2\right) \left(1+n \right)}-\frac{\left(720\,\mathrm{e}^{-1} n^{2}+720\,\mathrm{e}^{-1} n \right) a_{1+n}}{720 \left(n +2\right) \left(1+n \right)}\\
&\quad -\frac{\sin \left(\mathrm{e}\right) a_{n -6}}{720 \left(n +2\right) \left(1+n \right)}+\frac{\cos \left(\mathrm{e}\right) a_{n -5}}{120 \left(n +2\right) \left(1+n \right)}\\
&\quad -\frac{\left(-120\,\mathrm{e}^{-6} n^{2}+1080\,\mathrm{e}^{-6} n -30 \sin \left(\mathrm{e}\right)-2400\,\mathrm{e}^{-6}\right) a_{n -4}}{720 \left(n +2\right) \left(1+n \right)}\\
&\quad -\frac{\left(144\,\mathrm{e}^{-5} n^{2}-1008\,\mathrm{e}^{-5} n +1728\,\mathrm{e}^{-5}+120 \cos \left(\mathrm{e}\right)\right) a_{n -3}}{720 \left(n +2\right) \left(1+n \right)}\\
&\quad -\frac{\left(-180\,\mathrm{e}^{-4} n^{2}+900\,\mathrm{e}^{-4} n +360 \sin \left(\mathrm{e}\right)-1080\,\mathrm{e}^{-4}\right) a_{n -2}}{720 \left(n +2\right) \left(1+n \right)}\\
&\quad -\frac{\left(240\,\mathrm{e}^{-3} n^{2}-720\,\mathrm{e}^{-3} n -720 \cos \left(\mathrm{e}\right)+480\,\mathrm{e}^{-3}\right) a_{n -1}}{720 \left(n +2\right) \left(1+n \right)}
\end{aligned}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_{n} t^{n}\\
&= a_{3} t^{3}+a_{2} t^{2}+a_{1} t +a_{0}+\dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y(t) &= a_{0}+a_{1} t +\frac{a_{0} \sin \left(\mathrm{e}\right) t^{2}}{2}+\left(-\frac{a_{0} \sin \left(\mathrm{e}\right) \mathrm{e}^{-1}}{6}+\frac{a_{0} \cos \left(\mathrm{e}\right)}{6}+\frac{a_{1} \sin \left(\mathrm{e}\right)}{6}\right) t^{3}\\
&\quad +\left(\frac{a_{0} \sin \left(\mathrm{e}\right) \mathrm{e}^{-2}}{8}-\frac{\mathrm{e}^{-1} a_{0} \cos \left(\mathrm{e}\right)}{12}+\frac{a_{0} \sin \left(\mathrm{e}\right)^{2}}{24}-\frac{a_{0} \sin \left(\mathrm{e}\right)}{24}-\frac{\mathrm{e}^{-1} a_{1} \sin \left(\mathrm{e}\right)}{12}\right.\\
&\quad \left.+\frac{a_{1} \cos \left(\mathrm{e}\right)}{12}\right) t^{4}+\left(-\frac{a_{0} \sin \left(\mathrm{e}\right) \mathrm{e}^{-3}}{15}-\frac{\mathrm{e}^{-2} a_{0} \sin \left(\mathrm{e}\right) \mathrm{e}^{-1}}{20}+\frac{3\,\mathrm{e}^{-2} a_{0} \cos \left(\mathrm{e}\right)}{40}\right.\\
&\quad -\frac{\mathrm{e}^{-1} a_{0} \sin \left(\mathrm{e}\right)^{2}}{30}+\frac{a_{0} \sin \left(\mathrm{e}\right) \mathrm{e}^{-1}}{40}+\frac{a_{0} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)}{30}-\frac{a_{0} \cos \left(\mathrm{e}\right)}{120}+\frac{3\,\mathrm{e}^{-2} a_{1} \sin \left(\mathrm{e}\right)}{40}\\
&\quad \left.-\frac{\mathrm{e}^{-1} a_{1} \cos \left(\mathrm{e}\right)}{20}+\frac{a_{1} \sin \left(\mathrm{e}\right)^{2}}{120}-\frac{a_{1} \sin \left(\mathrm{e}\right)}{40}\right) t^{5}+\dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y(t) = {} & \left(1+\frac{\sin(\mathrm{e})\, t^2}{2}+\left(-\frac{\mathrm{e}^{-1}\sin(\mathrm{e})}{6}+\frac{\cos(\mathrm{e})}{6}\right)t^3\right. \\
& +\left(\frac{\mathrm{e}^{-2}\sin(\mathrm{e})}{8}-\frac{\cos(\mathrm{e})\,\mathrm{e}^{-1}}{12}+\frac{\sin(\mathrm{e})^2}{24}-\frac{\sin(\mathrm{e})}{24}\right)t^4 \\
& +\left(-\frac{\sin(\mathrm{e})\,\mathrm{e}^{-3}}{15}-\frac{\mathrm{e}^{-2}\sin(\mathrm{e})\,\mathrm{e}^{-1}}{20}+\frac{3\,\mathrm{e}^{-2}\cos(\mathrm{e})}{40}-\frac{\sin(\mathrm{e})^2\,\mathrm{e}^{-1}}{30}+\frac{\mathrm{e}^{-1}\sin(\mathrm{e})}{40}\right. \\
& \left.\left.+\frac{\sin(\mathrm{e})\cos(\mathrm{e})}{30}-\frac{\cos(\mathrm{e})}{120}\right)t^5\right)a_0+\left(t+\frac{\sin(\mathrm{e})\, t^3}{6}+\left(-\frac{\mathrm{e}^{-1}\sin(\mathrm{e})}{12}+\frac{\cos(\mathrm{e})}{12}\right)t^4\right. \\
& \left.+\left(\frac{3\,\mathrm{e}^{-2}\sin(\mathrm{e})}{40}-\frac{\cos(\mathrm{e})\,\mathrm{e}^{-1}}{20}+\frac{\sin(\mathrm{e})^2}{120}-\frac{\sin(\mathrm{e})}{40}\right)t^5\right)a_1+O(t^6)
\end{aligned} \tag{3}
$$

At $t=0$ the solution above becomes

$$
\begin{aligned}
y(t) = {} & \left(1+\frac{\sin(\mathrm{e})\, t^2}{2}+\left(-\frac{\mathrm{e}^{-1}\sin(\mathrm{e})}{6}+\frac{\cos(\mathrm{e})}{6}\right)t^3\right. \\
& +\left(\frac{\mathrm{e}^{-2}\sin(\mathrm{e})}{8}-\frac{\cos(\mathrm{e})\,\mathrm{e}^{-1}}{12}+\frac{\sin(\mathrm{e})^2}{24}-\frac{\sin(\mathrm{e})}{24}\right)t^4 \\
& +\left(-\frac{\sin(\mathrm{e})\,\mathrm{e}^{-3}}{15}-\frac{\mathrm{e}^{-2}\sin(\mathrm{e})\,\mathrm{e}^{-1}}{20}+\frac{3\,\mathrm{e}^{-2}\cos(\mathrm{e})}{40}-\frac{\sin(\mathrm{e})^2\,\mathrm{e}^{-1}}{30}+\frac{\mathrm{e}^{-1}\sin(\mathrm{e})}{40}\right. \\
& \left.\left.+\frac{\sin(\mathrm{e})\cos(\mathrm{e})}{30}-\frac{\cos(\mathrm{e})}{120}\right)t^5\right)c_1+\left(t+\frac{\sin(\mathrm{e})\, t^3}{6}+\left(-\frac{\mathrm{e}^{-1}\sin(\mathrm{e})}{12}+\frac{\cos(\mathrm{e})}{12}\right)t^4\right. \\
& \left.+\left(\frac{3\,\mathrm{e}^{-2}\sin(\mathrm{e})}{40}-\frac{\cos(\mathrm{e})\,\mathrm{e}^{-1}}{20}+\frac{\sin(\mathrm{e})^2}{120}-\frac{\sin(\mathrm{e})}{40}\right)t^5\right)c_2+O(t^6)
\end{aligned}
$$

$$
\begin{aligned}
y(t) = {} & \mathrm{e}^{-1}+\frac{\mathrm{e}^{-1}\sin(\mathrm{e})\, t^2}{2}-\frac{(\mathrm{e}^{-1})^2\, t^3\sin(\mathrm{e})}{6}+\frac{t^3\cos(\mathrm{e})\,\mathrm{e}^{-1}}{6}+\frac{\mathrm{e}^{-1}t^4\mathrm{e}^{-2}\sin(\mathrm{e})}{8} \\
& -\frac{(\mathrm{e}^{-1})^2\, t^4\cos(\mathrm{e})}{12}+\frac{t^4\sin(\mathrm{e})^2\,\mathrm{e}^{-1}}{24}-\frac{t^4\mathrm{e}^{-1}\sin(\mathrm{e})}{24}-\frac{\mathrm{e}^{-1}t^5\sin(\mathrm{e})\,\mathrm{e}^{-3}}{15} \\
& -\frac{(\mathrm{e}^{-1})^2\, t^5\mathrm{e}^{-2}\sin(\mathrm{e})}{20}+\frac{3\,\mathrm{e}^{-1}t^5\mathrm{e}^{-2}\cos(\mathrm{e})}{40}-\frac{(\mathrm{e}^{-1})^2\, t^5\sin(\mathrm{e})^2}{30} \\
& +\frac{(\mathrm{e}^{-1})^2\, t^5\sin(\mathrm{e})}{40}+\frac{t^5\mathrm{e}^{-1}\sin(\mathrm{e})\cos(\mathrm{e})}{30}-\frac{t^5\cos(\mathrm{e})\,\mathrm{e}^{-1}}{120}+O(t^6)
\end{aligned}
$$

Replacing t in the above with the original independent variable $x$susing $t = x-\mathrm{e}$ results in

$$
\begin{aligned}
y = \mathrm{e}^{-1}+O\left(\left(x-\mathrm{e}\right)^{6}\right)+\frac{\mathrm{e}^{-1} \sin \left(\mathrm{e}\right) \left(x-\mathrm{e}\right)^{2}}{2}-\frac{\left(x-\mathrm{e}\right)^{3} \mathrm{e}^{-2} \sin \left(\mathrm{e}\right)}{6}+\frac{\left(x-\mathrm{e}\right)^{3} \cos \left(\mathrm{e}\right) \mathrm{e}^{-1}}{6} \\
+\frac{\left(x-\mathrm{e}\right)^{4} \sin \left(\mathrm{e}\right)^{2} \mathrm{e}^{-1}}{24}+\frac{\left(x-\mathrm{e}\right)^{4} \sin \left(\mathrm{e}\right) \mathrm{e}^{-3}}{8}-\frac{\left(x-\mathrm{e}\right)^{4} \mathrm{e}^{-1} \sin \left(\mathrm{e}\right)}{24} \\
-\frac{\left(x-\mathrm{e}\right)^{4} \mathrm{e}^{-2} \cos \left(\mathrm{e}\right)}{12}-\frac{\left(x-\mathrm{e}\right)^{5} \sin \left(\mathrm{e}\right)^{2} \mathrm{e}^{-2}}{30}+\frac{3\left(x-\mathrm{e}\right)^{5} \cos \left(\mathrm{e}\right) \mathrm{e}^{-3}}{40} \\
-\frac{\left(x-\mathrm{e}\right)^{5} \cos \left(\mathrm{e}\right) \mathrm{e}^{-1}}{120}-\frac{7\left(x-\mathrm{e}\right)^{5} \sin \left(\mathrm{e}\right) \mathrm{e}^{-4}}{60}+\frac{\left(x-\mathrm{e}\right)^{5} \mathrm{e}^{-2} \sin \left(\mathrm{e}\right)}{40} \\
+\frac{5\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right)^{2} \mathrm{e}^{-3}}{144}+\frac{11\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-1} \cos \left(\mathrm{e}\right)^{2}}{720}+\frac{11\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right) \mathrm{e}^{-5}}{90} \\
-\frac{7\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-4} \cos \left(\mathrm{e}\right)}{90}-\frac{\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right) \mathrm{e}^{-3}}{40}+\frac{\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-2} \cos \left(\mathrm{e}\right)}{180} \\
+\frac{\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-1} \sin \left(\mathrm{e}\right)}{360}-\frac{7\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-1}}{720}+\frac{\left(x-\mathrm{e}\right)^{5} \mathrm{e}^{-1} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)}{30} \\
-\frac{\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)^{2} \mathrm{e}^{-1}}{720}-\frac{11\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-2} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)}{360}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \mathrm{e}^{-1}+O\left(\left(x-\mathrm{e}\right)^{6}\right)+\frac{\mathrm{e}^{-1} \sin \left(\mathrm{e}\right) \left(x-\mathrm{e}\right)^{2}}{2}-\frac{\left(x-\mathrm{e}\right)^{3} \mathrm{e}^{-2} \sin \left(\mathrm{e}\right)}{6} \\
+\frac{\left(x-\mathrm{e}\right)^{3} \cos \left(\mathrm{e}\right) \mathrm{e}^{-1}}{6}+\frac{\left(x-\mathrm{e}\right)^{4} \sin \left(\mathrm{e}\right)^{2} \mathrm{e}^{-1}}{24}+\frac{\left(x-\mathrm{e}\right)^{4} \sin \left(\mathrm{e}\right) \mathrm{e}^{-3}}{8} \\
-\frac{\left(x-\mathrm{e}\right)^{4} \mathrm{e}^{-1} \sin \left(\mathrm{e}\right)}{24}-\frac{\left(x-\mathrm{e}\right)^{4} \mathrm{e}^{-2} \cos \left(\mathrm{e}\right)}{12}-\frac{\left(x-\mathrm{e}\right)^{5} \sin \left(\mathrm{e}\right)^{2} \mathrm{e}^{-2}}{30} \\
+\frac{3\left(x-\mathrm{e}\right)^{5} \cos \left(\mathrm{e}\right) \mathrm{e}^{-3}}{40}-\frac{\left(x-\mathrm{e}\right)^{5} \cos \left(\mathrm{e}\right) \mathrm{e}^{-1}}{120}-\frac{7\left(x-\mathrm{e}\right)^{5} \sin \left(\mathrm{e}\right) \mathrm{e}^{-4}}{60} \\
+\frac{\left(x-\mathrm{e}\right)^{5} \mathrm{e}^{-2} \sin \left(\mathrm{e}\right)}{40}+\frac{5\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right)^{2} \mathrm{e}^{-3}}{144} \\
+\frac{11\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-1} \cos \left(\mathrm{e}\right)^{2}}{720}+\frac{11\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right) \mathrm{e}^{-5}}{90} \\
-\frac{7\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-4} \cos \left(\mathrm{e}\right)}{90}-\frac{\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right) \mathrm{e}^{-3}}{40}+\frac{\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-2} \cos \left(\mathrm{e}\right)}{180} \\
+\frac{\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-1} \sin \left(\mathrm{e}\right)}{360}-\frac{7\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-1}}{720}+\frac{\left(x-\mathrm{e}\right)^{5} \mathrm{e}^{-1} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)}{30} \\
-\frac{\left(x-\mathrm{e}\right)^{6} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)^{2} \mathrm{e}^{-1}}{720}-\frac{11\left(x-\mathrm{e}\right)^{6} \mathrm{e}^{-2} \sin \left(\mathrm{e}\right) \cos \left(\mathrm{e}\right)}{360}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-1} + O\left((x-\mathrm{e})^{6}\right) + \frac{\mathrm{e}^{-1}\sin(\mathrm{e})(x-\mathrm{e})^{2}}{2} - \frac{(x-\mathrm{e})^{3}\mathrm{e}^{-2}\sin(\mathrm{e})}{6} + \frac{(x-\mathrm{e})^{3}\cos(\mathrm{e})\mathrm{e}^{-1}}{6} + \frac{(x-\mathrm{e})^{4}\sin(\mathrm{e})^{2}\mathrm{e}^{-1}}{24} + \frac{(x-\mathrm{e})^{4}\sin(\mathrm{e})\mathrm{e}^{-3}}{8} - \frac{(x-\mathrm{e})^{4}\mathrm{e}^{-1}\sin(\mathrm{e})}{24} - \frac{(x-\mathrm{e})^{4}\mathrm{e}^{-2}\cos(\mathrm{e})}{12} - \frac{(x-\mathrm{e})^{5}\sin(\mathrm{e})^{2}\mathrm{e}^{-2}}{30} + \frac{3(x-\mathrm{e})^{5}\cos(\mathrm{e})\mathrm{e}^{-3}}{40} - \frac{(x-\mathrm{e})^{5}\cos(\mathrm{e})\mathrm{e}^{-1}}{120} - \frac{7(x-\mathrm{e})^{5}\sin(\mathrm{e})\mathrm{e}^{-4}}{60} + \frac{(x-\mathrm{e})^{5}\mathrm{e}^{-2}\sin(\mathrm{e})}{40} + \frac{5(x-\mathrm{e})^{6}\sin(\mathrm{e})^{2}\mathrm{e}^{-3}}{144} + \frac{11(x-\mathrm{e})^{6}\mathrm{e}^{-1}\cos(\mathrm{e})^{2}}{720} + \frac{11(x-\mathrm{e})^{6}\sin(\mathrm{e})\mathrm{e}^{-5}}{90} - \frac{7(x-\mathrm{e})^{6}\mathrm{e}^{-4}\cos(\mathrm{e})}{90} - \frac{(x-\mathrm{e})^{6}\sin(\mathrm{e})\mathrm{e}^{-3}}{40} + \frac{(x-\mathrm{e})^{6}\mathrm{e}^{-2}\cos(\mathrm{e})}{180} + \frac{(x-\mathrm{e})^{6}\mathrm{e}^{-1}\sin(\mathrm{e})}{360} - \frac{7(x-\mathrm{e})^{6}\mathrm{e}^{-1}}{720} + \frac{(x-\mathrm{e})^{5}\mathrm{e}^{-1}\sin(\mathrm{e})\cos(\mathrm{e})}{30} - \frac{(x-\mathrm{e})^{6}\sin(\mathrm{e})\cos(\mathrm{e})^{2}\mathrm{e}^{-1}}{720} - \frac{11(x-\mathrm{e})^{6}\mathrm{e}^{-2}\sin(\mathrm{e})\cos(\mathrm{e})}{360}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 142

```
Order:=6;
dsolve([ln(x)*diff(y(x),x$2)-y(x)*sin(x)=0,y(exp(1)) = 1/exp(1), D(y)(exp(1)) = 0],y(x),type
```

$$y(x) = \mathrm{e}^{-1} + \frac{1}{2}\sin(\mathrm{e})\,\mathrm{e}^{-1}(x-\mathrm{e})^2 + \frac{1}{6}(\cos(\mathrm{e})\,\mathrm{e} - \sin(\mathrm{e}))\,\mathrm{e}^{-2}(x-\mathrm{e})^3 + \left(\frac{\mathrm{e}^{-3}\mathrm{e}^2\sin(\mathrm{e})^2}{24} - \frac{(\mathrm{e}^2-3)\,\mathrm{e}^{-3}\sin(\mathrm{e})}{24} - \frac{\mathrm{e}^{-3}\cos(\mathrm{e})\,\mathrm{e}}{12}\right)(x-\mathrm{e})^4 + \left(-\frac{\mathrm{e}^{-4}\mathrm{e}^2\sin(\mathrm{e})^2}{30} + \frac{(4\cos(\mathrm{e})\,\mathrm{e}^3 + 3\,\mathrm{e}^2 - 14)\,\mathrm{e}^{-4}\sin(\mathrm{e})}{120} + \frac{3\cos(\mathrm{e})\,\mathrm{e}^{-4}\left(\mathrm{e} - \frac{\mathrm{e}^3}{9}\right)}{40}\right)(x-\mathrm{e})^5 + \mathrm{O}\left((x-\mathrm{e})^6\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
AsymptoticDSolveValue[{Log[x]*y''[x]-Sin[x]*y[x]==0,{y[Exp[1]]==1/Exp[1],y'[Exp[1]]==0}},y[x
```

Not solved

23.8 problem 731

Internal problem ID [15475]
Internal file name [`OUTPUT/15475_Friday_May_10_2024_03_32_48_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 731.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[NONE]`

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
trying 3rd order, integrating factor of the form mu(y) for some mu
Trying the formal computation of integrating factors depending on any 2 of [x, y, y, y]
differential order: 3; looking for linear symmetries
--- Trying Lie symmetry methods, high order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$3)+x*sin(y(x))=0,y(0) = 1/2*Pi, D(y)(0) = 0, (D@@2)(y)(0) = 0],y(x),type
```

$$y = \frac{\pi}{2} - \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 16

```
AsymptoticDSolveValue[{y'''[x]+x*Sin[y[x]]==0,{y[0]==Pi/2,y'[0]==0,y''[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{\pi}{2} - \frac{x^4}{24}$$

23.9 problem 732

Internal problem ID [15476]
Internal file name [`OUTPUT/15476_Friday_May_10_2024_05_47_24_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 732.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup", "first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2yx = 0}$$

With initial conditions

$$[y(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

23.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - 2yx = 0$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

23.9.2 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$F_0 = f(x,y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= 2yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= (4x^2+2)\, y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= (8x^3+12x)\, y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 4y(4x^4+12x^2+3) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= 32yx\left(x^4+5x^2+\frac{15}{4}\right)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 2 \\
F_2 &= 0 \\
F_3 &= 12 \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = x^2 + 1 + \frac{1}{2}x^4$$

Hence the solution can be written as

$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6)$$

which simplifies to

$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - 2yx &= 0 \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -2x \\ p(x) &= 0 \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-2x^{1+n} a_n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n \\
\sum_{n=0}^{\infty} \left(-2x^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-2a_{n-1} x^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^n\right) = 0 \tag{3}
$$

For $1 \leq n$, the recurrence equation is

$$
(1+n) a_{1+n} - 2a_{n-1} = 0 \tag{4}
$$

Solving for a_{1+n}, gives

$$
a_{1+n} = \frac{2a_{n-1}}{1+n} \tag{5}
$$

For $n = 1$ the recurrence equation gives

$$2a_2 - 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = a_0$$

For $n = 2$ the recurrence equation gives

$$3a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$4a_4 - 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 - 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{6}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x^2 + \frac{1}{2} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^2 + 1 + \frac{1}{2} x^4\right) a_0 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y(0) = a_0$$

Therefore the solution in Eq(3) now can be written as

$$y = \left(x^2 + 1 + \frac{1}{2} x^4\right) y(0) + O\left(x^6\right)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = 1$$

Therefore the solution becomes

$$y = x^2 + 1 + \frac{1}{2} x^4$$

Hence the solution can be written as

$$y = x^2 + 1 + \frac{x^4}{2} + O\left(x^6\right)$$

which simplifies to

$$y = x^2 + 1 + \frac{x^4}{2} + O\left(x^6\right)$$

Summary
The solution(s) found are the following

$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6) \tag{1}$$
$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6) \tag{2}$$

Verification of solutions

$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6)$$

Verified OK.

$$y = x^2 + 1 + \frac{x^4}{2} + O(x^6)$$

Verified OK.

23.9.3 Maple step by step solution

Let's solve

$[y' - 2yx = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 2x dx + c_1$

- Evaluate integral

  $\ln(y) = x^2 + c_1$

- Solve for y

  $y = e^{x^2 + c_1}$

- Use initial condition $y(0) = 1$

  $1 = e^{c_1}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = e^{x^2}$

- Solution to the IVP

  $y = e^{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x)-2*x*y(x)=0,y(0) = 1],y(x),type='series',x=0);
```

$$y = 1 + x^2 + \frac{1}{2}x^4 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 15

```
AsymptoticDSolveValue[{y'[x]-2*x*y[x]==0,{y[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{2} + x^2 + 1$$

23.10 problem 733

Internal problem ID [15477]
Internal file name [`OUTPUT/15477_Friday_May_10_2024_05_47_24_PM_93797961/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 733.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + y'x + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1224) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (1225) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y'x^2 + yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y'x^3 - yx^2 + 5y'x + 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^2 + 8\right) y' + yx\left(x^2 - 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right) y' - y\left(x^4 - 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= -15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

23.10.1 Maple step by step solution

Let's solve

$y'' = -y'x - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) + a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y = \left(1-\frac{1}{2}x^2+\frac{1}{8}x^4\right)y(0)+\left(x-\frac{1}{3}x^3+\frac{1}{15}x^5\right)D(y)\left(0\right)+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{15}-\frac{x^3}{3}+x\right)+c_1\left(\frac{x^4}{8}-\frac{x^2}{2}+1\right)$$

23.11 problem 734

Internal problem ID [15478]
Internal file name [`OUTPUT/15478_Friday_May_10_2024_05_47_25_PM_13727061/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 734.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y'x + y = 1}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

23.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= 1 \\ F &= 1 \end{aligned}$$

Hence the ode is

$$y'' - y'x + y = 1$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1227)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (1228)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y'x - y + 1 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (y'x - y + 1)\, x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^2 + 1\right)(y'x - y + 1) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x\left(x^2 + 3\right)(y'x - y + 1) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (y'x - y + 1)\left(x^4 + 6x^2 + 3\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 0$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 0 \\
F_2 &= 1 \\
F_3 &= 0 \\
F_4 &= 3
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{240} + O\left(x^6\right)
$$

$$
y = \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{240} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) + 1 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 1 \tag{3}$$

$n = 0$ gives

$$(2a_2 + a_0) x^0 = 1$$
$$2a_2 + a_0 = 1$$

$$a_2 = -\frac{a_0}{2} + \frac{1}{2}$$

For $1 \leq n$, the recurrence equation is

$$\left(\left(n+2\right) a_{n+2}\left(n+1\right)-n a_{n}+a_{n}\right) x^{n} = 1 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(6a_3)\, x &= 0\\ 6a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(12a_4 - a_2)\, x^2 &= 0\\ 12a_4 - a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24} - \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 - 2a_3)\, x^3 &= 0\\ 20a_5 - 2a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 - 3a_4)\, x^4 &= 0\\ 30a_6 - 3a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{240} - \frac{a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 - 4a_5)\, x^5 &= 0 \\ 42a_7 - 4a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{2} + \frac{1}{2}\right) x^2 + \left(\frac{1}{24} - \frac{a_0}{24}\right) x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) a_0 + a_1 x + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + c_2 x + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

$$y = \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{240} + O(x^6) \tag{1}$$

$$y = \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{240} + O(x^6)$$

Verified OK.

$$y = \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
Order:=6;
dsolve([diff(y(x),x$2)-x*diff(y(x),x)+y(x)=1,y(0) = 0, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y = \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
AsymptoticDSolveValue[{y''[x]-x*y'[x]+y[x]==1,{y[0]==0,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{24} + \frac{x^2}{2}$$

23.12 problem 735

Internal problem ID [15479]
Internal file name [`OUTPUT/15479_Friday_May_10_2024_05_47_25_PM_31591721/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 735.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel__ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - (x^2 + 1)\, y = 0}$$

With initial conditions

$$[y(0) = -2, y'(0) = 2]$$

With the expansion point for the power series method at $x = 0$.

23.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -x^2 - 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + (-x^2 - 1)\,y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -x^2 - 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1230}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1231}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \left(x^2+1\right) y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 2yx + \left(x^2+1\right) y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4y'x + y\left(x^4+2x^2+3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 8yx\left(x^2+1\right) + \left(x^4+2x^2+7\right) y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12\left(x^3+x\right) y' + y\left(x^6+3x^4+33x^2+15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=-2$ and $y'(0)=2$ gives

$$
\begin{aligned}
F_0 &= -2 \\
F_1 &= 2 \\
F_2 &= -6 \\
F_3 &= 14 \\
F_4 &= -30
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = -x^2 + 2x - 2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{7x^5}{60} - \frac{x^6}{24} + O\left(x^6\right)
$$

$$
y = -x^2 + 2x - 2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{7x^5}{60} - \frac{x^6}{24} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = (x^2+1) \left(\sum_{n=0}^{\infty} a_n x^n \right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n \right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) + \sum_{n=0}^{\infty} \left(-a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - a_{n-2} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{a_{n-2} + a_n}{(n+2)\,(n+1)} \\ &= \frac{a_n}{(n+2)\,(n+1)} + \frac{a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_0 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_1 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_2 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_3 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3a_1}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{7}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x + \frac{1}{6}x^3 + \frac{7}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{7}{120}x^5\right) c_2 + O(x^6)$$

$$y = -2 - x^2 - \frac{x^4}{4} + 2x + \frac{x^3}{3} + \frac{7x^5}{60} + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -x^2 + 2x - 2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{7x^5}{60} - \frac{x^6}{24} + O(x^6) \tag{1}$$

$$y = -2 - x^2 - \frac{x^4}{4} + 2x + \frac{x^3}{3} + \frac{7x^5}{60} + O(x^6) \tag{2}$$

Verification of solutions

$$y = -x^2 + 2x - 2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{7x^5}{60} - \frac{x^6}{24} + O(x^6)$$

Verified OK.

$$y = -2 - x^2 - \frac{x^4}{4} + 2x + \frac{x^3}{3} + \frac{7x^5}{60} + O(x^6)$$

Verified OK.

23.12.2 Maple step by step solution

Let's solve

$\left[y'' = (x^2+1)\, y, y(0) = -2, y'\Big|_{\{x=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + (-x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k->k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$$2a_2 - a_0 + (6a_3 - a_1)x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - a_k - a_{k-2})x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $[2a_2 - a_0 = 0, 6a_3 - a_1 = 0]$

- Solve for the dependent coefficient(s)

  $\left\{a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)a_{k+2} - a_k - a_{k-2} = 0$

- Shift index using $k- >k + 2$

  $\left((k+2)^2 + 3k + 8\right)a_{k+4} - a_{k+2} - a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{a_{k+2}+a_k}{k^2+7k+12}, a_2 = \frac{a_0}{2}, a_3 = \frac{a_1}{6}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([diff(y(x),x$2)-(1+x^2)*y(x)=0,y(0) = -2, D(y)(0) = 2],y(x),type='series',x=0);
```

$$y = -2 + 2x - x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{7}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[{y''[x]-(1+x^2)*y[x]==0,{y[0]==-2,y'[0]==2}},y[x],{x,0,5}]
```

$$y(x) \to \frac{7x^5}{60} - \frac{x^4}{4} + \frac{x^3}{3} - x^2 + 2x - 2$$

23.13 problem 736

Internal problem ID [15480]
Internal file name [`OUTPUT/15480_Friday_May_10_2024_05_47_25_PM_5344487/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 736.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - yx^2 + y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

23.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -x^2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - yx^2 + y' = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1234}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1235}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= yx^2 - y' \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^2+1\right) y' - yx(x-2) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-2x^2+4x-1\right) y' + y\left(x^4+x^2-2x+2\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4+3x^2-10x+7\right) y' - 2y\left(x^4-4x^3+\frac{1}{2}x^2-x+1\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-3x^4+12x^3-4x^2+18x-19\right) y' + y\left(x^6+3x^4-18x^3+31x^2-2x+2\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=1$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 2 \\
F_3 &= -2 \\
F_4 &= 2
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360} + O\left(x^6\right)
$$

$$
y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$
$$\sum_{n=0}^{\infty} \left(-x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 = 0$$

$$a_2 = -\frac{a_1}{2}$$

$n = 1$ gives

$$6a_3 + 2a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$6a_3 - a_1 = 0$$

Or

$$a_3 = \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + (n+1)\, a_{n+1} - a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_{n+1} - a_{n-2} + a_{n+1}}{(n+2)\,(n+1)} \\ &= \frac{a_{n-2}}{(n+2)\,(n+1)} - \frac{a_{n+1}}{n+2} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_3 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{24} + \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_4 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120} - \frac{a_0}{60}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_5 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{19a_1}{720} + \frac{a_0}{360}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_6 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_1}{1680} - \frac{a_0}{2520}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{a_1 x^2}{2} + \frac{a_1 x^3}{6} + \left(-\frac{a_1}{24} + \frac{a_0}{12}\right) x^4 + \left(\frac{7a_1}{120} - \frac{a_0}{60}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{7}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{12}x^4 - \frac{1}{60}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{7}{120}x^5\right) c_2 + O(x^6)$$

$$y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + O(x^6)$$

<u>Summary</u>

The solution(s) found are the following

$$y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360} + O(x^6) \tag{1}$$

$$y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + O(x^6) \tag{2}$$

<u>Verification of solutions</u>

$$y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360} + O(x^6)$$

Verified OK.

$$y = 1 + \frac{x^4}{12} - \frac{x^5}{60} + O(x^6)$$

Verified OK.

23.13.2 Maple step by step solution

Let's solve

$$\left[y'' = yx^2 - y', y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - yx^2 + y' = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  ◦ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- Convert y' to series expansion

$$y' = \sum_{k=1}^{\infty} a_k k\, x^{k-1}$$

- Shift index using $k- >k+1$

$$y' = \sum_{k=0}^{\infty} a_{k+1}(k+1)\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_1 + (6a_3 + 2a_2)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k+1}(k+1) - a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_1 = 0, 6a_3 + 2a_2 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_1}{2}, a_3 = \frac{a_1}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_{k+1} k - a_{k-2} + a_{k+1} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_{k+3}(k+2) - a_k + a_{k+3} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{k a_{k+3} - a_k + 3a_{k+3}}{k^2 + 7k + 12}, a_2 = -\frac{a_1}{2}, a_3 = \frac{a_1}{6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)=x^2*y(x)-diff(y(x),x),y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y = 1 + \frac{1}{12}x^4 - \frac{1}{60}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]==x^2*y[x]-y'[x],{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{60} + \frac{x^4}{12} + 1$$

23.14 problem 737

Internal problem ID [15481]
Internal file name [`OUTPUT/15481_Friday_May_10_2024_05_47_25_PM_3851012/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 737.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode_form_A", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - y\,e^{x} = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{1238}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{1239}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y\,\mathrm{e}^x \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \mathrm{e}^x (y' + y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \mathrm{e}^x (y\,\mathrm{e}^x + y + 2y') \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (4y + y')\,\mathrm{e}^{2x} + \mathrm{e}^x (y + 3y') \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= (11y + 6y')\,\mathrm{e}^{2x} + y\,\mathrm{e}^{3x} + \mathrm{e}^x (y + 4y')
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) + y(0) \\
F_2 &= 2y(0) + 2y'(0) \\
F_3 &= 5y(0) + 4y'(0) \\
F_4 &= 13y(0) + 10y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{13}{720}x^6\right) y(0) \\
& + \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \frac{1}{72}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \left(\sum_{n=0}^{\infty} a_n x^n\right) \mathrm{e}^{x} \tag{1}$$

Expanding $-\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -\mathrm{e}^{x} &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots \\ &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(-1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the second term in (1) gives

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + -1 \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - x \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^2}{2} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^3}{6} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^4}{24} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^5}{120} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) - \frac{x^6}{720} \cdot \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n\,=0}^{\infty} \left(-a_n x^n\right) + \sum_{n\,=0}^{\infty} \left(-x^{1+n} a_n\right) \\
&\quad + \sum_{n\,=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) + \sum_{n\,=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) + \sum_{n\,=0}^{\infty} \left(-\frac{x^{n+4} a_n}{24}\right) \\
&\quad + \sum_{n\,=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) + \sum_{n\,=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) = 0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n\,=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n \\
\sum_{n\,=0}^{\infty} \left(-x^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) \\
\sum_{n\,=0}^{\infty} \left(-\frac{x^{n+2} a_n}{2}\right) &= \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right) \\
\sum_{n\,=0}^{\infty} \left(-\frac{x^{n+3} a_n}{6}\right) &= \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) \\
\sum_{n\,=0}^{\infty} \left(-\frac{x^{n+4} a_n}{24}\right) &= \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^n}{24}\right) \\
\sum_{n\,=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) &= \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right) \\
\sum_{n\,=0}^{\infty} \left(-\frac{x^{n+6} a_n}{720}\right) &= \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) + \sum_{n=1}^{\infty} (-a_{n-1} x^n) + \sum_{n=2}^{\infty} \left(-\frac{a_{n-2} x^n}{2}\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^n}{6}\right) + \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^n}{24}\right) + \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right) + \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^n}{720}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 - a_1 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} + \frac{a_1}{6}$$

$n = 2$ gives

$$12a_4 - a_2 - a_1 - \frac{a_0}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_0}{12} + \frac{a_1}{12}$$

$n = 3$ gives

$$20a_5 - a_3 - a_2 - \frac{a_1}{2} - \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{24} + \frac{a_1}{30}$$

$n = 4$ gives

$$30a_6 - a_4 - a_3 - \frac{a_2}{2} - \frac{a_1}{6} - \frac{a_0}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{13a_0}{720} + \frac{a_1}{72}$$

$n = 5$ gives

$$42a_7 - a_5 - a_4 - \frac{a_3}{2} - \frac{a_2}{6} - \frac{a_1}{24} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{140} + \frac{29a_1}{5040}$$

For $6 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(1+n) - a_n - a_{n-1} - \frac{a_{n-2}}{2} - \frac{a_{n-3}}{6} - \frac{a_{n-4}}{24} - \frac{a_{n-5}}{120} - \frac{a_{n-6}}{720} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{720a_n + 720a_{n-1} + 360a_{n-2} + 120a_{n-3} + 30a_{n-4} + 6a_{n-5} + a_{n-6}}{720\,(n+2)\,(1+n)} \qquad (5) \\ &= \frac{a_n}{(n+2)\,(1+n)} + \frac{a_{n-6}}{720\,(n+2)\,(1+n)} + \frac{a_{n-5}}{120\,(n+2)\,(1+n)} \\ &\quad + \frac{a_{n-4}}{24\,(n+2)\,(1+n)} + \frac{a_{n-3}}{6\,(n+2)\,(1+n)} + \frac{a_{n-2}}{2\,(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + \frac{a_0x^2}{2} + \left(\frac{a_0}{6} + \frac{a_1}{6}\right)x^3 + \left(\frac{a_0}{12} + \frac{a_1}{12}\right)x^4 + \left(\frac{a_0}{24} + \frac{a_1}{30}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)a_{0}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)a_{1}+O\left(x^{6}\right) \quad (3)$$

At $x=0$ the solution above becomes

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned}y&=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}+\frac{13}{720}x^{6}\right)y(0)\\&+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}+\frac{1}{72}x^{6}\right)y'(0)+O\left(x^{6}\right)\end{aligned} \quad (1)$$

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right) \quad (2)$$

Verification of solutions

$$\begin{aligned}y&=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}+\frac{13}{720}x^{6}\right)y(0)\\&+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}+\frac{1}{72}x^{6}\right)y'(0)+O\left(x^{6}\right)\end{aligned}$$

Verified OK.

$$y=\left(1+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{24}x^{5}\right)c_{1}+\left(x+\frac{1}{6}x^{3}+\frac{1}{12}x^{4}+\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
   Change of variables used:
      [x = ln(t)]
   Linear ODE actually solved:
      -u(t)+diff(u(t),t)+t*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)-y(x)*exp(x)=0,y(x),type='series',x=0);
```

$$y = \left(1+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{24}x^5\right)y(0)+\left(x+\frac{1}{6}x^3+\frac{1}{12}x^4+\frac{1}{30}x^5\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[y''[x]-y[x]*Exp[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{30}+\frac{x^4}{12}+\frac{x^3}{6}+x\right)+c_1\left(\frac{x^5}{24}+\frac{x^4}{12}+\frac{x^3}{6}+\frac{x^2}{2}+1\right)$$

23.15 problem 738

Internal problem ID [15482]
Internal file name [`OUTPUT/15482_Friday_May_10_2024_05_47_26_PM_28168364/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.1 Integration of differential equation in series. Power series. Exercises page 171
Problem number: 738.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y' - e^{y} - yx = 0}$$

With initial conditions

$$[y(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

23.15.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$
\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}
$$

$$
\begin{aligned}
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f
\end{aligned} \tag{3}
$$

$$
\vdots
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
F_0 = f(x,y) \tag{4}
$$

$$
\begin{aligned}
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0
\end{aligned} \tag{5}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= e^{y} + yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}F_0 \\
&= e^{2y} + (y+1)\,x\,e^{y} + \left(x^2+1\right)y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}F_1 \\
&= x(3y+2)\,e^{2y} + 2\,e^{3y} + \left(x^2y^2 + y\left(2x^2+1\right) + x^2 + 2\right)e^{y} + \left(x^3+3x\right)y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}F_2 \\
&= \left(7x^2y^2 + 11yx^2 + 3x^2 + 4y + 5\right)e^{2y} + x(12y+7)\,e^{3y} + 6\,e^{4y} + \left(x^3y^3 + \left(4x^3+3x\right)y^2 + \left(3x^3+7x\right)y \right. \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}F_3 \\
&= \left(15x^3y^3 + \left(43x^3+25x\right)y^2 + 7\left(4x^3+7x\right)y + 4x^3 + 18x\right)e^{2y} + \left(50x^2y^2 + 69yx^2 + 17x^2 + 20y + 21\right.
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 0$ gives

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_2 &= 4 \\ F_3 &= 11 \\ F_4 &= 53 \end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{24} + \frac{53x^5}{120} + O(x^6)$$

Now we substitute the given initial conditions in the above to solve for $y(0)$. Solving for $y(0)$ from initial conditions gives

$$y(0) = y(0)$$

Therefore the solution becomes

$$y = x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{11}{24}x^4 + \frac{53}{120}x^5$$

Hence the solution can be written as

$$y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{24} + \frac{53x^5}{120} + O(x^6)$$

which simplifies to

$$y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{24} + \frac{53x^5}{120} + O(x^6)$$

Unable to also solve using normal power series since not linear ode. Not currently supported.

Summary

The solution(s) found are the following

$$y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{24} + \frac{53x^5}{120} + O(x^6) \tag{1}$$

Verification of solutions

$$y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{24} + \frac{53x^5}{120} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([diff(y(x),x)=exp(y(x))+x*y(x),y(0) = 0],y(x),type='series',x=0);
```

$$y = x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{11}{24}x^4 + \frac{53}{120}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 33

```
AsymptoticDSolveValue[{y'[x]==Exp[y[x]]+x*y[x],{y[0]==0}},y[x],{x,0,5}]
```

$$y(x) \to \frac{53x^5}{120} + \frac{11x^4}{24} + \frac{2x^3}{3} + \frac{x^2}{2} + x$$

24 Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177

24.1 problem 739

Internal problem ID [15483]
Internal file name [`OUTPUT/15483_Friday_May_10_2024_05_47_26_PM_97534733/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 739.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{4xy'' + 2y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + 2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{4x}$$

Table 715: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + 2y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(4x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2 - 2r\right) x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 2r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 2r\right) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2(2n^2 + 4nr + 2r^2 - n - r)} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{4n^2 + 2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4r^2 + 6r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16r^4 + 80r^3 + 140r^2 + 100r + 24}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64r^6 + 672r^5 + 2800r^4 + 5880r^3 + 6496r^2 + 3528r + 720}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$
a_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{5040}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16 (8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90) (2r^2 + 15r + 28)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{362880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$
a_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{5040}$
a_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{362880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32 (8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90) (2r^2 + 15r + 28) (2r^2 + 19r + 45)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{39916800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$
a_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{5040}$
a_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{362880}$
a_5	$-\frac{1}{32(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)(2r^2+19r+45)}$	$-\frac{1}{39916800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\ &= \sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) + 2(n+r)\,b_n + b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{2(2n^2 + 4nr + 2r^2 - n - r)} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-1}}{4n^2 - 2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{4r^2 + 6r + 2}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{16r^4 + 80r^3 + 140r^2 + 100r + 24}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{64r^6 + 672r^5 + 2800r^4 + 5880r^3 + 6496r^2 + 3528r + 720}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$
b_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{720}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16\left(8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90\right)\left(2r^2 + 15r + 28\right)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{40320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$
b_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{720}$
b_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{40320}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32 \left(8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90\right) \left(2r^2 + 15r + 28\right) \left(2r^2 + 19r + 45\right)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{3628800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$
b_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{720}$
b_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{40320}$
b_5	$-\frac{1}{32(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)(2r^2+19r+45)}$	$-\frac{1}{3628800}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O\left(x^6\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\frac{x^4}{362880}-\frac{x^5}{39916800}+O\left(x^6\right)\right) \\
&\quad + c_2\left(1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}+\frac{x^4}{40320}-\frac{x^5}{3628800}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\frac{x^4}{362880}-\frac{x^5}{39916800}+O\left(x^6\right)\right) \\
&\quad + c_2\left(1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}+\frac{x^4}{40320}-\frac{x^5}{3628800}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\frac{x^4}{362880}-\frac{x^5}{39916800}+O\left(x^6\right)\right) \\
&\quad + c_2\left(1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}+\frac{x^4}{40320}-\frac{x^5}{3628800}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1-\frac{x}{6}+\frac{x^2}{120}-\frac{x^3}{5040}+\frac{x^4}{362880}-\frac{x^5}{39916800}+O\left(x^6\right)\right) \\
&\quad + c_2\left(1-\frac{x}{2}+\frac{x^2}{24}-\frac{x^3}{720}+\frac{x^4}{40320}-\frac{x^5}{3628800}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

24.1.1 Maple step by step solution

Let's solve

$4y''x + 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{4x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{4x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4y''x + 2y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1}$

  - Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)\,(2k+1+2r) + a_k)\, x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{2(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{2(2k+1)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = -\frac{a_k}{2(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_kx^k\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}}\right),a_{k+1}=-\frac{a_k}{2(2k+1)(k+1)},b_{k+1}=-\frac{b_k}{2(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y=c_1\sqrt{x}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5040}x^3+\frac{1}{362880}x^4-\frac{1}{39916800}x^5+\mathrm{O}\left(x^6\right)\right)\\+c_2\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{40320}x^4-\frac{1}{3628800}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 85

```
AsymptoticDSolveValue[4*x*y''[x]+2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\sqrt{x}\left(-\frac{x^5}{39916800}+\frac{x^4}{362880}-\frac{x^3}{5040}+\frac{x^2}{120}-\frac{x}{6}+1\right)\\+c_2\left(-\frac{x^5}{3628800}+\frac{x^4}{40320}-\frac{x^3}{720}+\frac{x^2}{24}-\frac{x}{2}+1\right)$$

24.2 problem 740

Internal problem ID [15484]
Internal file name [`OUTPUT/15484_Friday_May_10_2024_05_47_26_PM_66644115/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 740.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x+1)\,y'-ny=0}$$

With the expansion point for the power series method at $x=0$.

24.2.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f\right)f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{ny}{x+1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}F_0\\
&= \frac{ny(-1+n)}{(x+1)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}F_1\\
&= \frac{ny(-2+n)(-1+n)}{(x+1)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}F_2\\
&= \frac{y(-3+n)(-2+n)n(-1+n)}{(x+1)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}F_3\\
&= \frac{y(-4+n)(-2+n)n(-3+n)(-1+n)}{(x+1)^5}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= y(0)\,n \\
F_1 &= y(0)\,n(-1+n) \\
F_2 &= y(0)\,n\left(n^2-3n+2\right) \\
F_3 &= y(0)\,n\left(n^3-6n^2+11n-6\right) \\
F_4 &= y(0)\,n\left(n^4-10n^3+35n^2-50n+24\right)
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
\begin{aligned}
y = \bigg(&1+nx-\frac{1}{2}n\,x^2+\frac{1}{2}n^2x^2+\frac{1}{6}n^3x^3-\frac{1}{2}n^2x^3+\frac{1}{3}n\,x^3+\frac{1}{24}n^4x^4-\frac{1}{4}n^3x^4+\frac{11}{24}n^2x^4 \\
&-\frac{1}{4}n\,x^4+\frac{1}{120}n^5x^5-\frac{1}{12}n^4x^5+\frac{7}{24}n^3x^5-\frac{5}{12}n^2x^5+\frac{1}{5}n\,x^5\bigg)\,y(0)+O\left(x^6\right)
\end{aligned}
$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y'+q(x)y &= p(x) \\
y'-\frac{ny}{x+1} &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -\frac{n}{x+1} \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$
(x+1)\,y'-ny = 0
$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x+1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - n\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-n a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-n a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$\begin{aligned} -n a_0 + a_1 &= 0 \\ a_1 &= n a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$n a_n + (n+1) a_{n+1} - n a_n = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{a_n(n-n)}{n+1} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$-na_1 + a_1 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{1}{2}n^2 a_0 - \frac{1}{2}na_0$$

For $n = 2$ the recurrence equation gives

$$-na_2 + 2a_2 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{1}{6}n^3 a_0 - \frac{1}{2}n^2 a_0 + \frac{1}{3}na_0$$

For $n = 3$ the recurrence equation gives

$$-na_3 + 3a_3 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24}n^4 a_0 - \frac{1}{4}n^3 a_0 + \frac{11}{24}n^2 a_0 - \frac{1}{4}na_0$$

For $n = 4$ the recurrence equation gives

$$-na_4 + 4a_4 + 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120}n^5 a_0 - \frac{1}{12}n^4 a_0 + \frac{7}{24}n^3 a_0 - \frac{5}{12}n^2 a_0 + \frac{1}{5}na_0$$

For $n = 5$ the recurrence equation gives

$$-na_5 + 5a_5 + 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{720}n^6 a_0 - \frac{1}{48}n^5 a_0 + \frac{17}{144}n^4 a_0 - \frac{5}{16}n^3 a_0 + \frac{137}{360}n^2 a_0 - \frac{1}{6}n a_0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + n a_0 x &+ \left(\frac{1}{2}n^2 a_0 - \frac{1}{2}n a_0\right) x^2 + \left(\frac{1}{6}n^3 a_0 - \frac{1}{2}n^2 a_0 + \frac{1}{3}n a_0\right) x^3 \\ &+ \left(\frac{1}{24}n^4 a_0 - \frac{1}{4}n^3 a_0 + \frac{11}{24}n^2 a_0 - \frac{1}{4}n a_0\right) x^4 \\ &+ \left(\frac{1}{120}n^5 a_0 - \frac{1}{12}n^4 a_0 + \frac{7}{24}n^3 a_0 - \frac{5}{12}n^2 a_0 + \frac{1}{5}n a_0\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \Bigg(1 + nx + \left(\frac{1}{2}n^2 - \frac{1}{2}n\right) x^2 + \left(\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n\right) x^3 + \left(\frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{11}{24}n^2 - \frac{1}{4}n\right) x^4 \\ + \left(\frac{1}{120}n^5 - \frac{1}{12}n^4 + \frac{7}{24}n^3 - \frac{5}{12}n^2 + \frac{1}{5}n\right) x^5\Bigg) a_0 + O(x^6) \end{aligned} \tag{3}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = \Bigg(1 + nx - \frac{1}{2}n\,x^2 + \frac{1}{2}n^2 x^2 + \frac{1}{6}n^3 x^3 - \frac{1}{2}n^2 x^3 + \frac{1}{3}n\,x^3 + \frac{1}{24}n^4 x^4 - \frac{1}{4}n^3 x^4 + \frac{11}{24}n^2 x^4 \\ - \frac{1}{4}n\,x^4 + \frac{1}{120}n^5 x^5 - \frac{1}{12}n^4 x^5 + \frac{7}{24}n^3 x^5 - \frac{5}{12}n^2 x^5 + \frac{1}{5}n\,x^5\Bigg) y(0) + O(x^6) \end{aligned} \tag{1}$$

$$\begin{aligned} y = \Bigg(1 + nx + \left(\frac{1}{2}n^2 - \frac{1}{2}n\right) x^2 + \left(\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n\right) x^3 + \left(\frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{11}{24}n^2 - \frac{1}{4}n\right) x^4 \\ + \left(\frac{1}{120}n^5 - \frac{1}{12}n^4 + \frac{7}{24}n^3 - \frac{5}{12}n^2 + \frac{1}{5}n\right) x^5\Bigg) c_1 + O(x^6) \end{aligned} \tag{2}$$

Verification of solutions

$$y = \left(1 + nx - \frac{1}{2} n\, x^2 + \frac{1}{2} n^2 x^2 + \frac{1}{6} n^3 x^3 - \frac{1}{2} n^2 x^3 + \frac{1}{3} n\, x^3 + \frac{1}{24} n^4 x^4 - \frac{1}{4} n^3 x^4 + \frac{11}{24} n^2 x^4 - \frac{1}{4} n\, x^4 + \frac{1}{120} n^5 x^5 - \frac{1}{12} n^4 x^5 + \frac{7}{24} n^3 x^5 - \frac{5}{12} n^2 x^5 + \frac{1}{5} n\, x^5\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1+nx+\left(\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(\frac{1}{6} n^3 - \frac{1}{2} n^2 + \frac{1}{3} n\right) x^3 + \left(\frac{1}{24} n^4 - \frac{1}{4} n^3 + \frac{11}{24} n^2 - \frac{1}{4} n\right) x^4 + \left(\frac{1}{120} n^5 - \frac{1}{12} n^4 + \frac{7}{24} n^3 - \frac{5}{12} n^2 + \frac{1}{5} n\right) x^5\right) c_1 + O(x^6)$$

Verified OK.

