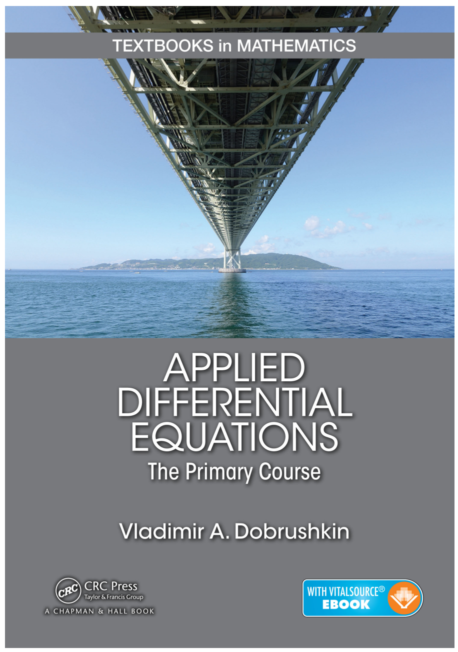
A Solution Manual For

# APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015

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## 1.1 problem Problem 1(a)



Internal problem ID [12212]
Internal file name [`OUTPUT/10864_Thursday_September_21_2023_05_48_15_AM_38450971/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - y\,\mathrm{e}^{y+x}\left(x^2+1\right) = 0}$$

### 1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \left(x^2+1\right)\mathrm{e}^x\mathrm{e}^y y \end{aligned}$$

Where $f(x) = (x^2+1)\,\mathrm{e}^x$ and $g(y) = \mathrm{e}^y y$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^y y}\,dy = \left(x^2+1\right)\mathrm{e}^x\,dx$$

$$\int \frac{1}{\mathrm{e}^y y}\,dy = \int \left(x^2+1\right)\mathrm{e}^x\,dx$$

$$-\operatorname{expIntegral}_1(y) = \left(x^2-2x+3\right)\mathrm{e}^x + c_1$$

Which results in

$$y = \mathrm{RootOf}\left(x^2 e^x - 2x\,e^x + 3\,e^x + \mathrm{expIntegral}_1\left(\_Z\right) + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{RootOf}\left(x^2 e^x - 2x\,e^x + 3\,e^x + \mathrm{expIntegral}_1\left(\_Z\right) + c_1\right) \tag{1}$$

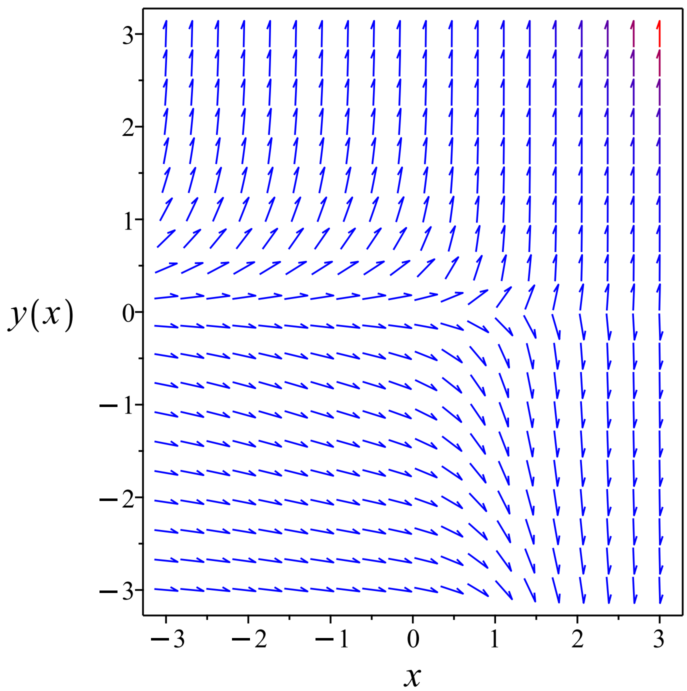



Figure 1: Slope field plot

Verification of solutions

$$y = \mathrm{RootOf}\left(x^2 e^x - 2x\,e^x + 3\,e^x + \mathrm{expIntegral}_1\left(\_Z\right) + c_1\right)$$

Verified OK.

### 1.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\,\mathrm{e}^{y+x}(x^2+1)\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\,g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\,e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$ | $\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\,y + g(x)\,y^n$ | 0 | $e^{-\int (n-1)f(x)dx} y^n$ |
| Reduced Riccati | $y' = f_1(x)\,y + f_2(x)\,y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{e^{-x}}{x^2+1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x,y) \to (R,S)$ where $(R,S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

$S$ is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{e^{-x}}{x^2+1}} dx
\end{aligned}
$$

Which results in

$$
S = \left(x^2 - 2x + 3\right) e^{x}
$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = y\, e^{y+x} \left(x^2 + 1\right)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= (x^2 + 1)\,\mathrm{e}^x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\mathrm{e}^{-y}}{y} \tag{2A}
$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\mathrm{e}^{-R}}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$
S(R) = -\operatorname{expIntegral}_1(R) + c_1 \tag{4}
$$

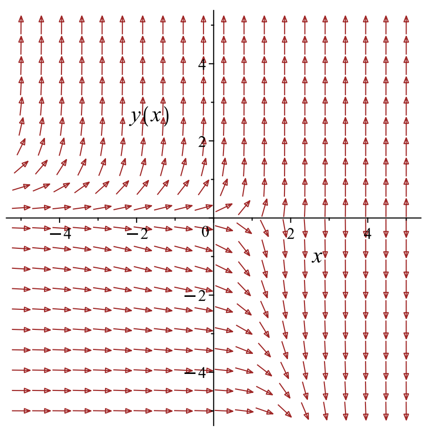
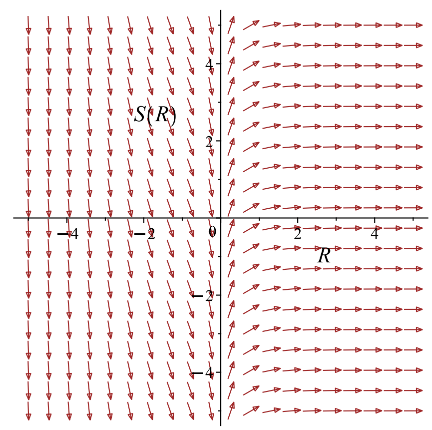
To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$
(x^2 - 2x + 3)\,\mathrm{e}^x = -\operatorname{expIntegral}_1(y) + c_1
$$

Which simplifies to

$$
(x^2 - 2x + 3)\,\mathrm{e}^x = -\operatorname{expIntegral}_1(y) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = y\,\mathrm{e}^{y+x}(x^2+1)$   | $R = y$<br>$S = (x^2 - 2x + 3)\,\mathrm{e}^{x}$ | $\frac{dS}{dR} = \frac{\mathrm{e}^{-R}}{R}$   |

Summary

The solution(s) found are the following

$$\left(x^2 - 2x + 3\right) \mathrm{e}^{x} = -\operatorname{expIntegral}_1\left(y\right) + c_1 \tag{1}$$

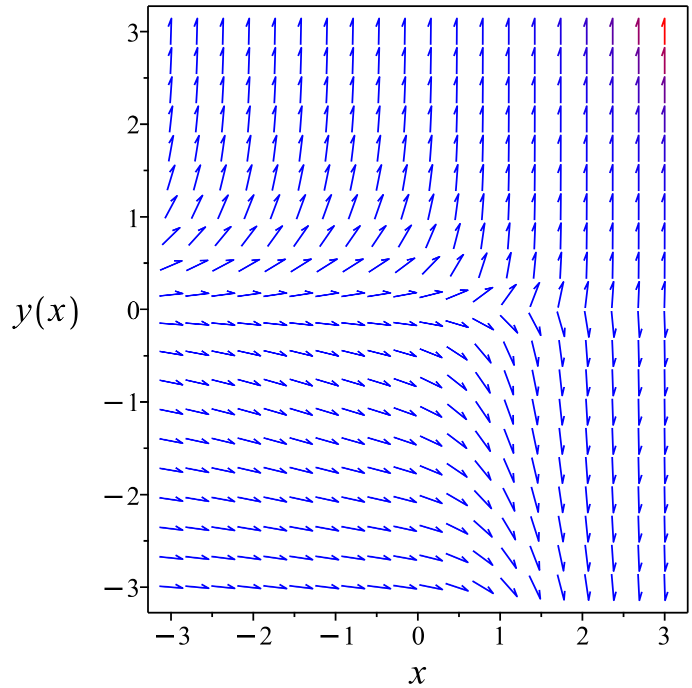


Figure 2: Slope field plot

Verification of solutions

$$\left(x^2-2x+3\right)\mathrm{e}^{x} = -\operatorname{expIntegral}_1\left(y\right)+c_1$$

Verified OK.

### 1.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{\mathrm{e}^{-y}}{y}\right) \mathrm{d}y = \left(\left(x^2+1\right) \mathrm{e}^x\right) \mathrm{d}x$$
$$\left(-\left(x^2+1\right) \mathrm{e}^x\right) \mathrm{d}x + \left(\frac{\mathrm{e}^{-y}}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\left(x^2+1\right) \mathrm{e}^x$$
$$N(x, y) = \frac{\mathrm{e}^{-y}}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\left(x^2+1\right) \mathrm{e}^x\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{e^{-y}}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -(x^2+1)\,e^x\,\mathrm{d}x\end{aligned}$$

$$\phi = -(x^2 - 2x + 3)\,e^x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{e^{-y}}{y}$. Therefore equation (4) becomes

$$\frac{e^{-y}}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{e^{-y}}{y}$$

Integrating the above w.r.t $y$ gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{e^{-y}}{y}\right)\mathrm{d}y\\ f(y) &= -\operatorname{expIntegral}_1(y) + c_1\end{aligned}$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = -(x^2 - 2x + 3)\,\mathrm{e}^x - \mathrm{expIntegral}_1(y) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -(x^2 - 2x + 3)\,\mathrm{e}^x - \mathrm{expIntegral}_1(y)$$

Summary

The solution(s) found are the following

$$-(x^2 - 2x + 3)\,\mathrm{e}^x - \mathrm{expIntegral}_1(y) = c_1 \tag{1}$$

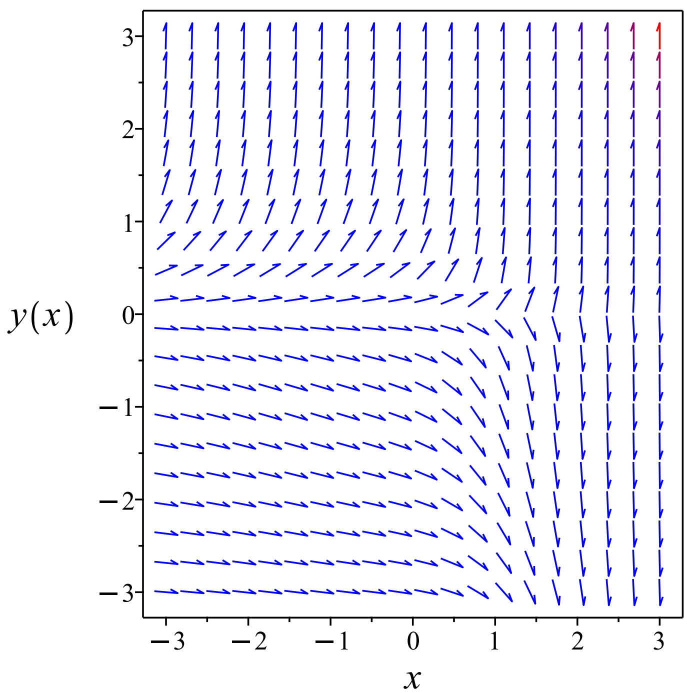



Figure 3: Slope field plot

Verification of solutions

$$-(x^2 - 2x + 3)\,\mathrm{e}^x - \mathrm{expIntegral}_1(y) = c_1$$

Verified OK.

### 1.1.4 Maple step by step solution

Let's solve

$y' - y\,\mathrm{e}^{y+x}(x^2+1) = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{\mathrm{e}^{y}y} = (x^2+1)\,\mathrm{e}^{x}$

- Integrate both sides with respect to $x$

  $\int \frac{y'}{\mathrm{e}^{y}y}dx = \int (x^2+1)\,\mathrm{e}^{x}dx + c_1$

- Evaluate integral

  $-\mathrm{Ei}_1(y) = (x^2-2x+3)\,\mathrm{e}^{x} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)=y(x)*exp(x+y(x))*(x^2+1),y(x), singsol=all)
```

$$(x^2-2x+3)\,\mathrm{e}^{x} + \mathrm{expIntegral}_1\,(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.859 (sec). Leaf size: 32

```
DSolve[y'[x]==y[x]*Exp[x+y[x]]*(x^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}[\text{ExpIntegralEi}(-\#1)\&]\left[e^x\left(x^2-2x+3\right)+c_1\right]$$
$$y(x) \to 0$$

## 1.2 problem Problem 1(b)



Internal problem ID [12213]
Internal file name [`OUTPUT/10865_Thursday_September_21_2023_05_48_16_AM_24830329/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2y' - y^2 = 1}$$

### 1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{x^2} \end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = \frac{1}{x^2}\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int \frac{1}{x^2}\,dx$$

$$\arctan(y) = -\frac{1}{x} + c_1$$

Which results in

$$y = \tan\left(\frac{c_1 x - 1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{c_1 x - 1}{x}\right) \tag{1}$$

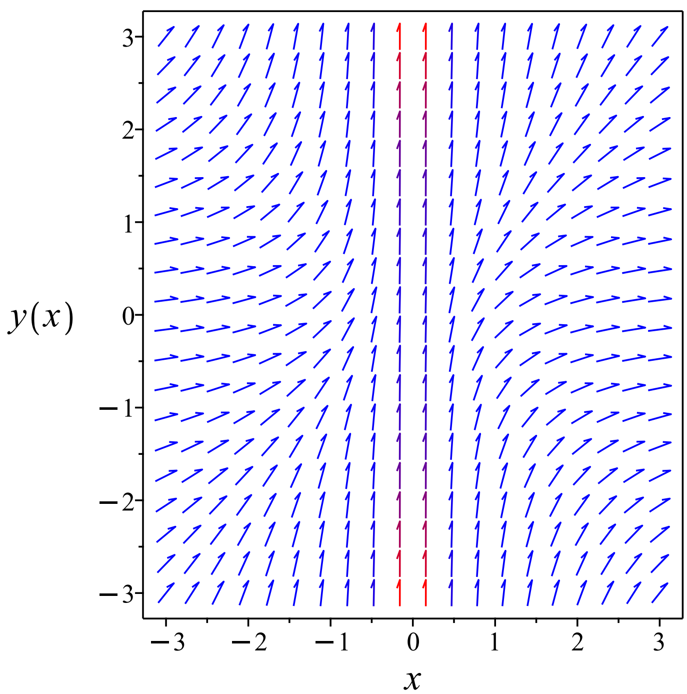



Figure 4: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{c_1 x - 1}{x}\right)$$

Verified OK.

### 1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2+1}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\,g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\,e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$ | $\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\,y + g(x)\,y^n$ | 0 | $e^{-\int(n-1)f(x)dx}y^n$ |
| Reduced Riccati | $y' = f_1(x)\,y + f_2(x)\,y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x^2 \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x,y) \to (R,S)$ where $(R,S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

$S$ is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x^2} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{1}{x}
$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y^2+1}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}
$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2+1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$
S(R) = \arctan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$
-\frac{1}{x} = \arctan(y) + c_1
$$

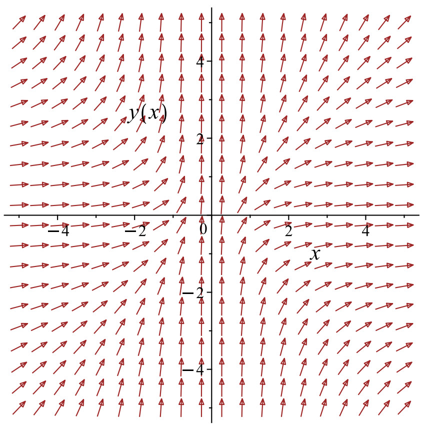
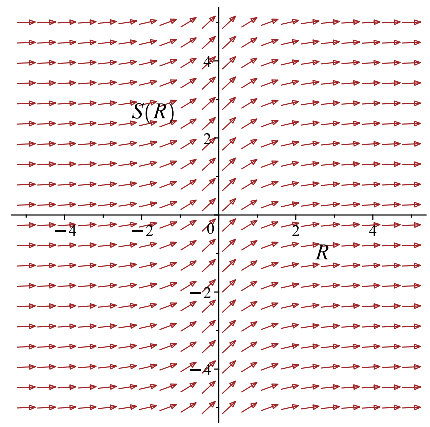
Which simplifies to

$$
-\frac{1}{x} = \arctan(y) + c_1
$$

Which gives

$$
y = -\tan\left(\frac{c_1 x + 1}{x}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = \frac{y^2+1}{x^2}$   | $R = y$ $S = -\frac{1}{x}$ | $\frac{dS}{dR} = \frac{1}{R^2+1}$   |

Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{c_1 x + 1}{x}\right) \tag{1}$$

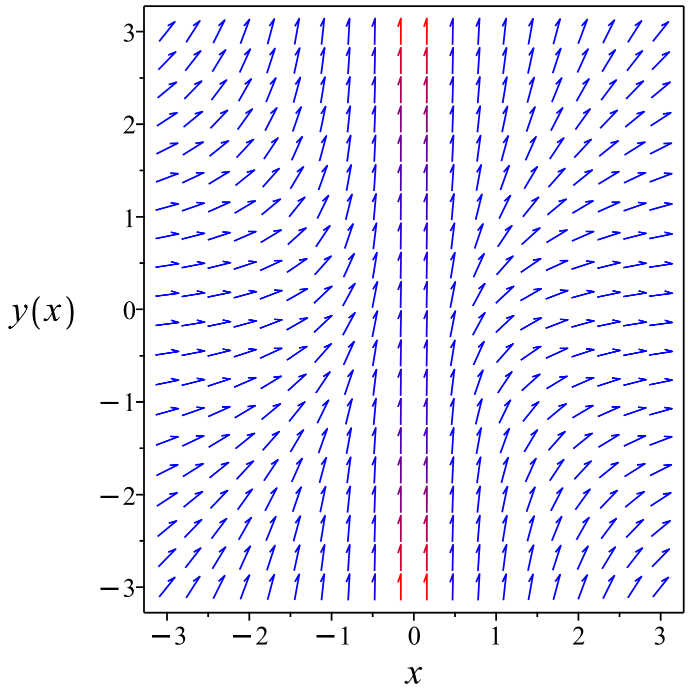



Figure 5: Slope field plot

Verification of solutions

$$y = -\tan\left(\frac{c_1 x + 1}{x}\right)$$

Verified OK.

### 1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= \left(\frac{1}{x^2}\right)\mathrm{d}x \\
\left(-\frac{1}{x^2}\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x^2} \\
N(x, y) &= \frac{1}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t $y$ gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2+1}\right)\mathrm{d}y$$
$$f(y) = \arctan(y) + c_1$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = \frac{1}{x} + \arctan(y) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = \frac{1}{x} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{c_1 x - 1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{c_1 x - 1}{x}\right) \tag{1}$$

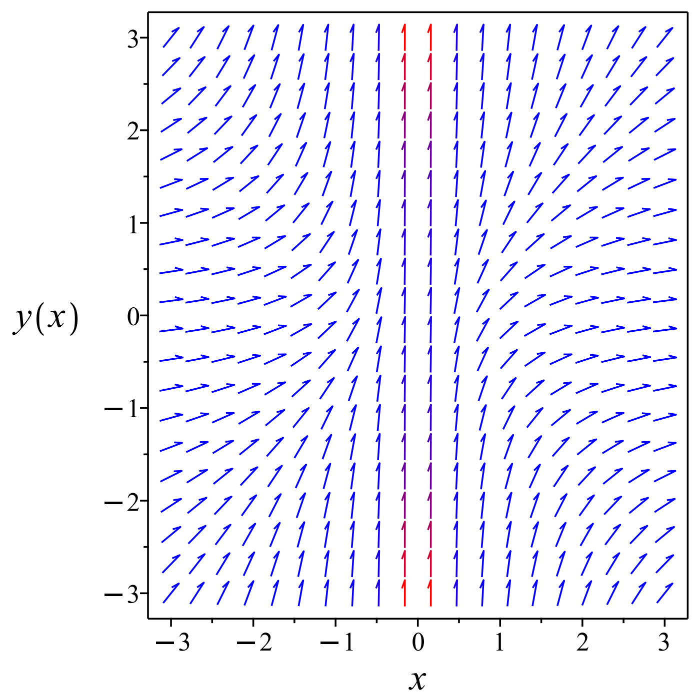



Figure 6: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{c_1 x - 1}{x}\right)$$

Verified OK.

### 1.2.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2 + 1}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2} + \frac{1}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2}$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{1}{x^6} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{2u'(x)}{x^3} + \frac{u(x)}{x^6} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{1}{x}\right) + c_2 \cos\left(\frac{1}{x}\right)$$

The above shows that

$$u'(x) = \frac{-c_1 \cos\left(\frac{1}{x}\right) + c_2 \sin\left(\frac{1}{x}\right)}{x^2}$$

Using the above in (1) gives the solution

$$y = -\frac{-c_1 \cos\left(\frac{1}{x}\right) + c_2 \sin\left(\frac{1}{x}\right)}{c_1 \sin\left(\frac{1}{x}\right) + c_2 \cos\left(\frac{1}{x}\right)}$$

Dividing both numerator and denominator by $c_1$ gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)}{c_3 \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_3 \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)}{c_3 \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)} \tag{1}$$

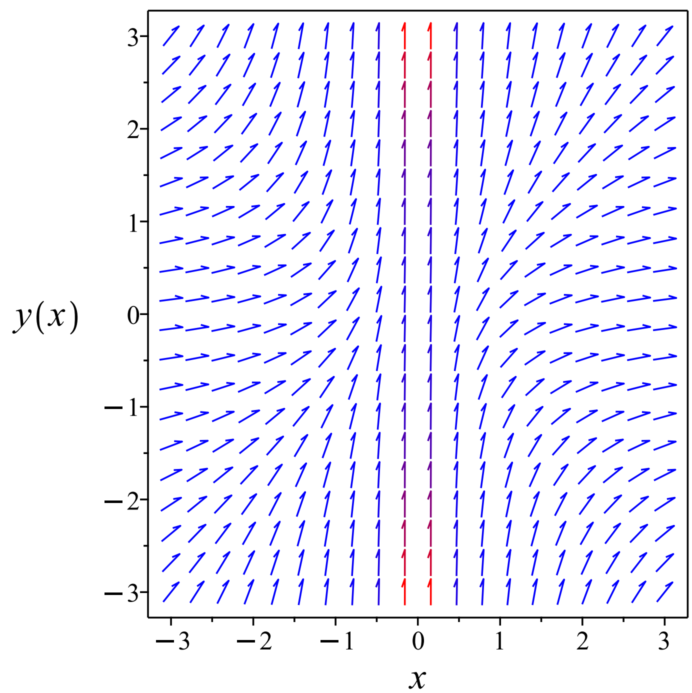



Figure 7: Slope field plot

Verification of solutions

$$y = \frac{c_3 \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right)}{c_3 \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)}$$

Verified OK.

### 1.2.5 Maple step by step solution

Let's solve

$x^2 y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{1+y^2} = \frac{1}{x^2}$

- Integrate both sides with respect to $x$

  $\int \frac{y'}{1+y^2} dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

$\arctan(y) = -\frac{1}{x} + c_1$

- Solve for $y$

$y = \tan\left(\frac{c_1 x - 1}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x)=1+y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{c_1 x - 1}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.354 (sec). Leaf size: 30

```
DSolve[x^2*y'[x]==1+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{-1 + c_1 x}{x}\right)$$
$$y(x) \to -i$$
$$y(x) \to i$$

## 1.3 problem Problem 1(c)

Internal problem ID [12214]
Internal file name [`OUTPUT/10866_Thursday_September_21_2023_05_48_16_AM_84499487/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - \sin(yx) = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=sin(x*y(x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==Sin[x*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 1.4 problem Problem 1(d)



Internal problem ID [12215]
Internal file name [`OUTPUT/10867_Thursday_September_21_2023_05_48_17_AM_25065571/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x(e^{y}+4)-e^{y+x}y'=0}$$

### 1.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\,e^{-x}\left(1+4\,e^{-y}\right) \end{aligned}$$

Where $f(x)=x\,e^{-x}$ and $g(y)=1+4\,e^{-y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1+4\,e^{-y}}\,dy &= x\,e^{-x}\,dx \\ \int \frac{1}{1+4\,e^{-y}}\,dy &= \int x\,e^{-x}\,dx \\ \ln\left(1+4\,e^{-y}\right)-\ln\left(e^{-y}\right) &= -(x+1)\,e^{-x}+c_1 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(1+4\,e^{-y})-\ln(e^{-y})} = e^{-(x+1)e^{-x}+c_1}$$

Which simplifies to

$$e^y + 4 = c_2 e^{-(x+1)e^{-x}}$$

Summary

The solution(s) found are the following

$$y = \ln\left(-4 + c_2 e^{-(x+1)e^{-x}}\right) \tag{1}$$

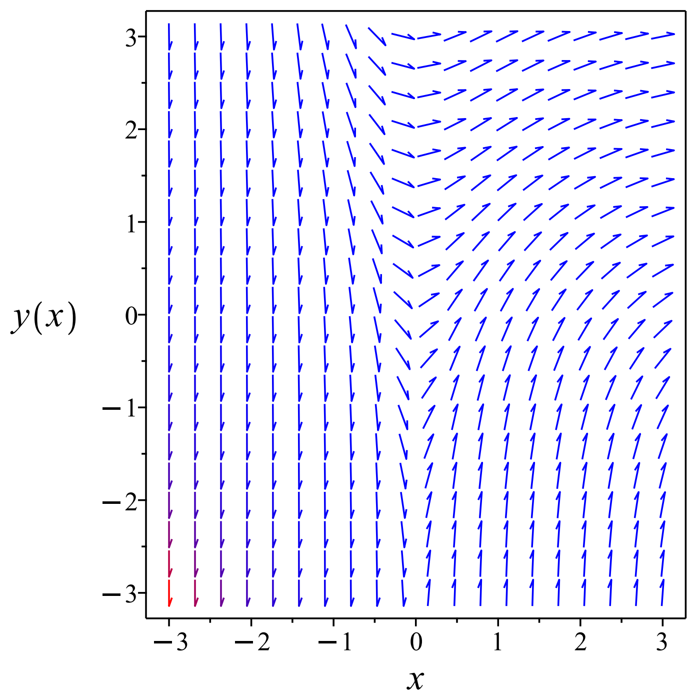



Figure 8: Slope field plot

Verification of solutions

$$y = \ln\left(-4 + c_2 e^{-(x+1)e^{-x}}\right)$$

Verified OK.

### 1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x(\mathrm{e}^{y}+4)\,\mathrm{e}^{-y-x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\,g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a+bx+cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\,e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$ | $\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\,y + g(x)\,y^n$ | 0 | $e^{-\int(n-1)f(x)dx}y^n$ |
| Reduced Riccati | $y' = f_1(x)\,y + f_2(x)\,y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{\mathrm{e}^{x}}{x} \\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x,y) \to (R,S)$ where $(R,S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

$S$ is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{\mathrm{e}^{x}}{x}} dx\end{aligned}$$

Which results in

$$S = -(x+1)\,\mathrm{e}^{-x}$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x(\mathrm{e}^{y} + 4)\,\mathrm{e}^{-y-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\ R_y &= 1 \\ S_x &= x\,\mathrm{e}^{-x} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^y}{e^y + 4} \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^R}{e^R + 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$S(R) = \ln\left(e^R + 4\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$-(x+1)\, e^{-x} = \ln\left(e^y + 4\right) + c_1$$

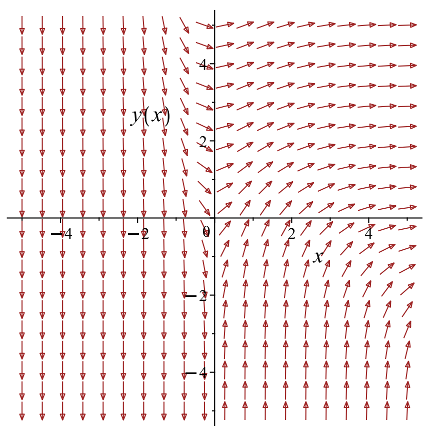
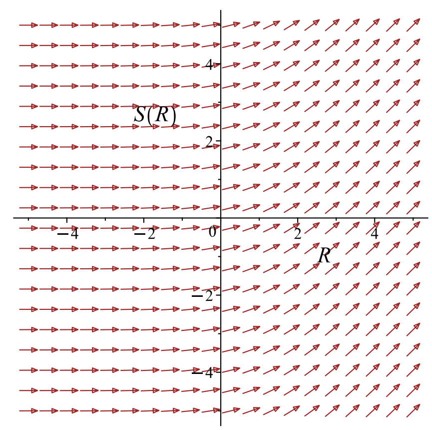
Which simplifies to

$$-(x+1)\, e^{-x} = \ln\left(e^y + 4\right) + c_1$$

Which gives

$$y = \ln\left(e^{-(e^x c_1 + x + 1)e^{-x}} - 4\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = x(\mathrm{e}^{y} + 4)\,\mathrm{e}^{-y-x}$ <br>   | $R = y$ <br> $S = -(x+1)\,\mathrm{e}^{-x}$ | $\frac{dS}{dR} = \frac{\mathrm{e}^{R}}{\mathrm{e}^{R}+4}$ <br>   |

Summary

The solution(s) found are the following

$$y = \ln\left(\mathrm{e}^{-(\mathrm{e}^{x} c_1 + x + 1)\mathrm{e}^{-x}} - 4\right) \tag{1}$$

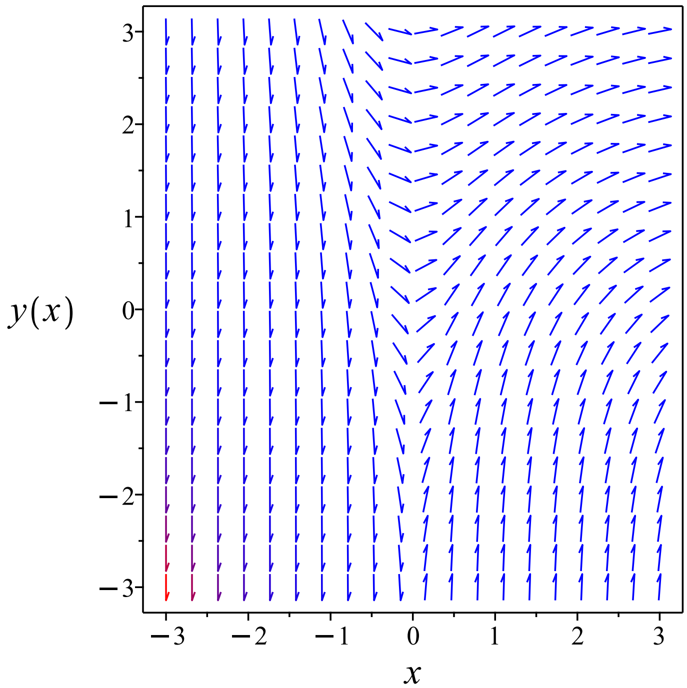



Figure 9: Slope field plot

Verification of solutions

$$y = \ln\left(e^{-(e^x c_1 + x + 1)e^{-x}} - 4\right)$$

Verified OK.

### 1.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{1 + 4 \, \mathrm{e}^{-y}}\right) \mathrm{d}y = \left(x \, \mathrm{e}^{-x}\right) \mathrm{d}x$$
$$\left(-x \, \mathrm{e}^{-x}\right) \mathrm{d}x + \left(\frac{1}{1 + 4 \, \mathrm{e}^{-y}}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x \, \mathrm{e}^{-x}$$
$$N(x, y) = \frac{1}{1 + 4 \, \mathrm{e}^{-y}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-x \, \mathrm{e}^{-x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1+4\,\mathrm{e}^{-y}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. $x$ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{e}^{-x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = (x+1)\,\mathrm{e}^{-x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{1+4\,\mathrm{e}^{-y}}$. Therefore equation (4) becomes

$$
\frac{1}{1+4\,\mathrm{e}^{-y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{1+4\,\mathrm{e}^{-y}}
$$

Integrating the above w.r.t $y$ gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{1+4\,\mathrm{e}^{-y}}\right)\mathrm{d}y \\
f(y) &= \ln\left(1+4\,\mathrm{e}^{-y}\right) - \ln\left(\mathrm{e}^{-y}\right) + c_1
\end{aligned}
$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = (x+1)\,\mathrm{e}^{-x} + \ln\left(1+4\,\mathrm{e}^{-y}\right) - \ln\left(\mathrm{e}^{-y}\right) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = (x+1)\,\mathrm{e}^{-x} + \ln\left(1+4\,\mathrm{e}^{-y}\right) - \ln\left(\mathrm{e}^{-y}\right)$$

The solution becomes

$$y = \ln\left(-4 + \mathrm{e}^{(\mathrm{e}^{x} c_1 - x - 1)\mathrm{e}^{-x}}\right)$$

Summary

The solution(s) found are the following

$$y = \ln\left(-4 + \mathrm{e}^{(\mathrm{e}^{x} c_1 - x - 1)\mathrm{e}^{-x}}\right) \tag{1}$$

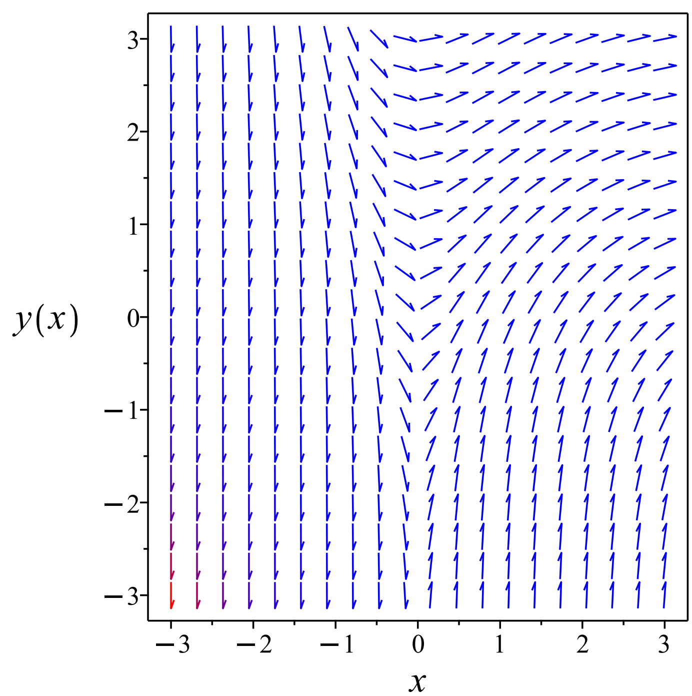



Figure 10: Slope field plot

Verification of solutions

$$y = \ln \left(-4 + \mathrm{e}^{(\mathrm{e}^x c_1 - x - 1)\mathrm{e}^{-x}}\right)$$

Verified OK.

### 1.4.4 Maple step by step solution

Let's solve

$x(\mathrm{e}^y + 4) - \mathrm{e}^{y+x} y' = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'\mathrm{e}^y}{\mathrm{e}^y+4} = \frac{x}{\mathrm{e}^x}$

- Integrate both sides with respect to $x$

  $\int \frac{y'\mathrm{e}^y}{\mathrm{e}^y+4} dx = \int \frac{x}{\mathrm{e}^x} dx + c_1$

- Evaluate integral

  $\ln (\mathrm{e}^y + 4) = -\frac{x+1}{\mathrm{e}^x} + c_1$

- Solve for $y$

  $y = \ln \left(-4 + \mathrm{e}^{\frac{\mathrm{e}^x c_1 - x - 1}{\mathrm{e}^x}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 19

```
dsolve(x*(exp(y(x))+4)=exp(x+y(x))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \ln\left(-4 + c_1 \mathrm{e}^{-\mathrm{e}^{-x}(1+x)}\right)$$

✓ Solution by Mathematica

Time used: 4.746 (sec). Leaf size: 51

```
DSolve[x*(Exp[y[x]]+4)==Exp[x+y[x]]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to \log\left(-4 + e^{e^{-x}(-x+c_1 e^x-1)}\right) \\ y(x) &\to \log(4) + i\pi \\ y(x) &\to \log(4) + i\pi \end{aligned}$$

## 1.5 problem Problem 1(e)



Internal problem ID [12216]
Internal file name [`OUTPUT/10868_Thursday_September_21_2023_05_48_18_AM_95651510/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(e).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \cos(y + x) = 0}$$

### 1.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \cos(y + x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine $\xi, \eta$ then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and $\omega$ into (A) gives

$$\begin{aligned} &b_2 + \cos(y+x)(b_3 - a_2) - \cos(y+x)^2 a_3 \\ &+ \sin(y+x)(xa_2 + ya_3 + a_1) + \sin(y+x)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} &\sin(y+x)\,xa_2 + \sin(y+x)\,xb_2 + \sin(y+x)\,ya_3 + \sin(y+x)\,yb_3 - \cos(y+x)^2 a_3 \\ &+ \sin(y+x)\,a_1 + \sin(y+x)\,b_1 - \cos(y+x)\,a_2 + \cos(y+x)\,b_3 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} &\sin(y+x)\,xa_2 + \sin(y+x)\,xb_2 + \sin(y+x)\,ya_3 + \sin(y+x)\,yb_3 - \cos(y+x)^2 a_3 \\ &+ \sin(y+x)\,a_1 + \sin(y+x)\,b_1 - \cos(y+x)\,a_2 + \cos(y+x)\,b_3 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &b_2 - \frac{a_3}{2} + \sin(y+x)\,xa_2 + \sin(y+x)\,xb_2 + \sin(y+x)\,ya_3 \\ &+ \sin(y+x)\,yb_3 - \frac{a_3\cos(2y+2x)}{2} + \sin(y+x)\,a_1 \\ &+ \sin(y+x)\,b_1 - \cos(y+x)\,a_2 + \cos(y+x)\,b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(y+x), \cos(2y+2x), \sin(y+x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(y+x) = v_3, \cos(2y+2x) = v_4, \sin(y+x) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2}a_3 + v_5 v_1 a_2 + v_5 v_1 b_2 + v_5 v_2 a_3 + v_5 v_2 b_3 - \frac{1}{2}a_3 v_4 + v_5 a_1 + v_5 b_1 - v_3 a_2 + v_3 b_3 = 0 \tag{7E}$$

Collecting the above on the terms $v_i$ introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (b_3 - a_2)\, v_3 - \frac{a_3 v_4}{2} + (a_1 + b_1)\, v_5 + (a_2 + b_2)\, v_1 v_5 + (a_3 + b_3)\, v_2 v_5 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -\frac{a_3}{2} &= 0 \\ a_1 + b_1 &= 0 \\ a_2 + b_2 &= 0 \\ a_3 + b_3 &= 0 \\ b_2 - \frac{a_3}{2} &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - (\cos(y + x))\,(-1) \\ &= 1 + \cos(y)\cos(x) - \sin(y)\sin(x) \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x, y) \to (R, S)$ where $(R, S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 + \cos(y)\cos(x) - \sin(y)\sin(x)} dy \end{aligned}$$

Which results in

$$S = \tan\left(\frac{y}{2} + \frac{x}{2}\right)$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \cos(y + x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\sec\left(\frac{y}{2} + \frac{x}{2}\right)^2}{2} \\ S_y &= \frac{\sec\left(\frac{y}{2} + \frac{x}{2}\right)^2}{2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec\left(\frac{y}{2} + \frac{x}{2}\right)^2 \cos\left(\frac{y}{2} + \frac{x}{2}\right)^2 \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates $R, S$. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$\tan\left(\frac{y}{2} + \frac{x}{2}\right) = x + c_1$$

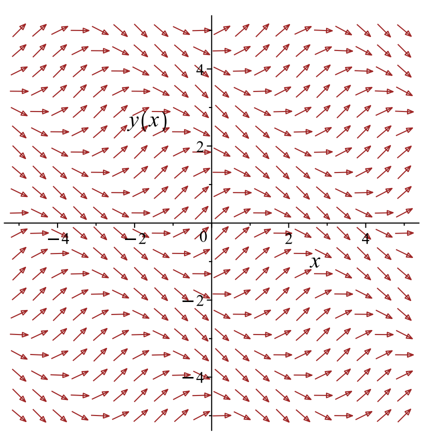
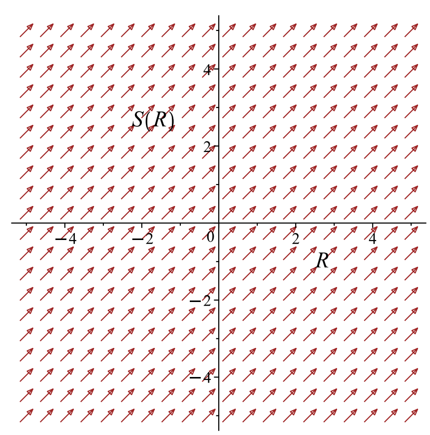
Which simplifies to

$$\tan\left(\frac{y}{2} + \frac{x}{2}\right) = x + c_1$$

Which gives

$$y = -x + 2\arctan(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = \cos(y + x)$   | $R = x$ <br> $S = \tan\left(\frac{y}{2} + \frac{x}{2}\right)$ | $\frac{dS}{dR} = 1$   |

Summary

The solution(s) found are the following

$$y = -x + 2 \arctan (x + c_1) \tag{1}$$

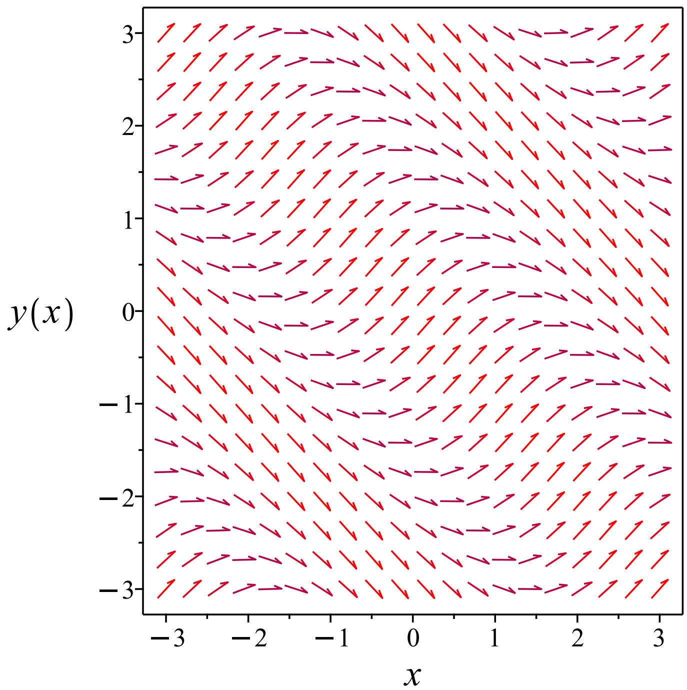



Figure 11: Slope field plot

Verification of solutions

$$y = -x + 2 \arctan (x + c_1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=cos(x+y(x)),y(x), singsol=all)
```

$$y(x) = -x - 2\arctan(c_1 - x)$$

✓ Solution by Mathematica

Time used: 1.551 (sec). Leaf size: 59

```
DSolve[y'[x]==Cos[x+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + 2\arctan\left(x + \frac{c_1}{2}\right)$$
$$y(x) \to -x + 2\arctan\left(x + \frac{c_1}{2}\right)$$
$$y(x) \to -x - \pi$$
$$y(x) \to \pi - x$$

## 1.6 problem Problem 1(f)



Internal problem ID [12217]
Internal file name [`OUTPUT/10869_Thursday_September_21_2023_05_48_19_AM_92027360/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(f).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x + y - y^2 x = 0}$$

### 1.6.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(xy - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 10: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\, g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\, e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$ | $\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\, y + g(x)\, y^n$ | 0 | $e^{-\int (n-1)f(x)dx} y^n$ |
| Reduced Riccati | $y' = f_1(x)\, y + f_2(x)\, y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x\, y^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x, y) \to (R, S)$ where $(R, S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x\,y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{xy}$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(xy-1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^2y} \\ S_y &= \frac{1}{x\,y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates $R, S$. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$-\frac{1}{yx} = \ln(x) + c_1$$

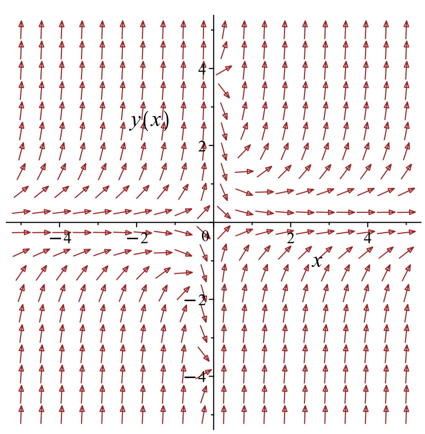
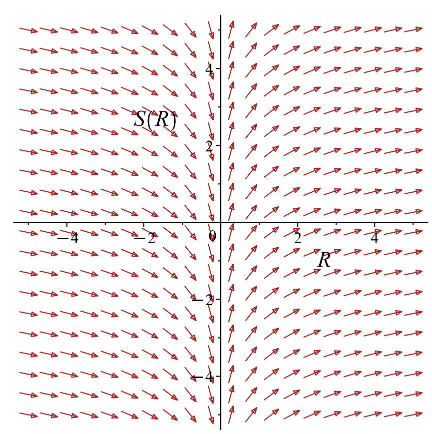
Which simplifies to

$$-\frac{1}{yx} = \ln(x) + c_1$$

Which gives

$$y = -\frac{1}{x(\ln(x) + c_1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = \frac{y(xy-1)}{x}$ <br>   | $R = x$ <br> $S = -\frac{1}{xy}$ | $\frac{dS}{dR} = \frac{1}{R}$ <br>   |

Summary
The solution(s) found are the following

$$y = -\frac{1}{x\left(\ln \left(x\right)+c_1\right)} \tag{1}$$

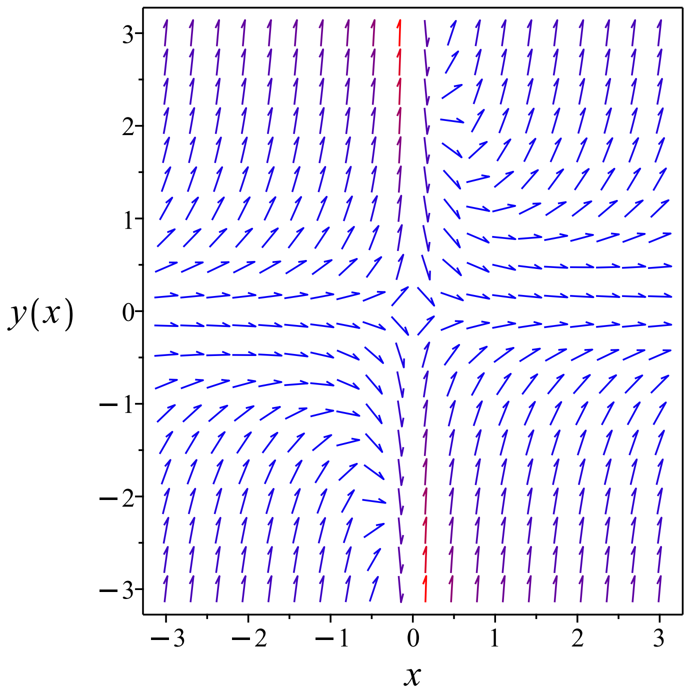



Figure 12: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(\ln \left(x\right)+c_1\right)}$$

Verified OK.

### 1.6.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(xy-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by $y^n$ which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= 1 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{xy} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t $x$ gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + 1 \\ w' &= \frac{w}{x} - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -1
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = -1
$$

The integrating factor $\mu$ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-1) \\
\mathrm{d}\left(\frac{w}{x}\right) &= \left(-\frac{1}{x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -\frac{1}{x}\,\mathrm{d}x \\
\frac{w}{x} &= -\ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = -x\ln(x) + c_1 x
$$

which simplifies to

$$
w(x) = x(-\ln(x) + c_1)
$$

Replacing $w$ in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = x(-\ln(x) + c_1)
$$

Or

$$y = \frac{1}{x\left(-\ln \left(x\right)+c_1\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x\left(-\ln \left(x\right)+c_1\right)} \tag{1}$$

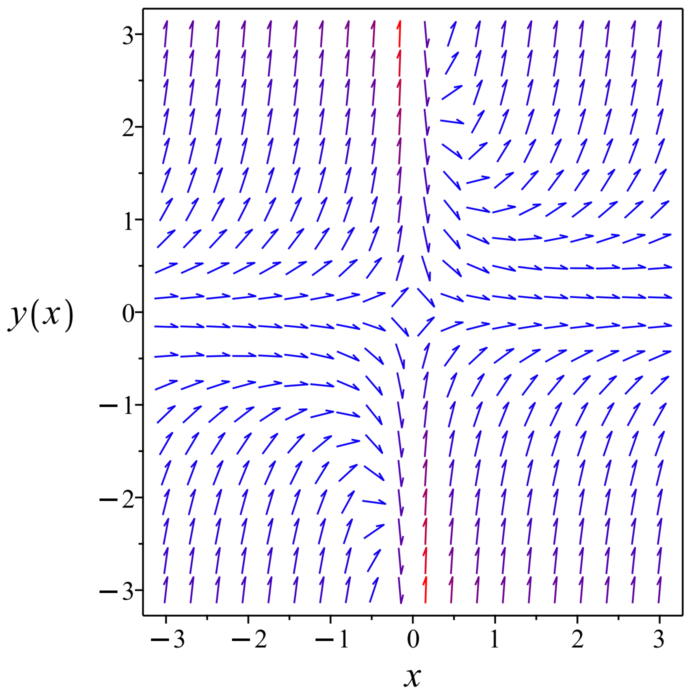



Figure 13: Slope field plot

Verification of solutions

$$y = \frac{1}{x\left(-\ln \left(x\right)+c_1\right)}$$

Verified OK.

### 1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(x\,y^2-y\right)\mathrm{d}x\\ \left(-x\,y^2+y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\,y^2+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x\,y^2 + y) \\ &= -2xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-2xy + 1) - (1)) \\ &= -2y\end{aligned}$$

Since $A$ depends on $y$, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{-x\,y^2 + y}((1) - (-2xy + 1)) \\ &= -\frac{2x}{xy - 1}\end{aligned}$$

Since $B$ depends on $x$, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

$R$ is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(1) - (-2xy + 1)}{x\left(-x\,y^2 + y\right) - y\,(x)} \\ &= -\frac{2}{xy} \end{aligned}$$

Replacing all powers of terms $xy$ by $t$ gives

$$R = -\frac{2}{t}$$

Since $R$ depends on $t$ only, then it can be used to find an integrating factor. Let the integrating factor be $\mu$ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2} \end{aligned}$$

Now $t$ is replaced back with $xy$ giving

$$\mu = \frac{1}{x^2 y^2}$$

Multiplying $M$ and $N$ by this integrating factor gives new $M$ and new $N$ which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original $M$ and $N$

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2 y^2}\left(-x\,y^2 + y\right) \\ &= \frac{-xy + 1}{x^2 y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2 y^2}(x) \\ &= \frac{1}{x\,y^2} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-xy+1}{x^2y}\right) + \left(\frac{1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-xy+1}{x^2y}\,\mathrm{d}x$$
$$\phi = -\frac{1}{xy} - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x\,y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{1}{x\,y^2} = \frac{1}{x\,y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where $c_1$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = -\frac{1}{xy} - \ln(x) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -\frac{1}{xy} - \ln(x)$$

The solution becomes

$$y = -\frac{1}{x(\ln(x) + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x(\ln(x) + c_1)} \tag{1}$$

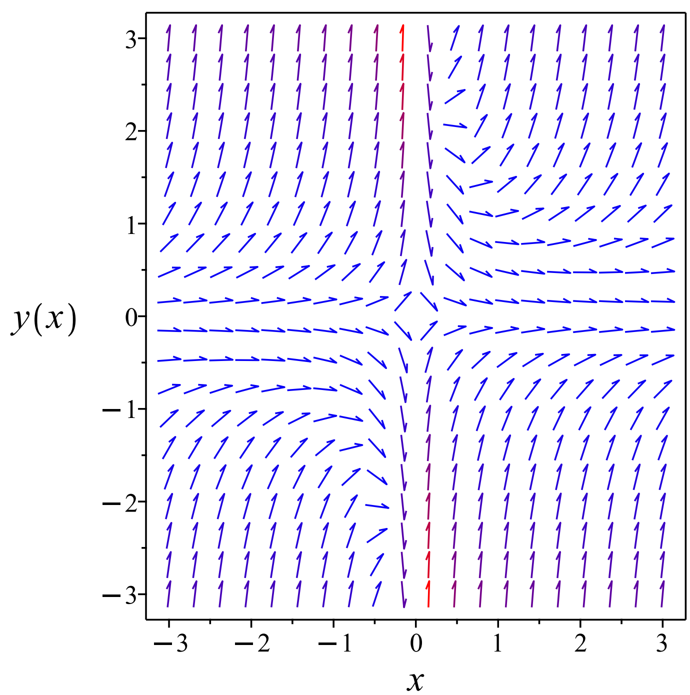



Figure 14: Slope field plot

Verification of solutions

$$y = -\frac{1}{x(\ln(x) + c_1)}$$

Verified OK.

### 1.6.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= \frac{y(xy - 1)}{x}
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 - \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = 1$. Let

$$\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{u}
\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= -\frac{1}{x} \\
f_2^2 f_0 &= 0
\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \frac{u'(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \ln(x)\, c_2$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{x\,(c_1 + \ln(x)\, c_2)}$$

Dividing both numerator and denominator by $c_1$ gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{x\,(c_3 + \ln(x))}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x\,(c_3 + \ln(x))} \tag{1}$$

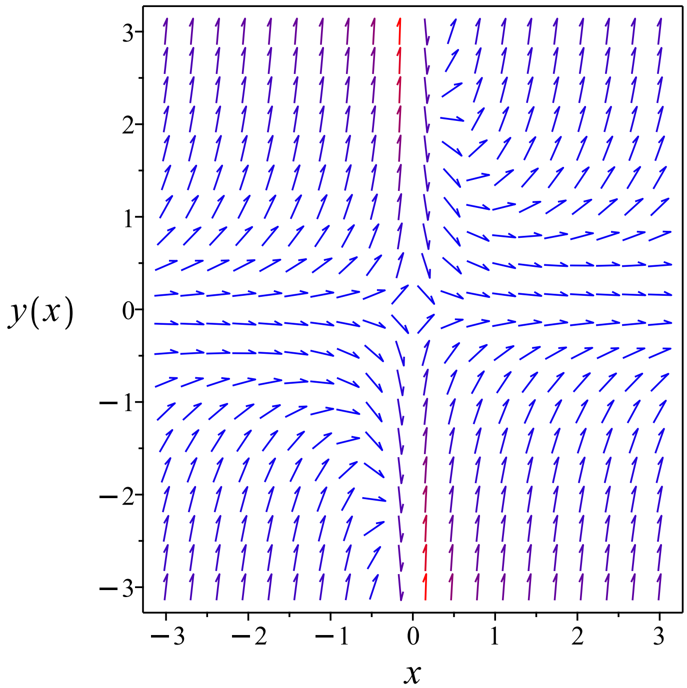



Figure 15: Slope field plot

Verification of solutions

$$y = -\frac{1}{x\left(c_3 + \ln\left(x\right)\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)+y(x)=x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{(-\ln(x) + c_1)\, x}$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 22

```
DSolve[x*y'[x]+y[x]==x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{-x\log(x) + c_1 x}$$

$$y(x) \to 0$$

## 1.7 problem Problem 1(g)

Internal problem ID [12218]
Internal file name [`OUTPUT/10870_Thursday_September_21_2023_05_48_20_AM_17240927/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(g).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - t\ln\left(y^{2t}\right) = t^2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(t),t)=t*ln(y(t)^(2*t))+t^2,y(t), singsol=all)
```

No solution found

✓ Solution by Mathematica

Time used: 0.47 (sec). Leaf size: 43

```
DSolve[y'[t]==t*Log[y[t]^(2*t)]+t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \text{InverseFunction}\left[\frac{\text{ExpIntegralEi}\left(\log(\#1) + \frac{1}{2}\right)}{2\sqrt{e}}\&\right]\left[\frac{t^3}{3} + c_1\right]$$
$$y(t) \to \frac{1}{\sqrt{e}}$$

## 1.8 problem Problem 1(h)



Internal problem ID [12219]
Internal file name [`OUTPUT/10871_Thursday_September_21_2023_05_48_21_AM_68953664/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(h).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x\,\mathrm{e}^{-x+y^2} = 0}$$

### 1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\,\mathrm{e}^{y^2}\mathrm{e}^{-x} \end{aligned}$$

Where $f(x) = x\,\mathrm{e}^{-x}$ and $g(y) = \mathrm{e}^{y^2}$. Integrating both sides gives

$$\frac{1}{\mathrm{e}^{y^2}}\,dy = x\,\mathrm{e}^{-x}\,dx$$

$$\int \frac{1}{\mathrm{e}^{y^2}}\,dy = \int x\,\mathrm{e}^{-x}\,dx$$

$$\frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} = -(x+1)\,\mathrm{e}^{-x} + c_1$$

Which results in

$$y = \mathrm{RootOf}\left(-\mathrm{erf}\left(\_Z\right)\sqrt{\pi}\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} c_1 - 2x - 2\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{RootOf}\left(-\mathrm{erf}\left(\_Z\right)\sqrt{\pi}\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} c_1 - 2x - 2\right) \tag{1}$$

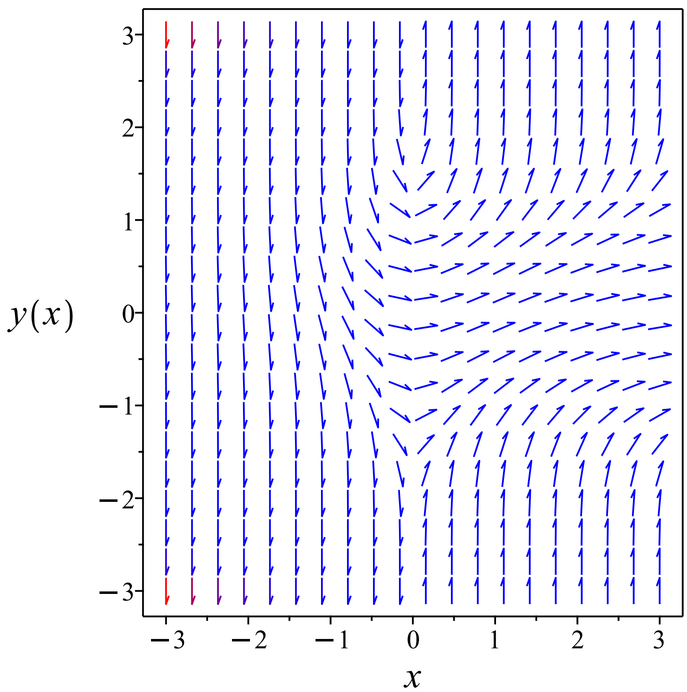



Figure 16: Slope field plot

Verification of solutions

$$y = \mathrm{RootOf}\left(-\mathrm{erf}\left(\_Z\right)\sqrt{\pi}\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} c_1 - 2x - 2\right)$$

Verified OK.