24.2.2 Maple step by step solution

Let's solve

$(x+1)\, y' - ny = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{n}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{n}{x+1} dx + c_1$

- Evaluate integral

  $\ln(y) = n \ln(x+1) + c_1$

- Solve for y

  $y = \mathrm{e}^{n \ln(x+1) + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 114

```
Order:=6;
dsolve((1+x)*diff(y(x),x)-n*y(x)=0,y(x),type='series',x=0);
```

$$y = \left(1 + nx + \frac{n(-1+n)\,x^2}{2} + \frac{n(n^2-3n+2)\,x^3}{6} + \frac{n(n^3-6n^2+11n-6)\,x^4}{24} + \frac{n(n^4-10n^3+35n^2-50n+24)\,x^5}{120}\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 143

```
AsymptoticDSolveValue[(1+x)*y'[x]-n*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{n^5x^5}{120} - \frac{n^4x^5}{12} + \frac{n^4x^4}{24} + \frac{7n^3x^5}{24} - \frac{n^3x^4}{4} + \frac{n^3x^3}{6} - \frac{5n^2x^5}{12} + \frac{11n^2x^4}{24} - \frac{n^2x^3}{2} + \frac{n^2x^2}{2} + \frac{nx^5}{5} - \frac{nx^4}{4} + \frac{nx^3}{3} - \frac{nx^2}{2} + nx + 1\right)$$

24.3 problem 741

Internal problem ID [15485]
Internal file name [`OUTPUT/15485_Friday_May_10_2024_05_47_26_PM_85755146/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 741.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[_Jacobi]`

$$\boxed{9x(1-x)\,y''-12y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-9x^2+9x\right)y''-12y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4}{3x\,(x-1)}$$
$$q(x)=-\frac{4}{9x\,(x-1)}$$

Table 718: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{3x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

$q(x) = -\frac{4}{9x(x-1)}$	
singularity	type
$x = 0$	"regular"
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 1, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-9y''x(x-1) - 12y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-9\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x(x-1) - 12\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-9x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}9x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-12(n+r)a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}4a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-9x^{n+r}a_n(n+r)(n+r-1)\right)=\sum_{n=1}^{\infty}\left(-9a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}4a_n x^{n+r}=\sum_{n=1}^{\infty}4a_{n-1}x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=1}^{\infty}\left(-9a_{n-1}(n+r-1)(n+r-2)x^{n+r-1}\right) \\ +\left(\sum_{n=0}^{\infty}9x^{n+r-1}a_n(n+r)(n+r-1)\right) \\ +\sum_{n=0}^{\infty}\left(-12(n+r)a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty}4a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$9x^{n+r-1}a_n(n+r)(n+r-1)-12(n+r)a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$9x^{-1+r}a_0r(-1+r)-12ra_0x^{-1+r}=0$$

Or

$$\left(9x^{-1+r}r(-1+r)-12r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(9r^2 - 21r\right) x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$9r^2 - 21r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{7}{3}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(9r^2 - 21r\right) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{7}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{7}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$-9a_{n-1}(n+r-1)(n+r-2) + 9a_n(n+r)(n+r-1) - 12a_n(n+r) + 4a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(3n+3r-2)\, a_{n-1}}{3n+3r} \quad (4)$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_n = \frac{(3n+5)\, a_{n-1}}{3n+7} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{7}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+3r}{3+3r}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_1 = \frac{4}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+3r}{3+3r}$	$\frac{4}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9r^2+15r+4}{9(1+r)(2+r)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_2 = \frac{44}{65}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+3r}{3+3r}$	$\frac{4}{5}$
a_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{44}{65}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{27r^3 + 108r^2 + 117r + 28}{27 (1 + r) (2 + r) (3 + r)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_3 = \frac{77}{130}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+3r}{3+3r}$	$\frac{4}{5}$
a_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{44}{65}$
a_3	$\frac{27r^3+108r^2+117r+28}{27(1+r)(2+r)(3+r)}$	$\frac{77}{130}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81r^4 + 594r^3 + 1431r^2 + 1254r + 280}{81 (1 + r) (2 + r) (3 + r) (4 + r)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_4 = \frac{1309}{2470}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+3r}{3+3r}$	$\frac{4}{5}$
a_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{44}{65}$
a_3	$\frac{27r^3+108r^2+117r+28}{27(1+r)(2+r)(3+r)}$	$\frac{77}{130}$
a_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{81(1+r)(2+r)(3+r)(4+r)}$	$\frac{1309}{2470}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{243r^5 + 2835r^4 + 12015r^3 + 22365r^2 + 17142r + 3640}{243 (1 + r) (2 + r) (3 + r) (4 + r) (5 + r)}$$

Which for the root $r = \frac{7}{3}$ becomes

$$a_5 = \frac{119}{247}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+3r}{3+3r}$	$\frac{4}{5}$
a_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{44}{65}$
a_3	$\frac{27r^3+108r^2+117r+28}{27(1+r)(2+r)(3+r)}$	$\frac{77}{130}$
a_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{81(1+r)(2+r)(3+r)(4+r)}$	$\frac{1309}{2470}$
a_5	$\frac{243r^5+2835r^4+12015r^3+22365r^2+17142r+3640}{243(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{119}{247}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{7}{3}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{7}{3}} \left(1 + \frac{4x}{5} + \frac{44x^2}{65} + \frac{77x^3}{130} + \frac{1309x^4}{2470} + \frac{119x^5}{247} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$-9b_{n-1}(n+r-1)(n+r-2) + 9b_n(n+r)(n+r-1) - 12(n+r)b_n + 4b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{(3n+3r-2)\, b_{n-1}}{3n+3r} \quad (4)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(3n-2)\, b_{n-1}}{3n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1 + 3r}{3 + 3r}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+3r}{3+3r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9r^2 + 15r + 4}{9\,(1 + r)\,(2 + r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{2}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+3r}{3+3r}$	$\frac{1}{3}$
b_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{2}{9}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{27r^3 + 108r^2 + 117r + 28}{27\,(1 + r)\,(2 + r)\,(3 + r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{14}{81}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+3r}{3+3r}$	$\frac{1}{3}$
b_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{2}{9}$
b_3	$\frac{27r^3+108r^2+117r+28}{27(1+r)(2+r)(3+r)}$	$\frac{14}{81}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81r^4 + 594r^3 + 1431r^2 + 1254r + 280}{81\left(1+r\right)\left(2+r\right)\left(3+r\right)\left(4+r\right)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{35}{243}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+3r}{3+3r}$	$\frac{1}{3}$
b_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{2}{9}$
b_3	$\frac{27r^3+108r^2+117r+28}{27(1+r)(2+r)(3+r)}$	$\frac{14}{81}$
b_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{81(1+r)(2+r)(3+r)(4+r)}$	$\frac{35}{243}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{243r^5 + 2835r^4 + 12015r^3 + 22365r^2 + 17142r + 3640}{243\left(1+r\right)\left(2+r\right)\left(3+r\right)\left(4+r\right)\left(5+r\right)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{91}{729}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1+3r}{3+3r}$	$\frac{1}{3}$
b_2	$\frac{9r^2+15r+4}{9(1+r)(2+r)}$	$\frac{2}{9}$
b_3	$\frac{27r^3+108r^2+117r+28}{27(1+r)(2+r)(3+r)}$	$\frac{14}{81}$
b_4	$\frac{81r^4+594r^3+1431r^2+1254r+280}{81(1+r)(2+r)(3+r)(4+r)}$	$\frac{35}{243}$
b_5	$\frac{243r^5+2835r^4+12015r^3+22365r^2+17142r+3640}{243(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{91}{729}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \frac{35x^4}{243} + \frac{91x^5}{729} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{7}{3}} \left(1 + \frac{4x}{5} + \frac{44x^2}{65} + \frac{77x^3}{130} + \frac{1309x^4}{2470} + \frac{119x^5}{247} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \frac{35x^4}{243} + \frac{91x^5}{729} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{7}{3}} \left(1 + \frac{4x}{5} + \frac{44x^2}{65} + \frac{77x^3}{130} + \frac{1309x^4}{2470} + \frac{119x^5}{247} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \frac{35x^4}{243} + \frac{91x^5}{729} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{7}{3}} \left(1 + \frac{4x}{5} + \frac{44x^2}{65} + \frac{77x^3}{130} + \frac{1309x^4}{2470} + \frac{119x^5}{247} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \frac{35x^4}{243} + \frac{91x^5}{729} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{7}{3}} \left(1+\frac{4x}{5}+\frac{44x^2}{65}+\frac{77x^3}{130}+\frac{1309x^4}{2470}+\frac{119x^5}{247}+O(x^6)\right) + c_2 \left(1+\frac{x}{3}+\frac{2x^2}{9}+\frac{14x^3}{81}+\frac{35x^4}{243}+\frac{91x^5}{729}+O(x^6)\right)
$$

Verified OK.

24.3.1 Maple step by step solution

Let's solve

$-9y''x(x-1)-12y'+4y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y}{9x(x-1)}-\frac{4y'}{3x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{4y'}{3x(x-1)}-\frac{4y}{9x(x-1)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{4}{3x(x-1)}, P_3(x)=-\frac{4}{9x(x-1)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x=0$

  $\left.(x \cdot P_2(x))\right|_{x=0}=-\frac{4}{3}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $9y''x(x-1)+12y'-4y=0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-3a_0r(-7+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (-3a_{k+1}(k+1+r)\,(3k+3r-4) + a_k(3k+3r+1)\,(3k+3r-4$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-3r(-7+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{7}{3}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $9\left((-k-r-1)\, a_{k+1} + a_k\left(k+r+\frac{1}{3}\right)\right)\left(k+r-\frac{4}{3}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(3k+3r+1)}{3(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(3k+1)}{3(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(3k+1)}{3(k+1)}\right]$

- Recursion relation for $r = \frac{7}{3}$

  $a_{k+1} = \frac{a_k(3k+8)}{3(k+\frac{10}{3})}$
- Solution for $r = \frac{7}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{7}{3}}, a_{k+1} = \frac{a_k(3k+8)}{3(k+\frac{10}{3})}\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{7}{3}}\right), a_{k+1} = \frac{a_k(3k+1)}{3(k+1)}, b_{k+1} = \frac{b_k(3k+8)}{3(k+\frac{10}{3})}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(9*x*(1-x)*diff(y(x),x$2)-12*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y = c_1 x^{\frac{7}{3}} \left(1 + \frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \frac{1309}{2470}x^4 + \frac{119}{247}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \frac{35}{243}x^4 + \frac{91}{729}x^5 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 85

```
AsymptoticDSolveValue[9*x*(1-x)*y''[x]-12*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{91x^5}{729} + \frac{35x^4}{243} + \frac{14x^3}{81} + \frac{2x^2}{9} + \frac{x}{3} + 1\right) + c_1 \left(\frac{119x^5}{247} + \frac{1309x^4}{2470} + \frac{77x^3}{130} + \frac{44x^2}{65} + \frac{4x}{5} + 1\right) x^{7/3}$$

24.4 problem 744

Internal problem ID [15486]
Internal file name [`OUTPUT/15486_Friday_May_10_2024_05_47_27_PM_59914383/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 744.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + y'x + \left(4x^2 - \frac{1}{9}\right)y = 0}$$

24.4.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + \left(4x^2 - \frac{1}{9}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 2 \\
n &= -\frac{1}{3} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, 2x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{3}, 2x\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, 2x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{3}, 2x\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 \operatorname{BesselJ}\left(-\frac{1}{3}, 2x\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{3}, 2x\right)
$$

Verified OK.

24.4.2 Maple step by step solution

Let's solve

$y''x^2 + y'x + \left(4x^2 - \frac{1}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(36x^2-1)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(36x^2-1)y}{9x^2} = 0$

- Simplify ODE

  $y''x^2 + y'x + 4yx^2 - \frac{y}{9} = 0$

- Make a change of variables

$$t = 2x$$

- Compute y'

$$y' = 2\frac{d}{dt}y(t)$$

- Compute second derivative

$$y'' = 4\frac{d^2}{dt^2}y(t)$$

- Apply change of variables to the ODE

$$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + \left(\frac{d}{dt}y(t)\right)t + y(t)\,t^2 - \frac{y(t)}{9} = 0$$

- ODE is now of the Bessel form
- Solution to Bessel ODE

$$y(t) = c_1 BesselJ\left(\frac{1}{3}, t\right) + c_2 BesselY\left(\frac{1}{3}, t\right)$$

- Make the change from t back to x

$$y = c_1 BesselJ\left(\frac{1}{3}, 2x\right) + c_2 BesselY\left(\frac{1}{3}, 2x\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(4*x^2-1/9)*y(x)=0,y(x), singsol=all)
```

$$y = c_1 \operatorname{BesselJ}\left(\frac{1}{3}, 2x\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, 2x\right)$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+x*y'[x]+(4*x^2-1/9)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(\frac{1}{3}, 2x\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, 2x\right)$$

24.5 problem 745

Internal problem ID [15487]
Internal file name [`OUTPUT/15487_Friday_May_10_2024_05_47_27_PM_22296256/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 745.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right)y = 0}$$

24.5.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{x^2 - \frac{1}{4}}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \frac{\left(\frac{1}{x}\right)'}{2} - \frac{\left(\frac{1}{x}\right)^2}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \frac{\left(-\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \left(-\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{1}{x}}{2}} \\
&= \frac{1}{\sqrt{x}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{\sqrt{x}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^{\frac{3}{2}}(v''(x) + v(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos(x) + c_2 \sin(x)}{\sqrt{x}}$$

Verified OK.

24.5.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + (-n^2 + x^2)\,y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 1$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\cos(x)}{\sqrt{\pi}\,\sqrt{x}} + \frac{c_2\sqrt{2}\,\sin(x)}{\sqrt{\pi}\,\sqrt{x}}$$

Verified OK.

24.5.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= x^2 - \frac{1}{4} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 721: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Verified OK.

24.5.4 Maple step by step solution

Let's solve

$y''x^2 + y'x + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{4}$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4y''x^2+4y'x+(4x^2-1)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m=0..2$

    $x^m \cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r+a_1(3+2r)\,(1+2r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1)+4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r)=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{-\frac{1}{2},\frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2+8kr+4r^2-1)+4a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}\left(4(k+2)^2+8(k+2)\,r+4r^2-1\right)+4a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r=-\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{4k^2+12k+8}$

- Solution for $r=-\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x), singsol=all)
```

$$y = \frac{c_1 \sin(x) + c_2 \cos(x)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 39

```
DSolve[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ix}(2c_1 - ic_2e^{2ix})}{2\sqrt{x}}$$

24.6 problem 746

Internal problem ID [15488]
Internal file name [`OUTPUT/15488_Friday_May_10_2024_05_47_27_PM_10656506/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 746.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{y'}{x} + \frac{y}{9} = 0}$$

24.6.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + y' x + \frac{y x^2}{9} = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = \frac{1}{3}$$
$$n = 0$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \text{BesselJ}\left(0, \frac{x}{3}\right) + c_2 \text{BesselY}\left(0, \frac{x}{3}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \text{BesselJ}\left(0, \frac{x}{3}\right) + c_2 \text{BesselY}\left(0, \frac{x}{3}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \text{BesselJ}\left(0, \frac{x}{3}\right) + c_2 \text{BesselY}\left(0, \frac{x}{3}\right)$$

Verified OK.

24.6.2 Maple step by step solution

Let's solve

$9y''x + yx + 9y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{9} = 0$

- Simplify ODE

  $y''x^2 + y'x + \frac{yx^2}{9} = 0$

- Make a change of variables

  $t = \frac{x}{3}$

- Compute y'

  $y' = \frac{\frac{d}{dt}y(t)}{3}$
- Compute second derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{9}$
- Apply change of variables to the ODE

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 + \left(\frac{d}{dt}y(t)\right)t + y(t)\,t^2 = 0$
- ODE is now of the Bessel form
- Solution to Bessel ODE

  $y(t) = c_1 \mathit{BesselJ}(0, t) + c_2 \mathit{BesselY}(0, t)$
- Make the change from t back to x

  $y = c_1 \mathit{BesselJ}\left(0, \frac{x}{3}\right) + c_2 \mathit{BesselY}\left(0, \frac{x}{3}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)+1/9*y(x)=0,y(x), singsol=all)
```

$$y = c_1 \operatorname{BesselJ}\left(0, \frac{x}{3}\right) + c_2 \operatorname{BesselY}\left(0, \frac{x}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 26

```
DSolve[y''[x]+1/x*y'[x]+1/9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}\left(0, \frac{x}{3}\right) + c_2 \operatorname{BesselY}\left(0, \frac{x}{3}\right)$$

24.7 problem 747

Internal problem ID [15489]
Internal file name [`OUTPUT/15489_Friday_May_10_2024_05_47_27_PM_90256140/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 747.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{y'}{x} + 4y = 0}$$

24.7.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + y' x + 4y x^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 0$$
$$\beta = 2$$
$$n = 0$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \operatorname{BesselJ}(0, 2x) + c_2 \operatorname{BesselY}(0, 2x)$$

Summary

The solution(s) found are the following

$$y = c_1 \operatorname{BesselJ}(0, 2x) + c_2 \operatorname{BesselY}(0, 2x) \tag{1}$$

Verification of solutions

$$y = c_1 \operatorname{BesselJ}(0, 2x) + c_2 \operatorname{BesselY}(0, 2x)$$

Verified OK.

24.7.2 Maple step by step solution

Let's solve

$y''x + 4yx + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + 4y = 0$

- Simplify ODE

  $y''x^2 + y'x + 4yx^2 = 0$

- Make a change of variables

  $t = 2x$

- Compute y'

$$y' = 2\frac{d}{dt}y(t)$$

- Compute second derivative

$$y'' = 4\frac{d^2}{dt^2}y(t)$$

- Apply change of variables to the ODE

$$\left(\frac{d^2}{dt^2}y(t)\right)t^2 + \left(\frac{d}{dt}y(t)\right)t + y(t)\,t^2 = 0$$

- ODE is now of the Bessel form
- Solution to Bessel ODE

$$y(t) = c_1 BesselJ(0, t) + c_2 BesselY(0, t)$$

- Make the change from t back to x

$$y = c_1 BesselJ(0, 2x) + c_2 BesselY(0, 2x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y = c_1 \operatorname{BesselJ}(0, 2x) + c_2 \operatorname{BesselY}(0, 2x)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[y''[x]+1/x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \operatorname{BesselJ}(0, 2x) + c_2 \operatorname{BesselY}(0, 2x)$$

24.8 problem 748

Internal problem ID [15490]
Internal file name [`OUTPUT/15490_Friday_May_10_2024_05_47_27_PM_88000253/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 748.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2y'x + 4\left(x^4 - 1\right)y = 0}$$

24.8.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' - 2y'x + \left(4x^4 - 4\right)y = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{3}{2} \\
\beta &= 1 \\
n &= -\frac{5}{4} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1 x^{\frac{3}{2}} \operatorname{BesselJ}\left(-\frac{5}{4}, x^2\right) + c_2 x^{\frac{3}{2}} \operatorname{BesselY}\left(-\frac{5}{4}, x^2\right)
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{3}{2}} \operatorname{BesselJ}\left(-\frac{5}{4}, x^2\right) + c_2 x^{\frac{3}{2}} \operatorname{BesselY}\left(-\frac{5}{4}, x^2\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{3}{2}} \operatorname{BesselJ}\left(-\frac{5}{4}, x^2\right) + c_2 x^{\frac{3}{2}} \operatorname{BesselY}\left(-\frac{5}{4}, x^2\right)
$$

Verified OK.

24.8.2 Maple step by step solution

Let's solve

$y''x^2 - 2y'x + (4x^4 - 4)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4(x^4-1)y}{x^2} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{4(x^4-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{4(x^4-1)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -4$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 - 2y'x + (4x^4 - 4)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-4+r)\,x^r + a_1(2+r)\,(-3+r)\,x^{1+r} + a_2(3+r)\,(-2+r)\,x^{2+r} + a_3(4+r)\,(-1+r)\,x$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)(-4+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 4\}$
- The coefficients of each power of x must be 0

  $[a_1(2+r)(-3+r)=0, a_2(3+r)(-2+r)=0, a_3(4+r)(-1+r)=0]$
- Solve for the dependent coefficient(s)

  $\{a_1=0, a_2=0, a_3=0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-4)+4a_{k-4}=0$
- Shift index using $k->k+4$

  $a_{k+4}(k+5+r)(k+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{4a_k}{(k+5+r)(k+r)}$
- Recursion relation for $r=-1$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k-1)}$
- Solution for $r=-1$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+4}=-\frac{4a_k}{(k+4)(k-1)}, a_1=0, a_2=0, a_3=0\right]$
- Recursion relation for $r=4$

  $a_{k+4}=-\frac{4a_k}{(k+9)(k+4)}$
- Solution for $r=4$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+4}=-\frac{4a_k}{(k+9)(k+4)}, a_1=0, a_2=0, a_3=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+4}\right), a_{k+4}=-\frac{4a_k}{(k+4)(k-1)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+9)(k+\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+4*(x^4-1)*y(x)=0,y(x), singsol=all)
```

$$y = -\frac{-\frac{\operatorname{BesselY}\left(\frac{1}{4}, x^2\right) c_2}{2} - \frac{\operatorname{BesselJ}\left(\frac{1}{4}, x^2\right) c_1}{2} + x^2 \left(c_1 \operatorname{BesselJ}\left(-\frac{3}{4}, x^2\right) + \operatorname{BesselY}\left(-\frac{3}{4}, x^2\right) c_2\right)}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 46

```
DSolve[x^2*y''[x]-2*x*y'[x]+4*(x^4-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^{3/2} \left(c_2 \operatorname{Gamma}\left(\frac{9}{4}\right) \operatorname{BesselJ}\left(\frac{5}{4}, x^2\right) - 4 c_1 \operatorname{Gamma}\left(\frac{3}{4}\right) \operatorname{BesselJ}\left(-\frac{5}{4}, x^2\right)\right)}{2^{3/4}}$$

24.9 problem 749

Internal problem ID [15491]
Internal file name [`OUTPUT/15491_Friday_May_10_2024_05_47_28_PM_34729963/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 749.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__x__method__1", "second__order__change__of__variable__on__x__method__2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{y''x + \frac{y'}{2} + \frac{y}{4} = 0}$$

24.9.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y''x + \frac{y'}{2} + \frac{y}{4} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{2x} \\
q(x) &= \frac{1}{4x}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\,\tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{2x} dx\right)} dx \\
&= \int e^{-\frac{\ln(x)}{2}} dx \\
&= \int \frac{1}{\sqrt{x}} dx \\
&= 2\sqrt{x}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{4x}}{\frac{1}{x}} \\
&= \frac{1}{4}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4} = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = \frac{1}{4}$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + \frac{e^{\lambda\tau}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \frac{1}{4}$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)\left(\frac{1}{4}\right)}$$
$$= \pm\frac{i}{2}$$

Hence

$$\lambda_1 = +\frac{i}{2}$$
$$\lambda_2 = -\frac{i}{2}$$

Which simplifies to

$$\lambda_1 = \frac{i}{2}$$
$$\lambda_2 = -\frac{i}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}\left(c_1 \cos\left(\frac{\tau}{2}\right) + c_2 \sin\left(\frac{\tau}{2}\right)\right)$$

Or

$$y(\tau) = c_1 \cos\left(\frac{\tau}{2}\right) + c_2 \sin\left(\frac{\tau}{2}\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

Verified OK.

24.9.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y''x + \frac{y'}{2} + \frac{y}{4} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{4x}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x}}}{2c} \\ \tau'' &= -\frac{1}{4c\sqrt{\frac{1}{x}}\,x^2} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{4c\sqrt{\frac{1}{x}}\,x^2} + \frac{1}{2x}\frac{\sqrt{\frac{1}{x}}}{2c}}{\left(\frac{\sqrt{\frac{1}{x}}}{2c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \frac{\sqrt{\frac{1}{x}}}{2}dx}{c} \\ &= \frac{x\sqrt{\frac{1}{x}}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

Verified OK.

24.9.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + \frac{y'x}{2} + \frac{yx}{4} = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{4}$$
$$\beta = 1$$
$$n = \frac{1}{2}$$
$$\gamma = \frac{1}{2}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sin\left(\sqrt{x}\right)}{\sqrt{\pi}} - \frac{c_2\sqrt{2}\,\cos\left(\sqrt{x}\right)}{\sqrt{\pi}}$$

Verified OK.

24.9.4 Solving using Kovacic algorithm

Writing the ode as

$$y''x + \frac{y'}{2} + \frac{y}{4} = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = x$$
$$B = \frac{1}{2} \tag{3}$$
$$C = \frac{1}{4}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4x - 3}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4x - 3 \\ t &= 16x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-4x - 3}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 726: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [2]$$

Attempting to find a solution using case $n = 2$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{16x^2} - \frac{1}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{3}{16}$. Hence

$$\begin{aligned} E_c &= \{2, 2 + 2\sqrt{1 + 4b}, 2 - 2\sqrt{1 + 4b}\} \\ &= \{1, 2, 3\} \end{aligned}$$

Since the order of r at ∞ is $1 < 2$ then

$$E_\infty = \{1\}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ for case 2 of Kovacic algorithm.

pole c location	pole order	E_c
0	2	$\{1, 2, 3\}$

Order of r at ∞	E_∞
1	$\{1\}$

Using the family $\{e_1, e_2, \dots, e_\infty\}$ given by

$$e_1 = 1, \; e_\infty = 1$$

Gives a non negative integer d (the degree of the polynomial $p(x)$), which is generated using

$$\begin{aligned} d &= \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right) \\ &= \frac{1}{2}(1-(1)) \\ &= 0 \end{aligned}$$

We now form the following rational function

$$\begin{aligned} \theta &= \frac{1}{2}\sum_{c\in\Gamma} \frac{e_c}{x-c} \\ &= \frac{1}{2}\left(\frac{1}{(x-(0))}\right) \\ &= \frac{1}{2x} \end{aligned}$$

Now we search for a monic polynomial $p(x)$ of degree $d=0$ such that

$$p''' + 3\theta p'' + \left(3\theta^2 + 3\theta' - 4r\right)p' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right)p = 0 \tag{1A}$$

Since $d=0$, then letting

$$p = 1 \tag{2A}$$

Substituting p and θ into Eq. (1A) gives

$$0 = 0$$

And solving for p gives

$$p = 1$$

Now that $p(x)$ is found let

$$\begin{aligned} \phi &= \theta + \frac{p'}{p} \\ &= \frac{1}{2x} \end{aligned}$$

Let ω be the solution of

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$

Substituting the values for ϕ and r into the above equation gives

$$w^2 - \frac{w}{2x} + \frac{1+4x}{16x^2} = 0$$

Solving for ω gives

$$\omega = \frac{1+2\sqrt{-x}}{4x}$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1+2\sqrt{-x}}{4x} dx} \\ &= x^{\frac{1}{4}} \mathrm{e}^{\sqrt{-x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{\frac{1}{2}}{x} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{4}} \\ &= z_1 \left(\frac{1}{x^{\frac{1}{4}}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{-x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{\frac{1}{2}}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\sqrt{-x}\left(-1+\mathrm{e}^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\sqrt{-x}}\right) + c_2\left(e^{\sqrt{-x}}\left(\frac{\sqrt{-x}\left(-1+e^{-2\sqrt{-x}}\right)}{\sqrt{x}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{\sqrt{-x}} - \frac{c_2\sqrt{-x}\left(e^{\sqrt{-x}} - e^{-\sqrt{-x}}\right)}{\sqrt{x}} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{\sqrt{-x}} - \frac{c_2\sqrt{-x}\left(e^{\sqrt{-x}} - e^{-\sqrt{-x}}\right)}{\sqrt{x}}
$$

Verified OK.

24.9.5 Maple step by step solution

Let's solve

$y''x + \frac{y'}{2} + \frac{y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4x} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{4x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{4x}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x + 2y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)\,(2k+1+2r) + a_k)\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

$$4\left(k+\frac{1}{2}+r\right)\left(k+1+r\right)a_{k+1}+a_k=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=-\frac{a_k}{2(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r=0$

$$a_{k+1}=-\frac{a_k}{2(2k+1)(k+1)}$$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+1}=-\frac{a_k}{2(2k+1)(k+1)}\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+1}=-\frac{a_k}{2(2k+2)\left(k+\frac{3}{2}\right)}$$

- Solution for $r=\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{a_k}{2(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k x^k\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+1}=-\frac{a_k}{2(2k+1)(k+1)}, b_{k+1}=-\frac{b_k}{2(2k+2)\left(k+\frac{3}{2}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)+1/2*diff(y(x),x)+1/4*y(x)=0,y(x), singsol=all)
```

$$y=c_1\sin\left(\sqrt{x}\right)+c_2\cos\left(\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 24

```
DSolve[x*y''[x]+1/2*y'[x]+1/4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\sqrt{x}\right) + c_2 \sin\left(\sqrt{x}\right)$$

24.10 problem 750

Internal problem ID [15492]
Internal file name [`OUTPUT/15492_Friday_May_10_2024_05_47_28_PM_80826676/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 750.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{y'' + \frac{5y'}{x} + y = 0}$$

24.10.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 5y'x + yx^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= -2\\ \beta &= 1\\ n &= 2\\ \gamma &= 1\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \operatorname{BesselJ}(2, x)}{x^2} + \frac{c_2 \operatorname{BesselY}(2, x)}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \operatorname{BesselJ}(2, x)}{x^2} + \frac{c_2 \operatorname{BesselY}(2, x)}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \operatorname{BesselJ}(2, x)}{x^2} + \frac{c_2 \operatorname{BesselY}(2, x)}{x^2}$$

Verified OK.

24.10.2 Maple step by step solution

Let's solve

$y''x + yx + 5y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{x} + y = 0$

- Simplify ODE

  $y''x^2 + 5y'x + yx^2 = 0$

- Make a change of variables

  $y = \frac{u(x)}{x^2}$

- Compute y'

  $y' = -\frac{2u(x)}{x^3} + \frac{u'(x)}{x^2}$

- Compute y''

  $y'' = \frac{6u(x)}{x^4} - \frac{4u'(x)}{x^3} + \frac{u''(x)}{x^2}$

- Apply change of variables to the ODE

  $u(x)\, x^2 + u''(x)\, x^2 + u'(x)\, x - 4u(x) = 0$

- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(x) = c_1 BesselJ(2, x) + c_2 BesselY(2, x)$

- Make the change from y back to y

  $y = \frac{c_1 BesselJ(2,x) + c_2 BesselY(2,x)}{x^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+5/x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y = \frac{-\operatorname{BesselY}(0,x)\, c_2 x - \operatorname{BesselJ}(0,x)\, c_1 x + 2\operatorname{BesselY}(1,x)\, c_2 + 2\operatorname{BesselJ}(1,x)\, c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+5/x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 \operatorname{BesselJ}(2,x) + c_2 \operatorname{BesselY}(2,x)}{x^2}$$

24.11 problem 751

Internal problem ID [15493]
Internal file name [`OUTPUT/15493_Friday_May_10_2024_05_47_28_PM_91538688/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.2. Expanding a solution in generalized power series. Bessels equation. Exercises page 177
Problem number: 751.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \frac{3y'}{x} + 4y = 0}$$

24.11.1 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 3y'x + 4yx^2 = 0 \tag{1}$$

Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = -1$$
$$\beta = 2$$
$$n = 1$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \operatorname{BesselJ}(1, 2x)}{x} + \frac{c_2 \operatorname{BesselY}(1, 2x)}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \operatorname{BesselJ}(1, 2x)}{x} + \frac{c_2 \operatorname{BesselY}(1, 2x)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \operatorname{BesselJ}(1, 2x)}{x} + \frac{c_2 \operatorname{BesselY}(1, 2x)}{x}$$

Verified OK.

24.11.2 Maple step by step solution

Let's solve

$y''x + 4yx + 3y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} - 4y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} + 4y = 0$

- Simplify ODE

  $y''x^2 + 3y'x + 4yx^2 = 0$

- Make a change of variables

  $t = 2x$

- Compute y'

  $y' = 2\frac{d}{dt}y(t)$
- Compute second derivative

  $y'' = 4\frac{d^2}{dt^2}y(t)$
- Apply change of variables to the ODE

  $\left(\frac{d^2}{dt^2}y(t)\right)t^2 + 3\left(\frac{d}{dt}y(t)\right)t + y(t)\,t^2 = 0$
- Make a change of variables

  $y(t) = \frac{u(t)}{t}$
- Compute $\frac{d}{dt}y(t)$

  $\frac{d}{dt}y(t) = -\frac{u(t)}{t^2} + \frac{\frac{d}{dt}u(t)}{t}$
- Compute $\frac{d^2}{dt^2}y(t)$

  $\frac{d^2}{dt^2}y(t) = \frac{2u(t)}{t^3} - \frac{2\left(\frac{d}{dt}u(t)\right)}{t^2} + \frac{\frac{d^2}{dt^2}u(t)}{t}$
- Apply change of variables to the ODE

  $t^2u(t) + \left(\frac{d^2}{dt^2}u(t)\right)t^2 + \left(\frac{d}{dt}u(t)\right)t - u(t) = 0$
- ODE is now of the Bessel form
- Solution to Bessel ODE

  $u(t) = c_1 BesselJ(1, t) + c_2 BesselY(1, t)$
- Make the change from y back to $y(t)$

  $y(t) = \frac{c_1 BesselJ(1,t)+c_2 BesselY(1,t)}{t}$
- Make the change from t back to x

  $y = \frac{c_1 BesselJ(1,2x)+c_2 BesselY(1,2x)}{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+3/x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y = \frac{c_1 \operatorname{BesselJ}(1, 2x) + c_2 \operatorname{BesselY}(1, 2x)}{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 26

```
DSolve[y''[x]+3/x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 \operatorname{BesselJ}(1, 2x) + c_2 \operatorname{BesselY}(1, 2x)}{x}$$

25 Chapter 2 (Higher order ODE's). Section 18.3. Finding periodic solutions of linear differential equations. Exercises page 187

25.1 problem 757

Internal problem ID [15494]
Internal file name [`OUTPUT/15494_Friday_May_10_2024_05_47_28_PM_54653337/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.3. Finding periodic solutions of linear differential equations. Exercises page 187
Problem number: 757.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \cos(x)^2}$$

25.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \cos(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\cos(2x) + A_3 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = \cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{8} + \frac{x \sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{1}{8} + \frac{x \sin(2x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{8} + \frac{x \sin(2x)}{8} \tag{1}$$

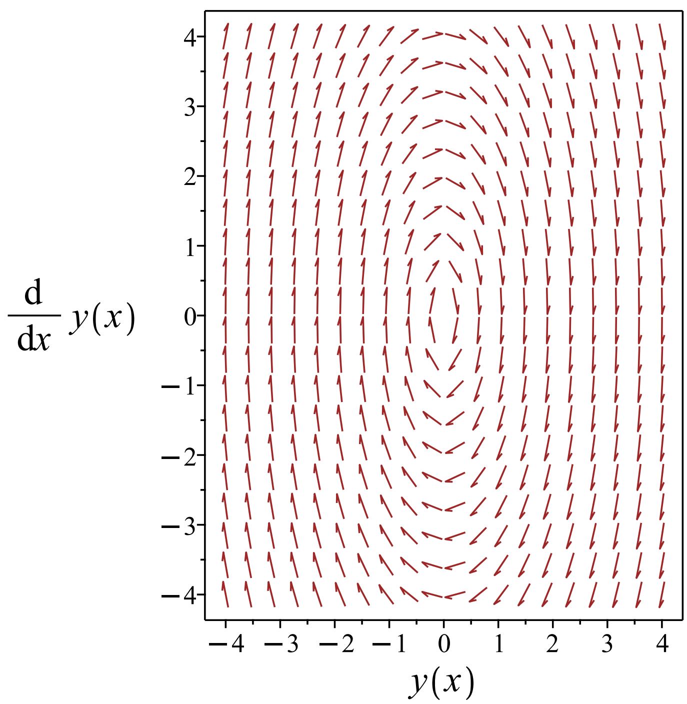


Figure 827: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{8} + \frac{x \sin(2x)}{8}$$

Verified OK.

25.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 730: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\cos(2x) + A_3 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = \cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{8} + \frac{x\sin(2x)}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{1}{8} + \frac{x \sin(2x)}{8}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{1}{8} + \frac{x \sin(2x)}{8} \tag{1}
$$

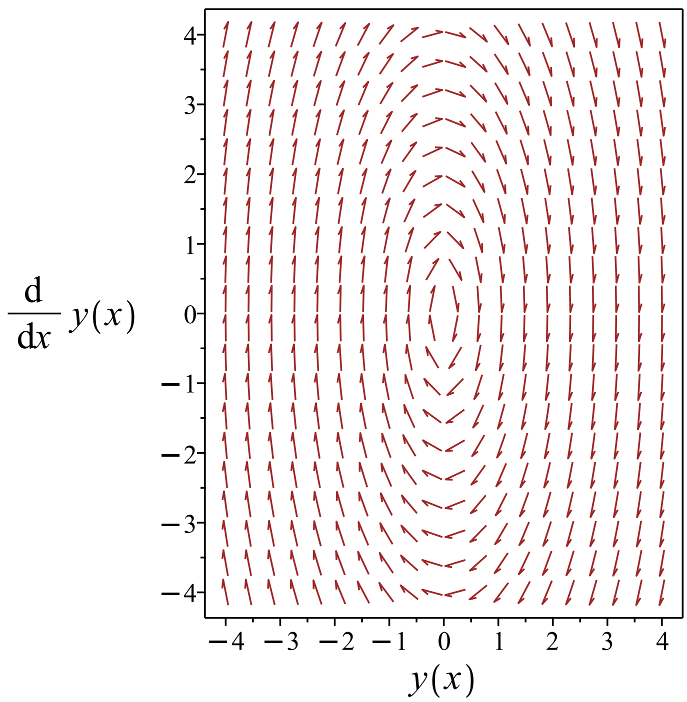


Figure 828: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{1}{8} + \frac{x \sin(2x)}{8}
$$

Verified OK.

25.1.3 Maple step by step solution

Let's solve

$y'' + 4y = \cos(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\cos(x)^2 dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\cos(x)^2 dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{x\sin(2x)}{8} + \frac{\cos(2x)}{8} + \frac{1}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x\sin(2x)}{8} + \frac{\cos(2x)}{8} + \frac{1}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+4*y(x)=cos(x)^2,y(x), singsol=all)
```

$$y = \frac{(8c_1 + 1)\cos(2x)}{8} + \frac{1}{8} + \frac{(x + 8c_2)\sin(2x)}{8}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Cos[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}((1 + 8c_1)\cos(2x) + (x + 8c_2)\sin(2x) + 1)$$

25.2 problem 758

Internal problem ID [15495]
Internal file name [`OUTPUT/15495_Friday_May_10_2024_05_47_28_PM_93908727/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.3. Finding periodic solutions of linear differential equations. Exercises page 187
Problem number: 758.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 4y = \pi^2 - x^2}$$

25.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = \pi^2 - x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda \, \mathrm{e}^{\lambda x} + 4 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)} \\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{2x} + c_2 x \, \mathrm{e}^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 x \, \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x \, \mathrm{e}^{2x}, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_3x^2 + 4A_2x - 8xA_3 + 4A_1 - 4A_2 + 2A_3 = \pi^2 - x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{\pi^2}{4} - \frac{3}{8}, A_2 = -\frac{1}{2}, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}\pi^2 - \frac{3}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + c_2 x\, \mathrm{e}^{2x}\right) + \left(-\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}\pi^2 - \frac{3}{8}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2x + c_1) - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_2x + c_1) - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8} \tag{1}$$

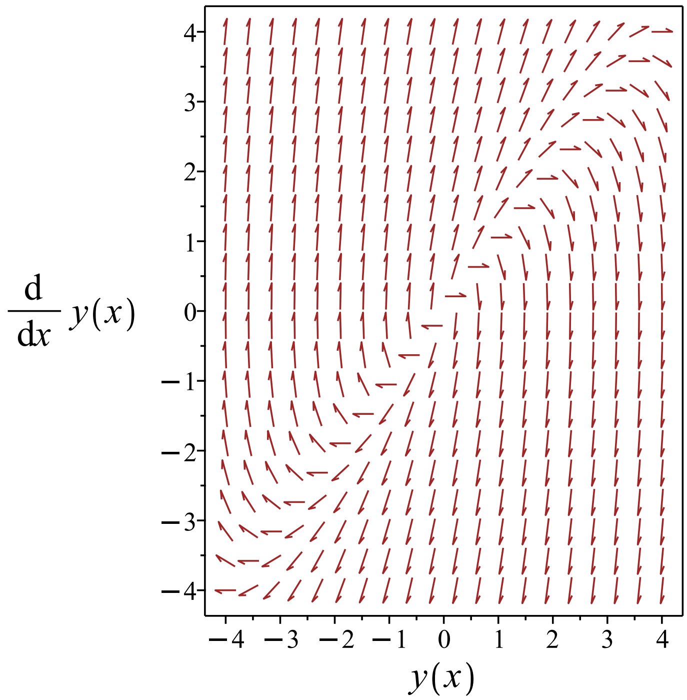


Figure 829: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}$$

Verified OK.

25.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4\, dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^{-2x}(\pi^2 - x^2) \\
(e^{-2x}y)'' &= e^{-2x}(\pi^2 - x^2)
\end{aligned}
$$

Integrating once gives

$$
(e^{-2x}y)' = -\frac{e^{-2x}(2\pi^2 - 2x^2 - 2x - 1)}{4} + c_1
$$

Integrating again gives

$$
(e^{-2x}y) = \frac{(2\pi^2 - 2x^2 - 4x - 3)\, e^{-2x}}{8} + c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{\frac{(2\pi^2 - 2x^2 - 4x - 3)\, e^{-2x}}{8} + c_1 x + c_2}{e^{-2x}}
$$

Or

$$
y = c_1 x\, e^{2x} + e^{2x} c_2 - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\, e^{2x} + e^{2x} c_2 - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8} \tag{1}
$$

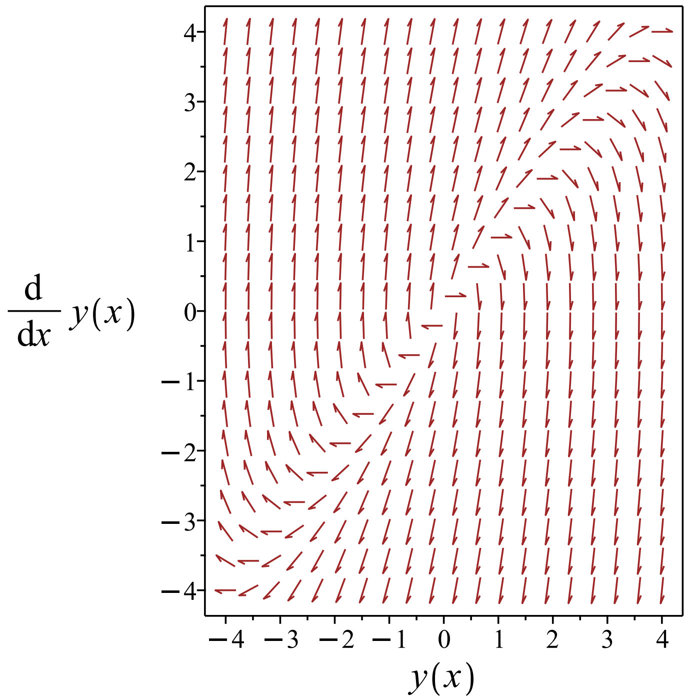


Figure 830: Slope field plot

Verification of solutions

$$y = c_1 x\,e^{2x} + e^{2x} c_2 - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}$$

Verified OK.

25.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 732: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1 \left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2x} + c_2 x\,\mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{2x}, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_3x^2 + 4A_2x - 8xA_3 + 4A_1 - 4A_2 + 2A_3 = \pi^2 - x^2
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{\pi^2}{4} - \frac{3}{8}, A_2 = -\frac{1}{2}, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}\pi^2 - \frac{3}{8}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + c_2 x\, \mathrm{e}^{2x}\right) + \left(-\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}\pi^2 - \frac{3}{8}\right)\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2 x + c_1) - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_2 x + c_1) - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8} \tag{1}$$

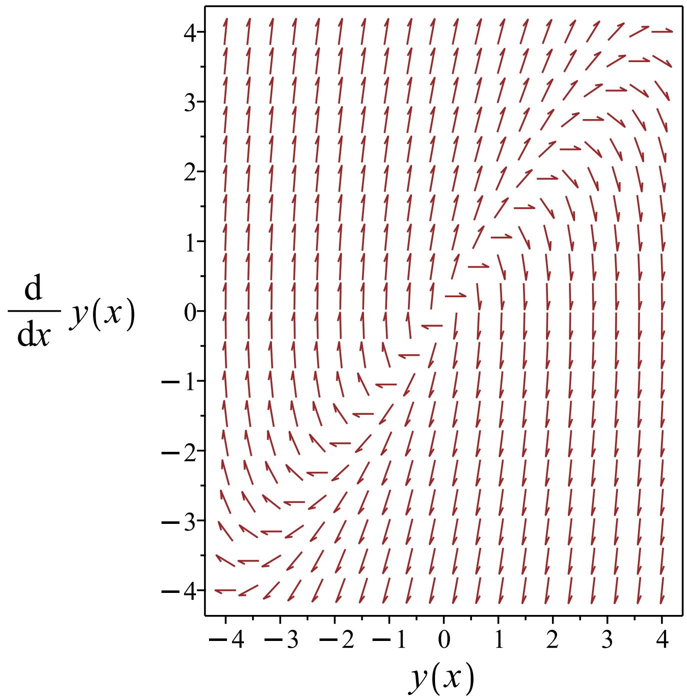


Figure 831: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}$$

Verified OK.

25.2.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = \pi^2 - x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

$r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2x} + c_2 x\,\mathrm{e}^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \pi^2 - x^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & x\,\mathrm{e}^{2x} \\ 2\,\mathrm{e}^{2x} & \mathrm{e}^{2x} + 2x\,\mathrm{e}^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{2x}\left(-\left(\int x\,\mathrm{e}^{-2x}(\pi^2 - x^2)\,dx\right) + \left(\int \mathrm{e}^{-2x}(\pi^2 - x^2)\,dx\right)x\right)$

  ◦ Compute integrals

  $y_p(x) = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4}\pi^2 - \frac{3}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{2x} + c_1 \mathrm{e}^{2x} - \frac{x^2}{4} - \frac{x}{2} + \frac{\pi^2}{4} - \frac{3}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=Pi^2-x^2,y(x), singsol=all)
```

$$y = -\frac{3}{8} + (c_1 x + c_2)\,\mathrm{e}^{2x} - \frac{x^2}{4} + \frac{\pi^2}{4} - \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 42

```
DSolve[y''[x]-4*y'[x]+4*y[x]==Pi^2-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(-2x^2 - 4x + 2\pi^2 - 3\right) + c_1 e^{2x} + c_2 e^{2x} x$$

25.3 problem 759

Internal problem ID [15496]
Internal file name [`OUTPUT/15496_Friday_May_10_2024_05_47_29_PM_62813000/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.3. Finding periodic solutions of linear differential equations. Exercises page 187
Problem number: 759.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \cos(\pi x)}$$

25.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = \cos(\pi x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(\pi x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\pi x), \sin(\pi x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(\pi x) + A_2 \sin(\pi x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\pi^2 \cos(\pi x) - A_2\pi^2 \sin(\pi x) - 4A_1 \cos(\pi x) - 4A_2 \sin(\pi x) = \cos(\pi x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{\pi^2 + 4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(\pi x)}{\pi^2 + 4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-2x}\right) + \left(-\frac{\cos(\pi x)}{\pi^2 + 4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{\cos(\pi x)}{\pi^2 + 4} \tag{1}$$

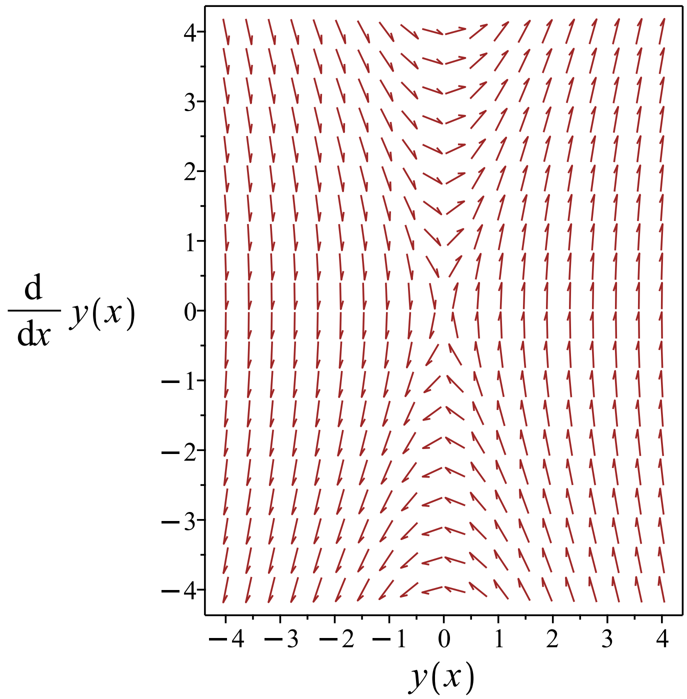


Figure 832: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{\cos(\pi x)}{\pi^2 + 4}$$

Verified OK.

25.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -4
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 734: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-2x}
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}} \, dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x} \right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{\mathrm{e}^{2x} c_2}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(\pi x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\pi x), \sin(\pi x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{2x}}{4}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(\pi x) + A_2 \sin(\pi x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\pi^2 \cos(\pi x) - A_2\pi^2 \sin(\pi x) - 4A_1 \cos(\pi x) - 4A_2 \sin(\pi x) = \cos(\pi x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{\pi^2 + 4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(\pi x)}{\pi^2 + 4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{e^{2x} c_2}{4}\right) + \left(-\frac{\cos(\pi x)}{\pi^2 + 4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{e^{2x} c_2}{4} - \frac{\cos(\pi x)}{\pi^2 + 4} \tag{1}$$

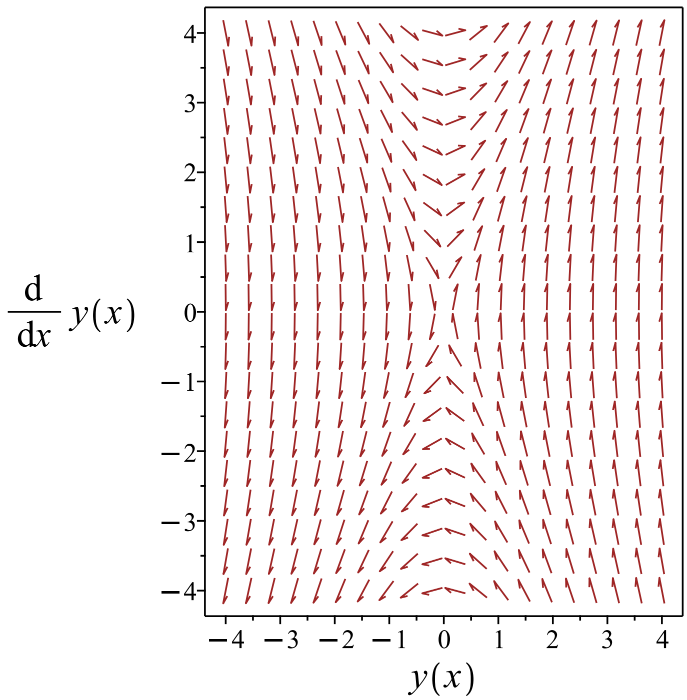


Figure 833: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{e^{2x} c_2}{4} - \frac{\cos(\pi x)}{\pi^2 + 4}$$

Verified OK.

25.3.3 Maple step by step solution

Let's solve

$y'' - 4y = \cos(\pi x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 2)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + e^{2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(\pi x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{2x} \\ -2 e^{-2x} & 2 e^{2x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-2x}\left(\int e^{2x} \cos(\pi x) dx\right)}{4} + \frac{e^{2x}\left(\int e^{-2x} \cos(\pi x) dx\right)}{4}$
  - Compute integrals

    $y_p(x) = -\frac{\cos(\pi x)}{\pi^2 + 4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + e^{2x} c_2 - \frac{\cos(\pi x)}{\pi^2 + 4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$2)-4*y(x)=cos(Pi*x),y(x), singsol=all)
```