### 1.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\,\mathrm{e}^{y^2-x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 12: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\,g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\,e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$ | $\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\,y + g(x)\,y^n$ | 0 | $e^{-\int (n-1)f(x)dx}y^n$ |
| Reduced Riccati | $y' = f_1(x)\,y + f_2(x)\,y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{e^x}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x,y) \to (R,S)$ where $(R,S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

$S$ is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{e^x}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -(x+1)\,e^{-x}
$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = x\,e^{y^2 - x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x\,e^{-x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{-y^2} \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{-R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$S(R) = \frac{\sqrt{\pi}\, \operatorname{erf}(R)}{2} + c_1 \tag{4}$$

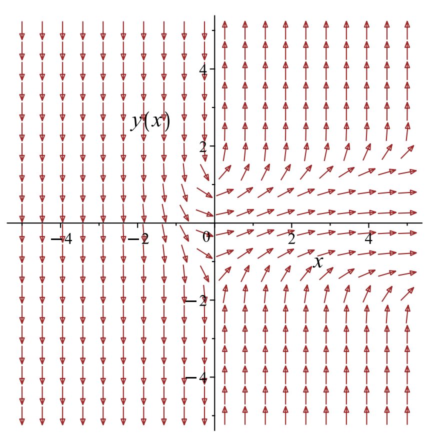
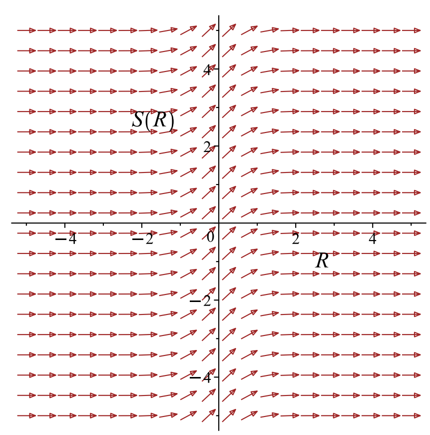
To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$-(x+1)\, e^{-x} = \frac{\sqrt{\pi}\, \operatorname{erf}(y)}{2} + c_1$$

Which simplifies to

$$-(x+1)\, e^{-x} = \frac{\sqrt{\pi}\, \operatorname{erf}(y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = x\, e^{y^2 - x}$<br> | $R = y$<br>$S = -(x+1)\, e^{-x}$ | $\frac{dS}{dR} = e^{-R^2}$<br> |

Summary
The solution(s) found are the following

$$-(x+1)\,\mathrm{e}^{-x} = \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} + c_1 \tag{1}$$

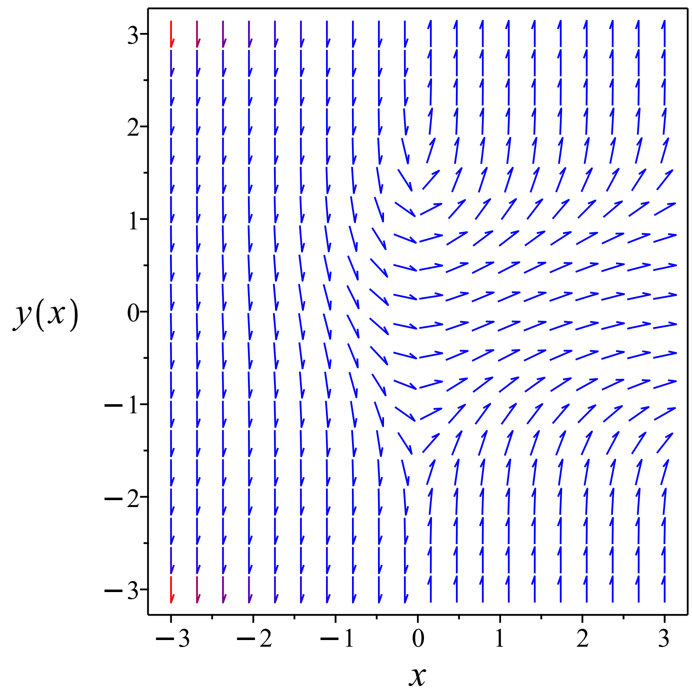



Figure 17: Slope field plot

Verification of solutions

$$-(x+1)\,\mathrm{e}^{-x} = \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} + c_1$$

Verified OK.

### 1.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{-y^2}\right)\mathrm{d}y&=\left(x\,\mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(-x\,\mathrm{e}^{-x}\right)\mathrm{d}x+\left(\mathrm{e}^{-y^2}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\,\mathrm{e}^{-x}\\ N(x,y)&=\mathrm{e}^{-y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x\,\mathrm{e}^{-x}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{-y^2}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. $x$ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{e}^{-x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = (x+1)\,\mathrm{e}^{-x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-y^2}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \mathrm{e}^{-y^2}
$$

Integrating the above w.r.t $y$ gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\mathrm{e}^{-y^2}\right)\mathrm{d}y\\
f(y) &= \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} + c_1
\end{aligned}
$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = (x+1)\,\mathrm{e}^{-x} + \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = (x+1)\,\mathrm{e}^{-x} + \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2}$$

Summary

The solution(s) found are the following

$$(x+1)\,\mathrm{e}^{-x} + \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} = c_1 \tag{1}$$

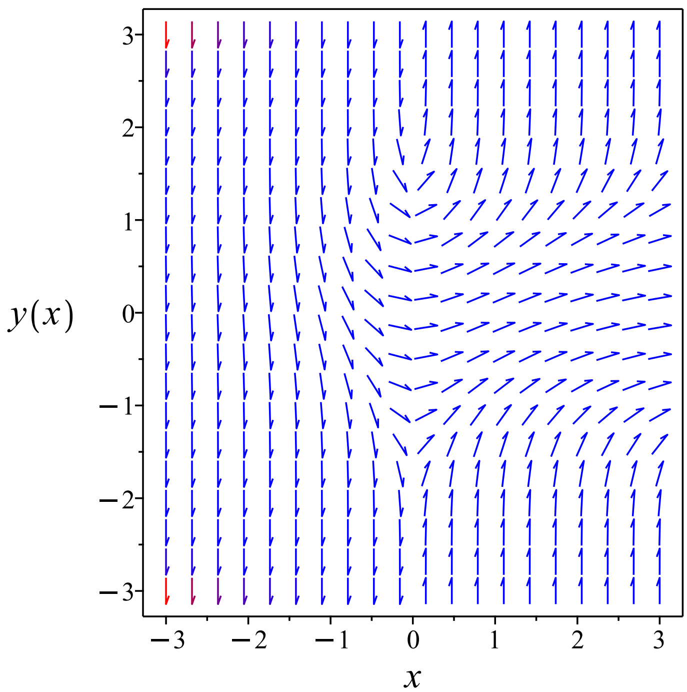



Figure 18: Slope field plot

Verification of solutions

$$(x+1)\,\mathrm{e}^{-x} + \frac{\sqrt{\pi}\,\operatorname{erf}(y)}{2} = c_1$$

Verified OK.

### 1.8.4 Maple step by step solution

Let's solve

$y' - x\,\mathrm{e}^{-x+y^2} = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{\mathrm{e}^{y^2}} = \frac{x}{\mathrm{e}^{x}}$

- Integrate both sides with respect to $x$

  $\int \frac{y'}{\mathrm{e}^{y^2}}dx = \int \frac{x}{\mathrm{e}^{x}}dx + c_1$

- Evaluate integral

  $\frac{\sqrt{\pi}\,\mathrm{erf}(y)}{2} = -\frac{x+1}{\mathrm{e}^{x}} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=x*exp(y(x)^2-x),y(x), singsol=all)
```

$$(-x-1)\,\mathrm{e}^{-x} - \frac{\sqrt{\pi}\,\mathrm{erf}\,(y(x))}{2} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 1.288 (sec). Leaf size: 28

```
DSolve[y'[x]==x*Exp[y[x]^2-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \operatorname{erf}^{-1}\left(-\frac{2e^{-x}(x - c_1 e^x + 1)}{\sqrt{\pi}}\right)$$

## 1.9 problem Problem 1(i)

Internal problem ID [12220]
Internal file name [`OUTPUT/10872_Thursday_September_21_2023_05_48_21_AM_28399815/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 1(i).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - \ln(yx) = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=ln(x*y(x)),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==Log[x*y[x]],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 1.10 problem Problem 2(a)



Internal problem ID [12221]
Internal file name [`OUTPUT/10873_Thursday_September_21_2023_05_48_22_AM_3019761/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 2, First Order Equations. Problems page 149
**Problem number**: Problem 2(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x(y+1)^2-\left(x^2+1\right) y\,\mathrm{e}^{y} y'=0}$$

### 1.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{x\,\mathrm{e}^{-y}(y+1)^2}{y\,(x^2+1)} \end{aligned}$$

Where $f(x)=\frac{x}{x^2+1}$ and $g(y)=\frac{(y+1)^2\mathrm{e}^{-y}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{(y+1)^2\mathrm{e}^{-y}}{y}}\,dy=\frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{\frac{(y+1)^2 \mathrm{e}^{-y}}{y}} dy = \int \frac{x}{x^2+1} dx$$

$$\frac{\mathrm{e}^{y}}{y+1} = \frac{\ln(x^2+1)}{2} + c_1$$

Which results in

$$y = -\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-1}}{2c_1 + \ln(x^2+1)}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-1}}{2c_1 + \ln(x^2+1)}\right) - 1 \tag{1}$$

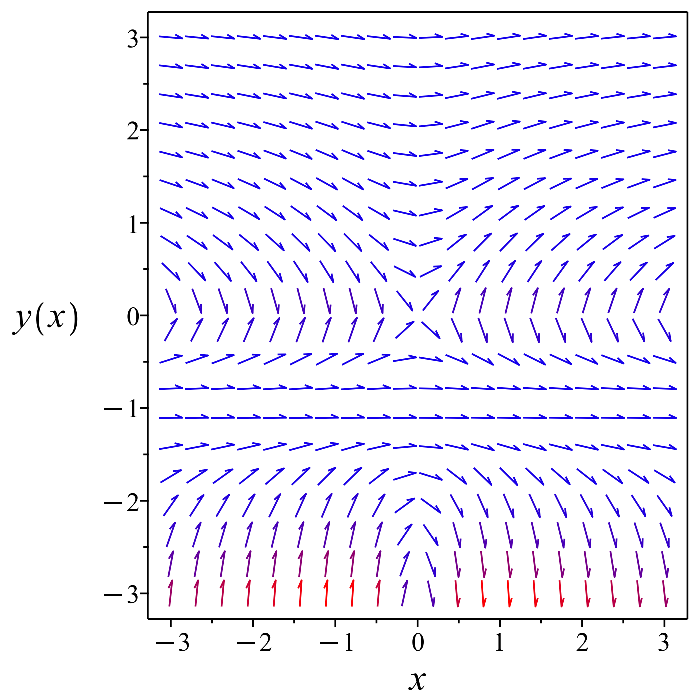



Figure 19: Slope field plot

Verification of solutions

$$y = -\mathrm{LambertW}\left(-\frac{2\,\mathrm{e}^{-1}}{2c_1 + \ln(x^2+1)}\right) - 1$$

Verified OK.

### 1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x(y^2 + 2y + 1)\, \mathrm{e}^{-y}}{y\,(x^2 + 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 15: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\, g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\, e^{h(x)+by} + f(x)$ | $\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$ | $\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$ | $\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$ |
| Bernoulli ode | $y' = f(x)\, y + g(x)\, y^n$ | 0 | $e^{-\int (n-1) f(x) dx} y^n$ |
| Reduced Riccati | $y' = f_1(x)\, y + f_2(x)\, y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2+1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x,y) \to (R,S)$ where $(R,S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

$S$ is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2+1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(x^2+1)}{2}
$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x(y^2+2y+1)\,\mathrm{e}^{-y}}{y\,(x^2+1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y\,\mathrm{e}^{y}}{(y+1)^2} \tag{2A}
$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R\,\mathrm{e}^{R}}{(R+1)^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates $R, S$. Integrating the above gives

$$
S(R) = \frac{\mathrm{e}^{R}}{R+1} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$
\frac{\ln(x^2+1)}{2} = \frac{\mathrm{e}^{y}}{y+1} + c_1
$$

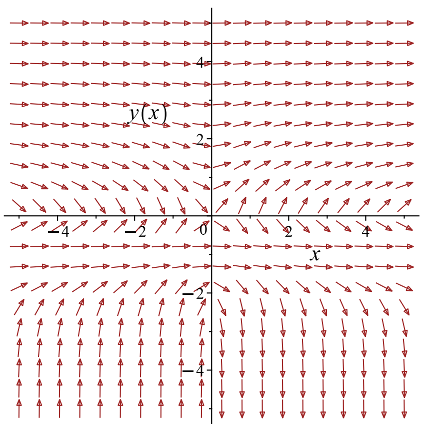
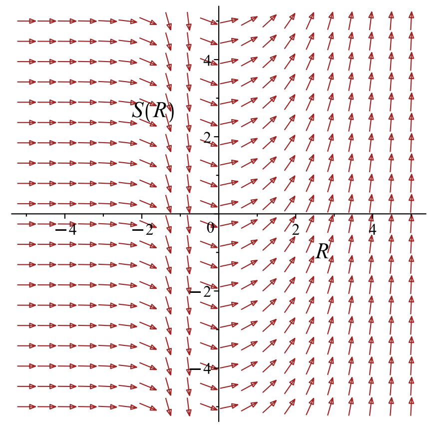
Which simplifies to

$$
\frac{\ln(x^2+1)}{2} = \frac{\mathrm{e}^{y}}{y+1} + c_1
$$

Which gives

$$
y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1}}{\frac{\ln(x^2+1)}{2} - c_1}\right) - 1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = \frac{x(y^2+2y+1)\,\mathrm{e}^{-y}}{y(x^2+1)}$   | $R = y$<br>$S = \frac{\ln(x^2+1)}{2}$ | $\frac{dS}{dR} = \frac{R\,\mathrm{e}^{R}}{(R+1)^2}$   |

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1}}{\frac{\ln(x^2+1)}{2} - c_1}\right) - 1 \tag{1}$$

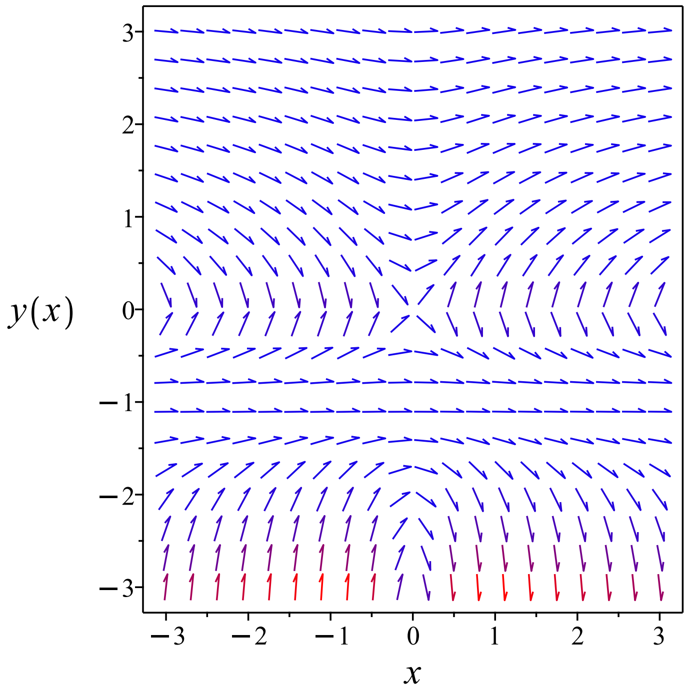



Figure 20: Slope field plot

Verification of solutions

$$y = -\mathrm{LambertW}\left(-\frac{e^{-1}}{\frac{\ln(x^2+1)}{2} - c_1}\right) - 1$$

Verified OK.

### 1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= M \\
\frac{\partial\phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{\mathrm{e}^{y} y}{y^2+2y+1}\right)\mathrm{d}y &= \left(\frac{x}{x^2+1}\right)\mathrm{d}x \\
\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(\frac{\mathrm{e}^{y} y}{y^2+2y+1}\right)\mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -\frac{x}{x^2+1} \\
N(x,y) &= \frac{\mathrm{e}^{y} y}{y^2+2y+1}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\mathrm{e}^{y} y}{y^{2}+2y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. $x$ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{x}{x^{2}+1} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln\left(x^{2}+1\right)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{y} y}{y^{2}+2y+1}$. Therefore equation (4) becomes

$$
\frac{\mathrm{e}^{y} y}{y^{2}+2y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{\mathrm{e}^{y} y}{y^{2}+2y+1} \\
&= \frac{y\,\mathrm{e}^{y}}{(y+1)^{2}}
\end{aligned}
$$

Integrating the above w.r.t $y$ results in

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{y\, \mathrm{e}^{y}}{(y+1)^{2}}\right) \mathrm{d}y$$

$$f(y) = \frac{\mathrm{e}^{y}}{y+1} + c_1$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{\mathrm{e}^{y}}{y+1} + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\mathrm{e}^{y}}{y+1}$$

The solution becomes

$$y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1}}{\frac{\ln(x^2+1)}{2} + c_1}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1}}{\frac{\ln(x^2+1)}{2} + c_1}\right) - 1 \tag{1}$$

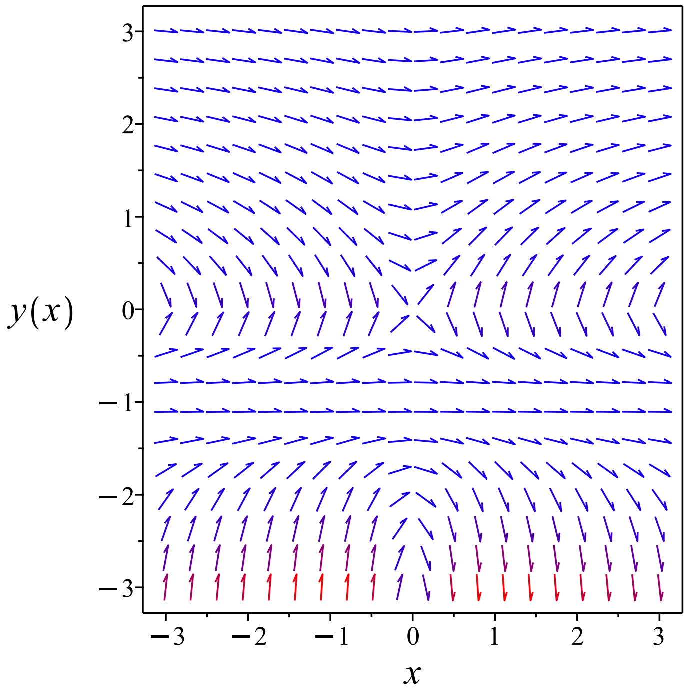



Figure 21: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1}}{\frac{\ln(x^2+1)}{2} + c_1}\right) - 1$$

Verified OK.

### 1.10.4 Maple step by step solution

Let's solve

$x(y+1)^2 - (x^2+1)\, y\, e^y y' = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y' y\, e^y}{(y+1)^2} = \frac{x}{x^2+1}$

- Integrate both sides with respect to $x$

  $\int \frac{y' y\, e^y}{(y+1)^2} dx = \int \frac{x}{x^2+1} dx + c_1$

- Evaluate integral

$$\frac{e^y}{y+1} = \frac{\ln(x^2+1)}{2} + c_1$$

- Solve for $y$

$$y = -\mathit{LambertW}\left(-\frac{e^{-1}}{\frac{\ln(x^2+1)}{2}+c_1}\right) - 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x*(y(x)+1)^2=(x^2+1)*y(x)*exp(y(x))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{2\,e^{-1}}{\ln(x^2+1)+2c_1}\right) - 1$$

✓ Solution by Mathematica

Time used: 1.003 (sec). Leaf size: 33

```
DSolve[x*(y[x]+1)^2==(x^2+1)*y[x]*Exp[y[x]]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 - W\left(-\frac{2}{e\log(x^2+1)+2ec_1}\right)$$

$$y(x) \to -1$$

# 2 Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221

## 2.1 problem Problem 1(a)



Internal problem ID [12222]
Internal file name [`OUTPUT/10874_Thursday_September_21_2023_05_48_23_AM_65554067/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + x^2 y = 0}$$

### 2.1.1 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + yx^4 = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for $\alpha, \beta, n, \gamma$ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{1}{2}$$
$$n = \frac{1}{4}$$
$$\gamma = 2$$

Substituting all the above into (4) gives the solution as

$$y = c_1 \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)$$

Verified OK.

### 2.1.2 Maple step by step solution

Let's solve

$y'' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Assume series solution for $y$

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

- Shift index using $k->k-2$

  $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2}x^k$

- Convert $y''$ to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- Shift index using $k->k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $6a_3x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of $x$ must be 0

  $[2a_2 = 0, 6a_3 = 0]$

- Solve for the dependent coefficient(s)

  $\{a_2 = 0, a_3 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_{k-2} = 0$

- Shift index using $k->k+2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\mathrm{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \mathrm{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1\right) \sqrt{x}$$

✓ Solution by Mathematica
Time used: 0.032 (sec). Leaf size: 30

```
DSolve[y''[x]+x^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \,\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, (-1+i)x\right) + c_1 \,\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, (1+i)x\right)$$

## 2.2 problem Problem 1(b)



Internal problem ID [12223]
Internal file name [OUTPUT/10875_Thursday_September_28_2023_01_05_41_AM_64152595/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(b).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{y''' + yx = \sin(x)}$$

Unable to solve this ODE.

### 2.2.1 Maple step by step solution

Let's solve

$y''' + yx = \sin(x)$

- Highest derivative means the order of the ODE is 3

  $y'''$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
<- pFq successful: received ODE is equivalent to the  0F2  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 1443

```
dsolve(diff(y(x),x$3)+x*y(x)=sin(x),y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'''[x]+x*y[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

Timed out

## 2.3 problem Problem 1(c)



Internal problem ID [12224]
Internal file name [`OUTPUT/10876_Thursday_September_28_2023_01_05_42_AM_88747455/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{y'' + yy' = 1}$$

### 2.3.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + yy')\, dx = \int 1 dx$$
$$\frac{y^2}{2} + y' = x + c_1$$

Which is now solved for $y$. In canonical form the ODE is

$$y' = F(x, y)$$
$$= -\frac{y^2}{2} + x + c_1$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{2} + x + c_1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x + c_1$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{x}{4} + \frac{c_1}{4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{2} + \left(\frac{x}{4} + \frac{c_1}{4}\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_3 \operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x + c_2)}{2}\right) + c_3 \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x + c_2)}{2}\right)$$

The above shows that

$$u'(x) = \frac{c_3 2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y = \frac{c_3 2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)\right)}{c_3 \operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right) + c_3 \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)}$$

Dividing both numerator and denominator by $c_2$ gives, after renaming the constant $\frac{c_3}{c_2} = c_4$ the following solution

$$y = \frac{2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)}$$

Verified OK.

### 2.3.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable $y$ an independent variable. Using

$$y' = p(y)$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y)\left(\frac{d}{dy}p(y)\right) + yp(y) = 1
$$

Which is now solved as first order ode for $p(y)$. Unable to determine ODE type.

Unable to solve. Terminating

### 2.3.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
y'' + yy' = 1
$$

Integrating both sides of the ODE w.r.t $x$ gives

$$
\begin{aligned}
\int (y'' + yy')\, dx &= \int 1 dx \\
\frac{y^2}{2} + y' &= x + c_1
\end{aligned}
$$

Which is now solved for $y$. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= -\frac{y^2}{2} + x + c_1
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -\frac{y^2}{2} + x + c_1
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = x + c_1$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \frac{x}{4} + \frac{c_1}{4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{u''(x)}{2} + \left(\frac{x}{4} + \frac{c_1}{4}\right) u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_3 \operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x + c_2)}{2}\right) + c_3 \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x + c_2)}{2}\right)
$$

The above shows that

$$
u'(x) = \frac{c_3 2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)\right)}{2}
$$

Using the above in (1) gives the solution

$$
y = \frac{c_3 2^{\frac{2}{3}} \left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)\right)}{c_3 \operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right) + c_3 \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)}
$$

Dividing both numerator and denominator by $c_2$ gives, after renaming the constant $\frac{c_3}{c_2} = c_4$ the following solution

$$y = \frac{2^{\frac{2}{3}}\left(\text{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \text{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\text{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \text{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{2}{3}}\left(\text{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \text{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\text{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \text{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2^{\frac{2}{3}}\left(\text{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \text{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\text{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \text{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)}$$

Verified OK.

### 2.3.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= 1\\ a_1 &= y\\ a_0 &= -1\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$
\begin{aligned}
\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx &= c_1 \\
\int 1\,dy' + \int y\,dy + \int -1\,dx &= c_1
\end{aligned}
$$

Which results in

$$
y' + \frac{y^2}{2} - x = c_1
$$

Which is now solved In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{y^2}{2} + x + c_1
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -\frac{y^2}{2} + x + c_1
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = x + c_1$, $f_1(x) = 0$ and $f_2(x) = -\frac{1}{2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{2}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= \frac{x}{4} + \frac{c_1}{4}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{2}+\left(\frac{x}{4}+\frac{c_1}{4}\right)u(x)=0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x)=c_3\,\mathrm{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)+c_3\,\mathrm{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)$$

The above shows that

$$u'(x)=\frac{c_3 2^{\frac{2}{3}}\left(\mathrm{AiryAi}\left(1,\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)+\mathrm{AiryBi}\left(1,\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)\right)}{2}$$

Using the above in (1) gives the solution

$$y=\frac{c_3 2^{\frac{2}{3}}\left(\mathrm{AiryAi}\left(1,\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)+\mathrm{AiryBi}\left(1,\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)\right)}{c_3\,\mathrm{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)+c_3\,\mathrm{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_2)}{2}\right)}$$

Dividing both numerator and denominator by $c_2$ gives, after renaming the constant $\frac{c_3}{c_2}=c_4$ the following solution

$$y=\frac{2^{\frac{2}{3}}\left(\mathrm{AiryAi}\left(1,\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)+\mathrm{AiryBi}\left(1,\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\mathrm{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)+\mathrm{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)}$$

Summary

The solution(s) found are the following

$$y=\frac{2^{\frac{2}{3}}\left(\mathrm{AiryAi}\left(1,\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)+\mathrm{AiryBi}\left(1,\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\mathrm{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)+\mathrm{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)} \tag{1}$$

Verification of solutions

$$y = \frac{2^{\frac{2}{3}}\left(\operatorname{AiryAi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(1, \frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)\right)}{\operatorname{AiryAi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right) + \operatorname{AiryBi}\left(\frac{2^{\frac{2}{3}}(x+c_4)}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)*_a-1 = 0, _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   trying Abel
   <- Abel successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 60

```
dsolve(diff(y(x),x$2)+y(x)*diff(y(x),x)=1,y(x), singsol=all)
```

$$-2\,2^{\frac{2}{3}}\left(\int^{y(x)}\frac{1}{2^{\frac{2}{3}}\_a^2-4\,\mathrm{RootOf}\left(\mathrm{AiryBi}\left(\_Z\right)2^{\frac{1}{3}}c_1\_a+2^{\frac{1}{3}}\_a\,\mathrm{AiryAi}\left(\_Z\right)-2\,\mathrm{AiryBi}\left(1,\_Z\right)c_1-2\,\mathrm{Air}\right.}\right.$$

$$-x-c_2=0$$

✓ Solution by Mathematica

Time used: 71.741 (sec). Leaf size: 73

```
DSolve[y''[x]+y[x]*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{2^{2/3}\left(c_2\,\mathrm{AiryAiPrime}\left(\frac{x-c_1}{\sqrt[3]{2}}\right)+\mathrm{AiryBiPrime}\left(\frac{x-c_1}{\sqrt[3]{2}}\right)\right)}{c_2\,\mathrm{AiryAi}\left(\frac{x-c_1}{\sqrt[3]{2}}\right)+\mathrm{AiryBi}\left(\frac{x-c_1}{\sqrt[3]{2}}\right)}$$

## 2.4 problem Problem 1(d)

Internal problem ID [12225]
Internal file name [`OUTPUT/10877_Thursday_September_28_2023_01_05_45_AM_32247685/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(d).
**ODE order**: 5.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y^{(5)} - y'''' + y' = 2x^2 + 3}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE And $y_p$ is a particular solution to the nonhomogeneous ODE. $y_h$ is the solution to

$$y^{(5)} - y'''' + y' = 0$$

The characteristic equation is

$$\lambda^5 - \lambda^4 + \lambda = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= \mathrm{RootOf}\left(_Z^4 - _Z^3 + 1, \mathrm{index} = 1\right) \\
\lambda_3 &= \mathrm{RootOf}\left(_Z^4 - _Z^3 + 1, \mathrm{index} = 2\right) \\
\lambda_4 &= \mathrm{RootOf}\left(_Z^4 - _Z^3 + 1, \mathrm{index} = 3\right) \\
\lambda_5 &= \mathrm{RootOf}\left(_Z^4 - _Z^3 + 1, \mathrm{index} = 4\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=1)x} c_2 + e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=4)x} c_3 + e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=2)x} c_4 + e^{\text{Root}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=1)x}$$
$$y_3 = e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=4)x}$$
$$y_4 = e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=2)x}$$
$$y_5 = e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=3)x}$$

Now the particular solution to the given ODE is found

$$y^{(5)} - y'''' + y' = 2x^2 + 3$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=1)x}, e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=2)x}, e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}=3)x}, e^{\text{RootOf}(\_Z^4 - \_Z^3 + 1, \text{index}}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra $x$. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^3 + A_2x^2 + A_1x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3x^2 A_3 + 2x A_2 + A_1 = 2x^2 + 3$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = 0, A_3 = \frac{2}{3}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = \frac{2}{3}x^3 + 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=1)x} c_2 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=4)x} c_3 \right. \\ &\quad \left. + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=2)x} c_4 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=3)x} c_5\right) + \left(\frac{2}{3}x^3 + 3x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=1)x} c_2 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=4)x} c_3 \\ &\quad + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=2)x} c_4 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=3)x} c_5 + \frac{2x^3}{3} + 3x \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=1)x} c_2 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=4)x} c_3 \\ &\quad + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=2)x} c_4 + e^{\mathrm{RootOf}(\_Z^4 - \_Z^3 + 1, \mathrm{index}=3)x} c_5 + \frac{2x^3}{3} + 3x \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 5; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(diff(_b(_a), _a), _a), _a), _a) = 2*_a^2-_b
   Methods for high order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 4; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 5; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 372

```
dsolve(diff(y(x),x$5)-diff(y(x),x$4) +diff(y(x),x)=2*x^2+3,y(x), singsol=all)
```

$$y(x) = \frac{2\,\mathrm{RootOf}\left(\_Z^4-\_Z^3+1,\mathrm{index}=2\right)^2\left(\frac{3c_1 e^{\mathrm{RootOf}\left(\_Z^4-\_Z^3+1,\mathrm{index}=1\right)x}\,\mathrm{RootOf}\left(\_Z^4-\_Z^3+1,\mathrm{index}=2\right)\mathrm{RootOf}\left(\_Z\right.}{2}\right.}{}$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 182

```
DSolve[y'''''[x]-y''''[x] +y'[x]==2*x^2+3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \exp\left(x\text{Root}\left[\#1^4-\#1^3+1\&,2\right]\right)}{\text{Root}\left[\#1^4-\#1^3+1\&,2\right]}+\frac{c_1 \exp\left(x\text{Root}\left[\#1^4-\#1^3+1\&,1\right]\right)}{\text{Root}\left[\#1^4-\#1^3+1\&,1\right]}+\frac{c_4 \exp\left(x\text{Root}\left[\#1^4-\#1^3+1\&,4\right]\right)}{\text{Root}\left[\#1^4-\#1^3+1\&,4\right]}+\frac{c_3 \exp\left(x\text{Root}\left[\#1^4-\#1^3+1\&,3\right]\right)}{\text{Root}\left[\#1^4-\#1^3+1\&,3\right]}+\frac{2x^3}{3}+3x+c_5$$

## 2.5 problem Problem 1(e)

Internal problem ID [12226]
Internal file name [`OUTPUT/10878_Thursday_September_28_2023_01_05_45_AM_84918379/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x], [_high_order,
   _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying 4th order ODE linearizable_by_differentiation
trying high order reducible
trying differential order: 4; mu polynomial in y
-> Calling odsolve with the ODE`, diff(diff(diff(diff(y(x), x), x), x), x) = (-(diff(diff(y(
   Integrating factor hint being investigated...
trying differential order: 4; exact nonlinear
trying differential order: 4; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(diff(diff(_b(_a), _a), _a), _a))*_b(_a)^3+(4*(diff(d
   symmetry methods on request
`, `high order, trying reduction of order with given symmetries:`[_a, 1/2*_b]
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+y(x)*diff(y(x),x$4)=1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+y[x]*y''''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.6 problem Problem 1(f)



Internal problem ID [12227]
Internal file name [`OUTPUT/10879_Thursday_September_28_2023_01_05_45_AM_18391819/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(f).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{y''' + yx = \cosh(x)}$$

Unable to solve this ODE.

### 2.6.1 Maple step by step solution

Let's solve

$y''' + yx = \cosh(x)$

- Highest derivative means the order of the ODE is 3

  $y'''$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
<- pFq successful: received ODE is equivalent to the  0F2  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 1443

```
dsolve(diff(y(x),x$3)+x*y(x)=cosh(x),y(x), singsol=all)
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'''[x]+x*y[x]==Cosh[x],y[x],x,IncludeSingularSolutions -> True]
```

Timed out

## 2.7 problem Problem 1(g)



Internal problem ID [12228]
Internal file name [`OUTPUT/10880_Thursday_September_28_2023_01_05_46_AM_85677221/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(g).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x)\, y' + y\,\mathrm{e}^{x^2} = \sinh(x)}$$

### 2.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \sec(x)\,\mathrm{e}^{x^2}\\ q(x) &= \sec(x)\sinh(x)\end{aligned}$$

Hence the ode is

$$y' + \sec(x)\,\mathrm{e}^{x^2} y = \sec(x)\sinh(x)$$

The integrating factor $\mu$ is

$$\mu = e^{\int \sec(x) e^{x^2} dx}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sec(x)\sinh(x)) \\
\frac{d}{dx}\left(e^{\int \sec(x) e^{x^2} dx} y\right) &= \left(e^{\int \sec(x) e^{x^2} dx}\right)(\sec(x)\sinh(x)) \\
d\left(e^{\int \sec(x) e^{x^2} dx} y\right) &= \left(\sec(x)\sinh(x)\, e^{\int \sec(x) e^{x^2} dx}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
e^{\int \sec(x) e^{x^2} dx} y &= \int \sec(x)\sinh(x)\, e^{\int \sec(x) e^{x^2} dx} \, dx \\
e^{\int \sec(x) e^{x^2} dx} y &= \int \sec(x)\sinh(x)\, e^{\int \sec(x) e^{x^2} dx} dx + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\int \sec(x) e^{x^2} dx}$ results in

$$y = e^{-\left(\int \sec(x) e^{x^2} dx\right)} \left(\int \sec(x)\sinh(x)\, e^{\int \sec(x) e^{x^2} dx} dx\right) + c_1 e^{-\left(\int \sec(x) e^{x^2} dx\right)}$$

which simplifies to

$$y = e^{-\left(\int \sec(x) e^{x^2} dx\right)} \left(\int \sec(x)\sinh(x)\, e^{\int \sec(x) e^{x^2} dx} dx + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\left(\int \sec(x) e^{x^2} dx\right)} \left(\int \sec(x)\sinh(x)\, e^{\int \sec(x) e^{x^2} dx} dx + c_1\right) \tag{1}$$

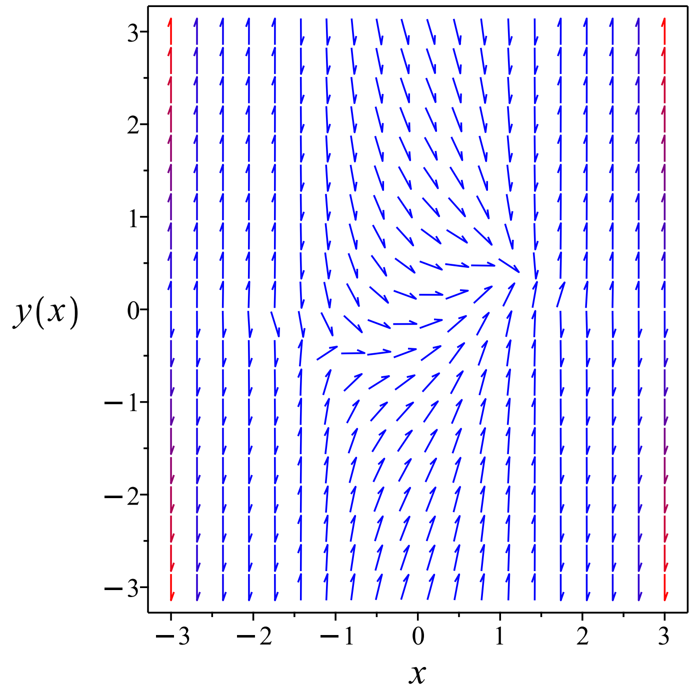



Figure 22: Slope field plot

Verification of solutions

$$y = e^{-\left(\int \sec(x)e^{x^2}dx\right)}\left(\int \sec(x)\sinh(x)\,e^{\int \sec(x)e^{x^2}dx}dx + c_1\right)$$

Verified OK.

### 2.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^{x^2}y - \sinh(x)}{\cos(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\, g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\, e^{h(x)+by} + f(x)$ | $\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$ | $\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ | $\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$ | $\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$ |
| Bernoulli ode | $y' = f(x)\, y + g(x)\, y^n$ | 0 | $e^{-\int (n-1)f(x)dx} y^n$ |
| Reduced Riccati | $y' = f_1(x)\, y + f_2(x)\, y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\int -\sec(x)\mathrm{e}^{x^2} dx}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x, y) \to (R, S)$ where $(R, S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\int -\sec(x)e^{x^2} dx}} dy \end{aligned}$$

### 2.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(x))\,\mathrm{d}y &= \left(-\mathrm{e}^{x^2}y + \sinh(x)\right)\mathrm{d}x\\ \left(\mathrm{e}^{x^2}y - \sinh(x)\right)\mathrm{d}x + (\cos(x))\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \mathrm{e}^{x^2}y - \sinh(x)\\ N(x,y) &= \cos(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x^2}y - \sinh(x)\right)\\ &= \mathrm{e}^{x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(x))\\ &= -\sin(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \sec(x)\left(\left(\mathrm{e}^{x^2}\right) - (-\sin(x))\right)\\ &= \sec(x)\,\mathrm{e}^{x^2} + \tan(x)\end{aligned}$$

Since $A$ does not depend on $y$, then it can be used to find an integrating factor. The integrating factor $\mu$ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \sec(x)\mathrm{e}^{x^2} + \tan(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\mu = e^{-ix+\int \frac{2\,e^{ix} e^{x^2}+2i}{e^{2ix}+1} dx}$$
$$= e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)}$$

$M$ and $N$ are multiplied by this integrating factor, giving new $M$ and new $N$ which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original $M$ and $N$.

$$\overline{M} = \mu M$$
$$= e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)} \left(e^{x^2} y - \sinh(x)\right)$$
$$= \left(e^{x^2} y - \sinh(x)\right) e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)}$$

And

$$\overline{N} = \mu N$$
$$= e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)} (\cos(x))$$
$$= \cos(x)\, e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{dy}{dx} = 0$$
$$\left(\left(e^{x^2} y - \sinh(x)\right) e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)}\right) + \left(\cos(x)\, e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)}\right) \frac{dy}{dx} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$
$$\int \frac{\partial \phi}{\partial x} dx = \int \left(e^{x^2} y - \sinh(x)\right) e^{-ix+2\left(\int \frac{e^{x(x+i)}+i}{e^{2ix}+1} dx\right)} dx$$
$$\phi = \int^x \left(e^{-a^2} y - \sinh(\_a)\right) e^{-i\_a+2\left(\int \frac{e^{-a(\_a+i)}+i}{e^{2i\_a}+1} d\_a\right)} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \int^{x} \mathrm{e}^{-\_a^2} \mathrm{e}^{-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a + f'(y) \qquad (4) \\
&= \int^{x} \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)}$. Therefore equation (4) becomes

$$
\cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} = \int^{x} \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} - \left(\int^{x} \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a\right)
$$

Integrating the above w.r.t $y$ gives

$$
\begin{aligned}
&\int f'(y)\,\mathrm{d}y \\
&= \int \left(\cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} - \left(\int^{x} \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a\right)\right) \mathrm{d}y \\
f(y) &= \int_{0}^{y} \left(\cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} - \left(\int^{x} \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a\right)\right) d\_a \\
&\quad + c_1
\end{aligned}
$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$
\begin{aligned}
\phi &= \int^{x} \left(\mathrm{e}^{-\_a^2} y - \sinh(\_a)\right) \mathrm{e}^{-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a \\
&\quad + \int_{0}^{y} \left(\cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} - \left(\int^{x} \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a\right)\right) d\_a \\
&\quad + c_1
\end{aligned}
$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = \int^x \left(\mathrm{e}^{-\_a^2} y - \sinh(\_a)\right) \mathrm{e}^{-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a + \int_0^y \left(\cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} - \left(\int^x \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a\right)\right) d\_a$$

<u>Summary</u>
The solution(s) found are the following

$$\int^x \left(\mathrm{e}^{-\_a^2} y - \sinh(\_a)\right) \mathrm{e}^{-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a + \int_0^y \left(\cos(x)\,\mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1} dx\right)} - \left(\int^x \mathrm{e}^{\_a^2-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1} d\_a\right)} d\_a\right)\right) d\_a = c_1 \tag{1}$$

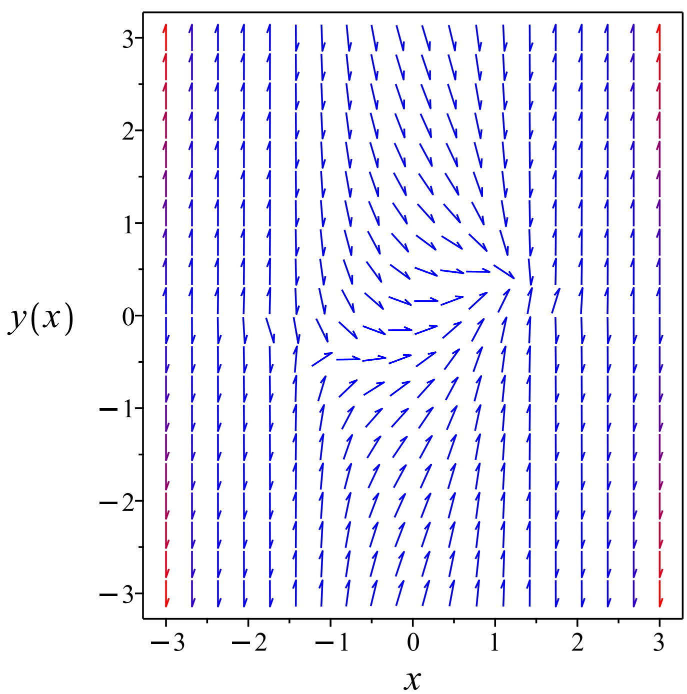



Figure 23: Slope field plot

Verification of solutions

$$\int^{x} \left(\mathrm{e}^{-\_a^{2}} y - \sinh(\_a)\right) \mathrm{e}^{-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1}d\_a\right)} d\_a + \int_{0}^{y} \left(\cos(x)\, \mathrm{e}^{-ix+2\left(\int \frac{\mathrm{e}^{x(x+i)}+i}{\mathrm{e}^{2ix}+1}dx\right)} - \left(\int^{x} \mathrm{e}^{\_a^{2}-i\_a+2\left(\int \frac{\mathrm{e}^{\_a(\_a+i)}+i}{\mathrm{e}^{2i\_a}+1}d\_a\right)} d\_a\right)\right) d\_a = c_1$$

Verified OK.

### 2.7.4 Maple step by step solution

Let's solve

$\cos(x)\, y' + y\, \mathrm{e}^{x^{2}} = \sinh(x)$

- Highest derivative means the order of the ODE is 1

  $y'$

- Isolate the derivative

  $y' = -\frac{y\,\mathrm{e}^{x^{2}}}{\cos(x)} + \frac{\sinh(x)}{\cos(x)}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y\,\mathrm{e}^{x^{2}}}{\cos(x)} = \frac{\sinh(x)}{\cos(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y\,\mathrm{e}^{x^{2}}}{\cos(x)}\right) = \frac{\mu(x)\sinh(x)}{\cos(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y\,\mathrm{e}^{x^{2}}}{\cos(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)\mathrm{e}^{x^{2}}}{\cos(x)}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\int \frac{\mathrm{e}^{x^{2}}}{\cos(x)}dx}$

- Integrate both sides with respect to $x$

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\sinh(x)}{\cos(x)}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\sinh(x)}{\cos(x)} dx + c_1$

- Solve for $y$

  $y = \frac{\int \frac{\mu(x)\sinh(x)}{\cos(x)} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\int \frac{\mathrm{e}^{x^2}}{\cos(x)} dx}$

  $y = \frac{\int \frac{\sinh(x)\mathrm{e}^{\int \frac{\mathrm{e}^{x^2}}{\cos(x)} dx}}{\cos(x)} dx + c_1}{\mathrm{e}^{\int \frac{\mathrm{e}^{x^2}}{\cos(x)} dx}}$

- Simplify

  $y = \mathrm{e}^{-\left(\int \sec(x)\mathrm{e}^{x^2} dx\right)} \left(\int \sec(x) \sinh(x) \mathrm{e}^{\int \sec(x)\mathrm{e}^{x^2} dx} dx + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(cos(x)*diff(y(x),x)+y(x)*exp(x^2)=sinh(x),y(x), singsol=all)
```

$$y(x) = \left(\int \sec(x) \sinh(x) \mathrm{e}^{\int \sec(x)\mathrm{e}^{x^2} dx} dx + c_1\right) \mathrm{e}^{-\left(\int \sec(x)\mathrm{e}^{x^2} dx\right)}$$

✓ Solution by Mathematica

Time used: 1.562 (sec). Leaf size: 66

```
DSolve[Cos[x]*y'[x]+y[x]*Exp[x^2]==Sinh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \exp\left(\int_1^x -e^{K[1]^2}\sec(K[1])dK[1]\right)\left(\int_1^x \exp\left(-\int_1^{K[2]} -e^{K[1]^2}\sec(K[1])dK[1]\right)\sec(K[2])\sinh(K[2])dK[2] + c_1\right)$$

## 2.8 problem Problem 1(h)



Internal problem ID [12229]
Internal file name [`OUTPUT/10881_Thursday_September_28_2023_01_07_02_AM_86373971/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(h).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{y''' + yx = \cosh(x)}$$

Unable to solve this ODE.

### 2.8.1 Maple step by step solution

Let's solve

$y''' + yx = \cosh(x)$

- Highest derivative means the order of the ODE is 3

  $y'''$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
<- pFq successful: received ODE is equivalent to the  0F2  ODE, case  c = 0 `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 1443

```
dsolve(diff(y(x),x$3)+x*y(x)=cosh(x),y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 91.544 (sec). Leaf size: 2230

```
DSolve[y'''[x]+x*y[x]==Cosh[x],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

## 2.9 problem Problem 1(i)



Internal problem ID [12230]
Internal file name [`OUTPUT/10882_Thursday_September_28_2023_01_07_02_AM_25852880/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(i).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{yy' = 1}$$

### 2.9.1 Solving as quadrature ode

Integrating both sides gives

$$\int y dy = x + c_1$$
$$\frac{y^2}{2} = x + c_1$$

Solving for $y$ gives these solutions

$$y_1 = \sqrt{2x + 2c_1}$$
$$y_2 = -\sqrt{2x + 2c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{2x + 2c_1} \tag{1}$$
$$y = -\sqrt{2x + 2c_1} \tag{2}$$

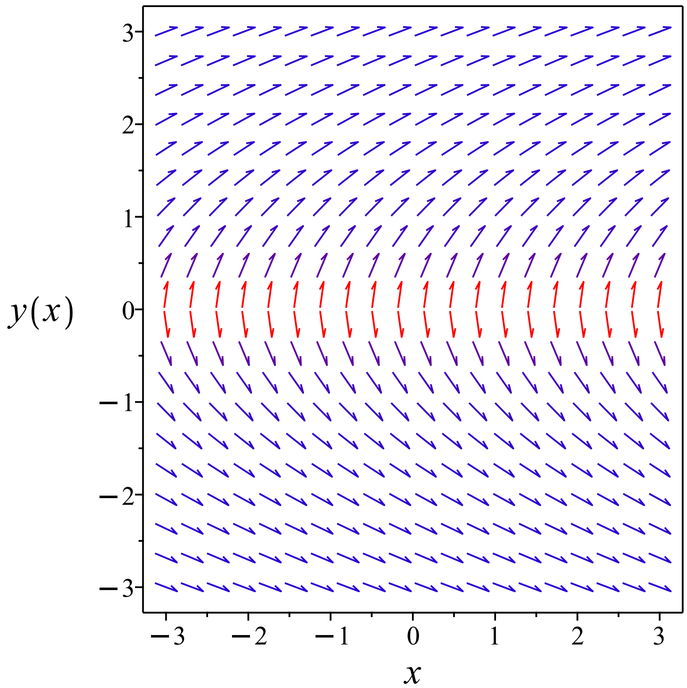



Figure 24: Slope field plot

Verification of solutions

$$y = \sqrt{2x + 2c_1}$$

Verified OK.

$$y = -\sqrt{2x + 2c_1}$$

Verified OK.

### 2.9.2 Maple step by step solution

Let's solve

$yy' = 1$

- Highest derivative means the order of the ODE is 1

  $y'$

- Integrate both sides with respect to $x$

  $\int yy'dx = \int 1dx + c_1$

- Evaluate integral

$$\frac{y^2}{2} = x + c_1$$

- Solve for $y$

$$\left\{y = \sqrt{2x + 2c_1}, y = -\sqrt{2x + 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(y(x)*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 + 2x}$$
$$y(x) = -\sqrt{c_1 + 2x}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 38

```
DSolve[y[x]*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{x + c_1}$$
$$y(x) \to \sqrt{2}\sqrt{x + c_1}$$

## 2.10 problem Problem 1(j)



Internal problem ID [12231]
Internal file name [`OUTPUT/10883_Thursday_September_28_2023_01_07_03_AM_89250507/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(j).
**ODE order**: 1.
**ODE degree**: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\sinh(x) y'^2 + 3y = 0}$$

### 2.10.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = -\frac{3}{\sinh(x)}, g = y$. Hence the ode is

$$(y')^2 = -\frac{3y}{\sinh(x)}$$

Solving for $y'$ from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$-\frac{3}{\sinh(x)} > 0$$
$$y > 0$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{-\frac{3}{\sinh\left(x\right)}}\right)\,dx \\
-\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{-\frac{3}{\sinh\left(x\right)}}\right)\,dx
\end{aligned}
$$

Integrating now gives the solutions.

$$
\begin{aligned}
\int \frac{1}{\sqrt{y}}dy &= \int \sqrt{-\frac{3}{\sinh\left(x\right)}}dx + c_1 \\
\int -\frac{1}{\sqrt{y}}dy &= \int \sqrt{-\frac{3}{\sinh\left(x\right)}}dx + c_1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
2\sqrt{y} &= \int \sqrt{-\frac{3}{\sinh\left(x\right)}}dx + c_1 \\
-2\sqrt{y} &= \int \sqrt{-\frac{3}{\sinh\left(x\right)}}dx + c_1
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y &= \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)^{2}}{4} + \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right) c_1}{2} + \frac{c_1^{2}}{4} \\
y &= \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)^{2}}{4} + \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right) c_1}{2} + \frac{c_1^{2}}{4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)^2}{4} + \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)c_1}{2} + \frac{c_1^2}{4} \tag{1}$$

$$y = \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)^2}{4} + \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)c_1}{2} + \frac{c_1^2}{4} \tag{2}$$

Verification of solutions

$$y = \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)^2}{4} + \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)c_1}{2} + \frac{c_1^2}{4}$$

Verified OK. `{0 < y, 0 < -3/sinh(x)}`

$$y = \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)^2}{4} + \frac{\left(\int \sqrt{-\frac{3}{\sinh(x)}}dx\right)c_1}{2} + \frac{c_1^2}{4}$$

Verified OK. `{0 < y, 0 < -3/sinh(x)}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      <- exact successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      <- exact successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 707

```
dsolve(sinh(x)*diff(y(x),x)^2+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = -\frac{\mathrm{RootOf}\left(-\mathrm{JacobiSN}\left(\frac{\mathrm{RootOf}\left(\_Z^2-\mathrm{e}^x,\mathrm{index}=1\right)\sqrt{-\mathrm{e}^x+1}\ \mathrm{RootOf}\left(\_Z^2-2\,\mathrm{e}^x-2,\mathrm{index}=1\right)\left(\sqrt{3}\,\sqrt{2}\,\sqrt{-\mathrm{e}^x\left(\mathrm{e}^{2x}-1\right)}\,c_1-2\_Z\right.}{12\,\mathrm{e}^{2x}-12}\right.\right.}{6\,\mathrm{e}^{2x}-6}$$

$$y(x) = -\frac{\mathrm{RootOf}\left(\mathrm{JacobiSN}\left(\frac{\mathrm{RootOf}\left(\_Z^2-\mathrm{e}^x,\mathrm{index}=1\right)\sqrt{-\mathrm{e}^x+1}\ \mathrm{RootOf}\left(\_Z^2-2\,\mathrm{e}^x-2,\mathrm{index}=1\right)\left(\sqrt{3}\,\sqrt{2}\,\sqrt{-\mathrm{e}^x\left(\mathrm{e}^{2x}-1\right)}\,c_1+2\_Z\right)\mathrm{e}}{12\,\mathrm{e}^{2x}-12}\right.\right.}{6\,\mathrm{e}^{2x}-6}$$

$$y(x) = -\frac{\mathrm{RootOf}\left(\mathrm{JacobiSN}\left(\frac{\mathrm{RootOf}\left(\_Z^2-2\,\mathrm{e}^x-2,\mathrm{index}=1\right)\mathrm{RootOf}\left(\_Z^2-\mathrm{e}^x,\mathrm{index}=1\right)\left(3\,\mathrm{e}^{2x}\,\mathrm{RootOf}\left(-1+\left(-6\,\mathrm{e}^{3x}+6\,\mathrm{e}^x\right)\_Z^2\right)c_1\right.}{6\sqrt{-\mathrm{e}^x+1}\left(\mathrm{e}^x+1\right)}\right.\right.}{6\,\mathrm{e}^{2x}-6}$$

$$y(x) = -\frac{\mathrm{RootOf}\left(\mathrm{JacobiSN}\left(\frac{\mathrm{RootOf}\left(\_Z^2-\mathrm{e}^x,\mathrm{index}=1\right)\sqrt{-\mathrm{e}^x+1}\ \mathrm{RootOf}\left(\_Z^2-2\,\mathrm{e}^x-2,\mathrm{index}=1\right)\left(\sqrt{3}\,\sqrt{2}\,\sqrt{-\mathrm{e}^x\left(\mathrm{e}^{2x}-1\right)}\,c_1-2\_Z\right)\mathrm{e}}{12\,\mathrm{e}^{2x}-12}\right.\right.}{6\,\mathrm{e}^{2x}-6}$$

$$y(x) = -\frac{\mathrm{RootOf}\left(-\mathrm{JacobiSN}\left(\frac{\mathrm{RootOf}\left(\_Z^2-\mathrm{e}^x,\mathrm{index}=1\right)\sqrt{-\mathrm{e}^x+1}\ \mathrm{RootOf}\left(\_Z^2-2\,\mathrm{e}^x-2,\mathrm{index}=1\right)\left(\sqrt{3}\,\sqrt{2}\,\sqrt{-\mathrm{e}^x\left(\mathrm{e}^{2x}-1\right)}\,c_1+2\_Z\right.}{12\,\mathrm{e}^{2x}-12}\right.\right.}{6\,\mathrm{e}^{2x}-6}$$

$$y(x) = -\frac{\mathrm{RootOf}\left(-\mathrm{JacobiSN}\left(\frac{\mathrm{RootOf}\left(\_Z^2-2\,\mathrm{e}^x-2,\mathrm{index}=1\right)\mathrm{RootOf}\left(\_Z^2-\mathrm{e}^x,\mathrm{index}=1\right)\left(3\,\mathrm{e}^{2x}\,\mathrm{RootOf}\left(-1+\left(-6\,\mathrm{e}^{3x}+6\,\mathrm{e}^x\right)\_Z^2\right)\right.}{6\sqrt{-\mathrm{e}^x+1}\left(\mathrm{e}^x+1\right)}\right.\right.}{6\,\mathrm{e}^{2x}-6}$$

✓ Solution by Mathematica

Time used: 0.648 (sec). Leaf size: 145

```
DSolve[Sinh[x]*y'[x]^2+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3i \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2ix), 2\right)^2 - \sqrt{3}c_1\sqrt{i\sinh(x)}\sqrt{\operatorname{csch}(x)} \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2ix), 2\right) + \frac{c_1{}^2}{4}$$

$$y(x) \to 3i \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2ix), 2\right)^2 + \sqrt{3}c_1\sqrt{i\sinh(x)}\sqrt{\operatorname{csch}(x)} \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2ix), 2\right) + \frac{c_1{}^2}{4}$$

$$y(x) \to 0$$

## 2.11 problem Problem 1(k)



Internal problem ID [12232]
Internal file name [`OUTPUT/10884_Thursday_September_28_2023_01_07_09_AM_40036526/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(k).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{5y' - yx = 0}$$

### 2.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{xy}{5} \end{aligned}$$

Where $f(x) = \frac{x}{5}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{x}{5}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{x}{5}\,dx \\
\ln(y) &= \frac{x^2}{10} + c_1 \\
y &= \mathrm{e}^{\frac{x^2}{10}+c_1} \\
&= c_1 \mathrm{e}^{\frac{x^2}{10}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{x^2}{10}} \tag{1}
$$

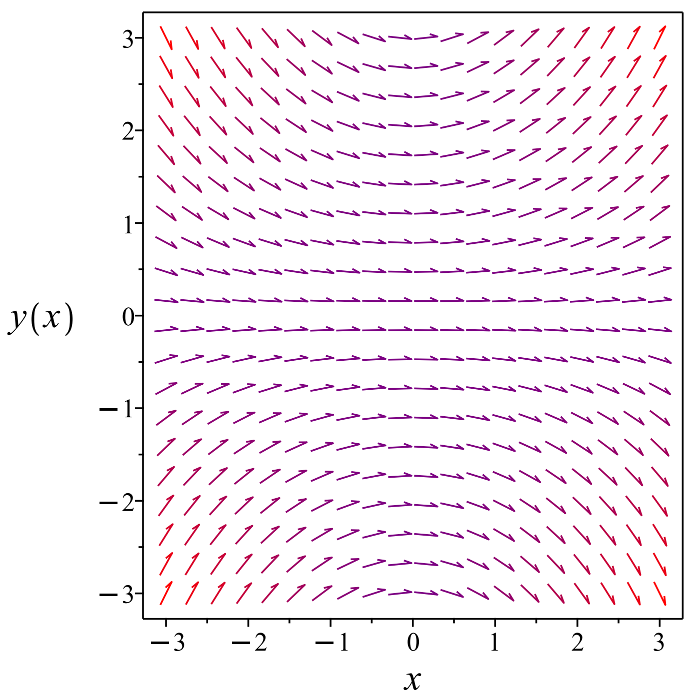



Figure 25: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{x^2}{10}}
$$

Verified OK.

### 2.11.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x}{5} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{yx}{5} = 0$$

The integrating factor $\mu$ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{x}{5}dx} \\ &= \mathrm{e}^{-\frac{x^2}{10}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x^2}{10}}y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{-\frac{x^2}{10}}y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x^2}{10}}$ results in

$$y = c_1\mathrm{e}^{\frac{x^2}{10}}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{\frac{x^2}{10}} \tag{1}$$

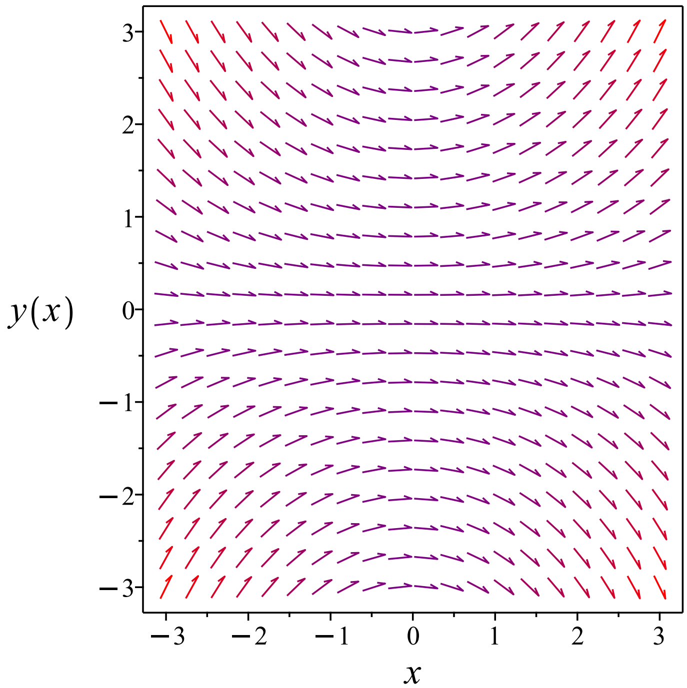



Figure 26: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x^2}{10}}$$

Verified OK.

### 2.11.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$5u'(x)\,x + 5u(x) - u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2-5)}{5x} \end{aligned}$$

Where $f(x) = \frac{x^2-5}{5x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{x^2-5}{5x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{x^2-5}{5x}\, dx \\
\ln(u) &= \frac{x^2}{10} - \ln(x) + c_2 \\
u &= \mathrm{e}^{\frac{x^2}{10} - \ln(x) + c_2} \\
&= c_2 \mathrm{e}^{\frac{x^2}{10} - \ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{\frac{x^2}{10}}}{x}
$$

Therefore the solution $y$ is

$$
\begin{aligned}
y &= xu \\
&= c_2 \mathrm{e}^{\frac{x^2}{10}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{\frac{x^2}{10}} \tag{1}
$$

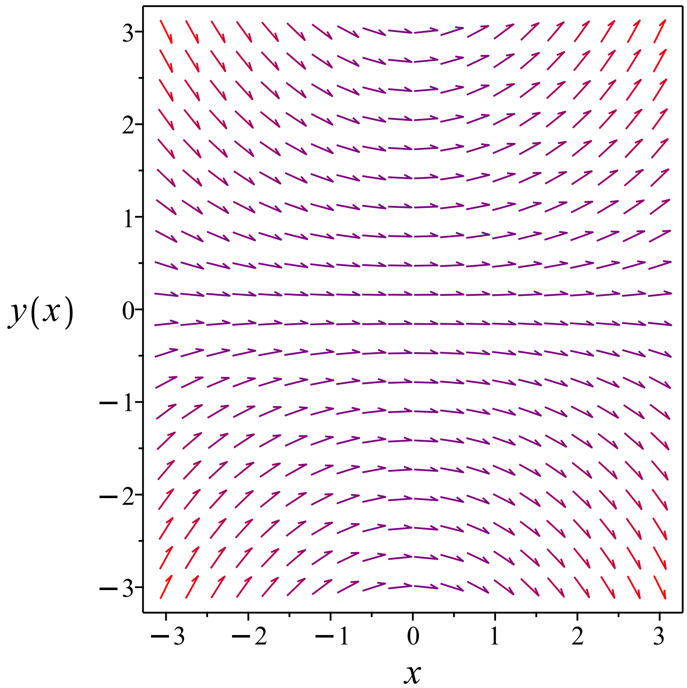



Figure 27: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{x^2}{10}}$$

Verified OK.

### 2.11.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{xy}{5}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find $\xi, \eta$

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

| ODE class | Form | $\xi$ | $\eta$ |
|---|---|---|---|
| linear ode | $y' = f(x)y(x) + g(x)$ | 0 | $e^{\int f dx}$ |
| separable ode | $y' = f(x)\, g(y)$ | $\frac{1}{f}$ | 0 |
| quadrature ode | $y' = f(x)$ | 0 | 1 |
| quadrature ode | $y' = g(y)$ | 1 | 0 |
| homogeneous ODEs of Class A | $y' = f\left(\frac{y}{x}\right)$ | $x$ | $y$ |
| homogeneous ODEs of Class C | $y' = (a + bx + cy)^{\frac{n}{m}}$ | 1 | $-\frac{b}{c}$ |
| homogeneous class D | $y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$ | $x^2$ | $xy$ |
| First order special form ID 1 | $y' = g(x)\, e^{h(x)+by} + f(x)$ | $\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$ | $\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$ |
| polynomial type ode | $y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$ | $\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$ | $\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$ |
| Bernoulli ode | $y' = f(x)\, y + g(x)\, y^n$ | 0 | $e^{-\int (n-1) f(x) dx} y^n$ |
| Reduced Riccati | $y' = f_1(x)\, y + f_2(x)\, y^2$ | 0 | $e^{-\int f_1 dx}$ |

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\frac{x^2}{10}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates $R, S$. The canonical coordinates map $(x, y) \to (R, S)$ where $(R, S)$ are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable $R$ in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

$S$ is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{10}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{10}} y$$

Now that $R, S$ are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above $R_x, R_y, S_x, S_y$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy}{5}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x\, e^{-\frac{x^2}{10}} y}{5} \\ S_y &= e^{-\frac{x^2}{10}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of $R$ only. This is done by solving for $x, y$ in terms of $R, S$ from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates $R, S$. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to $x, y$ coordinates. This results in

$$e^{-\frac{x^2}{10}} y = c_1$$

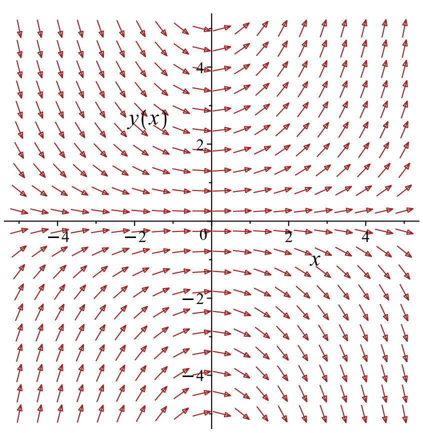
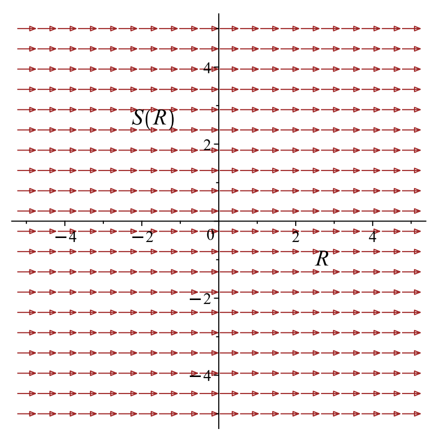
Which simplifies to

$$e^{-\frac{x^2}{10}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{x^2}{10}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

| Original ode in $x, y$ coordinates | Canonical coordinates transformation | ODE in canonical coordinates $(R, S)$ |
|---|---|---|
| $\frac{dy}{dx} = \frac{xy}{5}$  | $R = x$ <br> $S = e^{-\frac{x^2}{10}} y$ | $\frac{dS}{dR} = 0$  |

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x^2}{10}} \tag{1}$$

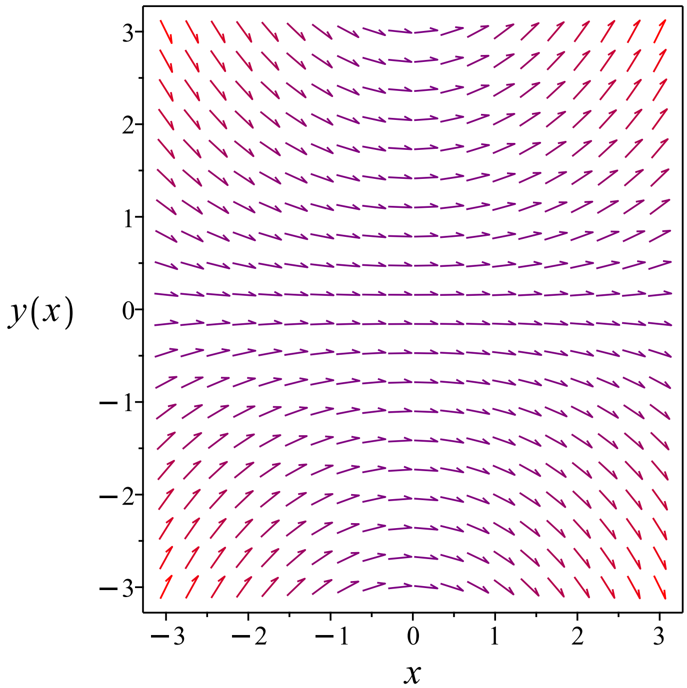



Figure 28: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{\frac{x^2}{10}}$$

Verified OK.

### 2.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{5}{y}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(\frac{5}{y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{5}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{5}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int -x \, dx$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{5}{y}$. Therefore equation (4) becomes

$$\frac{5}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{5}{y}$$

Integrating the above w.r.t $y$ gives

$$\int f'(y) \, dy = \int \left(\frac{5}{y}\right) dy$$

$$f(y) = 5 \ln(y) + c_1$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = -\frac{x^2}{2} + 5\ln(y) + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = -\frac{x^2}{2} + 5\ln(y)$$

The solution becomes

$$y = e^{\frac{x^2}{10}+\frac{c_1}{5}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{10}+\frac{c_1}{5}} \tag{1}$$

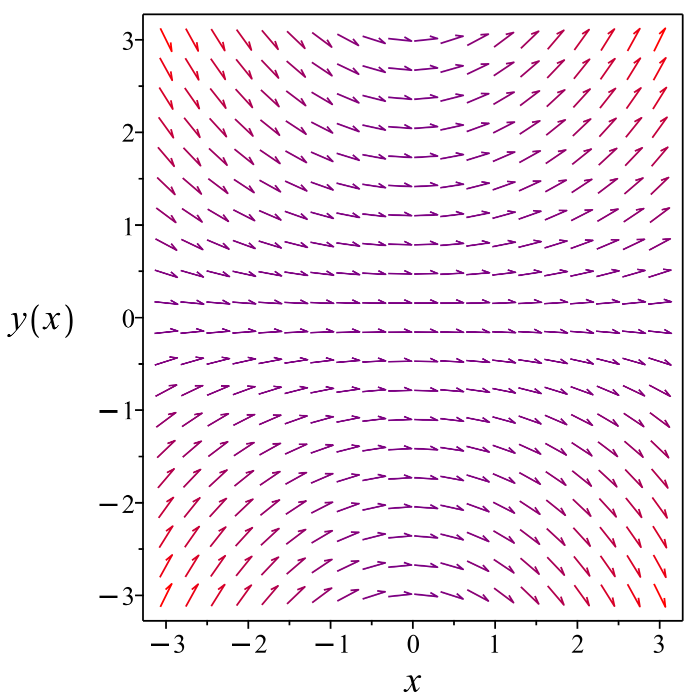



Figure 29: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{10}+\frac{c_1}{5}}$$

Verified OK.

### 2.11.6 Maple step by step solution

Let's solve

$5y' - yx = 0$

- Highest derivative means the order of the ODE is 1

  $y'$

- Separate variables

  $\frac{y'}{y} = \frac{x}{5}$

- Integrate both sides with respect to $x$

  $\int \frac{y'}{y} dx = \int \frac{x}{5} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^2}{10} + c_1$

- Solve for $y$

  $y = e^{\frac{x^2}{10}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(5*diff(y(x),x)-x*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^2}{10}}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 22

```
DSolve[5*y'[x]-x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{x^2}{10}}$$
$$y(x) \to 0$$

## 2.12 problem Problem 1(L)

Internal problem ID [12233]
Internal file name [`OUTPUT/10885_Thursday_September_28_2023_01_07_10_AM_98467938/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(L).
**ODE order**: 1.
**ODE degree**: 2.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)*y+H(x)]`]]
```

$$\boxed{{y'}^2\sqrt{y} = \sin(x)}$$

Solving the given ode for $y'$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}} \tag{1}$$

$$y' = -\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right)\mathrm{d}x\\ \left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right)\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right)\\ &= \frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{2y^{\frac{3}{2}}} - \frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\right) - (0)\right)\\ &= \frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\end{aligned}$$

Since $A$ depends on $y$, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{\sqrt{y}}{\sqrt{\sqrt{y}\,\sin(x)}}\left((0) - \left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{2y^{\frac{3}{2}}} - \frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\right)\right)\\ &= \frac{1}{4y}\end{aligned}$$

Since $B$ does not depend on $x$, it can be used to obtain an integrating factor. Let the integrating factor be $\mu$. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int \frac{1}{4y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{\ln(y)}{4}}\\ &= y^{\frac{1}{4}}\end{aligned}$$

$M$ and $N$ are now multiplied by this integrating factor, giving new $M$ and new $N$ which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original $M$ and $N$.

$$\begin{aligned}\overline{M} &= \mu M \\ &= y^{\frac{1}{4}}\left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right) \\ &= -\frac{\sqrt{\sqrt{y}\,\sin(x)}}{y^{\frac{1}{4}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= y^{\frac{1}{4}}(1) \\ &= y^{\frac{1}{4}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{y^{\frac{1}{4}}}\right) + \left(y^{\frac{1}{4}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sqrt{\sqrt{y}\,\sin(x)}}{y^{\frac{1}{4}}}\,\mathrm{d}x\end{aligned}$$

$$\phi = \int^{x} -\frac{\sqrt{\sqrt{y}\,\sin(\_a)}}{y^{\frac{1}{4}}}d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^{\frac{1}{4}}$. Therefore equation (4) becomes

$$y^{\frac{1}{4}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^{\frac{1}{4}}$$

Integrating the above w.r.t $y$ gives

$$\int f'(y)\, \mathrm{d}y = \int \left(y^{\frac{1}{4}}\right) \mathrm{d}y$$

$$f(y) = \frac{4y^{\frac{5}{4}}}{5} + c_1$$

Where $c_1$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = \int^x -\frac{\sqrt{\sqrt{y}\, \sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5} + c_1$$

But since $\phi$ itself is a constant function, then let $\phi = c_2$ where $c_2$ is new constant and combining $c_1$ and $c_2$ constants into new constant $c_1$ gives the solution as

$$c_1 = \int^x -\frac{\sqrt{\sqrt{y}\, \sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5}$$

<u>Summary</u>

The solution(s) found are the following

$$\int^x -\frac{\sqrt{\sqrt{y}\, \sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5} = c_1 \tag{1}$$

Verification of solutions

$$\int^{x} -\frac{\sqrt{\sqrt{y}\,\sin\left(\_a\right)}}{y^{\frac{1}{4}}}d\_a+\frac{4y^{\frac{5}{4}}}{5}=c_1$$

Verified OK.

Solving equation (2)

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$dy = \left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right) dx$$

$$\left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right) dx + dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right)\\ &= -\frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{2y^{\frac{3}{2}}} + \frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\right) - (0)\right)\\ &= -\frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\end{aligned}$$

Since $A$ depends on $y$, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{\sqrt{y}}{\sqrt{\sqrt{y}\,\sin(x)}}\left((0) - \left(-\frac{\sqrt{\sqrt{y}\,\sin(x)}}{2y^{\frac{3}{2}}} + \frac{\sin(x)}{4y\sqrt{\sqrt{y}\,\sin(x)}}\right)\right) \\
&= \frac{1}{4y}
\end{aligned}
$$

Since $B$ does not depend on $x$, it can be used to obtain an integrating factor. Let the integrating factor be $\mu$. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{1}{4y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\ln(y)}{4}} \\
&= y^{\frac{1}{4}}
\end{aligned}
$$

$M$ and $N$ are now multiplied by this integrating factor, giving new $M$ and new $N$ which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original $M$ and $N$.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y^{\frac{1}{4}}\left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{\sqrt{y}}\right) \\
&= \frac{\sqrt{\sqrt{y}\,\sin(x)}}{y^{\frac{1}{4}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y^{\frac{1}{4}}(1) \\
&= y^{\frac{1}{4}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\sqrt{\sqrt{y}\,\sin(x)}}{y^{\frac{1}{4}}}\right) + \left(y^{\frac{1}{4}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{\sqrt{\sqrt{y}\,\sin(x)}}{y^{\frac{1}{4}}}\,\mathrm{d}x$$
$$\phi = \int^{x} \frac{\sqrt{\sqrt{y}\,\sin(\_a)}}{y^{\frac{1}{4}}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = y^{\frac{1}{4}}$. Therefore equation (4) becomes

$$y^{\frac{1}{4}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^{\frac{1}{4}}$$

Integrating the above w.r.t $y$ gives

$$\int f'(y)\,\mathrm{d}y = \int \left(y^{\frac{1}{4}}\right)\mathrm{d}y$$
$$f(y) = \frac{4y^{\frac{5}{4}}}{5} + c_3$$

Where $c_3$ is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives $\phi$

$$\phi = \int^x \frac{\sqrt{\sqrt{y}\,\sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5} + c_3$$

But since $\phi$ itself is a constant function, then let $\phi = c_4$ where $c_4$ is new constant and combining $c_3$ and $c_4$ constants into new constant $c_3$ gives the solution as

$$c_3 = \int^x \frac{\sqrt{\sqrt{y}\,\sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5}$$

Summary

The solution(s) found are the following

$$\int^x \frac{\sqrt{\sqrt{y}\,\sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5} = c_3 \tag{1}$$

Verification of solutions

$$\int^x \frac{\sqrt{\sqrt{y}\,\sin(\_a)}}{y^{\frac{1}{4}}} d\_a + \frac{4y^{\frac{5}{4}}}{5} = c_3$$

Verified OK.

Maple trace

```
Warning: System is inconsistent
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 58

```
dsolve(diff(y(x),x)^2*sqrt(y(x))=sin(x),y(x), singsol=all)
```

$$\frac{4y(x)^{\frac{5}{4}}}{5} - \frac{\int^x \sqrt{\sqrt{y(x)}\,\sin(\_a)}d\_a}{y(x)^{\frac{1}{4}}} + c_1 = 0$$

$$\frac{4y(x)^{\frac{5}{4}}}{5} + \frac{\int^x \sqrt{\sqrt{y(x)}\,\sin(\_a)}d\_a}{y(x)^{\frac{1}{4}}} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.436 (sec). Leaf size: 77

```
DSolve[y'[x]^2*Sqrt[y[x]]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5^{4/5}\left(-2E\left(\frac{1}{4}(\pi-2x)\middle|\,2\right)+c_1\right)^{4/5}}{2\ 2^{3/5}}$$
$$y(x) \to \frac{5^{4/5}\left(2E\left(\frac{1}{4}(\pi-2x)\middle|\,2\right)+c_1\right)^{4/5}}{2\ 2^{3/5}}$$

## 2.13 problem Problem 1(m)

Internal problem ID [12234]
Internal file name [OUTPUT/10886_Thursday_September_28_2023_01_08_06_AM_94705325/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(m).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{2y'' + 3y' + 4x^2y = 1}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 376

```
dsolve(2*diff(y(x),x$2)+3*diff(y(x),x)+4*x^2*y(x)=1,y(x), singsol=all)
```

$$y(x) =$$

$$-48\left(-\frac{32\,\mathrm{KummerM}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)\left(i+\frac{3\sqrt{2}}{32}\right)\left(\int \frac{\mathrm{Kum}}{1563i\,\mathrm{KummerU}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)\mathrm{KummerM}\left(-\frac{9i\sqrt{2}}{128}-\right.}\right.}{3}\right.$$

$$+\left(\int \frac{\mathrm{KummerM}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)\mathrm{e}^{\frac{i\sqrt{}}{}}}{\left(192i\sqrt{2}-2048\right)\mathrm{KummerU}\left(-\frac{9i\sqrt{2}}{128}-\frac{1}{4},\frac{3}{2},i\sqrt{2}\,x^2\right)\mathrm{KummerM}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)-1563\,\mathrm{K}}\right.$$

$$\left.-\frac{32}{3}\right)\mathrm{KummerU}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)-\frac{\mathrm{KummerU}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)c_1}{48}$$

$$\left.-\frac{\mathrm{KummerM}\left(\frac{3}{4}-\frac{9i\sqrt{2}}{128},\frac{3}{2},i\sqrt{2}\,x^2\right)c_2}{48}\right)x\,\mathrm{e}^{-\frac{\left(i\sqrt{2}\,x+\frac{3}{2}\right)x}{2}}$$

✓ Solution by Mathematica

Time used: 11.093 (sec). Leaf size: 553

```
DSolve[2*y''[x]+3*y'[x]+4*x^2*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{1}{4}x\left(-3-2i\sqrt{2}x\right)}\left(\text{Hypergeometric1F1}\left(\frac{1}{4}\right.\right.$$

$$\left.-\frac{9i}{64\sqrt{2}},\frac{1}{2},i\sqrt{2}x^2\right)\int_1^x \frac{(8+8i)e^{\frac{1}{4}K}}{\left(9+16i\sqrt{2}\right)\left(\sqrt[4]{2}\,\text{HermiteH}\left(-\frac{3}{2}+\frac{9i}{32\sqrt{2}},\frac{(1+i)K[2]}{\sqrt[4]{2}}\right)\text{Hypergeometric1F1}\left(\frac{1}{4}-\right.\right.}$$

$$+\text{HermiteH}\left(-\frac{1}{2}\right.$$

$$\left.+\frac{9i}{32\sqrt{2}},\sqrt[4]{-2}x\right)\int_1^x \frac{16e^{\frac{1}{4}K[1]\left(2i\right.}}{\sqrt[4]{-2}\left(-32+9i\sqrt{2}\right)\text{HermiteH}\left(-\frac{3}{2}+\frac{9i}{32\sqrt{2}},\sqrt[4]{-2}K[1]\right)\text{Hypergeometric1F1}\left(\frac{1}{4}-\right.}$$

$$+c_1\,\text{HermiteH}\left(-\frac{1}{2}+\frac{9i}{32\sqrt{2}},\sqrt[4]{-2}x\right)$$

$$\left.+c_2\,\text{Hypergeometric1F1}\left(\frac{1}{4}-\frac{9i}{64\sqrt{2}},\frac{1}{2},i\sqrt{2}x^2\right)\right)$$

## 2.14 problem Problem 1(n)

Internal problem ID [12235]
Internal file name [`OUTPUT/10887_Thursday_September_28_2023_01_08_07_AM_11619658/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(n).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _quadrature]]
```

$$\boxed{y''' = 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE And $y_p$ is a particular solution to the nonhomogeneous ODE. $y_h$ is the solution to

$$y''' = 0$$

The characteristic equation is

$$\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= x^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$y''' = 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra $x$. The UC_set becomes

$$[\{x\}]$$

Since $x$ is duplicated in the UC_set, then this basis is multiplied by extra $x$. The UC_set becomes

$$[\{x^2\}]$$

Since $x^2$ is duplicated in the UC_set, then this basis is multiplied by extra $x$. The UC_set becomes

$$[\{x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = \frac{x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_3x^2 + c_2x + c_1) + \left(\frac{x^3}{6}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_3x^2 + c_2x + c_1 + \frac{1}{6}x^3 \tag{1}$$

Verification of solutions

$$y = c_3x^2 + c_2x + c_1 + \frac{1}{6}x^3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$3)=1,y(x), singsol=all)
```

$$y(x) = \frac{1}{6}x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[y'''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + c_3x^2 + c_2x + c_1$$

## 2.15 problem Problem 1(o)



Internal problem ID [12236]
Internal file name [`OUTPUT/10888_Thursday_September_28_2023_01_08_07_AM_94393377/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 1(o).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - y = \sin(x)^2}$$

### 2.15.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 0, C = -1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for $y_h$ from

$$x^2y'' - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by $x^r$ gives

$$r(r-1) + 0 - 1 = 0$$

Or

$$r^2 - r - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$
$$r_2 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + c_2 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

Next, we find the particular solution to the ODE

$$x^2 y'' - y = \sin(x)^2$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}$$

$$y_2 = x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} & x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \\ \frac{d}{dx}\left(x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\right) & \frac{d}{dx}\left(x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} & x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \\ \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}{x} & \frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}{x} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\right)\left(\frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}{x}\right) - \left(x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\right)\left(\frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}{x}\right)$$

Which simplifies to

$$W = \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \sqrt{5}}{x}$$

Which simplifies to

$$W = \sqrt{5}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \sin(x)^2}{\sqrt{5}\, x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{5}\, x^{\frac{\sqrt{5}}{2}-\frac{3}{2}} \sin(x)^2}{5} dx$$

Hence

$$u_1 = -\frac{2\sqrt{5}\, x^{\frac{3}{2}+\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}+\frac{\sqrt{5}}{4}\right], -x^2\right)}{15+5\sqrt{5}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \sin(x)^2}{\sqrt{5}\, x^2} dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{5}\, x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} \sin(x)^2}{5} dx$$

Hence

$$u_2 = -\frac{2\sqrt{5}\, x^{\frac{3}{2}-\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right)}{5\sqrt{5}-15}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2\sqrt{5}\, x^{\frac{3}{2}+\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}+\frac{\sqrt{5}}{4}\right], -x^2\right) x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{15+5\sqrt{5}} - \frac{2\sqrt{5}\, x^{\frac{3}{2}-\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5\sqrt{5}-15}$$

Which simplifies to

$$y_p(x) = \frac{x^2\sqrt{5}\left((3+\sqrt{5}) \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}\right.\right.\right.}{10}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \frac{x^2\sqrt{5}\left((3+\sqrt{5})\,\mathrm{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \mathrm{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2,\right.\right.\right.}{10}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2\sqrt{5}\left((3+\sqrt{5})\,\mathrm{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \mathrm{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\right.\right.\right.}{10} + c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + c_2 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \tag{1}$$

Verification of solutions

$$y = \frac{x^2\sqrt{5}\left((3+\sqrt{5})\,\mathrm{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \mathrm{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\right.\right.\right.}{10} + c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + c_2 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

Verified OK.

### 2.15.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where $r$ is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$
r = \frac{1}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 29: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{1}{x^2}$$

For the pole at $x = 0$ let $b$ be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = 1$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{5}}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{5}}{2} \end{aligned}$$

Since the order of $r$ at $\infty$ is 2 then $[\sqrt{r}]_\infty = 0$. Let $b$ be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of $r$ at $\infty$. which can be found by dividing the leading coefficient of $s$ by the leading coefficient of $t$ from

$$r = \frac{s}{t} = \frac{1}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 1$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{\sqrt{5}}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{\sqrt{5}}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = \frac{1}{x^2}$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^+$ | $\alpha_c^-$ |
|---|---|---|---|---|
| 0 | 2 | 0 | $\frac{1}{2}+\frac{\sqrt{5}}{2}$ | $\frac{1}{2}-\frac{\sqrt{5}}{2}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| 2 | 0 | $\frac{1}{2}+\frac{\sqrt{5}}{2}$ | $\frac{1}{2}-\frac{\sqrt{5}}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{\sqrt{5}}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \frac{\sqrt{5}}{2} - \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) \\
&= 0
\end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{x} + (-)(0) \\
&= \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{x} \\
&= -\frac{\sqrt{5}-1}{2x}
\end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x^2}\right) + \left(\frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x}\right)^2 - \left(\frac{1}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - \frac{\sqrt{5}}{2}}{x} dx} \\ &= x^{\frac{1}{2} - \frac{\sqrt{5}}{2}} \end{aligned}$$

The first solution to the original ode in $y$ is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{2} - \frac{\sqrt{5}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2} - \frac{\sqrt{5}}{2}}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \int \frac{1}{x^{-\sqrt{5}+1}}\, dx \\
&= x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \left(\frac{x^{\sqrt{5}}\sqrt{5}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \left(\frac{x^{\sqrt{5}}\sqrt{5}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. $y_h$ is the solution to

$$
x^2y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{c_2 \sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5}
$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}$$

$$y_2 = \frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} & \frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5} \\ \frac{d}{dx}\left(x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\right) & \frac{d}{dx}\left(\frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} & \frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5} \\ \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}{x} & \frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}{5x} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\right)\left(\frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}{5x}\right) - \left(\frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5}\right)\left(\frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}{x}\right)$$

Which simplifies to

$$W = \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \sin(x)^2}{5}}{x^2} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{5}\, x^{\frac{\sqrt{5}}{2}-\frac{3}{2}} \sin(x)^2}{5} dx$$

Hence

$$u_1 = -\frac{2\sqrt{5}\, x^{\frac{3}{2}+\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}+\frac{\sqrt{5}}{4}\right], -x^2\right)}{15+5\sqrt{5}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \sin(x)^2}{x^2} dx$$

Which simplifies to

$$u_2 = \int x^{-\frac{3}{2}-\frac{\sqrt{5}}{2}} \sin(x)^2\, dx$$

Hence

$$u_2 = -\frac{2x^{\frac{3}{2}-\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right)}{\sqrt{5}-3}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\frac{2\sqrt{5}\, x^{\frac{3}{2}+\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}+\frac{\sqrt{5}}{4}\right], -x^2\right) x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{15+5\sqrt{5}} \\ &- \frac{2x^{\frac{3}{2}-\frac{\sqrt{5}}{2}} \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) \sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5\left(\sqrt{5}-3\right)} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{x^2\sqrt{5}\left(\left(3+\sqrt{5}\right) \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4} - \right.\right.\right.}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{c_2\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5}\right) \\ &\quad + \left(\frac{x^2\sqrt{5}\left(\left(3+\sqrt{5}\right) \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, \right.\right.\right.}{10}\right. \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{c_2\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5} \\ &\quad + \frac{x^2\sqrt{5}\left(\left(3+\sqrt{5}\right) \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \right.\right.\right.}{10} \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{c_2\sqrt{5}\, x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{5} \\ &\quad + \frac{x^2\sqrt{5}\left(\left(3+\sqrt{5}\right) \operatorname{hypergeom}\left(\left[1, -\frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \frac{7}{4}-\frac{\sqrt{5}}{4}\right], -x^2\right) + \operatorname{hypergeom}\left(\left[1, \frac{\sqrt{5}}{4}+\frac{3}{4}\right], \left[\frac{3}{2}, 2, \right.\right.\right.}{10} \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 97

```
dsolve(x^2*diff(y(x),x$2)-y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{3\left(\sqrt{5}+\frac{5}{3}\right)x^2 \operatorname{hypergeom}\left(\left[1,-\frac{\sqrt{5}}{4}+\frac{3}{4}\right],\left[\frac{3}{2},2,\frac{7}{4}-\frac{\sqrt{5}}{4}\right],-x^2\right)}{10} - \frac{3x^2\left(\sqrt{5}-\frac{5}{3}\right)\operatorname{hypergeom}\left(\left[1,\frac{\sqrt{5}}{4}+\frac{3}{4}\right],\left[\frac{3}{2},2,\frac{7}{4}+\frac{\sqrt{5}}{4}\right],-x^2\right)}{10} + x^{-\frac{\sqrt{5}}{2}+\frac{1}{2}}c_1 + x^{\frac{\sqrt{5}}{2}+\frac{1}{2}}c_2$$

✓ Solution by Mathematica

Time used: 1.679 (sec). Leaf size: 445

```
DSolve[x^2*y''[x]-y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{10\sqrt{5}c_1x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + 10c_1x^{\frac{1}{2}-\frac{\sqrt{5}}{2}} + 10\sqrt{5}c_2x^{\frac{1}{2}\left(1+\sqrt{5}\right)} + 10c_2x^{\frac{1}{2}\left(1+\sqrt{5}\right)} + 2^{\frac{1}{2}\left(\sqrt{5}-1\right)}\left(5+\sqrt{5}\right)(-ix)^{\frac{1}{2}\left(1+\sqrt{5}\right)}\Gamma\Big(-}{}$$

## 2.16 problem Problem 2(a)



Internal problem ID [12237]
Internal file name [`OUTPUT/10889_Thursday_September_28_2023_01_08_17_AM_24827545/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y = x^2}$$

### 2.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. $y_h$ is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = e^x c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution $y_h$ is

$$y_h = e^x c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3x^2 - A_2x - A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -x^2 - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 e^{-x}) + (-x^2 - 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{-x} - x^2 - 2 \tag{1}$$

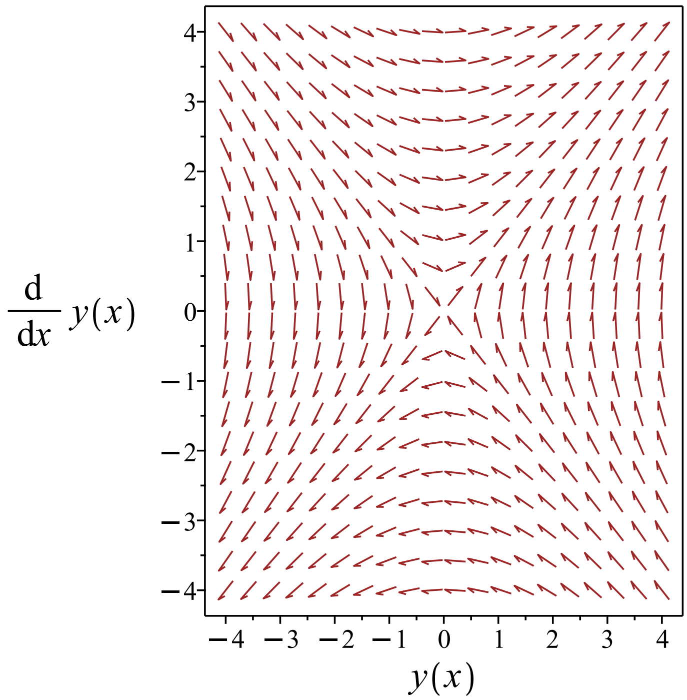



Figure 30: Slope field plot

Verification of solutions

$$y = e^x c_1 + c_2 e^{-x} - x^2 - 2$$

Verified OK.

### 2.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 30: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of $x$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}} \, dx \\ &= \mathrm{e}^{-x} \left( \frac{\mathrm{e}^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_2 \mathrm{e}^{x}}{2} + c_1 \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3x^2 - A_2x - A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -x^2 - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_2 \mathrm{e}^x}{2} + c_1 \mathrm{e}^{-x}\right) + \left(-x^2 - 2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 \mathrm{e}^x}{2} + c_1 \mathrm{e}^{-x} - x^2 - 2 \tag{1}$$

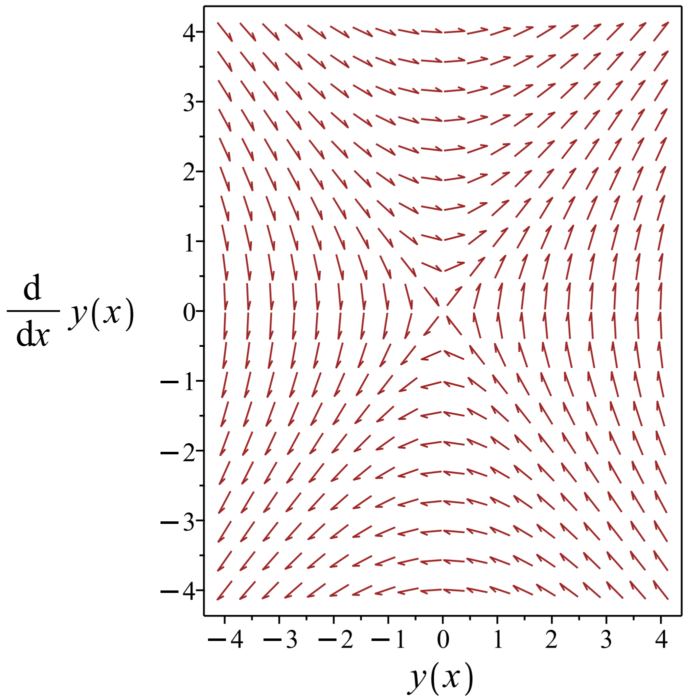



Figure 31: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^x}{2} + c_1 e^{-x} - x^2 - 2$$

Verified OK.

### 2.16.3 Maple step by step solution

Let's solve

$y'' - y = x^2$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-x}\left(\int x^2 e^{x} dx\right)}{2} + \frac{e^{x}\left(\int x^2 e^{-x} dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -x^2 - 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{x} - x^2 - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)=x^2+y(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + c_1 e^{x} - x^2 - 2$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 26

```
DSolve[y''[x]==x^2+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + c_1 e^x + c_2 e^{-x} - 2$$

## 2.17 problem Problem 2(b)

Internal problem ID [12238]
Internal file name [OUTPUT/10890_Thursday_September_28_2023_01_08_18_AM_47683329/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

[NONE]

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
trying 3rd order, integrating factor of the form mu(y) for some mu
Trying the formal computation of integrating factors depending on any 2 of [x, y, y, y]
differential order: 3; looking for linear symmetries
--- Trying Lie symmetry methods, high order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$3)+x*diff(y(x),x$2)-y(x)^2=sin(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'''[x]+x*y''[x]-y[x]^2==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.18 problem Problem 2(c)

Internal problem ID [12239]
Internal file name [`OUTPUT/10891_Thursday_September_28_2023_01_08_18_AM_53026221/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(c).
**ODE order**: 1.
**ODE degree**: 2.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y'^2 + yy'^2 x = \ln(x)}$$

Solving the given ode for $y'$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{(yx+1)\ln(x)}}{yx+1} \tag{1}$$

$$y' = -\frac{\sqrt{(yx+1)\ln(x)}}{yx+1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Unable to determine ODE type.

Unable to determine ODE type.

Solving equation (2)

Unable to determine ODE type.