$$y = \frac{c_1(\pi^2+4)\,\mathrm{e}^{-2x} + c_2(\pi^2+4)\,\mathrm{e}^{2x} - \cos(\pi x)}{\pi^2+4}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 35

```
DSolve[y''[x]-4*y[x]==Cos[Pi*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\cos(\pi x)}{4+\pi^2} + c_1e^{2x} + c_2e^{-2x}$$

25.4 problem 760

Internal problem ID [15497]
Internal file name [`OUTPUT/15497_Friday_May_10_2024_05_47_29_PM_20129129/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.3. Finding periodic solutions of linear differential equations. Exercises page 187
Problem number: 760.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 4y = \arcsin(\sin(x))}$$

25.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = \arcsin(\sin(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-4)^2 - (4)\,(1)\,(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x\, e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 x\, e^{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{2x}\\ y_2 &= x\, e^{2x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & x\,e^{2x} \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(x\,e^{2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})\left(e^{2x} + 2x\,e^{2x}\right) - \left(x\,e^{2x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,e^{2x} \arcsin(\sin(x))}{e^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \arcsin(\sin(x))\,x\,e^{-2x}dx$$

Hence

$$u_1 = -\left(\int_0^x \arcsin(\sin(\alpha))\, \alpha\, \mathrm{e}^{-2\alpha} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{2x} \arcsin(\sin(x))}{\mathrm{e}^{4x}} dx$$

Which simplifies to

$$u_2 = \int \arcsin(\sin(x))\, \mathrm{e}^{-2x} dx$$

Hence

$$u_2 = \int_0^x \arcsin(\sin(\alpha))\, \mathrm{e}^{-2\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \arcsin(\sin(\alpha))\, \alpha\, \mathrm{e}^{-2\alpha} d\alpha\right) \mathrm{e}^{2x} + \left(\int_0^x \arcsin(\sin(\alpha))\, \mathrm{e}^{-2\alpha} d\alpha\right) x\, \mathrm{e}^{2x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\, \alpha\, \mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\, \mathrm{e}^{-2\alpha} d\alpha\right) x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{2x} + c_2 x\, \mathrm{e}^{2x}) \\ &\quad + \left(\mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\, \alpha\, \mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\, \mathrm{e}^{-2\alpha} d\alpha\right) x\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y &= \mathrm{e}^{2x}(c_2 x + c_1) \\ &\quad + \mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\, \alpha\, \mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\, \mathrm{e}^{-2\alpha} d\alpha\right) x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\, e^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\, e^{-2\alpha} d\alpha\right) x\right) \tag{1}$$

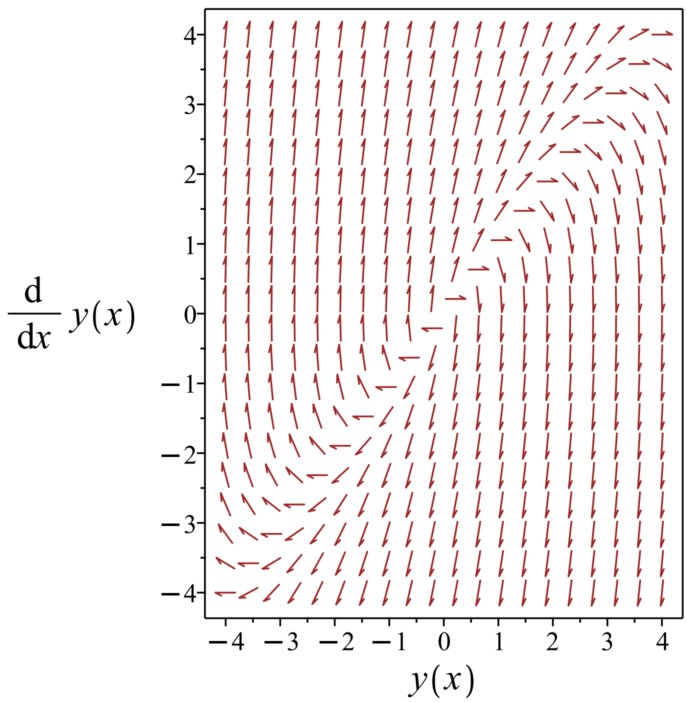


Figure 834: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + e^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\, e^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\, e^{-2\alpha} d\alpha\right) x\right)$$

Verified OK.

25.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4\, dx} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} \\ \left(\mathrm{e}^{-2x}y\right)'' &= \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-2x}y\right)' = \int \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} dx + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-2x}y\right) = \int \left(\int \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} dx + c_1\right) dx + c_2$$

Hence the solution is

$$y = \frac{\int \left(\int \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} dx + c_1\right) dx + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1 x\, \mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \mathrm{e}^{2x} \left(\int \int \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} dx dx\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \mathrm{e}^{2x} \left(\int \int \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} dx dx\right) \tag{1}$$

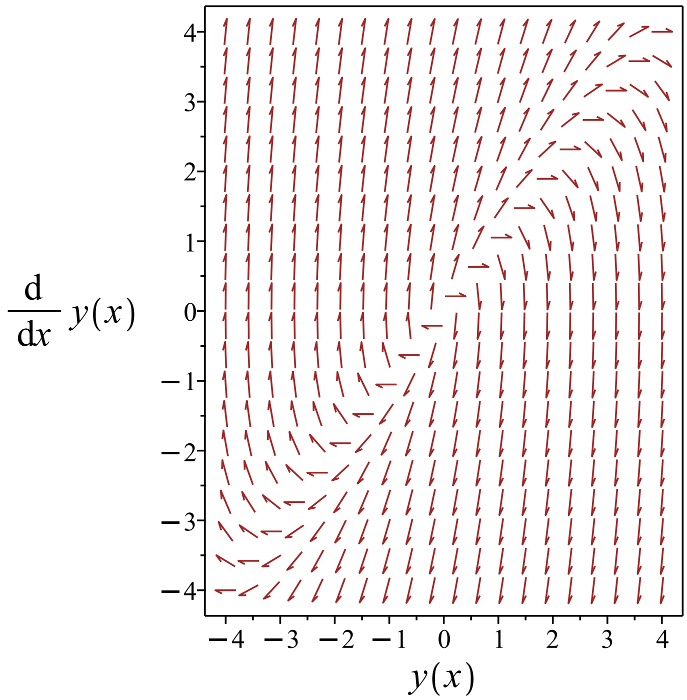


Figure 835: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{2x} + \mathrm{e}^{2x} c_2 + \mathrm{e}^{2x}\left(\int\int \arcsin\left(\sin\left(x\right)\right)\mathrm{e}^{-2x} dx dx\right)$$

Verified OK.

25.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 736: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1 \left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2x} + c_2 x \,\mathrm{e}^{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= x \,\mathrm{e}^{2x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & x\,e^{2x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(x\,e^{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(e^{2x} + 2x\,e^{2x}\right) - \left(x\,e^{2x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,e^{2x} \arcsin\left(\sin\left(x\right)\right)}{e^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \arcsin\left(\sin\left(x\right)\right) x\,e^{-2x} dx$$

Hence

$$u_1 = -\left(\int_0^x \arcsin\left(\sin\left(\alpha\right)\right) \alpha\,e^{-2\alpha} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \arcsin\left(\sin\left(x\right)\right)}{e^{4x}}\,dx$$

Which simplifies to

$$u_2 = \int \arcsin(\sin(x))\,\mathrm{e}^{-2x} dx$$

Hence

$$u_2 = \int_0^x \arcsin(\sin(\alpha))\,\mathrm{e}^{-2\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\,\mathrm{e}^{-2\alpha} d\alpha\right)\mathrm{e}^{2x} + \left(\int_0^x \arcsin(\sin(\alpha))\,\mathrm{e}^{-2\alpha} d\alpha\right) x\,\mathrm{e}^{2x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\,\mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\,\mathrm{e}^{-2\alpha} d\alpha\right) x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + c_2 x\,\mathrm{e}^{2x}\right) \\ &\quad + \left(\mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\,\mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\,\mathrm{e}^{-2\alpha} d\alpha\right) x\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y &= \mathrm{e}^{2x}(c_2 x + c_1) \\ &\quad + \mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\,\mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\,\mathrm{e}^{-2\alpha} d\alpha\right) x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= \mathrm{e}^{2x}(c_2 x + c_1) \\ &\quad + \mathrm{e}^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\,\mathrm{e}^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\,\mathrm{e}^{-2\alpha} d\alpha\right) x\right) \end{aligned} \tag{1}$$

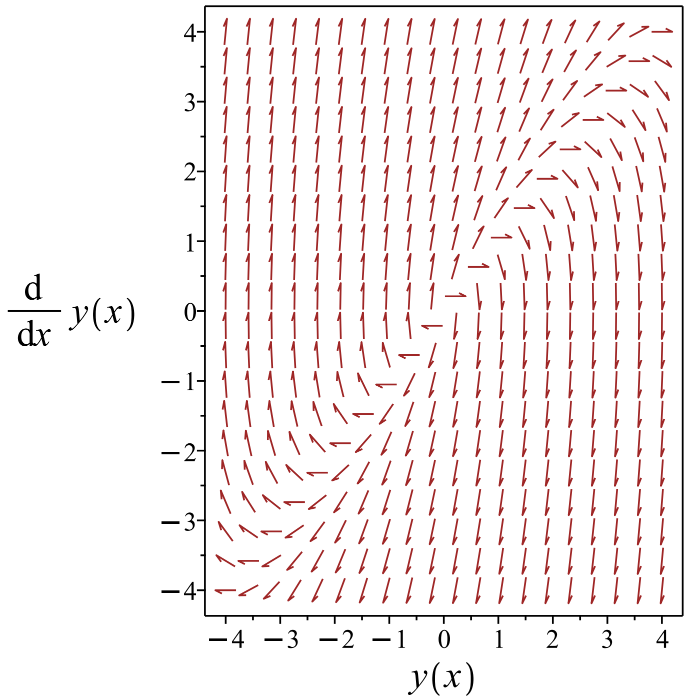


Figure 836: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + e^{2x}\left(-\left(\int_0^x \arcsin(\sin(\alpha))\,\alpha\, e^{-2\alpha} d\alpha\right) + \left(\int_0^x \arcsin(\sin(\alpha))\, e^{-2\alpha} d\alpha\right) x\right)$$

Verified OK.

25.4.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = \arcsin(\sin(x))$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 x\,e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \arcsin(\sin(x))\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{2x}\left(-\left(\int \arcsin(\sin(x))\,x\,e^{-2x} dx\right) + x\left(\int \arcsin(\sin(x))\,e^{-2x} dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = e^{2x}\left(-\left(\int \arcsin(\sin(x))\,x\,e^{-2x} dx\right) + \left(\int \arcsin(\sin(x))\,e^{-2x} dx\right) x\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + c_2 x\,e^{2x} + e^{2x}\left(-\left(\int \arcsin(\sin(x))\,x\,e^{-2x} dx\right) + \left(\int \arcsin(\sin(x))\,e^{-2x} dx\right) x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=arcsin(sin(x)),y(x), singsol=all)
```

$$y = \mathrm{e}^{2x}\left(c_2 + c_1 x - \left(\int \arcsin\left(\sin\left(x\right)\right) x\,\mathrm{e}^{-2x} dx\right) + x\left(\int \arcsin\left(\sin\left(x\right)\right) \mathrm{e}^{-2x} dx\right)\right)$$

✓ Solution by Mathematica

Time used: 1.363 (sec). Leaf size: 38

```
DSolve[y''[x]-4*y'[x]+4*y[x]==ArcSin[Sin[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(\arcsin(\sin(x)) + 4e^{2x}(c_2 x + c_1) + \sqrt{\cos^2(x)}\sec(x)\right)$$

25.5 problem 761

Internal problem ID [15498]
Internal file name [`OUTPUT/15498_Friday_May_10_2024_05_47_29_PM_78487314/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 2 (Higher order ODE's). Section 18.3. Finding periodic solutions of linear differential equations. Exercises page 187
Problem number: 761.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \sin(x)^3}$$

25.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \sin(x)^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x), \sin(x)\}, \{x\cos(3x), x\sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 x \cos(3x) + A_4 x \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) - 6A_3 \sin(3x) + 6A_4 \cos(3x) = \sin(x)^3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{3}{32}, A_3 = \frac{1}{24}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24} \tag{1}$$

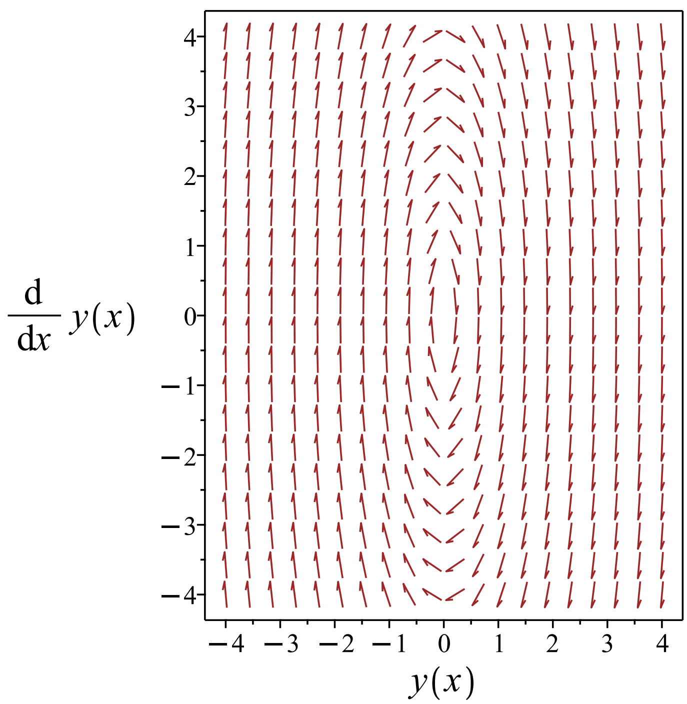


Figure 837: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24}$$

Verified OK.

25.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 738: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2} \, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2 \left(\cos(3x) \left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x), \sin(x)\}, \{x\cos(3x), x\sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 x \cos(3x) + A_4 x \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(x) + 8A_2 \sin(x) - 6A_3 \sin(3x) + 6A_4 \cos(3x) = \sin(x)^3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{3}{32}, A_3 = \frac{1}{24}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24} \tag{1}
$$

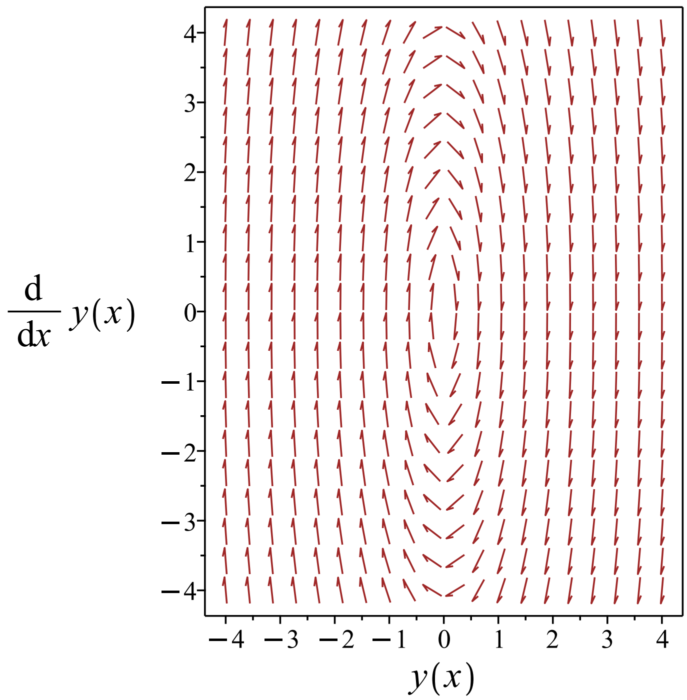


Figure 838: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{3\sin(x)}{32} + \frac{x\cos(3x)}{24}
$$

Verified OK.

25.5.3 Maple step by step solution

Let's solve

$y'' + 9y = \sin(x)^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^3\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ∘ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)\sin(x)^3 dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)\sin(x)^3 dx\right)}{3}$$

- Compute integrals

$$y_p(x) = -\frac{\cos(x)^2 \sin(x)}{36} + \frac{29\sin(x)}{288} + \frac{\cos(x)^3 x}{6} - \frac{\cos(x)x}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{\cos(x)^2 \sin(x)}{36} + \frac{29\sin(x)}{288} + \frac{\cos(x)^3 x}{6} - \frac{\cos(x)x}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+9*y(x)=sin(x)^3,y(x), singsol=all)
```

$$y = \frac{(x + 24c_1)\cos(3x)}{24} + \frac{(144c_2 - 1)\sin(3x)}{144} + \frac{3\sin(x)}{32}$$

✓ Solution by Mathematica

Time used: 0.205 (sec). Leaf size: 40

```
DSolve[y''[x]+9*y[x]==Sin[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3\sin(x)}{32} - \frac{1}{144}\sin(3x) + \left(\frac{x}{24} + c_1\right)\cos(3x) + c_2\sin(3x)$$

26 Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199

26.1 problem 767

Internal problem ID [15499]
Internal file name [`OUTPUT/15499_Friday_May_10_2024_05_47_29_PM_12105076/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 767.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x_1'(t) = -2tx_1(t)^2$$
$$x_2'(t) = \frac{x_2(t)}{t} + 1$$

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(x__1(t),t)=-2*t*x__1(t)^2,diff(x__2(t),t)=(x__2(t)+t)/t],singsol=all)
```

$$\left\{x_1(t) = \frac{1}{t^2 + c_2}\right\}$$
$$\{x_2(t) = (\ln(t) + c_1)\, t\}$$

✓ Solution by Mathematica
Time used: 0.143 (sec). Leaf size: 40

```
DSolve[{x1'[t]==-2*t*x1[t]^2,x2'[t]==(x2[t]+t)/t},{x1[t],x2[t]},t,IncludeSingularSolutions -
```

$$\text{x1}(t) \to \frac{1}{t^2 - c_1}$$
$$\text{x2}(t) \to t(\log(t) + c_2)$$
$$\text{x1}(t) \to 0$$
$$\text{x2}(t) \to t(\log(t) + c_2)$$

26.2 problem 768

Internal problem ID [15500]
Internal file name [`OUTPUT/15500_Friday_May_10_2024_05_47_29_PM_99659885/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 768.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x_1'(t) = e^t e^{-x_1(t)}$$
$$x_2'(t) = 2\, e^{x_1(t)}$$

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 24

```
dsolve([diff(x__1(t),t)=exp(t-x__1(t)),diff(x__2(t),t)=2*exp(x__1(t))],singsol=all)
```

$$\{x_1(t) = \ln\left(e^t + c_2\right)\}$$
$$\{x_2(t) = \int 2\, e^{x_1(t)} dt + c_1\}$$

✓ Solution by Mathematica
Time used: 0.15 (sec). Leaf size: 28

```
DSolve[{x1'[t]==Exp[t-x1[t]],x2'[t]==2*Exp[x1[t]]},{x1[t],x2[t]},t,IncludeSingularSolutions
```

$$\text{x1}(t) \to \log\left(e^t + c_1\right)$$
$$\text{x2}(t) \to 2e^t + 2c_1 t + c_2$$

26.3 problem 769

Internal problem ID [15501]
Internal file name [`OUTPUT/15501_Friday_May_10_2024_05_47_30_PM_27323432/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 769.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = y(t)$$
$$y'(t) = \frac{y(t)^2}{x(t)}$$

Does not currently support non linear system of equations. This is the phase plot of the system.

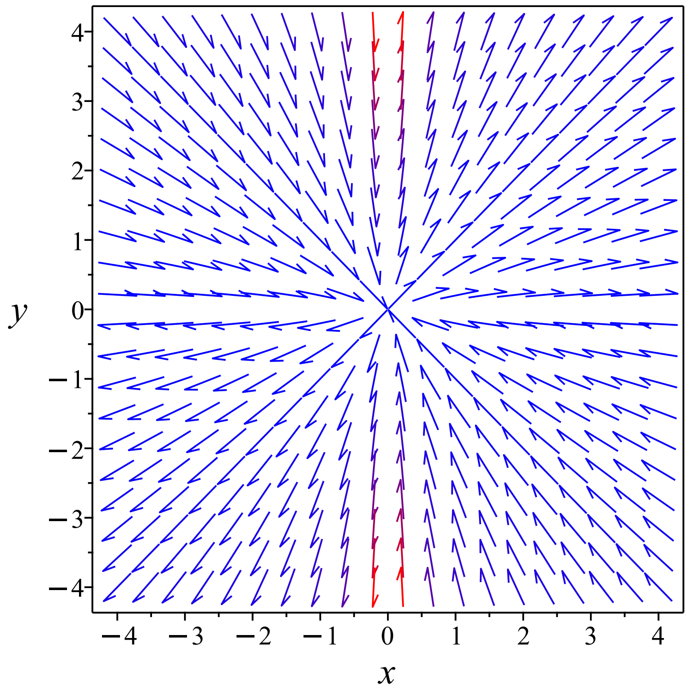


Figure 839: Phase plot

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)=y(t),diff(y(t),t)=y(t)^2/x(t)],singsol=all)
```

$$\{x(t) = e^{c_1 t} c_2\}$$
$$\{y(t) = \frac{d}{dt}x(t)\}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 28

```
DSolve[{x'[t]==y[t],y'[t]==y[t]^2/x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 c_2 e^{c_1 t}$$
$$x(t) \to c_2 e^{c_1 t}$$

26.4 problem 771

Internal problem ID [15502]
Internal file name [`OUTPUT/15502_Friday_May_10_2024_05_47_30_PM_81436689/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 771.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
x_1'(t) &= \frac{x_1(t)^2}{x_2(t)} \\
x_2'(t) &= x_2(t) - x_1(t)
\end{aligned}
$$

Does not currently support non linear system of equations. This is the phase plot of the system.

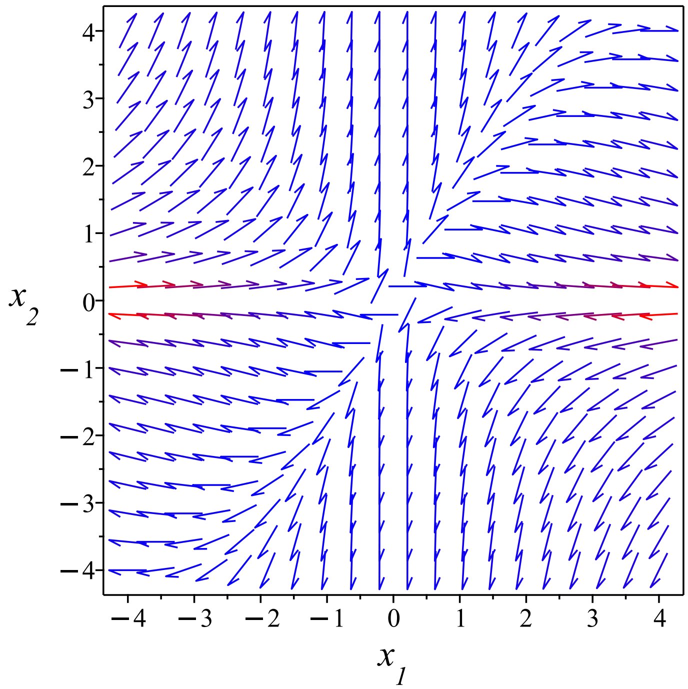


Figure 840: Phase plot

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 66

```
dsolve([diff(x__1(t),t)=x__1(t)^2/x__2(t),diff(x__2(t),t)=x__2(t)-x__1(t)],singsol=all)
```

$$[\{x_1(t) = 0\}, \{x_2(t) = c_1 e^t\}]$$

$$\left[\left\{x_1(t) = \frac{1}{\sqrt{2\,e^{-t}c_1 - 2c_2}}, x_1(t) = -\frac{1}{\sqrt{2\,e^{-t}c_1 - 2c_2}}\right\}, \left\{x_2(t) = \frac{x_1(t)^2}{\frac{d}{dt}x_1(t)}\right\}\right]$$

✓ Solution by Mathematica

Time used: 0.102 (sec). Leaf size: 143

```
DSolve[{x1'[t]==x1[t]^2/x2[t],x2'[t]==x2[t]-x1[t]},{x1[t],x2[t]},t,IncludeSingularSolutions
```

$$
\begin{aligned}
\mathrm{x2}(t) &\to 2ie^{\frac{t}{2}+c_2}\sqrt{-1+2c_1e^{t+2c_2}} \\
\mathrm{x1}(t) &\to -\frac{ie^{\frac{t}{2}+c_2}}{\sqrt{-1+2c_1e^{t+2c_2}}} \\
\mathrm{x2}(t) &\to -2ie^{\frac{t}{2}+c_2}\sqrt{-1+2c_1e^{t+2c_2}} \\
\mathrm{x1}(t) &\to \frac{ie^{\frac{t}{2}+c_2}}{\sqrt{-1+2c_1e^{t+2c_2}}}
\end{aligned}
$$

26.5 problem 772

Internal problem ID [15503]
Internal file name [`OUTPUT/15503_Friday_May_10_2024_05_47_30_PM_79597876/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 772.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = \frac{e^{-x(t)}}{t}$$
$$y'(t) = \frac{x(t)\,e^{-y(t)}}{t}$$

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 26

```
dsolve([diff(x(t),t)=exp(-x(t))/t,diff(y(t),t)=x(t)/t*exp(-y(t))],singsol=all)
```

$$\{x(t) = \ln(\ln(t) + c_2)\}$$
$$\left\{y(t) = \ln\left(\int \frac{x(t)}{t}dt + c_1\right)\right\}$$

✓ Solution by Mathematica
Time used: 0.107 (sec). Leaf size: 41

```
DSolve[{x'[t]==Exp[-x[t]],y'[t]==x[t]/t*Exp[-y[t]]},{x[t],y[t]},t,IncludeSingularSolutions -
```

$$x(t) \to \log(t + c_1)$$
$$y(t) \to \log\left(\text{PolyLog}\left(2, \frac{t}{c_1} + 1\right) + \log\left(-\frac{t}{c_1}\right)\log(t + c_1) + c_2\right)$$

26.6 problem 773

Internal problem ID [15504]
Internal file name [`OUTPUT/15504_Friday_May_10_2024_05_47_30_PM_95537058/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 773.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = \frac{y(t)}{x(t) + y(t)} + \frac{t}{x(t) + y(t)}$$
$$y'(t) = \frac{x(t)}{x(t) + y(t)} - \frac{t}{x(t) + y(t)}$$

✓ Solution by Maple
Time used: 0.062 (sec). Leaf size: 61

```
dsolve([diff(x(t),t)=(y(t)+t)/(x(t)+y(t)),diff(y(t),t)=(x(t)-t)/(x(t)+y(t))],singsol=all)
```

$$[\{x(t) = t\}, \{y(t) = c_1\}]$$
$$\left[\left\{x(t) = \frac{c_1 t^2 - c_2 t + 1}{c_1 t - c_2}\right\}, \left\{y(t) = \frac{-x(t)\left(\frac{d}{dt}x(t)\right) + t}{\frac{d}{dt}x(t) - 1}\right\}\right]$$

✓ Solution by Mathematica
Time used: 67.434 (sec). Leaf size: 45

```
DSolve[{x'[t]==(y[t]+t)/(x[t]+y[t]),y'[t]==(x[t]-t)/(x[t]+y[t])},{x[t],y[t]},t,IncludeSingul
```

$$x(t) \to \frac{t^2 + c_1 t + c_2}{t + c_1}$$
$$y(t) \to \frac{c_1 t + c_1{}^2 - c_2}{t + c_1}$$

26.7 problem 774

Internal problem ID [15505]
Internal file name [`OUTPUT/15505_Friday_May_10_2024_05_47_30_PM_48338425/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 774.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
x'(t) &= \frac{t}{y(t) - x(t)} - \frac{y(t)}{y(t) - x(t)} \\
y'(t) &= \frac{x(t)}{y(t) - x(t)} - \frac{t}{y(t) - x(t)}
\end{aligned}
$$

✓ Solution by Maple
Time used: 0.422 (sec). Leaf size: 132

```
dsolve([diff(x(t),t)=(t-y(t))/(y(t)-x(t)),diff(y(t),t)=(x(t)-t)/(y(t)-x(t))],singsol=all)
```

$$
\left\{x(t) = t + \mathrm{RootOf}\left(-t + \int^{\_Z} -\frac{2\left(e^{c_1}\_f^2 - 1\right)}{-4 + 3\,e^{c_1}\_f^2 - \sqrt{-3\,e^{c_1}\_f^2 + 4}\,e^{\frac{c_1}{2}}\_f} d\_f + c_2\right), x(t) = t + \mathrm{RootOf}\left(-t + \int^{\_Z} -\frac{2\left(e^{c_1}\_f^2 - 1\right)}{3\,e^{c_1}\_f^2 + \sqrt{-3\,e^{c_1}\_f^2 + 4}\,e^{\frac{c_1}{2}}\_f - 4} d\_f + c_2\right)\right\}
$$

$$
\left\{y(t) = \frac{x(t)\left(\frac{d}{dt}x(t)\right) + t}{\frac{d}{dt}x(t) + 1}\right\}
$$

✓ Solution by Mathematica

Time used: 14.351 (sec). Leaf size: 151

```
DSolve[{x'[t]==(t-y[t])/(y[t]-x[t]),y'[t]==(x[t]-t)/(y[t]-x[t])},{x[t],y[t]},t,IncludeSingul
```

$$x(t) \to \frac{1}{2}\left(-\sqrt{-3t^2 + 2c_1t + c_1{}^2 + 4c_2} - t + c_1\right)$$
$$y(t) \to \frac{1}{2}\left(\sqrt{-3t^2 + 2c_1t + c_1{}^2 + 4c_2} - t + c_1\right)$$
$$x(t) \to \frac{1}{2}\left(\sqrt{-3t^2 + 2c_1t + c_1{}^2 + 4c_2} - t + c_1\right)$$
$$y(t) \to \frac{1}{2}\left(-\sqrt{-3t^2 + 2c_1t + c_1{}^2 + 4c_2} - t + c_1\right)$$

26.8 problem 775

Internal problem ID [15506]
Internal file name [`OUTPUT/15506_Friday_May_10_2024_05_47_31_PM_7548041/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 19. Basic concepts and definitions. Exercises page 199
Problem number: 775.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete tl

Solve

$$x'(t) = \frac{y(t)}{x(t) + y(t)} + \frac{t}{x(t) + y(t)}$$
$$y'(t) = \frac{t}{x(t) + y(t)} + \frac{x(t)}{x(t) + y(t)}$$

✓ Solution by Maple
Time used: 1.656 (sec). Leaf size: 3853

```
dsolve([diff(x(t),t)=(t+y(t))/(y(t)+x(t)),diff(y(t),t)=(t+x(t))/(y(t)+x(t))],singsol=all)
```

Expression too large to display
Expression too large to display

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x'[t]==(t+y[t])/(y[t]+x[t]),y'[t]==(x[t]+t)/(y[t]+x[t])},{x[t],y[t]},t,IncludeSingul
```

Not solved

27 Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212

27.1 problem 776

Internal problem ID [15507]
Internal file name [`OUTPUT/15507_Friday_May_10_2024_05_47_31_PM_97786729/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 776.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -9y(t) \\
y'(t) &= x(t)
\end{aligned}
$$

27.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos(3t) & -3\sin(3t) \\ \frac{\sin(3t)}{3} & \cos(3t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(3t) & -3\sin(3t) \\ \frac{\sin(3t)}{3} & \cos(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(3t)\, c_1 - 3\sin(3t)\, c_2 \\ \frac{\sin(3t)c_1}{3} + \cos(3t)\, c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

27.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -9 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3i$	1	complex eigenvalue
$-3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3i & -9 \\ 1 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3i & -9 & 0 \\ 1 & 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{iR_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3i & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & -9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3it\}$

Hence the solution is

$$\begin{bmatrix} -3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -3it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} -3i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} -3i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & -9 \\ 1 & -3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3i & -9 & 0 \\ 1 & -3i & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{iR_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3i & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & -9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3it\}$

Hence the solution is

$$\begin{bmatrix} 3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 3it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3\,\mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 3i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3\,\mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 3i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3i$	1	1	No	$\begin{bmatrix} 3i \\ 1 \end{bmatrix}$
$-3i$	1	1	No	$\begin{bmatrix} -3i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 3i\mathrm{e}^{3it} \\ \mathrm{e}^{3it} \end{bmatrix} + c_2 \begin{bmatrix} -3i\mathrm{e}^{-3it} \\ \mathrm{e}^{-3it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3i(c_1 e^{3it} - c_2 e^{-3it}) \\ c_1 e^{3it} + c_2 e^{-3it} \end{bmatrix}$$

The following is the phase plot of the system.

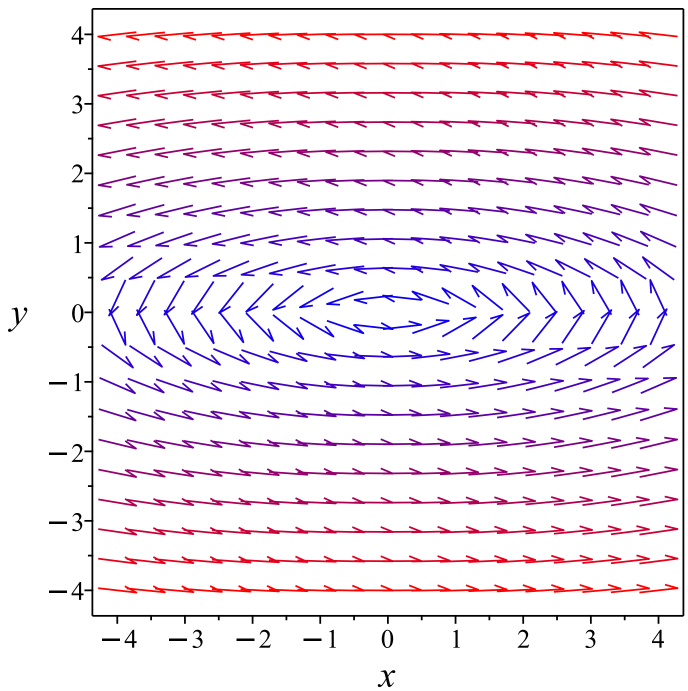


Figure 841: Phase plot

27.1.3 Maple step by step solution

Let's solve

$[x'(t) = -9y(t), y'(t) = x(t)]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -3\,\mathrm{I} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} 3\,\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -3\,\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I}t} \cdot \begin{bmatrix} -3\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} -3\,\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -3\,\mathrm{I}(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} -3\sin(3t) \\ \cos(3t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} -3\cos(3t) \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$
- Substitute solutions into the general solution

  $$\vec{x} = \begin{bmatrix} -3c_2\cos(3t) - 3c_1\sin(3t) \\ -c_2\sin(3t) + c_1\cos(3t) \end{bmatrix}$$
- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -3c_2\cos(3t) - 3c_1\sin(3t) \\ -c_2\sin(3t) + c_1\cos(3t) \end{bmatrix}$$
- Solution to the system of ODEs

  $\{x(t) = -3c_2\cos(3t) - 3c_1\sin(3t), y(t) = -c_2\sin(3t) + c_1\cos(3t)\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=-9*y(t),diff(y(t),t)=x(t)],singsol=all)
```

$$x(t) = c_1\sin(3t) + c_2\cos(3t)$$
$$y(t) = -\frac{c_1\cos(3t)}{3} + \frac{c_2\sin(3t)}{3}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 42

```
DSolve[{x'[t]==-9*y[t],y'[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1\cos(3t) - 3c_2\sin(3t)$$
$$y(t) \to c_2\cos(3t) + \frac{1}{3}c_1\sin(3t)$$

27.2 problem 777

Internal problem ID [15508]
Internal file name [`OUTPUT/15508_Friday_May_10_2024_05_47_31_PM_21740883/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 777.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y(t) + t \\
y'(t) &= x(t) - t
\end{aligned}
$$

27.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t \\ -t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-t}}{2} + \frac{e^t}{2}\right) c_1 + \left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) c_2 \\ \left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) c_1 + \left(\frac{e^{-t}}{2} + \frac{e^t}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1 - c_2)e^{-t}}{2} + \frac{e^t(c_2 + c_1)}{2} \\ \frac{(-c_1 + c_2)e^{-t}}{2} + \frac{e^t(c_2 + c_1)}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & -\frac{e^t}{2} + \frac{e^{-t}}{2} \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & -\frac{e^t}{2} + \frac{e^{-t}}{2} \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix} \begin{bmatrix} (t-1)\, e^t \\ -(t-1)\, e^t \end{bmatrix} \\
&= \begin{bmatrix} t - 1 \\ -t + 1 \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(c_1-c_2)\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{t}(c_2+c_1)}{2} + t - 1 \\ \frac{(-c_1+c_2)\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{t}(c_2+c_1)}{2} - t + 1 \end{bmatrix}$$

27.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t \\ -t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{e^{-t}}{2} & \frac{e^{-t}}{2} \\ -\frac{e^t}{2} & \frac{e^t}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-t}}{2} & \frac{e^{-t}}{2} \\ -\frac{e^t}{2} & \frac{e^t}{2} \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} 0 \\ -t\, e^t \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ -(t-1)\, e^t \end{bmatrix}$$

$$= \begin{bmatrix} t-1 \\ -t+1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_1 e^t \end{bmatrix} + \begin{bmatrix} -c_2 e^{-t} \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} t-1 \\ -t+1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t - c_2 e^{-t} + t - 1 \\ c_1 e^t + c_2 e^{-t} - t + 1 \end{bmatrix}$$

27.2.3 Maple step by step solution

Let's solve

$[x'(t) = y(t) + t, y'(t) = x(t) - t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t \\ -t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t \\ -t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} t \\ -t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} -e^{-t} & e^t \\ e^{-t} & e^t \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} -e^{-t} & e^t \\ e^{-t} & e^t \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & \frac{e^{-t}}{2} + \frac{e^t}{2} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \mathrm{e}^{-t} + t - 1 \\ -\mathrm{e}^{-t} - t + 1 \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} \mathrm{e}^{-t} + t - 1 \\ -\mathrm{e}^{-t} - t + 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + \mathrm{e}^{-t} + t - 1 \\ (c_1 - 1)\,\mathrm{e}^{-t} + c_2 \mathrm{e}^{t} - t + 1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} + \mathrm{e}^{-t} + t - 1, y(t) = (c_1 - 1)\,\mathrm{e}^{-t} + c_2 \mathrm{e}^{t} - t + 1\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)=y(t)+t,diff(y(t),t)=x(t)-t],singsol=all)
```

$$x(t) = c_2 \mathrm{e}^{t} + \mathrm{e}^{-t} c_1 + t - 1$$
$$y(t) = c_2 \mathrm{e}^{t} - \mathrm{e}^{-t} c_1 + 1 - t$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 78

```
DSolve[{x'[t]==y[t]+t,y'[t]==x[t]-t},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^{-t}\left(2e^{t}(t-1)+(c_1+c_2)e^{2t}+c_1-c_2\right)$$
$$y(t) \to \frac{1}{2}e^{-t}\left(-2e^{t}(t-1)+(c_1+c_2)e^{2t}-c_1+c_2\right)$$

27.3 problem 778

Internal problem ID [15509]
Internal file name [`OUTPUT/15509_Friday_May_10_2024_05_47_31_PM_76338301/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 778.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -3x(t) - 4y(t)$$
$$y'(t) = -2x(t) - 5y(t)$$

With initial conditions

$$[x(0) = 1, y(0) = 4]$$

27.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-7t}}{3} + \frac{2\,e^{-t}}{3} & -\frac{2\,e^{-t}}{3} + \frac{2\,e^{-7t}}{3} \\ -\frac{e^{-t}}{3} + \frac{e^{-7t}}{3} & \frac{2\,e^{-7t}}{3} + \frac{e^{-t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{e^{-7t}}{3} + \frac{2\,e^{-t}}{3} & -\frac{2\,e^{-t}}{3} + \frac{2\,e^{-7t}}{3} \\ -\frac{e^{-t}}{3} + \frac{e^{-7t}}{3} & \frac{2\,e^{-7t}}{3} + \frac{e^{-t}}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 3\,e^{-7t} - 2\,e^{-t} \\ e^{-t} + 3\,e^{-7t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

27.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -3 & -4 \\ -2 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -3-\lambda & -4 \\ -2 & -5-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + 8\lambda + 7 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -7$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -4 \\ -2 & -5 \end{bmatrix} - (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & -4 \\ -2 & -5 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -4 & 0 \\ -2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-7	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -7 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-7t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-7t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-7t} \\ e^{-7t} \end{bmatrix} + c_2 \begin{bmatrix} -2\,e^{-t} \\ e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-7t} - 2c_2 e^{-t} \\ c_1 e^{-7t} + c_2 e^{-t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 4 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 3 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3\,e^{-7t} - 2\,e^{-t} \\ e^{-t} + 3\,e^{-7t} \end{bmatrix}$$

The following is the phase plot of the system.

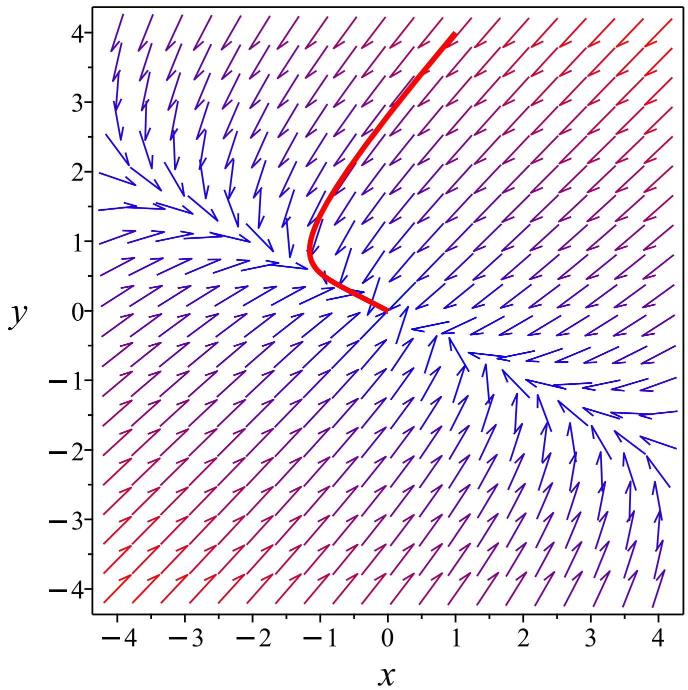
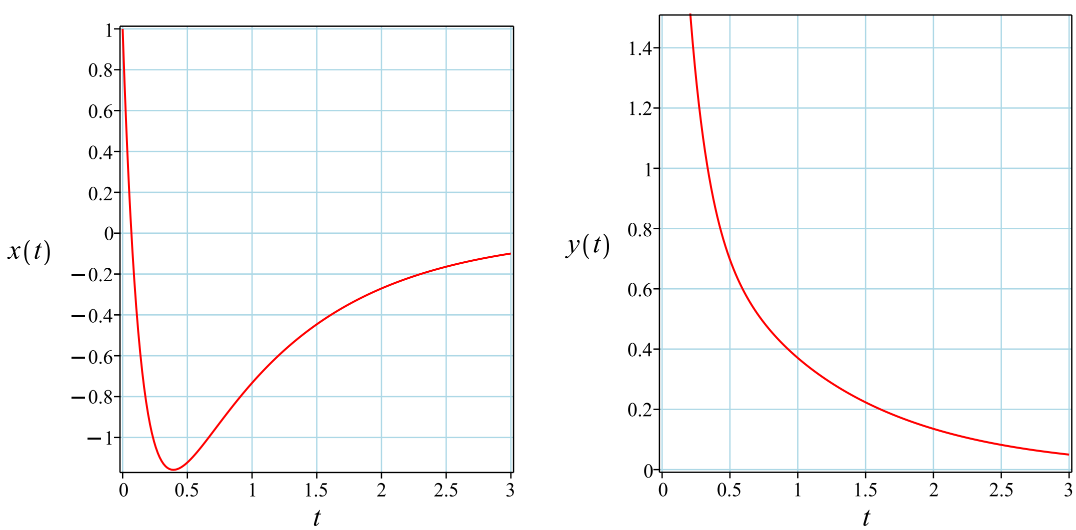


Figure 842: Phase plot

The following are plots of each solution.



✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)+3*x(t)+4*y(t) = 0, diff(y(t),t)+2*x(t)+5*y(t) = 0, x(0) = 1, y(0) = 4],
```

$$x(t) = 3\,\mathrm{e}^{-7t} - 2\,\mathrm{e}^{-t}$$
$$y(t) = 3\,\mathrm{e}^{-7t} + \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 36

```
DSolve[{x'[t]+3*x[t]+4*y[t]==0,y'[t]+2*x[t]+5*y[t]==0},{x[0]==1,y[0]==4},{x[t],y[t]},t,Inclu
```

$$x(t) \to e^{-7t}\left(3 - 2e^{6t}\right)$$
$$y(t) \to e^{-7t}\left(e^{6t} + 3\right)$$

27.4 problem 779

Internal problem ID [15510]
Internal file name [`OUTPUT/15510_Friday_May_10_2024_05_47_31_PM_5441983/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 779.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + 5y(t) \\
y'(t) &= -x(t) - 3y(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = -2, y(0) = 1]
$$

27.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(t) + 2\,e^{-t}\sin(t) & 5\,e^{-t}\sin(t) \\ -e^{-t}\sin(t) & e^{-t}\cos(t) - 2\,e^{-t}\sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & 5\,e^{-t}\sin(t) \\ -e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{-t}(\cos(t) + 2\sin(t)) & 5\,e^{-t}\sin(t) \\ -e^{-t}\sin(t) & e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2\,e^{-t}(\cos(t) + 2\sin(t)) + 5\,e^{-t}\sin(t) \\ 2\,e^{-t}\sin(t) + e^{-t}(\cos(t) - 2\sin(t)) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(-2\cos(t) + \sin(t)) \\ e^{-t}\cos(t) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

27.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 5 \\ -1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1+i$$
$$\lambda_2 = -1-i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1+i$	1	complex eigenvalue
$-1-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1-i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix} - (-1-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & 5 \\ -1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & 5 & 0 \\ -1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{2}{5} - \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-2+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-2+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -2+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -2+i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1+i$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix} - (-1+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & 5 \\ -1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & 5 & 0 \\ -1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-2-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+i$	1	1	No	$\begin{bmatrix} -2-i \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} -2+i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} (-2-i)\,\mathrm{e}^{(-1+i)t} \\ \mathrm{e}^{(-1+i)t} \end{bmatrix} + c_2 \begin{bmatrix} (-2+i)\,\mathrm{e}^{(-1-i)t} \\ \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (-2-i)\,c_1 \mathrm{e}^{(-1+i)t} + (-2+i)\,c_2 \mathrm{e}^{(-1-i)t} \\ c_1 \mathrm{e}^{(-1+i)t} + c_2 \mathrm{e}^{(-1-i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = -2 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-2-i)\,c_1 + (-2+i)\,c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-1 - \frac{i}{2}\right) \mathrm{e}^{(-1+i)t} + \left(-1 + \frac{i}{2}\right) \mathrm{e}^{(-1-i)t} \\ \frac{\mathrm{e}^{(-1+i)t}}{2} + \frac{\mathrm{e}^{(-1-i)t}}{2} \end{bmatrix}$$

The following is the phase plot of the system.

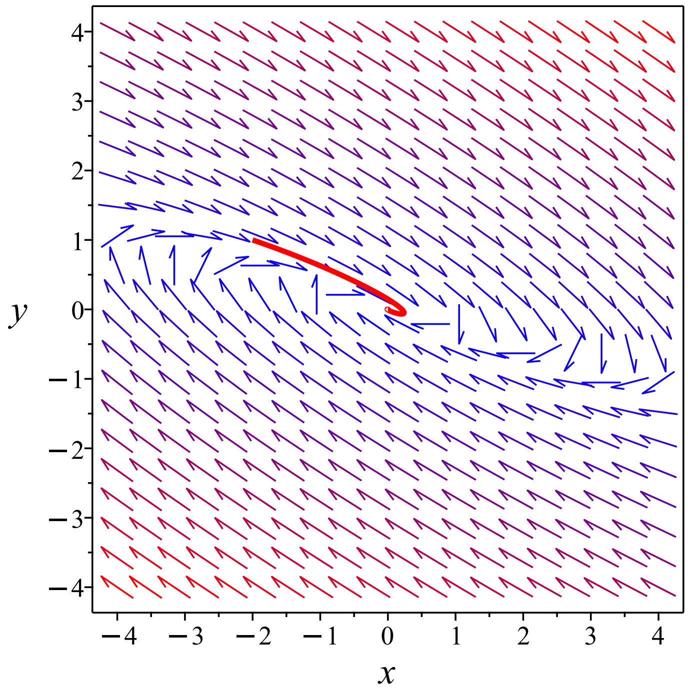


Figure 843: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve([diff(x(t),t) = x(t)+5*y(t), diff(y(t),t) = -x(t)-3*y(t), x(0) = -2, y(0) = 1], sings
```

$$x(t) = e^{-t}(\sin(t) - 2\cos(t))$$
$$y(t) = e^{-t}\cos(t)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 20

```
DSolve[{x'[t]+3*x[t]+4*y[t]==0,y'[t]+2*x[t]+5*y[t]==0},{x[0]==-2,y[0]==1},{x[t],y[t]},t,Incl
```

$$x(t) \to -2e^{-t}$$
$$y(t) \to e^{-t}$$

27.5 problem 780

Internal problem ID [15511]
Internal file name [`OUTPUT/15511_Friday_May_10_2024_05_47_31_PM_37563748/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 780.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -y(t) + \cos(t) \\
y'(t) &= -4y(t) + 4\cos(t) + 3x(t) - \sin(t)
\end{aligned}
$$

27.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 4\cos(t) - \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-3t}}{2}+\frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2}+\frac{e^{-3t}}{2} \\ \frac{3\,e^{-t}}{2}-\frac{3\,e^{-3t}}{2} & \frac{3\,e^{-3t}}{2}-\frac{e^{-t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-3t}}{2}+\frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2}+\frac{e^{-3t}}{2} \\ \frac{3\,e^{-t}}{2}-\frac{3\,e^{-3t}}{2} & \frac{3\,e^{-3t}}{2}-\frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-3t}}{2}+\frac{3\,e^{-t}}{2}\right)c_1+\left(-\frac{e^{-t}}{2}+\frac{e^{-3t}}{2}\right)c_2 \\ \left(\frac{3\,e^{-t}}{2}-\frac{3\,e^{-3t}}{2}\right)c_1+\left(\frac{3\,e^{-3t}}{2}-\frac{e^{-t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1+c_2)e^{-3t}}{2}+\frac{3\left(c_1-\frac{c_2}{3}\right)e^{-t}}{2} \\ \frac{(-3c_1+3c_2)e^{-3t}}{2}+\frac{3\left(c_1-\frac{c_2}{3}\right)e^{-t}}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} -\frac{(e^{2t}-3)e^{t}}{2} & \frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{3(e^{2t}-1)e^{t}}{2} & \frac{(3\,e^{2t}-1)e^{t}}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-3t}}{2}+\frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2}+\frac{e^{-3t}}{2} \\ \frac{3\,e^{-t}}{2}-\frac{3\,e^{-3t}}{2} & \frac{3\,e^{-3t}}{2}-\frac{e^{-t}}{2} \end{bmatrix} \int \begin{bmatrix} -\frac{(e^{2t}-3)e^{t}}{2} & \frac{(e^{2t}-1)e^{t}}{2} \\ -\frac{3(e^{2t}-1)e^{t}}{2} & \frac{(3\,e^{2t}-1)e^{t}}{2} \end{bmatrix} \begin{bmatrix} \cos(t) \\ 4\cos(t)-\sin(t) \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{e^{-3t}}{2}+\frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2}+\frac{e^{-3t}}{2} \\ \frac{3\,e^{-t}}{2}-\frac{3\,e^{-3t}}{2} & \frac{3\,e^{-3t}}{2}-\frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} \frac{(e^{2t}-1)e^{t}\cos(t)}{2} \\ \frac{\cos(t)(3\,e^{2t}-1)e^{t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(-c_1+c_2)e^{-3t}}{2} + \frac{3\left(c_1-\frac{c_2}{3}\right)e^{-t}}{2} \\ \frac{(-3c_1+3c_2)e^{-3t}}{2} + \frac{(3c_1-c_2)e^{-t}}{2} + \cos(t) \end{bmatrix}$$

27.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ 4\cos(t) - \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 3 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-3t} \\ &= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{\mathrm{e}^{-3t}}{3} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{-3t}}{3} \\ \mathrm{e}^{-t} & \mathrm{e}^{-3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{3\,\mathrm{e}^{t}}{2} & -\frac{\mathrm{e}^{t}}{2} \\ -\frac{3\,\mathrm{e}^{3t}}{2} & \frac{3\,\mathrm{e}^{3t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-t} & \frac{e^{-3t}}{3} \\ e^{-t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} \\ -\frac{3\,e^{3t}}{2} & \frac{3\,e^{3t}}{2} \end{bmatrix} \begin{bmatrix} \cos(t) \\ 4\cos(t) - \sin(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & \frac{e^{-3t}}{3} \\ e^{-t} & e^{-3t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{t}(\cos(t)-\sin(t))}{2} \\ \frac{3\,e^{3t}(3\cos(t)-\sin(t))}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & \frac{e^{-3t}}{3} \\ e^{-t} & e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{e^{t}\cos(t)}{2} \\ \frac{3\,e^{3t}\cos(t)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{-3t}}{3} \\ c_2 e^{-3t} \end{bmatrix} + \begin{bmatrix} 0 \\ \cos(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + \frac{c_2 e^{-3t}}{3} \\ c_1 e^{-t} + c_2 e^{-3t} + \cos(t) \end{bmatrix}$$

27.5.3 Maple step by step solution

Let's solve

$[x'(t) = -y(t) + \cos(t), y'(t) = -4y(t) + 4\cos(t) + 3x(t) - \sin(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \cos(t) \\ 4\cos(t) - \sin(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \cos(t) \\ 4\cos(t) - \sin(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \cos(t) \\ 4\cos(t) - \sin(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-3t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} \frac{e^{-3t}}{3} & e^{-t} \\ e^{-3t} & e^{-t} \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} \frac{e^{-3t}}{3} & e^{-t} \\ e^{-3t} & e^{-t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{bmatrix}}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} -\frac{e^{-3t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{e^{-3t}}{2} \\ \frac{3\,e^{-t}}{2} - \frac{3\,e^{-3t}}{2} & \frac{3\,e^{-3t}}{2} - \frac{e^{-t}}{2} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

    $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

  - Take the derivative of the particular solution

    $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

  - Cancel like terms

    $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

  - Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \left[\begin{array}{c} \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \cos(t) - \frac{3\,e^{-3t}}{2} + \frac{e^{-t}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \left[\begin{array}{c} \frac{e^{-t}}{2} - \frac{e^{-3t}}{2} \\ \cos(t) - \frac{3\,e^{-3t}}{2} + \frac{e^{-t}}{2} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{(2c_1-3)e^{-3t}}{6} + \frac{(6c_2+3)e^{-t}}{6} \\ \frac{(2c_1-3)e^{-3t}}{2} + \frac{(2c_2+1)e^{-t}}{2} + \cos(t) \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{(2c_1-3)e^{-3t}}{6} + \frac{(6c_2+3)e^{-t}}{6}, y(t) = \frac{(2c_1-3)e^{-3t}}{2} + \frac{(2c_2+1)e^{-t}}{2} + \cos(t)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([4*diff(x(t),t)-diff(y(t),t)+3*x(t)=sin(t),diff(x(t),t)+y(t)=cos(t)],singsol=all)
```

$$x(t) = \frac{c_2 e^{-3t}}{3} + e^{-t} c_1$$
$$y(t) = c_2 e^{-3t} + e^{-t} c_1 + \cos(t)$$

✓ Solution by Mathematica

Time used: 0.218 (sec). Leaf size: 76

```
DSolve[{4*x'[t]-y'[t]+3*x[t]==Sin[t],x'[t]+y[t]==Cos[t]},{x[t],y[t]},t,IncludeSingularSoluti
```

$$x(t) \to \frac{1}{2}e^{-3t}\left(c_1\left(3e^{2t}-1\right)-c_2\left(e^{2t}-1\right)\right)$$
$$y(t) \to \cos(t)+\frac{1}{2}e^{-3t}\left(3c_1\left(e^{2t}-1\right)-c_2\left(e^{2t}-3\right)\right)$$

27.6 problem 781

Internal problem ID [15512]
Internal file name [`OUTPUT/15512_Friday_May_10_2024_05_47_31_PM_21883334/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 781.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -y(t) + z(t) \\
y'(t) &= z(t) \\
z'(t) &= -x(t) + z(t)
\end{aligned}
$$

27.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & -\sin(t) & \sin(t) \\ -\frac{e^t}{2} + \frac{\cos(t)}{2} + \frac{\sin(t)}{2} & \frac{e^t}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{e^t}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ -\frac{e^t}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{e^t}{2} - \frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{e^t}{2} + \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) & -\sin(t) & \sin(t) \\ -\frac{e^t}{2} + \frac{\cos(t)}{2} + \frac{\sin(t)}{2} & \frac{e^t}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{e^t}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \\ -\frac{e^t}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{e^t}{2} - \frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{e^t}{2} + \frac{\cos(t)}{2} + \frac{\sin(t)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t)\,c_1 - \sin(t)\,c_2 + \sin(t)\,c_3 \\ \left(-\frac{e^t}{2} + \frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right)c_1 + \left(\frac{e^t}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2}\right)c_2 + \left(\frac{e^t}{2} - \frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right)c_3 \\ \left(-\frac{e^t}{2} + \frac{\cos(t)}{2} - \frac{\sin(t)}{2}\right)c_1 + \left(\frac{e^t}{2} - \frac{\cos(t)}{2} - \frac{\sin(t)}{2}\right)c_2 + \left(\frac{e^t}{2} + \frac{\cos(t)}{2} + \frac{\sin(t)}{2}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-c_2 + c_3)\sin(t) + \cos(t)\,c_1 \\ \frac{(c_2+c_1-c_3)\cos(t)}{2} + \frac{(c_2-c_1+c_3)e^t}{2} + \frac{\sin(t)(-c_2+c_1+c_3)}{2} \\ \frac{(-c_2+c_1+c_3)\cos(t)}{2} + \frac{(c_2-c_1+c_3)e^t}{2} - \frac{\sin(t)(c_2+c_1-c_3)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

27.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 & 1 \\ 0 & -\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & -1 & 1 \\ 0 & i & 1 \\ -1 & 0 & 1+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} i & -1 & 1 & 0 \\ 0 & i & 1 & 0 \\ -1 & 0 & 1+i & 0 \end{array}\right]$$

$$R_3 = -iR_1 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} i & -1 & 1 & 0 \\ 0 & i & 1 & 0 \\ 0 & i & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} i & -1 & 1 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & -1 & 1 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ \mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ \mathrm{I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ \mathrm{I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 & 1 \\ 0 & -i & 1 \\ -1 & 0 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -i & -1 & 1 & 0 \\ 0 & -i & 1 & 0 \\ -1 & 0 & 1-i & 0 \end{array}\right]$$

$$R_3 = iR_1 + R_3 \Longrightarrow \left[\begin{array}{ccc|c} -i & -1 & 1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & -i & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -i & -1 & 1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & -1 & 1 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-i)\,t, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-i)\,t \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\begin{bmatrix} 1-i \\ -i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 1+i \\ i \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} (1-i)\,\mathrm{e}^{it} \\ -i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} (1+i)\,\mathrm{e}^{-it} \\ i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} (1-i)\, c_1 \mathrm{e}^{it} + (1+i)\, c_2 \mathrm{e}^{-it} \\ -i c_1 \mathrm{e}^{it} + i c_2 \mathrm{e}^{-it} + c_3 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} + c_3 \mathrm{e}^{t} \end{bmatrix}$$

27.6.3 Maple step by step solution

Let's solve

$[x'(t) = -y(t) + z(t)\,, y'(t) = z(t)\,, z'(t) = -x(t) + z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} 1+\mathrm{I} \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 1-\mathrm{I} \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^t \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 1+\mathrm{I} \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} 1+\mathrm{I} \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 1+\mathrm{I} \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} (1+\mathrm{I})(\cos(t) - \mathrm{I}\sin(t)) \\ \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$
- Substitute solutions into the general solution
$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_3(\cos(t) - \sin(t)) + c_2(\cos(t) + \sin(t)) \\ c_2\sin(t) + c_3\cos(t) \\ c_2\cos(t) - c_3\sin(t) \end{bmatrix}$$
- Substitute in vector of dependent variables
$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos(t)(c_2 + c_3) + (c_2 - c_3)\sin(t) \\ c_1 e^t + c_3\cos(t) + c_2\sin(t) \\ c_1 e^t - c_3\sin(t) + c_2\cos(t) \end{bmatrix}$$
- Solution to the system of ODEs
$$\{x(t) = \cos(t)(c_2 + c_3) + (c_2 - c_3)\sin(t), y(t) = c_1 e^t + c_3\cos(t) + c_2\sin(t), z(t) = c_1 e^t - c_3\sin($$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 56

```
dsolve([diff(x(t),t)=-y(t)+z(t),diff(y(t),t)=z(t),diff(z(t),t)=-x(t)+z(t)],singsol=all)
```

$$x(t) = c_2\sin(t) + c_3\sin(t) - c_2\cos(t) + c_3\cos(t)$$
$$y(t) = c_1 e^t - c_2\cos(t) + c_3\sin(t)$$
$$z(t) = c_1 e^t + c_2\sin(t) + c_3\cos(t)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 112

```
DSolve[{x'[t]==-y[t]+z[t],y'[t]==z[t],z'[t]==-x[t]+z[t]},{x[t],y[t],z[t]},t,IncludeSingularS
```

$$x(t) \to c_1 \cos(t) + (c_3 - c_2) \sin(t)$$
$$y(t) \to \frac{1}{2}\left((-c_1 + c_2 + c_3)e^t + (c_1 + c_2 - c_3) \cos(t) + (c_1 - c_2 + c_3) \sin(t)\right)$$
$$z(t) \to \frac{1}{2}\left((-c_1 + c_2 + c_3)e^t + (c_1 - c_2 + c_3) \cos(t) - (c_1 + c_2 - c_3) \sin(t)\right)$$

27.7 problem 782

Internal problem ID [15513]
Internal file name [`OUTPUT/15513_Friday_May_10_2024_05_47_32_PM_35823396/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 782.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y(t) + z(t) \\
y'(t) &= x(t) + z(t) \\
z'(t) &= x(t) + y(t)
\end{aligned}
$$

27.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_3 \\ \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_2+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_3 \\ \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2+\left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2-c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_2+c_1+c_3)}{3} \\ \frac{(-c_1+2c_2-c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_2+c_1+c_3)}{3} \\ \frac{(-c_1-c_2+2c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_2+c_1+c_3)}{3} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

27.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array} \right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & -2 & 0 \end{array} \right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array} \right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	2	2	No	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

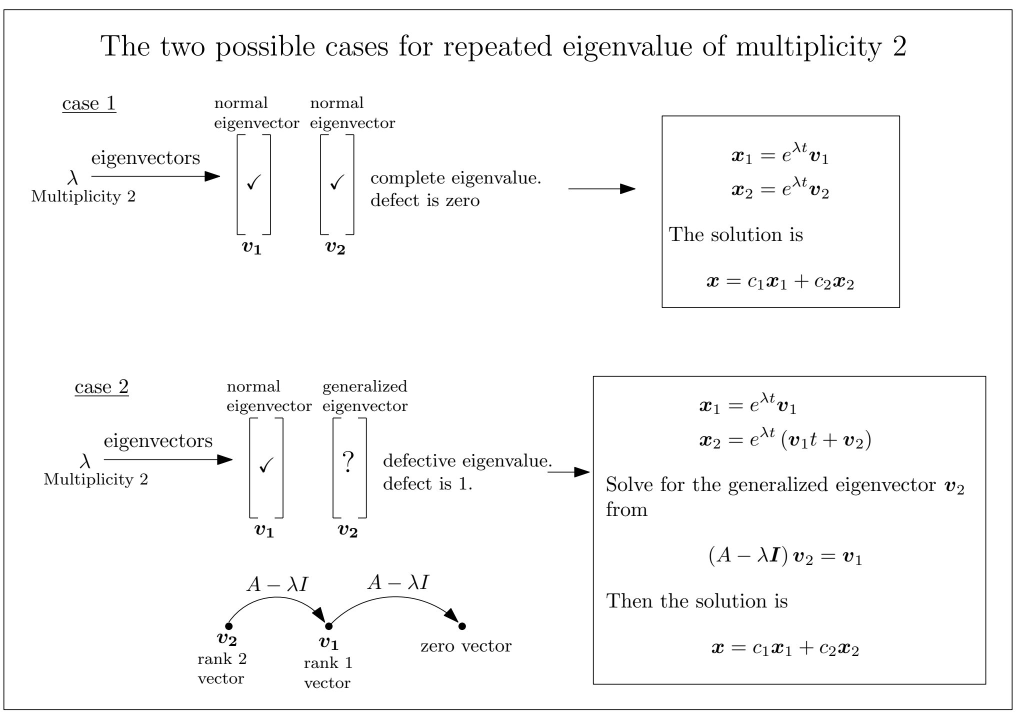


Figure 844: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} (-c_1 - c_2)\,\mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \\ c_2 \mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \end{bmatrix}$$

27.7.3 Maple step by step solution

Let's solve

$[x'(t) = y(t) + z(t), y'(t) = x(t) + z(t), z'(t) = x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{x}_1(t) = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$$\overrightarrow{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right] - (-1) \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{x}_2(t) = e^{-t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right) + c_3 e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} ((-t-1) c_2 - c_1) e^{-t} + c_3 e^{2t} \\ c_3 e^{2t} \\ (c_2 t + c_1) e^{-t} + c_3 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = ((-t-1) c_2 - c_1) e^{-t} + c_3 e^{2t}, y(t) = c_3 e^{2t}, z(t) = (c_2 t + c_1) e^{-t} + c_3 e^{2t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 64

```
dsolve([diff(x(t),t)=y(t)+z(t),diff(y(t),t)=x(t)+z(t),diff(z(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = c_2 e^{2t} + c_3 e^{-t}$$
$$y(t) = c_2 e^{2t} + c_3 e^{-t} + e^{-t} c_1$$
$$z(t) = c_2 e^{2t} - 2c_3 e^{-t} - e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 124

```
DSolve[{x'[t]==y[t]+z[t],y'[t]==x[t]+z[t],z'[t]==x[t]+y[t]},{x[t],y[t],z[t]},t,IncludeSingul
```

$$x(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} + 2 \right) + (c_2 + c_3) \left(e^{3t} - 1 \right) \right)$$
$$y(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1 \right) + c_2 \left(e^{3t} + 2 \right) + c_3 \left(e^{3t} - 1 \right) \right)$$
$$z(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1 \right) + c_2 \left(e^{3t} - 1 \right) + c_3 \left(e^{3t} + 2 \right) \right)$$

27.8 problem 783

Internal problem ID [15514]
Internal file name [`OUTPUT/15514_Friday_May_10_2024_05_47_32_PM_25585316/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 783.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"** Unable to solve or complete the solution.

Unable to parse ODE.

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 48

```
dsolve([diff(x(t),t$2)=y(t),diff(y(t),t$2)=x(t)],singsol=all)
```

$$x(t) = c_1 e^{t} + c_2 e^{-t} - c_3 \sin(t) - c_4 \cos(t)$$
$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \sin(t) + c_4 \cos(t)$$

✓ Solution by Mathematica
Time used: 0.017 (sec). Leaf size: 172

```
DSolve[{x''[t]==y[t],y''[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{4} e^{-t} \left(c_1 e^{2t} + c_2 e^{2t} + c_3 e^{2t} + c_4 e^{2t} + 2(c_1 - c_3) e^{t} \cos(t) + 2(c_2 - c_4) e^{t} \sin(t) + c_1 - c_2 + c_3 - c_4\right)$$
$$y(t) \to \frac{1}{4} e^{-t} \left(c_1 e^{2t} + c_2 e^{2t} + c_3 e^{2t} + c_4 e^{2t} - 2(c_1 - c_3) e^{t} \cos(t) - 2(c_2 - c_4) e^{t} \sin(t) + c_1 - c_2 + c_3 - c_4\right)$$

27.9 problem 784

Internal problem ID [15515]
Internal file name [`OUTPUT/15515_Friday_May_10_2024_05_47_32_PM_11055370/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 784.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"** Unable to solve or complete the solution.

Unable to parse ODE.

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 40

```
dsolve([diff(x(t),t$2)+diff(y(t),t)+x(t)=0,diff(x(t),t)+diff(y(t),t$2)=0],singsol=all)
```

$$x(t) = c_1 - \frac{1}{2}t^2c_1 - c_2t - c_3$$
$$y(t) = \frac{1}{6}t^3c_1 + \frac{1}{2}c_2t^2 + c_3t + c_4$$

✓ Solution by Mathematica
Time used: 0.006 (sec). Leaf size: 61

```
DSolve[{x''[t]+y'[t]+x[t]==0,x'[t]+y''[t]==0},{x[t],y[t]},t,IncludeSingularSolutions -> True
```

$$x(t) \to -\frac{c_1t^2}{2} - \frac{c_4t^2}{2} + c_2t + c_1$$
$$y(t) \to \frac{1}{6}(c_1 + c_4)t^3 - \frac{c_2t^2}{2} + c_4t + c_3$$

27.10 problem 785

Internal problem ID [15516]
Internal file name [`OUTPUT/15516_Friday_May_10_2024_05_47_32_PM_49721640/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 785.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"** Unable to solve or complete the solution.

Unable to parse ODE.

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 47

```
dsolve([diff(x(t),t$2)=3*x(t)+y(t),diff(y(t),t)=-2*x(t)],singsol=all)
```

$$x(t) = c_1 e^{-2t} - \frac{c_2 e^t}{2} - \frac{c_3 e^t t}{2} - \frac{c_3 e^t}{2}$$
$$y(t) = c_1 e^{-2t} + c_2 e^t + c_3 e^t t$$

✓ Solution by Mathematica
Time used: 0.052 (sec). Leaf size: 125

```
DSolve[{x''[t]==3*x[t]+y[t],y'[t]==-2*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{9}e^{-2t}\left(c_1\left(e^{3t}(3t+5)+4\right)+c_2\left(e^{3t}(3t+2)-2\right)+c_3\left(e^{3t}(3t-1)+1\right)\right)$$
$$y(t) \to \frac{1}{9}e^{-2t}\left(c_1\left(4-2e^{3t}(3t+2)\right)+c_2\left(e^{3t}(2-6t)-2\right)+c_3\left(e^{3t}(8-6t)+1\right)\right)$$

27.11 problem 786

Internal problem ID [15517]
Internal file name [`OUTPUT/15517_Friday_May_10_2024_05_47_32_PM_27558722/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 20. The method of elimination. Exercises page 212
Problem number: 786.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"** Unable to solve or complete the solution.

Unable to parse ODE.

✓ Solution by Maple
Time used: 0.063 (sec). Leaf size: 19

```
dsolve([diff(diff(x(t),t),t) = x(t)^2+y(t), diff(y(t),t) = -2*x(t)*diff(x(t),t)+x(t), x(0) =
```

$$x(t) = e^t$$
$$y(t) = -e^{2t} + e^t$$

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x''[t]==x[t]^2+y[t],y'[t]==-2*x[t]*x'[t]+x[t]},{x[0]==1,x'[0]==1,y[0]==0},{x[t],y[t]
```

Not solved

28 Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219

28.1 problem 787

Internal problem ID [15518]
Internal file name [`OUTPUT/15518_Friday_May_10_2024_05_47_32_PM_68303959/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 787.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = x(t)^2 + y(t)^2$$
$$y'(t) = 2x(t)\, y(t)$$

Does not currently support non linear system of equations. This is the phase plot of the system.

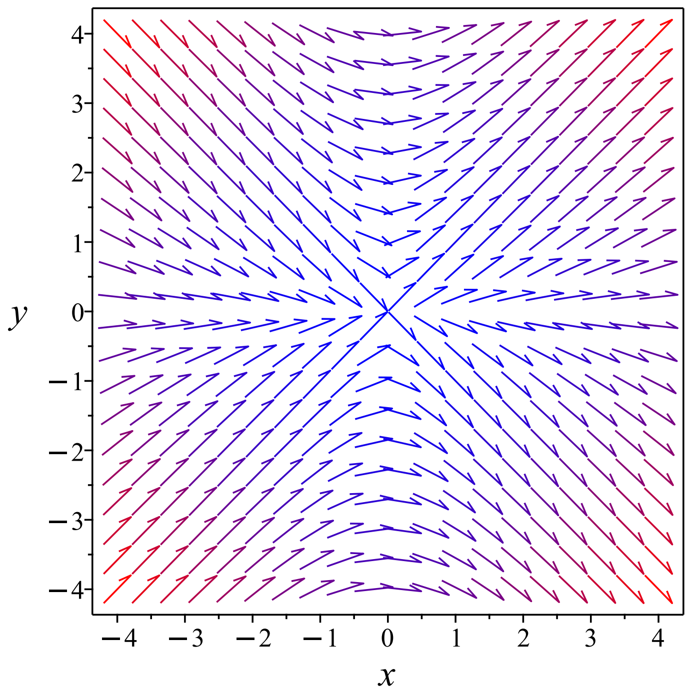


Figure 845: Phase plot

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 65

```
dsolve([diff(x(t),t)=x(t)^2+y(t)^2,diff(y(t),t)=2*x(t)*y(t)],singsol=all)
```

$$\left[\{y(t)=0\},\left\{x(t)=\frac{1}{-t+c_1}\right\}\right]$$
$$\left[\left\{y(t)=\frac{4c_1}{c_1^2c_2^2+2c_1^2c_2t+c_1^2t^2-16}\right\},\left\{x(t)=\frac{\frac{d}{dt}y(t)}{2y(t)}\right\}\right]$$

✓ Solution by Mathematica

Time used: 41.052 (sec). Leaf size: 3516

```
DSolve[{x'[t]==x[t]^2+y[t]^2,y'[t]==-2*x[t]*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

Too large to display

28.2 problem 788

Internal problem ID [15519]
Internal file name [`OUTPUT/15519_Friday_May_10_2024_05_47_33_PM_72649872/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 788.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = -\frac{1}{y(t)}$$
$$y'(t) = \frac{1}{x(t)}$$

Does not currently support non linear system of equations. This is the phase plot of the system.

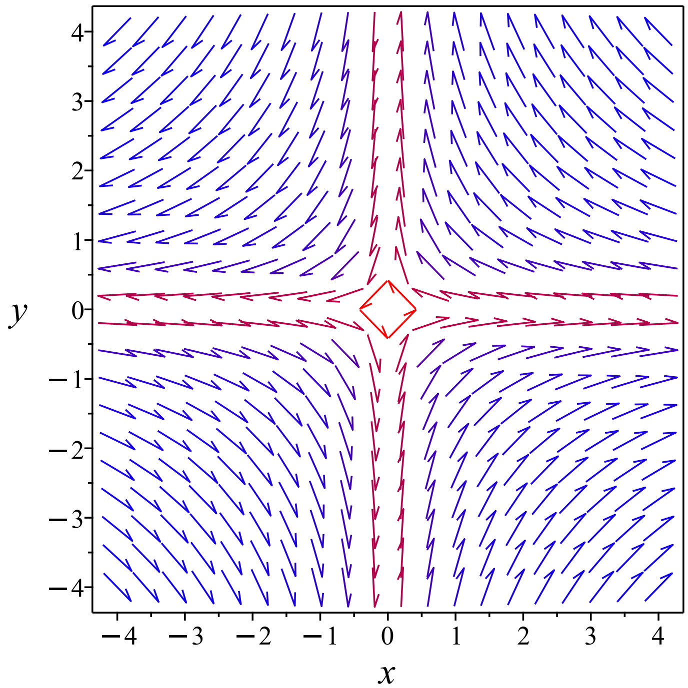


Figure 846: Phase plot

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 24

```
dsolve([diff(x(t),t)=-1/y(t),diff(y(t),t)=1/x(t)],singsol=all)
```

$$\{x(t) = e^{c_1 t} c_2\}$$
$$\left\{y(t) = -\frac{1}{\frac{d}{dt}x(t)}\right\}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 35

```
DSolve[{x'[t]==-1/y[t],y'[t]==1/x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{c_1 e^{\frac{t}{c_1}}}{c_2}$$
$$x(t) \to c_2 e^{-\frac{t}{c_1}}$$

28.3 problem 789

Internal problem ID [15520]
Internal file name [`OUTPUT/15520_Friday_May_10_2024_05_47_33_PM_93593442/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 789.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = \frac{x(t)}{y(t)}$$
$$y'(t) = \frac{y(t)}{x(t)}$$

Does not currently support non linear system of equations. This is the phase plot of the system.

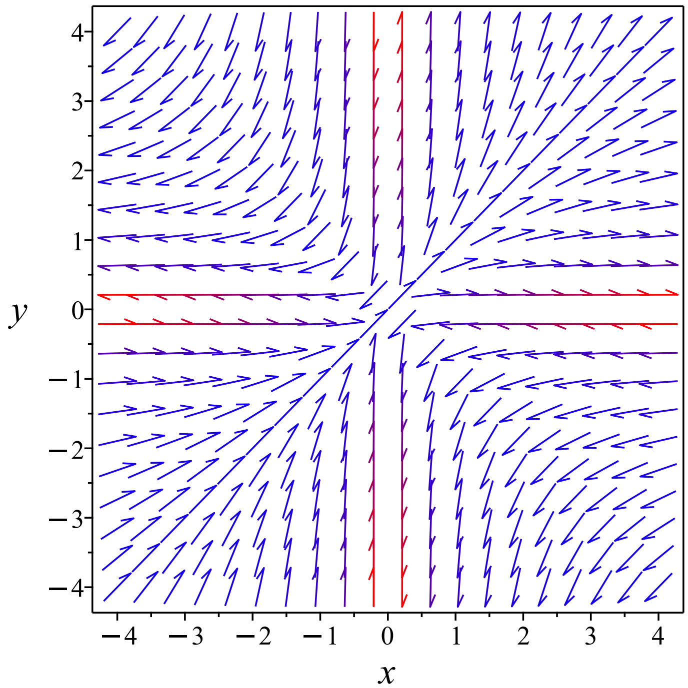


Figure 847: Phase plot

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 34

```
dsolve([diff(x(t),t)=x(t)/y(t),diff(y(t),t)=y(t)/x(t)],singsol=all)
```

$$\left\{x(t) = \frac{-1 + e^{c_2 c_1} e^{c_1 t}}{c_1}\right\}$$
$$\left\{y(t) = \frac{x(t)}{\frac{d}{dt}x(t)}\right\}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 45

```
DSolve[{x'[t]==x[t]/y[t],y'[t]==y[t]/x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{e^{c_1 t} + c_1 c_2}{c_1{}^2 c_2}$$
$$x(t) \to c_2 e^{c_1(-t)} + \frac{1}{c_1}$$

28.4 problem 790

Internal problem ID [15521]
Internal file name [`OUTPUT/15521_Friday_May_10_2024_05_47_33_PM_79270827/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 790.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete tl

Solve

$$x'(t) = -\frac{y(t)}{y(t) - x(t)}$$
$$y'(t) = -\frac{x(t)}{y(t) - x(t)}$$

Does not currently support non linear system of equations. This is the phase plot of the system.

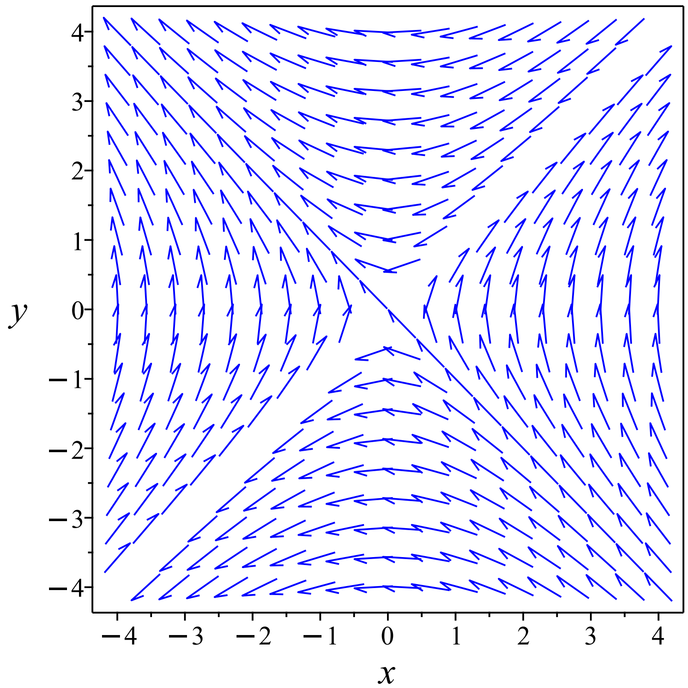


Figure 848: Phase plot

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)=y(t)/(x(t)-y(t)),diff(y(t),t)=x(t)/(x(t)-y(t))],singsol=all)
```

$$\left\{x(t)=\frac{-c_1t^2-2c_2t-2}{2c_1t+2c_2}\right\}$$
$$\left\{y(t)=\frac{x(t)\left(\frac{d}{dt}x(t)\right)}{\frac{d}{dt}x(t)+1}\right\}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 145

```
DSolve[{x'[t]==y[t]/(x[t]-y[t]),y'[t]==x[t]/(x[t]-y[t])},{x[t],y[t]},t,IncludeSingularSoluti
```

$$y(t) \to -\frac{1}{2}\sqrt{\frac{(t^2 - 2c_2t + c_2{}^2 + 2c_1)^2}{(t - c_2)^2}}$$
$$x(t) \to -\frac{t^2 - 2c_2t + c_2{}^2 - 2c_1}{2t - 2c_2}$$
$$y(t) \to \frac{1}{2}\sqrt{\frac{(t^2 - 2c_2t + c_2{}^2 + 2c_1)^2}{(t - c_2)^2}}$$
$$x(t) \to -\frac{t^2 - 2c_2t + c_2{}^2 - 2c_1}{2t - 2c_2}$$

28.5 problem 791

Internal problem ID [15522]
Internal file name [`OUTPUT/15522_Friday_May_10_2024_05_47_33_PM_68571445/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 791.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = \sin(x(t))\cos(y(t))$$
$$y'(t) = \cos(x(t))\sin(y(t))$$

Does not currently support non linear system of equations. This is the phase plot of the system.

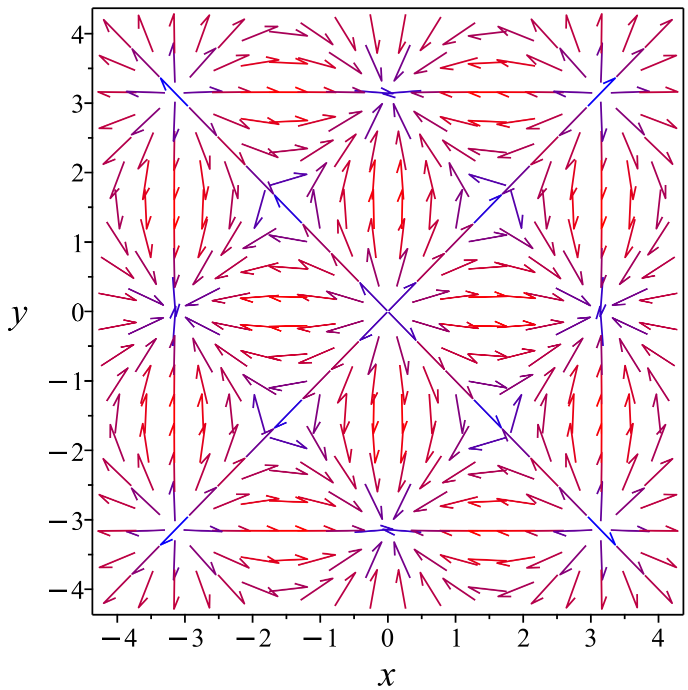


Figure 849: Phase plot

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 38

```
dsolve([diff(x(t),t)=sin(x(t))*cos(y(t)),diff(y(t),t)=cos(x(t))*sin(y(t))],singsol=all)
```

$$\left\{y(t) = \operatorname{arccot}\left(\frac{(c_1 e^{2t} - c_2) e^{-t}}{2}\right)\right\}$$
$$\left\{x(t) = \arccos\left(\frac{\frac{d}{dt}y(t)}{\sin(y(t))}\right)\right\}$$

✓ Solution by Mathematica

Time used: 0.492 (sec). Leaf size: 121

```
DSolve[{x'[t]==Sin[x[t]]*Cos[y[t]],y'[t]==Cos[x[t]]*Sin[y[t]]},{x[t],y[t]},t,IncludeSingular
```

$$y(t) \to \arcsin\left(e^{c_1}\sin\left(\text{InverseFunction}\left[-\operatorname{arctanh}\left(\frac{\sqrt{2}\cos(\#1)}{\sqrt{-e^{2c_1}\cos\left(2\left(\frac{\pi}{2}-\#1\right)\right)+2-e^{2c_1}}}\right)\&\right][t+c_2]\right)\right)$$

$$x(t) \to \text{InverseFunction}\left[-\operatorname{arctanh}\left(\frac{\sqrt{2}\cos(\#1)}{\sqrt{-e^{2c_1}\cos\left(2\left(\frac{\pi}{2}-\#1\right)\right)+2-e^{2c_1}}}\right)\&\right][t+c_2]$$

28.6 problem 792

Internal problem ID [15523]
Internal file name [`OUTPUT/15523_Friday_May_10_2024_05_47_33_PM_7763517/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 792.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$x'(t) = \frac{e^{-t}}{y(t)}$$
$$y'(t) = \frac{e^{-t}}{x(t)}$$

✓ Solution by Maple
Time used: 0.062 (sec). Leaf size: 52

```
dsolve([exp(t)*diff(x(t),t)=1/y(t),exp(t)*diff(y(t),t)=1/x(t)],singsol=all)
```

$$\left\{x(t) = \sqrt{-2\,e^{-t}c_1 + 2c_2}, x(t) = -\sqrt{-2\,e^{-t}c_1 + 2c_2}\right\}$$
$$\left\{y(t) = \frac{e^{-t}}{\frac{d}{dt}x(t)}\right\}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 125

```
DSolve[{Exp[t]*x'[t]==1/y[t],Exp[t]*y'[t]==1/x[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$y(t) \to -\sqrt{2}\sqrt{c_1}\sqrt{-e^{-t} + c_1c_2}$$
$$x(t) \to -\frac{\sqrt{-2e^{-t} + 2c_1c_2}}{\sqrt{c_1}}$$
$$y(t) \to \sqrt{2}\sqrt{c_1}\sqrt{-e^{-t} + c_1c_2}$$
$$x(t) \to \frac{\sqrt{-2e^{-t} + 2c_1c_2}}{\sqrt{c_1}}$$

28.7 problem 793

Internal problem ID [15524]
Internal file name [OUTPUT/15524_Friday_May_10_2024_05_47_33_PM_58516527/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 21. Finding integrable combinations. Exercises page 219
Problem number: 793.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
x'(t) &= \cos(x(t))^2 \cos(y(t))^2 + \sin(x(t))^2 \cos(y(t))^2 \\
y'(t) &= -2\sin(x(t))\cos(x(t))\sin(y(t))\cos(y(t))
\end{aligned}
$$

With initial conditions

$$[x(0) = 0, y(0) = 0]$$

Does not currently support non linear system of equations. This is the phase plot of the system.

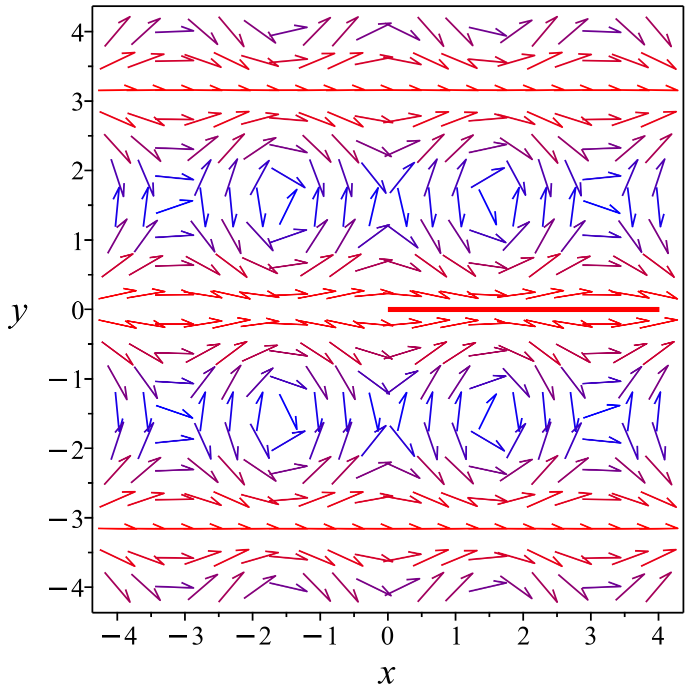


Figure 850: Phase plot

✗ Solution by Maple

```
dsolve([diff(x(t),t) = cos(x(t))^2*cos(y(t))^2+sin(x(t))^2*cos(y(t))^2, diff(y(t),t) = -1/2*
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x'[t]==Cos[x[t]]^2*Cos[y[t]]^2+Sin[x[t]]^2*Cos[y[t]]^2,y'[t]==-1/2*Sin[2*x[t]]*Sin[2
```

{}

29 Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230

29.1 problem 802

Internal problem ID [15525]
Internal file name [`OUTPUT/15525_Friday_May_10_2024_05_47_34_PM_50385180/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 802.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 8y(t) - x(t) \\
y'(t) &= x(t) + y(t)
\end{aligned}
$$

29.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-3t}}{3} + \frac{\mathrm{e}^{3t}}{3} & \frac{4\,\mathrm{e}^{3t}}{3} - \frac{4\,\mathrm{e}^{-3t}}{3} \\ \frac{\mathrm{e}^{3t}}{6} - \frac{\mathrm{e}^{-3t}}{6} & \frac{\mathrm{e}^{-3t}}{3} + \frac{2\,\mathrm{e}^{3t}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,e^{-3t}}{3}+\frac{e^{3t}}{3} & \frac{4\,e^{3t}}{3}-\frac{4\,e^{-3t}}{3} \\ \frac{e^{3t}}{6}-\frac{e^{-3t}}{6} & \frac{e^{-3t}}{3}+\frac{2\,e^{3t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,e^{-3t}}{3}+\frac{e^{3t}}{3}\right) c_1 + \left(\frac{4\,e^{3t}}{3}-\frac{4\,e^{-3t}}{3}\right) c_2 \\ \left(\frac{e^{3t}}{6}-\frac{e^{-3t}}{6}\right) c_1 + \left(\frac{e^{-3t}}{3}+\frac{2\,e^{3t}}{3}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-4c_2)e^{-3t}}{3}+\frac{e^{3t}(c_1+4c_2)}{3} \\ \frac{(-c_1+2c_2)e^{-3t}}{6}+\frac{e^{3t}(c_1+4c_2)}{6} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 8 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 8 & 0 \\ 1 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 8 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 8 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -4 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -4\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_1 \mathrm{e}^{3t} - 4c_2 \mathrm{e}^{-3t} \\ c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

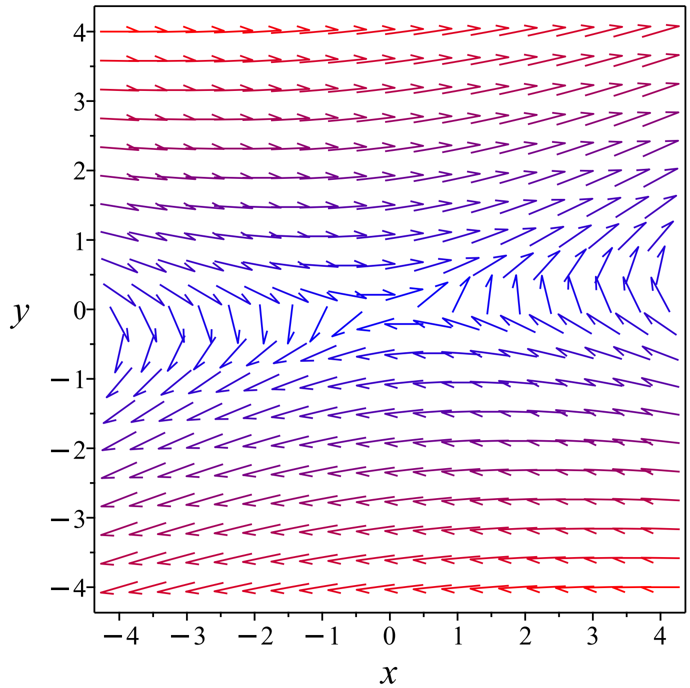


Figure 851: Phase plot

29.1.3 Maple step by step solution

Let's solve

$[x'(t) = 8y(t) - x(t), y'(t) = x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 8 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = e^{-3t} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = e^{3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^{-3t} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -4c_1 \mathrm{e}^{-3t} + 2c_2 \mathrm{e}^{3t} \\ c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -4c_1 \mathrm{e}^{-3t} + 2c_2 \mathrm{e}^{3t}, y(t) = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{3t}\}$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=8*y(t)-x(t),diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{-3t}$$
$$y(t) = \frac{c_1 \mathrm{e}^{3t}}{2} - \frac{c_2 \mathrm{e}^{-3t}}{4}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 72

```
DSolve[{x'[t]==8*y[t]-x[t],y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{3} e^{-3t} \left(c_1 \left(e^{6t} + 2\right) + 4c_2 \left(e^{6t} - 1\right)\right)$$
$$y(t) \to \frac{1}{6} e^{-3t} \left(c_1 \left(e^{6t} - 1\right) + 2c_2 \left(2e^{6t} + 1\right)\right)$$

29.2 problem 803

Internal problem ID [15526]
Internal file name [`OUTPUT/15526_Friday_May_10_2024_05_47_34_PM_48988675/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 803.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - y(t) \\
y'(t) &= y(t) - x(t)
\end{aligned}
$$

29.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & -\frac{e^{2t}}{2} + \frac{1}{2} \\ -\frac{e^{2t}}{2} + \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & -\frac{e^{2t}}{2}+\frac{1}{2} \\ -\frac{e^{2t}}{2}+\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{2}+\frac{e^{2t}}{2}\right)c_1 + \left(-\frac{e^{2t}}{2}+\frac{1}{2}\right)c_2 \\ \left(-\frac{e^{2t}}{2}+\frac{1}{2}\right)c_1 + \left(\frac{1}{2}+\frac{e^{2t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-c_2)e^{2t}}{2}+\frac{c_2}{2}+\frac{c_1}{2} \\ \frac{(-c_1+c_2)e^{2t}}{2}+\frac{c_2}{2}+\frac{c_1}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^0$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 - c_2 \mathrm{e}^{2t} \\ c_1 + c_2 \mathrm{e}^{2t} \end{bmatrix}$$

The following is the phase plot of the system.

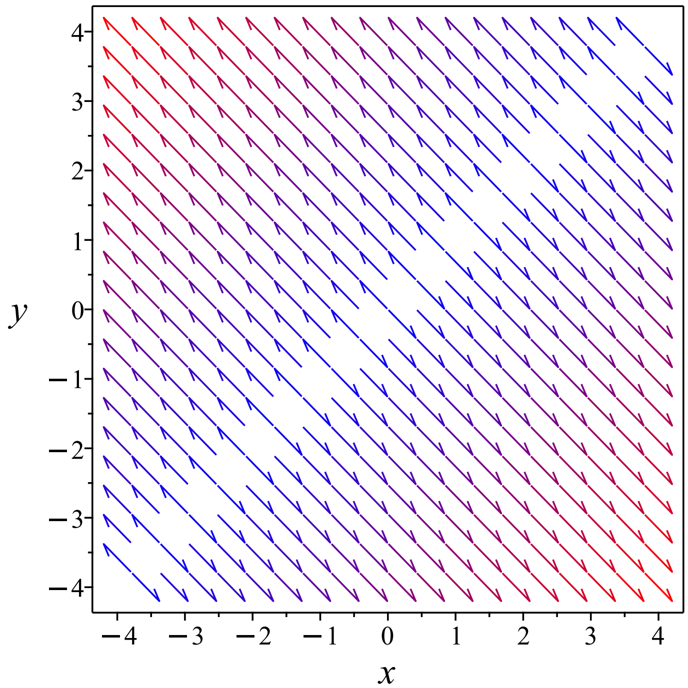


Figure 852: Phase plot

29.2.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - y(t), y'(t) = y(t) - x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_2 \mathrm{e}^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 - c_2 e^{2t} \\ c_1 + c_2 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_1 - c_2 e^{2t}, y(t) = c_1 + c_2 e^{2t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve([diff(x(t),t)=x(t)-y(t),diff(y(t),t)=y(t)-x(t)],singsol=all)
```

$$x(t) = c_1 + c_2 e^{2t}$$
$$y(t) = -c_2 e^{2t} + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 60

```
DSolve[{x'[t]==x[t]-y[t],y'[t]==y[t]-x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}\left(c_1 e^{2t} - c_2 e^{2t} + c_1 + c_2\right)$$
$$y(t) \to \frac{1}{2}\left(c_1\left(-e^{2t}\right) + c_2 e^{2t} + c_1 + c_2\right)$$

29.3 problem 804

Internal problem ID [15527]
Internal file name [`OUTPUT/15527_Friday_May_10_2024_05_47_34_PM_32247685/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 804.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) + y(t)$$
$$y'(t) = x(t) - 3y(t)$$

With initial conditions

$$[x(0) = 0, y(0) = 0]$$

29.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(5\sqrt{29}+29\right)e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}}{58} + \frac{\left(-5\sqrt{29}+29\right)e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}}{58} & -\frac{\left(-e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}+e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\right)\sqrt{29}}{29} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}+e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\right)\sqrt{29}}{29} & \frac{\left(-5\sqrt{29}+29\right)e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}}{58} + \frac{e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\left(5\sqrt{29}+29\right)}{58} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{\left(5\sqrt{29}+29\right)e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}}{58} + \frac{\left(-5\sqrt{29}+29\right)e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}}{58} & -\frac{\left(-e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}+e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\right)\sqrt{29}}{29} \\ -\frac{\left(-e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}+e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\right)\sqrt{29}}{29} & \frac{\left(-5\sqrt{29}+29\right)e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}}{58} + \frac{e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\left(5\sqrt{29}+29\right)}{58} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 7 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{\sqrt{29}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{\sqrt{29}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} + \frac{\sqrt{29}}{2}$	1	real eigenvalue
$-\frac{1}{2} - \frac{\sqrt{29}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{\sqrt{29}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} - \left(-\frac{1}{2} - \frac{\sqrt{29}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{5}{2} + \frac{\sqrt{29}}{2} & 1 \\ 1 & -\frac{5}{2} + \frac{\sqrt{29}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{5}{2} + \frac{\sqrt{29}}{2} & 1 & 0 \\ 1 & -\frac{5}{2} + \frac{\sqrt{29}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{5}{2} + \frac{\sqrt{29}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{5}{2} + \frac{\sqrt{29}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{5}{2}+\frac{\sqrt{29}}{2} & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{5+\sqrt{29}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{5+\sqrt{29}} \\ t \end{bmatrix}=\begin{bmatrix} -\frac{2t}{5+\sqrt{29}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{5+\sqrt{29}} \\ t \end{bmatrix}=t\begin{bmatrix} -\frac{2}{5+\sqrt{29}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{5+\sqrt{29}} \\ t \end{bmatrix}=\begin{bmatrix} -\frac{2}{5+\sqrt{29}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{5+\sqrt{29}} \\ t \end{bmatrix}=\begin{bmatrix} -\frac{2}{5+\sqrt{29}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2}+\frac{\sqrt{29}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}-\left(-\frac{1}{2}+\frac{\sqrt{29}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{5}{2}-\frac{\sqrt{29}}{2} & 1 \\ 1 & -\frac{5}{2}-\frac{\sqrt{29}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{5}{2}-\frac{\sqrt{29}}{2} & 1 & 0 \\ 1 & -\frac{5}{2}-\frac{\sqrt{29}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{\frac{5}{2} - \frac{\sqrt{29}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{5}{2} - \frac{\sqrt{29}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{5}{2} - \frac{\sqrt{29}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{-5+\sqrt{29}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{-5+\sqrt{29}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{-5+\sqrt{29}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{-5+\sqrt{29}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{-5+\sqrt{29}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{-5+\sqrt{29}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{-5+\sqrt{29}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{1}{2}+\frac{\sqrt{29}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-\frac{5}{2}+\frac{\sqrt{29}}{2}}\\ 1\end{array}\right]$
$-\frac{1}{2}-\frac{\sqrt{29}}{2}$	1	1	No	$\left[\begin{array}{c}\frac{1}{-\frac{5}{2}-\frac{\sqrt{29}}{2}}\\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{2}+\frac{\sqrt{29}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{1}{2}+\frac{\sqrt{29}}{2}\right)t}\\ &= \left[\begin{array}{c}\frac{1}{-\frac{5}{2}+\frac{\sqrt{29}}{2}}\\ 1\end{array}\right] e^{\left(-\frac{1}{2}+\frac{\sqrt{29}}{2}\right)t}\end{aligned}$$

Since eigenvalue $-\frac{1}{2}-\frac{\sqrt{29}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{1}{2}-\frac{\sqrt{29}}{2}\right)t}\\ &= \left[\begin{array}{c}\frac{1}{-\frac{5}{2}-\frac{\sqrt{29}}{2}}\\ 1\end{array}\right] e^{\left(-\frac{1}{2}-\frac{\sqrt{29}}{2}\right)t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right] = c_1\left[\begin{array}{c}\frac{\mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{29}}{2}\right)t}}{-\frac{5}{2}+\frac{\sqrt{29}}{2}}\\ \mathrm{e}^{\left(-\frac{1}{2}+\frac{\sqrt{29}}{2}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}\frac{\mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{29}}{2}\right)t}}{-\frac{5}{2}-\frac{\sqrt{29}}{2}}\\ \mathrm{e}^{\left(-\frac{1}{2}-\frac{\sqrt{29}}{2}\right)t}\end{array}\right]$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1\left(5+\sqrt{29}\right)e^{\frac{\left(-1+\sqrt{29}\right)t}{2}}}{2} - \frac{c_2 e^{-\frac{\left(1+\sqrt{29}\right)t}{2}}\left(-5+\sqrt{29}\right)}{2} \\ c_1 e^{\frac{\left(-1+\sqrt{29}\right)t}{2}} + c_2 e^{-\frac{\left(1+\sqrt{29}\right)t}{2}} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 0 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{(c_1-c_2)\sqrt{29}}{2} + \frac{5c_1}{2} + \frac{5c_2}{2} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The following is the phase plot of the system.

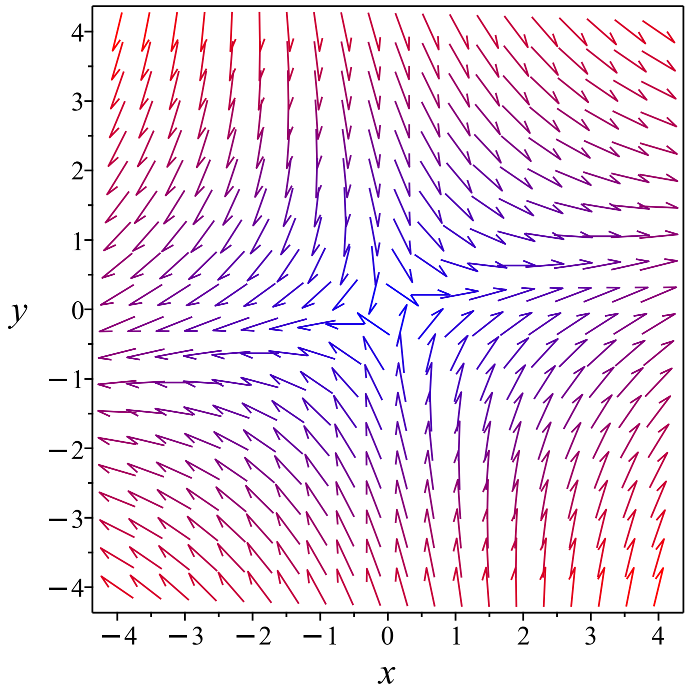
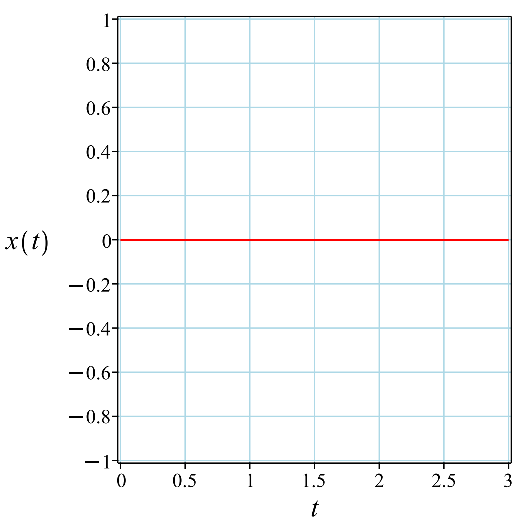
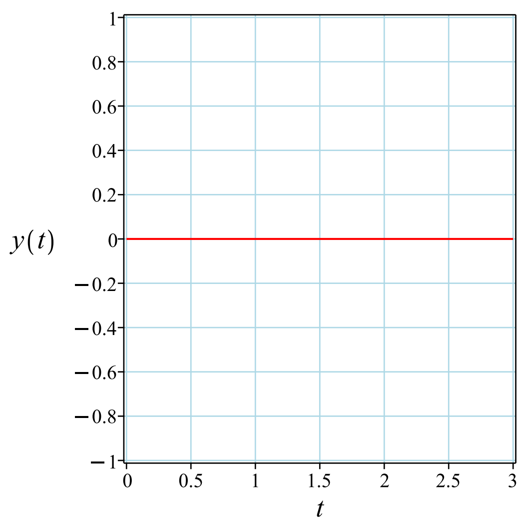


Figure 853: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(x(t),t) = 2*x(t)+y(t), diff(y(t),t) = x(t)-3*y(t), x(0) = 0, y(0) = 0], singsol
```

$$x(t) = 0$$
$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 10

```
DSolve[{x'[t]==2*x[t]+y[t],y'[t]==x[t]-3*y[t]},{x[0]==0,y[0]==0},{x[t],y[t]},t,IncludeSingul
```

$$x(t) \to 0$$
$$y(t) \to 0$$

29.4 problem 805

Internal problem ID [15528]
Internal file name [`OUTPUT/15528_Friday_May_10_2024_05_47_34_PM_84918379/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 805.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) + y(t)$$
$$y'(t) = 4y(t) - 2x(t)$$

With initial conditions

$$[x(0) = 0, y(0) = -1]$$

29.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 2\,\mathrm{e}^{2t} - \mathrm{e}^{3t} & \mathrm{e}^{3t} - \mathrm{e}^{2t} \\ -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{2t} & -\mathrm{e}^{2t} + 2\,\mathrm{e}^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} 2\,\mathrm{e}^{2t}-\mathrm{e}^{3t} & \mathrm{e}^{3t}-\mathrm{e}^{2t} \\ -2\,\mathrm{e}^{3t}+2\,\mathrm{e}^{2t} & -\mathrm{e}^{2t}+2\,\mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{3t}+\mathrm{e}^{2t} \\ \mathrm{e}^{2t}-2\,\mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A-\lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 5\lambda + 6 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{3t}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{3t}}{2} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{3t}}{2} + c_2 e^{2t} \\ c_1 e^{3t} + c_2 e^{2t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{2} + c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -2 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^{3t} + e^{2t} \\ e^{2t} - 2\,e^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

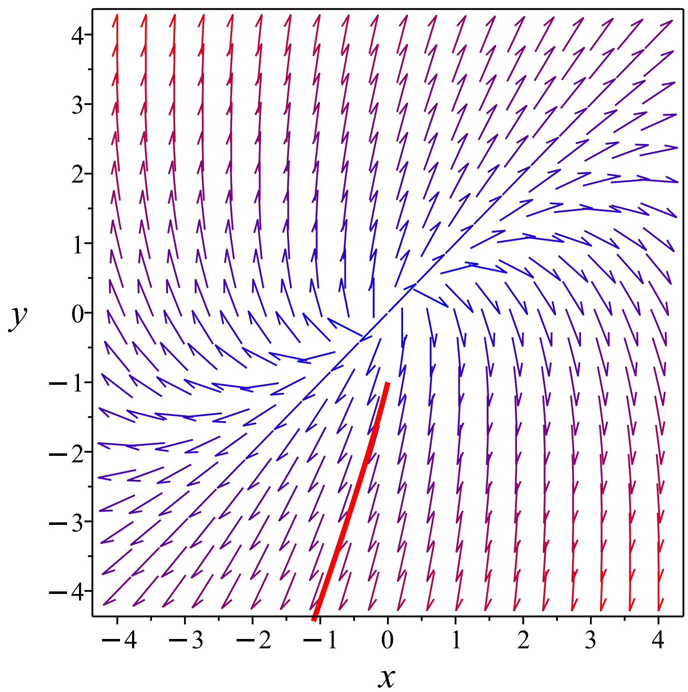
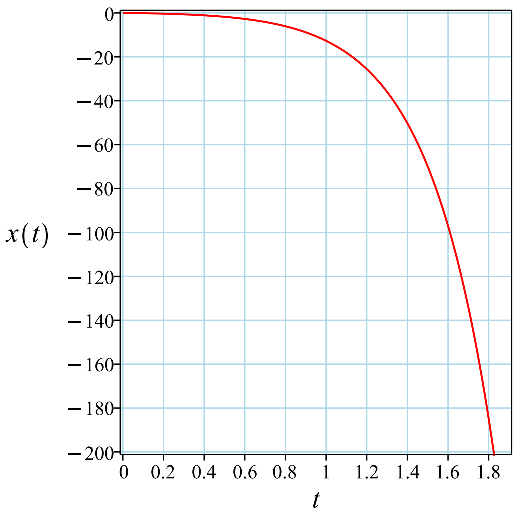
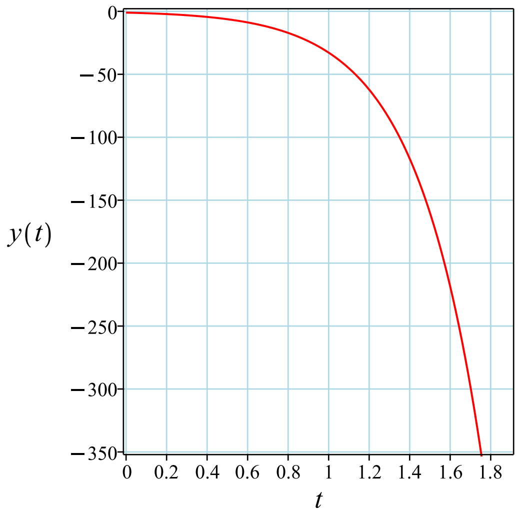


Figure 854: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve([diff(x(t),t) = x(t)+y(t), diff(y(t),t) = 4*y(t)-2*x(t), x(0) = 0, y(0) = -1], singso
```

$$x(t) = -e^{3t} + e^{2t}$$
$$y(t) = -2\, e^{3t} + e^{2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 33

```
DSolve[{x'[t]==x[t]+y[t],y'[t]==4*y[t]-2*x[t]},{x[0]==0,y[0]==-1},{x[t],y[t]},t,IncludeSingu
```

$$x(t) \to -e^{2t}\left(e^{t} - 1\right)$$
$$y(t) \to e^{2t} - 2e^{3t}$$

29.5 problem 806

Internal problem ID [15529]
Internal file name [`OUTPUT/15529_Friday_May_10_2024_05_47_34_PM_18391819/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 806.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4x(t) - 5y(t) \\
y'(t) &= x(t)
\end{aligned}
$$

With initial conditions

$$[x(0) = 0, y(0) = 1]$$

29.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{2t}\cos(t) + 2\,e^{2t}\sin(t) & -5\,e^{2t}\sin(t) \\ e^{2t}\sin(t) & e^{2t}\cos(t) - 2\,e^{2t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}(\cos(t) + 2\sin(t)) & -5\,e^{2t}\sin(t) \\ e^{2t}\sin(t) & e^{2t}(\cos(t) - 2\sin(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{2t}(\cos(t) + 2\sin(t)) & -5\,e^{2t}\sin(t) \\ e^{2t}\sin(t) & e^{2t}(\cos(t) - 2\sin(t)) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5\,e^{2t}\sin(t) \\ e^{2t}(\cos(t) - 2\sin(t)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -5 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2+i$$
$$\lambda_2 = 2-i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2+i$	1	complex eigenvalue
$2-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2-i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix} - (2-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 1 & -2+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2+i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (2-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2+i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix} - (2+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2-i & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (2+i)\,t\}$

Hence the solution is

$$
\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (2+i)\,t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 2+i \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} (2+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+i$	1	1	No	$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$
$2-i$	1	1	No	$\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} (2+i)\,\mathrm{e}^{(2+i)t} \\ \mathrm{e}^{(2+i)t} \end{bmatrix} + c_2 \begin{bmatrix} (2-i)\,\mathrm{e}^{(2-i)t} \\ \mathrm{e}^{(2-i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1 \mathrm{e}^{(2+i)t} + (2-i)\,c_2 \mathrm{e}^{(2-i)t} \\ c_1 \mathrm{e}^{(2+i)t} + c_2 \mathrm{e}^{(2-i)t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (2+i)\,c_1 + (2-i)\,c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{1}{2} + i \\ c_2 = \frac{1}{2} - i \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{5i\mathrm{e}^{(2-i)t}}{2} + \frac{5i\mathrm{e}^{(2+i)t}}{2} \\ \left(\frac{1}{2} - i\right) \mathrm{e}^{(2-i)t} + \left(\frac{1}{2} + i\right) \mathrm{e}^{(2+i)t} \end{bmatrix}$$

The following is the phase plot of the system.