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
-> Calling odsolve with the ODE`, diff(y(x), x) = (2*y(x)*exp(x^2-LambertW(-x^2*y(x)*exp(x^2
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   trying exact
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(y(x), x) = (-2*(-x^2+ln(y(x)))*y(x)/x-2*y(x)*x)/(y(x)
   Methods for first order ODEs:
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)^2+y(x)*diff(y(x),x)^2*x=ln(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]^2+y[x]*y'[x]^2*x==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.19 problem Problem 2(d)

Internal problem ID [12240]
Internal file name [`OUTPUT/10892_Thursday_September_28_2023_01_08_20_AM_14008183/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x], [_high_order,
   _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying 4th order ODE linearizable_by_differentiation
trying high order reducible
trying differential order: 4; mu polynomial in y
-> Calling odsolve with the ODE`, diff(diff(diff(diff(y(x), x), x), x), x) = (-sin(diff(diff
   Integrating factor hint being investigated...
trying differential order: 4; exact nonlinear
trying differential order: 4; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(diff(diff(_b(_a), _a), _a), _a))*_b(_a)^3+(4*(diff(d
   symmetry methods on request
`, `high order, trying reduction of order with given symmetries:`[_a, 1/2*_b]
```

✗ Solution by Maple

```
dsolve(sin(diff(y(x),x$2))+y(x)*diff(y(x),x$4)=1,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[Sin[y''[x]]+y[x]*y''''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.20 problem Problem 2(e)

Internal problem ID [12241]
Internal file name [`OUTPUT/10893_Thursday_September_28_2023_01_08_20_AM_61627412/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[NONE]`

Unable to solve or complete the solution.

$$\boxed{\sinh(x)\, {y'}^2 + y'' - yx = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integratin
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for the S-
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dy
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dy
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dy
      -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integrating
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, only a reduction of order through one integrat
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = formal`
```

✗ Solution by Maple

```
dsolve(sinh(x)*diff(y(x),x)^2+diff(y(x),x$2)=x*y(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[Sinh[x]*y'[x]^2+y''[x]==x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.21 problem Problem 2(f)



Internal problem ID [12242]
Internal file name [OUTPUT/10894_Thursday_September_28_2023_01_08_21_AM_73243775/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]
```

$$\boxed{yy'' = 1}$$

### 2.21.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable $y$ an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) = 1$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{1}{yp}
\end{aligned}
$$

Where $f(y) = \frac{1}{y}$ and $g(p) = \frac{1}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{p}}\, dp &= \frac{1}{y}\, dy \\
\int \frac{1}{\frac{1}{p}}\, dp &= \int \frac{1}{y}\, dy \\
\frac{p^2}{2} &= \ln(y) + c_1
\end{aligned}
$$

The solution is

$$
\frac{p(y)^2}{2} - \ln(y) - c_1 = 0
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\frac{y'^2}{2} - \ln(y) - c_1 = 0
$$

Solving the given ode for $y'$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
\begin{aligned}
y' &= \sqrt{2\ln(y) + 2c_1} \quad (1) \\
y' &= -\sqrt{2\ln(y) + 2c_1} \quad (2)
\end{aligned}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{2\ln(y) + 2c_1}} dy &= \int dx \\
\int^y \frac{1}{\sqrt{2\ln(\_a) + 2c_1}} d\_a &= x + c_2
\end{aligned}
$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{2\ln(y)+2c_1}}dy = \int dx$$

$$-\left(\int^{y}\frac{1}{\sqrt{2\ln(\_a)+2c_1}}d\_a\right) = c_3 + x$$

Summary

The solution(s) found are the following

$$\int^{y}\frac{1}{\sqrt{2\ln(\_a)+2c_1}}d\_a = x + c_2 \tag{1}$$

$$-\left(\int^{y}\frac{1}{\sqrt{2\ln(\_a)+2c_1}}d\_a\right) = c_3 + x \tag{2}$$

Verification of solutions

$$\int^{y}\frac{1}{\sqrt{2\ln(\_a)+2c_1}}d\_a = x + c_2$$

Verified OK.

$$-\left(\int^{y}\frac{1}{\sqrt{2\ln(\_a)+2c_1}}d\_a\right) = c_3 + x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-1/_a = 0, _b(_a), HINT = [[_a, 0
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 51

```
dsolve(y(x)*diff(y(x),x$2)=1,y(x), singsol=all)
```

$$\int^{y(x)} \frac{1}{\sqrt{2\ln(\_a) - c_1}} d\_a - x - c_2 = 0$$

$$-\left(\int^{y(x)} \frac{1}{\sqrt{2\ln(\_a) - c_1}} d\_a\right) - x - c_2 = 0$$

✓ Solution by Mathematica

Time used: 60.104 (sec). Leaf size: 93

```
DSolve[y[x]*y''[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \exp\left(-\text{erf}^{-1}\left(-i\sqrt{\frac{2}{\pi}}\sqrt{e^{c_1}(x+c_2)^2}\right)^2 - \frac{c_1}{2}\right)$$

$$y(x) \to \exp\left(-\text{erf}^{-1}\left(i\sqrt{\frac{2}{\pi}}\sqrt{e^{c_1}(x+c_2)^2}\right)^2 - \frac{c_1}{2}\right)$$

## 2.22 problem Problem 2(h)

Internal problem ID [12243]
Internal file name [OUTPUT/10895_Thursday_September_28_2023_01_08_21_AM_1427532/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 2(h).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

[NONE]

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
Successful isolation of d^3y/dx^3: 2 solutions were found. Trying to solve each resulting OD
   *** Sublevel 2 ***
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying 3rd order ODE linearizable_by_differentiation
   differential order: 3; trying a linearization to 4th order
   trying differential order: 3; exact nonlinear
   trying 3rd order, integrating factor of the form mu(y) for some mu
   Trying the formal computation of integrating factors depending on any 2 of [x, y, y, y]
   differential order: 3; looking for linear symmetries`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$3)^2+sqrt(y(x))=sin(x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'''[x]^2+Sqrt[y[x]]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.23 problem Problem 3(a)



Internal problem ID [12244]
Internal file name [`OUTPUT/10896_Thursday_September_28_2023_01_08_21_AM_85711767/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 3(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + y = 0}$$

### 2.23.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(1)} \\ &= -2 \pm \sqrt{3}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + \sqrt{3} \\ \lambda_2 &= -2 - \sqrt{3}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{3} - 2 \\ \lambda_2 &= -2 - \sqrt{3}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\sqrt{3}-2\right)x} + c_2 e^{\left(-2-\sqrt{3}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\left(\sqrt{3}-2\right)x} + c_2 \mathrm{e}^{\left(-2-\sqrt{3}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(\sqrt{3}-2\right)x} + c_2 \mathrm{e}^{\left(-2-\sqrt{3}\right)x} \tag{1}$$

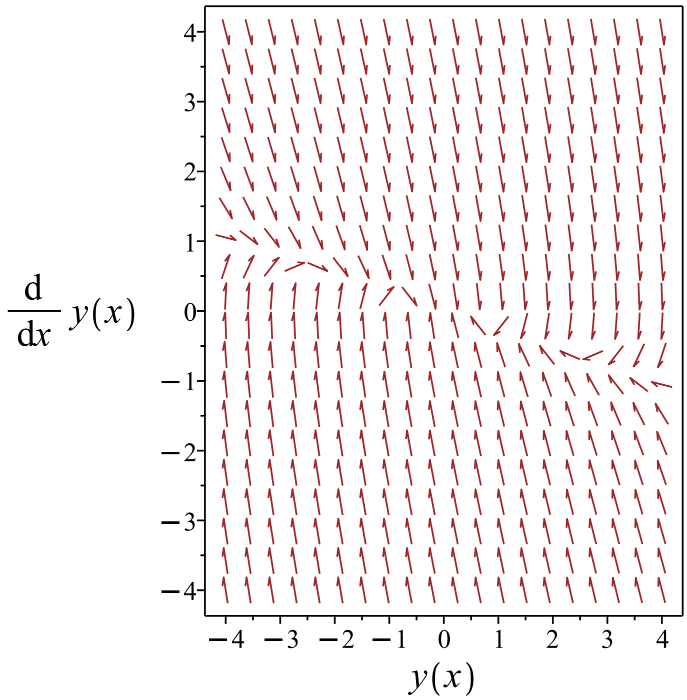



Figure 32: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(\sqrt{3}-2\right)x} + c_2 e^{\left(-2-\sqrt{3}\right)x}$$

Verified OK.

### 2.23.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 3z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 32: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 3$ is not a function of $x$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\sqrt{3}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\left(2+\sqrt{3}\right)x}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{\left(y_1\right)^2}\,dx \\ &= y_1 \left(\frac{\sqrt{3}\,\mathrm{e}^{2\sqrt{3}\,x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left( e^{-\left(2+\sqrt{3}\right)x} \right) + c_2 \left( e^{-\left(2+\sqrt{3}\right)x} \left( \frac{\sqrt{3}\, e^{2\sqrt{3}\,x}}{6} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\left(2+\sqrt{3}\right)x} + \frac{c_2 \sqrt{3}\, e^{\left(\sqrt{3}-2\right)x}}{6} \tag{1}$$

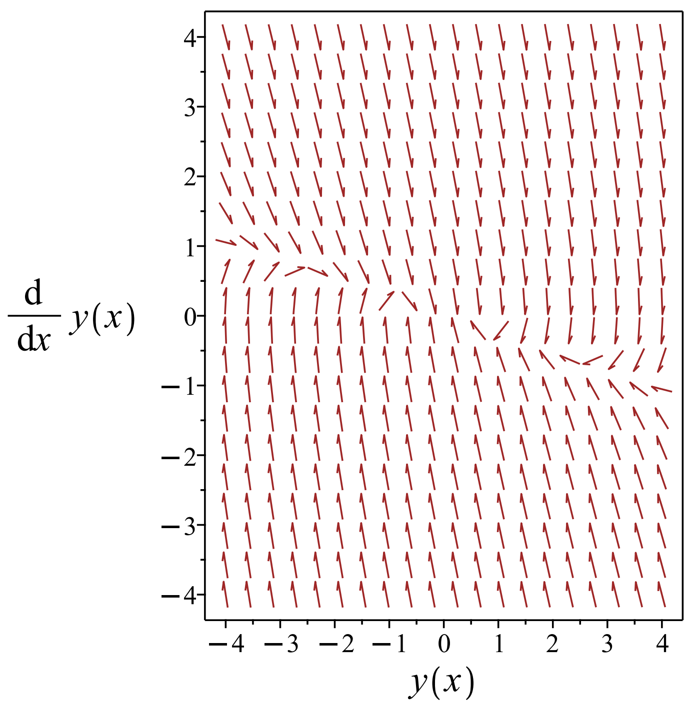



Figure 33: Slope field plot

Verification of solutions

$$y = c_1 e^{-\left(2+\sqrt{3}\right)x} + \frac{c_2 \sqrt{3}\, e^{\left(\sqrt{3}-2\right)x}}{6}$$

Verified OK.

### 2.23.3 Maple step by step solution

Let's solve

$y'' + 4y' + y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 + 4r + 1 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-4)\pm\left(\sqrt{12}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2-\sqrt{3}, \sqrt{3}-2\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\left(-2-\sqrt{3}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\left(\sqrt{3}-2\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\left(-2-\sqrt{3}\right)x} + c_2 e^{\left(\sqrt{3}-2\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\left(-2+\sqrt{3}\right)x} + c_2 e^{-\left(2+\sqrt{3}\right)x}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 34

```
DSolve[y''[x]+4*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\left(\left(2+\sqrt{3}\right)x\right)}\left(c_2 e^{2\sqrt{3}x} + c_1\right)$$

## 2.24 problem Problem 3(b)



Internal problem ID [12245]
Internal file name [OUTPUT/10897_Thursday_September_28_2023_01_08_23_AM_35475611/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 3(b).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 5y'' + y' - y = 0}$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{\left(116 + 6\sqrt{78}\right)^{\frac{1}{3}}}{3} + \frac{22}{3\left(116 + 6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3}$$

$$\lambda_2 = -\frac{\left(116 + 6\sqrt{78}\right)^{\frac{1}{3}}}{6} - \frac{11}{3\left(116 + 6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3} + \frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3} - \frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(116 + 6\sqrt{78}\right)^{\frac{1}{3}}}{6} - \frac{11}{3\left(116 + 6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3} - \frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3} - \frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_1 + \mathrm{e}^{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} c_2 -$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_2 = \mathrm{e}^{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x}$$

$$y_3 = \mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}+\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_1 \\ & + \mathrm{e}^{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} c_2 \\ & + \mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}+\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_3 \end{aligned} \tag{1}$$

Verification of solutions

$$y = e^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_1 + e^{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} c_2 + e^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}+\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_3$$

Verified OK.

### 2.24.1 Maple step by step solution

Let's solve

$y''' - 5y'' + y' - y = 0$

- Highest derivative means the order of the ODE is 3

  $y'''$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 5y_3(x) - y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 5y_3(x) - y_2(x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 5 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 5 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3} + \frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3}, \left[\begin{array}{c} \frac{1}{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3} + \frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3}\right)^2} \\ \frac{1}{\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3} + \frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3}} \\ 1 \end{array}\right]\right], \left[-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6} - \frac{}{3\left(116+\right.}\right.\right.$$

- Consider eigenpair

$$\left[\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}, \left[\begin{array}{c}\frac{1}{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}} \\ \frac{1}{\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1}=\mathrm{e}^{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}} \\ \frac{1}{\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}} \\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}, \left[\begin{array}{c}\frac{}{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right.} \\ \frac{}{-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}}\end{array}\right.\right.$$

- Solution from eigenpair

$$
\mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{3}\cdots}{\cdots}\right.} \\ \frac{1}{-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{\cdots}}{\cdots}} \\ 1 \end{array}\right.
$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$
\mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} \cdot \left(\cos\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}\cdots\right.}{2\cdots}\right.\right.
$$

- Simplify expression

$$e^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{\cdots}{3(\cdots}\right.}{2}\right.}{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\cdots\right.}\right.}{2}\right.} \\ \frac{\cos\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{\cdots}{3(\cdots}\right.}{2}\right.}{-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{\mathrm{I}\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\cdots\right.}\right.}{2}} \\ \cos\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}\cdots\right.}{\cdots}\right. \end{array}\right.$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x)=e^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} \cdot \left[-\frac{9\left(116+6\sqrt{78}\right)^{\frac{2}{3}}\left(\left(116+6\sqrt{78}\right)^{\frac{4}{3}}\sqrt{3}\sin\left(\frac{\sqrt{3}\left(\left(116+6\sqrt{78}\right)^{\frac{2}{3}}-22\right)\cdots}{6\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right.\right.}{\cdots}\right.\right.$$

- General solution to the system of ODEs

$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}} \\ \frac{1}{\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{}{3(11}\right.}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{9\left(\left(\left(\frac{\left(\left(-c_3-\frac{c_2\sqrt{3}}{3}\right)\sqrt{26}+7c_2+7c_3\sqrt{3}\right)\left(116+6\sqrt{3}\sqrt{26}\right)^{\frac{2}{3}}}{9}+c_2\left(\frac{58}{3}+\sqrt{3}\sqrt{26}\right)\left(116+6\sqrt{3}\sqrt{26}\right)^{\frac{1}{3}}+\frac{94\left(-c_3+\frac{c_2\sqrt{3}}{3}\right)\sqrt{26}}{9}-\frac{310c_3\sqrt{}}{9}\right.\right.\right.}{}$$

<u>Maple trace</u>

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 183

```
dsolve(diff(y(x),x$3)-5*diff(y(x),x$2)+diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{\left(\left(116+6\sqrt{78}\right)^{\frac{2}{3}}+5\left(116+6\sqrt{78}\right)^{\frac{1}{3}}+22\right)x}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}} - c_2 e^{-\frac{\left(22+\left(116+6\sqrt{78}\right)^{\frac{2}{3}}-10\left(116+6\sqrt{78}\right)^{\frac{1}{3}}\right)x}{6\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}} \sin\left(\frac{\sqrt{3}\left(\left(116+6\sqrt{3}\sqrt{26}\right)^{\frac{2}{3}}-22\right)x}{6\left(116+6\sqrt{3}\sqrt{26}\right)^{\frac{1}{3}}}\right) + c_3 e^{-\frac{\left(22+\left(116+6\sqrt{78}\right)^{\frac{2}{3}}-10\left(116+6\sqrt{78}\right)^{\frac{1}{3}}\right)x}{6\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}} \cos\left(\frac{\sqrt{3}\left(\left(116+6\sqrt{3}\sqrt{26}\right)^{\frac{2}{3}}-22\right)x}{6\left(116+6\sqrt{3}\sqrt{26}\right)^{\frac{1}{3}}}\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 81

```
DSolve[y'''[x]-5*y''[x]+y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 \exp\left(x\text{Root}\left[\#1^3 - 5\#1^2 + \#1 - 1\&, 2\right]\right) + c_3 \exp\left(x\text{Root}\left[\#1^3 - 5\#1^2 + \#1 - 1\&, 3\right]\right) + c_1 \exp\left(x\text{Root}\left[\#1^3 - 5\#1^2 + \#1 - 1\&, 1\right]\right)$$

## 2.25 problem Problem 3(c)



Internal problem ID [12246]
Internal file name [`OUTPUT/10898_Thursday_September_28_2023_01_08_24_AM_64232980/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 3(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' - 3y' - 2y = 0}$$

### 2.25.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -3, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 3\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -3, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{-3^2 - (4)\,(2)\,(-2)}\\ &= \frac{3}{4} \pm \frac{5}{4}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{4} + \frac{5}{4}$$
$$\lambda_2 = \frac{3}{4} - \frac{5}{4}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

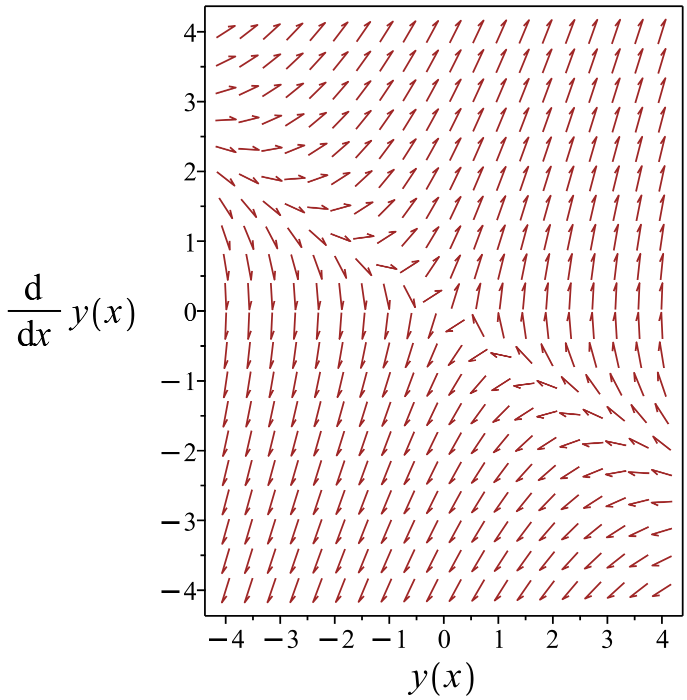



Figure 34: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-\frac{x}{2}}$$

Verified OK.

### 2.25.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 3y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -3 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{25}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 35: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{16}$ is not a function of $x$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{4}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{2}\,dx} \\
&= z_1 e^{\frac{3x}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{3x}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}\left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{2c_2 \mathrm{e}^{2x}}{5} \tag{1}
$$

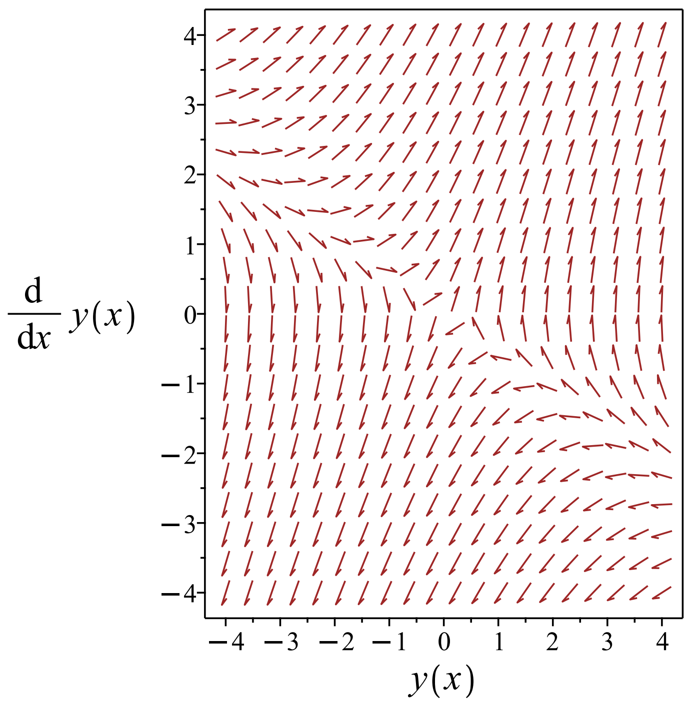



Figure 35: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{2c_2 \mathrm{e}^{2x}}{5}
$$

Verified OK.

### 2.25.3 Maple step by step solution

Let's solve

$2y'' - 3y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2} + y$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2} - y = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{3}{2}r - 1 = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)(r-2)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(2, -\frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{2x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(2*diff(y(x),x$2)-3*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-\frac{x}{2}} + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[2*y''[x]-3*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/2} + c_2 e^{2x}$$

## 2.26 problem Problem 3(d)



Internal problem ID [12247]
Internal file name [`OUTPUT/10899_Thursday_September_28_2023_01_08_25_AM_88166950/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 3(d).
**ODE order**: 4.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{3y'''' - 2y'' + y' = 0}$$

The characteristic equation is

$$3\lambda^4 - 2\lambda^2 + \lambda = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -1\\ \lambda_3 &= \frac{i\sqrt{3}}{6} + \frac{1}{2}\\ \lambda_4 &= -\frac{i\sqrt{3}}{6} + \frac{1}{2}\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{\left(-\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x} c_3 + \mathrm{e}^{\left(\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= \mathrm{e}^{\left(-\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x} \\
y_4 &= \mathrm{e}^{\left(\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} + c_2 + \mathrm{e}^{\left(-\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x}c_3 + \mathrm{e}^{\left(\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x}c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-x} + c_2 + \mathrm{e}^{\left(-\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x}c_3 + \mathrm{e}^{\left(\frac{i\sqrt{3}}{6}+\frac{1}{2}\right)x}c_4
$$

Verified OK.

### 2.26.1 Maple step by step solution

Let's solve

$3y'''' - 2y'' + y' = 0$

- Highest derivative means the order of the ODE is 4

  $y''''$

- Isolate 4th derivative

  $y'''' = \frac{2y''}{3} - \frac{y'}{3}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{2y''}{3} + \frac{y'}{3} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$
- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \frac{2y_3(x)}{3} - \frac{y_2(x)}{3}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{2y_3(x)}{3} - \frac{y_2(x)}{3}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$
- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\frac{I\sqrt{3}}{6}+\frac{1}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^3} \\ \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^2} \\ \frac{1}{-\frac{I\sqrt{3}}{6}+\frac{1}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{I\sqrt{3}}{6}+\frac{1}{2}, \begin{bmatrix} \frac{1}{\left(\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^3} \\ \frac{1}{\left(\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^2} \\ \frac{1}{\frac{I\sqrt{3}}{6}+\frac{1}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{I\sqrt{3}}{6}+\frac{1}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{2}} \\ \frac{1}{-\frac{I\sqrt{3}}{6}+\frac{1}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{2}} \\ \frac{1}{-\frac{I\sqrt{3}}{6}+\frac{1}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{6}\right) - I\sin\left(\frac{\sqrt{3}\,x}{6}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{2}} \\ \frac{1}{-\frac{I\sqrt{3}}{6}+\frac{1}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{6}\right)-I\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{6}\right)-I\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{\left(-\frac{I\sqrt{3}}{6}+\frac{1}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{6}\right)-I\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{-\frac{I\sqrt{3}}{6}+\frac{1}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{6}\right)-I\sin\left(\frac{\sqrt{3}\,x}{6}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} 3\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right) \\ \frac{3\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} + \frac{3\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ \frac{3\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{6}\right) \end{bmatrix}, \vec{y}_4(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} 3\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right) \\ \frac{3\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} - \frac{3\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} - \frac{3\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{6}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs
  $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$
- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \begin{bmatrix} 3\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right) \\ \frac{3\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} + \frac{3\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ \frac{3\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{6}\right) \end{bmatrix} + c_4 e^{\frac{x}{2}} \cdot \begin{bmatrix} 3\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right) \\ \frac{3\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} - \frac{3\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right)}{2} - \frac{3\sin\left(\frac{\sqrt{3}\,x}{6}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{6}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE
  $y = \left(3c_3 e^{\frac{3x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{6}\right) + 3c_4 e^{\frac{3x}{2}}\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{6}\right) + c_2 e^{x} - c_1\right) e^{-x}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(3*diff(y(x),x$4)-2*diff(y(x),x$2)+diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{6}\right) + c_4 e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{6}\right) + c_1 e^{x} + c_2\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 1.175 (sec). Leaf size: 87

```
DSolve[3*y''''[x]-2*y''[x]+y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3\left(-e^{-x}\right) - \frac{1}{2}\left(\sqrt{3}c_1 - 3c_2\right)e^{x/2}\cos\left(\frac{x}{2\sqrt{3}}\right) + \frac{1}{2}\left(3c_1 + \sqrt{3}c_2\right)e^{x/2}\sin\left(\frac{x}{2\sqrt{3}}\right) + c_4$$

## 2.27 problem Problem 5(a)



Internal problem ID [12248]
Internal file name [`OUTPUT/10900_Thursday_September_28_2023_01_08_25_AM_66455539/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 5(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{(x-3)\,y''+y\ln(x)=x^2}$$

With initial conditions

$$[y(1)=1,y'(1)=2]$$

### 2.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=0\\ q(x)&=\frac{\ln(x)}{x-3}\\ F&=\frac{x^2}{x-3}\end{aligned}$$

Hence the ode is

$$y'' + \frac{\ln(x)\, y}{x-3} = \frac{x^2}{x-3}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{\ln(x)}{x-3}$ is

$$\{0 < x < 3, 3 < x \leq \infty\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = \frac{x^2}{x-3}$ is

$$\{x < 3 \vee 3 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
         trying Riccati sub-methods:
            trying Riccati_symmetries
            -> trying a symmetry pattern of the form [F(x)*G(y), 0]
            -> trying a symmetry pattern of the form [0, F(x)*G(y)]
            -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`
```

✗ Solution by Maple

```
dsolve([(x-3)*diff(y(x),x$2)+ln(x)*y(x)=x^2,y(1) = 1, D(y)(1) = 2],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(x-3)*y''[x]+log[x]*y[x]==x^2,{y[1]==1,y'[1]==2}},y[x],x,IncludeSingularSolutions ->
```

Not solved

## 2.28 problem Problem 5(b)



Internal problem ID [12249]
Internal file name [`OUTPUT/10901_Thursday_September_28_2023_01_08_25_AM_74340670/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 5(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + y' \tan (x) + y \cot (x) = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = 1, y'\left(\frac{\pi}{4}\right) = 0\right]$$

### 2.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \tan (x) \\ q(x) &= \cot (x) \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' \tan(x) + y \cot(x) = 0$$

The domain of $p(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \vee \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = \cot(x)$ is

$$\{x < \pi\_Z118 \vee \pi\_Z118 < x\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
```

✓ Solution by Maple

Time used: 6.454 (sec). Leaf size: 46435

```
dsolve([diff(y(x),x$2)+tan(x)*diff(y(x),x)+cot(x)*y(x)=0,y(1/4*Pi) = 1, D(y)(1/4*Pi) = 0],y(
```

Expression too large to display

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+Tan[x]*y'[x]+Cot[x]*y[x]==0,{y[Pi/4]==1,y'[Pi/4]==0}},y[x],x,IncludeSingularS
```

Not solved

## 2.29 problem Problem 5(c)



Internal problem ID [12250]
Internal file name [`OUTPUT/10902_Thursday_September_28_2023_01_08_26_AM_88437475/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 5(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\left(x^{2}+1\right) y''+y'(x-1)+y=0}$$

With initial conditions

$$[y(0)=0, y'(0)=1]$$

### 2.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{x-1}{x^{2}+1} \\ q(x) &= \frac{1}{x^{2}+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{(x-1)\,y'}{x^2+1} + \frac{y}{x^2+1} = 0$$

The domain of $p(x) = \frac{x-1}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 157

```
dsolve([(x^2+1)*diff(y(x),x$2)+(x-1)*diff(y(x),x)+y(x)=0,y(0) = 0, D(y)(0) = 1],y(x), singso
```

$$y(x) = \frac{-20\,\mathrm{hypergeom}\left([i,-i],\left[\frac{1}{2}-\frac{i}{2}\right],\frac{1}{2}\right)\mathrm{e}^{\left(\frac{1}{4}-\frac{i}{4}\right)\pi}(x+i)^{\frac{1}{2}+\frac{i}{2}}\,\mathrm{hypergeom}\left(\left[\frac{1}{2}-\frac{i}{2},\frac{1}{2}+\frac{3i}{2}\right],\left[\frac{3}{2}+\right.\right.}{(10-10i)\left(\mathrm{hypergeom}\left([1-i,1+i],\left[\frac{3}{2}-\frac{i}{2}\right],\frac{1}{2}\right)-\mathrm{hypergeom}\left([i,-i],\left[\frac{1}{2}-\frac{i}{2}\right],\frac{1}{2}\right)\right)\mathrm{hypergeom}\left(\left[\frac{1}{2}-\right.\right.}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(x^2+1)*y''[x]+(x-1)*y'[x]+y[x]==0,{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutio
```

Not solved

## 2.30 problem Problem 5(d)



Internal problem ID [12251]
Internal file name [`OUTPUT/10903_Thursday_September_28_2023_01_08_26_AM_73359447/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 5(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{xy'' + 2x^2y' + \sin(x)\, y = \sinh(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

### 2.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= \frac{\sin(x)}{x} \\ F &= \frac{\sinh(x)}{x} \end{aligned}$$

Hence the ode is

$$y'' + 2y'x + \frac{\sin(x)\, y}{x} = \frac{\sinh(x)}{x}$$

The domain of $p(x) = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{\sin(x)}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all. The domain of $F = \frac{\sinh(x)}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         -> trying with_periodic_functions in the coefficients
            --- Trying Lie symmetry methods, 2nd order ---
            `, `-> Computing symmetries using: way = 5
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         -> trying with_periodic_functions in the coefficients
            --- Trying Lie symmetry methods, 2nd order ---
            `, `-> Computing symmetries using: way = 5
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
```

✗ Solution by Maple

```
dsolve([x*diff(y(x),x$2)+2*x^2*diff(y(x),x)+y(x)*sin(x)=sinh(x),y(0) = 1, D(y)(0) = 1],y(x),
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^2*y''[x]+2*x^2*y'[x]+y[x]*Sin[x]==Sinh[x],{y[0]==1,y'[0]==1}},y[x],x,IncludeSingul
```

Not solved

## 2.31 problem Problem 5(e)



Internal problem ID [12252]
Internal file name [`OUTPUT/10904_Thursday_September_28_2023_01_08_27_AM_34661333/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 5(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{\sin(x)\, y'' + y'x + 7y = 1}$$

With initial conditions

$$[y(1) = 1, y'(1) = 0]$$

### 2.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}
p(x) &= \frac{x}{\sin(x)} \\
q(x) &= \frac{7}{\sin(x)} \\
F &= \frac{1}{\sin(x)}
\end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{\sin(x)} + \frac{7y}{\sin(x)} = \frac{1}{\sin(x)}$$

The domain of $p(x) = \frac{x}{\sin(x)}$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{7}{\sin(x)}$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = \frac{1}{\sin(x)}$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
Try integration with the canonical coordinates of the symmetry [0, -1+7*y]
-> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-(diff(y(x), x))*x-7*y(x)+1)/sin
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   trying symmetries linear in x and y(x)
   -> Try solving first the homogeneous part of the ODE
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
```

✗ Solution by Maple

```
dsolve([sin(x)*diff(y(x),x$2)+x*diff(y(x),x)+7*y(x)=1,y(1) = 1, D(y)(1) = 0],y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{Sin[x]*y''[x]+x*y'[x]+7*y[x]==1,{y[1]==1,y'[1]==0}},y[x],x,IncludeSingularSolutions
```

Not solved

## 2.32 problem Problem 5(f)



Internal problem ID [12253]
Internal file name [`OUTPUT/10905_Thursday_September_28_2023_01_08_27_AM_4852287/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 5(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{y'' - y'(x-1) + x^2 y = \tan(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 2.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 - x \\ q(x) &= x^2 \\ F &= \tan(x) \end{aligned}$$

Hence the ode is

$$y'' + (1 - x)\, y' + x^2 y = \tan(x)$$

The domain of $p(x) = 1 - x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \lor \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         <- hyper3 successful: indirect Equivalence to 0F1 under \`\^ @ Moebius\`\` is reso
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 1.125 (sec). Leaf size: 522

```
dsolve([diff(y(x),x$2)-(x-1)*diff(y(x),x)+x^2*y(x)=tan(x),y(0) = 0, D(y)(0) = 0],y(x), sings
```

Expression too large to display

✓ Solution by Mathematica

Time used: 90.104 (sec). Leaf size: 4228

```
DSolve[{y''[x]-(x-1)*y'[x]+x^2*y[x]==Tan[x],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSoluti
```

Too large to display

## 2.33 problem Problem 10



Internal problem ID [12254]
Internal file name [`OUTPUT/10906_Thursday_September_28_2023_01_08_27_AM_7286131/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 10.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x-1)\,y''-y'x+y=0}$$

### 2.33.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x-1)\,y''-y'x+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{x}{x-1} \\ q(x) &= \frac{1}{x-1} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $y$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x-1} + \frac{1}{x-1} = 0 \tag{5}$$

Solving (5) for $n$ gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2-2x+2)}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-2x+2}{x(x-1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^2-2x+2}{x(x-1)}\,dx \\
\ln(u) &= x + \ln(x-1) - 2\ln(x) + c_1 \\
u &= e^{x+\ln(x-1)-2\ln(x)+c_1} \\
&= c_1 e^{x+\ln(x-1)-2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{e^x c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{e^x c_1}{x} + c_2\right)x \\
&= e^x c_1 + c_2 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{e^x c_1}{x} + c_2\right)x \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{e^x c_1}{x} + c_2\right)x
$$

Verified OK.

### 2.33.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for $u$. Now a new ode $v' = u$ is solved for $v$ as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x - 1 \\
B &= -x \\
C &= 1 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x - 1)(0) + (-x)(-1) + (1)(-x) \\
&= 0
\end{aligned}$$

Hence the ode in $v$ given in (1) now simplifies to

$$-x(x-1)\,v''+\left(x^2-2x+2\right)v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-x^2+x\right)u'(x)+\left(x^2-2x+2\right)u(x)=0$$

Which is now solved for $u$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u(x^2-2x+2)}{x\,(x-1)}\end{aligned}$$

Where $f(x)=\frac{x^2-2x+2}{x(x-1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{x^2-2x+2}{x\,(x-1)}\,dx\\ \ln(u) &= x+\ln(x-1)-2\ln(x)+c_1\\ u &= \mathrm{e}^{x+\ln(x-1)-2\ln(x)+c_1}\\ &= c_1\mathrm{e}^{x+\ln(x-1)-2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^x}{x^2}\right)$$

The ode for $v$ now becomes

$$\begin{aligned}v' &= u\\ &= c_1\left(\frac{\mathrm{e}^x}{x}-\frac{\mathrm{e}^x}{x^2}\right)\end{aligned}$$

Which is now solved for $v$. Integrating both sides gives

$$\begin{aligned}v(x) &= \int\frac{(x-1)\,\mathrm{e}^x c_1}{x^2}\,\mathrm{d}x\\ &= \frac{\mathrm{e}^x c_1}{x}+c_2\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-x)\left(\frac{e^{x} c_1}{x} + c_2\right) \\ &= -e^{x} c_1 - c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -e^{x} c_1 - c_2 x \tag{1}$$

Verification of solutions

$$y = -e^{x} c_1 - c_2 x$$

Verified OK.

### 2.33.3 Solving using Kovacic algorithm

Writing the ode as

$$(x-1)\, y'' - y'x + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x - 1 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 38: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{1}{4} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let $b$ be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of $r$ at $\infty$ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i \qquad (8)
\end{aligned}$$

Let $a$ be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in $r$. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of $\frac{1}{x}$ in $r$ will be the coefficient in $R$ of the term in $x$ of degree of $t$ minus one, divided by the leading coefficient in $t$. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of $t$ is 2, then we see that the coefficient of the term $x$ in the remainder $R$ is $-2$. Dividing this by leading coefficient in $t$ which is 4 gives $-\frac{1}{2}$. Now $b$ can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$
r = \frac{x^2 - 4x + 6}{4\left(x - 1\right)^2}
$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^+$ | $\alpha_c^-$ |
|---|---|---|---|---|
| 1 | 2 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$
\omega = \sum_{c \in \Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\,(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2\,(x-1)} + \frac{1}{2} \\
&= \frac{x-2}{2x-2}
\end{aligned}
$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2\,(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2\,(x-1)^2}\right) + \left(-\frac{1}{2\,(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4\,(x-1)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-x\,\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-x\,\mathrm{e}^{-x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x} c_1 - c_2 x \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{x} c_1 - c_2 x$$

Verified OK.

### 2.33.4 Maple step by step solution

Let's solve

$(x-1)\, y'' - y' x + y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

☐ Check to see if $x_0 = 1$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$

- ○ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\bigg|_{x=1} = -1$

- ○ $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

  $\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$

- ○ $x = 1$is a regular singular point

  Check to see if $x_0 = 1$ is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $(x-1)\,y'' - y'x + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2}y(u)\right) + (-u-1)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ○ Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$

- Recursion relation that defines series solution to ODE

$a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

$a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$a_{k+1} = \frac{a_k}{k+3}$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x-1)^{k+2}\right), a_{1+k}=\frac{a_k}{1+k}, b_{1+k}=\frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x)=c_2\mathrm{e}^x+c_1x$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1e^x-c_2x$$

## 2.34 problem Problem 13



Internal problem ID [12255]
Internal file name [OUTPUT/10907_Thursday_September_28_2023_01_08_28_AM_87767529/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 13.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$x^2y'' - 4x^2y' + (x^2 + 1)\, y = 0$$

### 2.34.1 Maple step by step solution

Let's solve

$y''x^2 - 4x^2y' + (x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = 4y' - \frac{(x^2+1)y}{x^2}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 4y' + \frac{(x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -4, P_3(x) = \frac{x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

• Multiply by denominators

$y''x^2 - 4x^2y' + (x^2 + 1)\, y = 0$

• Assume series solution for $y$

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^2 \cdot y'$ to series expansion

  $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r+1}$

- Shift index using $k- >k - 1$

  $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k - 1 + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k - 1 + r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(r^2-r+1)\,x^r+\left((r^2+r+1)\,a_1-4a_0r\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(a_k(k^2+2kr+r^2-k-r+1)-4a_{k-1}\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2-r+1=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2},\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right\}$$

- Each term must be 0

  $$(r^2+r+1)\,a_1-4a_0r=0$$

- Solve for the dependent coefficient(s)

  $$a_1=\frac{4a_0r}{r^2+r+1}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2+(2r-1)\,k+r^2-r+1)\,a_k-4a_{k-1}k-4a_{k-1}r+a_{k-2}+4a_{k-1}=0$$

- Shift index using $k->k+2$

  $$\left((k+2)^2+(2r-1)\,(k+2)+r^2-r+1\right)a_{k+2}-4a_{k+1}(k+2)-4a_{k+1}r+a_k+4a_{k+1}=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=\frac{4ka_{k+1}+4a_{k+1}r-a_k+4a_{k+1}}{k^2+2kr+r^2+3k+3r+3}$$

- Recursion relation for $r=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

  $$a_{k+2}=\frac{4ka_{k+1}+4a_{k+1}\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)-a_k+4a_{k+1}}{k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+3k+\frac{9}{2}-\frac{3\,\mathrm{I}\sqrt{3}}{2}}$$

- Solution for $r=\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}},a_{k+2}=\frac{4ka_{k+1}+4a_{k+1}\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)-a_k+4a_{k+1}}{k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+3k+\frac{9}{2}-\frac{3\,\mathrm{I}\sqrt{3}}{2}},a_1=\frac{4a_0\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+\frac{3}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\right]$$

- Recursion relation for $r=\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}$

  $$a_{k+2}=\frac{4ka_{k+1}+4a_{k+1}\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)-a_k+4a_{k+1}}{k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+3k+\frac{9}{2}+\frac{3\,\mathrm{I}\sqrt{3}}{2}}$$

- Solution for $r=\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}$

  $$\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}},a_{k+2}=\frac{4ka_{k+1}+4a_{k+1}\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)-a_k+4a_{k+1}}{k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+3k+\frac{9}{2}+\frac{3\,\mathrm{I}\sqrt{3}}{2}},a_1=\frac{4a_0\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)}{\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2+\frac{3}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}-\frac{I\sqrt{3}}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}+\frac{I\sqrt{3}}{2}}\right), a_{k+2} = \frac{4k a_{1+k} + 4a_{1+k}\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right) - a_k + 4a_{1+k}}{k^2 + 2k\left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right) + \left(\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2 + 3k + \frac{9}{2} - \frac{3I\sqrt{3}}{2}}, a_1 = \frac{\ }{\left(\frac{1}{2}\right.}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(x^2*diff(y(x),x$2)-4*x^2*diff(y(x),x)+(x^2+1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\,e^{2x}\left(c_1 \operatorname{BesselI}\left(\frac{i\sqrt{3}}{2}, \sqrt{3}\,x\right) + c_2 \operatorname{BesselK}\left(\frac{i\sqrt{3}}{2}, \sqrt{3}\,x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 67

```
DSolve[x^2*y''[x]-4*x^2*y'[x]+(x^2+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}\sqrt{x}\left(c_1 \operatorname{BesselJ}\left(\frac{i\sqrt{3}}{2}, -i\sqrt{3}x\right) + c_2 \operatorname{BesselY}\left(\frac{i\sqrt{3}}{2}, -i\sqrt{3}x\right)\right)$$

## 2.35 problem Problem 15

Internal problem ID [12256]
Internal file name [`OUTPUT/10908_Thursday_September_28_2023_01_08_28_AM_72433297/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 15.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_Emden, _Fowler], [_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{y'' + \frac{kx}{y^4} = 0}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
-> Calling odsolve with the ODE`, -(_y1^3*k+16*x^2)*y(x)/(k*_y1^3)-(4/3)*x^2*(3*(diff(y(x),
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
differential order: 2; found: 1 linear symmetries. Trying reduction of order
`, `2nd order, trying reduction of order with given symmetries:`[5/3*x, y]
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 97

```
dsolve(diff(y(x),x$2)+k*x/(y(x)^4)=0,y(x), singsol=all)
```

$$y(x) = \text{RootOf}\left(15\sqrt{3}\left(\int^{\_Z} \frac{\sqrt{-\_f^4 c_1 + 50\_fk}\,\_f}{\_f^3 c_1 - 50k} d\_f\right) x - 5c_2 x - 3\right) x$$
$$y(x) = \text{RootOf}\left(15\sqrt{3}\left(\int^{\_Z} \frac{\sqrt{-\_f^4 c_1 + 50\_fk}\,\_f}{\_f^3 c_1 - 50k} d\_f\right) x + 5c_2 x + 3\right) x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+k*x/(y[x]^4)==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.36 problem Problem 18(a)



Internal problem ID [12257]
Internal file name [`OUTPUT/10909_Thursday_September_28_2023_01_08_29_AM_30621015/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + 2y'x + 2y = 0}$$

### 2.36.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + 2y'x + 2y)\, dx = 0$$
$$2yx + y' = c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 2x \\
q(x) &= c_1
\end{aligned}
$$

Hence the ode is

$$
2yx + y' = c_1
$$

The integrating factor $\mu$ is

$$
\begin{aligned}
\mu &= e^{\int 2x dx} \\
&= e^{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(c_1) \\
\frac{d}{dx}\left(e^{x^2} y\right) &= \left(e^{x^2}\right)(c_1) \\
d\left(e^{x^2} y\right) &= \left(c_1 e^{x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x^2} y &= \int c_1 e^{x^2} dx \\
e^{x^2} y &= \frac{c_1 \sqrt{\pi}\, \mathrm{erfi}(x)}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$
y = \frac{e^{-x^2} c_1 \sqrt{\pi}\, \mathrm{erfi}(x)}{2} + c_2 e^{-x^2}
$$

which simplifies to

$$
y = e^{-x^2} \left( \frac{c_1 \sqrt{\pi}\, \mathrm{erfi}(x)}{2} + c_2 \right)
$$

Summary

The solution(s) found are the following

$$
y = e^{-x^2} \left( \frac{c_1 \sqrt{\pi}\, \mathrm{erfi}(x)}{2} + c_2 \right) \tag{1}
$$

Verification of solutions

$$
y = e^{-x^2} \left( \frac{c_1 \sqrt{\pi}\, \mathrm{erfi}(x)}{2} + c_2 \right)
$$

Verified OK.

### 2.36.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2y'x + 2y = 0$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + 2y'x + 2y)\,dx = 0$$
$$2yx + y' = c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = c_1$$

Hence the ode is

$$2yx + y' = c_1$$

The integrating factor $\mu$ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(c_1)$$
$$\frac{d}{dx}\left(e^{x^2}y\right) = \left(e^{x^2}\right)(c_1)$$
$$d\left(e^{x^2}y\right) = \left(c_1 e^{x^2}\right) dx$$

Integrating gives

$$e^{x^2}y = \int c_1 e^{x^2}\, dx$$
$$e^{x^2}y = \frac{c_1\sqrt{\pi}\,\mathrm{erfi}(x)}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = \frac{e^{-x^2} c_1 \sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_2 e^{-x^2}$$

which simplifies to

$$y = e^{-x^2} \left( \frac{c_1 \sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_2 \right)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2} \left( \frac{c_1 \sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_2 \right) \tag{1}$$

Verification of solutions

$$y = e^{-x^2} \left( \frac{c_1 \sqrt{\pi}\, \operatorname{erfi}(x)}{2} + c_2 \right)$$

Verified OK.

### 2.36.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x^2 - 1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 - 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 - 1)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 41: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $-2$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of $r$ at $\infty$ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let $a$ be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$\sqrt{r} \approx x - \frac{1}{2x} - \frac{1}{8x^3} - \frac{1}{16x^5} - \frac{5}{128x^7} - \frac{7}{256x^9} - \frac{21}{1024x^{11}} - \frac{33}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^0 = 1$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in $r$. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of 1 in $r$ will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2-1}{1} \\
&= Q + \frac{R}{1} \\
&= (x^2-1) + (0) \\
&= x^2 - 1
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-1$. Now $b$ can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 1\right) &= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 1\right) &= 0
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$
r = x^2 - 1
$$

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^{+}$ | $\alpha_\infty^{-}$ |
|---|---|---|---|
| $-2$ | $x$ | $-1$ | $0$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^{-} = 0$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^{-} \\
&= 0
\end{aligned}
$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c \in \Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)(x) \\
&= -x \\
&= -x
\end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2(-x)(0) + \left((-1) + (-x)^2 - \left(x^2 - 1\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -x dx} \\
&= e^{-\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in $y$ is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left( e^{-\frac{x^2}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x^2}\right) + c_2 \left(\mathrm{e}^{-x^2} \left(\frac{\sqrt{\pi}\, \operatorname{erfi}(x)}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^2} + \frac{c_2 \sqrt{\pi}\, \mathrm{e}^{-x^2} \operatorname{erfi}(x)}{2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x^2} + \frac{c_2 \sqrt{\pi}\, \mathrm{e}^{-x^2} \operatorname{erfi}(x)}{2}$$

Verified OK.

### 2.36.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2x \\ r(x) &= 2 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 2 \end{aligned}$$

Therefore (1) becomes

$$0 - (2) + (2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for $p, q, r, s$ gives

$$2yx + y' = c_1$$

We now have a first order ode to solve which is

$$2yx + y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = c_1$$

Hence the ode is

$$2yx + y' = c_1$$

The integrating factor $\mu$ is

$$\mu = e^{\int 2x dx}$$
$$= e^{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(c_1)$$
$$\frac{d}{dx}\left(e^{x^2} y\right) = \left(e^{x^2}\right)(c_1)$$
$$d\left(e^{x^2} y\right) = \left(c_1 e^{x^2}\right) dx$$

Integrating gives

$$e^{x^2} y = \int c_1 e^{x^2} dx$$
$$e^{x^2} y = \frac{c_1 \sqrt{\pi}\, \text{erfi}(x)}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = \frac{e^{-x^2} c_1 \sqrt{\pi}\, \text{erfi}(x)}{2} + c_2 e^{-x^2}$$

which simplifies to

$$y = e^{-x^2}\left(\frac{c_1 \sqrt{\pi}\, \text{erfi}(x)}{2} + c_2\right)$$

Summary

The solution(s) found are the following

$$y = e^{-x^2}\left(\frac{c_1 \sqrt{\pi}\, \text{erfi}(x)}{2} + c_2\right) \tag{1}$$

Verification of solutions

$$y = e^{-x^2}\left(\frac{c_1 \sqrt{\pi}\, \text{erfi}(x)}{2} + c_2\right)$$

Verified OK.

### 2.36.5 Maple step by step solution

Let's solve

$y'' + 2y'x + 2y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Assume series solution for $y$

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $y''$ to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- > k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + 2a_k(k+1))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\,(a_{k+2}(k+2) + 2a_k) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x^2}(\operatorname{erfi}(x) c_1 + c_2)$$

✓ Solution by Mathematica
Time used: 0.033 (sec). Leaf size: 31

```
DSolve[y''[x]+2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x^2}\left(\sqrt{\pi}c_1\operatorname{erfi}(x) + 2c_2\right)$$

## 2.37 problem Problem 18(b)



Internal problem ID [12258]
Internal file name [`OUTPUT/10910_Thursday_September_28_2023_01_08_30_AM_54738996/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{xy'' + y' \sin(x) + y \cos(x) = 0}$$

### 2.37.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (xy'' + y' \sin(x) + y \cos(x)) \, dx = 0$$
$$(-1 + \sin(x)) \, y + y'x = c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-\sin(x)+1}{x} \\
q(x) &= \frac{c_1}{x}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{(-\sin(x)+1)\,y}{x} = \frac{c_1}{x}
$$

The integrating factor $\mu$ is

$$
\mu = \mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx} y\right) &= \left(\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx}\right)\left(\frac{c_1}{x}\right) \\
\mathrm{d}\left(\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx} y\right) &= \left(\frac{c_1 \mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx} y &= \int \frac{c_1 \mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}\,\mathrm{d}x \\
\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx} y &= \int \frac{c_1 \mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx}$ results in

$$
y = \mathrm{e}^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(\int \frac{c_1 \mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}dx\right) + c_2 \mathrm{e}^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}
$$

which simplifies to

$$
y = \mathrm{e}^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}dx\right) + c_2\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \mathrm{e}^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(c_1\left(\int \frac{\mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}dx\right) + c_2\right) \tag{1}
$$

Verification of solutions

$$y = \mathrm{e}^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)} \left(c_1 \left(\int \frac{\mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}dx\right) + c_2\right)$$

Verified OK.

### 2.37.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' + y' \sin(x) + y \cos(x) = 0$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (xy'' + y' \sin(x) + y \cos(x))\, dx = 0$$
$$(-1 + \sin(x))\, y + y'x = c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-\sin(x) + 1}{x}$$
$$q(x) = \frac{c_1}{x}$$

Hence the ode is

$$y' - \frac{(-\sin(x) + 1)\, y}{x} = \frac{c_1}{x}$$

The integrating factor $\mu$ is

$$\mu = \mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx} y\right) = \left(\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx}\right)\left(\frac{c_1}{x}\right)$$
$$\mathrm{d}\left(\mathrm{e}^{\int -\frac{-\sin(x)+1}{x}dx} y\right) = \left(\frac{c_1 \mathrm{e}^{\int \frac{-1+\sin(x)}{x}dx}}{x}\right) \mathrm{d}x$$

Integrating gives

$$
\begin{aligned}
e^{\int -\frac{-\sin(x)+1}{x}dx} y &= \int \frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x}\, \mathrm{d}x \\
e^{\int -\frac{-\sin(x)+1}{x}dx} y &= \int \frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-\sin(x)+1}{x}dx}$ results in

$$
y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)} \left(\int \frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2 e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}
$$

which simplifies to

$$
y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)} \left(c_1 \left(\int \frac{e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2\right)
$$

Summary

The solution(s) found are the following

$$
y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)} \left(c_1 \left(\int \frac{e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2\right) \tag{1}
$$

Verification of solutions

$$
y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)} \left(c_1 \left(\int \frac{e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2\right)
$$

Verified OK.