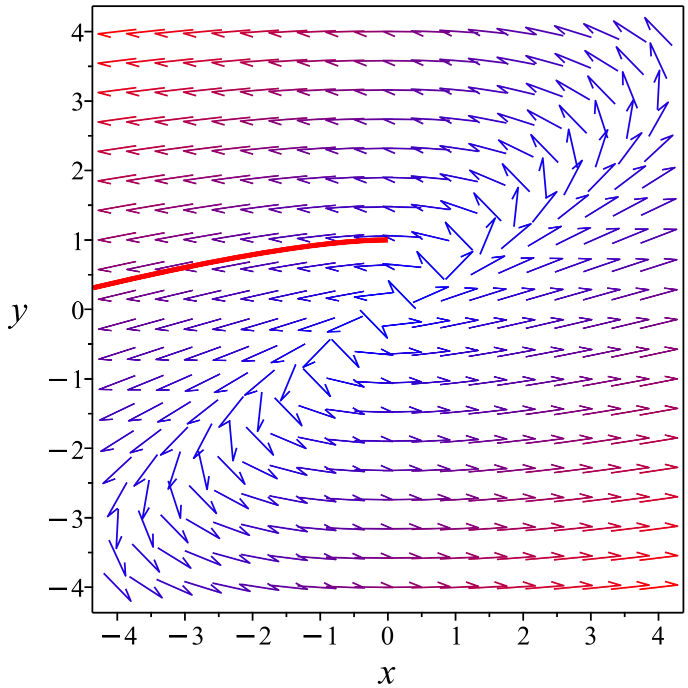


Figure 855: Phase plot

The following are plots of each solution.

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 28

```
dsolve([diff(x(t),t) = 4*x(t)-5*y(t), diff(y(t),t) = x(t), x(0) = 0, y(0) = 1], singsol=all)
```

$$x(t) = -5\,e^{2t}\sin(t)$$
$$y(t) = e^{2t}(-2\sin(t) + \cos(t))$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 27

```
DSolve[{x'[t]==4*x[t]-4*y[t],y'[t]==x[t]},{x[0]==0,y[0]==1},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to -4e^{2t}t$$
$$y(t) \to e^{2t}(1 - 2t)$$

29.6 problem 807

Internal problem ID [15530]
Internal file name [`OUTPUT/15530_Friday_May_10_2024_05_47_34_PM_85677221/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 807.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -x(t) + y(t) + z(t) \\ y'(t) &= x(t) - y(t) + z(t) \\ z'(t) &= x(t) + y(t) - z(t) \end{aligned}$$

29.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{3t}+2)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}+2)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}+2)e^{-2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(e^{3t}+2)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}+2)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} \\ \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}-1)e^{-2t}}{3} & \frac{(e^{3t}+2)e^{-2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(e^{3t}+2)e^{-2t}c_1}{3} + \frac{(e^{3t}-1)e^{-2t}c_2}{3} + \frac{(e^{3t}-1)e^{-2t}c_3}{3} \\ \frac{(e^{3t}-1)e^{-2t}c_1}{3} + \frac{(e^{3t}+2)e^{-2t}c_2}{3} + \frac{(e^{3t}-1)e^{-2t}c_3}{3} \\ \frac{(e^{3t}-1)e^{-2t}c_1}{3} + \frac{(e^{3t}-1)e^{-2t}c_2}{3} + \frac{(e^{3t}+2)e^{-2t}c_3}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-2t}(e^{3t}(c_2+c_1+c_3)+2c_1-c_2-c_3)}{3} \\ \frac{(e^{3t}(c_2+c_1+c_3)-c_1+2c_2-c_3)e^{-2t}}{3} \\ \frac{(e^{3t}(c_2+c_1+c_3)-c_1-c_2+2c_3)e^{-2t}}{3} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda & 1 & 1\\ 1 & -1-\lambda & 1\\ 1 & 1 & -1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix} - (-2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
-2	2	2	No	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

eigenvalue -2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

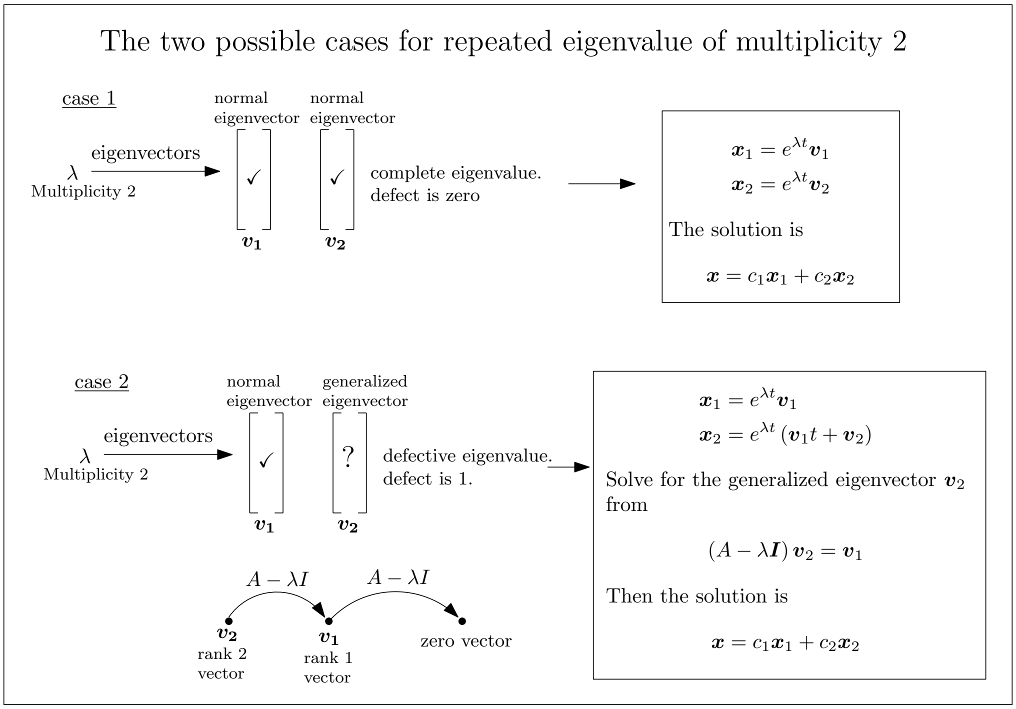


Figure 856: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\
&= \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right] e^{-2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{-2t} \\
&= \left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right] e^{-2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix} + c_3 \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} (c_1 e^{3t} - c_2 - c_3)\, e^{-2t} \\ (c_1 e^{3t} + c_3)\, e^{-2t} \\ (c_1 e^{3t} + c_2)\, e^{-2t} \end{bmatrix}$$

29.6.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + y(t) + z(t)\,, y'(t) = x(t) - y(t) + z(t)\,, z'(t) = x(t) + y(t) - z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -2

$$\vec{x}_1(t) = e^{-2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, an

$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

  $$\left(\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} - (-2) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$
- Second solution from eigenvalue -2

  $$\vec{x}_2(t) = \mathrm{e}^{-2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$
- Consider eigenpair

  $$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair

  $$\vec{x}_3 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3$
- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-2t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -e^{-2t}(-c_3 e^{3t} + c_2(t+1) + c_1) \\ c_3 e^{t} \\ (c_3 e^{3t} + c_2 t + c_1)\, e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -e^{-2t}(-c_3 e^{3t} + c_2(t+1) + c_1)\,, y(t) = c_3 e^{t}, z(t) = (c_3 e^{3t} + c_2 t + c_1)\, e^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 58

```
dsolve([diff(x(t),t)=-x(t)+y(t)+z(t),diff(y(t),t)=x(t)-y(t)+z(t),diff(z(t),t)=x(t)+y(t)-z(t)
```

$$x(t) = c_2 e^{t} + c_3 e^{-2t}$$
$$y(t) = c_2 e^{t} + c_3 e^{-2t} + c_1 e^{-2t}$$
$$z(t) = c_2 e^{t} - 2c_3 e^{-2t} - c_1 e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 124

```
DSolve[{x'[t]==-x[t]+y[t]+z[t],y'[t]==x[t]-y[t]+z[t],z'[t]==x[t]+y[t]-z[t]},{x[t],y[t],z[t]}
```

$$x(t) \to \frac{1}{3} e^{-2t} \left(c_1 \left(e^{3t} + 2\right) + (c_2 + c_3)\left(e^{3t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3} e^{-2t} \left(c_1 \left(e^{3t} - 1\right) + c_2 \left(e^{3t} + 2\right) + c_3 \left(e^{3t} - 1\right)\right)$$
$$z(t) \to \frac{1}{3} e^{-2t} \left(c_1 \left(e^{3t} - 1\right) + c_2 \left(e^{3t} - 1\right) + c_3 \left(e^{3t} + 2\right)\right)$$

29.7 problem 808

Internal problem ID [15531]
Internal file name [`OUTPUT/15531_Friday_May_10_2024_05_47_35_PM_7665776/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 808.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 2x(t) - y(t) + z(t) \\ y'(t) &= x(t) + 2y(t) - z(t) \\ z'(t) &= x(t) - y(t) + 2z(t) \end{aligned}$$

29.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & -e^{3t} + e^{2t} & e^{3t} - e^{2t} \\ e^{2t} - e^{t} & e^{2t} & -e^{2t} + e^{t} \\ e^{2t} - e^{t} & -e^{3t} + e^{2t} & e^{t} + e^{3t} - e^{2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & -e^{3t}+e^{2t} & e^{3t}-e^{2t} \\ e^{2t}-e^{t} & e^{2t} & -e^{2t}+e^{t} \\ e^{2t}-e^{t} & -e^{3t}+e^{2t} & e^{t}+e^{3t}-e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 + (-e^{3t}+e^{2t})\, c_2 + (e^{3t}-e^{2t})\, c_3 \\ (e^{2t}-e^{t})\, c_1 + e^{2t}c_2 + (-e^{2t}+e^{t})\, c_3 \\ (e^{2t}-e^{t})\, c_1 + (-e^{3t}+e^{2t})\, c_2 + (e^{t}+e^{3t}-e^{2t})\, c_3 \end{bmatrix} \\
&= \begin{bmatrix} (c_2+c_1-c_3)\, e^{2t} - e^{3t}(c_2-c_3) \\ (c_2+c_1-c_3)\, e^{2t} - e^{t}(c_1-c_3) \\ (c_2+c_1-c_3)\, e^{2t} + (-c_2+c_3)\, e^{3t} - e^{t}(c_1-c_3) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 & 1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^t \\
&= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t
\end{aligned}
$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{3t}$$
$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{3t} \\ 0 \\ \mathrm{e}^{3t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_2 \mathrm{e}^{3t} + c_3 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^t + c_3 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^t + c_2 \mathrm{e}^{3t} + c_3 \mathrm{e}^{2t} \end{bmatrix}$$

29.7.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - y(t) + z(t), y'(t) = x(t) + 2y(t) - z(t), z'(t) = x(t) - y(t) + 2z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^t \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{2t} + c_3 e^{3t} \\ c_1 e^t + c_2 e^{2t} \\ c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = c_2 e^{2t} + c_3 e^{3t}, y(t) = c_1 e^t + c_2 e^{2t}, z(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}\}$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 52

```
dsolve([diff(x(t),t)=2*x(t)-y(t)+z(t),diff(y(t),t)=x(t)+2*y(t)-z(t),diff(z(t),t)=x(t)-y(t)+2
```

$$x(t) = c_2 e^{3t} + c_3 e^{2t}$$
$$y(t) = c_3 e^{2t} + c_1 e^t$$
$$z(t) = c_3 e^{2t} + c_2 e^{3t} + c_1 e^t$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 99

```
DSolve[{x'[t]==2*x[t]-y[t]+z[t],y'[t]==x[t]+2*y[t]-z[t],z'[t]==x[t]-y[t]+2*z[t]},{x[t],y[t],
```

$$x(t) \to e^{2t}(c_1 - (c_2 - c_3)(e^t - 1))$$
$$y(t) \to e^t(c_1(e^t - 1) + (c_2 - c_3)e^t + c_3)$$
$$z(t) \to e^t(c_1(e^t - 1) + (c_2 - c_3)e^t + (c_3 - c_2)e^{2t} + c_3)$$

29.8 problem 809

Internal problem ID [15532]
Internal file name [`OUTPUT/15532_Friday_May_10_2024_05_47_35_PM_79807819/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 22. Integration of homogeneous linear systems with constant coefficients. Eulers method. Exercises page 230
Problem number: 809.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) - y(t) + z(t) \\
y'(t) &= x(t) + z(t) \\
z'(t) &= y(t) - 2z(t) - 3x(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 0, y(0) = 0, z(0) = 1]
$$

29.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{-t} + e^{t} + 1 & -e^{t} + 1 & 1 - e^{-t} \\ 1 - e^{-t} & 1 & 1 - e^{-t} \\ -e^{t} - 1 + 2e^{-t} & e^{t} - 1 & -1 + 2e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} -e^{-t} + e^{t} + 1 & -e^{t} + 1 & 1 - e^{-t} \\ 1 - e^{-t} & 1 & 1 - e^{-t} \\ -e^{t} - 1 + 2e^{-t} & e^{t} - 1 & -1 + 2e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - e^{-t} \\ 1 - e^{-t} \\ -1 + 2e^{-t} \end{bmatrix} \end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

29.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ -3 & 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
0	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ -3 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & \frac{4}{3} & \frac{2}{3} & 0 \\ -3 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 & 1 \\ 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}, v_2 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -3 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -3 & 1 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{t} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} \\ -\frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -\mathrm{e}^{t} \\ 0 \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-t}}{2} - c_2 - c_3 \mathrm{e}^{t} \\ -\frac{c_1 \mathrm{e}^{-t}}{2} - c_2 \\ c_1 \mathrm{e}^{-t} + c_2 + c_3 \mathrm{e}^{t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 0 \\ y(0) = 0 \\ z(0) = 1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{2} - c_2 - c_3 \\ -\frac{c_1}{2} - c_2 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

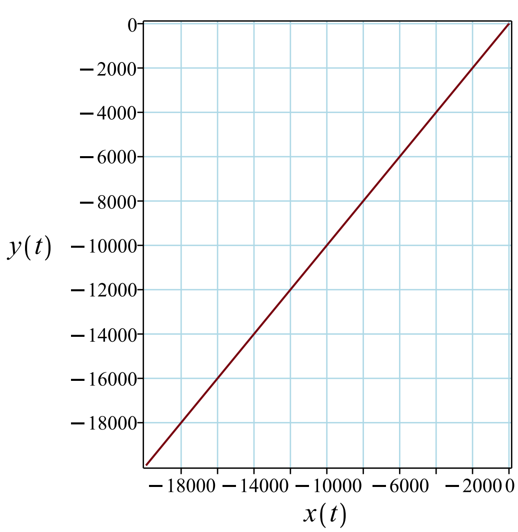
Solving for the constants of integrations gives

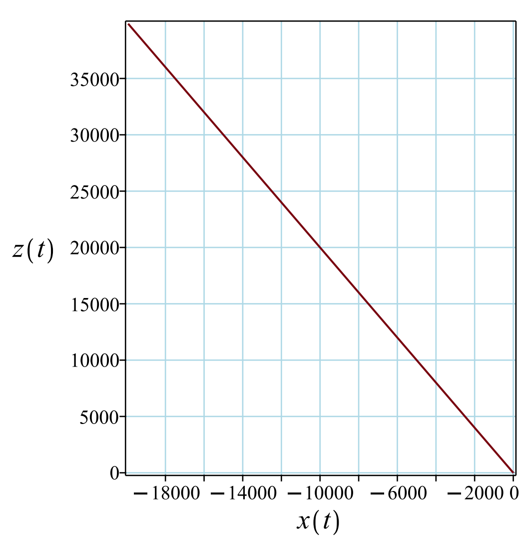
$$\begin{bmatrix} c_1 = 2 \\ c_2 = -1 \\ c_3 = 0 \end{bmatrix}$$

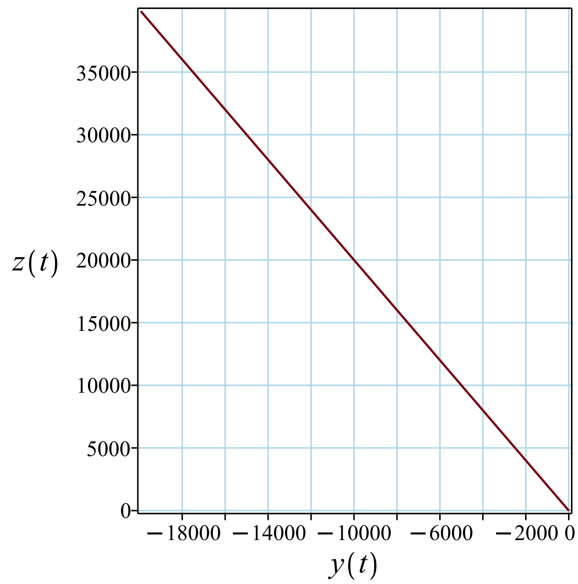
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 - \mathrm{e}^{-t} \\ 1 - \mathrm{e}^{-t} \\ -1 + 2\,\mathrm{e}^{-t} \end{bmatrix}$$

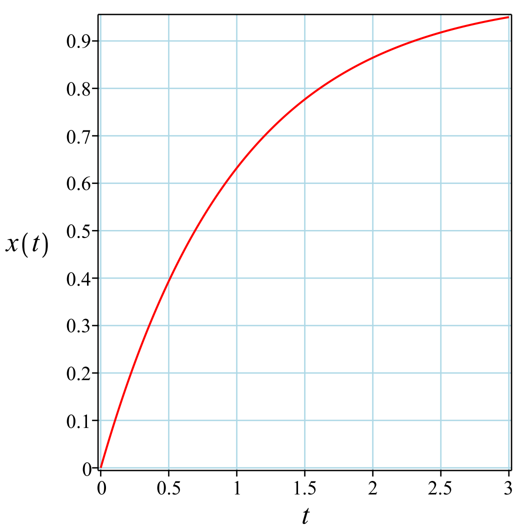
The following are plots of each solution against another.

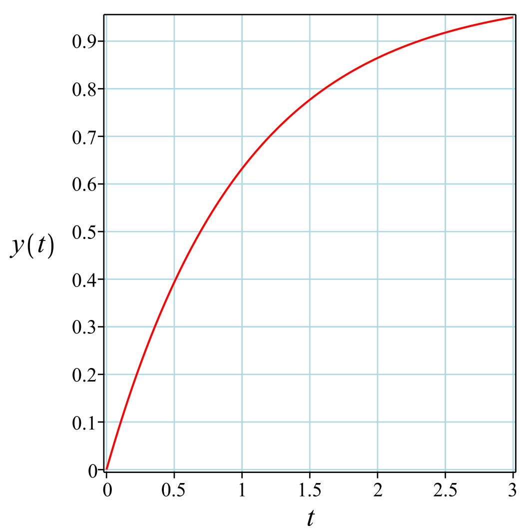


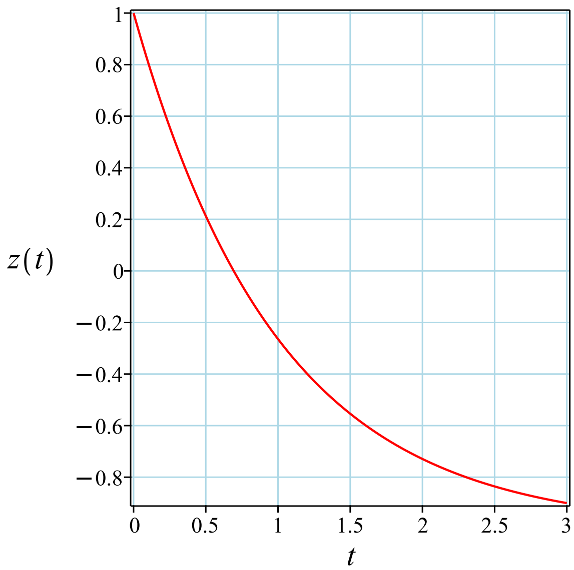




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve([diff(x(t),t) = 2*x(t)-y(t)+z(t), diff(y(t),t) = x(t)+z(t), diff(z(t),t) = y(t)-2*z(t
```

$$x(t) = 1 - e^{-t}$$
$$y(t) = 1 - e^{-t}$$
$$z(t) = 2\, e^{-t} - 1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 38

```
DSolve[{x'[t]==2*x[t]-y[t]+z[t],y'[t]==x[t]+z[t],z'[t]==y[t]-2*z[t]-3*x[t]},{x[0]==0,y[0]==0
```

$$x(t) \to 1 - e^{-t}$$
$$y(t) \to 1 - e^{-t}$$
$$z(t) \to 2e^{-t} - 1$$

30 Chapter 3 (Systems of differential equations). Section 23. Methods of integrating nonhomogeneous linear systems with constant coefficients. Exercises page 234

30.1 problem 810

Internal problem ID [15533]
Internal file name [`OUTPUT/15533_Friday_May_10_2024_08_58_46_PM_67136864/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23. Methods of integrating nonhomogeneous linear systems with constant coefficients. Exercises page 234
Problem number: 810.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -2x(t) + y(t) - e^{2t}$$
$$y'(t) = -3x(t) + 2y(t) + 6\,e^{2t}$$

30.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ 6\,e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{3\,e^{-t}}{2} - \frac{e^{t}}{2}\right) c_1 + \left(\frac{e^{t}}{2} - \frac{e^{-t}}{2}\right) c_2 \\ \left(-\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2}\right) c_1 + \left(-\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_2+3c_1)e^{-t}}{2} - \frac{e^{t}(c_1-c_2)}{2} \\ \frac{(-c_2+3c_1)e^{-t}}{2} - \frac{3\,e^{t}(c_1-c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} & -\frac{e^{t}}{2} + \frac{e^{-t}}{2} \\ \frac{3\,e^{t}}{2} - \frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} -e^{2t} \\ 6\,e^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^{t}}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^{t}}{2} \end{bmatrix} \begin{bmatrix} -\frac{3\,e^{3t}}{2} + \frac{7\,e^{t}}{2} \\ -\frac{3\,e^{3t}}{2} + \frac{21\,e^{t}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2\,e^{2t} \\ 9\,e^{2t} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(-c_2+3c_1)e^{-t}}{2} + 2\,e^{2t} + \frac{(-c_1+c_2)e^{t}}{2} \\ \frac{(-c_2+3c_1)e^{-t}}{2} + 9\,e^{2t} + \frac{(-3c_1+3c_2)e^{t}}{2} \end{bmatrix}
\end{aligned}
$$

30.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ 6\,e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ -3 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ -3 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{t} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{t}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{t}}{3} \\ e^{t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{e^{t}}{3} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{3 e^{-t}}{2} & \frac{3 e^{-t}}{2} \\ \frac{3 e^{t}}{2} & -\frac{e^{t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^t}{3} & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} -\frac{3\,e^{-t}}{2} & \frac{3\,e^{-t}}{2} \\ \frac{3\,e^t}{2} & -\frac{e^t}{2} \end{bmatrix} \begin{bmatrix} -e^{2t} \\ 6\,e^{2t} \end{bmatrix} dt$$
$$= \begin{bmatrix} \frac{e^t}{3} & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{21\,e^t}{2} \\ -\frac{9\,e^{3t}}{2} \end{bmatrix} dt$$
$$= \begin{bmatrix} \frac{e^t}{3} & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{21\,e^t}{2} \\ -\frac{3\,e^{3t}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 2\,e^{2t} \\ 9\,e^{2t} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^t}{3} \\ c_1 e^t \end{bmatrix} + \begin{bmatrix} c_2 e^{-t} \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 2\,e^{2t} \\ 9\,e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^t}{3} + c_2 e^{-t} + 2\,e^{2t} \\ c_1 e^t + c_2 e^{-t} + 9\,e^{2t} \end{bmatrix}$$

30.1.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -2x(t) + y(t) - (e^t)^2, y'(t) = -3x(t) + 2y(t) + 6(e^t)^2\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -(e^t)^2 \\ 6(e^t)^2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -(\mathrm{e}^t)^2 \\ 6(\mathrm{e}^t)^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -(\mathrm{e}^t)^2 \\ 6(\mathrm{e}^t)^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^{-t} & \frac{e^t}{3} \\ e^{-t} & e^t \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-t} & \frac{e^t}{3} \\ e^{-t} & e^t \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{3} \\ 1 & 1 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{3\,e^{-t}}{2} - \frac{e^t}{2} & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ -\frac{3\,e^t}{2} + \frac{3\,e^{-t}}{2} & -\frac{e^{-t}}{2} + \frac{3\,e^t}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

$\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

$\vec{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{3\,e^{-t}}{2} + 2\,e^{2t} - \frac{7\,e^{t}}{2} \\ -\frac{21\,e^{t}}{2} + 9\,e^{2t} + \frac{3\,e^{-t}}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} \frac{3\,e^{-t}}{2} + 2\,e^{2t} - \frac{7\,e^{t}}{2} \\ -\frac{21\,e^{t}}{2} + 9\,e^{2t} + \frac{3\,e^{-t}}{2} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + \frac{c_2 e^{t}}{3} + \frac{3\,e^{-t}}{2} + 2\,e^{2t} - \frac{7\,e^{t}}{2} \\ c_1 e^{-t} + c_2 e^{t} - \frac{21\,e^{t}}{2} + 9\,e^{2t} + \frac{3\,e^{-t}}{2} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = c_1 e^{-t} + \frac{c_2 e^{t}}{3} + \frac{3\,e^{-t}}{2} + 2\,e^{2t} - \frac{7\,e^{t}}{2}, y(t) = c_1 e^{-t} + c_2 e^{t} - \frac{21\,e^{t}}{2} + 9\,e^{2t} + \frac{3\,e^{-t}}{2}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 43

```
dsolve([diff(x(t),t)+2*x(t)-y(t)=-exp(2*t),diff(y(t),t)+3*x(t)-2*y(t)=6*exp(2*t)],singsol=al
```

$$x(t) = c_2 e^{t} + e^{-t} c_1 + 2\,e^{2t}$$
$$y(t) = 3c_2 e^{t} + e^{-t} c_1 + 9\,e^{2t}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 85

```
DSolve[{x'[t]+2*x[t]-y[t]==-Exp[2*t],y'[t]+3*x[t]-2*y[t]==6*Exp[2*t]},{x[t],y[t]},t,IncludeS
```

$$x(t) \to \frac{1}{2} e^{-t} \left(4e^{3t} + (c_2 - c_1) e^{2t} + 3c_1 - c_2\right)$$
$$y(t) \to \frac{1}{2} e^{-t} \left(18e^{3t} - 3(c_1 - c_2) e^{2t} + 3c_1 - c_2\right)$$

30.2 problem 811

Internal problem ID [15534]
Internal file name [`OUTPUT/15534_Friday_May_10_2024_09_47_50_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23. Methods of integrating nonhomogeneous linear systems with constant coefficients. Exercises page 234
Problem number: 811.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + y(t) - \cos(t) \\
y'(t) &= -y(t) - 2x(t) + \cos(t) + \sin(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = -2]
$$

30.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t) - \sin(t) \\ -2\cos(t) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix} \begin{bmatrix} -\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix} \\
&= \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix} \begin{bmatrix} -t \\ t - 1 \end{bmatrix} \\
&= \begin{bmatrix} -\sin(t) - t\cos(t) \\ (t-1)\cos(t) + (t+1)\sin(t) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \cos(t) - 2\sin(t) - t\cos(t) \\ (t-3)\cos(t) + (t+1)\sin(t) \end{bmatrix}$$

30.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -\cos(t) \\ \cos(t) + \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & 1 & 0 \\ -2 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} 1-i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} i e^{-it} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ -i e^{it} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} i e^{-it} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ -i e^{it} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{it} \end{bmatrix} \begin{bmatrix} -\cos(t) \\ \cos(t)+\sin(t) \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-it}((-1+i)\cos(t)+(-1-i)\sin(t))}{2} \\ \frac{((1+i)\cos(t)+(1-i)\sin(t))e^{it}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} -\frac{e^{-it}((-1+i)\cos(t)+(-1-i)\sin(t))t}{2} \\ \frac{((1+i)\cos(t)+(1-i)\sin(t))e^{it}t}{2} \end{bmatrix} \\
&= \begin{bmatrix} -t\cos(t) \\ (\cos(t)+\sin(t))\, t \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{it} \\ c_1 \mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{-it} \\ c_2 \mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} -t\cos(t) \\ (\cos(t)+\sin(t))\, t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{((-1+i)c_2-t)\mathrm{e}^{-it}}{2} - \frac{((1+i)c_1+t)\mathrm{e}^{it}}{2} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} + (\cos(t)+\sin(t))\, t \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = -2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

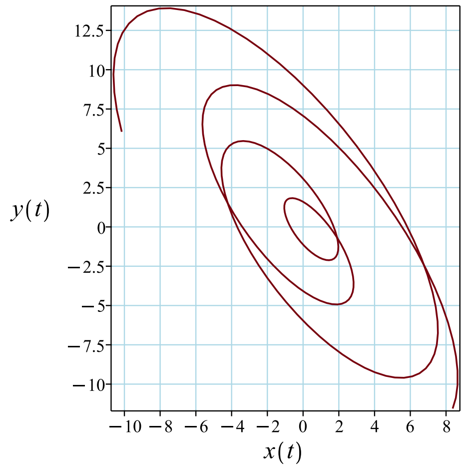
$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_1 + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -1 \\ c_2 = -1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(1-i-t)\mathrm{e}^{-it}}{2} - \frac{(-1-i+t)\mathrm{e}^{it}}{2} \\ -\mathrm{e}^{it} - \mathrm{e}^{-it} + (\cos(t)+\sin(t))\, t \end{bmatrix}$$



The following are plots of each solution.

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 33

```
dsolve([diff(x(t),t) = x(t)+y(t)-cos(t), diff(y(t),t) = -y(t)-2*x(t)+cos(t)+sin(t), x(0) = 1
```

$$x(t) = -\sin(t) + \cos(t) - \cos(t)\, t$$
$$y(t) = -2\cos(t) + \sin(t)\, t + \cos(t)\, t$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 31

```
DSolve[{x'[t]==x[t]+y[t]-Cos[t],y'[t]==-y[t]-2*x[t]+Cos[t]+Sin[t]},{x[0]==1,y[0]==-2},{x[t],
```

$$x(t) \to -\sin(t) - t\cos(t) + \cos(t)$$
$$y(t) \to t\sin(t) + (t-2)\cos(t)$$

30.3 problem 812

Internal problem ID [15535]
Internal file name [`OUTPUT/15535_Friday_May_10_2024_09_47_51_PM_45960389/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23. Methods of integrating nonhomogeneous linear systems with constant coefficients. Exercises page 234
Problem number: 812.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= y(t) + \tan(t)^2 - 1 \\
y'(t) &= \tan(t) - x(t)
\end{aligned}
$$

30.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \tan(t)^2 - 1 \\ \tan(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 \\ -\sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \tan(t)^2 - 1 \\ \tan(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} -\sin(t) \\ \cos(t) + \sec(t) \end{bmatrix} \\ &= \begin{bmatrix} \tan(t) \\ 2 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 + \tan(t) \\ -\sin(t)\, c_1 + \cos(t)\, c_2 + 2 \end{bmatrix} \end{aligned}$$

30.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \tan(t)^2 - 1 \\ \tan(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -i e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} i e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{i\mathrm{e}^{-it}}{2} & \frac{\mathrm{e}^{-it}}{2} \\ -\frac{i\mathrm{e}^{it}}{2} & \frac{\mathrm{e}^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{i\mathrm{e}^{-it}}{2} & \frac{\mathrm{e}^{-it}}{2} \\ -\frac{i\mathrm{e}^{it}}{2} & \frac{\mathrm{e}^{it}}{2} \end{bmatrix} \begin{bmatrix} \tan(t)^2 - 1 \\ \tan(t) \end{bmatrix} dt \\
&= \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{-it}\left(i\sec(t)^2 - 2i + \tan(t)\right)}{2} \\ -\frac{\mathrm{e}^{it}\left(i\sec(t)^2 - 2i - \tan(t)\right)}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{-it}(i\tan(t)+2)}{2} \\ -\frac{\mathrm{e}^{it}(i\tan(t)-2)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \tan(t) \\ 2 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -ic_1\mathrm{e}^{it} \\ c_1\mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} ic_2\mathrm{e}^{-it} \\ c_2\mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} \tan(t) \\ 2 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -ic_1\mathrm{e}^{it} + ic_2\mathrm{e}^{-it} + \tan(t) \\ c_1\mathrm{e}^{it} + c_2\mathrm{e}^{-it} + 2 \end{bmatrix}$$

30.3.3 Maple step by step solution

Let's solve

$\left[x'(t) = y(t) + \tan(t)^2 - 1, y'(t) = \tan(t) - x(t)\right]$

- Define vector

$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]$

- Convert system into a vector equation

$\vec{x}'(t) = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} \tan(t)^2 - 1 \\ \tan(t) \end{array}\right]$

- System to solve

$\vec{x}'(t) = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} \tan(t)^2 - 1 \\ \tan(t) \end{array}\right]$

- Define the forcing function

$\vec{f}(t) = \left[\begin{array}{c} \tan(t)^2 - 1 \\ \tan(t) \end{array}\right]$

- Define the coefficient matrix

$A = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$\left[\left[-\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$\left[-\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ 1 \end{array}\right]\right]$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} \sin(t)\left(-\cos(t) + \int_0^t \sin(s)\tan(s)^2\, ds\right) \\ \sin(t)^2 + \cos(t)\left(\int_0^t \sin(s)\tan(s)^2\, ds\right) \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \left[\begin{array}{c} \sin(t)\left(-\cos(t) + \int_0^t \sin(s)\tan(s)^2\, ds\right) \\ \sin(t)^2 + \cos(t)\left(\int_0^t \sin(s)\tan(s)^2\, ds\right) \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \sin(t)\left(\int_0^t \sin(s)\tan(s)^2\, ds\right) + (c_1 - \cos(t))\sin(t) + c_2\cos(t) \\ \sin(t)^2 + \cos(t)\left(\int_0^t \sin(s)\tan(s)^2\, ds\right) - c_2\sin(t) + c_1\cos(t) \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = \sin(t)\left(\int_0^t \sin(s)\tan(s)^2 ds\right) + (c_1 - \cos(t))\sin(t) + c_2\cos(t), y(t) = \sin(t)^2 + \cos(t)\right.$$

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 30

```
dsolve([diff(x(t),t)=y(t)+tan(t)^2-1,diff(y(t),t)=tan(t)-x(t)],singsol=all)
```

$$x(t) = c_2\sin(t) + c_1\cos(t) + \tan(t)$$
$$y(t) = c_2\cos(t) - c_1\sin(t) + 2$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 34

```
DSolve[{x'[t]==y[t]+Tan[t]^2-1,y'[t]==Tan[t]-x[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \tan(t) + c_1\cos(t) + c_2\sin(t)$$
$$y(t) \to c_2\cos(t) - c_1\sin(t) + 2$$

30.4 problem 813

Internal problem ID [15536]
Internal file name [`OUTPUT/15536_Friday_May_10_2024_09_47_53_PM_96363/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23. Methods of integrating nonhomogeneous linear systems with constant coefficients. Exercises page 234
Problem number: 813.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
x'(t) &= -\frac{4x(t)\,\mathrm{e}^t}{\mathrm{e}^t-1} - \frac{2y(t)\,\mathrm{e}^t}{\mathrm{e}^t-1} + \frac{4x(t)}{\mathrm{e}^t-1} + \frac{2y(t)}{\mathrm{e}^t-1} + \frac{2}{\mathrm{e}^t-1} \\
y'(t) &= \frac{6x(t)\,\mathrm{e}^t}{\mathrm{e}^t-1} + \frac{3y(t)\,\mathrm{e}^t}{\mathrm{e}^t-1} - \frac{6x(t)}{\mathrm{e}^t-1} - \frac{3y(t)}{\mathrm{e}^t-1} - \frac{3}{\mathrm{e}^t-1}
\end{aligned}
$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✓ Solution by Maple
Time used: 0.094 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=-4*x(t)-2*y(t)+2/(exp(t)-1),diff(y(t),t)=6*x(t)+3*y(t)-3/(exp(t)-1)],si
```

$$
\begin{aligned}
x(t) &= 2\,\mathrm{e}^{-t}\ln\left(\mathrm{e}^t-1\right) - \mathrm{e}^{-t}c_1 + 2\,\mathrm{e}^{-t} + c_2 \\
y(t) &= \frac{6\,\mathrm{e}^{-t}\ln\left(\mathrm{e}^t-1\right) - 4c_2\mathrm{e}^t - 3\,\mathrm{e}^{-t}c_1 - 6\ln\left(\mathrm{e}^t-1\right) + 6\,\mathrm{e}^{-t} + 3c_1 + 4c_2 - 6}{2\,\mathrm{e}^t-2}
\end{aligned}
$$

✓ Solution by Mathematica
Time used: 0.023 (sec). Leaf size: 76

```
DSolve[{x'[t]==-4*x[t]-2*y[t]+2/(Exp[t]-1),y'[t]==6*x[t]+3*y[t]-3/(Exp[t]-1)},{x[t],y[t]},t,
```

$$
\begin{aligned}
x(t) &\to e^{-t}\left(2\log\left(e^t-1\right) + c_1\left(4-3e^t\right) - 2c_2\left(e^t-1\right)\right) \\
y(t) &\to e^{-t}\left(-3\log\left(e^t-1\right) + 6c_1\left(e^t-1\right) + c_2\left(4e^t-3\right)\right)
\end{aligned}
$$

30.5 problem 814

Internal problem ID [15537]
Internal file name [`OUTPUT/15537_Friday_May_10_2024_09_47_53_PM_16935513/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23. Methods of integrating nonhomogeneous linear systems with constant coefficients. Exercises page 234
Problem number: 814.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = y(t)$$
$$y'(t) = -x(t) + \frac{1}{\cos(t)}$$

30.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\cos(t)} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 \\ -\sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\cos(t)} \end{bmatrix} dt \\
&= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \ln(\cos(t)) \\ t \end{bmatrix} \\
&= \begin{bmatrix} \cos(t)\ln(\cos(t)) + \sin(t)\, t \\ -\sin(t)\ln(\cos(t)) + t\cos(t) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \cos(t)\ln(\cos(t)) + \cos(t)\, c_1 + \sin(t)\,(t + c_2) \\ -\sin(t)\ln(\cos(t)) + (t + c_2)\cos(t) - \sin(t)\, c_1 \end{bmatrix}
\end{aligned}$$

30.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\cos(t)} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -i e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} i e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ -\frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ -\frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\cos(t)} \end{bmatrix} dt \\
&= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-it} \sec(t)}{2} \\ \frac{e^{it} \sec(t)}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} -\frac{i(2\ln(e^{it}) - \ln(e^{2it}+1))}{2} \\ -\frac{i \ln(e^{2it}+1)}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{(e^{-it}+e^{it}) \ln(e^{2it}+1)}{2} - e^{it} \ln(e^{it}) \\ -\frac{i(-e^{it}+e^{-it}) \ln(e^{2it}+1)}{2} - i e^{it} \ln(e^{it}) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -i c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} i c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} \frac{(e^{-it}+e^{it}) \ln(e^{2it}+1)}{2} - e^{it} \ln(e^{it}) \\ -\frac{i(-e^{it}+e^{-it}) \ln(e^{2it}+1)}{2} - i e^{it} \ln(e^{it}) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(e^{-it}+e^{it}) \ln(e^{2it}+1)}{2} + i c_2 e^{-it} - e^{it}(i c_1 + \ln(e^{it})) \\ \frac{i(e^{it}-e^{-it}) \ln(e^{2it}+1)}{2} + c_2 e^{-it} + (c_1 - i \ln(e^{it}))\, e^{it} \end{bmatrix}$$

30.5.3 Maple step by step solution

Let's solve

$\left[x'(t)=y(t), y'(t)=-x(t)+\frac{1}{\cos(t)}\right]$

- Define vector

$$\vec{x}(t)=\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t)=\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]\cdot\vec{x}(t)+\left[\begin{array}{c} 0 \\ -\frac{x(t)\cos(t)-1}{\cos(t)}+x(t) \end{array}\right]$$

- System to solve

$$\vec{x}'(t)=\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]\cdot\vec{x}(t)$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t)=A\cdot\vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_2\cos(t) + c_1\sin(t) \\ -c_2\sin(t) + c_1\cos(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2\cos(t) + c_1\sin(t) \\ -c_2\sin(t) + c_1\cos(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = c_2\cos(t) + c_1\sin(t), y(t) = -c_2\sin(t) + c_1\cos(t)\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)=y(t),diff(y(t),t)=-x(t)+1/cos(t)],singsol=all)
```

$$x(t) = c_2\sin(t) + c_1\cos(t) + \sin(t)\,t + \cos(t)\ln(\cos(t))$$
$$y(t) = c_2\cos(t) - c_1\sin(t) + \cos(t)\,t - \sin(t)\ln(\cos(t))$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 43

```
DSolve[{x'[t]==y[t],y'[t]==-x[t]+1/Cos[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to (t + c_2) \sin(t) + \cos(t)(\log(\cos(t)) + c_1)$$
$$y(t) \to (t + c_2) \cos(t) - \sin(t)(\log(\cos(t)) + c_1)$$

31 Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239

31.1 problem 815

Internal problem ID [15538]
Internal file name [`OUTPUT/15538_Friday_May_10_2024_09_47_54_PM_49443378/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 815.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = y(t)$$
$$y'(t) = 1 - x(t)$$

31.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 \\ -\sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \cos(t)\, c_1 + \sin(t)\, c_2 + 1 \\ -\sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix} \end{aligned}$$

31.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -i\mathrm{e}^{it} & i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ -\frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ -\frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt \\ &= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-it}}{2} \\ \frac{e^{it}}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} -i e^{it} & i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \frac{i e^{-it}}{2} \\ -\frac{i e^{it}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} -i c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} i c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -i c_1 e^{it} + i c_2 e^{-it} + 1 \\ c_1 e^{it} + c_2 e^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

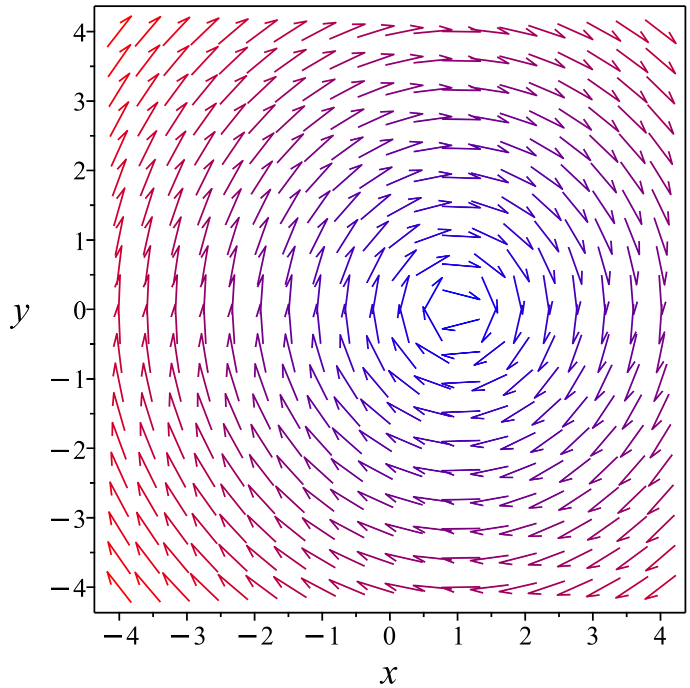


Figure 857: Phase plot

31.1.3 Maple step by step solution

Let's solve

$[x'(t) = y(t), y'(t) = 1 - x(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \Phi(t)^{-1} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$\vec{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} 1 - \cos(t) \\ \sin(t) \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \begin{bmatrix} 1 - \cos(t) \\ \sin(t) \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (c_2 - 1)\cos(t) + c_1 \sin(t) + 1 \\ \sin(t) - c_2 \sin(t) + c_1 \cos(t) \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = (c_2 - 1)\cos(t) + c_1 \sin(t) + 1, y(t) = \sin(t) - c_2 \sin(t) + c_1 \cos(t)\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve([diff(x(t),t)=y(t),diff(y(t),t)=1-x(t)],singsol=all)
```

$$x(t) = c_2 \sin(t) + c_1 \cos(t) + 1$$
$$y(t) = c_2 \cos(t) - c_1 \sin(t)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 32

```
DSolve[{x'[t]==y[t],y'[t]==1-x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to c_1 \cos(t) + c_2 \sin(t) + 1$$
$$y(t) \to c_2 \cos(t) - c_1 \sin(t)$$

31.2 problem 816

Internal problem ID [15539]
Internal file name [`OUTPUT/15539_Friday_May_10_2024_09_47_55_PM_68452352/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 816.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3 - 2y(t)$$
$$y'(t) = 2x(t) - 2t$$

31.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 3 \\ -2t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(2t)\, c_1 - \sin(2t)\, c_2 \\ \sin(2t)\, c_1 + \cos(2t)\, c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \int \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} \sin(2t) + \cos(2t)\, t \\ \cos(2t) - \sin(2t)\, t \end{bmatrix} \\ &= \begin{bmatrix} t \\ 1 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \cos(2t)\, c_1 - \sin(2t)\, c_2 + t \\ \sin(2t)\, c_1 + \cos(2t)\, c_2 + 1 \end{bmatrix} \end{aligned}$$

31.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 3 \\ -2t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -2 \\ 2 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2i$	1	complex eigenvalue
$2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - (-2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - (2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i e^{2it} \\ e^{2it} \end{bmatrix} + c_2 \begin{bmatrix} -i e^{-2it} \\ e^{-2it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} i e^{2it} & -i e^{-2it} \\ e^{2it} & e^{-2it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i\mathrm{e}^{-2it}}{2} & \frac{\mathrm{e}^{-2it}}{2} \\ \frac{i\mathrm{e}^{2it}}{2} & \frac{\mathrm{e}^{2it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} i\mathrm{e}^{2it} & -i\mathrm{e}^{-2it} \\ \mathrm{e}^{2it} & \mathrm{e}^{-2it} \end{bmatrix} \int \begin{bmatrix} -\frac{i\mathrm{e}^{-2it}}{2} & \frac{\mathrm{e}^{-2it}}{2} \\ \frac{i\mathrm{e}^{2it}}{2} & \frac{\mathrm{e}^{2it}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt \\
&= \begin{bmatrix} i\mathrm{e}^{2it} & -i\mathrm{e}^{-2it} \\ \mathrm{e}^{2it} & \mathrm{e}^{-2it} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{-2it}(3i+2t)}{2} \\ \mathrm{e}^{2it}\left(\frac{3i}{2} - t\right) \end{bmatrix} dt \\
&= \begin{bmatrix} i\mathrm{e}^{2it} & -i\mathrm{e}^{-2it} \\ \mathrm{e}^{2it} & \mathrm{e}^{-2it} \end{bmatrix} \begin{bmatrix} -\frac{\mathrm{e}^{-2it}(it-1)}{2} \\ \frac{\mathrm{e}^{2it}(it+1)}{2} \end{bmatrix} \\
&= \begin{bmatrix} t \\ 1 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} ic_1\mathrm{e}^{2it} \\ c_1\mathrm{e}^{2it} \end{bmatrix} + \begin{bmatrix} -ic_2\mathrm{e}^{-2it} \\ c_2\mathrm{e}^{-2it} \end{bmatrix} + \begin{bmatrix} t \\ 1 \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ic_1\mathrm{e}^{2it} - ic_2\mathrm{e}^{-2it} + t \\ c_1\mathrm{e}^{2it} + c_2\mathrm{e}^{-2it} + 1 \end{bmatrix}$$

31.2.3 Maple step by step solution

Let's solve

$[x'(t) = 3 - 2y(t), y'(t) = 2x(t) - 2t]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3 \\ -2t \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3 \\ -2t \end{bmatrix}$

- Define the forcing function

$\vec{f}(t) = \begin{bmatrix} 3 \\ -2t \end{bmatrix}$

- Define the coefficient matrix

$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$\left[\left[-2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$\left[-2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$

- Solution from eigenpair

  $e^{-2\mathrm{I}t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$

- Use Euler identity to write solution in terms of sin and cos

  $(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$

- Simplify expression

  $\begin{bmatrix} -\mathrm{I}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$

- Both real and imaginary parts are solutions to the homogeneous system

  $\left[\vec{x}_1(t) = \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} -\sin(2t) & -\cos(2t) \\ \cos(2t) & -\sin(2t) \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} -\sin(2t) & -\cos(2t) \\ \cos(2t) & -\sin(2t) \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} t + \sin(2t) \\ 1 - \cos(2t) \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t) + \left[\begin{array}{c} t + \sin(2t) \\ 1 - \cos(2t) \end{array}\right]$$
- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} t + \sin(2t) - c_2 \cos(2t) - c_1 \sin(2t) \\ (c_1 - 1)\cos(2t) - c_2 \sin(2t) + 1 \end{array}\right]$$
- Solution to the system of ODEs

  $\{x(t) = t + \sin(2t) - c_2 \cos(2t) - c_1 \sin(2t), y(t) = (c_1 - 1)\cos(2t) - c_2 \sin(2t) + 1\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)=3-2*y(t),diff(y(t),t)=2*x(t)-2*t],singsol=all)
```

$$x(t) = c_2 \sin(2t) + c_1 \cos(2t) + t$$
$$y(t) = -c_2 \cos(2t) + c_1 \sin(2t) + 1$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 41

```
DSolve[{x'[t]==3-2*y[t],y'[t]==2*x[t]-2*t},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to t + c_1 \cos(2t) - c_2 \sin(2t)$$
$$y(t) \to c_2 \cos(2t) + c_1 \sin(2t) + 1$$

31.3 problem 817

Internal problem ID [15540]
Internal file name [`OUTPUT/15540_Friday_May_10_2024_09_47_57_PM_84437884/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 817.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -y(t) + \sin(t)$$
$$y'(t) = x(t) + \cos(t)$$

31.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)\, c_1 - \sin(t)\, c_2 \\ \sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \sin(t)^2 \\ \frac{\sin(2t)}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \cos(t)\, c_1 - \sin(t)\, c_2 \\ \sin(t)\, c_1 + \cos(t)\, c_2 + \sin(t) \end{bmatrix} \end{aligned}$$

31.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} -i e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ \frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ \frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} dt \\
&= \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-it}(i\sin(t)-\cos(t))}{2} \\ \frac{e^{it}(i\sin(t)+\cos(t))}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \frac{e^{-it}\sin(t)}{2} \\ \frac{e^{it}\sin(t)}{2} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} i c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} -i c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} i(c_1 e^{it} - c_2 e^{-it}) \\ c_1 e^{it} + c_2 e^{-it} + \sin(t) \end{bmatrix}$$

31.3.3 Maple step by step solution

Let's solve

$[x'(t) = -y(t) + \sin(t), y'(t) = x(t) + \cos(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos(t) - \mathrm{I}\sin(t)\right) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} 0 \\ \sin(t) \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \left[\begin{array}{c} 0 \\ \sin(t) \end{array}\right]$$
- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -c_2\cos(t) - c_1\sin(t) \\ \sin(t) - c_2\sin(t) + c_1\cos(t) \end{array}\right]$$
- Solution to the system of ODEs

  $\{x(t) = -c_2\cos(t) - c_1\sin(t), y(t) = \sin(t) - c_2\sin(t) + c_1\cos(t)\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([diff(x(t),t)=-y(t)+sin(t),diff(y(t),t)=x(t)+cos(t)],singsol=all)
```

$$x(t) = c_1 \sin(t) + c_2 \cos(t)$$
$$y(t) = -c_1 \cos(t) + c_2 \sin(t) + \sin(t)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 41

```
DSolve[{x'[t]==-y[t]+Sin[t],y'[t]==x[t]+Cos[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \left(-\frac{1}{2} + c_1\right)\cos(t) - c_2 \sin(t)$$
$$y(t) \to \frac{\sin(t)}{2} + c_2 \cos(t) + c_1 \sin(t)$$

31.4 problem 818

Internal problem ID [15541]
Internal file name [`OUTPUT/15541_Friday_May_10_2024_09_47_58_PM_51749197/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 818.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + y(t) + \mathrm{e}^{t} \\
y'(t) &= x(t) + y(t) - \mathrm{e}^{t}
\end{aligned}
$$

31.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{2}+\frac{e^{2t}}{2}\right)c_1+\left(\frac{e^{2t}}{2}-\frac{1}{2}\right)c_2 \\ \left(\frac{e^{2t}}{2}-\frac{1}{2}\right)c_1+\left(\frac{1}{2}+\frac{e^{2t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1+c_2)e^{2t}}{2}+\frac{c_1}{2}-\frac{c_2}{2} \\ \frac{(c_1+c_2)e^{2t}}{2}-\frac{c_1}{2}+\frac{c_2}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{e^{-2t}}{2} \\ -\frac{1}{2}+\frac{e^{-2t}}{2} & \frac{e^{-2t}}{2}+\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2t}}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{e^{-2t}}{2} \\ -\frac{1}{2}+\frac{e^{-2t}}{2} & \frac{e^{-2t}}{2}+\frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{1}{2}+\frac{e^{2t}}{2} & \frac{e^{2t}}{2}-\frac{1}{2} \\ \frac{e^{2t}}{2}-\frac{1}{2} & \frac{1}{2}+\frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} \\
&= \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(c_1+c_2)\mathrm{e}^{2t}}{2} + \frac{c_1}{2} - \frac{c_2}{2} + \mathrm{e}^t \\ \frac{(c_1+c_2)\mathrm{e}^{2t}}{2} - \frac{c_1}{2} + \frac{c_2}{2} - \mathrm{e}^t \end{bmatrix}
\end{aligned}
$$

31.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ -\mathrm{e}^t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 2\lambda = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^0 \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} & -1 \\ \mathrm{e}^{2t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\mathrm{e}^{-2t}}{2} & \frac{\mathrm{e}^{-2t}}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{2t} & -1 \\ e^{2t} & 1 \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{2t} & -1 \\ e^{2t} & 1 \end{bmatrix} \int \begin{bmatrix} 0 \\ -e^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{2t} & -1 \\ e^{2t} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} - c_2 + e^{t} \\ c_1 e^{2t} + c_2 - e^{t} \end{bmatrix}$$

31.4.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + y(t) + e^{t}, y'(t) = x(t) + y(t) - e^{t}]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -1 & e^{2t} \\ 1 & e^{2t} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} - \frac{1}{2} \\ \frac{e^{2t}}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

$\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

$\vec{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\vec{x}_p(t) = \begin{bmatrix} e^t - 1 \\ 1 - e^t \end{bmatrix}$

- Plug particular solution back into general solution

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} e^t - 1 \\ 1 - e^t \end{bmatrix}$

- Substitute in vector of dependent variables

  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2e^{2t} + e^t - 1 - c_1 \\ c_2e^{2t} + 1 - e^t + c_1 \end{bmatrix}$

- Solution to the system of ODEs

  $\{x(t) = c_2e^{2t} + e^t - 1 - c_1, y(t) = c_2e^{2t} + 1 - e^t + c_1\}$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 34

```
dsolve([diff(x(t),t)=x(t)+y(t)+exp(t),diff(y(t),t)=x(t)+y(t)-exp(t)],singsol=all)
```

$$x(t) = \frac{c_1e^{2t}}{2} + e^t + c_2$$
$$y(t) = \frac{c_1e^{2t}}{2} - e^t - c_2$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 62

```
DSolve[{x'[t]==x[t]+y[t]+Exp[t],y'[t]==x[t]+y[t]-Exp[t]},{x[t],y[t]},t,IncludeSingularSoluti
```

$$
\begin{aligned}
x(t) &\to \frac{1}{2}\left(2e^t + (c_1 + c_2)e^{2t} + c_1 - c_2\right) \\
y(t) &\to \frac{1}{2}\left(-2e^t + (c_1 + c_2)e^{2t} - c_1 + c_2\right)
\end{aligned}
$$

31.5 problem 819

Internal problem ID [15542]
Internal file name [`OUTPUT/15542_Friday_May_10_2024_09_47_59_PM_11103093/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 819.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4x(t) - 5y(t) + 4t - 1 \\
y'(t) &= x(t) - 2y(t) + t
\end{aligned}
$$

With initial conditions

$$
[x(0) = 0, y(0) = 0]
$$

31.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4t - 1 \\ t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-t}}{4}+\frac{5\,e^{3t}}{4} & -\frac{5\,e^{3t}}{4}+\frac{5\,e^{-t}}{4} \\ \frac{e^{3t}}{4}-\frac{e^{-t}}{4} & \frac{5\,e^{-t}}{4}-\frac{e^{3t}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} -\frac{e^{-t}}{4}+\frac{5\,e^{3t}}{4} & -\frac{5\,e^{3t}}{4}+\frac{5\,e^{-t}}{4} \\ \frac{e^{3t}}{4}-\frac{e^{-t}}{4} & \frac{5\,e^{-t}}{4}-\frac{e^{3t}}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} -\frac{(e^{4t}-5)e^{-3t}}{4} & \frac{5(e^{4t}-1)e^{-3t}}{4} \\ -\frac{(e^{4t}-1)e^{-3t}}{4} & \frac{(5\,e^{4t}-1)e^{-3t}}{4} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-t}}{4}+\frac{5\,e^{3t}}{4} & -\frac{5\,e^{3t}}{4}+\frac{5\,e^{-t}}{4} \\ \frac{e^{3t}}{4}-\frac{e^{-t}}{4} & \frac{5\,e^{-t}}{4}-\frac{e^{3t}}{4} \end{bmatrix} \int \begin{bmatrix} -\frac{(e^{4t}-5)e^{-3t}}{4} & \frac{5(e^{4t}-1)e^{-3t}}{4} \\ -\frac{(e^{4t}-1)e^{-3t}}{4} & \frac{(5\,e^{4t}-1)e^{-3t}}{4} \end{bmatrix} \begin{bmatrix} 4t-1 \\ t \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{e^{-t}}{4}+\frac{5\,e^{3t}}{4} & -\frac{5\,e^{3t}}{4}+\frac{5\,e^{-t}}{4} \\ \frac{e^{3t}}{4}-\frac{e^{-t}}{4} & \frac{5\,e^{-t}}{4}-\frac{e^{3t}}{4} \end{bmatrix} \begin{bmatrix} \frac{t(e^{4t}-5)e^{-3t}}{4} \\ \frac{(e^{4t}-1)e^{-3t}t}{4} \end{bmatrix} \\ &= \begin{bmatrix} -t \\ 0 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} -t \\ 0 \end{bmatrix}$$

31.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4t-1 \\ t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 5 & -5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 5 & -5 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 1 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 5t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 5t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{3t}$$
$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{-t} & \mathrm{e}^{3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{t}}{4} & \frac{5\,\mathrm{e}^{t}}{4} \\ \frac{\mathrm{e}^{-3t}}{4} & -\frac{\mathrm{e}^{-3t}}{4} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{-t} & 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{-t} & \mathrm{e}^{3t} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{t}}{4} & \frac{5\,\mathrm{e}^{t}}{4} \\ \frac{\mathrm{e}^{-3t}}{4} & -\frac{\mathrm{e}^{-3t}}{4} \end{bmatrix} \begin{bmatrix} 4t-1 \\ t \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{-t} & 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{-t} & \mathrm{e}^{3t} \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{t}(t+1)}{4} \\ \frac{\mathrm{e}^{-3t}(3t-1)}{4} \end{bmatrix} dt \\
&= \begin{bmatrix} \mathrm{e}^{-t} & 5\,\mathrm{e}^{3t} \\ \mathrm{e}^{-t} & \mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{t} t}{4} \\ -\frac{\mathrm{e}^{-3t} t}{4} \end{bmatrix} \\
&= \begin{bmatrix} -t \\ 0 \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} 5c_2 \mathrm{e}^{3t} \\ c_2 \mathrm{e}^{3t} \end{bmatrix} + \begin{bmatrix} -t \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} + 5c_2 \mathrm{e}^{3t} - t \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{3t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 0 \\ y(0) = 0 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

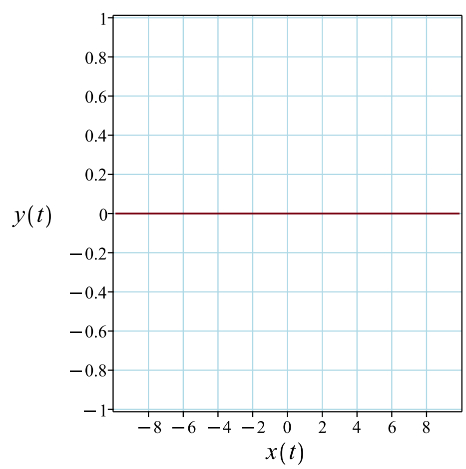
$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 5c_2 \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

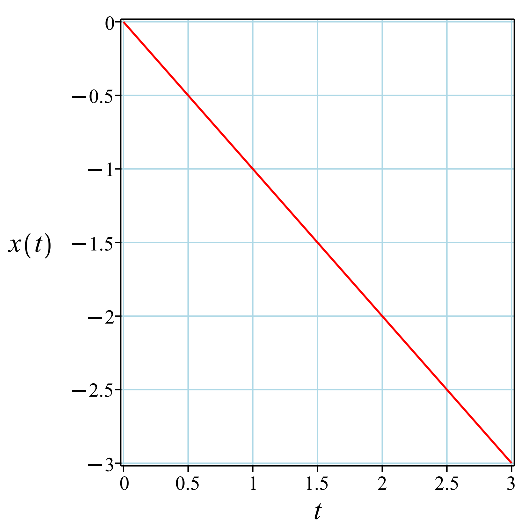
$$
\begin{bmatrix} c_1 = 0 \\ c_2 = 0 \end{bmatrix}
$$

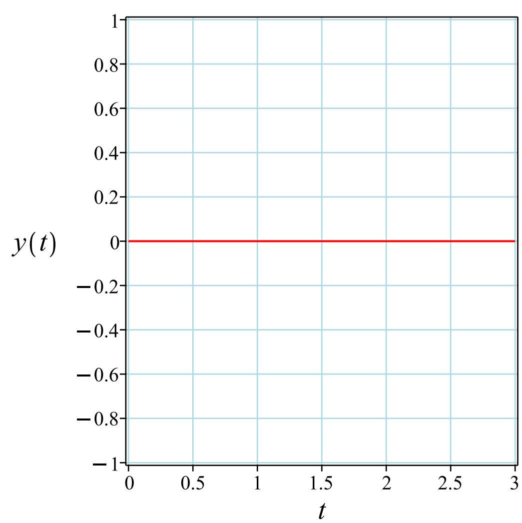
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -t \\ 0 \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 12

```
dsolve([diff(x(t),t) = 4*x(t)-5*y(t)+4*t-1, diff(y(t),t) = x(t)-2*y(t)+t, x(0) = 0, y(0) = 0
```

$$x(t) = -t$$
$$y(t) = 0$$

✓ Solution by Mathematica

Time used: 0.148 (sec). Leaf size: 12

```
DSolve[{x'[t]==4*x[t]-5*y[t]+4*t-1,y'[t]==x[t]-2*y[t]+t},{x[0]==0,y[0]==0},{x[t],y[t]},t,Inc
```

$$x(t) \to -t$$
$$y(t) \to 0$$

31.6 problem 820

Internal problem ID [15543]
Internal file name [OUTPUT/15543_Friday_May_10_2024_09_47_59_PM_20573560/index.tex]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 820.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = y(t) - x(t) + \mathrm{e}^{t}$$
$$y'(t) = x(t) - y(t) + \mathrm{e}^{t}$$

With initial conditions

$$[x(0) = 0, y(0) = 1]$$

31.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{1}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ \frac{1}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{1}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{1}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ \frac{1}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} - \frac{e^{-2t}}{2} \\ \frac{e^{-2t}}{2} + \frac{1}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & -\frac{e^{2t}}{2} + \frac{1}{2} \\ -\frac{e^{2t}}{2} + \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{1}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ \frac{1}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{1}{2} \end{bmatrix} \int \begin{bmatrix} \frac{1}{2} + \frac{e^{2t}}{2} & -\frac{e^{2t}}{2} + \frac{1}{2} \\ -\frac{e^{2t}}{2} + \frac{1}{2} & \frac{1}{2} + \frac{e^{2t}}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{1}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ \frac{1}{2} - \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} \\ &= \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(2\,e^{3t}+e^{2t}-1)e^{-2t}}{2} \\ \frac{(2\,e^{3t}+e^{2t}+1)e^{-2t}}{2} \end{bmatrix}$$

31.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^0 \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-2t} & 1 \\ \mathrm{e}^{-2t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{2} & \frac{\mathrm{e}^{2t}}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\mathrm{e}^{-2t} & 1 \\ \mathrm{e}^{-2t} & 1 \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{2} & \frac{\mathrm{e}^{2t}}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-2t} & 1 \\ \mathrm{e}^{-2t} & 1 \end{bmatrix} \int \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-2t} & 1 \\ \mathrm{e}^{-2t} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} -c_1 \mathrm{e}^{-2t} \\ c_1 \mathrm{e}^{-2t} \end{bmatrix} + \begin{bmatrix} c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (\mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} - c_1)\, \mathrm{e}^{-2t} \\ (\mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} + c_1)\, \mathrm{e}^{-2t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 0 \\ y(0) = 1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

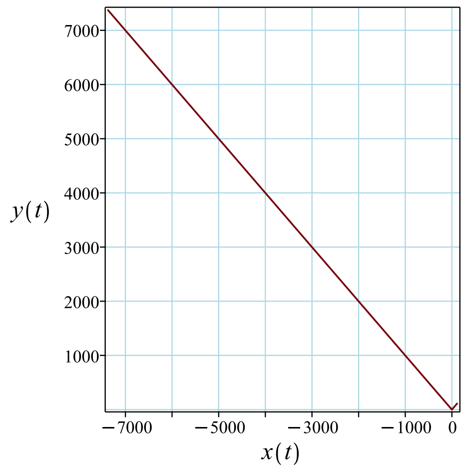
$$
\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + c_2 - c_1 \\ 1 + c_2 + c_1 \end{bmatrix}
$$

Solving for the constants of integrations gives

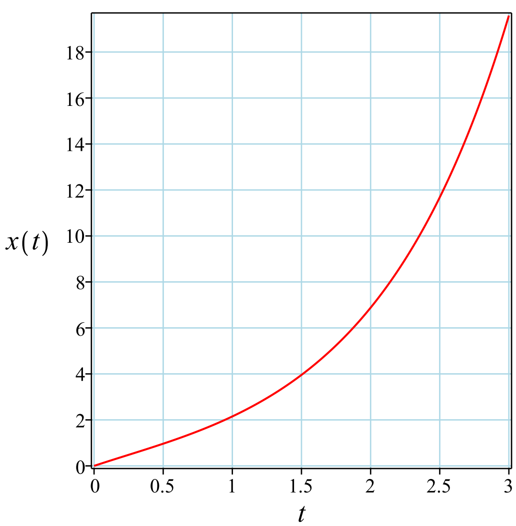
$$
\begin{bmatrix} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{bmatrix}
$$

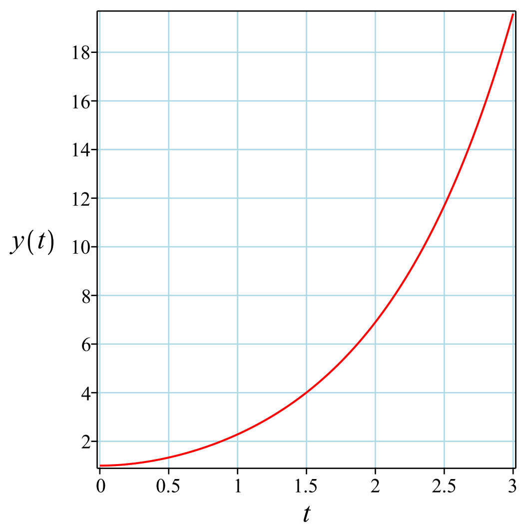
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(e^{3t} - \frac{e^{2t}}{2} - \frac{1}{2}\right) e^{-2t} \\ \left(e^{3t} - \frac{e^{2t}}{2} + \frac{1}{2}\right) e^{-2t} \end{bmatrix}$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 28

```
dsolve([diff(x(t),t) = y(t)-x(t)+exp(t), diff(y(t),t) = x(t)-y(t)+exp(t), x(0) = 0, y(0) = 1
```

$$x(t) = -\frac{e^{-2t}}{2} + e^{t} - \frac{1}{2}$$
$$y(t) = \frac{e^{-2t}}{2} + e^{t} - \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 40

```
DSolve[{x'[t]==y[t]-x[t]+Exp[t],y'[t]==x[t]-y[t]+Exp[t]},{x[0]==0,y[0]==1},{x[t],y[t]},t,Inc
```

$$x(t) \to -\frac{e^{-2t}}{2} + e^{t} - \frac{1}{2}$$
$$y(t) \to \frac{e^{-2t}}{2} + e^{t} - \frac{1}{2}$$

31.7 problem 821

Internal problem ID [15544]
Internal file name [`OUTPUT/15544_Friday_May_10_2024_09_48_00_PM_1140586/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 821.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= t^2 - y(t) \\
y'(t) &= x(t) + t
\end{aligned}
$$

31.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t^2 \\ t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)\, c_1 - \sin(t)\, c_2 \\ \sin(t)\, c_1 + \cos(t)\, c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} t^2 \\ t \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} \sin(t)\, t^2 - \sin(t) + t\cos(t) \\ -\cos(t) - \sin(t)\, t + \cos(t)\, t^2 \end{bmatrix} \\ &= \begin{bmatrix} t \\ t^2 - 1 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \cos(t)\, c_1 - \sin(t)\, c_2 + t \\ \sin(t)\, c_1 + \cos(t)\, c_2 + t^2 - 1 \end{bmatrix} \end{aligned}$$

31.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t^2 \\ t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} i\mathrm{e}^{it} & -i\mathrm{e}^{-it} \\ \mathrm{e}^{it} & \mathrm{e}^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ \frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{i e^{-it}}{2} & \frac{e^{-it}}{2} \\ \frac{i e^{it}}{2} & \frac{e^{it}}{2} \end{bmatrix} \begin{bmatrix} t^2 \\ t \end{bmatrix} dt \\
&= \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -\frac{t\,e^{-it}(it-1)}{2} \\ \frac{t\,e^{it}(it+1)}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} i e^{it} & -i e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} -\frac{(-t^2+it+1)e^{-it}}{2} \\ \frac{(t^2+it-1)e^{it}}{2} \end{bmatrix} \\
&= \begin{bmatrix} t \\ t^2 - 1 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} i c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} -i c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} t \\ t^2 - 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} i c_1 e^{it} - i c_2 e^{-it} + t \\ c_1 e^{it} + c_2 e^{-it} + t^2 - 1 \end{bmatrix}$$

31.7.3 Maple step by step solution

Let's solve

$[x'(t) = t^2 - y(t), y'(t) = x(t) + t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t^2 \\ t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t^2 \\ t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} t^2 \\ t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $e^{-It} \cdot \begin{bmatrix} -I \\ 1 \end{bmatrix}$

- Use Euler identity to write solution in terms of sin and cos

  $(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} -I \\ 1 \end{bmatrix}$

- Simplify expression

  $\begin{bmatrix} -I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$

- Both real and imaginary parts are solutions to the homogeneous system

  $\left[\vec{x}_1(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} -\cos(t) \\ -\sin(t) \end{bmatrix}\right]$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$

□ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} -\sin(t) & -\cos(t) \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$
  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution
  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms
  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}'(t) = \Phi(t)^{-1} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$
  $\vec{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution
  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute
  $$\vec{x}_p(t) = \left[\begin{array}{c} t - \sin(t) \\ t^2 + \cos(t) - 1 \end{array}\right]$$

- Plug particular solution back into general solution
  $$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \left[\begin{array}{c} t - \sin(t) \\ t^2 + \cos(t) - 1 \end{array}\right]$$
- Substitute in vector of dependent variables
  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} (-c_1 - 1)\sin(t) - c_2\cos(t) + t \\ t^2 + \cos(t) - 1 - c_2\sin(t) + c_1\cos(t) \end{array}\right]$$
- Solution to the system of ODEs
  $\{x(t) = (-c_1 - 1)\sin(t) - c_2\cos(t) + t, y(t) = t^2 + \cos(t) - 1 - c_2\sin(t) + c_1\cos(t)\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)+y(t)=t^2,diff(y(t),t)-x(t)=t],singsol=all)
```

$$x(t) = c_2 \sin(t) + c_1 \cos(t) + t$$
$$y(t) = t^2 - c_2 \cos(t) + c_1 \sin(t) - 1$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 36

```
DSolve[{x'[t]+y[t]==t^2,y'[t]-x[t]==t},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to t + c_1 \cos(t) - c_2 \sin(t)$$
$$y(t) \to t^2 + c_2 \cos(t) + c_1 \sin(t) - 1$$

31.8 problem 822

Internal problem ID [15545]
Internal file name [`OUTPUT/15545_Friday_May_10_2024_09_48_01_PM_75078509/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 822.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \sin(t) - \mathrm{e}^{-t} - y(t) \\
y'(t) &= -\sin(t) + 2\,\mathrm{e}^{-t}
\end{aligned}
$$

31.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \sin(t) - \mathrm{e}^{-t} \\ -\sin(t) + 2\,\mathrm{e}^{-t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} -tc_2 + c_1 \\ c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \int \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(t) - \mathrm{e}^{-t} \\ -\sin(t) + 2\,\mathrm{e}^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (-1-2t)\,\mathrm{e}^{-t} + (t-1)\cos(t) - \sin(t) \\ -2\,\mathrm{e}^{-t} + \cos(t) \end{bmatrix} \\
&= \begin{bmatrix} -\sin(t) - \cos(t) - \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-t} + \cos(t) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} -tc_2 + c_1 - \sin(t) - \cos(t) - \mathrm{e}^{-t} \\ c_2 - 2\,\mathrm{e}^{-t} + \cos(t) \end{bmatrix}
\end{aligned}$$

31.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \sin(t) - \mathrm{e}^{-t} \\ -\sin(t) + 2\,\mathrm{e}^{-t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 0 & -\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	2	1	Yes	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

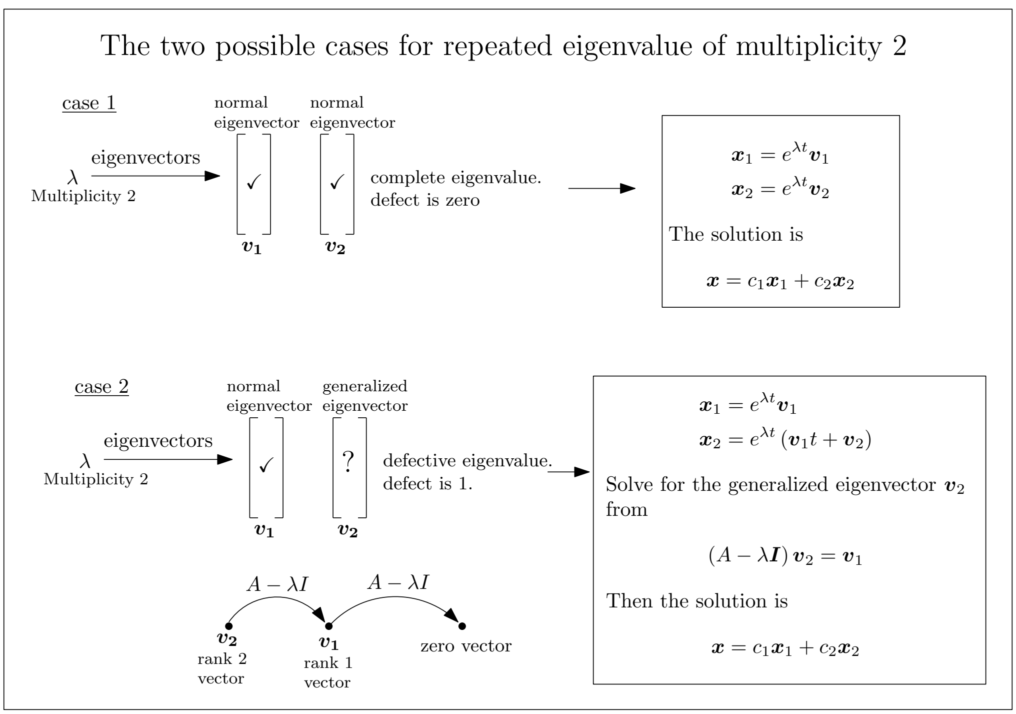


Figure 858: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) 1 \\ &= \begin{bmatrix} t+1 \\ -1 \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t+1 \\ -1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 1 & t+1 \\ 0 & -1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 1 & t+1 \\ 0 & -1 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 1 & t+1 \\ 0 & -1 \end{bmatrix} \int \begin{bmatrix} 1 & t+1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sin(t) - e^{-t} \\ -\sin(t) + 2\,e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} 1 & t+1 \\ 0 & -1 \end{bmatrix} \int \begin{bmatrix} -\sin(t)\, t + 2t\,e^{-t} + e^{-t} \\ \sin(t) - 2\,e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} 1 & t+1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} (-3-2t)\,e^{-t} + t\cos(t) - \sin(t) \\ 2\,e^{-t} - \cos(t) \end{bmatrix} \\
&= \begin{bmatrix} -\sin(t) - \cos(t) - e^{-t} \\ -2\,e^{-t} + \cos(t) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2(t+1) \\ -c_2 \end{bmatrix} + \begin{bmatrix} -\sin(t) - \cos(t) - e^{-t} \\ -2\,e^{-t} + \cos(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 t - \sin(t) - e^{-t} - \cos(t) + c_1 + c_2 \\ -c_2 - 2\,e^{-t} + \cos(t) \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 39

```
dsolve([diff(x(t),t)+diff(y(t),t)+y(t)=exp(-t),2*diff(x(t),t)+diff(y(t),t)+2*y(t)=sin(t)],si
```

$$x(t) = -\sin(t) - e^{-t} - \cos(t) + c_1 t + c_2$$
$$y(t) = \cos(t) - 2\,e^{-t} - c_1$$

✓ Solution by Mathematica

Time used: 0.244 (sec). Leaf size: 43

```
DSolve[{x'[t]+y'[t]+y[t]==Exp[-t],2*x'[t]+y'[t]+2*y[t]==Sin[t]},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to -e^{-t} - \sin(t) - \cos(t) - c_2 t + c_1$$
$$y(t) \to -2e^{-t} + \cos(t) + c_2$$

31.9 problem 823

Internal problem ID [15546]
Internal file name [`OUTPUT/15546_Friday_May_10_2024_09_48_02_PM_89153353/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 823.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + y(t) - 2z(t) + 2 - t \\
y'(t) &= 1 - x(t) \\
z'(t) &= x(t) + y(t) - z(t) + 1 - t
\end{aligned}
$$

31.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 2-t \\ 1 \\ 1-t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{ccc} \mathrm{e}^{t}+\sin(t) & \sin(t) & -\mathrm{e}^{t}+\cos(t)-\sin(t) \\ -\mathrm{e}^{t}+\cos(t) & \cos(t) & \mathrm{e}^{t}-\cos(t)-\sin(t) \\ \sin(t) & \sin(t) & \cos(t)-\sin(t) \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{ccc} \mathrm{e}^{t}+\sin(t) & \sin(t) & -\mathrm{e}^{t}+\cos(t)-\sin(t) \\ -\mathrm{e}^{t}+\cos(t) & \cos(t) & \mathrm{e}^{t}-\cos(t)-\sin(t) \\ \sin(t) & \sin(t) & \cos(t)-\sin(t) \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] \\
&= \left[\begin{array}{c} (\mathrm{e}^{t}+\sin(t))\,c_1+\sin(t)\,c_2+(-\mathrm{e}^{t}+\cos(t)-\sin(t))\,c_3 \\ (-\mathrm{e}^{t}+\cos(t))\,c_1+\cos(t)\,c_2+(\mathrm{e}^{t}-\cos(t)-\sin(t))\,c_3 \\ \sin(t)\,c_1+\sin(t)\,c_2+(\cos(t)-\sin(t))\,c_3 \end{array}\right] \\
&= \left[\begin{array}{c} (c_1+c_2-c_3)\sin(t)+(c_1-c_3)\,\mathrm{e}^{t}+c_3\cos(t) \\ (c_1+c_2-c_3)\cos(t)+(-c_1+c_3)\,\mathrm{e}^{t}-c_3\sin(t) \\ (c_1+c_2-c_3)\sin(t)+c_3\cos(t) \end{array}\right]
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \left[\begin{array}{ccc} -\sin(t)+\mathrm{e}^{-t} & -\sin(t) & \sin(t)+\cos(t)-\mathrm{e}^{-t} \\ \cos(t)-\mathrm{e}^{-t} & \cos(t) & \sin(t)-\cos(t)+\mathrm{e}^{-t} \\ -\sin(t) & -\sin(t) & \cos(t)+\sin(t) \end{array}\right]
\end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^t + \sin(t) & \sin(t) & -e^t + \cos(t) - \sin(t) \\ -e^t + \cos(t) & \cos(t) & e^t - \cos(t) - \sin(t) \\ \sin(t) & \sin(t) & \cos(t) - \sin(t) \end{bmatrix} \int \begin{bmatrix} -\sin(t) + e^{-t} & -\sin(t) & \sin(t) + \cos(t) \\ \cos(t) - e^{-t} & \cos(t) & \sin(t) - \cos(t) \\ -\sin(t) & -\sin(t) & \cos(t) + \sin \end{bmatrix}$$

$$= \begin{bmatrix} e^t + \sin(t) & \sin(t) & -e^t + \cos(t) - \sin(t) \\ -e^t + \cos(t) & \cos(t) & e^t - \cos(t) - \sin(t) \\ \sin(t) & \sin(t) & \cos(t) - \sin(t) \end{bmatrix} \begin{bmatrix} \sin(t) - e^{-t} - \sin(t)\, t + \cos(t) \\ e^{-t} + (t-1)\cos(t) + \sin(t) \\ -\sin(t)\, t + \sin(t) + \cos(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} (c_1 + c_2 - c_3)\sin(t) + (c_1 - c_3)e^t + c_3\cos(t) \\ (c_1 + c_2 - c_3)\cos(t) + (-c_1 + c_3)e^t - c_3\sin(t) + t \\ (c_1 + c_2 - c_3)\sin(t) + c_3\cos(t) + 1 \end{bmatrix} \end{aligned}$$

31.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 2-t \\ 1 \\ 1-t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -2 \\ -1 & -\lambda & 0 \\ 1 & 1 & -1-\lambda \end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3-\lambda^2+\lambda-1=0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= i \\ \lambda_3 &= -i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \\ v_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \\ v_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2+i & 1 & -2 \\ -1 & i & 0 \\ 1 & 1 & -1+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2+i & 1 & -2 & 0 \\ -1 & i & 0 & 0 \\ 1 & 1 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{2}{5} - \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2+i & 1 & -2 & 0 \\ 0 & \frac{2}{5} + \frac{4i}{5} & -\frac{4}{5} + \frac{2i}{5} & 0 \\ 1 & 1 & -1+i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{2}{5} + \frac{i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2+i & 1 & -2 & 0 \\ 0 & \frac{2}{5} + \frac{4i}{5} & -\frac{4}{5} + \frac{2i}{5} & 0 \\ 0 & \frac{3}{5} + \frac{i}{5} & -\frac{1}{5} + \frac{3i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2+i & 1 & -2 & 0 \\ 0 & \frac{2}{5}+\frac{4i}{5} & -\frac{4}{5}+\frac{2i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2+i & 1 & -2 \\ 0 & \frac{2}{5}+\frac{4i}{5} & -\frac{4}{5}+\frac{2i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} t \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-i & 1 & -2 \\ -1 & -i & 0 \\ 1 & 1 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2-i & 1 & -2 & 0 \\ -1 & -i & 0 & 0 \\ 1 & 1 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{2}{5} + \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2-i & 1 & -2 & 0 \\ 0 & \frac{2}{5} - \frac{4i}{5} & -\frac{4}{5} - \frac{2i}{5} & 0 \\ 1 & 1 & -1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{2}{5} - \frac{i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2-i & 1 & -2 & 0 \\ 0 & \frac{2}{5} - \frac{4i}{5} & -\frac{4}{5} - \frac{2i}{5} & 0 \\ 0 & \frac{3}{5} - \frac{i}{5} & -\frac{1}{5} - \frac{3i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right)R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2-i & 1 & -2 & 0 \\ 0 & \frac{2}{5} - \frac{4i}{5} & -\frac{4}{5} - \frac{2i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2-i & 1 & -2 \\ 0 & \frac{2}{5} - \frac{4i}{5} & -\frac{4}{5} - \frac{2i}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} t \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
i	1	1	No	$\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^t\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^t \\ \mathrm{e}^t \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{it} \\ i\mathrm{e}^{it} \\ \mathrm{e}^{it} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-it} \\ -i\mathrm{e}^{-it} \\ \mathrm{e}^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -e^{t} & e^{it} & e^{-it} \\ e^{t} & i e^{it} & -i e^{-it} \\ 0 & e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -e^{-t} & 0 & e^{-t} \\ -\frac{i e^{-it}}{2} & -\frac{i e^{-it}}{2} & \left(\frac{1}{2} + \frac{i}{2}\right) e^{-it} \\ \frac{i e^{it}}{2} & \frac{i e^{it}}{2} & \left(\frac{1}{2} - \frac{i}{2}\right) e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -e^{t} & e^{it} & e^{-it} \\ e^{t} & i e^{it} & -i e^{-it} \\ 0 & e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -e^{-t} & 0 & e^{-t} \\ -\frac{i e^{-it}}{2} & -\frac{i e^{-it}}{2} & \left(\frac{1}{2} + \frac{i}{2}\right) e^{-it} \\ \frac{i e^{it}}{2} & \frac{i e^{it}}{2} & \left(\frac{1}{2} - \frac{i}{2}\right) e^{it} \end{bmatrix} \begin{bmatrix} 2-t \\ 1 \\ 1-t \end{bmatrix} dt \\ &= \begin{bmatrix} -e^{t} & e^{it} & e^{-it} \\ e^{t} & i e^{it} & -i e^{-it} \\ 0 & e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -e^{-t} \\ -\frac{e^{-it}(-1+2i+t)}{2} \\ \frac{e^{it}(1+2i-t)}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} -e^{t} & e^{it} & e^{-it} \\ e^{t} & i e^{it} & -i e^{-it} \\ 0 & e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} e^{-t} \\ -\frac{e^{-it}(it-i-1)}{2} \\ \frac{e^{it}(it-i+1)}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix} \end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{t} \\ c_1 e^{t} \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 e^{it} \\ i c_2 e^{it} \\ c_2 e^{it} \end{bmatrix} + \begin{bmatrix} c_3 e^{-it} \\ -i c_3 e^{-it} \\ c_3 e^{-it} \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{t} + c_2 e^{it} + c_3 e^{-it} \\ c_1 e^{t} + i c_2 e^{it} - i c_3 e^{-it} + t \\ c_2 e^{it} + c_3 e^{-it} + 1 \end{bmatrix}$$

31.9.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + y(t) - 2z(t) + 2 - t, y'(t) = 1 - x(t), z'(t) = x(t) + y(t) - z(t) + 1 - t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2-t \\ 1 \\ 1-t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2-t \\ 1 \\ 1-t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2-t \\ 1 \\ 1-t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} 1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^t \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} 1 \\ -I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I\sin(t)) \cdot \begin{bmatrix} 1 \\ -I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \cos(t) - I\sin(t) \\ -I(\cos(t) - I\sin(t)) \\ \cos(t) - I\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ \cos(t) \end{bmatrix}, \vec{x}_3(t) = \begin{bmatrix} -\sin(t) \\ -\cos(t) \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -e^t & \cos(t) & -\sin(t) \\ e^t & -\sin(t) & -\cos(t) \\ 0 & \cos(t) & -\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -e^t & \cos(t) & -\sin(t) \\ e^t & -\sin(t) & -\cos(t) \\ 0 & \cos(t) & -\sin(t) \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{t} + \sin(t) & \sin(t) & -e^{t} + \cos(t) - \sin(t) \\ -e^{t} + \cos(t) & \cos(t) & e^{t} - \cos(t) - \sin(t) \\ \sin(t) & \sin(t) & \cos(t) - \sin(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \begin{bmatrix} \sin(t)^2 + \cos(t)^2 - 1 + \sin(t) - \cos(t) + e^{t} \\ (t-1)\cos(t)^2 + \cos(t) + (t-1)\sin(t)^2 + \sin(t) - e^{t} + 1 \\ 1 + \sin(t) - \cos(t) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \left[\begin{array}{c} \sin(t)^2 + \cos(t)^2 - 1 + \sin(t) - \cos(t) + e^t \\ (t-1)\cos(t)^2 + \cos(t) + (t-1)\sin(t)^2 + \sin(t) - e^t + 1 \\ 1 + \sin(t) - \cos(t) \end{array}\right.$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} (c_2-1)\cos(t) + (1-c_1)e^t - \sin(t)(c_3-1) \\ (-c_3+1)\cos(t) + (c_1-1)e^t + (-c_2+1)\sin(t) + t \\ (c_2-1)\cos(t) + 1 + (-c_3+1)\sin(t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = (c_2-1)\cos(t) + (1-c_1)e^t - \sin(t)(c_3-1), y(t) = (-c_3+1)\cos(t) + (c_1-1)e^t + (-$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve([diff(x(t),t)=2*x(t)+y(t)-2*z(t)+2-t,diff(y(t),t)=1-x(t),diff(z(t),t)=x(t)+y(t)-z(t)+
```

$$x(t) = c_1\sin(t) - c_2e^t - c_3\cos(t)$$
$$y(t) = t + c_1\cos(t) + c_2e^t + c_3\sin(t)$$
$$z(t) = 1 + c_1\sin(t) - c_3\cos(t)$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 92

```
DSolve[{x'[t]==2*x[t]+y[t]-2*z[t]+2-t,y'[t]==1-x[t],z'[t]==x[t]+y[t]-z[t]+1-t},{x[t],y[t],z[
```

$$x(t) \to (c_1-c_3)e^t + c_3\cos(t) + (c_1+c_2-c_3)\sin(t)$$
$$y(t) \to t - c_1e^t + c_3e^t + (c_1+c_2-c_3)\cos(t) - c_3\sin(t)$$
$$z(t) \to c_3\cos(t) + (c_1+c_2-c_3)\sin(t) + 1$$

31.10 problem 824

Internal problem ID [15547]
Internal file name [`OUTPUT/15547_Friday_May_10_2024_09_48_04_PM_18554227/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.2 The method of undetermined coefficients. Exercises page 239
Problem number: 824.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) - 2y(t) + 2\,\mathrm{e}^{-t}$$
$$y'(t) = -y(t) - z(t) + 1$$
$$z'(t) = -z(t) + 1$$

With initial conditions

$$[x(0) = 1, y(0) = 1, z(0) = 1]$$

31.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ 1 \\ 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & -2t\,\mathrm{e}^{-t} & t^2\mathrm{e}^{-t} \\ 0 & \mathrm{e}^{-t} & -t\,\mathrm{e}^{-t} \\ 0 & 0 & \mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} \mathrm{e}^{-t} & -2t\,\mathrm{e}^{-t} & t^2\mathrm{e}^{-t} \\ 0 & \mathrm{e}^{-t} & -t\,\mathrm{e}^{-t} \\ 0 & 0 & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{-t} - 2t\,\mathrm{e}^{-t} + t^2\mathrm{e}^{-t} \\ \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{-t}(t-1)^2 \\ \mathrm{e}^{-t}(1-t) \\ \mathrm{e}^{-t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \mathrm{e}^{t} & 2\,\mathrm{e}^{t}t & t^2\mathrm{e}^{t} \\ 0 & \mathrm{e}^{t} & \mathrm{e}^{t}t \\ 0 & 0 & \mathrm{e}^{t} \end{bmatrix} \end{aligned}$$

Hence

$$
\vec{x}_p(t) = \begin{bmatrix} e^{-t} & -2t\,e^{-t} & t^2 e^{-t} \\ 0 & e^{-t} & -t\,e^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix} \int \left[\begin{bmatrix} e^{t} & 2\,e^{t} t & t^2 e^{t} \\ 0 & e^{t} & e^{t} t \\ 0 & 0 & e^{t} \end{bmatrix}\right] \left[\begin{bmatrix} 2\,e^{-t} \\ 1 \\ 1 \end{bmatrix}\right] dt
$$

$$
= \begin{bmatrix} e^{-t} & -2t\,e^{-t} & t^2 e^{-t} \\ 0 & e^{-t} & -t\,e^{-t} \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} t(e^{t} t + 2) \\ e^{t} t \\ e^{t} \end{bmatrix}
$$

$$
= \begin{bmatrix} 2t\,e^{-t} \\ 0 \\ 1 \end{bmatrix}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} e^{-t}(t^2+1) \\ -e^{-t}(t-1) \\ e^{-t}+1 \end{bmatrix}
\end{aligned}
$$

31.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 2\,e^{-t} \\ 1 \\ 1 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix}-1&-2&0\\0&-1&-1\\0&0&-1\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix}-1-\lambda&-2&0\\0&-1-\lambda&-1\\0&0&-1-\lambda\end{bmatrix}\right)=0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(-1-\lambda)(-1-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=-1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-1$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-1&-2&0\\0&-1&-1\\0&0&-1\end{bmatrix}-(-1)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&-2&0\\0&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	3	1	Yes	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

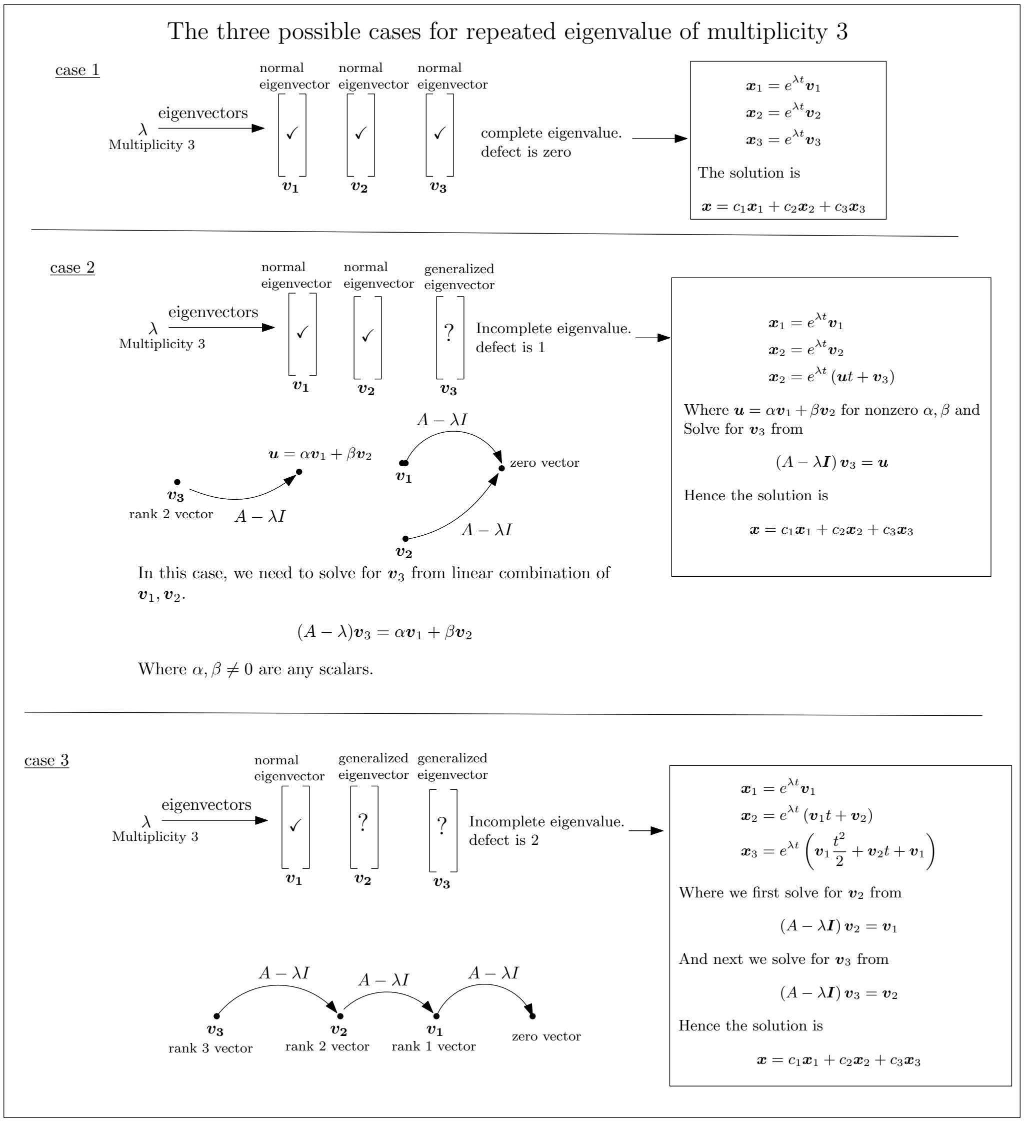


Figure 859: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector $\vec{v}_3$ of rank 3.$\vec{v}_2$ is found by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & -2 & 0\\ 0 & -1 & -1\\ 0 & 0 & -1\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & -2 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}1\\ -\frac{1}{2}\\ 0\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A - \lambda I)\,\vec{v}_3 = \vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}-1 & -2 & 0\\ 0 & -1 & -1\\ 0 & 0 & -1\end{bmatrix} - (-1)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ -\frac{1}{2}\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & -2 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}1\\ -\frac{1}{2}\\ 0\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3 = \begin{bmatrix}1\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue -1. Therefore the three basis solutions associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{-t} \\
&= \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2) \\
&= \mathrm{e}^{-t}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ -\frac{\mathrm{e}^{-t}}{2} \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= \left(\vec{v}_1 \frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right) e^{\lambda t} \\
&= \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right) \mathrm{e}^{-t} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-t}(t^2+2t+2)}{2} \\ -\frac{\mathrm{e}^{-t}(t+1)}{2} \\ \frac{\mathrm{e}^{-t}}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-t}(t+1) \\ -\frac{\mathrm{e}^{-t}}{2} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{-t}\left(\frac{1}{2}t^2+t+1\right) \\ \mathrm{e}^{-t}\left(-\frac{t}{2}-\frac{1}{2}\right) \\ \frac{\mathrm{e}^{-t}}{2} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-t}(t+1) & \mathrm{e}^{-t}\left(\frac{1}{2}t^2+t+1\right) \\ 0 & -\frac{\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t}\left(-\frac{t}{2}-\frac{1}{2}\right) \\ 0 & 0 & \frac{\mathrm{e}^{-t}}{2} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \mathrm{e}^{t} & 2\,\mathrm{e}^{t}(t+1) & \mathrm{e}^{t}t(t+2) \\ 0 & -2\,\mathrm{e}^{t} & -2\,\mathrm{e}^{t}(t+1) \\ 0 & 0 & 2\,\mathrm{e}^{t} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-t} & e^{-t}(t+1) & e^{-t}\left(\frac{1}{2}t^2+t+1\right) \\ 0 & -\frac{e^{-t}}{2} & e^{-t}\left(-\frac{t}{2}-\frac{1}{2}\right) \\ 0 & 0 & \frac{e^{-t}}{2} \end{bmatrix} \int \begin{bmatrix} e^{t} & 2e^{t}(t+1) & e^{t}t(t+2) \\ 0 & -2e^{t} & -2e^{t}(t+1) \\ 0 & 0 & 2e^{t} \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 1 \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & e^{-t}(t+1) & e^{-t}\left(\frac{1}{2}t^2+t+1\right) \\ 0 & -\frac{e^{-t}}{2} & e^{-t}\left(-\frac{t}{2}-\frac{1}{2}\right) \\ 0 & 0 & \frac{e^{-t}}{2} \end{bmatrix} \int \begin{bmatrix} 2+(t^2+4t+2)e^{t} \\ -2e^{t}(t+2) \\ 2e^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{-t} & e^{-t}(t+1) & e^{-t}\left(\frac{1}{2}t^2+t+1\right) \\ 0 & -\frac{e^{-t}}{2} & e^{-t}\left(-\frac{t}{2}-\frac{1}{2}\right) \\ 0 & 0 & \frac{e^{-t}}{2} \end{bmatrix} \begin{bmatrix} (2+e^{t}(t+2))\,t \\ -2e^{t}(t+1) \\ 2e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 2t\,e^{-t} \\ 0 \\ 1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_2 e^{-t}(t+1) \\ -\frac{c_2 e^{-t}}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} c_3 e^{-t}\left(\frac{1}{2}t^2+t+1\right) \\ c_3 e^{-t}\left(-\frac{t}{2}-\frac{1}{2}\right) \\ \frac{c_3 e^{-t}}{2} \end{bmatrix} + \begin{bmatrix} 2t\,e^{-t} \\ 0 \\ 1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{-t}(c_3t^2+(2c_2+2c_3+4)t+2c_1+2c_2+2c_3)}{2} \\ -\frac{e^{-t}(c_3t+c_2+c_3)}{2} \\ \frac{c_3 e^{-t}}{2}+1 \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0)=1 \\ y(0)=1 \\ z(0)=1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 \\ -\frac{c_2}{2} - \frac{c_3}{2} \\ \frac{c_3}{2} + 1 \end{bmatrix}$$

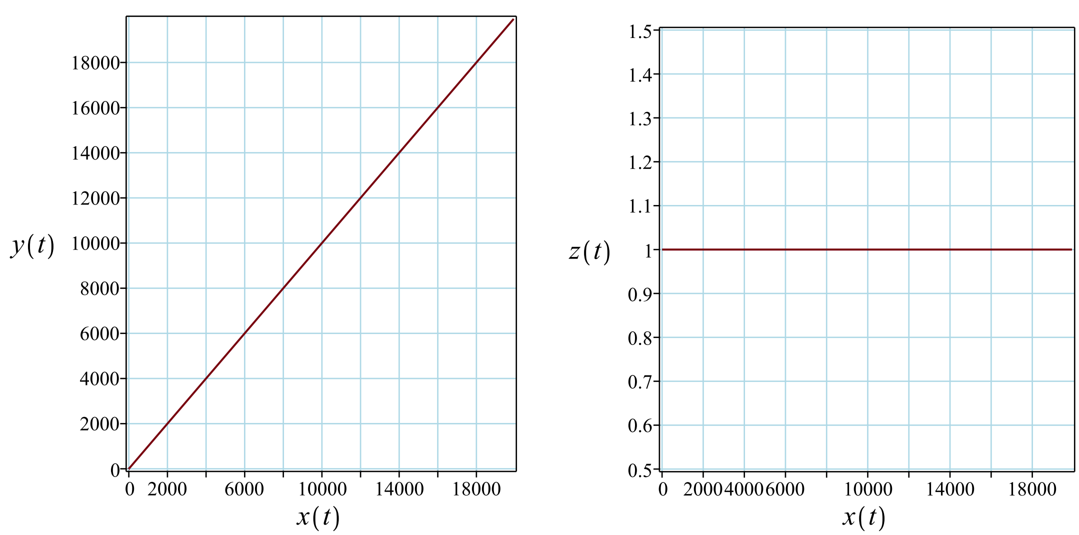
Solving for the constants of integrations gives

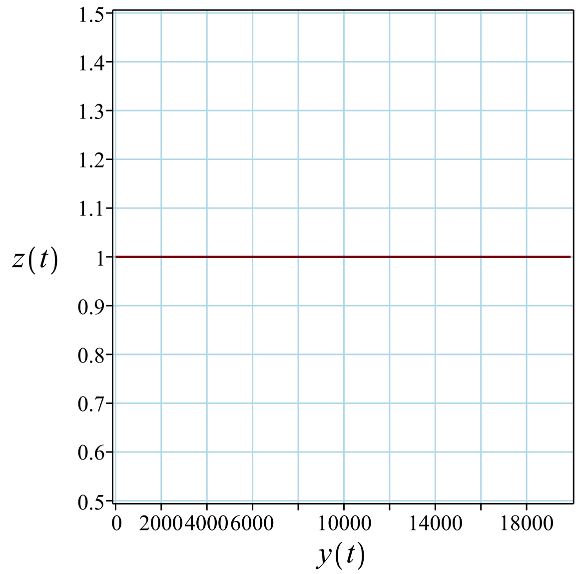
$$\begin{bmatrix} c_1 = 3 \\ c_2 = -2 \\ c_3 = 0 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

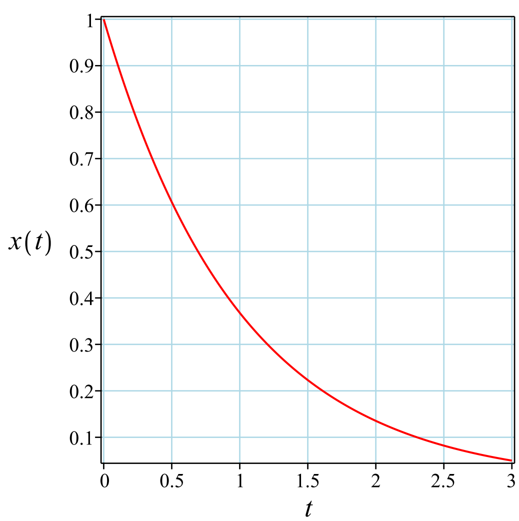
$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 1 \end{bmatrix}$$

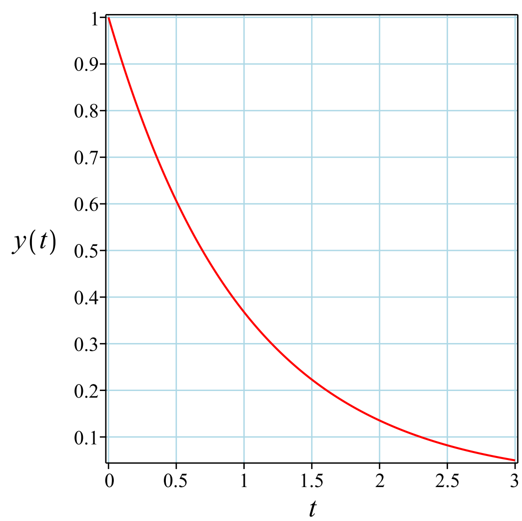
The following are plots of each solution against another.