### 2.37.3 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x \\
q(x) &= \sin(x) \\
r(x) &= \cos(x) \\
s(x) &= 0
\end{aligned}
$$

Hence

$$p''(x) = 0$$
$$q'(x) = \cos(x)$$

Therefore (1) becomes

$$0 - (\cos(x)) + (\cos(x)) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for $p, q, r, s$ gives

$$(-1 + \sin(x))\, y + y'x = c_1$$

We now have a first order ode to solve which is

$$(-1 + \sin(x))\, y + y'x = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-\sin(x) + 1}{x}$$
$$q(x) = \frac{c_1}{x}$$

Hence the ode is

$$y' - \frac{(-\sin(x) + 1)\, y}{x} = \frac{c_1}{x}$$

The integrating factor $\mu$ is

$$\mu = e^{\int -\frac{-\sin(x)+1}{x} dx}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x}\right)$$
$$\frac{d}{dx}\left(e^{\int -\frac{-\sin(x)+1}{x}dx} y\right) = \left(e^{\int -\frac{-\sin(x)+1}{x}dx}\right)\left(\frac{c_1}{x}\right)$$
$$d\left(e^{\int -\frac{-\sin(x)+1}{x}dx} y\right) = \left(\frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x}\right) dx$$

Integrating gives

$$e^{\int -\frac{-\sin(x)+1}{x}dx} y = \int \frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x}\, dx$$
$$e^{\int -\frac{-\sin(x)+1}{x}dx} y = \int \frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-\sin(x)+1}{x}dx}$ results in

$$y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(\int \frac{c_1 e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2 e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}$$

which simplifies to

$$y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(c_1\left(\int \frac{e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(c_1\left(\int \frac{e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2\right) \tag{1}$$

Verification of solutions

$$y = e^{-\left(\int \frac{-1+\sin(x)}{x}dx\right)}\left(c_1\left(\int \frac{e^{\int \frac{-1+\sin(x)}{x}dx}}{x} dx\right) + c_2\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of integral
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
No hypergeometric solution was found.
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(x*diff(y(x),x$2)+sin(x)*diff(y(x),x)+cos(x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \left(\int \frac{e^{\mathrm{Si}(x)}}{x^2} dx\right) + c_2\right) e^{-\mathrm{Si}(x)} x$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+Sin[x]*y'[x]+Cos[x]*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.38 problem Problem 18(c)



Internal problem ID [12259]
Internal file name [`OUTPUT/10911_Thursday_September_28_2023_01_08_34_AM_71650361/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2x^2y' + 4yx = 2x}$$

### 2.38.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + 2x^2y' + 4yx)\, dx = \int 2x dx$$
$$2x^2y + y' = x^2 + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x^2$$
$$q(x) = x^2 + c_1$$

Hence the ode is

$$2x^2 y + y' = x^2 + c_1$$

The integrating factor $\mu$ is

$$\mu = e^{\int 2x^2 dx} = e^{\frac{2x^3}{3}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x^2 + c_1)$$
$$\frac{d}{dx}\left(e^{\frac{2x^3}{3}} y\right) = \left(e^{\frac{2x^3}{3}}\right)(x^2 + c_1)$$
$$d\left(e^{\frac{2x^3}{3}} y\right) = \left((x^2 + c_1) e^{\frac{2x^3}{3}}\right) dx$$

Integrating gives

$$e^{\frac{2x^3}{3}} y = \int (x^2 + c_1) e^{\frac{2x^3}{3}} dx$$
$$e^{\frac{2x^3}{3}} y = -\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{2x^3}{3}}$ results in

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}} \quad (1)$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{2x^3}{3}} \left( -\frac{1}{2} + \frac{\mathrm{e}^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left( \frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}} \right)}{18} \right) + c_2 \mathrm{e}^{-\frac{2x^3}{3}}$$

Verified OK.

### 2.38.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 2x^2y' + 4yx = 2x$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + 2x^2y' + 4yx)\, dx = \int 2x dx$$
$$2x^2y + y' = x^2 + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x^2$$
$$q(x) = x^2 + c_1$$

Hence the ode is

$$2x^2y + y' = x^2 + c_1$$

The integrating factor $\mu$ is

$$\mu = \mathrm{e}^{\int 2x^2 dx}$$
$$= \mathrm{e}^{\frac{2x^3}{3}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2 + c_1\right)$$

$$\frac{d}{dx}\left(e^{\frac{2x^3}{3}} y\right) = \left(e^{\frac{2x^3}{3}}\right)\left(x^2 + c_1\right)$$

$$d\left(e^{\frac{2x^3}{3}} y\right) = \left(\left(x^2 + c_1\right) e^{\frac{2x^3}{3}}\right) dx$$

Integrating gives

$$e^{\frac{2x^3}{3}} y = \int \left(x^2 + c_1\right) e^{\frac{2x^3}{3}} \, dx$$

$$e^{\frac{2x^3}{3}} y = -\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{2x^3}{3}}$ results in

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}}$$

Verified OK.

### 2.38.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2x^2y' + 4yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x^2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 2)}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 - 2) \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x(x^3 - 2)\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 43: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 4 \\ &= -4 \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $-4$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of $r$ at $\infty$ is $O_r(\infty) = -4$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$.

Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{2} a_i x^i \qquad (8)
\end{aligned}
$$

Let $a$ be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$
\sqrt{r} \approx x^2 - \frac{1}{x} - \frac{1}{2x^4} - \frac{1}{2x^7} - \frac{5}{8x^{10}} - \frac{7}{8x^{13}} - \frac{21}{16x^{16}} - \frac{33}{16x^{19}} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 1
$$

From Eq. (9) the sum up to $v = 2$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\
&= x^2 \qquad (10)
\end{aligned}
$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^1 = x$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = x^4
$$

This shows that the coefficient of $x$ in the above is 0. Now we need to find the coefficient of $x$ in $r$. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of $x$ in $r$ will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x(x^3 - 2)}{1} \\
&= Q + \frac{R}{1} \\
&= \left(x^4 - 2x\right) + (0) \\
&= x^4 - 2x
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-2$. Now $b$ can be found.

$$\begin{aligned} b &= (-2) - (0) \\ &= -2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x^2 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-2}{1} - 2\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-2}{1} - 2\right) &&= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = x(x^3 - 2)$$

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| $-4$ | $x^2$ | $-2$ | $0$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(x^2\right) \\ &= -x^2 \\ &= -x^2 \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(-x^2)\,(0) + \left((-2x) + (-x^2)^2 - (x(x^3 - 2))\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -x^2 dx} \\ &= \mathrm{e}^{-\frac{x^3}{3}} \end{aligned}$$

The first solution to the original ode in $y$ is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2x^2}{1}\, dx} \\ &= z_1 e^{-\frac{x^3}{3}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^3}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{2x^3}{3}}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2x^3}{3}}}{\left(y_1\right)^2}\,dx \\
&= y_1 \left( \frac{18^{\frac{2}{3}} x \left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{2x^3}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{2x^3}{3}} \left( \frac{18^{\frac{2}{3}} x \left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$
y'' + 2x^2 y' + 4yx = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{2x^3}{3}} + \frac{c_2 \mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)}
$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-\frac{2x^3}{3}} \\
y_2 &= \frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)}
\end{aligned}
$$

In the Variation of parameters $u_1, u_2$ are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \mathrm{e}^{-\frac{2x^3}{3}} & \frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54(-x^3)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{2x^3}{3}}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54(-x^3)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \mathrm{e}^{-\frac{2x^3}{3}} & \frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54(-x^3)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} \\ -2x^2\mathrm{e}^{-\frac{2x^3}{3}} & -\frac{x^3\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} \left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{27(-x^3)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} + \frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} \left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54(-x^3)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x^3 \mathrm{e}^{\frac{2x^3}{3}}}{9\left(-\frac{2x^3}{3}\right)^{\frac{2}{3}}(-x^3)^{\frac{1}{3}}} +
\end{vmatrix}
$$

Therefore

$$W = \left(\mathrm{e}^{-\frac{2x^3}{3}}\right)\left(-\frac{x^3\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{27\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} + \frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}x^3\mathrm{e}^{\frac{2x^3}{3}}}{9\left(-\frac{2x^3}{3}\right)^{\frac{2}{3}}\left(-x^3\right)^{\frac{1}{3}}} + \frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}x^3\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54\left(-x^3\right)^{\frac{4}{3}}\Gamma\left(\frac{2}{3}\right)}\right) - \left(\frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}x\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right)\left(-2x^2\mathrm{e}^{-\frac{2x^3}{3}}\right)$$

Which simplifies to

$$W = \frac{\mathrm{e}^{-\frac{4x^3}{3}}18^{\frac{2}{3}}x^6\mathrm{e}^{\frac{2x^3}{3}}}{9\left(-\frac{2x^3}{3}\right)^{\frac{2}{3}}\left(-x^3\right)^{\frac{4}{3}}}$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{2x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}x^2\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{27\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}}{\mathrm{e}^{-\frac{2x^3}{3}}}dx$$

Which simplifies to

$$u_1 = -\int \frac{18^{\frac{2}{3}}x^2\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{27\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{18^{\frac{2}{3}}\alpha^2\left(2\pi\sqrt{3}-3\Gamma\left(\frac{1}{3},-\frac{2\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{27\left(-\alpha^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x\,\mathrm{e}^{-\frac{2x^3}{3}}}{\mathrm{e}^{-\frac{2x^3}{3}}}\,dx$$

Which simplifies to

$$u_2 = \int 2x dx$$

Hence

$$u_2 = x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \frac{18^{\frac{2}{3}}\alpha^2\left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{27\left(-\alpha^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}d\alpha\right)\mathrm{e}^{-\frac{2x^3}{3}} + \frac{x^3\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}\left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}\left(2x^3\sqrt{3}\,\pi - 3x^3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right) - 2\left(\int_0^x \frac{\alpha^2\left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}}d\alpha\right)\left(-x^3\right)^{\frac{1}{3}}\right)}{54\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}$$

Therefore the general solution is

$$y = y_h + y_p = \left(c_1\mathrm{e}^{-\frac{2x^3}{3}} + \frac{c_2\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}x\left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{54\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right) + \left(\frac{\mathrm{e}^{-\frac{2x^3}{3}}18^{\frac{2}{3}}\left(2x^3\sqrt{3}\,\pi - 3x^3\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)\Gamma\left(\frac{2}{3}\right) - 2\left(\int_0^x \frac{\alpha^2\left(2\pi\sqrt{3} - 3\Gamma\left(\frac{1}{3}, -\frac{2\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}}d\alpha\right)\left(-x^3\right)^{\frac{1}{3}}\right)}{54\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right)$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= c_1 \mathrm{e}^{-\frac{2x^3}{3}}+\frac{c_2 \mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi \sqrt{3}-3\Gamma \left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma \left(\frac{2}{3}\right)\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma \left(\frac{2}{3}\right)} \tag{1} \\
&+\frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} \left(2x^3 \sqrt{3}\, \pi -3x^3 \Gamma \left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma \left(\frac{2}{3}\right)-2\left(\int_0^x \frac{\alpha^2 \left(2\pi \sqrt{3}-3\Gamma \left(\frac{1}{3}, -\frac{2\alpha^3}{3}\right) \Gamma \left(\frac{2}{3}\right)\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}}d\alpha \right) \left(-x^3\right)^{\frac{1}{3}}\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma \left(\frac{2}{3}\right)}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= c_1 \mathrm{e}^{-\frac{2x^3}{3}}+\frac{c_2 \mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} x \left(2\pi \sqrt{3}-3\Gamma \left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma \left(\frac{2}{3}\right)\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma \left(\frac{2}{3}\right)} \\
&+\frac{\mathrm{e}^{-\frac{2x^3}{3}} 18^{\frac{2}{3}} \left(2x^3 \sqrt{3}\, \pi -3x^3 \Gamma \left(\frac{1}{3}, -\frac{2x^3}{3}\right) \Gamma \left(\frac{2}{3}\right)-2\left(\int_0^x \frac{\alpha^2 \left(2\pi \sqrt{3}-3\Gamma \left(\frac{1}{3}, -\frac{2\alpha^3}{3}\right) \Gamma \left(\frac{2}{3}\right)\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}}d\alpha \right) \left(-x^3\right)^{\frac{1}{3}}\right)}{54 \left(-x^3\right)^{\frac{1}{3}} \Gamma \left(\frac{2}{3}\right)}
\end{align*}
$$

Verified OK.

### 2.38.4 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y''+q(x)\, y'+r(x)\, y=s(x)
$$

is exact if

$$
p''(x)-q'(x)+r(x)=0 \tag{1}
$$

For the given ode we have

$$
\begin{align*}
p(x) &= 1 \\
q(x) &= 2x^2 \\
r(x) &= 4x \\
s(x) &= 2x
\end{align*}
$$

Hence

$$
\begin{align*}
p''(x) &= 0 \\
q'(x) &= 4x
\end{align*}
$$

Therefore (1) becomes

$$0-(4x)+(4x)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for $p,q,r,s$ gives

$$2x^2y+y'=\int 2x\,dx$$

We now have a first order ode to solve which is

$$2x^2y+y'=x^2+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=2x^2\\ q(x)&=x^2+c_1\end{aligned}$$

Hence the ode is

$$2x^2y+y'=x^2+c_1$$

The integrating factor $\mu$ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int 2x^2dx}\\ &=\mathrm{e}^{\frac{2x^3}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(x^2+c_1\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{2x^3}{3}}y\right)&=\left(\mathrm{e}^{\frac{2x^3}{3}}\right)\left(x^2+c_1\right)\\ \mathrm{d}\left(\mathrm{e}^{\frac{2x^3}{3}}y\right)&=\left(\left(x^2+c_1\right)\mathrm{e}^{\frac{2x^3}{3}}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$e^{\frac{2x^3}{3}} y = \int \left(x^2 + c_1\right) e^{\frac{2x^3}{3}} \, dx$$

$$e^{\frac{2x^3}{3}} y = -\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{2x^3}{3}}$ results in

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{2x^3}{3}} \left(-\frac{1}{2} + \frac{e^{\frac{2x^3}{3}}}{2} - \frac{18^{\frac{2}{3}} (-1)^{\frac{2}{3}} c_1 \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}} \pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{18}\right) + c_2 e^{-\frac{2x^3}{3}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 69

```
dsolve(diff(y(x),x$2)+2*x^2*diff(y(x),x)+4*x*y(x)=2*x,y(x), singsol=all)
```

$$y(x) = \frac{6x\Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right) c_1\Gamma\left(\frac{2}{3}\right) e^{-\frac{2x^3}{3}} + \left((2c_2 - 1)\left(-x^3\right)^{\frac{1}{3}} - 4x\sqrt{3}\,\pi c_1\right) e^{-\frac{2x^3}{3}} + \left(-x^3\right)^{\frac{1}{3}}}{2\left(-x^3\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.063 (sec). Leaf size: 66

```
DSolve[y''[x]+2*x^2*y'[x]+4*x*y[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^{-\frac{2x^3}{3}} + \frac{c_1 e^{-\frac{2x^3}{3}} \left(-x^3\right)^{2/3} \Gamma\left(\frac{1}{3}, -\frac{2x^3}{3}\right)}{\sqrt[3]{2}3^{2/3}x^2} + \frac{1}{2}$$

## 2.39 problem Problem 18(d)



Internal problem ID [12260]
Internal file name [`OUTPUT/10912_Thursday_September_28_2023_01_08_36_AM_25886009/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(-x^2+1)\,y''+(1-x)\,y'+y=-2x+1}$$

### 2.39.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left((-x^2+1)\,y''+(1-x)\,y'+y\right)dx=\int (-2x+1)\,dx$$
$$-(-x-1)\,y-\left(x^2-1\right)y'=-x^2+x+c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{1}{x-1} \\
q(x) &= \frac{x^2 - c_1 - x}{x^2 - 1}
\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x-1} = \frac{x^2 - c_1 - x}{x^2 - 1}$$

The integrating factor $\mu$ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x-1}dx} \\
&= \frac{1}{x-1}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2 - c_1 - x}{x^2 - 1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x-1}\right) &= \left(\frac{1}{x-1}\right)\left(\frac{x^2 - c_1 - x}{x^2 - 1}\right) \\
\mathrm{d}\left(\frac{y}{x-1}\right) &= \left(\frac{x^2 - c_1 - x}{(x+1)(x-1)^2}\right) \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x-1} &= \int \frac{x^2 - c_1 - x}{(x+1)(x-1)^2} \mathrm{d}x \\
\frac{y}{x-1} &= \frac{c_1}{2x-2} + \left(\frac{1}{2} + \frac{c_1}{4}\right)\ln(x-1) + \left(\frac{1}{2} - \frac{c_1}{4}\right)\ln(x+1) + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$y = (x-1)\left(\frac{c_1}{2x-2} + \left(\frac{1}{2} + \frac{c_1}{4}\right)\ln(x-1) + \left(\frac{1}{2} - \frac{c_1}{4}\right)\ln(x+1)\right) + c_2(x-1)$$

which simplifies to

$$y = -\frac{(-2 + c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1 + 2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

Summary
The solution(s) found are the following

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

Verified OK.

### 2.39.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for $u$. Now a new ode $v' = u$ is solved for $v$ as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= -x^2 + 1 \\
B &= 1 - x \\
C &= 1 \\
F &= -2x + 1
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(-x^2 + 1\right)(0) + (1 - x)(-1) + (1)(1 - x) \\
&= 0
\end{aligned}
$$

Hence the ode in $v$ given in (1) now simplifies to

$$
(x + 1)(x - 1)^2 v'' + \left(3x^2 - 2x - 1\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(\left(x^2 - 1\right) u'(x) + u(x)(3x + 1)\right)(x - 1) = 0
$$

Which is now solved for $u$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u(3x + 1)}{x^2 - 1}
\end{aligned}
$$

Where $f(x) = -\frac{3x+1}{x^2-1}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{3x + 1}{x^2 - 1}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3x + 1}{x^2 - 1}\, dx \\
\ln(u) &= -2\ln(x - 1) - \ln(x + 1) + c_1 \\
u &= \mathrm{e}^{-2\ln(x-1) - \ln(x+1) + c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x-1) - \ln(x+1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1}{(x - 1)^2 (x + 1)}
$$

The ode for $v$ now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{(x-1)^2 (x+1)}
\end{aligned}
$$

Which is now solved for $v$. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{(x-1)^2 (x+1)}\,\mathrm{d}x \\
&= c_1\left(-\frac{1}{2(x-1)}-\frac{\ln(x-1)}{4}+\frac{\ln(x+1)}{4}\right)+c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (1-x)\left(c_1\left(-\frac{1}{2(x-1)}-\frac{\ln(x-1)}{4}+\frac{\ln(x+1)}{4}\right)+c_2\right) \\
&= -\frac{(x-1)\,c_1\ln(x+1)}{4}+\frac{(x-1)\,c_1\ln(x-1)}{4}-c_2x+\frac{c_1}{2}+c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1-x \\
y_2 &= -\frac{\ln(x+1)\,x}{4}+\frac{\ln(x-1)\,x}{4}+\frac{\ln(x+1)}{4}-\frac{\ln(x-1)}{4}+\frac{1}{2}
\end{aligned}
$$

In the Variation of parameters $u_1, u_2$ are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1-x & -\frac{\ln(x+1)x}{4} + \frac{\ln(x-1)x}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} + \frac{1}{2} \\ \frac{d}{dx}(1-x) & \frac{d}{dx}\left(-\frac{\ln(x+1)x}{4} + \frac{\ln(x-1)x}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} + \frac{1}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1-x & -\frac{\ln(x+1)x}{4} + \frac{\ln(x-1)x}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} + \frac{1}{2} \\ -1 & -\frac{x}{4(x+1)} - \frac{\ln(x+1)}{4} + \frac{x}{4x-4} + \frac{\ln(x-1)}{4} + \frac{1}{4+4x} - \frac{1}{4(x-1)} \end{vmatrix}$$

Therefore

$$W = (1-x)\left(-\frac{x}{4(x+1)} - \frac{\ln(x+1)}{4} + \frac{x}{4x-4} + \frac{\ln(x-1)}{4} + \frac{1}{4+4x} - \frac{1}{4(x-1)}\right) - \left(-\frac{\ln(x+1)x}{4} + \frac{\ln(x-1)x}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} + \frac{1}{2}\right)(-1)$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\ln(x+1)x}{4} + \frac{\ln(x-1)x}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} + \frac{1}{2}\right)(-2x+1)}{\frac{-x^2+1}{x+1}} dx$$

Which simplifies to

$$u_1 = -\int -\frac{(2x-1)(\ln(x+1)x - \ln(x-1)x - \ln(x+1) + \ln(x-1) - 2)}{4x-4} dx$$

Hence

$$
\begin{aligned}
u_1 = &-\frac{(x-1)^2 \ln(x-1)}{4} + \frac{1}{2} - \frac{(x-1)\ln(x-1)}{4} \\
&+ \frac{(x+1)^2 \ln(x+1)}{4} - \frac{x}{2} - \frac{3(x+1)\ln(x+1)}{4} - \frac{\ln(x-1)}{2}
\end{aligned}
$$

And Eq. (3) becomes

$$
u_2 = \int \frac{(1-x)(-2x+1)}{\frac{-x^2+1}{x+1}}\, dx
$$

Which simplifies to

$$
u_2 = \int (-2x+1)\, dx
$$

Hence

$$
u_2 = -x^2 + x
$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{(x^2-x-2)\ln(x+1)}{4} + \frac{(-x^2+x-2)\ln(x-1)}{4} - \frac{x}{2} + \frac{1}{2} \\
u_2 &= -x^2 + x
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = &\left(\frac{(x^2-x-2)\ln(x+1)}{4} + \frac{(-x^2+x-2)\ln(x-1)}{4} - \frac{x}{2} + \frac{1}{2}\right)(1-x) \\
&+ (-x^2+x)\left(-\frac{\ln(x+1)\, x}{4} + \frac{\ln(x-1)\, x}{4} + \frac{\ln(x+1)}{4} - \frac{\ln(x-1)}{4} + \frac{1}{2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = \frac{(\ln(x+1) + \ln(x-1) - 1)(x-1)}{2}
$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(-\frac{(x-1)\, c_1 \ln(x+1)}{4} + \frac{(x-1)\, c_1 \ln(x-1)}{4} - c_2 x + \frac{c_1}{2} + c_2\right) + \left(\frac{(\ln(x+1) + \ln(x-1) - 1)}{2}\right. \\
&= -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + \frac{(-4c_2-2)\, x}{4} + \frac{c_1}{2} + c_2 + \frac{1}{2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= -\frac{(-2+c_1)(x-1)\ln(x+1)}{4}+\frac{(c_1+2)(x-1)\ln(x-1)}{4} \\
&\quad +\frac{(-4c_2-2)x}{4}+\frac{c_1}{2}+c_2+\frac{1}{2} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= -\frac{(-2+c_1)(x-1)\ln(x+1)}{4}+\frac{(c_1+2)(x-1)\ln(x-1)}{4} \\
&\quad +\frac{(-4c_2-2)x}{4}+\frac{c_1}{2}+c_2+\frac{1}{2}
\end{align*}
$$

Verified OK.

### 2.39.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
(-x^2+1)\,y''+(1-x)\,y'+y=-2x+1
$$

Integrating both sides of the ODE w.r.t $x$ gives

$$
\begin{align*}
\int\left(\left(-x^2+1\right)y''+(1-x)\,y'+y\right)dx &= \int(-2x+1)\,dx \\
-(-x-1)\,y-\left(x^2-1\right)y' &= -x^2+x+c_1
\end{align*}
$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{align*}
p(x) &= -\frac{1}{x-1} \\
q(x) &= \frac{x^2-c_1-x}{x^2-1}
\end{align*}
$$

Hence the ode is

$$
y'-\frac{y}{x-1}=\frac{x^2-c_1-x}{x^2-1}
$$

The integrating factor $\mu$ is

$$\mu = e^{\int -\frac{1}{x-1}dx}$$
$$= \frac{1}{x-1}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x^2 - c_1 - x}{x^2 - 1}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x-1}\right) = \left(\frac{1}{x-1}\right)\left(\frac{x^2 - c_1 - x}{x^2 - 1}\right)$$
$$d\left(\frac{y}{x-1}\right) = \left(\frac{x^2 - c_1 - x}{(x+1)(x-1)^2}\right) dx$$

Integrating gives

$$\frac{y}{x-1} = \int \frac{x^2 - c_1 - x}{(x+1)(x-1)^2} dx$$
$$\frac{y}{x-1} = \frac{c_1}{2x-2} + \left(\frac{1}{2} + \frac{c_1}{4}\right)\ln(x-1) + \left(\frac{1}{2} - \frac{c_1}{4}\right)\ln(x+1) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$y = (x-1)\left(\frac{c_1}{2x-2} + \left(\frac{1}{2} + \frac{c_1}{4}\right)\ln(x-1) + \left(\frac{1}{2} - \frac{c_1}{4}\right)\ln(x+1)\right) + c_2(x-1)$$

which simplifies to

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2 \quad (1)$$

Verification of solutions

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

Verified OK.

### 2.39.4 Solving using Kovacic algorithm

Writing the ode as

$$(-x^2+1)\,y''+(1-x)\,y'+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^2+1 \\ B &= 1-x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r=\frac{3x+5}{(4x-4)\,(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x+5 \\ t &= (4x-4)\,(x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x+5}{(4x-4)\,(x+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 44: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 3 - 1 \\
&= 2
\end{aligned}
$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = (4x - 4)(x + 1)^2$. There is a pole at $x = -1$ of order 2. There is a pole at $x = 1$ of order 1. Since there is no odd order pole larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 1. For the pole at $x = 1$ of order 1 then

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= 1 \\
\alpha_c^- &= 1
\end{aligned}
$$

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{1}{2x-2} - \frac{1}{4(x+1)^2} - \frac{1}{2(x+1)}$$

For the pole at $x = -1$ let $b$ be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of $r$ at $\infty$ is 2 then $[\sqrt{r}]_\infty = 0$. Let $b$ be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of $r$ at $\infty$. which can be found by dividing the leading coefficient of $s$ by the leading coefficient of $t$ from

$$r = \frac{s}{t} = \frac{3x+5}{(4x-4)(x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = \frac{3x+5}{(4x-4)(x+1)^2}$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^+$ | $\alpha_c^-$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 1 |
| $-1$ | 2 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| 2 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^{+} = \frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= \frac{3}{2} - \left(\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c \in \Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left( (-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left( (+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{1}{x - 1} + \frac{1}{2x + 2} + (0) \\ &= \frac{1}{x - 1} + \frac{1}{2x + 2} \\ &= \frac{3x + 1}{2x^2 - 2} \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{x - 1} + \frac{1}{2x + 2}\right)(0) + \left(\left(-\frac{1}{(x - 1)^2} - \frac{1}{2(x + 1)^2}\right) + \left(\frac{1}{x - 1} + \frac{1}{2x + 2}\right)^2 - \left(\frac{3x + 5}{(4x - 4)(x + 1}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{x-1} + \frac{1}{2x+2}\right) dx} \\
&= (x-1)\sqrt{x+1}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1-x}{-x^2+1}\, dx} \\
&= z_1 e^{-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x - 1
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1-x}{-x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x+1)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{2x-2} - \frac{\ln(x-1)}{4} + \frac{\ln(x+1)}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x-1) + c_2\left(x - 1\left(-\frac{1}{2x-2} - \frac{\ln(x-1)}{4} + \frac{\ln(x+1)}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$\left(-x^2+1\right) y'' + (1-x)\, y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1(x-1) + c_2\left(\frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4}\right)$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x-1 \\ y_2 &= \frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4} \end{aligned}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x-1 & \frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4} \\ \frac{d}{dx}(x-1) & \frac{d}{dx}\left(\frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x-1 & \frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4} \\ 1 & \frac{\ln(x+1)}{4} + \frac{x-1}{4+4x} - \frac{\ln(x-1)}{4} + \frac{1-x}{4x-4} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & (x-1)\left(\frac{\ln(x+1)}{4} + \frac{x-1}{4+4x} - \frac{\ln(x-1)}{4} + \frac{1-x}{4x-4}\right) \\ & - \left(\frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4}\right)(1) \end{aligned}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4}\right)(-2x+1)}{\frac{-x^2+1}{x+1}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(2x-1)\left(\ln(x+1)\,x - \ln(x-1)\,x - \ln(x+1) + \ln(x-1) - 2\right)}{4x-4}\,dx$$

Hence

$$\begin{aligned} u_1 = {} & \frac{(x-1)^2\ln(x-1)}{4} - \frac{1}{2} + \frac{(x-1)\ln(x-1)}{4} \\ & - \frac{(x+1)^2\ln(x+1)}{4} + \frac{x}{2} + \frac{3(x+1)\ln(x+1)}{4} + \frac{\ln(x-1)}{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x-1)(-2x+1)}{\frac{-x^2+1}{x+1}}\, dx$$

Which simplifies to

$$u_2 = \int (2x-1)\, dx$$

Hence

$$u_2 = x^2 - x$$

Which simplifies to

$$\begin{aligned} u_1 &= \frac{(-x^2+x+2)\ln(x+1)}{4} + \frac{(x^2-x+2)\ln(x-1)}{4} + \frac{x}{2} - \frac{1}{2} \\ u_2 &= x^2 - x \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(\frac{(-x^2+x+2)\ln(x+1)}{4} + \frac{(x^2-x+2)\ln(x-1)}{4} + \frac{x}{2} - \frac{1}{2}\right)(x-1) \\ &+ (x^2-x)\left(\frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{(\ln(x+1) + \ln(x-1) - 1)(x-1)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1(x-1) + c_2\left(\frac{(x-1)\ln(x+1)}{4} - \frac{1}{2} + \frac{(1-x)\ln(x-1)}{4}\right)\right) \\ &\quad + \left(\frac{(\ln(x+1) + \ln(x-1) - 1)(x-1)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{align*}
y &= c_1(x-1)+c_2\left(\frac{(x-1)\ln\left(x+1\right)}{4}-\frac{1}{2}+\frac{(1-x)\ln\left(x-1\right)}{4}\right) \\
&+\frac{\left(\ln\left(x+1\right)+\ln\left(x-1\right)-1\right)(x-1)}{2} \tag{1}
\end{align*}
$$

Verification of solutions

$$
\begin{align*}
y &= c_1(x-1)+c_2\left(\frac{(x-1)\ln\left(x+1\right)}{4}-\frac{1}{2}+\frac{(1-x)\ln\left(x-1\right)}{4}\right) \\
&+\frac{\left(\ln\left(x+1\right)+\ln\left(x-1\right)-1\right)(x-1)}{2}
\end{align*}
$$

Verified OK.

### 2.39.5 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)
$$

is exact if

$$
p''(x)-q'(x)+r(x)=0 \tag{1}
$$

For the given ode we have

$$
\begin{align*}
p(x) &= -x^2+1 \\
q(x) &= 1-x \\
r(x) &= 1 \\
s(x) &= -2x+1
\end{align*}
$$

Hence

$$
\begin{align*}
p''(x) &= -2 \\
q'(x) &= -1
\end{align*}
$$

Therefore (1) becomes

$$
-2-(-1)+(1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)
$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\ dx$$

Substituting the above values for $p, q, r, s$ gives

$$\left(-x^2+1\right) y' + y(x+1) = \int -2x + 1\, dx$$

We now have a first order ode to solve which is

$$\left(-x^2+1\right) y' + y(x+1) = -x^2 + c_1 + x$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x-1} \\ q(x) &= \frac{x^2 - c_1 - x}{x^2 - 1} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x-1} = \frac{x^2 - c_1 - x}{x^2 - 1}$$

The integrating factor $\mu$ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x-1}dx} \\ &= \frac{1}{x-1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2 - c_1 - x}{x^2 - 1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x-1}\right) &= \left(\frac{1}{x-1}\right)\left(\frac{x^2 - c_1 - x}{x^2 - 1}\right) \\ \mathrm{d}\left(\frac{y}{x-1}\right) &= \left(\frac{x^2 - c_1 - x}{(x+1)(x-1)^2}\right)\ \mathrm{d}x \end{aligned}$$

Integrating gives

$$\frac{y}{x-1} = \int \frac{x^2 - c_1 - x}{(x+1)(x-1)^2}\, dx$$

$$\frac{y}{x-1} = \frac{c_1}{2x-2} + \left(\frac{1}{2} + \frac{c_1}{4}\right)\ln(x-1) + \left(\frac{1}{2} - \frac{c_1}{4}\right)\ln(x+1) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$y = (x-1)\left(\frac{c_1}{2x-2} + \left(\frac{1}{2} + \frac{c_1}{4}\right)\ln(x-1) + \left(\frac{1}{2} - \frac{c_1}{4}\right)\ln(x+1)\right) + c_2(x-1)$$

which simplifies to

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2 \quad (1)$$

Verification of solutions

$$y = -\frac{(-2+c_1)(x-1)\ln(x+1)}{4} + \frac{(c_1+2)(x-1)\ln(x-1)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve((1-x^2)*diff(y(x),x$2)+(1-x)*diff(y(x),x)+y(x)=1-2*x,y(x), singsol=all)
```

$$y(x) = \frac{(c_1+2)(-1+x)\ln(-1+x)}{4} - \frac{(-2+c_1)(-1+x)\ln(1+x)}{4} + c_2 x + \frac{c_1}{2} - c_2$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 74

```
DSolve[(1-x^2)*y''[x]+(1-x)*y'[x]+y[x]==1-2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}((x-1)\log(1-x) + 2x\log(x+1) - 2\log(x+1) - 4c_1x + (1+c_2)(x-1)\log(x-1) - c_2x\log(x+1) + c_2\log(x+1) + 4c_1 + 2c_2)$$

## 2.40 problem Problem 18(e)



Internal problem ID [12261]
Internal file name [`OUTPUT/10913_Thursday_September_28_2023_01_08_38_AM_57152358/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y'x + \left(4x^2 + 2\right) y = 0}$$

### 2.40.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4x$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4x\, dx} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(e^{x^2}y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(e^{x^2}y\right)' = c_1
$$

Integrating again gives

$$
\left(e^{x^2}y\right) = c_1x + c_2
$$

Hence the solution is

$$
y = \frac{c_1x + c_2}{e^{x^2}}
$$

Or

$$
y = c_1x\,e^{-x^2} + c_2e^{-x^2}
$$

Summary

The solution(s) found are the following

$$
y = c_1x\,e^{-x^2} + c_2e^{-x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1x\,e^{-x^2} + c_2e^{-x^2}
$$

Verified OK.

### 2.40.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= 4x \\
q(x) &= 4x^2 + 2
\end{aligned}
$$

Calculating the Liouville ode invariant $Q$ given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4x^2 + 2 - \frac{(4x)'}{2} - \frac{(4x)^2}{4} \\
&= 4x^2 + 2 - \frac{(4)}{2} - \frac{(16x^2)}{4} \\
&= 4x^2 + 2 - (2) - 4x^2 \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable $x$ then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in $v$. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{4x}{2}} \\
&= \mathrm{e}^{-x^2}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, \mathrm{e}^{-x^2} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x)\, \mathrm{e}^{-x^2} = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\, (z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = \mathrm{e}^{-x^2}
$$

Hence (7) becomes

$$y = e^{-x^2}(c_1 x + c_2)$$

Summary
The solution(s) found are the following

$$y = e^{-x^2}(c_1 x + c_2) \tag{1}$$

Verification of solutions

$$y = e^{-x^2}(c_1 x + c_2)$$

Verified OK.

### 2.40.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' + 4y'x + (4x^2 + 2)\, y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= 4x \tag{3} \\ C &= 4x^2 + 2 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 45: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of $x$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{1}\,dx} \\
&= z_1 e^{-x^2} \\
&= z_1 \left(\mathrm{e}^{-x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x^2}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x^2}\right) + c_2\left(\mathrm{e}^{-x^2}(x)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^2} + c_2 x\,\mathrm{e}^{-x^2} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x^2} + c_2 x \,\mathrm{e}^{-x^2}$$

Verified OK.

### 2.40.4 Maple step by step solution

Let's solve

$y'' + 4y'x + (4x^2 + 2)\, y = 0$

- Highest derivative means the order of the ODE is 2

$y''$

- Assume series solution for $y$

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert $y''$ to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 + 2a_0 + (6a_3 + 6a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k(2k+1) + 4a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of $x$ must be 0

$$[2a_2 + 2a_0 = 0, 6a_3 + 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = -a_0, a_3 = -a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 4a_k k + 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 4a_{k+2}(k+2) + 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2(2ka_{k+2} + 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = -a_0, a_3 = -a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)+4*x*diff(y(x),x)+(2+4*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x^2}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 20

```
DSolve[y''[x]+4*x*y'[x]+(2+4*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x^2}(c_2x + c_1)$$

## 2.41 problem Problem 18(f)



Internal problem ID [12262]
Internal file name [OUTPUT/10914_Thursday_September_28_2023_01_08_38_AM_56532464/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$\boxed{x^2y'' + x^2y' + 2(1-x)\,y = 0}$$

### 2.41.1 Maple step by step solution

Let's solve

$y''x^2 + x^2y' + (-2x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -y' + \frac{2(x-1)y}{x^2}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{2(x-1)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = 1, P_3(x) = -\frac{2(x-1)}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + x^2y' + (-2x + 2)\, y = 0$

- Assume series solution for $y$

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r+1}$

  - Shift index using $k- >k - 1$

    $x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1} (k - 1 + r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k - 1 + r)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(r^2-r+2)\,x^r+\left(\sum_{k=1}^{\infty}\left(a_k(k^2+2kr+r^2-k-r+2)+a_{k-1}(k-3+r)\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2-r+2=0$

- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2},\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+(2r-1)\,k+r^2-r+2)\,a_k+a_{k-1}(k-3+r)=0$

- Shift index using $k->k+1$

  $\left((k+1)^2+(2r-1)\,(k+1)+r^2-r+2\right)a_{k+1}+a_k(k+r-2)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r-2)}{k^2+2kr+r^2+k+r+2}$

- Recursion relation for $r=\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}$

  $a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)}{k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)^2+k+\frac{5}{2}-\frac{\mathrm{I}\sqrt{7}}{2}}$

- Solution for $r=\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}},a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)}{k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)^2+k+\frac{5}{2}-\frac{\mathrm{I}\sqrt{7}}{2}}\right]$

- Recursion relation for $r=\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}$

  $a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)}{k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)^2+k+\frac{5}{2}+\frac{\mathrm{I}\sqrt{7}}{2}}$

- Solution for $r=\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}},a_{k+1}=-\frac{a_k\left(k-\frac{3}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)}{k^2+2k\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)+\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)^2+k+\frac{5}{2}+\frac{\mathrm{I}\sqrt{7}}{2}}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_kx^{k+\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}}\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}}\right),a_{1+k}=-\frac{a_k\left(k-\frac{3}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)}{k^2+2k\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)+\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)^2+k+\frac{5}{2}-\frac{\mathrm{I}\sqrt{7}}{2}},b_{1+k}=\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple
Time used: 0.063 (sec). Leaf size: 93

```
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+2*(1-x)*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\,\mathrm{e}^{-\frac{x}{2}}\left(x c_1(x+2)\,\mathrm{BesselI}\left(\frac{i\sqrt{7}}{2}+1, \frac{x}{2}\right) - x c_2(x+2)\,\mathrm{BesselK}\left(\frac{i\sqrt{7}}{2}+1, \frac{x}{2}\right) + \left(\mathrm{BesselI}\left(\frac{i\sqrt{7}}{2}, \frac{x}{2}\right) c_1 + \mathrm{BesselK}\left(\frac{i\sqrt{7}}{2}, \frac{x}{2}\right) c_2\right)\left(-2+i(x+2)\sqrt{7}+x^2 + 3x\right)\right)$$

✓ Solution by Mathematica
Time used: 0.085 (sec). Leaf size: 89

```
DSolve[x^2*y''[x]+x^2*y'[x]+2*(1-x)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} x^{\frac{1}{2}+\frac{i\sqrt{7}}{2}}\left(c_1\,\mathrm{HypergeometricU}\left(\frac{5}{2}+\frac{i\sqrt{7}}{2}, 1+i\sqrt{7}, x\right) + c_2 L_{-\frac{1}{2}i\left(-5i+\sqrt{7}\right)}^{i\sqrt{7}}(x)\right)$$

## 2.42 problem Problem 18(g)



Internal problem ID [12263]
Internal file name [`OUTPUT/10915_Thursday_September_28_2023_01_08_39_AM_63653667/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(g).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + x^2 y' + 2yx = 2x}$$

### 2.42.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + x^2 y' + 2yx)\, dx = \int 2x dx$$
$$x^2 y + y' = x^2 + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x^2$$
$$q(x) = x^2 + c_1$$

Hence the ode is

$$x^2 y + y' = x^2 + c_1$$

The integrating factor $\mu$ is

$$\begin{aligned}\mu &= e^{\int x^2 dx} \\ &= e^{\frac{x^3}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^2 + c_1\right) \\ \frac{d}{dx}\left(e^{\frac{x^3}{3}} y\right) &= \left(e^{\frac{x^3}{3}}\right)\left(x^2 + c_1\right) \\ d\left(e^{\frac{x^3}{3}} y\right) &= \left(\left(x^2 + c_1\right) e^{\frac{x^3}{3}}\right) dx\end{aligned}$$

Integrating gives

$$e^{\frac{x^3}{3}} y = \int \left(x^2 + c_1\right) e^{\frac{x^3}{3}} dx$$

$$e^{\frac{x^3}{3}} y = -1 + e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^3}{3}}$ results in

$$y = e^{-\frac{x^3}{3}} \left(-1 + e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-\frac{x^3}{3}} \left(-1 + e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} \left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{x^3}{3}} \left( -1 + \mathrm{e}^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} \left( \frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}} \right)}{3} \right) + c_2 \mathrm{e}^{-\frac{x^3}{3}}$$

Verified OK.

### 2.42.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + x^2 y' + 2yx = 2x$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + x^2 y' + 2yx)\, dx = \int 2x dx$$
$$x^2 y + y' = x^2 + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x^2$$
$$q(x) = x^2 + c_1$$

Hence the ode is

$$x^2 y + y' = x^2 + c_1$$

The integrating factor $\mu$ is

$$\mu = \mathrm{e}^{\int x^2 dx}$$
$$= \mathrm{e}^{\frac{x^3}{3}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2+c_1\right)$$
$$\frac{d}{dx}\left(e^{\frac{x^3}{3}}y\right) = \left(e^{\frac{x^3}{3}}\right)\left(x^2+c_1\right)$$
$$d\left(e^{\frac{x^3}{3}}y\right) = \left(\left(x^2+c_1\right)e^{\frac{x^3}{3}}\right)\,dx$$

Integrating gives

$$e^{\frac{x^3}{3}}y = \int \left(x^2+c_1\right)e^{\frac{x^3}{3}}\,dx$$
$$e^{\frac{x^3}{3}}y = -1+e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^3}{3}}$ results in

$$y = e^{-\frac{x^3}{3}}\left(-1+e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x^3}{3}}\left(-1+e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}} \tag{1}$$

Verification of solutions

$$y = e^{-\frac{x^3}{3}}\left(-1+e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}}$$

Verified OK.

### 2.42.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + x^2y' + 2yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x^2 \\ C &= 2x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x(x^3 - 4)}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x(x^3 - 4) \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x(x^3 - 4)}{4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 48: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}
$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $-4$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of $r$ at $\infty$ is $O_r(\infty) = -4$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{4}{2} = 2
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$.

Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{2} a_i x^i \end{aligned} \tag{8}$$

Let $a$ be the coefficient of $x^v = x^2$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$\sqrt{r} \approx \frac{x^2}{2} - \frac{1}{x} - \frac{1}{x^4} - \frac{2}{x^7} - \frac{5}{x^{10}} - \frac{14}{x^{13}} - \frac{42}{x^{16}} - \frac{132}{x^{19}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 2$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{2} a_i x^i \\ &= \frac{x^2}{2} \end{aligned} \tag{10}$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^1 = x$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^4}{4}$$

This shows that the coefficient of $x$ in the above is 0. Now we need to find the coefficient of $x$ in $r$. How this is done depends on if $v = 0$ or not. Since $v = 2$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of $x$ in $r$ will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x(x^3 - 4)}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^4 - x\right) + (0) \\ &= \frac{1}{4}x^4 - x \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-1$. Now $b$ can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 2\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 2\right) &= 0 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = \frac{x(x^3 - 4)}{4}$$

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| $-4$ | $\frac{x^2}{2}$ | $-2$ | $0$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^- = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c \in \Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x^2}{2}\right) \\ &= -\frac{x^2}{2} \\ &= -\frac{x^2}{2}\end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{x^2}{2}\right)(0) + \left((-x) + \left(-\frac{x^2}{2}\right)^2 - \left(\frac{x(x^3-4)}{4}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{x^2}{2} dx} \\ &= \mathrm{e}^{-\frac{x^3}{6}}\end{aligned}$$

The first solution to the original ode in $y$ is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x^2}{1} \, dx} \\ &= z_1 e^{-\frac{x^3}{6}} \\ &= z_1\left(\mathrm{e}^{-\frac{x^3}{6}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^3}{3}}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^3}{3}}}{(y_1)^2}\, dx \\
&= y_1 \left( -\frac{x\left(-2\, 3^{\frac{5}{6}}\pi + 3\, 3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^3}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{x^3}{3}}\left(-\frac{x\left(-2\, 3^{\frac{5}{6}}\pi + 3\, 3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$y'' + x^2y' + 2yx = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-\frac{x^3}{3}} - \frac{c_2\mathrm{e}^{-\frac{x^3}{3}}x\left(-2\, 3^{\frac{5}{6}}\pi + 3\, 3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= e^{-\frac{x^3}{3}} \\
y_2 &= -\frac{e^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)}
\end{aligned}
$$

In the Variation of parameters $u_1, u_2$ are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} e^{-\frac{x^3}{3}} & -\frac{e^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \\ \frac{d}{dx}\left(e^{-\frac{x^3}{3}}\right) & \frac{d}{dx}\left(-\frac{e^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} e^{-\frac{x^3}{3}} & -\frac{e^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \\ -x^2 e^{-\frac{x^3}{3}} & \frac{x^3 e^{-\frac{x^3}{3}} \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} - \frac{e^{-\frac{x^3}{3}} \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} - \frac{e^{-\frac{x^3}{3}} x^3 3^{\frac{1}{3}} e^{\frac{x^3}{3}}}{3 \left(-\frac{x^3}{3}\right)^{\frac{2}{3}} \left(-x^3\right)^{\frac{1}{3}}} - \frac{e^{-\frac{x^3}{3}} x^3 \cdots}{\cdots} \end{vmatrix}
$$

Therefore

$$W = \left(\mathrm{e}^{-\frac{x^3}{3}}\right)\left(\frac{x^3 \mathrm{e}^{-\frac{x^3}{3}}\left(-2\,3^{\frac{5}{6}}\pi + 3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{\mathrm{e}^{-\frac{x^3}{3}}\left(-2\,3^{\frac{5}{6}}\pi + 3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} - \frac{\mathrm{e}^{-\frac{x^3}{3}} x^3 3^{\frac{1}{3}} \mathrm{e}^{\frac{x^3}{3}}}{3\left(-\frac{x^3}{3}\right)^{\frac{2}{3}}\left(-x^3\right)^{\frac{1}{3}}} - \frac{\mathrm{e}^{-\frac{x^3}{3}} x^3\left(-2\,3^{\frac{5}{6}}\pi + 3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{4}{3}}\Gamma\left(\frac{2}{3}\right)}\right) - \left(-\frac{\mathrm{e}^{-\frac{x^3}{3}} x\left(-2\,3^{\frac{5}{6}}\pi + 3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}\right)\left(-x^2 \mathrm{e}^{-\frac{x^3}{3}}\right)$$

Which simplifies to

$$W = \frac{\mathrm{e}^{-\frac{2x^3}{3}} x^6 3^{\frac{1}{3}} \mathrm{e}^{\frac{x^3}{3}}}{3\left(-x^3\right)^{\frac{4}{3}}\left(-\frac{x^3}{3}\right)^{\frac{2}{3}}}$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{x^3}{3}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{2\,\mathrm{e}^{-\frac{x^3}{3}} x^2\left(-2\,3^{\frac{5}{6}}\pi + 3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}}{\mathrm{e}^{-\frac{x^3}{3}}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{4x^2\left(-\frac{3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)}{2} + 3^{\frac{5}{6}}\pi\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{4\alpha^2\left(-\frac{3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)}{2} + 3^{\frac{5}{6}}\pi\right)}{9\left(-\alpha^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,e^{-\frac{x^3}{3}} x}{e^{-\frac{x^3}{3}}}\, dx$$

Which simplifies to

$$u_2 = \int 2x dx$$

Hence

$$u_2 = x^2$$

Which simplifies to

$$u_1 = \frac{2\left(\int_0^x -\frac{2\alpha^2\left(-\frac{3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)}{2}+3^{\frac{5}{6}}\pi\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}} d\alpha\right)}{9\Gamma\left(\frac{2}{3}\right)}$$

$$u_2 = x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2\left(\int_0^x -\frac{2\alpha^2\left(-\frac{3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)}{2}+3^{\frac{5}{6}}\pi\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}} d\alpha\right) e^{-\frac{x^3}{3}}}{9\Gamma\left(\frac{2}{3}\right)} - \frac{x^3 e^{-\frac{x^3}{3}}\left(-2\,3^{\frac{5}{6}}\pi + 3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right)\right)}{9\left(-x^3\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}$$

Which simplifies to

$$y_p(x) = \frac{e^{-\frac{x^3}{3}}\left(2x^3 3^{\frac{5}{6}}\pi - 3x^3 3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)\Gamma\left(\frac{2}{3}\right) + 2\left(\int_0^x -\frac{2\alpha^2\left(-\frac{3\,3^{\frac{1}{3}}\Gamma\left(\frac{1}{3},-\frac{\alpha^3}{3}\right)\Gamma\left(\frac{2}{3}\right)}{2}+3^{\frac{5}{6}}\pi\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}} d\alpha\right)\left(-x^3\right)^{\frac{1}{3}}\right)}{9\Gamma\left(\frac{2}{3}\right)\left(-x^3\right)^{\frac{1}{3}}}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{x^3}{3}} - \frac{c_2 \mathrm{e}^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)}\right) \\
&\quad + \left(\frac{\mathrm{e}^{-\frac{x^3}{3}} \left(2x^3 3^{\frac{5}{6}} \pi - 3x^3 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right) + 2\left(\int_0^x -\frac{2\alpha^2 \left(-\frac{3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{\alpha^3}{3}\right) \Gamma\left(\frac{2}{3}\right)}{2} + 3^{\frac{5}{6}} \pi\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}} d\alpha\right) \left(-x^3\right)^{\frac{1}{3}}\right)}{9\Gamma\left(\frac{2}{3}\right) \left(-x^3\right)^{\frac{1}{3}}}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \mathrm{e}^{-\frac{x^3}{3}} - \frac{c_2 \mathrm{e}^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \\
&\quad + \frac{\mathrm{e}^{-\frac{x^3}{3}} \left(2x^3 3^{\frac{5}{6}} \pi - 3x^3 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right) + 2\left(\int_0^x -\frac{2\alpha^2 \left(-\frac{3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{\alpha^3}{3}\right) \Gamma\left(\frac{2}{3}\right)}{2} + 3^{\frac{5}{6}} \pi\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}} d\alpha\right) \left(-x^3\right)^{\frac{1}{3}}\right)}{9\Gamma\left(\frac{2}{3}\right) \left(-x^3\right)^{\frac{1}{3}}}
\end{aligned}
\tag{1}
$$

<u>Verification of solutions</u>

$$
\begin{aligned}
y &= c_1 \mathrm{e}^{-\frac{x^3}{3}} - \frac{c_2 \mathrm{e}^{-\frac{x^3}{3}} x \left(-2\, 3^{\frac{5}{6}} \pi + 3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right)\right)}{9 \left(-x^3\right)^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \\
&\quad + \frac{\mathrm{e}^{-\frac{x^3}{3}} \left(2x^3 3^{\frac{5}{6}} \pi - 3x^3 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) \Gamma\left(\frac{2}{3}\right) + 2\left(\int_0^x -\frac{2\alpha^2 \left(-\frac{3\, 3^{\frac{1}{3}} \Gamma\left(\frac{1}{3}, -\frac{\alpha^3}{3}\right) \Gamma\left(\frac{2}{3}\right)}{2} + 3^{\frac{5}{6}} \pi\right)}{\left(-\alpha^3\right)^{\frac{1}{3}}} d\alpha\right) \left(-x^3\right)^{\frac{1}{3}}\right)}{9\Gamma\left(\frac{2}{3}\right) \left(-x^3\right)^{\frac{1}{3}}}
\end{aligned}
$$

Verified OK.