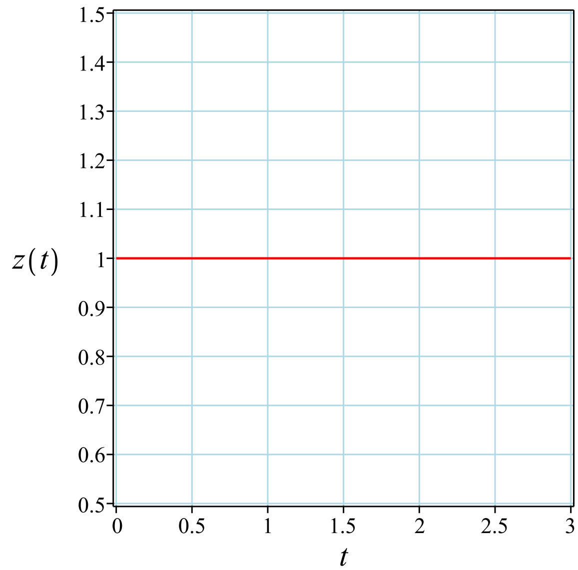




The following are plots of each solution.







✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 20

```
dsolve([diff(x(t),t)+x(t)+2*y(t) = 2*exp(-t), diff(y(t),t)+y(t)+z(t) = 1, diff(z(t),t)+z(t)
```

$$x(t) = e^{-t}$$
$$y(t) = e^{-t}$$
$$z(t) = 1$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 22

```
DSolve[{x'[t]+x[t]+2*y[t]==2*Exp[-t],y'[t]+y[t]+z[t]==1,z'[t]+z[t]==1},{x[0]==1,y[0]==1,z[0]
```

$$x(t) \to e^{-t}$$
$$y(t) \to e^{-t}$$
$$z(t) \to 1$$

32 Chapter 3 (Systems of differential equations). Section 23.3 dAlemberts method. Exercises page 243

32.1 problem 825

Internal problem ID [15548]
Internal file name [`OUTPUT/15548_Friday_May_10_2024_09_48_05_PM_57110052/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.3 dAlemberts method. Exercises page 243
Problem number: 825.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 5x(t) + 4y(t) \\
y'(t) &= x(t) + 2y(t)
\end{aligned}
$$

32.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{e^t}{5} + \frac{4\,e^{6t}}{5} & \frac{4\,e^{6t}}{5} - \frac{4\,e^t}{5} \\ \frac{e^{6t}}{5} - \frac{e^t}{5} & \frac{4\,e^t}{5} + \frac{e^{6t}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^t}{5}+\frac{4\,e^{6t}}{5} & \frac{4\,e^{6t}}{5}-\frac{4\,e^t}{5} \\ \frac{e^{6t}}{5}-\frac{e^t}{5} & \frac{4\,e^t}{5}+\frac{e^{6t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^t}{5}+\frac{4\,e^{6t}}{5}\right)c_1 + \left(\frac{4\,e^{6t}}{5}-\frac{4\,e^t}{5}\right)c_2 \\ \left(\frac{e^{6t}}{5}-\frac{e^t}{5}\right)c_1 + \left(\frac{4\,e^t}{5}+\frac{e^{6t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(4c_1+4c_2)e^{6t}}{5} + \frac{e^t(c_1-4c_2)}{5} \\ \frac{(c_1+c_2)e^{6t}}{5} - \frac{e^t(c_1-4c_2)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

32.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 7\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 4t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	1	1	No	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{6t}$$
$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{6t}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 4\,\mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4c_1 \mathrm{e}^{6t} - c_2 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{6t} + c_2 \mathrm{e}^{t} \end{bmatrix}$$

The following is the phase plot of the system.

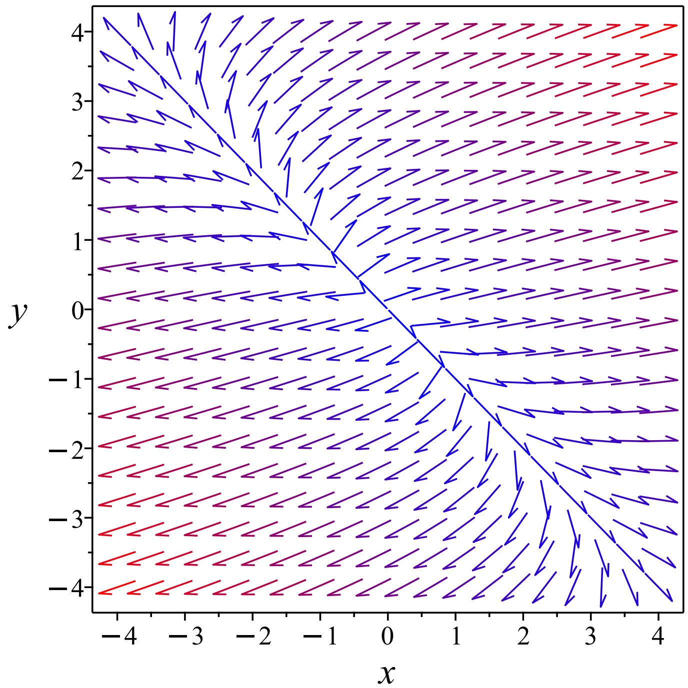


Figure 860: Phase plot

32.1.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) + 4y(t), y'(t) = x(t) + 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{6t} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{6t} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^t + 4c_2 e^{6t} \\ c_1 e^t + c_2 e^{6t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -c_1 e^t + 4c_2 e^{6t}, y(t) = c_1 e^t + c_2 e^{6t}\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)=5*x(t)+4*y(t),diff(y(t),t)=x(t)+2*y(t)],singsol=all)
```

$$x(t) = c_1 e^t + c_2 e^{6t}$$
$$y(t) = -c_1 e^t + \frac{c_2 e^{6t}}{4}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 67

```
DSolve[{x'[t]==5*x[t]+4*y[t],y'[t]==x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{5} e^t \left(c_1 \left(4e^{5t} + 1\right) + 4c_2 \left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5} e^t \left(c_1 \left(e^{5t} - 1\right) + c_2 \left(e^{5t} + 4\right)\right)$$

32.2 problem 826

Internal problem ID [15549]
Internal file name [`OUTPUT/15549_Saturday_May_11_2024_01_35_29_AM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.3 dAlemberts method. Exercises page 243
Problem number: 826.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 6x(t) + y(t)$$
$$y'(t) = 4x(t) + 3y(t)$$

32.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{2t}}{5} + \frac{4\,e^{7t}}{5} & \frac{e^{7t}}{5} - \frac{e^{2t}}{5} \\ \frac{4\,e^{7t}}{5} - \frac{4\,e^{2t}}{5} & \frac{4\,e^{2t}}{5} + \frac{e^{7t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{2t}}{5}+\frac{4\,e^{7t}}{5} & \frac{e^{7t}}{5}-\frac{e^{2t}}{5} \\ \frac{4\,e^{7t}}{5}-\frac{4\,e^{2t}}{5} & \frac{4\,e^{2t}}{5}+\frac{e^{7t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{2t}}{5}+\frac{4\,e^{7t}}{5}\right)c_1+\left(\frac{e^{7t}}{5}-\frac{e^{2t}}{5}\right)c_2 \\ \left(\frac{4\,e^{7t}}{5}-\frac{4\,e^{2t}}{5}\right)c_1+\left(\frac{4\,e^{2t}}{5}+\frac{e^{7t}}{5}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_2+c_1)e^{2t}}{5}+\frac{4\left(c_1+\frac{c_2}{4}\right)e^{7t}}{5} \\ \frac{(4c_2-4c_1)e^{2t}}{5}+\frac{4\left(c_1+\frac{c_2}{4}\right)e^{7t}}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

32.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 9\lambda + 14 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 7$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
7	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{7t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{7t} \\ \mathrm{e}^{7t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{4} \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{7t} - \frac{c_2 \mathrm{e}^{2t}}{4} \\ c_1 \mathrm{e}^{7t} + c_2 \mathrm{e}^{2t} \end{bmatrix}
$$

The following is the phase plot of the system.

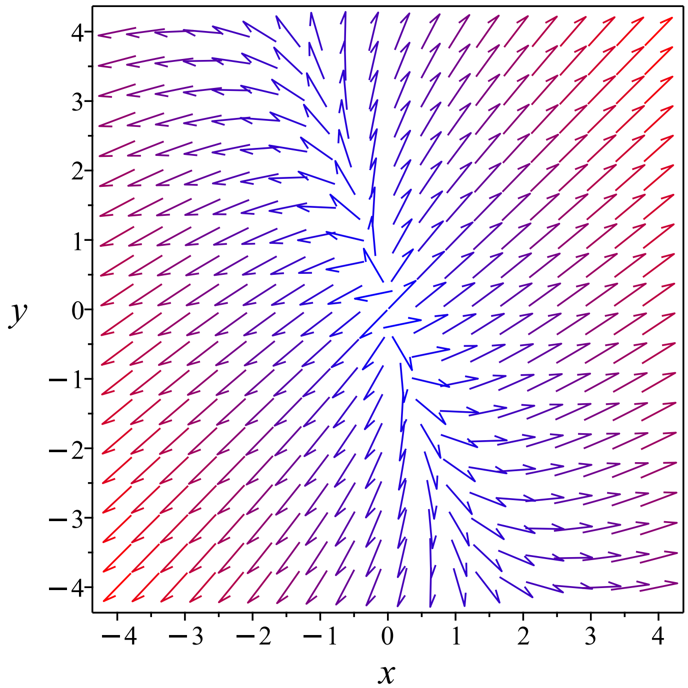


Figure 861: Phase plot

32.2.3 Maple step by step solution

Let's solve

$$[x'(t) = 6x(t) + y(t), y'(t) = 4x(t) + 3y(t)]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[7, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{2t} \cdot \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[7, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{7t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} + c_2 e^{7t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{2t}}{4} + c_2 e^{7t} \\ c_1 e^{2t} + c_2 e^{7t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = -\frac{c_1 e^{2t}}{4} + c_2 e^{7t}, y(t) = c_1 e^{2t} + c_2 e^{7t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=6*x(t)+y(t),diff(y(t),t)=4*x(t)+3*y(t)],singsol=all)
```

$$x(t) = c_1 e^{7t} + c_2 e^{2t}$$
$$y(t) = c_1 e^{7t} - 4c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 71

```
DSolve[{x'[t]==6*x[t]+y[t],y'[t]==4*x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{5} e^{2t} \left(c_1 \left(4e^{5t} + 1\right) + c_2 \left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5} e^{2t} \left(4c_1 \left(e^{5t} - 1\right) + c_2 \left(e^{5t} + 4\right)\right)$$

32.3 problem 827

Internal problem ID [15550]
Internal file name [`OUTPUT/15550_Saturday_May_11_2024_01_35_29_AM_31591721/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.3 dAlemberts method. Exercises page 243
Problem number: 827.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 2x(t) - 4y(t) + 1$$
$$y'(t) = -x(t) + 5y(t)$$

32.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{cc} \frac{4\,e^{t}}{5}+\frac{e^{6t}}{5} & -\frac{4\,e^{6t}}{5}+\frac{4\,e^{t}}{5} \\ -\frac{e^{6t}}{5}+\frac{e^{t}}{5} & \frac{e^{t}}{5}+\frac{4\,e^{6t}}{5} \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cc} \frac{4\,e^{t}}{5}+\frac{e^{6t}}{5} & -\frac{4\,e^{6t}}{5}+\frac{4\,e^{t}}{5} \\ -\frac{e^{6t}}{5}+\frac{e^{t}}{5} & \frac{e^{t}}{5}+\frac{4\,e^{6t}}{5} \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \left(\frac{4\,e^{t}}{5}+\frac{e^{6t}}{5}\right)c_1+\left(-\frac{4\,e^{6t}}{5}+\frac{4\,e^{t}}{5}\right)c_2 \\ \left(-\frac{e^{6t}}{5}+\frac{e^{t}}{5}\right)c_1+\left(\frac{e^{t}}{5}+\frac{4\,e^{6t}}{5}\right)c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{(c_1-4c_2)e^{6t}}{5}+\frac{4\,e^{t}(c_1+c_2)}{5} \\ \frac{(-c_1+4c_2)e^{6t}}{5}+\frac{e^{t}(c_1+c_2)}{5} \end{array}\right]
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \left[\begin{array}{cc} \frac{e^{-6t}(4\,e^{5t}+1)}{5} & \frac{4\,e^{-6t}(e^{5t}-1)}{5} \\ \frac{e^{-6t}(e^{5t}-1)}{5} & \frac{e^{-6t}(e^{5t}+4)}{5} \end{array}\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{cc} \frac{4\,e^{t}}{5}+\frac{e^{6t}}{5} & -\frac{4\,e^{6t}}{5}+\frac{4\,e^{t}}{5} \\ -\frac{e^{6t}}{5}+\frac{e^{t}}{5} & \frac{e^{t}}{5}+\frac{4\,e^{6t}}{5} \end{array}\right]\int\left[\begin{array}{cc} \frac{e^{-6t}(4\,e^{5t}+1)}{5} & \frac{4\,e^{-6t}(e^{5t}-1)}{5} \\ \frac{e^{-6t}(e^{5t}-1)}{5} & \frac{e^{-6t}(e^{5t}+4)}{5} \end{array}\right]\left[\begin{array}{c} 1 \\ 0 \end{array}\right]dt \\
&= \left[\begin{array}{cc} \frac{4\,e^{t}}{5}+\frac{e^{6t}}{5} & -\frac{4\,e^{6t}}{5}+\frac{4\,e^{t}}{5} \\ -\frac{e^{6t}}{5}+\frac{e^{t}}{5} & \frac{e^{t}}{5}+\frac{4\,e^{6t}}{5} \end{array}\right]\left[\begin{array}{c} -\frac{e^{-6t}}{30}-\frac{4\,e^{-t}}{5} \\ -\frac{e^{-t}}{5}+\frac{e^{-6t}}{30} \end{array}\right] \\
&= \left[\begin{array}{c} -\frac{5}{6} \\ -\frac{1}{6} \end{array}\right]
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(c_1-4c_2)e^{6t}}{5} + \frac{4\,e^t(c_1+c_2)}{5} - \frac{5}{6} \\ \frac{(-c_1+4c_2)e^{6t}}{5} + \frac{e^t(c_1+c_2)}{5} - \frac{1}{6} \end{bmatrix}$$

32.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -4 \\ -1 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 7\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -4 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ -1 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 4t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ -1 & -1 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -4 & -4 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
6	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{6t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{6t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^t$$
$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{6t} \\ e^{6t} \end{bmatrix} + c_2 \begin{bmatrix} 4\,e^t \\ e^t \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -e^{6t} & 4\,e^t \\ e^{6t} & e^t \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{e^{-6t}}{5} & \frac{4\,e^{-6t}}{5} \\ \frac{e^{-t}}{5} & \frac{e^{-t}}{5} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{6t} & 4\,e^{t} \\ e^{6t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-6t}}{5} & \frac{4\,e^{-6t}}{5} \\ \frac{e^{-t}}{5} & \frac{e^{-t}}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{6t} & 4\,e^{t} \\ e^{6t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-6t}}{5} \\ \frac{e^{-t}}{5} \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{6t} & 4\,e^{t} \\ e^{6t} & e^{t} \end{bmatrix} \begin{bmatrix} \frac{e^{-6t}}{30} \\ -\frac{e^{-t}}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{6} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{6t} \\ c_1 e^{6t} \end{bmatrix} + \begin{bmatrix} 4c_2 e^{t} \\ c_2 e^{t} \end{bmatrix} + \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{6} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{6t} + 4c_2 e^{t} - \frac{5}{6} \\ c_1 e^{6t} + c_2 e^{t} - \frac{1}{6} \end{bmatrix}$$

The following is the phase plot of the system.

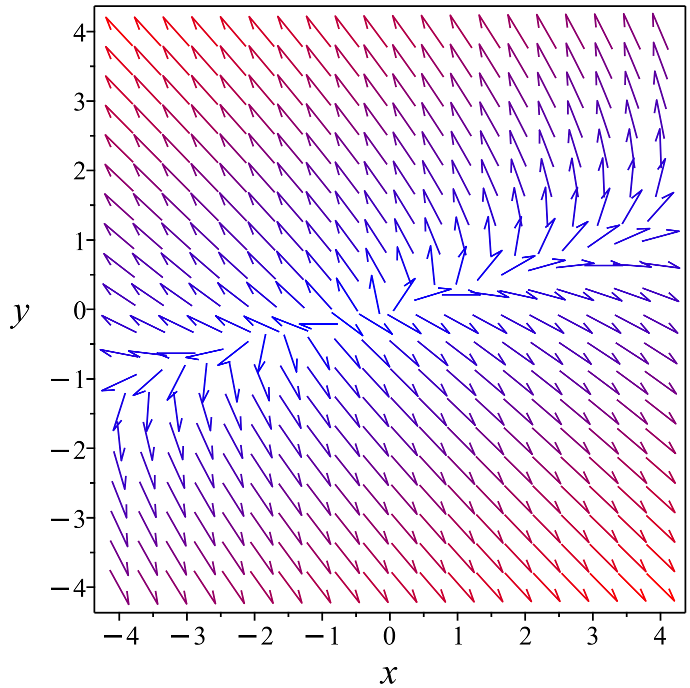


Figure 862: Phase plot

32.3.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - 4y(t) + 1, y'(t) = -x(t) + 5y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{6t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\begin{bmatrix} 4\,e^{t} & -e^{6t} \\ e^{t} & e^{6t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

  $\Phi(t)=\phi(t)\cdot\phi(0)^{-1}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t)=\begin{bmatrix} 4\,e^{t} & -e^{6t} \\ e^{t} & e^{6t} \end{bmatrix}\cdot\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t)=\begin{bmatrix} \frac{4\,e^{t}}{5}+\frac{e^{6t}}{5} & -\frac{4\,e^{6t}}{5}+\frac{4\,e^{t}}{5} \\ -\frac{e^{6t}}{5}+\frac{e^{t}}{5} & \frac{e^{t}}{5}+\frac{4\,e^{6t}}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t)=\Phi(t)\cdot\vec{v}(t)$
- Take the derivative of the particular solution

  $\vec{x}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\Phi(t)^{-1}\cdot\vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \Phi(s)^{-1}\cdot\vec{f}(s)\,ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \Phi(s)^{-1}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{4\,e^t}{5} - \frac{5}{6} + \frac{e^{6t}}{30} \\ -\frac{e^{6t}}{30} - \frac{1}{6} + \frac{e^t}{5} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} \frac{4\,e^t}{5} - \frac{5}{6} + \frac{e^{6t}}{30} \\ -\frac{e^{6t}}{30} - \frac{1}{6} + \frac{e^t}{5} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 4c_1 e^t - c_2 e^{6t} + \frac{4\,e^t}{5} - \frac{5}{6} + \frac{e^{6t}}{30} \\ c_1 e^t + c_2 e^{6t} - \frac{e^{6t}}{30} - \frac{1}{6} + \frac{e^t}{5} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = 4c_1 e^t - c_2 e^{6t} + \frac{4\,e^t}{5} - \frac{5}{6} + \frac{e^{6t}}{30}, y(t) = c_1 e^t + c_2 e^{6t} - \frac{e^{6t}}{30} - \frac{1}{6} + \frac{e^t}{5}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve([diff(x(t),t)=2*x(t)-4*y(t)+1,diff(y(t),t)=-x(t)+5*y(t)],singsol=all)
```

$$x(t) = c_2 e^t + e^{6t} c_1 - \frac{5}{6}$$
$$y(t) = \frac{c_2 e^t}{4} - e^{6t} c_1 - \frac{1}{6}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 67

```
DSolve[{x'[t]==2*x[t]-4*y[t],y'[t]==-x[t]+5*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{5} e^t \left(c_1\left(e^{5t} + 4\right) - 4c_2\left(e^{5t} - 1\right)\right)$$
$$y(t) \to \frac{1}{5} e^t \left(c_1\left(-e^{5t}\right) + 4c_2 e^{5t} + c_1 + c_2\right)$$

32.4 problem 828

Internal problem ID [15551]
Internal file name [`OUTPUT/15551_Saturday_May_11_2024_01_35_30_AM_28168364/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISE-LYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.3 dAlemberts method. Exercises page 243
Problem number: 828.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) + y(t) + \mathrm{e}^{t} \\
y'(t) &= x(t) + 3y(t) - \mathrm{e}^{t}
\end{aligned}
$$

32.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t} \\ -\mathrm{e}^{t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{2t}}{2} + \frac{e^{4t}}{2}\right) c_1 + \left(\frac{e^{4t}}{2} - \frac{e^{2t}}{2}\right) c_2 \\ \left(\frac{e^{4t}}{2} - \frac{e^{2t}}{2}\right) c_1 + \left(\frac{e^{2t}}{2} + \frac{e^{4t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_2+c_1)e^{2t}}{2} + \frac{e^{4t}(c_1+c_2)}{2} \\ \frac{(c_2-c_1)e^{2t}}{2} + \frac{e^{4t}(c_1+c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{e^{-4t}(1+e^{2t})}{2} & -\frac{e^{-4t}(e^{2t}-1)}{2} \\ -\frac{e^{-4t}(e^{2t}-1)}{2} & \frac{e^{-4t}(1+e^{2t})}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-4t}(1+e^{2t})}{2} & -\frac{e^{-4t}(e^{2t}-1)}{2} \\ -\frac{e^{-4t}(e^{2t}-1)}{2} & \frac{e^{-4t}(1+e^{2t})}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -e^{t} \\ e^{t} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(-c_2+c_1)\mathrm{e}^{2t}}{2} + \frac{\mathrm{e}^{4t}(c_1+c_2)}{2} - \mathrm{e}^t \\ \frac{(c_2-c_1)\mathrm{e}^{2t}}{2} + \frac{\mathrm{e}^{4t}(c_1+c_2)}{2} + \mathrm{e}^t \end{bmatrix}$$

32.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^t \\ -\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{2t} & \mathrm{e}^{4t} \\ \mathrm{e}^{2t} & \mathrm{e}^{4t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{2} & \frac{\mathrm{e}^{-2t}}{2} \\ \frac{\mathrm{e}^{-4t}}{2} & \frac{\mathrm{e}^{-4t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} \\ \frac{e^{-4t}}{2} & \frac{e^{-4t}}{2} \end{bmatrix} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} dt$$
$$= \begin{bmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \int \begin{bmatrix} -e^{-t} \\ 0 \end{bmatrix} dt$$
$$= \begin{bmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -e^{t} \\ e^{t} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{2t} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} c_2 e^{4t} \\ c_2 e^{4t} \end{bmatrix} + \begin{bmatrix} -e^{t} \\ e^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{2t} + c_2 e^{4t} - e^{t} \\ c_1 e^{2t} + c_2 e^{4t} + e^{t} \end{bmatrix}$$

32.4.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) + y(t) + e^{t}, y'(t) = x(t) + 3y(t) - e^{t}]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -e^{2t} & e^{4t} \\ e^{2t} & e^{4t} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^{4t}}{2} & \frac{e^{4t}}{2} - \frac{e^{2t}}{2} \\ \frac{e^{4t}}{2} - \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \Phi(t)^{-1} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

$\vec{v}(t) = \int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \Phi(s)^{-1} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} -e^t + e^{2t} \\ -e^{2t} + e^t \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} -e^t + e^{2t} \\ -e^{2t} + e^t \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{2t} + c_2 e^{4t} - e^t + e^{2t} \\ (c_1 - 1) e^{2t} + c_2 e^{4t} + e^t \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = -c_1 e^{2t} + c_2 e^{4t} - e^t + e^{2t}, y(t) = (c_1 - 1) e^{2t} + c_2 e^{4t} + e^t\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 43

```
dsolve([diff(x(t),t)=3*x(t)+y(t)+exp(t),diff(y(t),t)=x(t)+3*y(t)-exp(t)],singsol=all)
```

$$x(t) = \frac{c_1 e^{4t}}{2} - e^t + c_2 e^{2t}$$
$$y(t) = \frac{c_1 e^{4t}}{2} + e^t - c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 70

```
DSolve[{x'[t]==3*x[t]+y[t]+Exp[t],y'[t]==x[t]+3*y[t]-Exp[t]},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to \frac{1}{2}e^t\left((c_1 - c_2)e^t + (c_1 + c_2)e^{3t} - 2\right)$$
$$y(t) \to \frac{1}{2}e^t\left((c_2 - c_1)e^t + (c_1 + c_2)e^{3t} + 2\right)$$

32.5 problem 829

Internal problem ID [15552]
Internal file name [`OUTPUT/15552_Saturday_May_11_2024_01_35_31_AM_85755146/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3 (Systems of differential equations). Section 23.3 dAlemberts method. Exercises page 243
Problem number: 829.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 4y(t) + \cos(t) \\
y'(t) &= -x(t) - 2y(t) + \sin(t)
\end{aligned}
$$

32.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (1+2t)\,c_1 + 4tc_2 \\ -tc_1 + (1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1 + 4c_2)\,t + c_1 \\ (-c_1 - 2c_2)\,t + c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} 1-2t & -4t \\ t & 1+2t \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} \int \begin{bmatrix} 1-2t & -4t \\ t & 1+2t \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} dt \\
&= \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} \begin{bmatrix} (4t-2)\cos(t) + (-2t-3)\sin(t) \\ (t+2)\sin(t) - 2t\cos(t) \end{bmatrix} \\
&= \begin{bmatrix} -3\sin(t) - 2\cos(t) \\ 2\sin(t) \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} 2(c_1 + 2c_2)\, t + c_1 - 3\sin(t) - 2\cos(t) \\ -(c_1 + 2c_2)\, t + c_2 + 2\sin(t) \end{bmatrix}\end{aligned}$$

32.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 4 \\ -1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
0	2	1	Yes	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

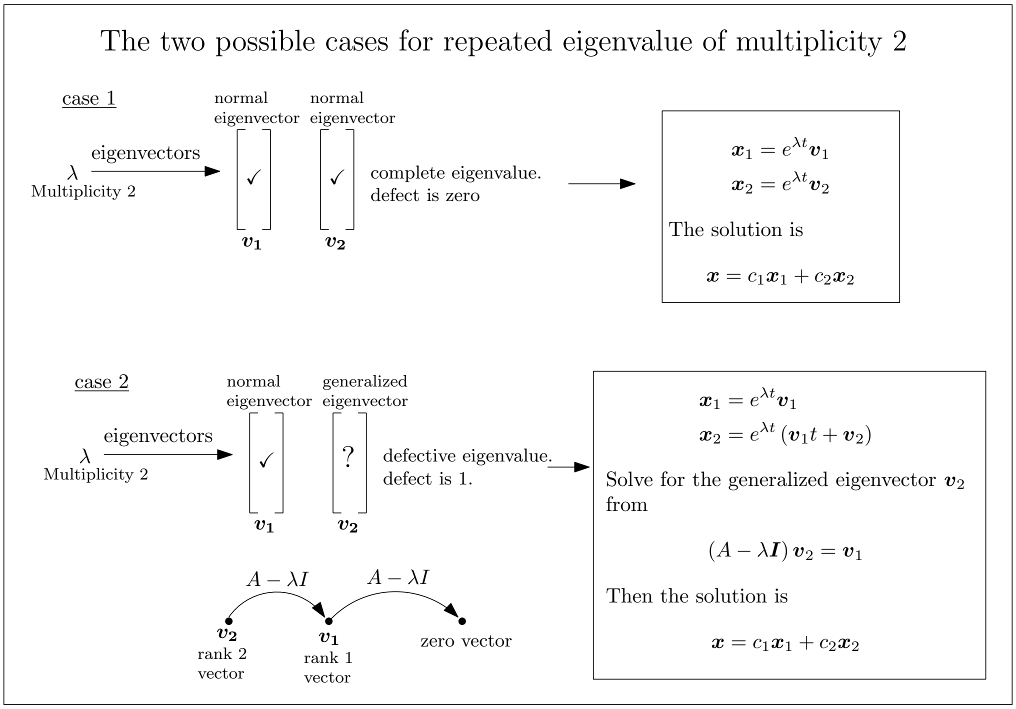


Figure 863: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} 1 \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right) 1 \\ &= \begin{bmatrix} -2t - 3 \\ t + 1 \end{bmatrix}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2t - 3 \\ t + 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -2 & -2t - 3 \\ 1 & t + 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} t+1 & 2t+3 \\ -1 & -2 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -2 & -2t-3 \\ 1 & t+1 \end{bmatrix} \int \begin{bmatrix} t+1 & 2t+3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} dt \\
&= \begin{bmatrix} -2 & -2t-3 \\ 1 & t+1 \end{bmatrix} \int \begin{bmatrix} (t+1)\cos(t) + (2t+3)\sin(t) \\ -\cos(t) - 2\sin(t) \end{bmatrix} dt \\
&= \begin{bmatrix} -2 & -2t-3 \\ 1 & t+1 \end{bmatrix} \begin{bmatrix} (-2t-2)\cos(t) + \sin(t)(t+3) \\ -\sin(t) + 2\cos(t) \end{bmatrix} \\
&= \begin{bmatrix} -3\sin(t) - 2\cos(t) \\ 2\sin(t) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2(-2t-3) \\ c_2(t+1) \end{bmatrix} + \begin{bmatrix} -3\sin(t) - 2\cos(t) \\ 2\sin(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 + c_2(-2t-3) - 3\sin(t) - 2\cos(t) \\ c_2t + 2\sin(t) + c_1 + c_2 \end{bmatrix}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 36

```
dsolve([diff(x(t),t)=2*x(t)+4*y(t)+cos(t),diff(y(t),t)=-x(t)-2*y(t)+sin(t)],singsol=all)
```

$$\begin{aligned}
x(t) &= -2\cos(t) - 3\sin(t) + c_1t + c_2 \\
y(t) &= 2\sin(t) + \frac{c_1}{4} - \frac{c_1t}{2} - \frac{c_2}{2}
\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 46

```
DSolve[{x'[t]==2*x[t]+4*y[t]+Cos[t],y'[t]==-x[t]-2*y[t]+Sin[t]},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to -3\sin(t) - 2\cos(t) + 2c_1t + 4c_2t + c_1$$
$$y(t) \to 2\sin(t) - (c_1 + 2c_2)t + c_2$$

33 Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249

33.1 problem 830

Internal problem ID [15553]
Internal file name [`OUTPUT/15553_Saturday_May_11_2024_01_35_32_AM_99659885/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 830.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + 3x = e^{-2t}}$$

With initial conditions

$$[x(0) = 0]$$

33.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= e^{-2t} \end{aligned}$$

Hence the ode is

$$x' + 3x = e^{-2t}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

33.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + 3Y(s) = \frac{1}{s+2} \tag{1}$$

Replacing initial condition gives

$$sY(s) + 3Y(s) = \frac{1}{s+2}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{(s+2)(s+3)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

The inverse Laplace of each term above is now found, which gives

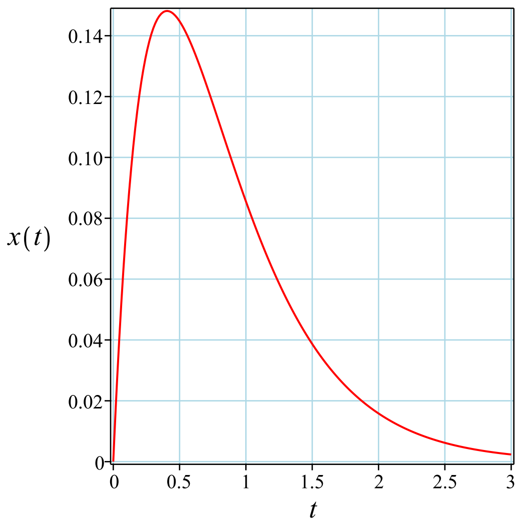
$$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+3}\right) = -e^{-3t}$$

Adding the above results and simplifying gives

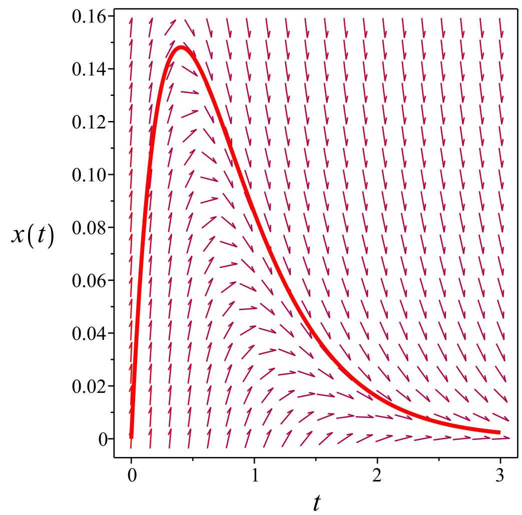
$$x = -e^{-3t} + e^{-2t}$$

Summary
The solution(s) found are the following

$$x = -e^{-3t} + e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -e^{-3t} + e^{-2t}$$

Verified OK.

33.1.3 Maple step by step solution

Let's solve

$[x' + 3x = e^{-2t}, x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -3x + e^{-2t}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + 3x = \mathrm{e}^{-2t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) \left(x' + 3x\right) = \mu(t)\,\mathrm{e}^{-2t}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t) \left(x' + 3x\right) = \mu'(t)\,x + \mu(t)\,x'$
- Isolate $\mu'(t)$

  $\mu'(t) = 3\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{3t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\,\mathrm{e}^{-2t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\,\mathrm{e}^{-2t} dt + c_1$
- Solve for x

  $x = \frac{\int \mu(t)\mathrm{e}^{-2t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{3t}$

  $x = \frac{\int \mathrm{e}^{3t}\mathrm{e}^{-2t} dt + c_1}{\mathrm{e}^{3t}}$
- Evaluate the integrals on the rhs

  $x = \frac{\mathrm{e}^{t} + c_1}{\mathrm{e}^{3t}}$
- Simplify

  $x = \mathrm{e}^{-3t}(\mathrm{e}^{t} + c_1)$
- Use initial condition $x(0) = 0$

  $0 = 1 + c_1$
- Solve for c_1

  $c_1 = -1$
- Substitute $c_1 = -1$ into general solution and simplify

  $x = \mathrm{e}^{-3t}(\mathrm{e}^{t} - 1)$
- Solution to the IVP

$$x = e^{-3t}(e^t - 1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.454 (sec). Leaf size: 15

```
dsolve([diff(x(t),t)+3*x(t)=exp(-2*t),x(0) = 0],x(t), singsol=all)
```

$$x(t) = e^{-2t} - e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 16

```
DSolve[{x'[t]+3*x[t]==Exp[-2*t],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-3t}(e^t - 1)$$

33.2 problem 831

Internal problem ID [15554]
Internal file name [`OUTPUT/15554_Saturday_May_11_2024_01_35_33_AM_23389732/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 831.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' - 3x = 3t^3 + 3t^2 + 2t + 1}$$

With initial conditions

$$[x(0) = -1]$$

33.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 3t^3 + 3t^2 + 2t + 1 \end{aligned}$$

Hence the ode is

$$x' - 3x = 3t^3 + 3t^2 + 2t + 1$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3t^3 + 3t^2 + 2t + 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

33.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) - 3Y(s) = \frac{s^3 + 2s^2 + 6s + 18}{s^4} \tag{1}$$

Replacing initial condition gives

$$sY(s) + 1 - 3Y(s) = \frac{s^3 + 2s^2 + 6s + 18}{s^4}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{s^3 + 2s^2 + 4s + 6}{s^4}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s} - \frac{4}{s^3} - \frac{2}{s^2} - \frac{6}{s^4}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s}\right) = -1$$

$$\mathcal{L}^{-1}\left(-\frac{4}{s^3}\right) = -2t^2$$

$$\mathcal{L}^{-1}\left(-\frac{2}{s^2}\right) = -2t$$

$$\mathcal{L}^{-1}\left(-\frac{6}{s^4}\right) = -t^3$$

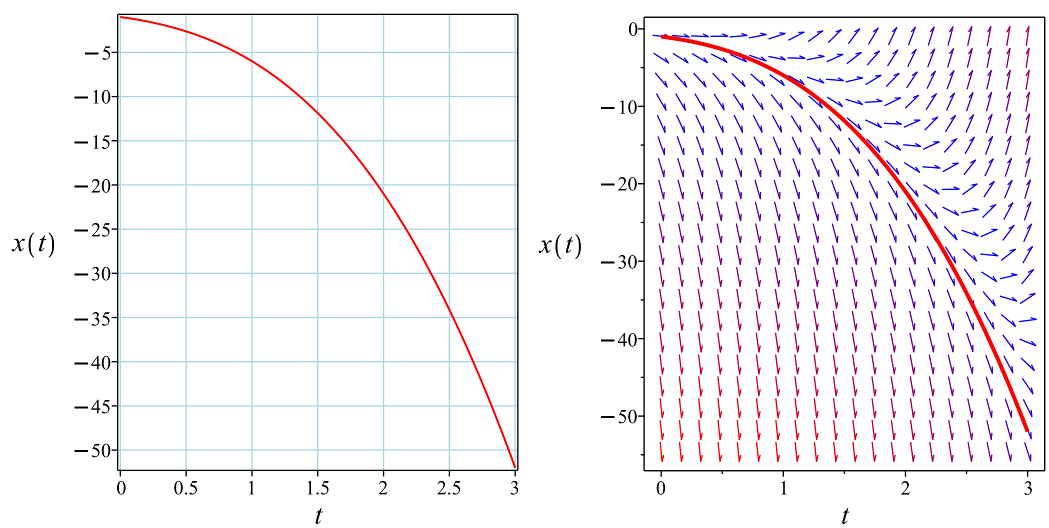
Adding the above results and simplifying gives

$$x = -(t+1)\left(t^2+t+1\right)$$

<u>Summary</u>

The solution(s) found are the following

$$x = -(t+1)\left(t^2+t+1\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = -(t+1)\left(t^2+t+1\right)$$

Verified OK.

33.2.3 Maple step by step solution

Let's solve

$[x' - 3x = 3t^3 + 3t^2 + 2t + 1, x(0) = -1]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = 3x + 3t^3 + 3t^2 + 2t + 1$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - 3x = 3t^3 + 3t^2 + 2t + 1$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(x' - 3x) = \mu(t)\,(3t^3 + 3t^2 + 2t + 1)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\,(x' - 3x) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = -3\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-3t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\,(3t^3 + 3t^2 + 2t + 1)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\,(3t^3 + 3t^2 + 2t + 1)\,dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)(3t^3+3t^2+2t+1)dt+c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-3t}$

  $x = \frac{\int \mathrm{e}^{-3t}(3t^3+3t^2+2t+1)dt+c_1}{\mathrm{e}^{-3t}}$

- Evaluate the integrals on the rhs

  $x = \frac{-(t^3+2t^2+2t+1)\mathrm{e}^{-3t}+c_1}{\mathrm{e}^{-3t}}$

- Simplify

  $x = -t^3 + c_1\mathrm{e}^{3t} - 2t^2 - 2t - 1$

- Use initial condition $x(0) = -1$

  $-1 = c_1 - 1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $x = -t^3 - 2t^2 - 2t - 1$
- Solution to the IVP

  $x = -t^3 - 2t^2 - 2t - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 15

```
dsolve([diff(x(t),t)-3*x(t)=3*t^3+3*t^2+2*t+1,x(0) = -1],x(t), singsol=all)
```

$$x(t) = -(t+1)\left(t^2+t+1\right)$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 20

```
DSolve[{x'[t]-3*x[t]==3*t^3+3*t^2+2*t+1,{x[0]==-1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \rightarrow -t^3 - 2t^2 - 2t - 1$$

33.3 problem 832

Internal problem ID [15555]
Internal file name [`OUTPUT/15555_Saturday_May_11_2024_01_35_33_AM_6112988/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 832.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' - x = \cos(t) - \sin(t)}$$

With initial conditions

$$[x(0) = 0]$$

33.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \cos(t) - \sin(t) \end{aligned}$$

Hence the ode is

$$x' - x = \cos(t) - \sin(t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \cos(t) - \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

33.3.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) - Y(s) = \frac{s-1}{s^2+1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - Y(s) = \frac{s-1}{s^2+1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s^2+1}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{2(s-i)} + \frac{i}{2s+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{2(s-i)}\right) = -\frac{i\mathrm{e}^{it}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{i}{2s+2i}\right) = \frac{i\mathrm{e}^{-it}}{2}$$

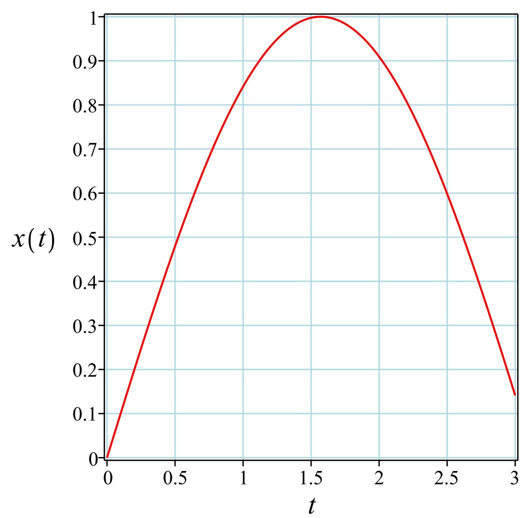
Adding the above results and simplifying gives

$$x = \sin(t)$$

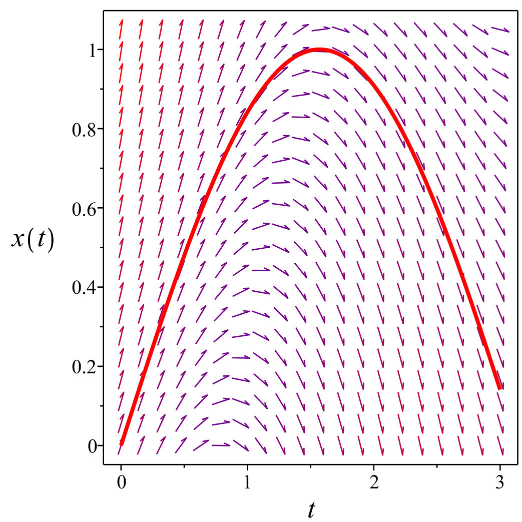
Summary

The solution(s) found are the following

$$x = \sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \sin(t)$$

Verified OK.

33.3.3 Maple step by step solution

Let's solve

$[x' - x = \cos(t) - \sin(t), x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = x + \cos(t) - \sin(t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' - x = \cos(t) - \sin(t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(x' - x) = \mu(t)(\cos(t) - \sin(t))$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) x)$

  $\mu(t)(x' - x) = \mu'(t) x + \mu(t) x'$
- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) x)\right) dt = \int \mu(t)(\cos(t) - \sin(t)) dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t) x = \int \mu(t)(\cos(t) - \sin(t)) dt + c_1$
- Solve for x

  $x = \frac{\int \mu(t)(\cos(t) - \sin(t))dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $x = \frac{\int \mathrm{e}^{-t}(\cos(t) - \sin(t))dt + c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $x = \frac{\mathrm{e}^{-t}\sin(t) + c_1}{\mathrm{e}^{-t}}$
- Simplify

  $x = c_1 \mathrm{e}^{t} + \sin(t)$
- Use initial condition $x(0) = 0$

  $0 = c_1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $x = \sin(t)$
- Solution to the IVP

$$x = \sin(t)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 6

```
dsolve([diff(x(t),t)-x(t)=cos(t)-sin(t),x(0) = 0],x(t), singsol=all)
```

$$x(t) = \sin(t)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 7

```
DSolve[{x'[t]-x[t]==Cos[t]-Sin[t],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \sin(t)$$

33.4 problem 833

Internal problem ID [15556]
Internal file name [`OUTPUT/15556_Saturday_May_11_2024_01_35_34_AM_34637621/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 833.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{2x' + 6x = t\,\mathrm{e}^{-3t}}$$

With initial conditions

$$\left[x(0) = -\frac{1}{2}\right]$$

33.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = \frac{t\,\mathrm{e}^{-3t}}{2}$$

Hence the ode is

$$x' + 3x = \frac{t\,\mathrm{e}^{-3t}}{2}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{t\,\mathrm{e}^{-3t}}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

33.4.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2sY(s) - 2x(0) + 6Y(s) = \frac{1}{(s+3)^2} \tag{1}$$

Replacing initial condition gives

$$2sY(s) + 1 + 6Y(s) = \frac{1}{(s+3)^2}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{s^2 + 6s + 8}{2(s+3)^3}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{2(s+3)} + \frac{1}{2(s+3)^3}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{2\,(s+3)}\right) = -\frac{e^{-3t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{2\,(s+3)^3}\right) = \frac{t^2 e^{-3t}}{4}$$

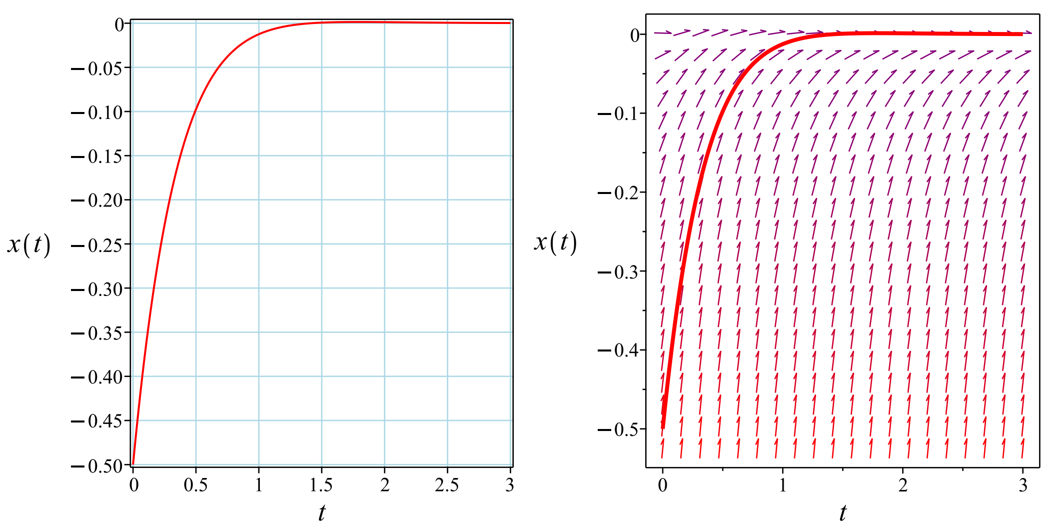
Adding the above results and simplifying gives

$$x = \frac{e^{-3t}(t^2-2)}{4}$$

Summary

The solution(s) found are the following

$$x = \frac{e^{-3t}(t^2-2)}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{e^{-3t}(t^2-2)}{4}$$

Verified OK.

33.4.3 Maple step by step solution

Let's solve

$\left[2x'+6x=t\,\mathrm{e}^{-3t}, x(0)=-\frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x'=-3x+\frac{t\,\mathrm{e}^{-3t}}{2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x'+3x=\frac{t\,\mathrm{e}^{-3t}}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\,(x'+3x)=\frac{\mu(t)t\,\mathrm{e}^{-3t}}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)\,(x'+3x)=\mu'(t)\,x+\mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t)=3\mu(t)$

- Solve to find the integrating factor

  $\mu(t)=\mathrm{e}^{3t}$

- Integrate both sides with respect to t

  $\int\left(\frac{d}{dt}(\mu(t)\,x)\right)dt=\int\frac{\mu(t)t\,\mathrm{e}^{-3t}}{2}dt+c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x=\int\frac{\mu(t)t\,\mathrm{e}^{-3t}}{2}dt+c_1$

- Solve for x

  $x=\frac{\int\frac{\mu(t)t\,\mathrm{e}^{-3t}}{2}dt+c_1}{\mu(t)}$

- Substitute $\mu(t)=\mathrm{e}^{3t}$

  $x=\frac{\int\frac{\mathrm{e}^{3t}t\,\mathrm{e}^{-3t}}{2}dt+c_1}{\mathrm{e}^{3t}}$

- Evaluate the integrals on the rhs

  $x=\frac{\frac{t^2}{4}+c_1}{\mathrm{e}^{3t}}$

- Simplify

$x = \frac{e^{-3t}(t^2+4c_1)}{4}$

- Use initial condition $x(0) = -\frac{1}{2}$

  $-\frac{1}{2} = c_1$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $x = \frac{e^{-3t}(t^2-2)}{4}$

- Solution to the IVP

  $x = \frac{e^{-3t}(t^2-2)}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.391 (sec). Leaf size: 15

```
dsolve([2*diff(x(t),t)+6*x(t)=t*exp(-3*t),x(0) = -1/2],x(t), singsol=all)
```

$$x(t) = \frac{e^{-3t}(t^2-2)}{4}$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 19

```
DSolve[{2*x'[t]+6*x[t]==t*Exp[-3*t],{x[0]==-1/2}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{4}e^{-3t}(t^2-2)$$

33.5 problem 834

Internal problem ID [15557]
Internal file name [`OUTPUT/15557_Saturday_May_11_2024_01_35_34_AM_12648957/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 834.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + x = 2\sin(t)}$$

With initial conditions

$$[x(0) = 0]$$

33.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 2\sin(t) \end{aligned}$$

Hence the ode is

$$x' + x = 2\sin(t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2\sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

33.5.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - x(0) + Y(s) = \frac{2}{s^2+1} \tag{1}$$

Replacing initial condition gives

$$sY(s) + Y(s) = \frac{2}{s^2+1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2}{(s^2+1)(s+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s+1} + \frac{-\frac{1}{2} - \frac{i}{2}}{s-i} + \frac{-\frac{1}{2} + \frac{i}{2}}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = \mathrm{e}^{-t}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}-\frac{i}{2}}{s-i}\right) = \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{it}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}+\frac{i}{2}}{s+i}\right) = \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{-it}$$

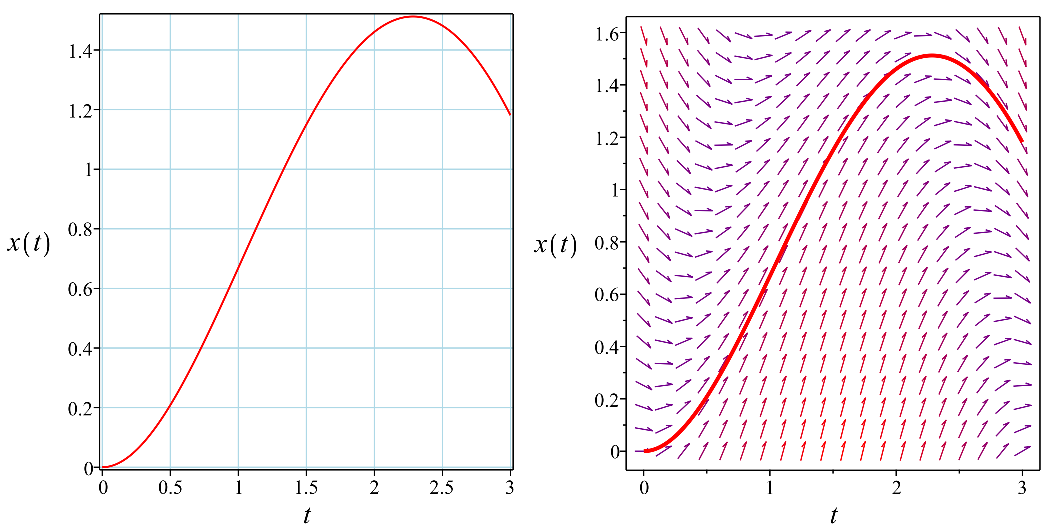
Adding the above results and simplifying gives

$$x = \mathrm{e}^{-t}-\cos\left(t\right)+\sin\left(t\right)$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-t}-\cos\left(t\right)+\sin\left(t\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \mathrm{e}^{-t}-\cos\left(t\right)+\sin\left(t\right)$$

Verified OK.