### 2.42.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 \\ r(x) &= 2x \\ s(x) &= 2x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 2x \end{aligned}$$

Therefore (1) becomes

$$0 - (2x) + (2x) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for $p, q, r, s$ gives

$$x^2 y + y' = \int 2x\,dx$$

We now have a first order ode to solve which is

$$x^2 y + y' = x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= x^2\\ q(x) &= x^2 + c_1\end{aligned}$$

Hence the ode is

$$x^2y + y' = x^2 + c_1$$

The integrating factor $\mu$ is

$$\begin{aligned}\mu &= e^{\int x^2dx}\\ &= e^{\frac{x^3}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^2 + c_1\right)\\ \frac{d}{dx}\left(e^{\frac{x^3}{3}}y\right) &= \left(e^{\frac{x^3}{3}}\right)\left(x^2 + c_1\right)\\ d\left(e^{\frac{x^3}{3}}y\right) &= \left(\left(x^2 + c_1\right)e^{\frac{x^3}{3}}\right)\, dx\end{aligned}$$

Integrating gives

$$e^{\frac{x^3}{3}}y = \int \left(x^2 + c_1\right)e^{\frac{x^3}{3}}\, dx$$

$$e^{\frac{x^3}{3}}y = -1 + e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^3}{3}}$ results in

$$y = e^{-\frac{x^3}{3}}\left(-1 + e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x^3}{3}}\left(-1 + e^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}}(-1)^{\frac{2}{3}}\left(\frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}}\right)}{3}\right) + c_2 e^{-\frac{x^3}{3}} \quad (1)$$

Verification of solutions

$$y = \mathrm{e}^{-\frac{x^3}{3}} \left( -1 + \mathrm{e}^{\frac{x^3}{3}} - \frac{c_1 3^{\frac{1}{3}} (-1)^{\frac{2}{3}} \left( \frac{2x\sqrt{3}\,(-1)^{\frac{1}{3}}\pi}{3\Gamma\left(\frac{2}{3}\right)(-x^3)^{\frac{1}{3}}} - \frac{x(-1)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{(-x^3)^{\frac{1}{3}}} \right)}{3} \right) + c_2 \mathrm{e}^{-\frac{x^3}{3}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 66

```
dsolve(diff(y(x),x$2)+x^2*diff(y(x),x)+2*x*y(x)=2*x,y(x), singsol=all)
```

$$y(x) = \frac{3x\Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right) c_1 \Gamma\left(\frac{2}{3}\right) \mathrm{e}^{-\frac{x^3}{3}} + \left((c_2 - 1)\,(-x^3)^{\frac{1}{3}} - 2x\sqrt{3}\,\pi c_1\right) \mathrm{e}^{-\frac{x^3}{3}} + (-x^3)^{\frac{1}{3}}}{(-x^3)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 59

```
DSolve[y''[x]+x^2*y'[x]+2*x*y[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 e^{-\frac{x^3}{3}} + \frac{c_1 e^{-\frac{x^3}{3}} (-x^3)^{2/3}\, \Gamma\left(\frac{1}{3}, -\frac{x^3}{3}\right)}{3^{2/3} x^2} + 1$$

## 2.43 problem Problem 18(h)

Internal problem ID [12264]
Internal file name [`OUTPUT/10916_Thursday_September_28_2023_01_08_41_AM_95843298/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(h).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\ln\left(x^2+1\right) y''+\frac{4xy'}{x^2+1}+\frac{\left(-x^2+1\right) y}{\left(x^2+1\right)^2}=0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✗ Solution by Maple

```
dsolve(ln(1+x^2)*diff(y(x),x$2)+4*x/(1+x^2)*diff(y(x),x)+(1-x^2)/(1+x^2)^2*y(x)=0,y(x), sing
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[Log[1+x^2]*y''[x]+4*x/(1+x^2)*y'[x]+(1-x^2)/(1+x^2)^2*y[x]==0,y[x],x,IncludeSingularS
```

Not solved

## 2.44 problem Problem 18(i)



Internal problem ID [12265]
Internal file name [OUTPUT/10917_Thursday_September_28_2023_01_08_41_AM_10721492/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(i).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _exact, _linear, _homogeneous]]`

$$\boxed{xy'' + x^2y' + 2yx = 0}$$

### 2.44.1 Solving using Kovacic algorithm

Writing the ode as

$$y'x + 2y + y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x^2 - 6}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 6 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2}{4} - \frac{3}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 49: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $-2$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of $r$ at $\infty$ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let $a$ be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$\sqrt{r} \approx \frac{x}{2} - \frac{3}{2x} - \frac{9}{4x^3} - \frac{27}{4x^5} - \frac{405}{16x^7} - \frac{1701}{16x^9} - \frac{15309}{32x^{11}} - \frac{72171}{32x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} \end{aligned} \tag{10}$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^0 = 1$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{x^2}{4}$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in $r$. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of 1 in $r$ will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 6}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{x^2}{4} - \frac{3}{2}\right) + (0) \\
&= \frac{x^2}{4} - \frac{3}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{2}$. Now $b$ can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = \frac{x^2}{4} - \frac{3}{2}$$

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| $-2$ | $\frac{x}{2}$ | $-2$ | $1$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned}d &= \alpha_\infty^- \\ &= 1\end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}\omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}\right) \\ &= -\frac{x}{2} \\ &= -\frac{x}{2}\end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-\frac{x}{2}\right)^2 - \left(\frac{x^2}{4} - \frac{3}{2}\right)\right) &= 0 \\ a_0 &= 0\end{aligned}$$

Solving for the coefficients $a_i$ in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= (x)\,\mathrm{e}^{\int -\frac{x}{2} dx} \\ &= (x)\,\mathrm{e}^{-\frac{x^2}{4}} \\ &= x\,\mathrm{e}^{-\frac{x^2}{4}} \end{aligned}$$

The first solution to the original ode in $y$ is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\,\mathrm{e}^{-\frac{x^2}{2}}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left( \frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{-\frac{x^2}{2}}\left(\frac{-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x - 2\,\mathrm{e}^{\frac{x^2}{2}}}{2x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right) \tag{1}
$$

Verification of solutions

$$
y = c_1x\,\mathrm{e}^{-\frac{x^2}{2}} + c_2\left(-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)x\,\mathrm{e}^{-\frac{x^2}{2}}}{2} - 1\right)
$$

Verified OK.

### 2.44.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$
y'x + 2y + y'' = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= x \\
q(x) &= 2 \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\,p)' + \xi q &= 0 \\
\xi'' - (x\xi(x))' + (2\xi(x)) &= 0 \\
\xi''(x) + \xi(x) - x\xi'(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the ode

$$\xi''(x) + \xi(x) - x\xi'(x) = 0 \tag{1}$$

Becomes

$$\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -x \\ q(x) &= 1 \end{aligned}$$

Applying change of variables on the depndent variable $\xi(x) = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - n + 1 = 0 \tag{5}$$

Solving (5) for $n$ gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - x\right) v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} - x\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - x\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 2)}{x} \end{aligned}$$

Where $f(x) = \frac{x^2-2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{x^2 - 2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{x^2 - 2}{x}\, dx \\ \ln(u) &= \frac{x^2}{2} - 2\ln(x) + c_1 \\ u &= e^{\frac{x^2}{2} - 2\ln(x) + c_1} \\ &= c_1 e^{\frac{x^2}{2} - 2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 e^{\frac{x^2}{2}}}{x^2}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \left( -\frac{e^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\, x}{2}\right)}{2} \right) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
\xi(x) &= v(x)\, x^n \\
&= \left(c_1 \left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right) + c_2\right) x \\
&= -c_1 \mathrm{e}^{\frac{x^2}{2}} - \frac{x\left(i c_1 \sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) - 2c_2\right)}{2}
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}
$$

$$
y' + y\left(x - \frac{\frac{c_3 \mathrm{e}^{\frac{x^2}{2}}}{x} + c_3\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right) + c_2}{\left(c_3\left(-\frac{\mathrm{e}^{\frac{x^2}{2}}}{x} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)}{2}\right) + c_2\right) x}\right) = 0
$$

Which is now a first order ode. This is now solved for $y$. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{y\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) c_3 x^2 - i c_3 \sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) + 2\,\mathrm{e}^{\frac{x^2}{2}} c_3 x - 2c_2 x^2 + 2c_2\right)}{i\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,x}{2}\right) c_3 x + 2c_3 \mathrm{e}^{\frac{x^2}{2}} - 2c_2 x}
\end{aligned}
$$

Where $f(x) = -\frac{i\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x^2-ic_3\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2\,\mathrm{e}^{\frac{x^2}{2}}c_3x-2c_2x^2+2c_2}{i\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x+2c_3\mathrm{e}^{\frac{x^2}{2}}-2c_2x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{i\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x^2-ic_3\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2\,\mathrm{e}^{\frac{x^2}{2}}c_3x-2c_2x^2+2c_2}{i\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x+2c_3\mathrm{e}^{\frac{x^2}{2}}-2c_2x}\,dx$$

$$\int\frac{1}{y}\,dy = \int -\frac{i\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x^2-ic_3\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)+2\,\mathrm{e}^{\frac{x^2}{2}}c_3x-2c_2x^2+2c_2}{i\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x+2c_3\mathrm{e}^{\frac{x^2}{2}}-2c_2x}\,dx$$

$$\ln(y) = -\frac{x^2}{2}+\ln\left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x-2ic_3\mathrm{e}^{\frac{x^2}{2}}+2ic_2x\right)\sqrt{2}}{2\sqrt{\pi}}\right)+c_3$$

$$y = \mathrm{e}^{-\frac{x^2}{2}+\ln\left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x-2ic_3\mathrm{e}^{\frac{x^2}{2}}+2ic_2x\right)\sqrt{2}}{2\sqrt{\pi}}\right)+c_3}$$

$$= c_3\mathrm{e}^{-\frac{x^2}{2}+\ln\left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x-2ic_3\mathrm{e}^{\frac{x^2}{2}}+2ic_2x\right)\sqrt{2}}{2\sqrt{\pi}}\right)}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3\mathrm{e}^{-\frac{x^2}{2}+\ln\left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x-2ic_3\mathrm{e}^{\frac{x^2}{2}}+2ic_2x\right)\sqrt{2}}{2\sqrt{\pi}}\right)}$$

Summary

The solution(s) found are the following

$$y = c_3\mathrm{e}^{-\frac{x^2}{2}+\ln\left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x-2ic_3\mathrm{e}^{\frac{x^2}{2}}+2ic_2x\right)\sqrt{2}}{2\sqrt{\pi}}\right)} \tag{1}$$

Verification of solutions

$$y = c_3\mathrm{e}^{-\frac{x^2}{2}+\ln\left(\frac{\left(\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)c_3x-2ic_3\mathrm{e}^{\frac{x^2}{2}}+2ic_2x\right)\sqrt{2}}{2\sqrt{\pi}}\right)}$$

Verified OK.

### 2.44.3 Maple step by step solution

Let's solve

$y'x + 2y + y'' = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Assume series solution for $y$

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ○ Convert $y''$ to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ○ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k+2))\, x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+2)\,(k a_{k+2} + a_k + a_{k+2}) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+1}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(x*diff(y(x),x$2)+x^2*diff(y(x),x)+2*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(ic_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,x}{2}\right)\sqrt{\pi}\,\sqrt{2} + c_1\right) x\,\mathrm{e}^{-\frac{x^2}{2}} + 2c_2$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 69

```
DSolve[x*y''[x]+x^2*y'[x]+2*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\frac{\pi}{2}} c_2 e^{-\frac{x^2}{2}} \sqrt{x^2} \operatorname{erfi}\left(\frac{\sqrt{x^2}}{\sqrt{2}}\right) + \sqrt{2} c_1 e^{-\frac{x^2}{2}} x + c_2$$

## 2.45 problem Problem 18(j)



Internal problem ID [12266]
Internal file name [`OUTPUT/10918_Thursday_September_28_2023_01_08_44_AM_64807657/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(j).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' \sin(x) + y \cos(x) = \cos(x)}$$

### 2.45.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + y' \sin(x) + y \cos(x))\, dx = \int \cos(x)\, dx$$
$$\sin(x)\, y + y' = \sin(x) + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \sin(x) \\
q(x) &= \sin(x) + c_1
\end{aligned}
$$

Hence the ode is

$$
\sin(x)\,y + y' = \sin(x) + c_1
$$

The integrating factor $\mu$ is

$$
\begin{aligned}
\mu &= e^{\int \sin(x)dx} \\
&= e^{-\cos(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\,(\sin(x) + c_1) \\
\frac{d}{dx}\left(e^{-\cos(x)} y\right) &= \left(e^{-\cos(x)}\right)(\sin(x) + c_1) \\
d\left(e^{-\cos(x)} y\right) &= \left((\sin(x) + c_1)\, e^{-\cos(x)}\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\cos(x)} y &= \int (\sin(x) + c_1)\, e^{-\cos(x)}\, dx \\
e^{-\cos(x)} y &= \int (\sin(x) + c_1)\, e^{-\cos(x)} dx + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\cos(x)}$ results in

$$
y = e^{\cos(x)} \left( \int (\sin(x) + c_1)\, e^{-\cos(x)} dx \right) + c_2 e^{\cos(x)}
$$

which simplifies to

$$
y = e^{\cos(x)} \left( \int (\sin(x) + c_1)\, e^{-\cos(x)} dx + c_2 \right)
$$

Summary

The solution(s) found are the following

$$
y = e^{\cos(x)} \left( \int (\sin(x) + c_1)\, e^{-\cos(x)} dx + c_2 \right) \tag{1}
$$

Verification of solutions

$$
y = e^{\cos(x)} \left( \int (\sin(x) + c_1)\, e^{-\cos(x)} dx + c_2 \right)
$$

Verified OK.

### 2.45.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' \sin(x) + y \cos(x) = \cos(x)$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (y'' + y' \sin(x) + y \cos(x))\, dx = \int \cos(x)\, dx$$
$$\sin(x)\, y + y' = \sin(x) + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \sin(x)$$
$$q(x) = \sin(x) + c_1$$

Hence the ode is

$$\sin(x)\, y + y' = \sin(x) + c_1$$

The integrating factor $\mu$ is

$$\mu = e^{\int \sin(x) dx}$$
$$= e^{-\cos(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x) + c_1)$$
$$\frac{d}{dx}\left(e^{-\cos(x)} y\right) = \left(e^{-\cos(x)}\right)(\sin(x) + c_1)$$
$$d\left(e^{-\cos(x)} y\right) = \left((\sin(x) + c_1)\, e^{-\cos(x)}\right)\, dx$$

Integrating gives

$$e^{-\cos(x)} y = \int (\sin(x) + c_1)\, e^{-\cos(x)}\, dx$$
$$e^{-\cos(x)} y = \int (\sin(x) + c_1)\, e^{-\cos(x)} dx + c_2$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\cos(x)}$ results in

$$y = \mathrm{e}^{\cos(x)} \left( \int (\sin(x) + c_1)\, \mathrm{e}^{-\cos(x)} dx \right) + c_2 \mathrm{e}^{\cos(x)}$$

which simplifies to

$$y = \mathrm{e}^{\cos(x)} \left( \int (\sin(x) + c_1)\, \mathrm{e}^{-\cos(x)} dx + c_2 \right)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\cos(x)} \left( \int (\sin(x) + c_1)\, \mathrm{e}^{-\cos(x)} dx + c_2 \right) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\cos(x)} \left( \int (\sin(x) + c_1)\, \mathrm{e}^{-\cos(x)} dx + c_2 \right)$$

Verified OK.

### 2.45.3 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= \sin(x) \\ r(x) &= \cos(x) \\ s(x) &= \cos(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= \cos(x) \end{aligned}$$

Therefore (1) becomes

$$0 - (\cos(x)) + (\cos(x)) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for $p, q, r, s$ gives

$$\sin(x)\,y + y' = \int \cos(x)\;dx$$

We now have a first order ode to solve which is

$$\sin(x)\,y + y' = \sin(x) + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \sin(x)\\
q(x) &= \sin(x) + c_1
\end{aligned}$$

Hence the ode is

$$\sin(x)\,y + y' = \sin(x) + c_1$$

The integrating factor $\mu$ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int \sin(x)dx}\\
&= \mathrm{e}^{-\cos(x)}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(\sin(x) + c_1)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\cos(x)}y\right) &= \left(\mathrm{e}^{-\cos(x)}\right)(\sin(x) + c_1)\\
\mathrm{d}\left(\mathrm{e}^{-\cos(x)}y\right) &= \left((\sin(x) + c_1)\,\mathrm{e}^{-\cos(x)}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$e^{-\cos(x)}y = \int (\sin(x) + c_1) e^{-\cos(x)} \, dx$$
$$e^{-\cos(x)}y = \int (\sin(x) + c_1) e^{-\cos(x)} dx + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-\cos(x)}$ results in

$$y = e^{\cos(x)} \left( \int (\sin(x) + c_1) e^{-\cos(x)} dx \right) + c_2 e^{\cos(x)}$$

which simplifies to

$$y = e^{\cos(x)} \left( \int (\sin(x) + c_1) e^{-\cos(x)} dx + c_2 \right)$$

Summary
The solution(s) found are the following

$$y = e^{\cos(x)} \left( \int (\sin(x) + c_1) e^{-\cos(x)} dx + c_2 \right) \tag{1}$$

Verification of solutions

$$y = e^{\cos(x)} \left( \int (\sin(x) + c_1) e^{-\cos(x)} dx + c_2 \right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+sin(x)*diff(y(x),x)+cos(x)*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \left( c_2 + \int (\sin(x) + c_1) e^{-\cos(x)} dx \right) e^{\cos(x)}$$

✓ Solution by Mathematica

Time used: 1.199 (sec). Leaf size: 34

```
DSolve[y''[x]+Sin[x]*y'[x]+Cos[x]*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\cos(x)} \left( \int_1^x e^{-\cos(K[1])}(c_1 + \sin(K[1]))dK[1] + c_2 \right)$$

## 2.46 problem Problem 18(k)



Internal problem ID [12267]
Internal file name [`OUTPUT/10919_Thursday_September_28_2023_01_08_46_AM_24811125/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(k).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' \cot(x) - \csc(x)^2 y = \cos(x)}$$

### 2.46.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. $y_h$ is the solution to

$$y'' + y' \cot(x) - \csc(x)^2 y = 0$$

In normal form the ode

$$y'' + y' \cot(x) - \csc(x)^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= -\csc(x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where $\tau$ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \cot(x)dx\right)}dx \\ &= \int e^{-\ln(\sin(x))}\,dx \\ &= \int \csc(x)\,dx \\ &= -\ln(\csc(x)+\cot(x)) \end{aligned} \tag{6}$$

Using (6) to evaluate $q_1$ from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\csc(x)^2}{\csc(x)^2} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}$$
$$= \pm 1$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to $y$ using (6) which results in

$$y = -((c_1 - c_2)\cos(x) - c_1 - c_2)\csc(x)$$

Therefore the homogeneous solution $y_h$ is

$$y_h = -((c_1 - c_2)\cos(x) - c_1 - c_2)\csc(x)$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -\cos(x)\csc(x) + \csc(x)$$
$$y_2 = \cos(x)\csc(x) + \csc(x)$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -\cos(x)\csc(x) + \csc(x) & \cos(x)\csc(x) + \csc(x) \\ \frac{d}{dx}(-\cos(x)\csc(x) + \csc(x)) & \frac{d}{dx}(\cos(x)\csc(x) + \csc(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -\cos(x)\csc(x) + \csc(x) & \cos(x)\csc(x) + \csc(x) \\ \sin(x)\csc(x) + \cos(x)\csc(x)\cot(x) - \csc(x)\cot(x) & -\sin(x)\csc(x) - \cos(x)\csc(x)\cot(x) - \mathrm{c} \end{vmatrix}$$

Therefore

$$W = (-\cos(x)\csc(x) + \csc(x))(-\sin(x)\csc(x) - \cos(x)\csc(x)\cot(x) - \csc(x)\cot(x)) - (\cos(x)\csc(x) + \csc(x))(\sin(x)\csc(x) + \cos(x)\csc(x)\cot(x) - \csc(x)\cot(x))$$

Which simplifies to

$$W = -2\csc(x)^2 \sin(x)$$

Which simplifies to

$$W = -2\csc(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(\cos(x)\csc(x) + \csc(x))\cos(x)}{-2\csc(x)}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\cos(x)(\cos(x)+1)}{2} dx$$

Hence

$$u_1 = \frac{\cos(x)\sin(x)}{4} + \frac{x}{4} + \frac{\sin(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(-\cos(x)\csc(x) + \csc(x))\cos(x)}{-2\csc(x)}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(\cos(x)-1)\cos(x)}{2} dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{4} + \frac{x}{4} - \frac{\sin(x)}{2}$$

Which simplifies to

$$u_1 = \frac{\sin(x)(2+\cos(x))}{4} + \frac{x}{4}$$
$$u_2 = \frac{\sin(x)(-2+\cos(x))}{4} + \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(x)(2+\cos(x))}{4} + \frac{x}{4}\right)(-\cos(x)\csc(x) + \csc(x)) + \left(\frac{\sin(x)(-2+\cos(x))}{4} + \frac{x}{4}\right)(\cos(x)\csc(x) + \csc(x))$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{2} + \frac{\csc(x)\,x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (-((c_1 - c_2)\cos(x) - c_1 - c_2)\csc(x)) + \left(-\frac{\cos(x)}{2} + \frac{\csc(x)\,x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -((c_1 - c_2)\cos(x) - c_1 - c_2)\csc(x) - \frac{\cos(x)}{2} + \frac{\csc(x)\,x}{2} \tag{1}$$

Verification of solutions

$$y = -((c_1 - c_2)\cos(x) - c_1 - c_2)\csc(x) - \frac{\cos(x)}{2} + \frac{\csc(x)\,x}{2}$$

Verified OK.

### 2.46.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = \cot(x), C = -\csc(x)^2, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for $y_h$ from

$$y'' + y'\cot(x) - \csc(x)^2\,y = 0$$

In normal form the ode

$$y'' + y'\cot(x) - \csc(x)^2\,y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \cot(x) \\
q(x) &= -\csc(x)^2
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where $\tau$ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where $c$ is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{-\csc(x)^2}}{c} \\
\tau'' &= \frac{\csc(x)^2\cot(x)}{c\sqrt{-\csc(x)^2}}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{\csc(x)^2\cot(x)}{c\sqrt{-\csc(x)^2}} + \cot(x)\frac{\sqrt{-\csc(x)^2}}{c}}{\left(\frac{\sqrt{-\csc(x)^2}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0
$$

$$
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) = 0 \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{-\csc(x)^2} dx}{c} \\
&= \frac{\sqrt{-\csc(x)^2}\, \ln(\csc(x) - \cot(x)) \sin(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = -i\cot(x)\, c_2 + c_1 \cosh(\ln(\csc(x) - \cot(x)))$$

Now the particular solution to this ODE is found

$$y'' + y' \cot(x) - \csc(x)^2 y = \cos(x)$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= -\cos(x)\csc(x) + \csc(x) \\
y_2 &= \cos(x)\csc(x) + \csc(x)
\end{aligned}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -\cos(x)\csc(x) + \csc(x) & \cos(x)\csc(x) + \csc(x) \\ \frac{d}{dx}(-\cos(x)\csc(x) + \csc(x)) & \frac{d}{dx}(\cos(x)\csc(x) + \csc(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -\cos(x)\csc(x) + \csc(x) & \cos(x)\csc(x) + \csc(x) \\ \sin(x)\csc(x) + \cos(x)\csc(x)\cot(x) - \csc(x)\cot(x) & -\sin(x)\csc(x) - \cos(x)\csc(x)\cot(x) - c \end{vmatrix}$$

Therefore

$$\begin{aligned} W = &(-\cos(x)\csc(x)+\csc(x))(-\sin(x)\csc(x)-\cos(x)\csc(x)\cot(x)-\csc(x)\cot(x)) \\ &- (\cos(x)\csc(x) + \csc(x))(\sin(x)\csc(x) + \cos(x)\csc(x)\cot(x) - \csc(x)\cot(x)) \end{aligned}$$

Which simplifies to

$$W = -2\csc(x)^2 \sin(x)$$

Which simplifies to

$$W = -2\csc(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(\cos(x)\csc(x) + \csc(x))\cos(x)}{-2\csc(x)}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\cos(x)(\cos(x)+1)}{2}dx$$

Hence

$$u_1 = \frac{\cos(x)\sin(x)}{4} + \frac{x}{4} + \frac{\sin(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(-\cos(x)\csc(x) + \csc(x))\cos(x)}{-2\csc(x)}\,dx$$

Which simplifies to

$$u_2 = \int \frac{(\cos(x) - 1)\cos(x)}{2} dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{4} + \frac{x}{4} - \frac{\sin(x)}{2}$$

Which simplifies to

$$u_1 = \frac{\sin(x)(2 + \cos(x))}{4} + \frac{x}{4}$$
$$u_2 = \frac{\sin(x)(-2 + \cos(x))}{4} + \frac{x}{4}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(\frac{\sin(x)(2 + \cos(x))}{4} + \frac{x}{4}\right)(-\cos(x)\csc(x) + \csc(x)) \\ &+ \left(\frac{\sin(x)(-2 + \cos(x))}{4} + \frac{x}{4}\right)(\cos(x)\csc(x) + \csc(x)) \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{2} + \frac{\csc(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (-i\cot(x)\, c_2 + c_1 \cosh(\ln(\csc(x) - \cot(x)))) + \left(-\frac{\cos(x)}{2} + \frac{\csc(x)\, x}{2}\right) \\ &= -\frac{\cos(x)}{2} + \frac{\csc(x)\, x}{2} - i\cot(x)\, c_2 + c_1 \cosh(\ln(\csc(x) - \cot(x))) \end{aligned}$$

Which simplifies to

$$y = \frac{(2c_1 + x)\csc(x)}{2} - i\cot(x)\, c_2 - \frac{\cos(x)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{(2c_1 + x)\csc(x)}{2} - i\cot(x)\,c_2 - \frac{\cos(x)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(2c_1 + x)\csc(x)}{2} - i\cot(x)\,c_2 - \frac{\cos(x)}{2}$$

Verified OK.

### 2.46.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left(y'' + y'\cot(x) - \csc(x)^2 y\right) dx = \int \cos(x)\, dx$$
$$y\cot(x) + y' = \sin(x) + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = \sin(x) + c_1$$

Hence the ode is

$$y\cot(x) + y' = \sin(x) + c_1$$

The integrating factor $\mu$ is

$$\mu = e^{\int \cot(x)dx}$$
$$= \sin(x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(x) + c_1)$$
$$\frac{d}{dx}(\sin(x)\,y) = (\sin(x))(\sin(x) + c_1)$$
$$d(\sin(x)\,y) = ((\sin(x) + c_1)\sin(x))\, dx$$

Integrating gives

$$
\begin{aligned}
\sin(x)\, y &= \int (\sin(x) + c_1) \sin(x)\, \mathrm{d}x \\
\sin(x)\, y &= -\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} - c_1 \cos(x) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$
y = \csc(x) \left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} - c_1 \cos(x)\right) + \csc(x)\, c_2
$$

which simplifies to

$$
y = \frac{(x + 2c_2)\csc(x)}{2} - \cot(x)\, c_1 - \frac{\cos(x)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x + 2c_2)\csc(x)}{2} - \cot(x)\, c_1 - \frac{\cos(x)}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{(x + 2c_2)\csc(x)}{2} - \cot(x)\, c_1 - \frac{\cos(x)}{2}
$$

Verified OK.

### 2.46.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right)v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right)u = 0
$$

The above ode is first order ode which is solved for $u$. Now a new ode $v' = u$ is solved for $v$ as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 1 \\
B &= \cot(x) \\
C &= -\csc(x)^2 \\
F &= \cos(x)
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (1)\left(-2\cot(x)\left(-1-\cot(x)^2\right)\right) + (\cot(x))\left(-1-\cot(x)^2\right) + \left(-\csc(x)^2\right)(\cot(x)) \\
&= -\cot(x)\left(-1-\cot(x)^2\right) - \csc(x)^2\cot(x) \\
&= 0
\end{aligned}
$$

Hence the ode in $v$ given in (1) now simplifies to

$$
\cot(x)v'' + \left(-\cot(x)^2 - 2\right)v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-\cot(x)^2 u(x) + \cot(x)u'(x) - 2u(x) = 0
$$

Which is now solved for $u$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u\left(\cot(x)^2 + 2\right)}{\cot(x)}
\end{aligned}
$$

Where $f(x) = \frac{\cot(x)^2+2}{\cot(x)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u} du &= \frac{\cot(x)^2 + 2}{\cot(x)} dx \\
\int \frac{1}{u} du &= \int \frac{\cot(x)^2 + 2}{\cot(x)} dx \\
\ln(u) &= -2\ln(\cos(x)) + \ln(\sin(x)) + c_1 \\
u &= \mathrm{e}^{-2\ln(\cos(x))+\ln(\sin(x))+c_1} \\
&= c_1 \mathrm{e}^{-2\ln(\cos(x))+\ln(\sin(x))}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 \sin(x)}{\cos(x)^2}
$$

The ode for $v$ now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1 \sin(x)}{\cos(x)^2}
\end{aligned}
$$

Which is now solved for $v$. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1 \sin(x)}{\cos(x)^2} \mathrm{d}x \\
&= \frac{c_1}{\cos(x)} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (\cot(x)) \left(\frac{c_1}{\cos(x)} + c_2\right) \\
&= \cot(x) c_2 + \csc(x) c_1
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cot(x)$$
$$y_2 = \csc(x)$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cot(x) & \csc(x) \\ \frac{d}{dx}(\cot(x)) & \frac{d}{dx}(\csc(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cot(x) & \csc(x) \\ -1-\cot(x)^2 & -\csc(x)\cot(x) \end{vmatrix}$$

Therefore

$$W = (\cot(x))(-\csc(x)\cot(x)) - (\csc(x))\left(-1-\cot(x)^2\right)$$

Which simplifies to

$$W = \csc(x)$$

Which simplifies to

$$W = \csc(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\cos(x)\csc(x)}{\csc(x)}\,dx$$

Which simplifies to

$$u_1 = -\int \cos(x)\, dx$$

Hence

$$u_1 = -\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\cot(x)}{\csc(x)}\, dx$$

Which simplifies to

$$u_2 = \int \cos(x)^2\, dx$$

Hence

$$u_2 = \frac{\cos(x)\sin(x)}{2} + \frac{x}{2}$$

Which simplifies to

$$\begin{aligned} u_1 &= -\sin(x) \\ u_2 &= \frac{\sin(2x)}{4} + \frac{x}{2} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cot(x)\sin(x) + \left(\frac{\sin(2x)}{4} + \frac{x}{2}\right)\csc(x)$$

Which simplifies to

$$y_p(x) = -\frac{\cos(x)}{2} + \frac{\csc(x)\, x}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (\cot(x)\, c_2 + \csc(x)\, c_1) + \left(-\frac{\cos(x)}{2} + \frac{\csc(x)\, x}{2}\right) \\ &= \frac{(2c_1 + x)\csc(x)}{2} + \cot(x)\, c_2 - \frac{\cos(x)}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(2c_1 + x)\csc(x)}{2} + \cot(x)\, c_2 - \frac{\cos(x)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(2c_1 + x)\csc(x)}{2} + \cot(x)\, c_2 - \frac{\cos(x)}{2}$$

Verified OK.

### 2.46.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y'\cot(x) - \csc(x)^2 y = \cos(x)$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left(y'' + y'\cot(x) - \csc(x)^2 y\right) dx = \int \cos(x)\, dx$$
$$y\cot(x) + y' = \sin(x) + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = \sin(x) + c_1$$

Hence the ode is

$$y\cot(x) + y' = \sin(x) + c_1$$

The integrating factor $\mu$ is

$$\mu = e^{\int \cot(x)dx}$$
$$= \sin(x)$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\sin\left(x\right)+c_1\right) \\
\frac{d}{dx}\left(\sin\left(x\right) y\right) &= \left(\sin\left(x\right)\right)\left(\sin\left(x\right)+c_1\right) \\
d\left(\sin\left(x\right) y\right) &= \left(\left(\sin\left(x\right)+c_1\right)\sin\left(x\right)\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin\left(x\right) y &= \int \left(\sin\left(x\right)+c_1\right)\sin\left(x\right)\, dx \\
\sin\left(x\right) y &= -\frac{\cos\left(x\right)\sin\left(x\right)}{2}+\frac{x}{2}-c_1\cos\left(x\right)+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin\left(x\right)$ results in

$$
y = \csc\left(x\right)\left(-\frac{\cos\left(x\right)\sin\left(x\right)}{2}+\frac{x}{2}-c_1\cos\left(x\right)\right)+\csc\left(x\right) c_2
$$

which simplifies to

$$
y = \frac{\left(x+2c_2\right)\csc\left(x\right)}{2}-\cot\left(x\right) c_1-\frac{\cos\left(x\right)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(x+2c_2\right)\csc\left(x\right)}{2}-\cot\left(x\right) c_1-\frac{\cos\left(x\right)}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{\left(x+2c_2\right)\csc\left(x\right)}{2}-\cot\left(x\right) c_1-\frac{\cos\left(x\right)}{2}
$$

Verified OK.

### 2.46.6 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y''+q(x)\, y'+r(x)\, y = s(x)
$$

is exact if

$$
p''(x)-q'(x)+r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \cot(x) \\
r(x) &= -\csc(x)^2 \\
s(x) &= \cos(x)
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= -1 - \cot(x)^2
\end{aligned}
$$

Therefore (1) becomes

$$
0 - \left(-1 - \cot(x)^2\right) + \left(-\csc(x)^2\right) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\, y' + \left(q(x) - p'(x)\right) y\right)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + \left(q(x) - p'(x)\right) y = \int s(x)\, dx
$$

Substituting the above values for $p, q, r, s$ gives

$$
y \cot(x) + y' = \int \cos(x)\, dx
$$

We now have a first order ode to solve which is

$$
y \cot(x) + y' = \sin(x) + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \cot(x) \\
q(x) &= \sin(x) + c_1
\end{aligned}
$$

Hence the ode is

$$y\cot(x) + y' = \sin(x) + c_1$$

The integrating factor $\mu$ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int \cot(x)dx} \\
&= \sin(x)
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\sin(x) + c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)\, y) &= (\sin(x))(\sin(x) + c_1) \\
\mathrm{d}(\sin(x)\, y) &= ((\sin(x) + c_1)\sin(x))\, \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sin(x)\, y &= \int (\sin(x) + c_1)\sin(x)\, \mathrm{d}x \\
\sin(x)\, y &= -\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} - c_1\cos(x) + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = \csc(x)\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} - c_1\cos(x)\right) + \csc(x)\, c_2$$

which simplifies to

$$y = \frac{(x + 2c_2)\csc(x)}{2} - \cot(x)\, c_1 - \frac{\cos(x)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{(x + 2c_2)\csc(x)}{2} - \cot(x)\, c_1 - \frac{\cos(x)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(x + 2c_2)\csc(x)}{2} - \cot(x)\, c_1 - \frac{\cos(x)}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+cot(x)*diff(y(x),x)-csc(x)^2*y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = -\cos\left(\frac{x}{2}\right)^2 + \frac{1}{2} + \frac{\sec\left(\frac{x}{2}\right)\csc\left(\frac{x}{2}\right)x}{4} + \cot\left(\frac{x}{2}\right)c_2 + \tan\left(\frac{x}{2}\right)c_1$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 45

```
DSolve[y''[x]+Cot[x]*y'[x]-Csc[x]^2*y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x\csc(x) + \frac{2c_1}{\sqrt{\sin^2(x)}} + \cos(x)\left(-1 - \frac{2ic_2}{\sqrt{\sin^2(x)}}\right)\right)$$

## 2.47 problem Problem 18(L)



Internal problem ID [12268]
Internal file name [`OUTPUT/10920_Thursday_September_28_2023_01_08_51_AM_21653611/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 18(L).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x \ln(x) y'' + 2y' - \frac{y}{x} = 1}$$

### 2.47.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = \frac{2}{x \ln(x)}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int \frac{2}{x \ln(x)}\, dx} \\ &= \ln(x) \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{1}{x} \\ (y \ln(x))'' &= \frac{1}{x} \end{aligned}$$

Integrating once gives

$$(y \ln(x))' = \ln(x) + c_1$$

Integrating again gives

$$(y \ln(x)) = x(\ln(x) + c_1 - 1) + c_2$$

Hence the solution is

$$y = \frac{x(\ln(x) + c_1 - 1) + c_2}{\ln(x)}$$

Or

$$y = \frac{c_1 x}{\ln(x)} + x + \frac{c_2}{\ln(x)} - \frac{x}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{\ln(x)} + x + \frac{c_2}{\ln(x)} - \frac{x}{\ln(x)} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x}{\ln(x)} + x + \frac{c_2}{\ln(x)} - \frac{x}{\ln(x)}$$

Verified OK.

### 2.47.2 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. $y_h$ is the solution to

$$x \ln(x) y'' + 2y' - \frac{y}{x} = 0$$

In normal form the given ode is written as

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x \ln(x)} \\ q(x) &= -\frac{1}{x^2 \ln(x)} \end{aligned}$$

Calculating the Liouville ode invariant $Q$ given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= -\frac{1}{x^2 \ln(x)} - \frac{\left(\frac{2}{x\ln(x)}\right)'}{2} - \frac{\left(\frac{2}{x\ln(x)}\right)^2}{4} \\ &= -\frac{1}{x^2 \ln(x)} - \frac{\left(-\frac{2}{x^2\ln(x)} - \frac{2}{x^2\ln(x)^2}\right)}{2} - \frac{\left(\frac{4}{x^2\ln(x)^2}\right)}{4} \\ &= -\frac{1}{x^2 \ln(x)} - \left(-\frac{1}{x^2 \ln(x)} - \frac{1}{x^2 \ln(x)^2}\right) - \frac{1}{x^2 \ln(x)^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable $x$ then the transformation

$$y = v(x) z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in $v$. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{\frac{2}{x \ln(x)}}{2}} \\ &= \frac{1}{\ln(x)} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{\ln(x)} \tag{4}$$

Applying this change of variable to the original ode results in

$$x v''(x) = 1$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) = \frac{1}{x}$$

Integrating once gives

$$v'(x) = \ln(x) + c_1$$

Integrating again gives

$$v(x) = x \ln(x) - x + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + x \ln(x) - x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\ln(x)}$$

Hence (7) becomes

$$y = \frac{c_1 x + x \ln(x) - x + c_2}{\ln(x)}$$

Therefore the homogeneous solution $y_h$ is

$$y_h = \frac{c_1 x + x \ln(x) - x + c_2}{\ln(x)}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{x}{\ln(x)}$$

$$y_2 = \frac{1}{\ln(x)}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{x}{\ln(x)} & \frac{1}{\ln(x)} \\ \frac{d}{dx}\left(\frac{x}{\ln(x)}\right) & \frac{d}{dx}\left(\frac{1}{\ln(x)}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{x}{\ln(x)} & \frac{1}{\ln(x)} \\ \frac{1}{\ln(x)} - \frac{1}{\ln(x)^2} & -\frac{1}{\ln(x)^2 x} \end{vmatrix}$$

Therefore

$$W = \left(\frac{x}{\ln(x)}\right)\left(-\frac{1}{\ln(x)^2 x}\right) - \left(\frac{1}{\ln(x)}\right)\left(\frac{1}{\ln(x)} - \frac{1}{\ln(x)^2}\right)$$

Which simplifies to

$$W = -\frac{1}{\ln(x)^2}$$

Which simplifies to

$$W = -\frac{1}{\ln(x)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{\ln(x)}}{-\frac{x}{\ln(x)}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{x} dx$$

Hence

$$u_1 = \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x}{\ln(x)}}{-\frac{x}{\ln(x)}}\, dx$$

Which simplifies to

$$u_2 = \int (-1)\, dx$$

Hence

$$u_2 = -x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x - \frac{x}{\ln(x)}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 x + x \ln(x) - x + c_2}{\ln(x)}\right) + \left(x - \frac{x}{\ln(x)}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \frac{x \ln(x) + (-1 + c_1) x + c_2}{\ln(x)} + x - \frac{x}{\ln(x)}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{x \ln(x) + (-1 + c_1) x + c_2}{\ln(x)} + x - \frac{x}{\ln(x)} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{x \ln(x) + (-1 + c_1) x + c_2}{\ln(x)} + x - \frac{x}{\ln(x)}
$$

Warning, solution could not be verified

### 2.47.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$
\begin{aligned}
\int \left(x \ln(x) y'' + 2y' - \frac{y}{x}\right) dx &= \int 1 dx \\
-\frac{(x \ln(x) - x) y}{x} + x \ln(x) y' &= x + c_1
\end{aligned}
$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{\ln(x) - 1}{x \ln(x)} \\
q(x) &= \frac{x + c_1}{x \ln(x)}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{(\ln(x) - 1) y}{x \ln(x)} = \frac{x + c_1}{x \ln(x)}$$

The integrating factor $\mu$ is

$$\begin{aligned}\mu &= e^{\int -\frac{\ln(x)-1}{x\ln(x)}dx} \\ &= e^{-\ln(x)+\ln(\ln(x))}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\ln(x)}{x}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x + c_1}{x\ln(x)}\right) \\ \frac{d}{dx}\left(\frac{\ln(x) y}{x}\right) &= \left(\frac{\ln(x)}{x}\right)\left(\frac{x + c_1}{x\ln(x)}\right) \\ d\left(\frac{\ln(x) y}{x}\right) &= \left(\frac{x + c_1}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\ln(x) y}{x} &= \int \frac{x + c_1}{x^2} dx \\ \frac{\ln(x) y}{x} &= -\frac{c_1}{x} + \ln(x) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\ln(x)}{x}$ results in

$$y = \frac{x\left(-\frac{c_1}{x} + \ln(x)\right)}{\ln(x)} + \frac{c_2 x}{\ln(x)}$$

which simplifies to

$$y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)} \tag{1}$$

Verification of solutions

$$y = \frac{x \ln(x) + c_2 x - c_1}{\ln(x)}$$

Verified OK.

### 2.47.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x \ln(x) y'' + 2y' - \frac{y}{x} = 1$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left(x \ln(x) y'' + 2y' - \frac{y}{x}\right) dx = \int 1 dx$$
$$-\frac{(x \ln(x) - x) y}{x} + x \ln(x) y' = x + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{\ln(x) - 1}{x \ln(x)}$$
$$q(x) = \frac{x + c_1}{x \ln(x)}$$

Hence the ode is

$$y' - \frac{(\ln(x) - 1) y}{x \ln(x)} = \frac{x + c_1}{x \ln(x)}$$

The integrating factor $\mu$ is

$$\mu = e^{\int -\frac{\ln(x)-1}{x \ln(x)} dx}$$
$$= e^{-\ln(x) + \ln(\ln(x))}$$

Which simplifies to

$$\mu = \frac{\ln(x)}{x}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x+c_1}{x\ln(x)}\right) \\
\frac{d}{dx}\left(\frac{\ln(x)\, y}{x}\right) &= \left(\frac{\ln(x)}{x}\right)\left(\frac{x+c_1}{x\ln(x)}\right) \\
d\left(\frac{\ln(x)\, y}{x}\right) &= \left(\frac{x+c_1}{x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\ln(x)\, y}{x} &= \int \frac{x+c_1}{x^2}\, dx \\
\frac{\ln(x)\, y}{x} &= -\frac{c_1}{x} + \ln(x) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\ln(x)}{x}$ results in

$$
y = \frac{x\left(-\frac{c_1}{x} + \ln(x)\right)}{\ln(x)} + \frac{c_2 x}{\ln(x)}
$$

which simplifies to

$$
y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)} \tag{1}
$$

Verification of solutions

$$
y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)}
$$

Verified OK.

### 2.47.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
x\ln(x)\, y'' + 2y' - \frac{y}{x} &= 0 && (1) \\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x\ln(x) \\
B &= 2 \\
C &= -\frac{1}{x}
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where $r$ is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 51: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of $x$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{x\ln(x)}\,dx} \\
&= z_1 e^{-\ln(\ln(x))} \\
&= z_1 \left(\frac{1}{\ln(x)}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{1}{\ln(x)}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x\ln(x)}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2\ln(\ln(x))}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{\ln(x)}\right) + c_2\left(\frac{1}{\ln(x)}(x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$x\ln(x)\, y'' + 2y' - \frac{y}{x} = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{\ln(x)} + \frac{c_2 x}{\ln(x)}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{\ln(x)}$$
$$y_2 = \frac{x}{\ln(x)}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{\ln(x)} & \frac{x}{\ln(x)} \\ \frac{d}{dx}\left(\frac{1}{\ln(x)}\right) & \frac{d}{dx}\left(\frac{x}{\ln(x)}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{\ln(x)} & \frac{x}{\ln(x)} \\ -\frac{1}{\ln(x)^2 x} & \frac{1}{\ln(x)} - \frac{1}{\ln(x)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{\ln(x)}\right)\left(\frac{1}{\ln(x)} - \frac{1}{\ln(x)^2}\right) - \left(\frac{x}{\ln(x)}\right)\left(-\frac{1}{\ln(x)^2 x}\right)$$

Which simplifies to

$$W = \frac{1}{\ln(x)^2}$$

Which simplifies to

$$W = \frac{1}{\ln(x)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x}{\ln(x)}}{\frac{x}{\ln(x)}} dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{\ln(x)}}{\frac{x}{\ln(x)}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x - \frac{x}{\ln(x)}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{\ln(x)} + \frac{c_2 x}{\ln(x)}\right) + \left(x - \frac{x}{\ln(x)}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2 x + c_1}{\ln(x)} + x - \frac{x}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 x + c_1}{\ln(x)} + x - \frac{x}{\ln(x)} \tag{1}$$

Verification of solutions

$$y = \frac{c_2 x + c_1}{\ln(x)} + x - \frac{x}{\ln(x)}$$

Verified OK.

### 2.47.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \ln(x) \\ q(x) &= 2 \\ r(x) &= -\frac{1}{x} \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= \frac{1}{x} \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$\frac{1}{x} - (0) + \left(-\frac{1}{x}\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\; dx$$

Substituting the above values for $p, q, r, s$ gives

$$x \ln(x)\, y' + (-\ln(x) + 1)\, y = \int 1\, dx$$

We now have a first order ode to solve which is

$$x \ln(x)\, y' + (-\ln(x) + 1)\, y = x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{\ln(x) - 1}{x \ln(x)} \\ q(x) &= \frac{x + c_1}{x \ln(x)} \end{aligned}$$

Hence the ode is

$$y' - \frac{(\ln(x) - 1)\, y}{x \ln(x)} = \frac{x + c_1}{x \ln(x)}$$

The integrating factor $\mu$ is

$$\begin{aligned} \mu &= e^{\int -\frac{\ln(x)-1}{x \ln(x)} dx} \\ &= e^{-\ln(x)+\ln(\ln(x))} \end{aligned}$$

Which simplifies to

$$\mu = \frac{\ln(x)}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{x+c_1}{x\ln(x)}\right)$$
$$\frac{d}{dx}\left(\frac{\ln(x)\,y}{x}\right) = \left(\frac{\ln(x)}{x}\right)\left(\frac{x+c_1}{x\ln(x)}\right)$$
$$d\left(\frac{\ln(x)\,y}{x}\right) = \left(\frac{x+c_1}{x^2}\right)\,dx$$

Integrating gives

$$\frac{\ln(x)\,y}{x} = \int \frac{x+c_1}{x^2}\,dx$$
$$\frac{\ln(x)\,y}{x} = -\frac{c_1}{x} + \ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{\ln(x)}{x}$ results in

$$y = \frac{x\left(-\frac{c_1}{x} + \ln(x)\right)}{\ln(x)} + \frac{c_2 x}{\ln(x)}$$

which simplifies to

$$y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)} \tag{1}$$

Verification of solutions

$$y = \frac{x\ln(x) + c_2 x - c_1}{\ln(x)}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*ln(x)*diff(y(x),x$2)+2*diff(y(x),x)-y(x)/x=1,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\ln(x)} + x + \frac{c_2 x}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 23

```
DSolve[x*Log[x]*y''[x]+2*y'[x]-y[x]/x==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x\log(x) + (-1 + c_2)x + c_1}{\log(x)}$$

## 2.48 problem Problem 19(a)



Internal problem ID [12269]
Internal file name [`OUTPUT/10921_Thursday_September_28_2023_01_08_53_AM_70360323/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 19(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible,
   _mu_x_y1], [_2nd_order, _reducible, _mu_y_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

Unable to solve or complete the solution.

$$\boxed{xy'' + (6y^2x + 1)\, y' + 2y^3 = -1}$$

### 2.48.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left(xy'' + (6y^2x + 1)\, y' + 2y^3\right) dx = \int (-1)\, dx$$

$$2y^3x + y'x = -x + c_1$$

Which is now solved for $y$. This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = -2y^3 - \frac{x - c_1}{x} \tag{1}$$

Therefore

$$
\begin{aligned}
f_0(x) &= -1 + \frac{c_1}{x} \\
f_1(x) &= 0 \\
f_2(x) &= 0 \\
f_3(x) &= -2
\end{aligned}
$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on $x$ or not. The Abel invariant is given by

$$
-\frac{f_1^3}{f_0^2 f_3}
$$

Which when evaluating gives

$$
\frac{c_1^3}{54x^6 \left(-1 + \frac{c_1}{x}\right)^5}
$$

Since the Abel invariant depends on $x$ then unable to solve this ode at this time.

### 2.48.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
xy'' + \left(6y^2x + 1\right) y' + 2y^3 = -1
$$

Integrating both sides of the ODE w.r.t $x$ gives

$$
\begin{aligned}
\int \left(xy'' + \left(6y^2x + 1\right) y' + 2y^3\right) dx &= \int (-1)\, dx \\
2y^3x + y'x &= -x + c_1
\end{aligned}
$$

Which is now solved for $y$. This is Abel first kind ODE, it has the form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3
$$

Comparing the above to given ODE which is

$$
y' = -2y^3 - \frac{x - c_1}{x} \tag{1}
$$

Therefore

$$
\begin{aligned}
f_0(x) &= -1 + \frac{c_1}{x} \\
f_1(x) &= 0 \\
f_2(x) &= 0 \\
f_3(x) &= -2
\end{aligned}
$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on $x$ or not. The Abel invariant is given by

$$
-\frac{f_1^3}{f_0^2 f_3}
$$

Which when evaluating gives

$$
\frac{c_1^3}{54x^6 \left(-1 + \frac{c_1}{x}\right)^5}
$$

Since the Abel invariant depends on $x$ then unable to solve this ode at this time.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integrating f
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for dynam
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integratin
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cases
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
-> Calling odsolve with the ODE`, 2*_b(_a)^3*_a+(diff(_b(_a), _a))*_a+_a+c__1 = 0, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   trying Abel
   Looking for potential symmetries
   Looking for potential symmetries
   Looking for potential symmetries
   trying inverse_Riccati
   trying an equivalence to an Abel ODE
   differential order: 1; trying a linearization to 2nd order
   --- trying a change of variables {x -> y(x), y(x) -> x}
   differential order: 1; trying a linearization to 2nd order
   trying 1st order ODE linearizable_by_differentiation
   --- Trying Lie symmetry methods, 1st order ---
   `, `-> Computing symmetries using: way = 3
   `, `-> Computing symmetries using: way = 4
   `, `-> Computing symmetries using: way = 2
   trying symmetry patterns for 1st order ODEs
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
```

✗ Solution by Maple

```
dsolve(x*diff(y(x),x$2)+(6*x*y(x)^2+1)*diff(y(x),x)+2*y(x)^3+1=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[x*y''[x]+(6*x*y[x]^2+1)*y'[x]+2*y[x]^3+1==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

## 2.49 problem Problem 19(b)



Internal problem ID [12270]
Internal file name [`OUTPUT/10922_Thursday_September_28_2023_01_08_53_AM_71924558/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 19(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order,
   _with_linear_symmetries], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{\frac{xy''}{y+1}+\frac{yy'-xy'^2+y'}{(y+1)^2}=x\sin(x)}$$

### 2.49.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int\left(\frac{xy''}{y+1}+\left(\frac{y}{(y+1)^2}-\frac{xy'}{(y+1)^2}+\frac{1}{(y+1)^2}\right)y'\right)dx=\int x\sin(x)\,dx$$

$$\left(\frac{xy}{(y+1)^2}+\frac{x}{(y+1)^2}\right)y'=-\cos(x)\,x+\sin(x)+c_1$$

Which is now solved for $y$. In canonical form the ODE is

$$\begin{aligned}y'&=F(x,y)\\&=f(x)g(y)\\&=\frac{(-\cos(x)\,x+\sin(x)+c_1)\,(y+1)}{x}\end{aligned}$$

Where $f(x) = \frac{-\cos(x)x+\sin(x)+c_1}{x}$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\,dy = \frac{-\cos(x)\,x+\sin(x)+c_1}{x}\,dx$$
$$\int \frac{1}{y+1}\,dy = \int \frac{-\cos(x)\,x+\sin(x)+c_1}{x}\,dx$$
$$\ln(y+1) = -\sin(x) + \mathrm{Si}(x) + c_1\ln(x) + c_2$$

Raising both side to exponential gives

$$y+1 = \mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2}$$

Which simplifies to

$$y+1 = c_3\mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)}$$

Summary

The solution(s) found are the following

$$y = c_3\mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2} - 1 \tag{1}$$

Verification of solutions

$$y = c_3\mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2} - 1$$

Verified OK.

### 2.49.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\frac{xy''}{y+1} + \left(\frac{y}{(y+1)^2} - \frac{xy'}{(y+1)^2} + \frac{1}{(y+1)^2}\right)y' = x\sin(x)$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left(\frac{xy''}{y+1} + \left(\frac{y}{(y+1)^2} - \frac{xy'}{(y+1)^2} + \frac{1}{(y+1)^2}\right)y'\right)dx = \int x\sin(x)\,dx$$
$$\left(\frac{xy}{(y+1)^2} + \frac{x}{(y+1)^2}\right)y' = -\cos(x)\,x + \sin(x) + c_1$$

Which is now solved for $y$. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(-\cos(x)x+\sin(x)+c_1)(y+1)}{x} \end{aligned}$$

Where $f(x) = \frac{-\cos(x)x+\sin(x)+c_1}{x}$ and $g(y) = y+1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y+1}\,dy &= \frac{-\cos(x)x+\sin(x)+c_1}{x}\,dx \\ \int \frac{1}{y+1}\,dy &= \int \frac{-\cos(x)x+\sin(x)+c_1}{x}\,dx \\ \ln(y+1) &= -\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2 \end{aligned}$$

Raising both side to exponential gives

$$y+1 = \mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2}$$

Which simplifies to

$$y+1 = c_3\mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)}$$

Summary

The solution(s) found are the following

$$y = c_3\mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2} - 1 \tag{1}$$

Verification of solutions

$$y = c_3\mathrm{e}^{-\sin(x)+\mathrm{Si}(x)+c_1\ln(x)+c_2} - 1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.219 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x$2)/(1+y(x))+( y(x)*diff(y(x),x)-x* diff(y(x),x)^2+diff(y(x),x))/( 1+y(x
```

$$y(x) = c_1 x^{-c_2} \mathrm{e}^{\mathrm{Si}(x)-\sin(x)-\frac{\pi\ \mathrm{csgn}(x)}{2}} - 1$$

✓ Solution by Mathematica

Time used: 1.681 (sec). Leaf size: 28

```
DSolve[x*y''[x]/(1+y[x])+( y[x]*y'[x]-x* y'[x]^2+y'[x])/( 1+y[x])^2==x*Sin[x],y[x],x,Include
```

$$y(x) \to -1 + x^{c_2} e^{\mathrm{Si}(x)-\sin(x)+c_1}$$
$$y(x) \to -1$$

## 2.50 problem Problem 19(c)



Internal problem ID [12271]
Internal file name [`OUTPUT/10923_Thursday_September_28_2023_01_09_00_AM_30179084/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 19(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{(\cos(y)x+\sin(x))y''-xy'^2\sin(y)+2(\cos(y)+\cos(x))y'-\sin(x)y=0}$$

### 2.50.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int((\cos(y)x+\sin(x))y''+(-x\sin(y)y'+2\cos(x)+2\cos(y))y'-\sin(x)y)\,dx=0$$

$$\sin(y)+y\cos(x)+(\cos(y)x+\sin(x))y'=c_1$$

Which is now solved for $y$.

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y)\,x + \sin(x))\,\mathrm{d}y &= (-\sin(y) - \cos(x)\,y + c_1)\,\mathrm{d}x\\ (\cos(x)\,y + \sin(y) - c_1)\,\mathrm{d}x + (\cos(y)\,x + \sin(x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \cos(x)\,y + \sin(y) - c_1\\ N(x,y) &= \cos(y)\,x + \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\,y + \sin(y) - c_1) \\ &= \cos(y) + \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)\,x + \sin(x)) \\ &= \cos(y) + \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \cos(x)\,y + \sin(y) - c_1\,\mathrm{d}x$$

$$\phi = \sin(x)\,y - x(c_1 - \sin(y)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = \cos(y)\,x + \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y)\,x + \sin(x)$. Therefore equation (4) becomes

$$\cos(y)\,x + \sin(x) = \cos(y)\,x + \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_2$$

Where $c_2$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = \sin(x) y - x(c_1 - \sin(y)) + c_2$$

But since $\phi$ itself is a constant function, then let $\phi = c_3$ where $c_3$ is new constant and combining $c_2$ and $c_3$ constants into new constant $c_2$ gives the solution as

$$c_2 = \sin(x) y - x(c_1 - \sin(y))$$

Summary
The solution(s) found are the following

$$\sin(x) y - x(c_1 - \sin(y)) = c_2 \tag{1}$$

Verification of solutions

$$\sin(x) y - x(c_1 - \sin(y)) = c_2$$

Verified OK.

### 2.50.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(\cos(y) x + \sin(x)) y'' + (-x \sin(y) y' + 2\cos(x) + 2\cos(y)) y' - \sin(x) y = 0$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int ((\cos(y) x + \sin(x)) y'' + (-x \sin(y) y' + 2\cos(x) + 2\cos(y)) y' - \sin(x) y) dx = 0$$
$$\sin(y) + y\cos(x) + (\cos(y) x + \sin(x)) y' = c_1$$

Which is now solved for $y$.

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where $c$ is constant, that satisfies the ode. Taking derivative of $\phi$ w.r.t. $x$ gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(y)\,x + \sin(x))\,\mathrm{d}y &= (-\sin(y) - \cos(x)\,y + c_1)\,\mathrm{d}x \\ (\cos(x)\,y + \sin(y) - c_1)\,\mathrm{d}x + (\cos(y)\,x + \sin(x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \cos(x)\,y + \sin(y) - c_1 \\ N(x, y) &= \cos(y)\,x + \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x)\,y + \sin(y) - c_1)\\ &= \cos(y) + \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)\,x + \sin(x))\\ &= \cos(y) + \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. $x$ gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \cos(x)\,y + \sin(y) - c_1\,\mathrm{d}x$$

$$\phi = \sin(x)\,y - x(c_1 - \sin(y)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since $\phi$ is a function of both $x$ and $y$. Taking derivative of equation (3) w.r.t $y$ gives

$$\frac{\partial \phi}{\partial y} = \cos(y)\,x + \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y)\,x + \sin(x)$. Therefore equation (4) becomes

$$\cos(y)\,x + \sin(x) = \cos(y)\,x + \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_2$$

Where $c_2$ is constant of integration. Substituting this result for $f(y)$ into equation (3) gives $\phi$

$$\phi = \sin(x)\, y - x(c_1 - \sin(y)) + c_2$$

But since $\phi$ itself is a constant function, then let $\phi = c_3$ where $c_3$ is new constant and combining $c_2$ and $c_3$ constants into new constant $c_2$ gives the solution as

$$c_2 = \sin(x)\, y - x(c_1 - \sin(y))$$

Summary

The solution(s) found are the following

$$\sin(x)\, y - x(c_1 - \sin(y)) = c_2 \tag{1}$$

Verification of solutions

$$\sin(x)\, y - x(c_1 - \sin(y)) = c_2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve((x*cos(y(x))+sin(x))*diff(y(x),x$2)- x*diff(y(x),x)^2*sin(y(x)) + 2*(cos(y(x))+cos(x)
```

$$-x\sin(y(x)) - y(x)\sin(x) - c_1 x + c_2 = 0$$

✓ Solution by Mathematica

Time used: 0.292 (sec). Leaf size: 25

```
DSolve[(x*Cos[y[x]]+Sin[x])*y''[x]- x*y'[x]^2*Sin[y[x]] + 2*(Cos[y[x]]+Cos[x])*y'[x]==y[x]*S
```

$$\text{Solve}\left[\sin(y(x)) + \frac{y(x)\sin(x)}{x} - \frac{c_1}{x} = c_2, y(x)\right]$$

## 2.51 problem Problem 19(d)



Internal problem ID [12272]
Internal file name [`OUTPUT/10924_Thursday_September_28_2023_01_09_05_AM_43949395/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 19(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible,
  _mu_y_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' \sin(x) + (y' \sin(x) + y\cos(x))\, y' = \cos(x)}$$

### 2.51.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (yy'' \sin(x) + (y' \sin(x) + y\cos(x))\, y')\, dx = \int \cos(x)\, dx$$
$$yy' \sin(x) = \sin(x) + c_1$$

Which is now solved for $y$. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{\sin(x) + c_1}{y\sin(x)} \end{aligned}$$

Where $f(x) = \frac{\sin(x)+c_1}{\sin(x)}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{\sin(x)+c_1}{\sin(x)}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{\sin(x)+c_1}{\sin(x)}\,dx$$

$$\frac{y^2}{2} = x + c_1 \ln(\csc(x) - \cot(x)) + c_2$$

The solution is

$$\frac{y^2}{2} - x - c_1 \ln(\csc(x) - \cot(x)) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - x - c_1 \ln(\csc(x) - \cot(x)) - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - x - c_1 \ln(\csc(x) - \cot(x)) - c_2 = 0$$

Verified OK.

### 2.51.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' \sin(x) + (y' \sin(x) + y\cos(x))\, y' = \cos(x)$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int (yy'' \sin(x) + (y' \sin(x) + y\cos(x))\, y')\, dx = \int \cos(x)\, dx$$

$$yy' \sin(x) = \sin(x) + c_1$$

Which is now solved for $y$. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sin(x)+c_1}{y\sin(x)} \end{aligned}$$

Where $f(x) = \frac{\sin(x)+c_1}{\sin(x)}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{\sin(x)+c_1}{\sin(x)}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{\sin(x)+c_1}{\sin(x)}\,dx$$

$$\frac{y^2}{2} = x + c_1 \ln(\csc(x) - \cot(x)) + c_2$$

The solution is

$$\frac{y^2}{2} - x - c_1 \ln(\csc(x) - \cot(x)) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - x - c_1 \ln(\csc(x) - \cot(x)) - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - x - c_1 \ln(\csc(x) - \cot(x)) - c_2 = 0$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
   Change of variables used:
      [x = 1/2*arccos(t)]
   Linear ODE actually solved:
      u(t)+(t^2-2*t+1)*diff(u(t),t)+(2*t^3-2*t^2-2*t+2)*diff(diff(u(t),t),t) = 0
<- change of variables successful
`, `-> Computing symmetries using: way = HINT
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.282 (sec). Leaf size: 898

```
dsolve(y(x)*diff(y(x),x$2)*sin(x)+ ( diff(y(x),x)*sin(x)+y(x)*cos(x) )*diff(y(x),x)=cos(x),y
```

$y(x) =$

$$-\frac{\sqrt{12}\sqrt{\left(-\mathrm{e}^{2ix}+1\right)^{3}\left(-\frac{i}{3}+\frac{\mathrm{e}^{3ix}\pi\left(\mathrm{csgn}(i\mathrm{e}^{-ix})\,\mathrm{csgn}\left(i(\mathrm{e}^{ix}-1)^{2}\right)-1\right)\mathrm{csgn}\left(i\mathrm{e}^{-ix}(\mathrm{e}^{ix}-1)^{2}\right)}{2}-\frac{\mathrm{e}^{3ix}\pi\left(\mathrm{csgn}(i\mathrm{e}^{-ix})\,\mathrm{csgn}\left(i(\mathrm{e}^{ix}+1\right.\right.}{2}}}{}$$

$y(x)$

$$=\frac{\sqrt{12}\sqrt{\left(-\mathrm{e}^{2ix}+1\right)^{3}\left(-\frac{i}{3}+\frac{\mathrm{e}^{3ix}\pi\left(\mathrm{csgn}(i\mathrm{e}^{-ix})\,\mathrm{csgn}\left(i(\mathrm{e}^{ix}-1)^{2}\right)-1\right)\mathrm{csgn}\left(i\mathrm{e}^{-ix}(\mathrm{e}^{ix}-1)^{2}\right)}{2}-\frac{\mathrm{e}^{3ix}\pi\left(\mathrm{csgn}(i\mathrm{e}^{-ix})\,\mathrm{csgn}\left(i(\mathrm{e}^{ix}+1)^{2}\right)\right.}{2}}}{}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 50

```
DSolve[y[x]*y''[x]*Sin[x]+ ( y'[x]*Sin[x]+y[x]*Cos[x] )*y'[x]==Cos[x],y[x],x,IncludeSingular
```

$$y(x) \to -\sqrt{2}\sqrt{c_1 \mathrm{arctanh}(\cos(x)) + x + c_2}$$
$$y(x) \to \sqrt{2}\sqrt{c_1 \mathrm{arctanh}(\cos(x)) + x + c_2}$$

## 2.52 problem Problem 19(e)



Internal problem ID [12273]
Internal file name [`OUTPUT/10925_Thursday_September_28_2023_01_09_06_AM_20167140/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 19(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{(1-y)\,y''-y'^2=0}$$

### 2.52.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left((1-y)\,y''-y'^2\right) dx = 0$$
$$-(y-1)\,y' = c_1$$

Which is now solved for $y$. Integrating both sides gives

$$\int -\frac{y-1}{c_1}dy = x + c_2$$
$$-\frac{\frac{1}{2}y^2 - y}{c_1} = x + c_2$$

Solving for $y$ gives these solutions

$$y_1 = 1 - \sqrt{-2c_1c_2 - 2c_1x + 1}$$
$$y_2 = 1 + \sqrt{-2c_1c_2 - 2c_1x + 1}$$

Summary
The solution(s) found are the following

$$y = 1 - \sqrt{-2c_1c_2 - 2c_1x + 1} \tag{1}$$
$$y = 1 + \sqrt{-2c_1c_2 - 2c_1x + 1} \tag{2}$$

Verification of solutions

$$y = 1 - \sqrt{-2c_1c_2 - 2c_1x + 1}$$

Verified OK.

$$y = 1 + \sqrt{-2c_1c_2 - 2c_1x + 1}$$

Verified OK.

### 2.52.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable $y$ an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$(1-y)\,p(y)\left(\frac{d}{dy}p(y)\right)-p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p)\\
&= f(y)g(p)\\
&= -\frac{p}{y-1}
\end{aligned}$$

Where $f(y)=-\frac{1}{y-1}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{y-1}\,dy\\
\int\frac{1}{p}\,dp &= \int-\frac{1}{y-1}\,dy\\
\ln(p) &= -\ln(y-1)+c_1\\
p &= e^{-\ln(y-1)+c_1}\\
&= \frac{c_1}{y-1}
\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=\frac{c_1}{y-1}$$

Integrating both sides gives

$$\begin{aligned}
\int\frac{y-1}{c_1}dy &= x+c_2\\
\frac{\frac{1}{2}y^2-y}{c_1} &= x+c_2
\end{aligned}$$

Solving for $y$ gives these solutions

$$\begin{aligned}
y_1 &= 1-\sqrt{2c_1c_2+2c_1x+1}\\
y_2 &= 1+\sqrt{2c_1c_2+2c_1x+1}
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y=1-\sqrt{2c_1c_2+2c_1x+1} \tag{1}$$
$$y=1+\sqrt{2c_1c_2+2c_1x+1} \tag{2}$$

Verification of solutions

$$y = 1 - \sqrt{2c_1c_2 + 2c_1x + 1}$$

Verified OK.

$$y = 1 + \sqrt{2c_1c_2 + 2c_1x + 1}$$

Verified OK.

### 2.52.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(1 - y)\, y'' - y'^2 = 0$$

Integrating both sides of the ODE w.r.t $x$ gives

$$\int \left((1 - y)\, y'' - y'^2\right) dx = 0$$
$$-(y - 1)\, y' = c_1$$

Which is now solved for $y$. Integrating both sides gives

$$\int -\frac{y - 1}{c_1} dy = x + c_2$$
$$-\frac{\frac{1}{2}y^2 - y}{c_1} = x + c_2$$

Solving for $y$ gives these solutions

$$y_1 = 1 - \sqrt{-2c_1c_2 - 2c_1x + 1}$$
$$y_2 = 1 + \sqrt{-2c_1c_2 - 2c_1x + 1}$$

Summary

The solution(s) found are the following

$$y = 1 - \sqrt{-2c_1c_2 - 2c_1x + 1} \tag{1}$$
$$y = 1 + \sqrt{-2c_1c_2 - 2c_1x + 1} \tag{2}$$

Verification of solutions

$$y = 1 - \sqrt{-2c_1c_2 - 2c_1x + 1}$$

Verified OK.

$$y = 1 + \sqrt{-2c_1c_2 - 2c_1x + 1}$$

Verified OK.

### 2.52.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y'' + a_1(x,y,y')\,y' + a_0(x,y,y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}
a_2 &= 1-y \\
a_1 &= -y' \\
a_0 &= 0
\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}
\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx &= c_1 \\
\int 1-y\,dy' + \int -y'\,dy + \int 0\,dx &= c_1
\end{aligned}$$

Which results in

$$(1-y)\,y' - yy' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}
\int -\frac{2y-1}{c_1}dy &= x + c_2 \\
-\frac{y^2-y}{c_1} &= x + c_2
\end{aligned}$$

Solving for $y$ gives these solutions

$$\begin{aligned}
y_1 &= \frac{1}{2} - \frac{\sqrt{-4c_1c_2 - 4c_1x + 1}}{2} \\
y_2 &= \frac{1}{2} + \frac{\sqrt{-4c_1c_2 - 4c_1x + 1}}{2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2} - \frac{\sqrt{-4c_1c_2 - 4c_1x + 1}}{2} \tag{1}$$

$$y = \frac{1}{2} + \frac{\sqrt{-4c_1c_2 - 4c_1x + 1}}{2} \tag{2}$$

Verification of solutions

$$y = \frac{1}{2} - \frac{\sqrt{-4c_1c_2 - 4c_1x + 1}}{2}$$

Verified OK.

$$y = \frac{1}{2} + \frac{\sqrt{-4c_1c_2 - 4c_1x + 1}}{2}$$

Verified OK.

### 2.52.5 Maple step by step solution

Let's solve

$(1-y)\,y'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Define new dependent variable $u$

  $u(x) = y'$

- Compute $y''$

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of $u$

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $(1-y)\,u(y)\left(\frac{d}{dy}u(y)\right) - u(y)^2 = 0$

- Separate variables

$$\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1}{1-y}$$

- Integrate both sides with respect to $y$

$$\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int \frac{1}{1-y}dy + c_1$$

- Evaluate integral

$$\ln(u(y)) = -\ln(1-y) + c_1$$

- Solve for $u(y)$

$$u(y) = -\frac{e^{c_1}}{y-1}$$

- Solve 1st ODE for $u(y)$

$$u(y) = -\frac{e^{c_1}}{y-1}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{e^{c_1}}{y-1}$$

- Separate variables

$$(y-1)\, y' = -e^{c_1}$$

- Integrate both sides with respect to $x$

$$\int (y-1)\, y' dx = \int -e^{c_1} dx + c_2$$

- Evaluate integral

$$\frac{y^2}{2} - y = -e^{c_1}x + c_2$$

- Solve for $y$

$$\left\{y = 1 - \sqrt{1 - 2\,e^{c_1}x + 2c_2}, y = 1 + \sqrt{1 - 2\,e^{c_1}x + 2c_2}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 39

```
dsolve((1-y(x))*diff(y(x),x$2)-diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 1$$
$$y(x) = 1 - \sqrt{2c_1x + 2c_2 + 1}$$
$$y(x) = 1 + \sqrt{2c_1x + 2c_2 + 1}$$

✓ Solution by Mathematica

Time used: 0.881 (sec). Leaf size: 49

```
DSolve[(1-y[x])*y''[x]- y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - \sqrt{-2c_1x + 1 - 2c_2c_1}$$
$$y(x) \to 1 + \sqrt{-2c_1x + 1 - 2c_2c_1}$$

## 2.53 problem Problem 19(f)



Internal problem ID [12274]
Internal file name [`OUTPUT/10926_Thursday_September_28_2023_01_09_07_AM_72583590/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 19(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible,
   _mu_y_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{(\cos(y) - y\sin(y))\, y'' - {y'}^{2}(2\sin(y) + y\cos(y)) = \sin(x)}$$

### 2.53.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $x$ gives

$$\int ((\cos(y) - y\sin(y))\, y'' + (-y\cos(y)\, y' - 2\sin(y)\, y')\, y')\, dx = \int \sin(x)\, dx$$
$$(\cos(y) - y\sin(y))\, y' = -\cos(x) + c_1$$

Which is now solved for $y$. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{c_1 - \cos(x)}{\cos(y) - y\sin(y)} \end{aligned}$$

Where $f(x) = c_1 - \cos(x)$ and $g(y) = \frac{1}{\cos(y) - y\sin(y)}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{\cos(y)-y\sin(y)}}\, dy &= c_1 - \cos(x)\, dx \\
\int \frac{1}{\frac{1}{\cos(y)-y\sin(y)}}\, dy &= \int c_1 - \cos(x)\, dx \\
y\cos(y) &= c_1 x - \sin(x) + c_2
\end{aligned}
$$

The solution is

$$
y\cos(y) - c_1 x + \sin(x) - c_2 = 0
$$

Summary

The solution(s) found are the following

$$
y\cos(y) - c_1 x + \sin(x) - c_2 = 0 \tag{1}
$$

Verification of solutions

$$
y\cos(y) - c_1 x + \sin(x) - c_2 = 0
$$

Verified OK.

### 2.53.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
(\cos(y) - y\sin(y))\, y'' + (-y\cos(y)\, y' - 2\sin(y)\, y')\, y' = \sin(x)
$$

Integrating both sides of the ODE w.r.t $x$ gives

$$
\begin{aligned}
\int ((\cos(y) - y\sin(y))\, y'' + (-y\cos(y)\, y' - 2\sin(y)\, y')\, y')\, dx &= \int \sin(x)\, dx \\
(\cos(y) - y\sin(y))\, y' &= -\cos(x) + c_1
\end{aligned}
$$

Which is now solved for $y$. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{c_1 - \cos(x)}{\cos(y) - y\sin(y)}
\end{aligned}
$$

Where $f(x) = c_1 - \cos(x)$ and $g(y) = \frac{1}{\cos(y) - y\sin(y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\cos(y)-y\sin(y)}}\, dy = c_1 - \cos(x)\, dx$$

$$\int \frac{1}{\frac{1}{\cos(y)-y\sin(y)}}\, dy = \int c_1 - \cos(x)\, dx$$

$$y\cos(y) = c_1x - \sin(x) + c_2$$

The solution is

$$y\cos(y) - c_1x + \sin(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$y\cos(y) - c_1x + \sin(x) - c_2 = 0 \tag{1}$$

Verification of solutions

$$y\cos(y) - c_1x + \sin(x) - c_2 = 0$$

Verified OK.

### 2.53.3 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}
a_2 &= \cos(y) - y\sin(y) \\
a_1 &= -y\cos(y)\, y' - 2\sin(y)\, y' \\
a_0 &= -\sin(x)
\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx = c_1$$

$$\int \cos(y) - y\sin(y)\ dy' + \int -y\cos(y)\, y' - 2\sin(y)\, y'\, dy + \int -\sin(x)\ dx = c_1$$

Which results in

$$(\cos(y) - y\sin(y))\, y' + 2y'\cos(y) - y'(\cos(y) + y\sin(y)) + \cos(x) = c_1$$

Which is now solved In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\frac{c_1}{2} - \frac{\cos(x)}{2}}{\cos(y) - y\sin(y)} \end{aligned}$$

Where $f(x) = \frac{c_1}{2} - \frac{\cos(x)}{2}$ and $g(y) = \frac{1}{\cos(y) - y\sin(y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\cos(y)-y\sin(y)}}\, dy = \frac{c_1}{2} - \frac{\cos(x)}{2}\, dx$$

$$\int \frac{1}{\frac{1}{\cos(y)-y\sin(y)}}\, dy = \int \frac{c_1}{2} - \frac{\cos(x)}{2}\, dx$$

$$y\cos(y) = \frac{c_1 x}{2} - \frac{\sin(x)}{2} + c_2$$

The solution is

$$y\cos(y) - \frac{c_1 x}{2} + \frac{\sin(x)}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$y\cos(y) - \frac{c_1 x}{2} + \frac{\sin(x)}{2} - c_2 = 0 \tag{1}$$

Verification of solutions

$$y\cos(y) - \frac{c_1 x}{2} + \frac{\sin(x)}{2} - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 20

```
dsolve((cos(y(x))-y(x)*sin(y(x)))*diff(y(x),x$2)- diff(y(x),x)^2* (2*sin(y(x))+y(x)*cos(y(x)
```

$$-y(x)\cos(y(x)) - c_1 x - \sin(x) + c_2 = 0$$

✓ Solution by Mathematica

Time used: 0.531 (sec). Leaf size: 28

```
DSolve[(Cos[y[x]]-y[x]*Sin[y[x]])*y''[x]- y'[x]^2* (2*Sin[y[x]]+y[x]*Cos[y[x]])==Sin[x],y[x]
```

$$\text{Solve}\left[\frac{y(x)\cos(y(x))}{x} + \frac{\sin(x)}{x} + \frac{c_1}{x} = c_2, y(x)\right]$$

## 2.54 problem Problem 20(a)



Internal problem ID [12275]
Internal file name [`OUTPUT/10927_Thursday_September_28_2023_01_09_12_AM_59117329/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + \frac{2xy'}{2x-1} - \frac{4xy}{(2x-1)^2} = 0}$$

### 2.54.1 Solving using Kovacic algorithm

Writing the ode as

$$-4\left(x-\frac{1}{2}\right)^2 y'' + \left(-4x^2+2x\right) y' + 4yx = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -4\left(x-\frac{1}{2}\right)^2 \\ B &= -4x^2+2x \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 1}{(2x - 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 1 \\ t &= (2x - 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left( \frac{x^2 + 4x - 1}{(2x - 1)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 53: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x-1)^2$. There is a pole at $x = \frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{1}{4} + \frac{5}{16\left(x - \frac{1}{2}\right)^2} + \frac{5}{4\left(x - \frac{1}{2}\right)}$$

For the pole at $x = \frac{1}{2}$ let $b$ be the coefficient of $\frac{1}{\left(x-\frac{1}{2}\right)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b=\frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{5}{4} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{4}
\end{aligned}
$$

Since the order of $r$ at $\infty$ is $O_r(\infty)=0$ then

$$
v=\frac{-O_r(\infty)}{2}=\frac{0}{2}=0
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let $a$ be the coefficient of $x^v=x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$
\sqrt{r}\approx \frac{1}{2}+\frac{5}{4x}-\frac{5}{8x^2}+\frac{35}{16x^3}-\frac{175}{32x^4}+\frac{1065}{64x^5}-\frac{6795}{128x^6}+\frac{45445}{256x^7}+\dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a=\frac{1}{2}
$$

From Eq. (9) the sum up to $v=0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1}=x^{-1}=\frac{1}{x}$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2=\frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in $r$. How this is done depends on if $v=0$ or not. Since $v=0$ then starting from $r=\frac{s}{t}$ and doing long division in the form

$$
r=Q+\frac{R}{t}
$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of $\frac{1}{x}$ in $r$ will be the coefficient in $R$ of the term in $x$ of degree of $t$ minus one, divided by the leading coefficient in $t$. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t}\\
&= \frac{x^2+4x-1}{4x^2-4x+1}\\
&= Q + \frac{R}{4x^2-4x+1}\\
&= \left(\frac{1}{4}\right) + \left(\frac{5x-\frac{5}{4}}{4x^2-4x+1}\right)\\
&= \frac{1}{4} + \frac{5x-\frac{5}{4}}{4x^2-4x+1}
\end{aligned}$$

Since the degree of $t$ is 2, then we see that the coefficient of the term $x$ in the remainder $R$ is 5. Dividing this by leading coefficient in $t$ which is 4 gives $\frac{5}{4}$. Now $b$ can be found.

$$\begin{aligned}
b &= \left(\frac{5}{4}\right) - (0)\\
&= \frac{5}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2}\\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{\frac{5}{4}}{\frac{1}{2}} - 0\right) &&= \frac{5}{4}\\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{\frac{5}{4}}{\frac{1}{2}} - 0\right) &&= -\frac{5}{4}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = \frac{x^2+4x-1}{(2x-1)^2}$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^{+}$ | $\alpha_c^{-}$ |
|---|---|---|---|---|
| $\frac{1}{2}$ | 2 | 0 | $\frac{5}{4}$ | $-\frac{1}{4}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^{+}$ | $\alpha_\infty^{-}$ |
|---|---|---|---|
| 0 | $\frac{1}{2}$ | $\frac{5}{4}$ | $-\frac{5}{4}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^{+} = \frac{5}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{+}\right) \\ &= \frac{5}{4} - \left(\frac{5}{4}\right) \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left( (+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= \frac{5}{4\left(x-\frac{1}{2}\right)} + \left(\frac{1}{2}\right) \\ &= \frac{5}{4\left(x-\frac{1}{2}\right)} + \frac{1}{2} \\ &= \frac{x+2}{2x-1} \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{5}{4\left(x-\frac{1}{2}\right)} + \frac{1}{2}\right)(0) + \left(\left(-\frac{5}{4\left(x-\frac{1}{2}\right)^2}\right) + \left(\frac{5}{4\left(x-\frac{1}{2}\right)} + \frac{1}{2}\right)^2 - \left(\frac{x^2+4x-1}{(2x-1)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{5}{4\left(x-\frac{1}{2}\right)}+\frac{1}{2}\right) dx} \\
&= (2x-1)^{\frac{5}{4}} \mathrm{e}^{\frac{x}{2}}
\end{aligned}$$

The first solution to the original ode in $y$ is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2+2x}{-4\left(x-\frac{1}{2}\right)^2} \, dx} \\
&= z_1 e^{-\frac{x}{2}-\frac{\ln(2x-1)}{4}} \\
&= z_1 \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{(2x-1)^{\frac{1}{4}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 2x - 1$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+2x}{-4\left(x-\frac{1}{2}\right)^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x-\frac{\ln(2x-1)}{2}}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{-\frac{1}{2}} \left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{2x-1}}{2}\right) \sqrt{2}\,\sqrt{\pi}\,(2x-1)^{\frac{3}{2}} + 4\,\mathrm{e}^{-x+\frac{1}{2}} (x-1)\right)}{6\,(2x-1)^{\frac{3}{2}}}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1(2x-1)+c_2\left(2x-1\left(\frac{e^{-\frac{1}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{2x-1}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,(2x-1)^{\frac{3}{2}}+4\,e^{-x+\frac{1}{2}}(x-1)\right)}{6\,(2x-1)^{\frac{3}{2}}}\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1(2x-1)+\frac{c_2 e^{-\frac{1}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{2x-1}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,(2x-1)^{\frac{3}{2}}+4\,e^{-x+\frac{1}{2}}(x-1)\right)}{6\sqrt{2x-1}} \tag{1}$$

Verification of solutions

$$y = c_1(2x-1)+\frac{c_2 e^{-\frac{1}{2}}\left(\operatorname{erf}\left(\frac{\sqrt{2}\,\sqrt{2x-1}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,(2x-1)^{\frac{3}{2}}+4\,e^{-x+\frac{1}{2}}(x-1)\right)}{6\sqrt{2x-1}}$$

Verified OK.

### 2.54.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$-4\left(x-\frac{1}{2}\right)^2 y''+\left(-4x^2+2x\right)y'+4yx=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2x}{2x-1} \\ q(x) &= -\frac{4x}{(2x-1)^2} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi''-(\xi\,p)'+\xi q &= 0 \\ \xi''-\left(\frac{2x\xi(x)}{2x-1}\right)'+\left(-\frac{4x\xi(x)}{(2x-1)^2}\right) &= 0 \\ \xi''(x)-\frac{2\xi(x)}{2x-1}-\frac{2x\xi'(x)}{2x-1} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Integrating both sides of the ODE w.r.t $x$ gives

$$
\begin{aligned}
\int \left(\xi''(x)\,(2x-1)-2\xi(x)-2x\xi'(x)\right)dx &= 0\\
(-2x-2)\,\xi(x)+\xi'(x)\,(2x-1) &= c_1
\end{aligned}
$$

Which is now solved for $\xi(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\xi'(x)+p(x)\xi(x)=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2x+2}{2x-1}\\
q(x) &= \frac{c_1}{2x-1}
\end{aligned}
$$

Hence the ode is

$$
\xi'(x)-\frac{(2x+2)\,\xi(x)}{2x-1}=\frac{c_1}{2x-1}
$$

The integrating factor $\mu$ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2x+2}{2x-1}dx}\\
&= \mathrm{e}^{-x-\frac{3\ln(2x-1)}{2}}
\end{aligned}
$$

Which simplifies to

$$
\mu=\frac{\mathrm{e}^{-x}}{(2x-1)^{\frac{3}{2}}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu\xi) &= (\mu)\left(\frac{c_1}{2x-1}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{-x}\xi}{(2x-1)^{\frac{3}{2}}}\right) &= \left(\frac{\mathrm{e}^{-x}}{(2x-1)^{\frac{3}{2}}}\right)\left(\frac{c_1}{2x-1}\right)\\
\mathrm{d}\left(\frac{\mathrm{e}^{-x}\xi}{(2x-1)^{\frac{3}{2}}}\right) &= \left(\frac{c_1\mathrm{e}^{-x}}{(2x-1)^{\frac{5}{2}}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$\frac{e^{-x}\xi}{(2x-1)^{\frac{3}{2}}} = \int \frac{c_1 e^{-x}}{(2x-1)^{\frac{5}{2}}}\,dx$$

$$\frac{e^{-x}\xi}{(2x-1)^{\frac{3}{2}}} = c_1 e^{-\frac{1}{2}}\left(-\frac{e^{-x+\frac{1}{2}}}{3(2x-1)^{\frac{3}{2}}} + \frac{e^{-x+\frac{1}{2}}}{3\sqrt{2x-1}} + \frac{\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right)\sqrt{2}\sqrt{\pi}}{6}\right) + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-x}}{(2x-1)^{\frac{3}{2}}}$ results in

$$\xi(x) = (2x-1)^{\frac{3}{2}} e^{x} c_1 e^{-\frac{1}{2}}\left(-\frac{e^{-x+\frac{1}{2}}}{3(2x-1)^{\frac{3}{2}}} + \frac{e^{-x+\frac{1}{2}}}{3\sqrt{2x-1}} + \frac{\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right)\sqrt{2}\sqrt{\pi}}{6}\right) + c_2(2x-1)^{\frac{3}{2}} e^{x}$$

which simplifies to

$$\xi(x) = \frac{\left(x-\frac{1}{2}\right) c_1\sqrt{2}\, e^{x-\frac{1}{2}}\sqrt{2x-1}\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right)}{3} + 2\,e^{x}\left(x-\frac{1}{2}\right) c_2\sqrt{2x-1} + \frac{2c_1(x-1)}{3}$$

The original ode (2) now reduces to first order ode

$$y' + y\left(\frac{2x}{2x-1} - \frac{\frac{c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\sqrt{2x-1}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right)}{3} + \frac{\left(x-\frac{1}{2}\right)c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{2x-1}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right)}{3} + \frac{\left(x-\frac{1}{2}\right)c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\,\operatorname{erf}}{3\sqrt{2x-1}}}{\frac{\left(x-\frac{1}{2}\right)c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{2x-1}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right)}{3}}\right.$$

Which is now a first order ode. This is now solved for $y$. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2y\left(4x^2c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 4xc_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 4\sqrt{2x-1}\,e^{x-\frac{1}{2}}e^{-x+\frac{1}{2}}c_3x^2 + \right.}{(2x-1)^{\frac{3}{2}}\left(2xc_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\sqrt{2x-1}\,\operatorname{erf}\left(\right.\right.} \end{aligned}$$

Where $f(x) = \frac{8x^2c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8xc_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\,e^{x-\frac{1}{2}}e^{-x+\frac{1}{2}}c_3x^2 + 2c_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\,\operatorname{erf}\left(\right.}{(2x-1)^{\frac{3}{2}}\left(2xc_3\sqrt{2}\,e^{x-\frac{1}{2}}\sqrt{\pi}\sqrt{2x-1}\,\operatorname{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - c_3\sqrt{2}\,e^{x-\frac{1}{2}}\right.}$

and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 x^2 \cdots}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x-1} \cdots\right.} \\
\int \frac{1}{y}\, dy &= \int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 \cdots}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x - \cdots}\right.} \\
\ln(y) &= \int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 \cdots}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x - \cdots}\right.} \\
y &= \mathrm{e}^{\int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 x^2 + 2c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8 \cdots}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\sqrt{2x-1}\, \mathrm{er} \cdots\right.}} \\
&= c_3 \mathrm{e}^{\int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 x^2 + 2c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) \cdots}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\sqrt{2x-\cdots}\right.}}
\end{aligned}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_3 \mathrm{e}^{\int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 x^2 + 2c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8c_3 \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{\cdots}}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2 \cdots}}{2}\right.\right.}}
$$


Summary

The solution(s) found are the following

$$
y = c_3 \mathrm{e}^{\int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 x^2 + 2c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8c_3 \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{\cdots}}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2 \cdots}}{2}\right.\right.}} \tag{1}
$$


Verification of solutions

$$
y = c_3 \mathrm{e}^{\int \frac{8x^2 c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) + 8\sqrt{2x-1}\, \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{2}} c_3 x^2 + 2c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - 8c_3 \mathrm{e}^{x-\frac{1}{2}} \mathrm{e}^{\cdots}}{(2x-1)^{\frac{3}{2}} \left(2x c_3 \sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi} \sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2x-1}}{2}\right) - c_3\sqrt{2}\, \mathrm{e}^{x-\frac{1}{2}} \sqrt{\pi}\sqrt{2x-1}\, \mathrm{erf}\left(\frac{\sqrt{2}\sqrt{2 \cdots}}{2}\right.\right.}}
$$

Verified OK.

### 2.54.3 Maple step by step solution

Let's solve

$-4\left(x-\frac{1}{2}\right)^{2} y''+\left(-4x^{2}+2x\right) y'+4yx=0$

- Highest derivative means the order of the ODE is 2

$y''$

- Isolate 2nd derivative

$y''=-\frac{2xy'}{2x-1}+\frac{4xy}{(2x-1)^{2}}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{2xy'}{2x-1}-\frac{4xy}{(2x-1)^{2}}=0$

☐ Check to see if $x_0=\frac{1}{2}$ is a regular singular point

  - Define functions

  $\left[P_2(x)=\frac{2x}{2x-1}, P_3(x)=-\frac{4x}{(2x-1)^{2}}\right]$

  - $\left(x-\frac{1}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{2}$

  $\left(\left(x-\frac{1}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}}=\frac{1}{2}$

  - $\left(x-\frac{1}{2}\right)^{2}\cdot P_3(x)$ is analytic at $x=\frac{1}{2}$

  $\left(\left(x-\frac{1}{2}\right)^{2}\cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}}=-\frac{1}{2}$

  - $x=\frac{1}{2}$is a regular singular point

Check to see if $x_0=\frac{1}{2}$ is a regular singular point

$x_0=\frac{1}{2}$

- Multiply by denominators

$y''(2x-1)^{2}+2y'x(2x-1)-4yx=0$

- Change variables using $x=u+\frac{1}{2}$ so that the regular singular point is at $u=0$

$4u^{2}\left(\frac{d^{2}}{du^{2}}y(u)\right)+\left(4u^{2}+2u\right)\left(\frac{d}{du}y(u)\right)+(-4u-2)\,y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k - m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

  $$u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r}$$

  Rewrite ODE with series expansions

  $$2a_0(1+2r)\,(-1+r)\, u^r + \left(\sum_{k=1}^{\infty} \left(2a_k(2k+2r+1)\,(k+r-1) + 4a_{k-1}(k-2+r)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$2(1+2r)\,(-1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{1, -\frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$4(k+r-1)\left(k+r+\frac{1}{2}\right) a_k + 4a_{k-1}(k-2+r) = 0$$

- Shift index using $k- >k + 1$

  $$4(k+r)\left(k+\frac{3}{2}+r\right) a_{k+1} + 4a_k(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k(k+r-1)}{(k+r)(2k+3+2r)}$$

- Recursion relation for $r = 1$

  $$a_{k+1} = -\frac{2a_k k}{(k+1)(2k+5)}$$

- Solution for $r = 1$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+1}=-\frac{2a_k k}{(k+1)(2k+5)}\right]$$

- Revert the change of variables $u=x-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x-\frac{1}{2}\right)^{k+1}, a_{k+1}=-\frac{2a_k k}{(k+1)(2k+5)}\right]$$

- Recursion relation for $r=-\frac{1}{2}$

$$a_{k+1}=-\frac{2a_k\left(k-\frac{3}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}$$

- Solution for $r=-\frac{1}{2}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}}, a_{k+1}=-\frac{2a_k\left(k-\frac{3}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Revert the change of variables $u=x-\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty} a_k\left(x-\frac{1}{2}\right)^{k-\frac{1}{2}}, a_{k+1}=-\frac{2a_k\left(k-\frac{3}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k\left(x-\frac{1}{2}\right)^{1+k}\right)+\left(\sum_{k=0}^{\infty} b_k\left(x-\frac{1}{2}\right)^{k-\frac{1}{2}}\right), a_{1+k}=-\frac{2a_k k}{(1+k)(2k+5)}, b_{1+k}=-\frac{2b_k\left(k-\frac{3}{2}\right)}{\left(k-\frac{1}{2}\right)(2k+2)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Whittaker successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 77

```
dsolve(diff(y(x),x$2)+ (2*x)/(2*x-1)*diff(y(x),x)- 4*x/( (2*x-1)^2)*y(x)=0,y(x), singsol=all
```

$$y(x) = -\frac{2\,2^{\frac{3}{4}}\mathrm{e}^{-\frac{1}{4}}\left(\frac{\left(\frac{2\left(\mathrm{erf}\left(\frac{\sqrt{-2+4x}}{2}\right)-1\right)c_2}{3}+\mathrm{erf}\left(\frac{\sqrt{-2+4x}}{2}\right)c_1\right)\sqrt{-2+4x}\left(-\frac{1}{2}+x\right)\sqrt{\pi}}{2}+(-1+x)\,\mathrm{e}^{\frac{1}{2}-x}\left(c_1+\frac{2c_2}{3}\right)\right)}{\sqrt{-2+4x}}$$

✓ Solution by Mathematica

Time used: 0.508 (sec). Leaf size: 64

```
DSolve[y''[x]+ (2*x)/(2*x-1)*y'[x]- 4*x/( (2*x-1)^2)*y[x]==0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to c_1(2x-1) + \frac{1}{6}c_2\left(\frac{4e^{\frac{1}{2}-x}(x-1)}{\sqrt{2x-1}} + \sqrt{2}(1-2x)\Gamma\left(\frac{1}{2}, x-\frac{1}{2}\right)\right)$$

## 2.55 problem Problem 20(b)



Internal problem ID [12276]
Internal file name [`OUTPUT/10928_Thursday_September_28_2023_01_09_13_AM_2229465/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^{2}+2x\right)y''+\left(x^{2}+x+10\right)y'-\left(25-6x\right)y=0}$$

### 2.55.1 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^{2}+2x\right)y''+\left(x^{2}+x+10\right)y'+\left(6x-25\right)y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^{2}+2x \\ B &= x^{2}+x+10 \\ C &= 6x-25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{x^4 - 22x^3 + 75x^2 + 180x + 60}{4\left(x^2 + 2x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^4 - 22x^3 + 75x^2 + 180x + 60 \\ t &= 4\left(x^2 + 2x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^4 - 22x^3 + 75x^2 + 180x + 60}{4\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 55: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0\end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{1}{4} + \frac{15}{4x^2} + \frac{12}{(x+2)^2} - \frac{14}{x+2} + \frac{15}{2x}$$

For the pole at $x = -2$ let $b$ be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = 12$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3\end{aligned}$$

For the pole at $x = 0$ let $b$ be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of $r$ at $\infty$ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let $a$ be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{13}{2x} + \frac{3}{2x^2} - \frac{169}{2x^3} - \frac{3379}{4x^4} - \frac{45345}{4x^5} - \frac{602277}{4x^6} - \frac{8277417}{4x^7} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in $r$. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of $\frac{1}{x}$ in $r$ will be the coefficient in $R$ of the term in $x$ of degree of $t$ minus one, divided by the leading coefficient in $t$. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 22x^3 + 75x^2 + 180x + 60}{4x^4 + 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 + 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-26x^3 + 71x^2 + 180x + 60}{4x^4 + 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-26x^3 + 71x^2 + 180x + 60}{4x^4 + 16x^3 + 16x^2}
\end{aligned}$$

Since the degree of $t$ is 4, then we see that the coefficient of the term $x^3$ in the remainder $R$ is $-26$. Dividing this by leading coefficient in $t$ which is 4 gives $-\frac{13}{2}$. Now $b$ can be found.

$$\begin{aligned}
b &= \left(-\frac{13}{2}\right) - (0) \\
&= -\frac{13}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{13}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{13}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{13}{2}}{\frac{1}{2}} - 0\right) &&= \frac{13}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = \frac{x^4 - 22x^3 + 75x^2 + 180x + 60}{4\left(x^2 + 2x\right)^2}$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^+$ | $\alpha_c^-$ |
|---|---|---|---|---|
| $-2$ | 2 | 0 | 4 | $-3$ |
| 0 | 2 | 0 | $\frac{5}{2}$ | $-\frac{3}{2}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| 0 | $\frac{1}{2}$ | $-\frac{13}{2}$ | $\frac{13}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^{-} = \frac{13}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) \\ &= \frac{13}{2} - \left(\frac{13}{2}\right) \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left( (+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{+}}{x-c_1} \right) + \left( (+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{4}{x+2} + \frac{5}{2x} + (-)\left(\frac{1}{2}\right) \\ &= \frac{4}{x+2} + \frac{5}{2x} - \frac{1}{2} \\ &= \frac{4}{x+2} + \frac{5}{2x} - \frac{1}{2} \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{4}{x+2} + \frac{5}{2x} - \frac{1}{2}\right)(0) + \left(\left(-\frac{4}{(x+2)^2} - \frac{5}{2x^2}\right) + \left(\frac{4}{x+2} + \frac{5}{2x} - \frac{1}{2}\right)^2 - \left(\frac{x^4 - 22x^3 + 75x^2 + 1}{4\left(x^2+2x\right)^2}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{4}{x+2} + \frac{5}{2x} - \frac{1}{2}\right) dx} \\
&= x^{\frac{5}{2}} (x+2)^4 \, \mathrm{e}^{-\frac{x}{2}}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x^2+x+10}{x^2+2x} \, dx} \\
&= z_1 e^{-\frac{x}{2} - \frac{5 \ln(x)}{2} + 3 \ln(x+2)} \\
&= z_1 \left(\frac{(x+2)^3 \, \mathrm{e}^{-\frac{x}{2}}}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+2)^7 \, \mathrm{e}^{-x}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^2+x+10}{x^2+2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x-5\ln(x)+6\ln(x+2)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{88447 \, \mathrm{e}^{-2} x^4 (x+2)^7 \operatorname{expIntegral}_1(-x-2) - 11970 (x+2)^7 x^4 \operatorname{expIntegral}_1(-x) + \mathrm{e}^x (76477 x^{10} +}{}\right.
\end{aligned}
$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1\left(\left(x+2\right)^{7} \mathrm{e}^{-x}\right) + c_2\left(\left(x\right.\right.$$
$$\left.+2\right)^{7} \mathrm{e}^{-x}\left(\frac{88447\,\mathrm{e}^{-2} x^{4} \left(x+2\right)^{7} \mathrm{expIntegral}_{1}\left(-x-2\right) - 11970 \left(x+2\right)^{7} x^{4} \mathrm{expIntegral}_{1}\left(-x\right) + \mathrm{e}^{x} (764}{}\right.$$

Summary
The solution(s) found are the following

$$y = c_1 \left(x+2\right)^{7} \mathrm{e}^{-x} + \frac{c_2\left(88447 \left(x+2\right)^{7} x^{4} \mathrm{e}^{-x-2} \mathrm{expIntegral}_{1}\left(-x-2\right) - 11970\,\mathrm{e}^{-x} x^{4} \left(x+2\right)^{7} \mathrm{expIntegral}_{1}\left(-x\right) + 76477 x^{10}}{} \tag{1}$$

Verification of solutions

$$y = c_1 \left(x+2\right)^{7} \mathrm{e}^{-x} + \frac{c_2\left(88447 \left(x+2\right)^{7} x^{4} \mathrm{e}^{-x-2} \mathrm{expIntegral}_{1}\left(-x-2\right) - 11970\,\mathrm{e}^{-x} x^{4} \left(x+2\right)^{7} \mathrm{expIntegral}_{1}\left(-x\right) + 76477 x^{10}}{}$$

Verified OK.