33.5.3 Maple step by step solution

Let's solve

$[x' + x = 2\sin(t), x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -x + 2\sin(t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + x = 2\sin(t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(x' + x) = 2\mu(t)\sin(t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)(x' + x) = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int 2\mu(t)\sin(t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int 2\mu(t)\sin(t)\,dt + c_1$

- Solve for x

  $x = \frac{\int 2\mu(t)\sin(t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{t}$

  $x = \frac{\int 2\,\mathrm{e}^{t}\sin(t)dt + c_1}{\mathrm{e}^{t}}$

- Evaluate the integrals on the rhs

  $x = \frac{-\mathrm{e}^{t}\cos(t) + \mathrm{e}^{t}\sin(t) + c_1}{\mathrm{e}^{t}}$

- Simplify

  $x = \sin(t) - \cos(t) + c_1\mathrm{e}^{-t}$

- Use initial condition $x(0) = 0$

  $0 = c_1 - 1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $x = e^{-t} - \cos(t) + \sin(t)$
- Solution to the IVP

  $x = e^{-t} - \cos(t) + \sin(t)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.406 (sec). Leaf size: 15

```
dsolve([diff(x(t),t)+x(t)=2*sin(t),x(0) = 0],x(t), singsol=all)
```

$$x(t) = -\cos(t) + \sin(t) + e^{-t}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 17

```
DSolve[{x'[t]+x[t]==2*Sin[t],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t} + \sin(t) - \cos(t)$$

33.6 problem 835

Internal problem ID [15558]
Internal file name [`OUTPUT/15558_Saturday_May_11_2024_01_35_35_AM_58406224/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 835.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{x'' = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

33.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 0 \\
x'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = 0$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}(0) \\
&= 0
\end{aligned}$$

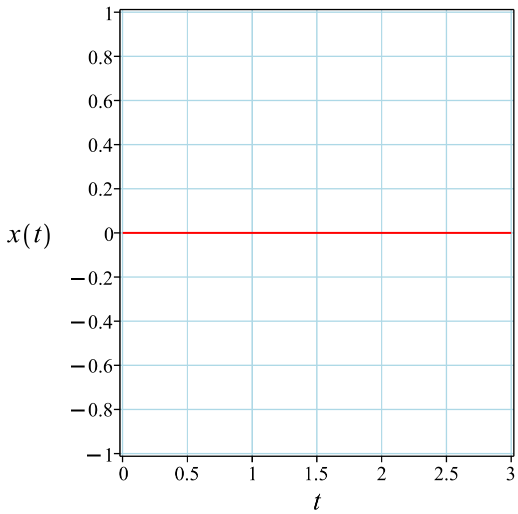
Simplifying the solution gives

$$x = 0$$

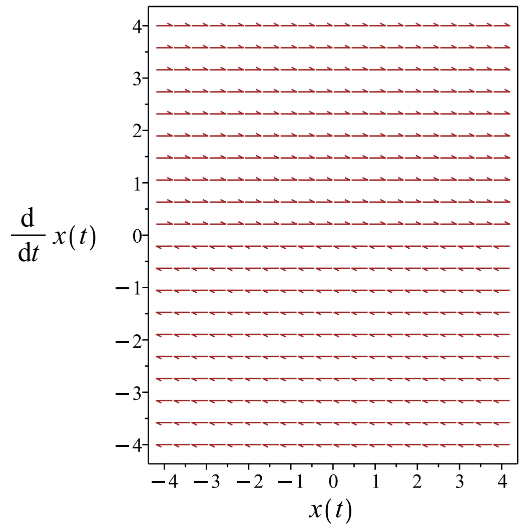
Summary

The solution(s) found are the following

$$x = 0 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 0$$

Verified OK.

33.6.2 Maple step by step solution

Let's solve

$$\left[x'' = 0, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $x_1(t) = 1$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_2 t + c_1$

- ☐ Check validity of solution $x = c_2 t + c_1$
  - Use initial condition $x(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $x' = c_2$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $x = 0$

- Solution to the IVP

  $x = 0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 5

```
dsolve([diff(x(t),t$2)=0,x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = 0$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 6

```
DSolve[{x''[t]==0,{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 0$$

33.7 problem 836

Internal problem ID [15559]
Internal file name [`OUTPUT/15559_Saturday_May_11_2024_01_35_35_AM_43331283/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 836.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__laplace", "exact linear second order ode", "second__order__integrable__as__is", "second__order__ode__missing__y", "second__order__ode__quadrature", "second__order__linear__constant__coeff", "second__order__ode__can__be__made__integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{x'' = 1}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

33.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 0 \\ F &= 1 \end{aligned}$$

Hence the ode is

$$x'' = 1$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) = \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) = \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s^3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}x &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{1}{s^3}\right)\\ &= \frac{t^2}{2}\end{aligned}$$

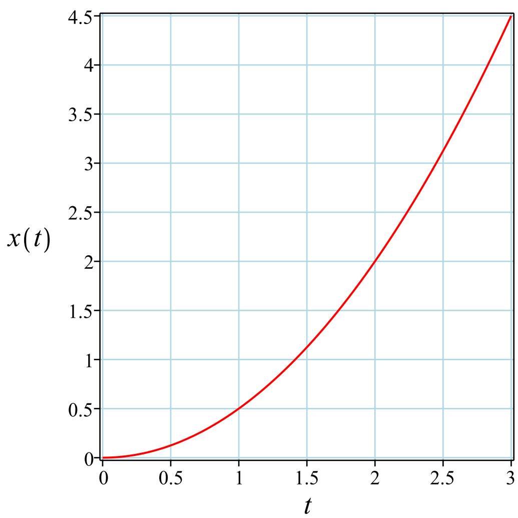
Simplifying the solution gives

$$x = \frac{t^2}{2}$$

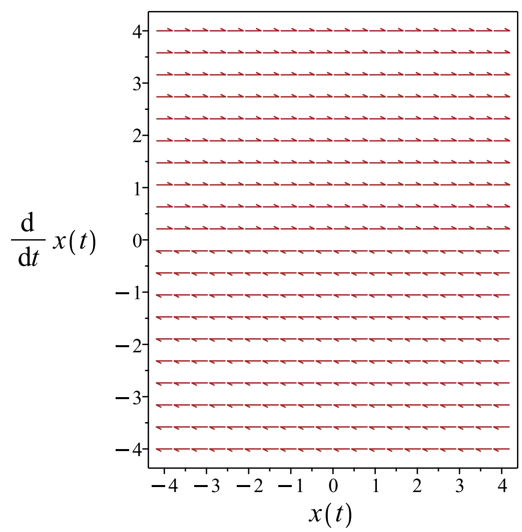
Summary

The solution(s) found are the following

$$x = \frac{t^2}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{t^2}{2}$$

Verified OK.

33.7.2 Maple step by step solution

Let's solve

$$\left[x'' = 1, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the homogeneous ODE

  $x_1(t) = 1$
- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 t + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 1\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$
  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\left(\int t dt\right) + t\left(\int 1 dt\right)$
  - Compute integrals

    $x_p(t) = \frac{t^2}{2}$
- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 t + \frac{1}{2}t^2$

- Check validity of solution $x = c_1 + c_2 t + \frac{1}{2}t^2$
  - Use initial condition $x(0) = 0$

    $0 = c_1$
  - Compute derivative of the solution

    $x' = c_2 + t$
  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $x = \frac{t^2}{2}$
- Solution to the IVP

  $x = \frac{t^2}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.343 (sec). Leaf size: 9

```
dsolve([diff(x(t),t$2)=1,x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{t^2}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 12

```
DSolve[{x''[t]==1,{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{t^2}{2}$$

33.8 problem 837

Internal problem ID [15560]
Internal file name [`OUTPUT/15560_Saturday_May_11_2024_01_35_36_AM_81779700/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 837.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_quadrature", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _quadrature]]
```

$$\boxed{x'' = \cos(t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

33.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 0 \\ F &= \cos(t) \end{aligned}$$

Hence the ode is

$$x'' = \cos(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0) \\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) = \frac{s}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0 \\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) = \frac{s}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s(s^2+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{2(s-i)} - \frac{1}{2(s+i)} + \frac{1}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{2(s-i)}\right) = -\frac{e^{it}}{2}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{2(s+i)}\right) = -\frac{e^{-it}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

Adding the above results and simplifying gives

$$x = 1-\cos(t)$$

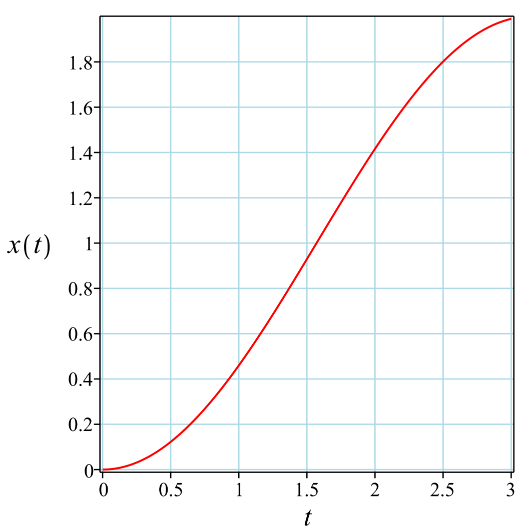
Simplifying the solution gives

$$x = 1-\cos(t)$$

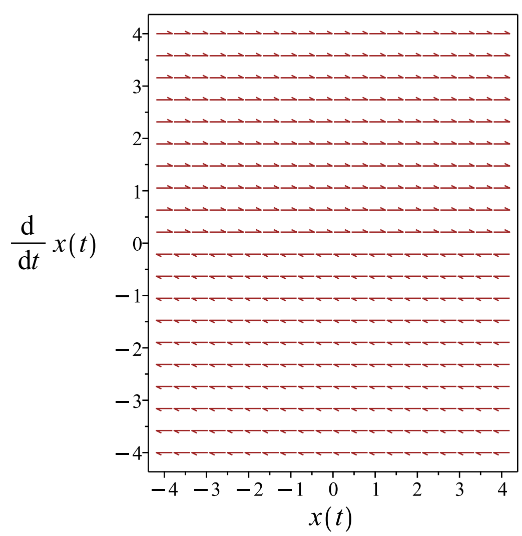
Summary

The solution(s) found are the following

$$x = 1-\cos(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1-\cos(t)$$

Verified OK.

33.8.2 Maple step by step solution

Let's solve

$\left[x'' = \cos(t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $x_1(t) = 1$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2t + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \cos(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

  ◦ Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$

  ◦ Substitute functions into equation for $x_p(t)$

$x_p(t) = -\left(\int t\cos(t)\,dt\right) + t\left(\int \cos(t)\,dt\right)$

- Compute integrals

  $x_p(t) = -\cos(t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 t - \cos(t)$

□ Check validity of solution $x = c_1 + c_2 t - \cos(t)$

- Use initial condition $x(0) = 0$

  $0 = c_1 - 1$

- Compute derivative of the solution

  $x' = c_2 + \sin(t)$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = 1 - \cos(t)$

- Solution to the IVP

  $x = 1 - \cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 10

```
dsolve([diff(x(t),t$2)=cos(t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = 1 - \cos(t)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 11

```
DSolve[{x''[t]==Cos[t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 1 - \cos(t)$$

33.9 problem 838

Internal problem ID [15561]
Internal file name [`OUTPUT/15561_Saturday_May_11_2024_01_35_36_AM_99754628/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 838.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + x' = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

33.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + x' = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + sY(s) - x(0) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 0\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = 0$$

Taking the inverse Laplace transform gives

$$\begin{aligned}x &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}(0)\\ &= 0\end{aligned}$$

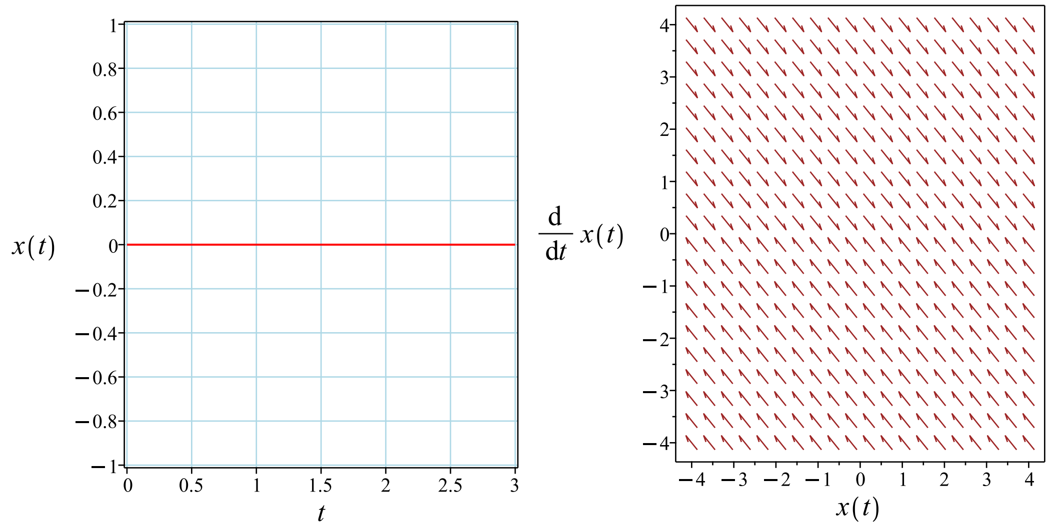
Simplifying the solution gives

$$x = 0$$

Summary

The solution(s) found are the following

$$x = 0 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = 0$$

Verified OK.

33.9.2 Maple step by step solution

Let's solve

$$\left[x'' + x' = 0, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of ODE
  $r^2 + r = 0$
- Factor the characteristic polynomial
  $r(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$
- 1st solution of the ODE

  $x_1(t) = e^{-t}$
- 2nd solution of the ODE

  $x_2(t) = 1$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$
- Substitute in solutions

  $x = c_1 e^{-t} + c_2$

□ Check validity of solution $x = c_1 e^{-t} + c_2$

  - Use initial condition $x(0) = 0$

    $0 = c_1 + c_2$
  - Compute derivative of the solution

    $x' = -c_1 e^{-t}$
  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = -c_1$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $x = 0$

- Solution to the IVP

  $x = 0$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 5

```
dsolve([diff(x(t),t$2)+diff(x(t),t)=0,x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = 0$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 6

```
DSolve[{x''[t]+x'[t]==0,{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 0$$

33.10 problem 839

Internal problem ID [15562]
Internal file name [`OUTPUT/15562_Saturday_May_11_2024_01_35_37_AM_34974537/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 839.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__laplace", "exact linear second order ode", "second__order__integrable__as__is", "second__order__ode__missing__y", "second__order__linear__constant__coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + x' = 0}$$

With initial conditions

$$[x(0) = 1, x'(0) = -1]$$

33.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + x' = 0$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + sY(s) - x(0) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 1\\ x'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s + sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s+1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}x &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\\ &= e^{-t}\end{aligned}$$

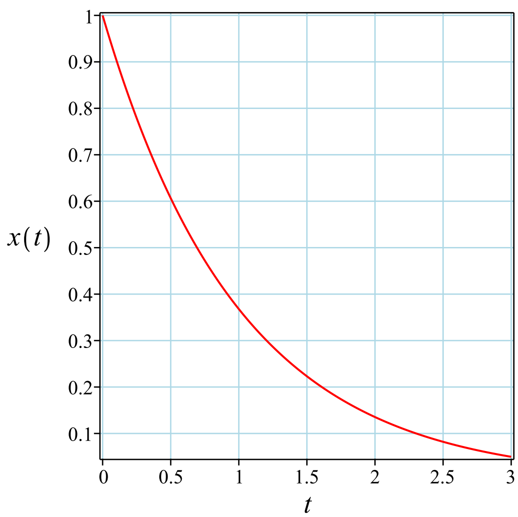
Simplifying the solution gives

$$x = e^{-t}$$

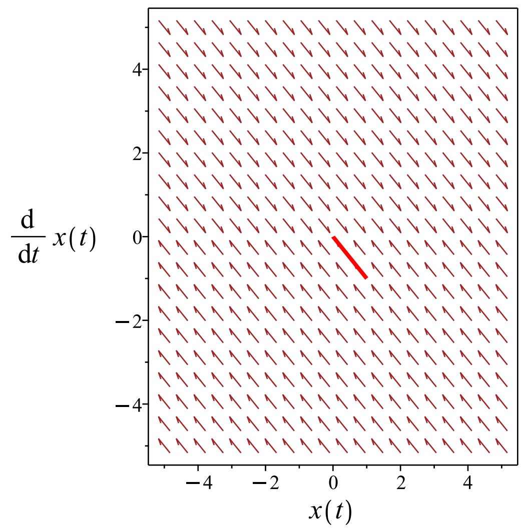
Summary

The solution(s) found are the following

$$x = e^{-t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-t}$$

Verified OK.

33.10.2 Maple step by step solution

Let's solve

$$\left[x'' + x' = 0, x(0) = 1, x'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-t}$

- 2nd solution of the ODE

  $x_2(t) = 1$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \mathrm{e}^{-t} + c_2$

☐ Check validity of solution $x = c_1 \mathrm{e}^{-t} + c_2$

  - Use initial condition $x(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $x' = -c_1 \mathrm{e}^{-t}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = -1$

    $-1 = -c_1$

  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $x = \mathrm{e}^{-t}$

- Solution to the IVP

  $x = \mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.282 (sec). Leaf size: 8

```
dsolve([diff(x(t),t$2)+diff(x(t),t)=0,x(0) = 1, D(x)(0) = -1],x(t), singsol=all)
```

$$x(t) = e^{-t}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 10

```
DSolve[{x''[t]+x'[t]==0,{x[0]==1,x'[0]==-1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{-t}$$

33.11 problem 840

Internal problem ID [15563]
Internal file name [`OUTPUT/15563_Saturday_May_11_2024_01_35_37_AM_84437884/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 840.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - x' = 1}$$

With initial conditions

$$[x(0) = -1, x'(0) = -1]$$

33.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 0 \\ F &= 1 \end{aligned}$$

Hence the ode is

$$x'' - x' = 1$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - sY(s) + x(0) = \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= -1\\ x'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + s - sY(s) = \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s+1}{s^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s} - \frac{1}{s^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s}\right) = -1$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s^2}\right) = -t$$

Adding the above results and simplifying gives

$$x = -t - 1$$

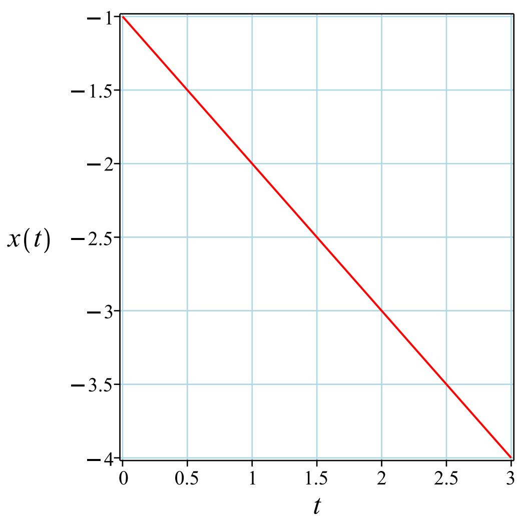
Simplifying the solution gives

$$x = -t - 1$$

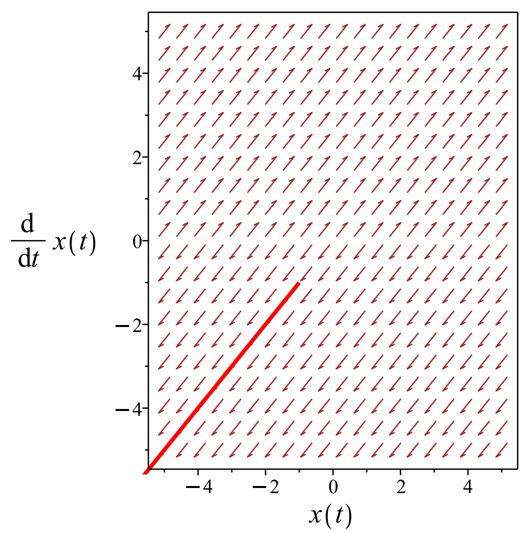
Summary

The solution(s) found are the following

$$x = -t - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -t - 1$$

Verified OK.

33.11.2 Maple step by step solution

Let's solve

$$\left[x''-x'=1, x(0)=-1, x'\Big|_{\{t=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2-r=0$

- Factor the characteristic polynomial

  $r(r-1)=0$

- Roots of the characteristic polynomial

  $r=(0,1)$

- 1st solution of the homogeneous ODE

  $x_1(t)=1$

- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^t$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1+c_2\mathrm{e}^t+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $$\left[x_p(t)=-x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=1\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(x_1(t), x_2(t))=\begin{bmatrix} 1 & \mathrm{e}^t \\ 0 & \mathrm{e}^t \end{bmatrix}$$

  - Compute Wronskian

    $W(x_1(t), x_2(t))=\mathrm{e}^t$

  - Substitute functions into equation for $x_p(t)$

$x_p(t) = -\left(\int 1dt\right) + e^t \left(\int e^{-t}dt\right)$

- Compute integrals

  $x_p(t) = -t - 1$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 e^t - t - 1$

□ Check validity of solution $x = c_1 + c_2 e^t - t - 1$

- Use initial condition $x(0) = -1$

  $-1 = c_1 + c_2 - 1$

- Compute derivative of the solution

  $x' = c_2 e^t - 1$

- Use the initial condition $x'\Big|_{\{t=0\}} = -1$

  $-1 = c_2 - 1$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = -t - 1$

- Solution to the IVP

  $x = -t - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)+1, _b(_a)`   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 9

```
dsolve([diff(x(t),t$2)-diff(x(t),t)=1,x(0) = -1, D(x)(0) = -1],x(t), singsol=all)
```

$$x(t) = -t - 1$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 10

```
DSolve[{x''[t]-x'[t]==1,{x[0]==-1,x'[0]==-1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -t - 1$$

33.12 problem 841

Internal problem ID [15564]
Internal file name [`OUTPUT/15564_Saturday_May_11_2024_01_35_37_AM_31944592/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 841.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x'' + x = t}$$

With initial conditions

$$[x(0) = 0, x'(0) = 1]$$

33.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= t \end{aligned}$$

Hence the ode is

$$x'' + x = t$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2 Y(s) - x'(0) - sx(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - x'(0) - sx(0) + Y(s) = \frac{1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 0 \\
x'(0) &= 1
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - 1 + Y(s) = \frac{1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s^2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} x &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) \\ &= t \end{aligned}$$

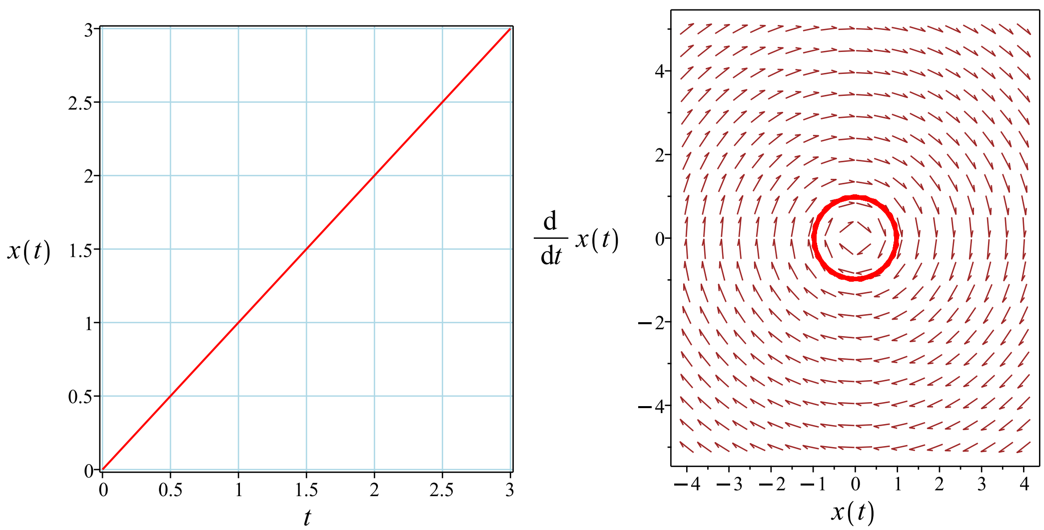
Simplifying the solution gives

$$x = t$$

Summary

The solution(s) found are the following

$$x = t \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = t$$

Verified OK.

33.12.2 Maple step by step solution

Let's solve

$$\left[x'' + x = t, x(0) = 0, x'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = t\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 1$

  - Substitute functions into equation for $x_p(t)$

$x_p(t) = -\cos(t)\left(\int t\sin(t)\,dt\right) + \sin(t)\left(\int t\cos(t)\,dt\right)$

- Compute integrals

  $x_p(t) = t$

- Substitute particular solution into general solution to ODE

  $x = c_1\cos(t) + c_2\sin(t) + t$

☐ Check validity of solution $x = c_1\cos(t) + c_2\sin(t) + t$

- Use initial condition $x(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $x' = -c_1\sin(t) + c_2\cos(t) + 1$

- Use the initial condition $x'\big|_{\{t=0\}} = 1$

  $1 = 1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = t$

- Solution to the IVP

  $x = t$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.329 (sec). Leaf size: 5

```
dsolve([diff(x(t),t$2)+x(t)=t,x(0) = 0, D(x)(0) = 1],x(t), singsol=all)
```

$$x(t) = t$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 6

```
DSolve[{x''[t]+x[t]==t,{x[0]==0,x'[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to t$$

33.13 problem 842

Internal problem ID [15565]
Internal file name [`OUTPUT/15565_Tuesday_May_14_2024_10_48_01_PM_54640770/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 842.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x'' + 6x' = 12t + 2}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

33.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 0 \\ F &= 12t + 2 \end{aligned}$$

Hence the ode is

$$x'' + 6x' = 12t + 2$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = 12t + 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(x') &= sY(s) - x(0) \\
\mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + 6sY(s) - 6x(0) = \frac{2}{s} + \frac{12}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
x(0) &= 0 \\
x'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 6sY(s) = \frac{2}{s} + \frac{12}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2}{s^3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2}{s^3}\right) \\
&= t^2
\end{aligned}$$

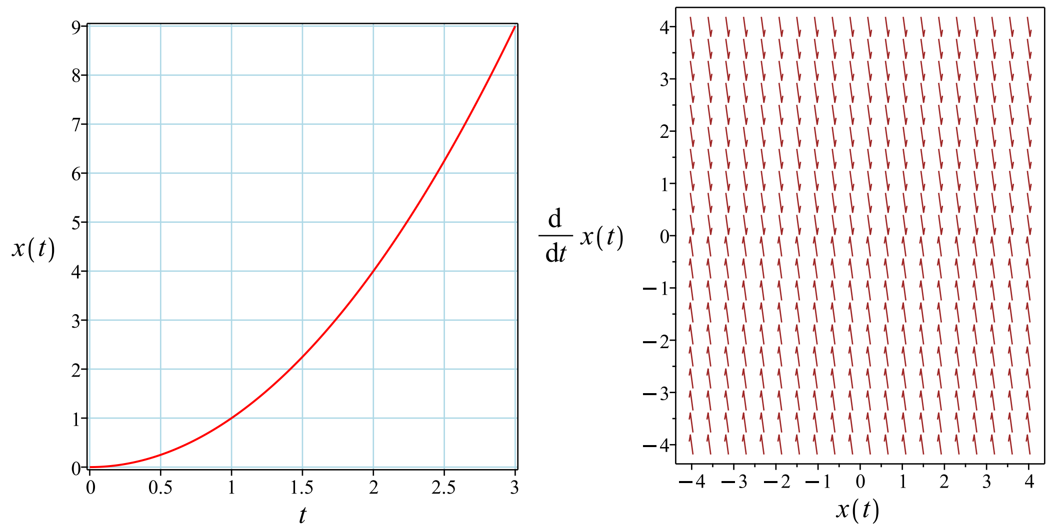
Simplifying the solution gives

$$x = t^2$$

Summary

The solution(s) found are the following

$$x = t^2 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = t^2$$

Verified OK.

33.13.2 Maple step by step solution

Let's solve

$\left[x'' + 6x' = 12t + 2, x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r = 0$

- Factor the characteristic polynomial

  $r(r+6)=0$
- Roots of the characteristic polynomial

  $r=(-6,0)$
- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{-6t}$
- 2nd solution of the homogeneous ODE

  $x_2(t)=1$
- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{-6t}+c_2+x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t)=-x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=12t+2\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t),x_2(t))=\begin{bmatrix} \mathrm{e}^{-6t} & 1 \\ -6\,\mathrm{e}^{-6t} & 0 \end{bmatrix}$
  - Compute Wronskian

    $W(x_1(t),x_2(t))=6\,\mathrm{e}^{-6t}$
  - Substitute functions into equation for $x_p(t)$

    $x_p(t)=-\frac{\mathrm{e}^{-6t}\left(\int(6t+1)\mathrm{e}^{6t}dt\right)}{3}+\frac{\left(\int(6t+1)dt\right)}{3}$
  - Compute integrals

    $x_p(t)=t^2$
- Substitute particular solution into general solution to ODE

  $x=c_1\mathrm{e}^{-6t}+c_2+t^2$

☐ Check validity of solution $x=c_1\mathrm{e}^{-6t}+c_2+t^2$

  - Use initial condition $x(0)=0$

    $0=c_1+c_2$

- Compute derivative of the solution

  $x' = -6c_1 e^{-6t} + 2t$

- Use the initial condition $x'\big|_{\{t=0\}} = 0$

  $0 = -6c_1$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $x = t^2$

- Solution to the IVP

  $x = t^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -6*_b(_a)+12*_a+2, _b(_a)`   *** Sublev
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 7

```
dsolve([diff(x(t),t$2)+6*diff(x(t),t)=12*t+2,x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x = t^2$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 8

```
DSolve[{x''[t]+6*x'[t]==12*t+2,{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \rightarrow t^2$$

33.14 problem 843

Internal problem ID [15566]
Internal file name [`OUTPUT/15566_Tuesday_May_14_2024_10_48_02_PM_31591721/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 843.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' - 2x' + 2x = 2}$$

With initial conditions

$$[x(0) = 1, x'(0) = 0]$$

33.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 2 \\ F &= 2 \end{aligned}$$

Hence the ode is

$$x'' - 2x' + 2x = 2$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) - 2sY(s) + 2x(0) + 2Y(s) = \frac{2}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= 1\\ x'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - 2sY(s) + 2Y(s) = \frac{2}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
x &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{1}{s}\right) \\
&= 1
\end{aligned}
$$

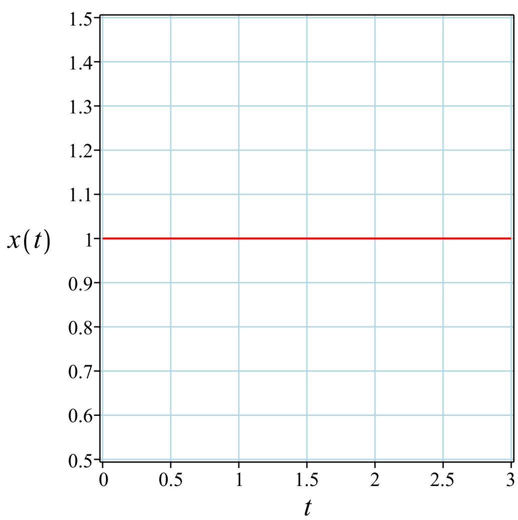
Simplifying the solution gives

$$x = 1$$

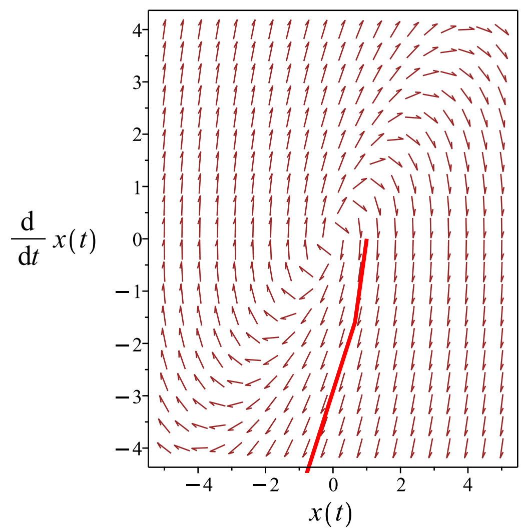
Summary

The solution(s) found are the following

$$x = 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1$$

Verified OK.

33.14.2 Maple step by step solution

Let's solve

$\left[x'' - 2x' + 2x = 2, x(0) = 1, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^t \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^t \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^t \cos(t) + c_2 \mathrm{e}^t \sin(t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ∘ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 2\right]$

  ∘ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^t \cos(t) & \mathrm{e}^t \sin(t) \\ \mathrm{e}^t \cos(t) - \mathrm{e}^t \sin(t) & \mathrm{e}^t \sin(t) + \mathrm{e}^t \cos(t) \end{bmatrix}$

  ∘ Compute Wronskian

  $W(x_1(t), x_2(t)) = \mathrm{e}^{2t}$

  ∘ Substitute functions into equation for $x_p(t)$

$$x_p(t) = -2\,\mathrm{e}^t \left(\cos(t) \left(\int \sin(t)\,\mathrm{e}^{-t} dt\right) - \sin(t) \left(\int \cos(t)\,\mathrm{e}^{-t} dt\right)\right)$$

- ◦ Compute integrals

  $x_p(t) = 1$

- Substitute particular solution into general solution to ODE

  $x = c_1\mathrm{e}^t \cos(t) + c_2\mathrm{e}^t \sin(t) + 1$

- □ Check validity of solution $x = c_1\mathrm{e}^t \cos(t) + c_2\mathrm{e}^t \sin(t) + 1$
  - ◦ Use initial condition $x(0) = 1$

    $1 = c_1 + 1$

  - ◦ Compute derivative of the solution

    $x' = c_1\mathrm{e}^t \cos(t) - c_1\mathrm{e}^t \sin(t) + c_2\mathrm{e}^t \sin(t) + c_2\mathrm{e}^t \cos(t)$

  - ◦ Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = c_1 + c_2$

  - ◦ Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$

  - ◦ Substitute constant values into general solution and simplify

    $x = 1$

- Solution to the IVP

  $x = 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(x(t),t$2)-2*diff(x(t),t)+2*x(t)=2,x(0) = 1, D(x)(0) = 0],x(t), singsol=all)
```

$$x = 1$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 6

```
DSolve[{x''[t]-2*x'[t]+2*x[t]==2,{x[0]==1,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True
```

$$x(t) \to 1$$

33.15 problem 844

Internal problem ID [15567]
Internal file name [`OUTPUT/15567_Tuesday_May_14_2024_10_48_02_PM_66644115/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 844.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 4x' + 4x = 4}$$

With initial conditions

$$[x(0) = 1, x'(0) = -4]$$

33.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 4 \\ F &= 4 \end{aligned}$$

Hence the ode is

$$x'' + 4x' + 4x = 4$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(x') &= sY(s) - x(0) \\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + 4sY(s) - 4x(0) + 4Y(s) = \frac{4}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned} x(0) &= 1 \\ x'(0) &= -4 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - s + 4sY(s) + 4Y(s) = \frac{4}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 4}{s\,(s^2 + 4s + 4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s} - \frac{4}{(s+2)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathcal{L}^{-1}\left(-\frac{4}{(s+2)^2}\right) = -4t\,e^{-2t}$$

Adding the above results and simplifying gives

$$x = 1 - 4t\,e^{-2t}$$

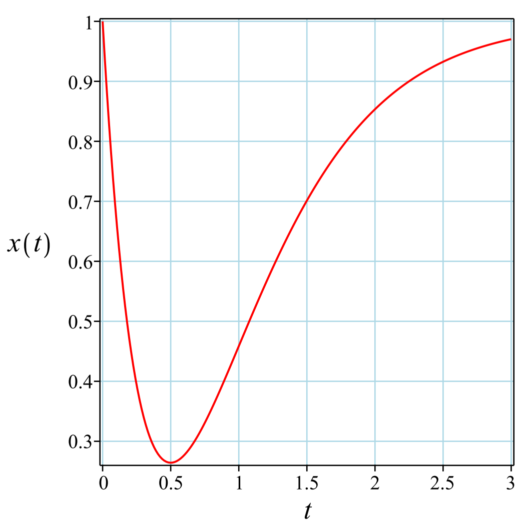
Simplifying the solution gives

$$x = 1 - 4t\,e^{-2t}$$

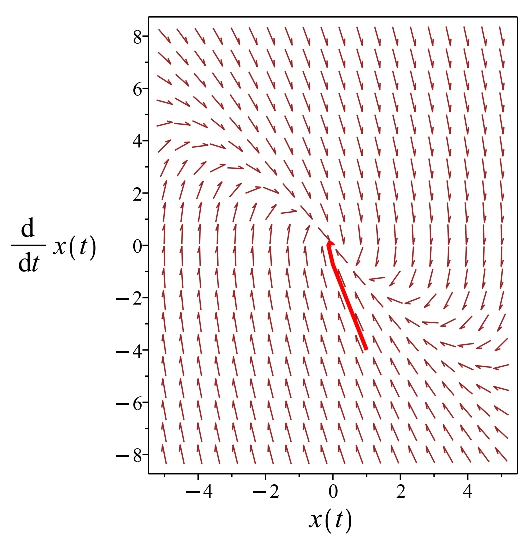
Summary

The solution(s) found are the following

$$x = 1 - 4t\,e^{-2t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 1 - 4t\,e^{-2t}$$

Verified OK.

33.15.2 Maple step by step solution

Let's solve

$\left[x''+4x'+4x=4, x(0)=1, x'\Big|_{\{t=0\}}=-4\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+4=0$

- Factor the characteristic polynomial

  $(r+2)^2=0$

- Root of the characteristic polynomial

  $r=-2$

- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{-2t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t)=t\,\mathrm{e}^{-2t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{-2t}+c_2t\,\mathrm{e}^{-2t}+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=4\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t))=\begin{bmatrix}\mathrm{e}^{-2t} & t\,\mathrm{e}^{-2t}\\ -2\,\mathrm{e}^{-2t} & \mathrm{e}^{-2t}-2t\,\mathrm{e}^{-2t}\end{bmatrix}$

  ◦ Compute Wronskian

    $W(x_1(t), x_2(t))=\mathrm{e}^{-4t}$

  ◦ Substitute functions into equation for $x_p(t)$

$$x_p(t) = -4\,e^{-2t}\left(\int t\,e^{2t}dt - \left(\int e^{2t}dt\right)t\right)$$

- Compute integrals

  $$x_p(t) = 1$$

- Substitute particular solution into general solution to ODE

  $$x = c_1 e^{-2t} + c_2 t\,e^{-2t} + 1$$

☐ Check validity of solution $x = c_1 e^{-2t} + c_2 t e^{-2t} + 1$

- Use initial condition $x(0) = 1$

  $$1 = c_1 + 1$$

- Compute derivative of the solution

  $$x' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t\,e^{-2t}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = -4$

  $$-4 = -2c_1 + c_2$$

- Solve for c_1 and c_2

  $$\{c_1 = 0, c_2 = -4\}$$

- Substitute constant values into general solution and simplify

  $$x = 1 - 4t\,e^{-2t}$$

- Solution to the IVP

  $$x = 1 - 4t\,e^{-2t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(x(t),t$2)+4*diff(x(t),t)+4*x(t)=4,x(0) = 1, D(x)(0) = -4],x(t), singsol=all)
```

$$x = 1 - 4t\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 15

```
DSolve[{x''[t]+4*x'[t]+4*x[t]==4,{x[0]==1,x'[0]==-4}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to 1 - 4e^{-2t}t$$

33.16 problem 845

Internal problem ID [15568]
Internal file name [`OUTPUT/15568_Tuesday_May_14_2024_10_48_03_PM_99659885/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 845.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{2x'' - 2x' = (1+t)\,\mathrm{e}^{t}}$$

With initial conditions

$$\left[x(0) = \frac{1}{2}, x'(0) = \frac{1}{2}\right]$$

33.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= 0 \\ F &= \frac{(1+t)\,\mathrm{e}^{t}}{2} \end{aligned}$$

Hence the ode is

$$x'' - x' = \frac{(1+t)\,\mathrm{e}^{t}}{2}$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $F = \frac{(1+t)\mathrm{e}^{t}}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(x') = sY(s) - x(0)$$
$$\mathcal{L}(x'') = s^2Y(s) - x'(0) - sx(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2s^2Y(s) - 2x'(0) - 2sx(0) - 2sY(s) + 2x(0) = \frac{s}{(s-1)^2} \tag{1}$$

But the initial conditions are

$$x(0) = \frac{1}{2}$$
$$x'(0) = \frac{1}{2}$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^2Y(s) - s - 2sY(s) = \frac{s}{(s-1)^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 - 2s + 2}{2\,(s-1)^3}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2(s-1)^3} + \frac{1}{2s-2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2(s-1)^3}\right) = \frac{t^2 e^t}{4}$$
$$\mathcal{L}^{-1}\left(\frac{1}{2s-2}\right) = \frac{e^t}{2}$$

Adding the above results and simplifying gives

$$x = \frac{e^t(t^2+2)}{4}$$

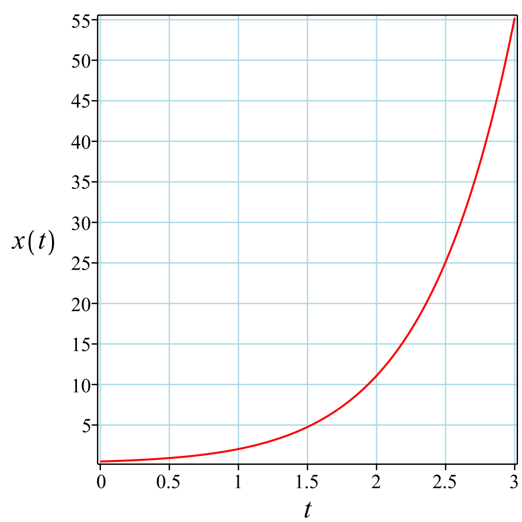
Simplifying the solution gives

$$x = \frac{e^t(t^2+2)}{4}$$

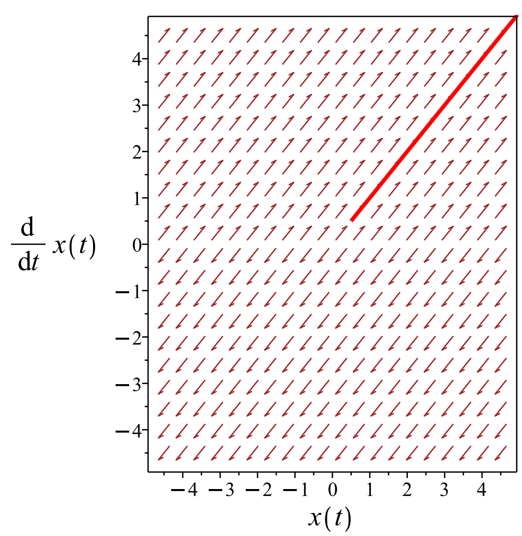
Summary

The solution(s) found are the following

$$x = \frac{e^t(t^2+2)}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^t (t^2 + 2)}{4}$$

Verified OK.

33.16.2 Maple step by step solution

Let's solve

$\left[2x'' - 2x' = (1+t)\,e^t, x(0) = \frac{1}{2}, x'\Big|_{\{t=0\}} = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 2

  x''
- Isolate 2nd derivative

  $x'' = \frac{e^t t}{2} + x' + \frac{e^t}{2}$
- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' - x' = \frac{(1+t)e^t}{2}$
- Characteristic polynomial of homogeneous ODE

  $r^2 - r = 0$
- Factor the characteristic polynomial

  $r(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (0, 1)$
- 1st solution of the homogeneous ODE

  $x_1(t) = 1$
- 2nd solution of the homogeneous ODE

  $x_2(t) = e^t$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x = c_1 + c_2 e^t + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \frac{(1+t)e^t}{2}\right]$
- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} 1 & e^t \\ 0 & e^t \end{bmatrix}$
- Compute Wronskian

  $W(x_1(t), x_2(t)) = e^t$
- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{\left(\int (1+t)e^t dt\right)}{2} + \frac{e^t\left(\int (1+t)dt\right)}{2}$
- Compute integrals

  $x_p(t) = \frac{t^2 e^t}{4}$

- Substitute particular solution into general solution to ODE

  $x = c_1 + c_2 e^t + \frac{t^2 e^t}{4}$

□ Check validity of solution $x = c_1 + c_2 e^t + \frac{t^2 e^t}{4}$

- Use initial condition $x(0) = \frac{1}{2}$

  $\frac{1}{2} = c_1 + c_2$
- Compute derivative of the solution

  $x' = c_2 e^t + \frac{e^t t}{2} + \frac{t^2 e^t}{4}$
- Use the initial condition $x'\Big|_{\{t=0\}} = \frac{1}{2}$

  $\frac{1}{2} = c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = \frac{1}{2}\right\}$
- Substitute constant values into general solution and simplify

  $x = \frac{e^t(t^2+2)}{4}$

- Solution to the IVP

  $x = \frac{e^t(t^2+2)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)+(1/2)*exp(_a)*_a+(1/2)*exp(_a),
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([2*diff(x(t),t$2)-2*diff(x(t),t)=(1+t)*exp(t),x(0) = 1/2, D(x)(0) = 1/2],x(t), singso
```

$$x = \frac{e^{t}(t^{2}+2)}{4}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 17

```
DSolve[{2*x''[t]-2*x'[t]==(1+t)*Exp[t],{x[0]==1/2,x'[0]==1/2}},x[t],t,IncludeSingularSolutio
```

$$x(t) \rightarrow \frac{1}{4}e^{t}\left(t^{2}+2\right)$$

33.17 problem 846

Internal problem ID [15569]
Internal file name [`OUTPUT/15569_Tuesday_May_14_2024_10_48_03_PM_85677221/index.tex`]

Book: A book of problems in ordinary differential equations. M.L. KRASNOV, A.L. KISELYOV, G.I. MARKARENKO. MIR, MOSCOW. 1983
Section: Chapter 3. Section 24.2. Solving the Cauchy problem for linear differential equation with constant coefficients. Exercises page 249
Problem number: 846.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + x = 2\cos(t)}$$

With initial conditions

$$[x(0) = -1, x'(0) = 1]$$

33.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 1 \\ F &= 2\cos(t) \end{aligned}$$

Hence the ode is

$$x'' + x = 2\cos(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(x) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(x') &= sY(s) - x(0)\\ \mathcal{L}(x'') &= s^2Y(s) - x'(0) - sx(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - x'(0) - sx(0) + Y(s) = \frac{2s}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}x(0) &= -1\\ x'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + s + Y(s) = \frac{2s}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^3 - s^2 - s - 1}{(s^2+1)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{2(s-i)^2} + \frac{i}{2(s+i)^2} + \frac{-\frac{1}{2}-\frac{i}{2}}{s-i} + \frac{-\frac{1}{2}+\frac{i}{2}}{s+i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{2(s-i)^2}\right) = -\frac{it\,e^{it}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{i}{2(s+i)^2}\right) = \frac{it\,e^{-it}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}-\frac{i}{2}}{s-i}\right) = \left(-\frac{1}{2}-\frac{i}{2}\right)e^{it}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2}+\frac{i}{2}}{s+i}\right) = \left(-\frac{1}{2}+\frac{i}{2}\right)e^{-it}$$

Adding the above results and simplifying gives

$$x = -\cos(t) + \sin(t)(1+t)$$

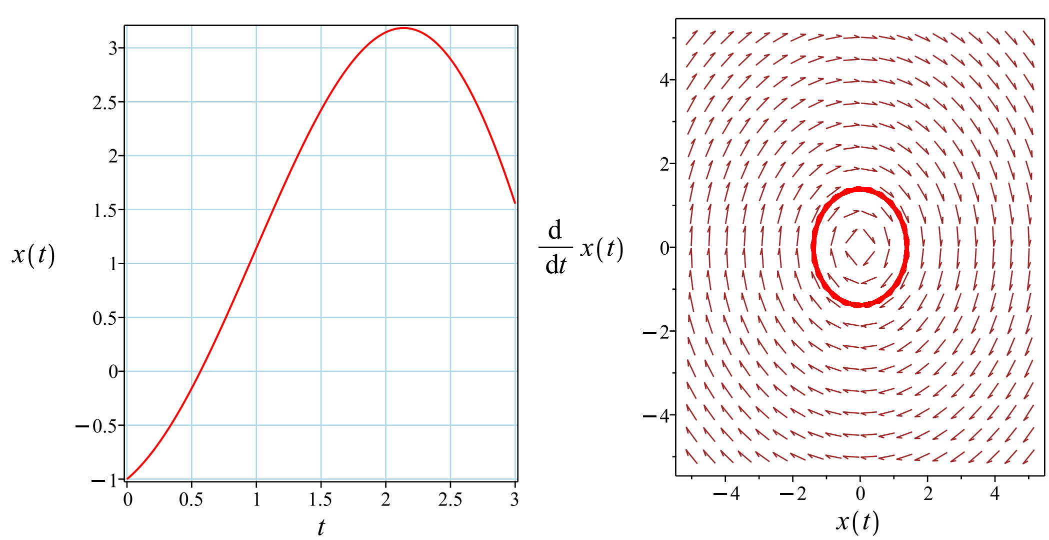
Simplifying the solution gives

$$x = -\cos(t) + \sin(t)(1+t)$$

Summary

The solution(s) found are the following

$$x = -\cos(t) + \sin(t)(1+t) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -\cos(t) + \sin(t)(1 + t)$$

Verified OK.

33.17.2 Maple step by step solution

Let's solve

$\left[x'' + x = 2\cos(t), x(0) = -1, x'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 2\cos(t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = 1$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\cos(t) \left(\int \sin(2t)\, dt\right) + 2\sin(t) \left(\int \cos(t)^2\, dt\right)$

- Compute integrals

  $x_p(t) = \frac{\cos(t)}{2} + \sin(t)\, t$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(t) + c_2 \sin(t) + \frac{\cos(t)}{2} + \sin(t)\, t$

☐ Check validity of solution $x = c_1 \cos(t) + c_2 \sin(t) + \frac{\cos(t)}{2} + \sin(t)\, t$

- Use initial condition $x(0) = -1$

  $-1 = c_1 + \frac{1}{2}$

- Compute derivative of the solution

  $x' = -c_1 \sin(t) + c_2 \cos(t) + \frac{\sin(t)}{2} + \cos(t)\, t$

- Use the initial condition $x'\Big|_{\{t=0\}} = 1$

  $1 = c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{3}{2}, c_2 = 1\right\}$

- Substitute constant values into general solution and simplify

  $x = \sin(t)\, t - \cos(t) + \sin(t)$

- Solution to the IVP

  $x = \sin(t)\, t - \cos(t) + \sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 15

```
dsolve([diff(x(t),t$2)+x(t)=2*cos(t),x(0) = -1, D(x)(0) = 1],x(t), singsol=all)
```

$$x = -\cos(t) + \sin(t)(1 + t)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 16

```
DSolve[{x''[t]+x[t]==2*Cos[t],{x[0]==-1,x'[0]==1}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \rightarrow (t + 1)\sin(t) - \cos(t)$$