### 2.55.2 Maple step by step solution

Let's solve

$\left(x^{2}+2x\right) y'' + \left(x^{2}+x+10\right) y' + \left(6x-25\right) y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{(6x-25)y}{x(x+2)} - \frac{(x^{2}+x+10)y'}{x(x+2)}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^{2}+x+10)y'}{x(x+2)} + \frac{(6x-25)y}{x(x+2)} = 0$

□ Check to see if $x_0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x^{2}+x+10}{x(x+2)}, P_3(x) = \frac{6x-25}{x(x+2)}\right]$

  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

$$\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2} = -6$$

- $(x+2)^2 \cdot P_3(x)$ is analytic at $x=-2$

  $$\left((x+2)^2 \cdot P_3(x)\right)\bigg|_{x=-2} = 0$$

- $x=-2$is a regular singular point

  Check to see if $x_0$ is a regular singular point

  $x_0 = -2$

- Multiply by denominators

  $y''x(x+2) + (x^2+x+10)\,y' + (6x-25)\,y = 0$

- Change variables using $x = u-2$ so that the regular singular point is at $u = 0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2-3u+12)\left(\frac{d}{du}y(u)\right) + (6u-37)\,y(u) = 0$

- Assume series solution for $y(u)$

  $$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..1$

  $$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-7+r)\,u^{-1+r} + \left(-2a_1(1+r)\,(-6+r) + a_0(r^2-4r-37)\right)u^r + \left(\sum_{k=1}^{\infty} (-2a_{k+1}(k+1+r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-2r(-7+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 7\}$$

- Each term must be 0

$$-2a_1(1+r)\,(-6+r) + a_0(r^2-4r-37) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(a_k - 2a_{k+1})\,k^2 + ((2a_k - 4a_{k+1})\,r - 4a_k + a_{k-1} + 10a_{k+1})\,k + (a_k - 2a_{k+1})\,r^2 + (-4a_k + a_{k-1} +$$

- Shift index using $k->k+1$

$$(a_{k+1} - 2a_{k+2})\,(k+1)^2 + ((2a_{k+1} - 4a_{k+2})\,r - 4a_{k+1} + a_k + 10a_{k+2})\,(k+1) + (a_{k+1} - 2a_{k+2})\,r^2$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{k^2 a_{k+1} + 2kra_{k+1} + r^2 a_{k+1} + ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} + 6a_k - 40a_{k+1}}{2(k^2+2kr+r^2-3k-3r-10)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{k^2 a_{k+1} + ka_k - 2ka_{k+1} + 6a_k - 40a_{k+1}}{2(k^2-3k-10)}$$

- Series not valid for $r = 0$ , division by 0 in the recursion relation at $k = 5$

$$a_{k+2} = \frac{k^2 a_{k+1} + ka_k - 2ka_{k+1} + 6a_k - 40a_{k+1}}{2(k^2-3k-10)}$$

- Recursion relation for $r = 7$

$$a_{k+2} = \frac{k^2 a_{k+1} + ka_k + 12ka_{k+1} + 13a_k - 5a_{k+1}}{2(k^2+11k+18)}$$

- Solution for $r = 7$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+7}, a_{k+2} = \frac{k^2 a_{k+1} + ka_k + 12ka_{k+1} + 13a_k - 5a_{k+1}}{2(k^2+11k+18)}, -16a_1 - 16a_0 = 0\right]$$

- Revert the change of variables $u = x+2$

$$\left[y = \sum_{k=0}^{\infty} a_k(x+2)^{k+7}, a_{k+2} = \frac{k^2 a_{k+1} + ka_k + 12ka_{k+1} + 13a_k - 5a_{k+1}}{2(k^2+11k+18)}, -16a_1 - 16a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 128

```
dsolve((2*x+x^2)*diff(y(x),x$2)+ (10+x+x^2)*diff(y(x),x)=(25-6*x)*y(x),y(x), singsol=all)
```

$$y(x) = \frac{88447x^4c_2\mathrm{e}^{-x-2}(x+2)^7\,\mathrm{expIntegral}_1(-x-2) - 11970x^4c_2\mathrm{e}^{-x}(x+2)^7\,\mathrm{expIntegral}_1(-x) + x^4c_1(x+2)}{}$$

✓ Solution by Mathematica

Time used: 1.158 (sec). Leaf size: 217

```
DSolve[(2*x+x^2)*y''[x]+ (10+x+x^2)*y'[x]==(25-6*x)*y[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-x-2}(11970e^2c_2x^4(x+2)^7\,\mathrm{ExpIntegralEi}(x) - 88447c_2x^4(x+2)^7\,\mathrm{ExpIntegralEi}(x+2) + e^2(322560c_1}{}$$

## 2.56 problem Problem 20(c)



Internal problem ID [12277]
Internal file name [OUTPUT/10929_Thursday_September_28_2023_01_09_13_AM_47902418/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + \frac{y'}{x+1} - \frac{(x+2)\,y}{x^2\,(x+1)} = 0}$$

### 2.56.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$\left(-x^3 - x^2\right) y'' - x^2 y' + (x+2)\,y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x+1}$$
$$q(x) = \frac{-x-2}{x^2\,(x+1)}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $y$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{(x+1)\, x} + \frac{-x-2}{x^2\, (x+1)} = 0 \tag{5}$$

Solving (5) for $n$ gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(-\frac{2}{x} + \frac{1}{x+1}\right) v'(x) &= 0 \\ v''(x) + \left(-\frac{2}{x} + \frac{1}{x+1}\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(-\frac{2}{x} + \frac{1}{x+1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u(x+2)}{x\, (x+1)} \end{aligned}$$

Where $f(x)=\frac{x+2}{x(x+1)}$ and $g(u)=u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x+2}{x\,(x+1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x+2}{x\,(x+1)}\,dx \\
\ln(u) &= -\ln(x+1)+2\ln(x)+c_1 \\
u &= \mathrm{e}^{-\ln(x+1)+2\ln(x)+c_1} \\
&= c_1\mathrm{e}^{-\ln(x+1)+2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x)=\frac{c_1x^2}{x+1}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1\left(\frac{x^2}{2}-x+\ln(x+1)\right)+c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \frac{c_1\left(\frac{x^2}{2}-x+\ln(x+1)\right)+c_2}{x} \\
&= \frac{2\ln(x+1)\,c_1+(x^2-2x)\,c_1+2c_2}{2x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y=\frac{c_1\left(\frac{x^2}{2}-x+\ln(x+1)\right)+c_2}{x} \tag{1}
$$

Verification of solutions

$$
y=\frac{c_1\left(\frac{x^2}{2}-x+\ln(x+1)\right)+c_2}{x}
$$

Verified OK.

### 2.56.2 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 + y'x + \left(-1 - \frac{2}{x}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$y''x^2 + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha) xy' + \left(\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for $\alpha, \beta, n, \gamma$ gives

$$\begin{aligned}
\alpha &= 0 \\
\beta &= 2i\sqrt{2} \\
n &= -2 \\
\gamma &= -\frac{1}{2}
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = -c_1 \operatorname{BesselI}\left(2, \frac{2\sqrt{2}}{\sqrt{x}}\right) + c_2 \operatorname{BesselY}\left(2, \frac{2i\sqrt{2}}{\sqrt{x}}\right)$$

Summary

The solution(s) found are the following

$$y = -c_1 \operatorname{BesselI}\left(2, \frac{2\sqrt{2}}{\sqrt{x}}\right) + c_2 \operatorname{BesselY}\left(2, \frac{2i\sqrt{2}}{\sqrt{x}}\right) \tag{1}$$

Verification of solutions

$$y = -c_1 \operatorname{BesselI}\left(2, \frac{2\sqrt{2}}{\sqrt{x}}\right) + c_2 \operatorname{BesselY}\left(2, \frac{2i\sqrt{2}}{\sqrt{x}}\right)$$

Verified OK.

### 2.56.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(-x^3-x^2\right)y''-x^2y'+(x+2)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= -x^3-x^2 \\ B &= -x^2 \\ C &= x+2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A,B,C$ from (3) in the above and simplifying gives

$$r=\frac{3x^2+12x+8}{4\left(x^2+x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3x^2+12x+8 \\ t &= 4\left(x^2+x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{3x^2+12x+8}{4\left(x^2+x\right)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0,1,2,4,6,8,\cdots\}$ | $\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1,2\}$ | $\{2,3,4,5,6,7,\cdots\}$ |

Table 57: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2+x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{2}{x^2} - \frac{1}{4(x+1)^2} + \frac{1}{x+1} - \frac{1}{x}$$

For the pole at $x=-1$ let $b$ be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

For the pole at $x=0$ let $b$ be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of $r$ at $\infty$ is 2 then $[\sqrt{r}]_\infty = 0$. Let $b$ be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of $r$ at $\infty$. which can be found by dividing the leading coefficient of $s$ by the leading coefficient of $t$ from

$$
r = \frac{s}{t} = \frac{3x^2+12x+8}{4\left(x^2+x\right)^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$
r = \frac{3x^2+12x+8}{4\left(x^2+x\right)^2}
$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^{+}$ | $\alpha_c^{-}$ |
|---|---|---|---|---|
| $-1$ | 2 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 0 | 2 | 0 | 2 | $-1$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^{+}$ | $\alpha_\infty^{-}$ |
|---|---|---|---|
| 2 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left( (+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1} \right) + \left( (-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x+2} - \frac{1}{x} + (-)(0) \\
&= \frac{1}{2x+2} - \frac{1}{x} \\
&= -\frac{x+2}{2x(x+1)}
\end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x+2} - \frac{1}{x}\right)(0) + \left(\left(-\frac{1}{2(x+1)^2} + \frac{1}{x^2}\right) + \left(\frac{1}{2x+2} - \frac{1}{x}\right)^2 - \left(\frac{3x^2+12x+8}{4(x^2+x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2x+2} - \frac{1}{x}\right) dx} \\
&= \frac{\sqrt{x+1}}{x}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2}{-x^3-x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2}{-x^3-x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2}{2} - x + \ln(x+1)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2} - x + \ln(x+1)\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 \left(\frac{x^2}{2} - x + \ln(x+1)\right)}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 \left(\frac{x^2}{2} - x + \ln(x+1)\right)}{x}$$

Verified OK.

### 2.56.4 Maple step by step solution

Let's solve

$(-x^3 - x^2) y'' - x^2 y' + (x+2) y = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x+1} + \frac{(x+2)y}{x^2(x+1)}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x+1} - \frac{(x+2)y}{x^2(x+1)} = 0$

□ Check to see if $x_0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x+1}, P_3(x) = -\frac{x+2}{x^2(x+1)}\right]$

  ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = 1$

  ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

  ◦ $x = -1$is a regular singular point

    Check to see if $x_0$ is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $y''x^2(x+1)+x^2y'+(-x-2)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-2u+1)\left(\frac{d}{du}y(u)\right)+(-u-1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

    $u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$

  - Shift index using $k->k-m$

    $u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

  - Shift index using $k->k+1-m$

    $u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

  - Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

  - Shift index using $k->k+2-m$

    $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_0r^2u^{-1+r}+\left(a_1(1+r)^2-a_0(2r^2+1)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)^2-a_k(2k^2+4kr+2r^2+1)\right.\right.$ -

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2=0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term must be 0

  $a_1(1+r)^2 - a_0(2r^2+1) = 0$

- Each term in the series must be 0, giving the recursion relation

  $(-2a_k + a_{k-1} + a_{k+1})\, k^2 + (-2a_{k-1} + 2a_{k+1})\, k - a_k + a_{k+1} = 0$

- Shift index using $k- >k+1$

  $(-2a_{k+1} + a_k + a_{k+2})\,(k+1)^2 + (-2a_k + 2a_{k+2})\,(k+1) - a_{k+1} + a_{k+2} = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - a_k - 3a_{k+1}}{k^2+4k+4}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - a_k - 3a_{k+1}}{k^2+4k+4}$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - a_k - 3a_{k+1}}{k^2+4k+4}, a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x+1$

  $$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 4k a_{k+1} - a_k - 3a_{k+1}}{k^2+4k+4}, a_1 - a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+diff(y(x),x)/(1+x)-(2+x)/(x^2*(1+x))*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2\ln(1+x)\, c_2 + (x^2-2x)\, c_2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 34

```
DSolve[y''[x]+y'[x]/(1+x)-(2+x)/(x^2*(1+x))*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2(x^2-2x+2\log(x+1)-3)+2c_1}{2x}$$

## 2.57 problem Problem 20(d)



Internal problem ID [12278]
Internal file name [`OUTPUT/10930_Thursday_September_28_2023_01_09_14_AM_3909741/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2-x)\,y''+(2x^2+4x-3)\,y'+8yx=0}$$

### 2.57.1 Solving using Kovacic algorithm

Writing the ode as

$$(x^2-x)\,y''+(2x^2+4x-3)\,y'+8yx=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A&=x^2-x\\ B&=2x^2+4x-3\\ C&=8x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{4x^4 - 16x^3 + 24x^2 - 12x + 3}{4\left(x^2 - x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 - 16x^3 + 24x^2 - 12x + 3 \\ t &= 4\left(x^2 - x\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 - 16x^3 + 24x^2 - 12x + 3}{4\left(x^2 - x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 59: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = 1 + \frac{3}{4x^2} - \frac{1}{2(x-1)} + \frac{3}{4(x-1)^2} - \frac{3}{2x}$$

For the pole at $x = 0$ let $b$ be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let $b$ be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of $r$ at $\infty$ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{0} a_i x^i \end{aligned} \tag{8}$$

Let $a$ be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$\sqrt{r} \approx 1 - \frac{1}{x} + \frac{1}{2x^3} + \frac{11}{8x^4} + \frac{21}{8x^5} + \frac{33}{8x^6} + \frac{87}{16x^7} + \frac{711}{128x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= 1 \end{aligned} \tag{10}$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = 1$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in $r$. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of $\frac{1}{x}$ in $r$ will be the coefficient in $R$ of the term in $x$ of degree of $t$ minus one, divided by the leading coefficient in $t$. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^4 - 16x^3 + 24x^2 - 12x + 3}{4x^4 - 8x^3 + 4x^2} \\
&= Q + \frac{R}{4x^4 - 8x^3 + 4x^2} \\
&= (1) + \left(\frac{-8x^3 + 20x^2 - 12x + 3}{4x^4 - 8x^3 + 4x^2}\right) \\
&= 1 + \frac{-8x^3 + 20x^2 - 12x + 3}{4x^4 - 8x^3 + 4x^2}
\end{aligned}
$$

Since the degree of $t$ is 4, then we see that the coefficient of the term $x^3$ in the remainder $R$ is $-8$. Dividing this by leading coefficient in $t$ which is 4 gives $-2$. Now $b$ can be found.

$$
\begin{aligned}
b &= (-2) - (0) \\
&= -2
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 1 \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-2}{1} - 0\right) = -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-2}{1} - 0\right) = 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$
r = \frac{4x^4 - 16x^3 + 24x^2 - 12x + 3}{4\left(x^2 - x\right)^2}
$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^{+}$ | $\alpha_c^{-}$ |
|---|---|---|---|---|
| 0 | 2 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ |
| 1 | 2 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^{+}$ | $\alpha_\infty^{-}$ |
|---|---|---|---|
| 0 | 1 | $-1$ | 1 |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c \in \Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left( (-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + \left( (-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x - c_2} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} - \frac{1}{2(x-1)} + (1) \\ &= -\frac{1}{2x} - \frac{1}{2(x-1)} + 1 \\ &= -\frac{1}{2x} - \frac{1}{2x-2} + 1 \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2(x-1)} + 1\right)(0) + \left( \left( \frac{1}{2x^2} + \frac{1}{2(x-1)^2} \right) + \left( -\frac{1}{2x} - \frac{1}{2(x-1)} + 1 \right)^2 - \left( \frac{4x^4 - 16x^3}{4(} \right. \right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-1)} + 1\right) dx} \\
&= \frac{\mathrm{e}^{x}}{\sqrt{x}\,\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^2+4x-3}{x^2-x} \, dx} \\
&= z_1 e^{-x - \frac{3\ln(x-1)}{2} - \frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-x}}{(x-1)^{\frac{3}{2}}\, x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^{\frac{3}{2}} (x-1)^{\frac{3}{2}} \sqrt{x(x-1)}}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2+4x-3}{x^2-x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-2x-3\ln(x-1)-3\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{x^2 \mathrm{e}^{-2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^{\frac{3}{2}} (x-1)^{\frac{3}{2}} \sqrt{x (x-1)}}\right) + c_2 \left(\frac{1}{x^{\frac{3}{2}} (x-1)^{\frac{3}{2}} \sqrt{x (x-1)}} \left(-\frac{x^2 \mathrm{e}^{-2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^{\frac{3}{2}} (x-1)^{\frac{3}{2}} \sqrt{x (x-1)}} - \frac{c_2 \sqrt{x}\, \mathrm{e}^{-2x}}{2 (x-1)^{\frac{3}{2}} \sqrt{x (x-1)}} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^{\frac{3}{2}} (x-1)^{\frac{3}{2}} \sqrt{x (x-1)}} - \frac{c_2 \sqrt{x}\, \mathrm{e}^{-2x}}{2 (x-1)^{\frac{3}{2}} \sqrt{x (x-1)}}
$$

Verified OK.

### 2.57.2 Maple step by step solution

Let's solve

$(x^2 - x) y'' + (2x^2 + 4x - 3) y' + 8yx = 0$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{8y}{x-1} - \frac{(2x^2+4x-3)y'}{x(x-1)}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(2x^2+4x-3)y'}{x(x-1)} + \frac{8y}{x-1} = 0$

□ Check to see if $x_0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{2x^2+4x-3}{x(x-1)}, P_3(x) = \frac{8}{x-1}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 3$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x(x-1) + (2x^2 + 4x - 3)\, y' + 8yx = 0$

- Assume series solution for $y$

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(2+r)\, x^{-1+r} + \left(-a_1(1+r)\,(3+r) + a_0 r(3+r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(-a_{k+1}(k+r+1)\,(k+r+3)\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$-r(2+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-2,0\}$

- Each term must be 0

  $-a_1(1+r)(3+r)+a_0r(3+r)=0$

- Each term in the series must be 0, giving the recursion relation

  $(k+r+3)\left((-k-r-1)a_{k+1}+ka_k+ra_k+2a_{k-1}\right)=0$

- Shift index using $k->k+1$

  $(k+r+4)\left((-k-r-2)a_{k+2}+(k+1)a_{k+1}+ra_{k+1}+2a_k\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{ka_{k+1}+ra_{k+1}+2a_k+a_{k+1}}{k+r+2}$

- Recursion relation for $r=-2$

  $a_{k+2}=\frac{ka_{k+1}+2a_k-a_{k+1}}{k}$

- Series not valid for $r=-2$ , division by 0 in the recursion relation at $k=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_k-a_{k+1}}{k}$

- Recursion relation for $r=0$

  $a_{k+2}=\frac{ka_{k+1}+2a_k+a_{k+1}}{k+2}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty}a_kx^k, a_{k+2}=\frac{ka_{k+1}+2a_k+a_{k+1}}{k+2}, -3a_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 24

```
dsolve((x^2-x)*diff(y(x),x$2)+(2*x^2+4*x-3)*diff(y(x),x)+8*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 e^{-2x} x^2 + c_1}{x^2 (-1+x)^2}$$

✓ Solution by Mathematica
Time used: 0.148 (sec). Leaf size: 30

```
DSolve[(x^2-x)*y''[x]+(2*x^2+4*x-3)*y'[x]+8*x*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{\frac{2c_1}{x^2} + c_2 e^{-2x}}{2(x-1)^2}$$

## 2.58 problem Problem 20(e)



Internal problem ID [12279]
Internal file name [OUTPUT/10931_Thursday_September_28_2023_01_09_15_AM_12253562/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{\frac{(x^2 - x)\, y''}{x} + \frac{(3x + 1)\, y'}{x} + \frac{y}{x} = 3x}$$

### 2.58.1 Solving using Kovacic algorithm

Writing the ode as

$$(x - 1)\, y'' + \left(3 + \frac{1}{x}\right) y' + \frac{y}{x} = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x - 1 \\ B &= 3 + \frac{1}{x} \\ C &= \frac{1}{x} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{-x^2 + 6x + 3}{4(x^2 - x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -x^2 + 6x + 3 \\ t &= 4(x^2 - x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-x^2 + 6x + 3}{4(x^2 - x)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 61: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$r = \frac{3}{4x^2} - \frac{3}{x-1} + \frac{2}{(x-1)^2} + \frac{3}{x}$$

For the pole at $x = 0$ let $b$ be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 1$ let $b$ be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of $r$ at $\infty$ is 2 then $[\sqrt{r}]_\infty = 0$. Let $b$ be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of $r$ at $\infty$. which can be found by dividing the leading coefficient of $s$ by the leading coefficient of $t$ from

$$r = \frac{s}{t} = \frac{-x^2 + 6x + 3}{4(x^2 - x)^2}$$

Since the gcd$(s,t)=1$. This gives $b=-\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$
r = \frac{-x^2+6x+3}{4\left(x^2-x\right)^2}
$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^{+}$ | $\alpha_c^{-}$ |
|---|---|---|---|---|
| 0 | 2 | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ |
| 1 | 2 | 0 | 2 | $-1$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^{+}$ | $\alpha_\infty^{-}$ |
|---|---|---|---|
| 2 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^{-}=\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{+}+\alpha_{c_2}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^-}{x-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{3}{2x} - \frac{1}{x-1} + (-)\,(0) \\
&= \frac{3}{2x} - \frac{1}{x-1} \\
&= \frac{x-3}{2x\,(x-1)}
\end{aligned}
$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{3}{2x} - \frac{1}{x-1}\right)(0) + \left(\left(-\frac{3}{2x^2} + \frac{1}{(x-1)^2}\right) + \left(\frac{3}{2x} - \frac{1}{x-1}\right)^2 - \left(\frac{-x^2+6x+3}{4\,(x^2-x)^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{3}{2x} - \frac{1}{x-1}\right) dx} \\
&= \frac{x^{\frac{3}{2}}}{x-1}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3+\frac{1}{x}}{x-1}\, dx} \\
&= z_1 e^{-2\ln(x-1) + \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\sqrt{x}}{(x-1)^2}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \frac{x^2}{(x-1)^3}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3+\frac{1}{x}}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x-1)+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{1}{2x^2} + \frac{2}{x} + \ln(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^2}{(x-1)^3}\right) + c_2 \left(\frac{x^2}{(x-1)^3} \left(-\frac{1}{2x^2} + \frac{2}{x} + \ln(x)\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$(x-1)\, y'' + \left(3 + \frac{1}{x}\right) y' + \frac{y}{x} = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 x^2}{(x-1)^3} + \frac{c_2(2x^2 \ln(x) + 4x - 1)}{2(x-1)^3}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{x^2}{(x-1)^3}$$

$$y_2 = \frac{2x^2 \ln(x) + 4x - 1}{2(x-1)^3}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{x^2}{(x-1)^3} & \frac{2x^2 \ln(x)+4x-1}{2(x-1)^3} \\ \frac{d}{dx}\left(\frac{x^2}{(x-1)^3}\right) & \frac{d}{dx}\left(\frac{2x^2 \ln(x)+4x-1}{2(x-1)^3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{x^2}{(x-1)^3} & \frac{2x^2 \ln(x)+4x-1}{2(x-1)^3} \\ \frac{2x}{(x-1)^3} - \frac{3x^2}{(x-1)^4} & \frac{4x \ln(x)+2x+4}{2(x-1)^3} - \frac{3(2x^2 \ln(x)+4x-1)}{2(x-1)^4} \end{vmatrix}$$

Therefore

$$W = \left(\frac{x^2}{(x-1)^3}\right)\left(\frac{4x\ln(x) + 2x + 4}{2(x-1)^3} - \frac{3(2x^2\ln(x) + 4x - 1)}{2(x-1)^4}\right) - \left(\frac{2x^2\ln(x) + 4x - 1}{2(x-1)^3}\right)\left(\frac{2x}{(x-1)^3} - \frac{3x^2}{(x-1)^4}\right)$$

Which simplifies to

$$W = \frac{x}{(x-1)^4}$$

Which simplifies to

$$W = \frac{x}{(x-1)^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3(2x^2 \ln(x)+4x-1)x}{2(x-1)^3}}{\frac{x}{(x-1)^3}}\, dx$$

Which simplifies to

$$u_1 = -\int \left(3x^2 \ln(x) + 6x - \frac{3}{2}\right) dx$$

Hence

$$u_1 = -3x^2 + \frac{3x}{2} - x^3 \ln(x) + \frac{x^3}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{3x^3}{(x-1)^3}}{\frac{x}{(x-1)^3}}\, dx$$

Which simplifies to

$$u_2 = \int 3x^2 dx$$

Hence

$$u_2 = x^3$$

Which simplifies to

$$u_1 = -x^3 \ln(x) + \frac{\left(x^2 - 9x + \frac{9}{2}\right) x}{3}$$

$$u_2 = x^3$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(-x^3 \ln(x) + \frac{\left(x^2-9x+\frac{9}{2}\right)x}{3}\right)x^2}{(x-1)^3} + \frac{x^3(2x^2\ln(x)+4x-1)}{2(x-1)^3}$$

Which simplifies to

$$y_p(x) = \frac{x^3(x^2-3x+3)}{3(x-1)^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^2}{(x-1)^3} + \frac{c_2(2x^2\ln(x)+4x-1)}{2(x-1)^3}\right) + \left(\frac{x^3(x^2-3x+3)}{3(x-1)^3}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1 x^2 + \frac{c_2(2x^2\ln(x)+4x-1)}{2}}{(x-1)^3} + \frac{x^3(x^2-3x+3)}{3(x-1)^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + \frac{c_2(2x^2\ln(x)+4x-1)}{2}}{(x-1)^3} + \frac{x^3(x^2-3x+3)}{3(x-1)^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + \frac{c_2(2x^2\ln(x)+4x-1)}{2}}{(x-1)^3} + \frac{x^3(x^2-3x+3)}{3(x-1)^3}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve((x^2-x)/x*diff(y(x),x$2)+(3*x+1)/x*diff(y(x),x)+y(x)/x=3*x,y(x), singsol=all)
```

$$y(x) = \frac{(2\ln(x)\,x^2 + 4x - 1)\,c_2 + c_1x^2 + \frac{x^3(x^2-3x+3)}{3}}{(-1+x)^3}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 55

```
DSolve[(x^2-x)/x*y''[x]+(3*x+1)/x*y'[x]+y[x]/x==3*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^5 - 6x^4 + 6x^3 - 6c_1x^2 - 6c_2x^2\log(x) - 12c_2x + 3c_2}{6(x-1)^3}$$

## 2.59 problem Problem 20(f)

Internal problem ID [12280]
Internal file name [`OUTPUT/10932_Thursday_September_28_2023_01_09_15_AM_7130705/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

Unable to solve or complete the solution.

$$(-\cos(x) + 2\sin(x))\, y'' + (7\sin(x) + 4\cos(x))\, y' + 10y\cos(x) = 0$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      -10*t*u(t)+(-8*t^2+7+6*(-t^2+1)^(1/2)*t)*diff(u(t),t)+(2*(-t^2+1)^(1/2)*t^2-t^3-2*(-t^
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 76

```
dsolve((2*sin(x)-cos(x))*diff(y(x),x$2)+(7*sin(x)+4*cos(x))*diff(y(x),x)+10*y(x)*cos(x)=0,y(
```

$$y(x) = -\mathrm{e}^{-\left(\int \frac{5\cos(x)\cot(x)-6\csc(x)}{-2\sin(x)+\cos(x)}dx\right)}\left(c_2\left(\int \frac{\csc(x)\,\mathrm{e}^{\int \frac{5\cos(x)\cot(x)-6\csc(x)}{-2\sin(x)+\cos(x)}dx}}{-2\sin(x)+\cos(x)}dx\right)-c_1\right)$$

✓ Solution by Mathematica

Time used: 3.823 (sec). Leaf size: 112

```
DSolve[(2*Sin[x]-Cos[x])*y''[x]+(7*Sin[x]+4*Cos[x])*y'[x]+10*y[x]*Cos[x]==0,y[x],x,IncludeSi
```

$$y(x) \to \frac{e^{2ix}\left(c_2\int_1^{e^{ix}} \frac{e^{3i\arctan\left(\frac{2-2K[1]^2}{K[1]^2+1}\right)}K[1]^{-2+2i}\left((1+2i)K[1]^2+(1-2i)\right)^4}{(5K[1]^4-6K[1]^2+5)^{3/2}}dK[1]+c_1\right)}{\left((1+2i)e^{2ix}+(1-2i)\right)^2}$$

## 2.60 problem Problem 20(g)

Internal problem ID [12281]
Internal file name [OUTPUT/10933_Thursday_September_28_2023_01_09_18_AM_44896323/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(g).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + \frac{(x-1)\,y'}{x} + \frac{y}{x^3} = \frac{e^{-\frac{1}{x}}}{x^3}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
```

✗ Solution by Maple

```
dsolve(diff(y(x),x$2)+(x-1)/x*diff(y(x),x)+y(x)/x^3=1/x^3*exp(-1/x),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[x]+(x-1)/x*y'[x]+y[x]/x^3==1/x^3*Exp[-1/x],y[x],x,IncludeSingularSolutions -> Tru
```

Not solved

## 2.61 problem Problem 20(h)



Internal problem ID [12282]
Internal file name [OUTPUT/10934_Thursday_September_28_2023_01_09_18_AM_43053175/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 4, Second and Higher Order Linear Differential Equations. Problems page 221
**Problem number**: Problem 20(h).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + (2x+5)\,y' + (4x+8)\,y = \mathrm{e}^{-2x}}$$

### 2.61.1 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (2x+5)\,y' + (4x+8)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x+5 \\ C &= 4x+8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where $r$ is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$
r = \frac{4x^2 + 4x - 3}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4x^2 + 4x - 3 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{3}{4} + x^2 + x\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then $y$ is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 62: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $-2$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of $r$ at $\infty$ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving $x^i$ for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at $\infty$. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let $a$ be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at $\infty$ is

$$\sqrt{r} \approx x + \frac{1}{2} - \frac{1}{2x} + \frac{1}{4x^2} - \frac{1}{4x^3} + \frac{1}{4x^4} - \frac{9}{32x^5} + \frac{21}{64x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x + \frac{1}{2} \qquad (10)
\end{aligned}$$

Now we need to find $b$, where $b$ be the coefficient of $x^{v-1} = x^0 = 1$ in $r$ minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2 + x + \frac{1}{4}$$

This shows that the coefficient of 1 in the above is $\frac{1}{4}$. Now we need to find the coefficient of 1 in $r$. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where $Q$ is the quotient and $R$ is the remainder. Then the coefficient of 1 in $r$ will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^2 + 4x - 3}{4} \\
&= Q + \frac{R}{4} \\
&= \left(-\frac{3}{4} + x^2 + x\right) + (0) \\
&= -\frac{3}{4} + x^2 + x
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{3}{4}$. Now $b$ can be found.

$$\begin{aligned}
b &= \left(-\frac{3}{4}\right) - \left(\frac{1}{4}\right) \\
&= -1
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= x + \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{1} - 1\right) &= -1 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{1} - 1\right) &= 0
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$r = -\frac{3}{4} + x^2 + x$$

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^{+}$ | $\alpha_\infty^{-}$ |
|---|---|---|---|
| $-2$ | $x+\frac{1}{2}$ | $-1$ | $0$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such $d$ is found to work in finding candidate $\omega$. Trying $\alpha_\infty^{-} = 0$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^{-} \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c\in\Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(x+\frac{1}{2}\right) \\ &= -\frac{1}{2} - x \\ &= -\frac{1}{2} - x \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2} - x\right)(0) + \left((-1) + \left(-\frac{1}{2} - x\right)^2 - \left(-\frac{3}{4} + x^2 + x\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int (-\frac{1}{2}-x) dx} \\
&= \mathrm{e}^{-\frac{x(x+1)}{2}}
\end{aligned}
$$

The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x+5}{1} \, dx} \\
&= z_1 e^{-\frac{1}{2}x^2 - \frac{5}{2}x} \\
&= z_1 \left(\mathrm{e}^{-\frac{x(x+5)}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x(x+3)}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x+5}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x^2-5x}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{i\sqrt{\pi}\, \mathrm{e}^{-\frac{1}{4}} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x(x+3)}\right) + c_2 \left(\mathrm{e}^{-x(x+3)} \left(-\frac{i\sqrt{\pi}\, \mathrm{e}^{-\frac{1}{4}} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. $y_h$ is the solution to

$$y'' + (2x + 5)\, y' + (4x + 8)\, y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x(x+3)} - \frac{i c_2 \sqrt{\pi}\, \mathrm{e}^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x + \frac{1}{2}\right)\right)}{2}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $x$ as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-x(x+3)} \\ y_2 &= -\frac{i\sqrt{\pi}\, \mathrm{e}^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x + \frac{1}{2}\right)\right)}{2} \end{aligned}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x(x+3)} & -\frac{i\sqrt{\pi}\, e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} \\ \frac{d}{dx}\left(e^{-x(x+3)}\right) & \frac{d}{dx}\left(-\frac{i\sqrt{\pi}\, e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x(x+3)} & -\frac{i\sqrt{\pi}\, e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} \\ (-2x-3)\, e^{-x(x+3)} & -\frac{i\sqrt{\pi}\, (-2x-3) e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} + e^{-\frac{1}{4}-x^2-3x} e^{\left(x+\frac{1}{2}\right)^2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x(x+3)}\right)\left(-\frac{i\sqrt{\pi}\, (-2x-3)\, e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} + e^{-\frac{1}{4}-x^2-3x} e^{\left(x+\frac{1}{2}\right)^2}\right) - \left(-\frac{i\sqrt{\pi}\, e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2}\right)\left((-2x-3)\, e^{-x(x+3)}\right)$$

Which simplifies to

$$W = e^{-x(x+3)} e^{-\frac{1}{4}-x^2-3x} e^{\frac{(2x+1)^2}{4}}$$

Which simplifies to

$$W = e^{-x(x+5)}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i\sqrt{\pi}\, e^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right) e^{-2x}}{2}}{e^{-x(x+5)}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i\sqrt{\pi}\, e^{-\frac{1}{4}} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} dx$$

Hence

$$u_1 = \frac{i\left(x+\frac{1}{2}\right)\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}}\,\mathrm{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}}\,\mathrm{erf}\left(\frac{i}{2}\right)}{4} + \frac{\mathrm{e}^{x(x+1)}}{2} - \frac{1}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-x(x+3)}\mathrm{e}^{-2x}}{\mathrm{e}^{-x(x+5)}}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(\frac{i\left(x+\frac{1}{2}\right)\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}}\,\mathrm{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}}\,\mathrm{erf}\left(\frac{i}{2}\right)}{4} + \frac{\mathrm{e}^{x(x+1)}}{2} - \frac{1}{2}\right)\mathrm{e}^{-x(x+3)} \\ & - \frac{ix\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}-x^2-3x}\,\mathrm{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{\sqrt{\pi}\left(i\,\mathrm{erf}\left(i\left(x+\frac{1}{2}\right)\right) + \mathrm{erfi}\left(\frac{1}{2}\right)\right)\mathrm{e}^{-\frac{1}{4}-x^2-3x}}{4} + \frac{\mathrm{e}^{-2x}}{2} - \frac{\mathrm{e}^{-x(x+3)}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x(x+3)} - \frac{ic_2\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}-x^2-3x}\,\mathrm{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2}\right) \\ &\quad + \left(\frac{\sqrt{\pi}\left(i\,\mathrm{erf}\left(i\left(x+\frac{1}{2}\right)\right) + \mathrm{erfi}\left(\frac{1}{2}\right)\right)\mathrm{e}^{-\frac{1}{4}-x^2-3x}}{4} + \frac{\mathrm{e}^{-2x}}{2} - \frac{\mathrm{e}^{-x(x+3)}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \mathrm{e}^{-x(x+3)} - \frac{i c_2 \sqrt{\pi}\, \mathrm{e}^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} \\
+ \frac{\sqrt{\pi}\left(i \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)+\operatorname{erfi}\left(\frac{1}{2}\right)\right) \mathrm{e}^{-\frac{1}{4}-x^2-3x}}{4} + \frac{\mathrm{e}^{-2x}}{2} - \frac{\mathrm{e}^{-x(x+3)}}{2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 \mathrm{e}^{-x(x+3)} - \frac{i c_2 \sqrt{\pi}\, \mathrm{e}^{-\frac{1}{4}-x^2-3x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)}{2} \\
+ \frac{\sqrt{\pi}\left(i \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right)+\operatorname{erfi}\left(\frac{1}{2}\right)\right) \mathrm{e}^{-\frac{1}{4}-x^2-3x}}{4} + \frac{\mathrm{e}^{-2x}}{2} - \frac{\mathrm{e}^{-x(x+3)}}{2}
\end{aligned}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+(2*x+5)*diff(y(x),x)+(4*x+8)*y(x)=exp(-2*x),y(x), singsol=all)
```

$$y(x) = e^{-(x+3)x} c_2 + e^{-(x+3)x} \operatorname{erf}\left(i\left(x+\frac{1}{2}\right)\right) c_1 + \frac{e^{-2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.333 (sec). Leaf size: 61

```
DSolve[y''[x]+(2*x+5)*y'[x]+(4*x+8)*y[x]==Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-x(x+3)-\frac{1}{4}} \left(\sqrt{\pi}(-1+2c_2)\operatorname{erfi}\left(x+\frac{1}{2}\right) + 2\left(e^{\left(x+\frac{1}{2}\right)^2} + 2\sqrt[4]{e} c_1\right)\right)$$

# 3 Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357

## 3.1 problem Problem 2



Internal problem ID [12283]
Internal file name [OUTPUT/10935_Thursday_September_28_2023_01_09_19_AM_97869762/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 2.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 9y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

### 3.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 9y = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2s + 9Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s}{s^2 + 9}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s - 3i} + \frac{1}{s + 3i}$$

The inverse Laplace of each term above is now found, which gives

$$
\mathcal{L}^{-1}\left(\frac{1}{s-3i}\right) = e^{3it}
$$
$$
\mathcal{L}^{-1}\left(\frac{1}{s+3i}\right) = e^{-3it}
$$

Adding the above results and simplifying gives

$$
y = 2\cos(3t)
$$

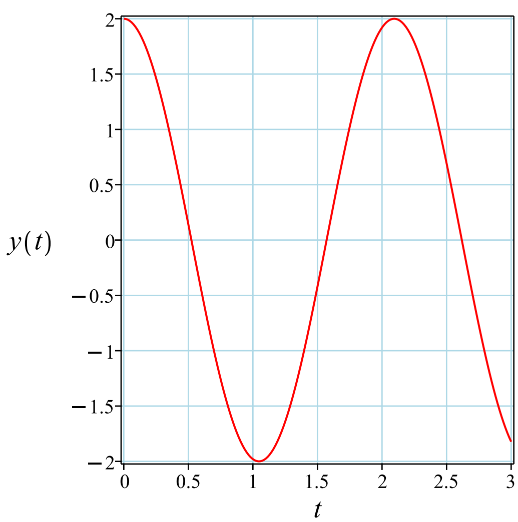
Simplifying the solution gives

$$
y = 2\cos(3t)
$$

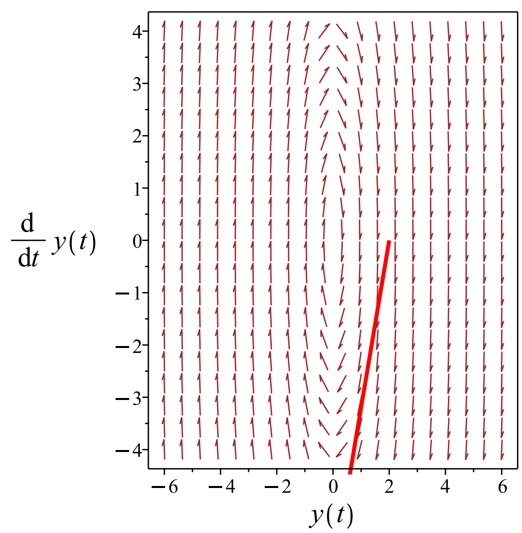
Summary

The solution(s) found are the following

$$
y = 2\cos(3t) \tag{1}
$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$
y = 2\cos(3t)
$$

Verified OK.

### 3.1.2 Maple step by step solution

Let's solve

$\left[y''+9y=0, y(0)=2, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2+9=0$

- Use quadratic formula to solve for $r$

  $r=\frac{0\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r=(-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t)=\cos(3t)$

- 2nd solution of the ODE

  $y_2(t)=\sin(3t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\cos(3t)+c_2\sin(3t)$

□ Check validity of solution $y=c_1\cos(3t)+c_2\sin(3t)$

  - Use initial condition $y(0)=2$

    $2=c_1$

  - Compute derivative of the solution

    $y'=-3c_1\sin(3t)+3c_2\cos(3t)$

  - Use the initial condition $y'\Big|_{\{t=0\}}=0$

    $0=3c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1=2, c_2=0\}$

- Substitute constant values into general solution and simplify

  $y = 2\cos(3t)$

- Solution to the IVP

  $y = 2\cos(3t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.469 (sec). Leaf size: 10

```
dsolve([diff(y(t),t$2)+9*y(t)=0,y(0) = 2, D(y)(0) = 0],y(t), singsol=all)
```

$$y(t) = 2\cos(3t)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 11

```
DSolve[{y''[t]+9*y[t]==0,{y[0]==2,y'[0]==0}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to 2\cos(3t)$$

## 3.2 problem Problem 3



Internal problem ID [12284]
Internal file name [`OUTPUT/10936_Thursday_September_28_2023_01_09_19_AM_72642257/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 3.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 4y' + 5y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 3]$$

### 3.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \frac{5}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' + \frac{5y}{4} = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{5}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) - 4sY(s) + 4y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) - 4 - 8s - 4sY(s) + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{8s + 4}{4s^2 - 4s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1 - i}{s - \frac{1}{2} - i} + \frac{1 + i}{s - \frac{1}{2} + i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1-i}{s-\frac{1}{2}-i}\right) = (1-i)\,e^{\left(\frac{1}{2}+i\right)t}$$
$$\mathcal{L}^{-1}\left(\frac{1+i}{s-\frac{1}{2}+i}\right) = (1+i)\,e^{\left(\frac{1}{2}-i\right)t}$$

Adding the above results and simplifying gives

$$y = 2\,e^{\frac{t}{2}}(\cos(t)+\sin(t))$$

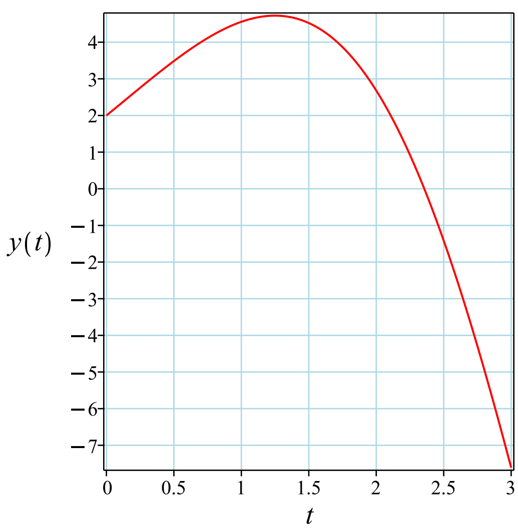
Simplifying the solution gives

$$y = 2\,e^{\frac{t}{2}}(\cos(t)+\sin(t))$$

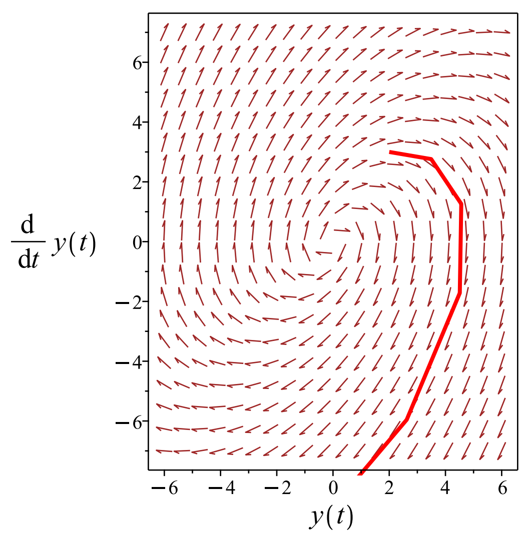
Summary

The solution(s) found are the following

$$y = 2\,e^{\frac{t}{2}}(\cos(t)+\sin(t)) \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = 2\,e^{\frac{t}{2}}(\cos(t)+\sin(t))$$

Verified OK.

### 3.2.2 Maple step by step solution

Let's solve

$$\left[4y''-4y'+5y=0, y(0)=2, y'\Big|_{\{t=0\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=y'-\frac{5y}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-y'+\frac{5y}{4}=0$

- Characteristic polynomial of ODE

  $r^2-r+\frac{5}{4}=0$

- Use quadratic formula to solve for $r$

  $r=\frac{1\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=\left(\frac{1}{2}-\mathrm{I}, \frac{1}{2}+\mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{\frac{t}{2}}\cos(t)$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{\frac{t}{2}}\sin(t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{\frac{t}{2}}\cos(t)+c_2\mathrm{e}^{\frac{t}{2}}\sin(t)$

☐ Check validity of solution $y=c_1\mathrm{e}^{\frac{t}{2}}\cos(t)+c_2\mathrm{e}^{\frac{t}{2}}\sin(t)$

  - Use initial condition $y(0)=2$

    $2=c_1$

  - Compute derivative of the solution

    $y'=\frac{c_1\mathrm{e}^{\frac{t}{2}}\cos(t)}{2}-c_1\mathrm{e}^{\frac{t}{2}}\sin(t)+\frac{c_2\mathrm{e}^{\frac{t}{2}}\sin(t)}{2}+c_2\mathrm{e}^{\frac{t}{2}}\cos(t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 3$

  $3 = \frac{c_1}{2} + c_2$
- Solve for $c_1$ and $c_2$

  $\{c_1 = 2, c_2 = 2\}$
- Substitute constant values into general solution and simplify

  $y = 2\,\mathrm{e}^{\frac{t}{2}}(\cos(t) + \sin(t))$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{\frac{t}{2}}(\cos(t) + \sin(t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.89 (sec). Leaf size: 15

```
dsolve([4*diff(y(t),t$2)-4*diff(y(t),t)+5*y(t)=0,y(0) = 2, D(y)(0) = 3],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{\frac{t}{2}}(\cos(t) + \sin(t))$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 19

```
DSolve[{4*y''[t]-4*y'[t]+5*y[t]==0,{y[0]==2,y'[0]==3}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to 2e^{t/2}(\sin(t) + \cos(t))$$

## 3.3 problem Problem 4



Internal problem ID [12285]
Internal file name [OUTPUT/10937_Saturday_September_30_2023_08_26_31_PM_64152595/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 4.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 2]$$

### 3.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -1\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + s + 2sY(s) + Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s}{s^2 + 2s + 1}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s+1} + \frac{1}{(s+1)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s+1}\right) = -\mathrm{e}^{-t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = t\,\mathrm{e}^{-t}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-t}(t-1)$$

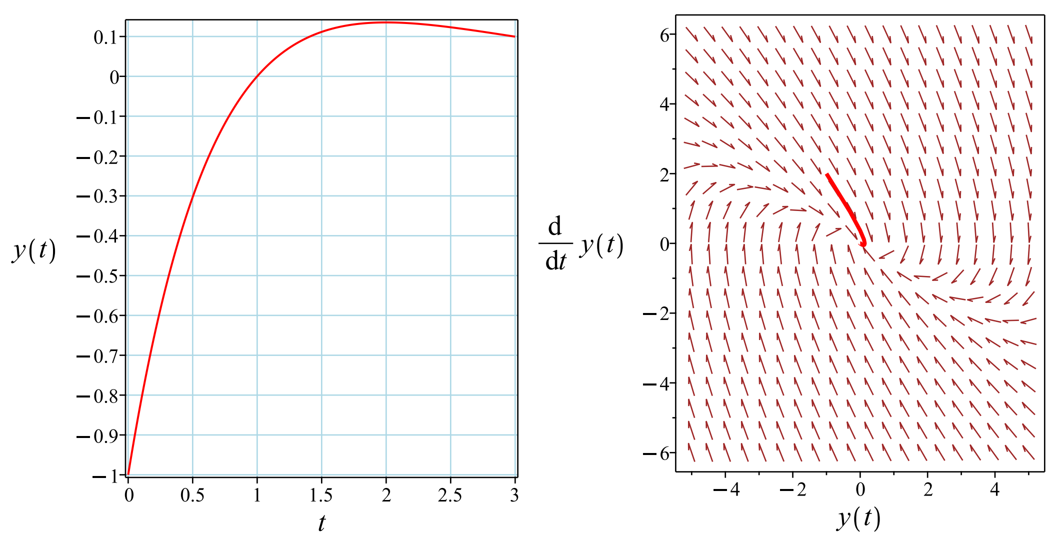
Simplifying the solution gives

$$y = \mathrm{e}^{-t}(t-1)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{-t}(t-1) \tag{1}$$




(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-t}(t-1)$$

Verified OK.

### 3.3.2 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + y = 0, y(0) = -1, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t) = t\,e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-t} + c_2 t\,e^{-t}$

☐ Check validity of solution $y = c_1 e^{-t} + c_2 t e^{-t}$

  - Use initial condition $y(0) = -1$

    $-1 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\,e^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 2$

    $2 = -c_1 + c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1 = -1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{-t}(t-1)$

- Solution to the IVP

  $y = \mathrm{e}^{-t}(t-1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.469 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=0,y(0) = -1, D(y)(0) = 2],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-t}(t-1)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 14

```
DSolve[{y''[t]+2*y'[t]+y[t]==0,{y[0]==-1,y'[0]==2}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \rightarrow e^{-t}(t-1)$$

## 3.4 problem Problem 5



Internal problem ID [12286]
Internal file name [`OUTPUT/10938_Saturday_September_30_2023_08_26_32_PM_31591721/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 5.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 5y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

### 3.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 5y = 0$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - 4sY(s) + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3}{s^2 - 4s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{3i}{2(s-2-i)} + \frac{3i}{2(s-2+i)}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{3i}{2(s-2-i)}\right) &= -\frac{3ie^{(2+i)t}}{2}\\ \mathcal{L}^{-1}\left(\frac{3i}{2(s-2+i)}\right) &= \frac{3ie^{(2-i)t}}{2}\end{aligned}$$

Adding the above results and simplifying gives

$$y = 3\,\mathrm{e}^{2t} \sin(t)$$

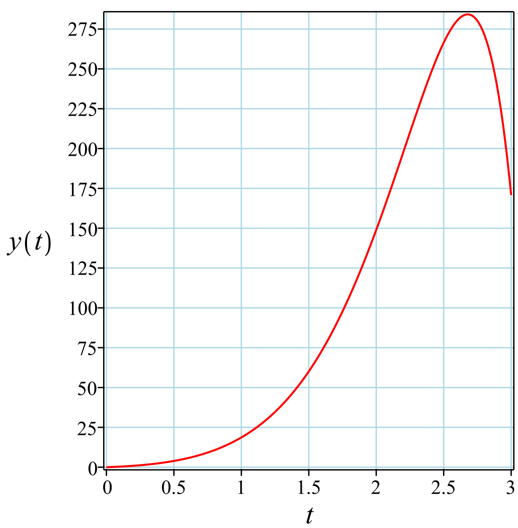
Simplifying the solution gives

$$y = 3\,\mathrm{e}^{2t} \sin(t)$$

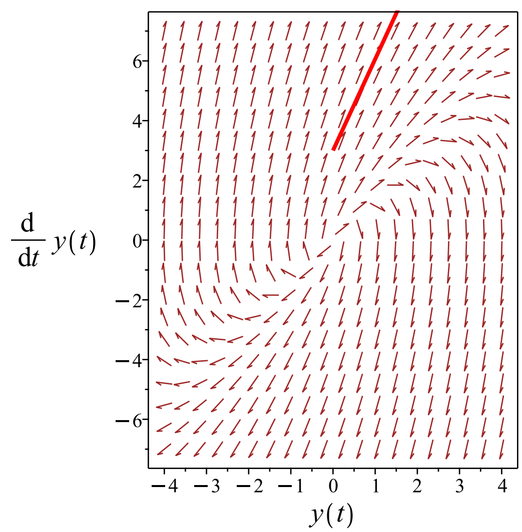
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{2t} \sin(t) \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{2t} \sin(t)$$

Verified OK.

### 3.4.2 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 5y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 - 4r + 5 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = e^{2t} \cos(t)$

- 2nd solution of the ODE

  $y_2(t) = e^{2t} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{2t} \cos(t) + c_2 e^{2t} \sin(t)$

☐ Check validity of solution $y = c_1 e^{2t} \cos(t) + c_2 e^{2t} \sin(t)$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = 2c_1 e^{2t} \cos(t) - c_1 e^{2t} \sin(t) + 2c_2 e^{2t} \sin(t) + c_2 e^{2t} \cos(t)$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 3$

    $3 = 2c_1 + c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1 = 0, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = 3\,\mathrm{e}^{2t} \sin(t)$

- Solution to the IVP

  $y = 3\,\mathrm{e}^{2t} \sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.485 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+5*y(t)=0,y(0) = 0, D(y)(0) = 3],y(t), singsol=all)
```

$$y(t) = 3\,\mathrm{e}^{2t} \sin(t)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 14

```
DSolve[{y''[t]-4*y'[t]+5*y[t]==0,{y[0]==0,y'[0]==3}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to 3e^{2t} \sin(t)$$

## 3.5 problem Problem 6



Internal problem ID [12287]
Internal file name [`OUTPUT/10939_Saturday_September_30_2023_08_26_33_PM_12105076/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 6.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 6y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 1]$$

### 3.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = -1$$
$$q(t) = -6$$
$$F = 0$$

Hence the ode is

$$y'' - y' - 6y = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 6Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - 2s - sY(s) - 6Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s-1}{s^2-s-6}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-3} + \frac{1}{s+2}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) &= e^{3t}\\ \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) &= e^{-2t}\end{aligned}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-2t} + \mathrm{e}^{3t}$$

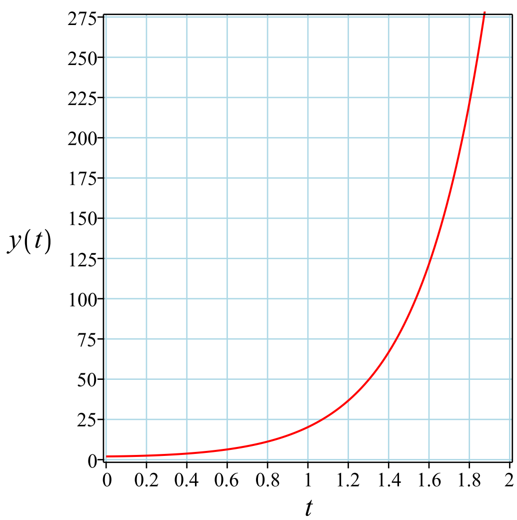
Simplifying the solution gives

$$y = \left(\mathrm{e}^{5t} + 1\right) \mathrm{e}^{-2t}$$

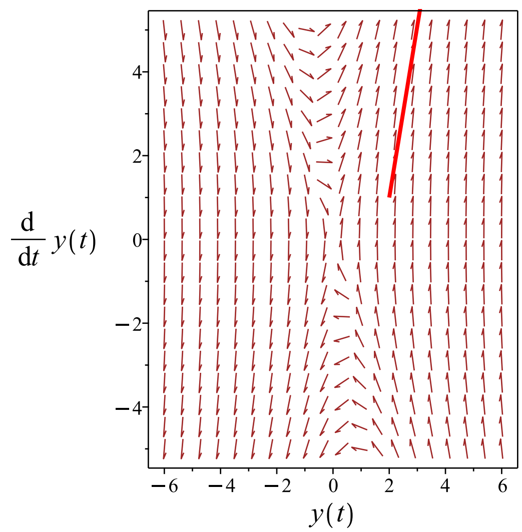
Summary

The solution(s) found are the following

$$y = \left(\mathrm{e}^{5t} + 1\right) \mathrm{e}^{-2t} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = \left(\mathrm{e}^{5t} + 1\right) \mathrm{e}^{-2t}$$

Verified OK.

### 3.5.2 Maple step by step solution

Let's solve

$$\left[y'' - y' - 6y = 0, y(0) = 2, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{-2t} + c_2 e^{3t}$

□ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{3t}$

  - Use initial condition $y(0) = 2$

    $2 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -2c_1 e^{-2t} + 3c_2 e^{3t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -2c_1 + 3c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1 = 1, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = (e^{5t} + 1) e^{-2t}$

- Solution to the IVP

  $y = (e^{5t} + 1) e^{-2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.562 (sec). Leaf size: 13

```
dsolve([diff(y(t),t$2)-diff(y(t),t)-6*y(t)=0,y(0) = 2, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \left(e^{5t} + 1\right) e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 16

```
DSolve[{y''[t]-y'[t]-6*y[t]==0,{y[0]==2,y'[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-2t} + e^{3t}$$

## 3.6 problem Problem 7



Internal problem ID [12288]
Internal file name [`OUTPUT/10940_Saturday_September_30_2023_08_26_33_PM_21740883/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 7.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 4y' + 37y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -3]$$

### 3.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \frac{37}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' + \frac{37y}{4} = 0$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{37}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) - 4sY(s) + 4y(0) + 37Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 2 \\ y'(0) &= -3 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) + 20 - 8s - 4sY(s) + 37Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{8s - 20}{4s^2 - 4s + 37}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1 + \frac{2i}{3}}{s - \frac{1}{2} - 3i} + \frac{1 - \frac{2i}{3}}{s - \frac{1}{2} + 3i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1+\frac{2i}{3}}{s-\frac{1}{2}-3i}\right) = \left(1+\frac{2i}{3}\right) e^{\left(\frac{1}{2}+3i\right)t}$$

$$\mathcal{L}^{-1}\left(\frac{1-\frac{2i}{3}}{s-\frac{1}{2}+3i}\right) = \left(1-\frac{2i}{3}\right) e^{\left(\frac{1}{2}-3i\right)t}$$

Adding the above results and simplifying gives

$$y = \frac{2\,e^{\frac{t}{2}}\left(3\cos\left(3t\right)-2\sin\left(3t\right)\right)}{3}$$

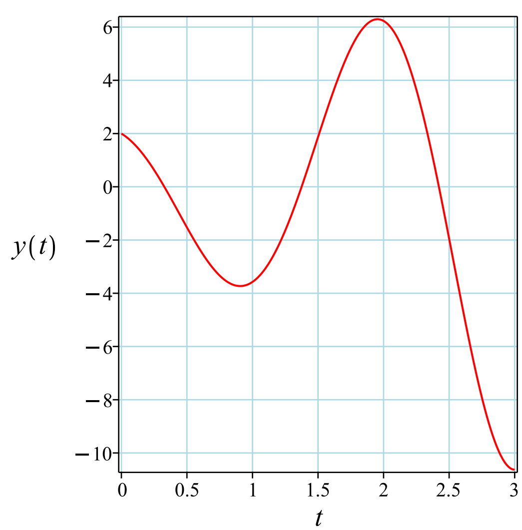
Simplifying the solution gives

$$y = \frac{2\,e^{\frac{t}{2}}\left(3\cos\left(3t\right)-2\sin\left(3t\right)\right)}{3}$$

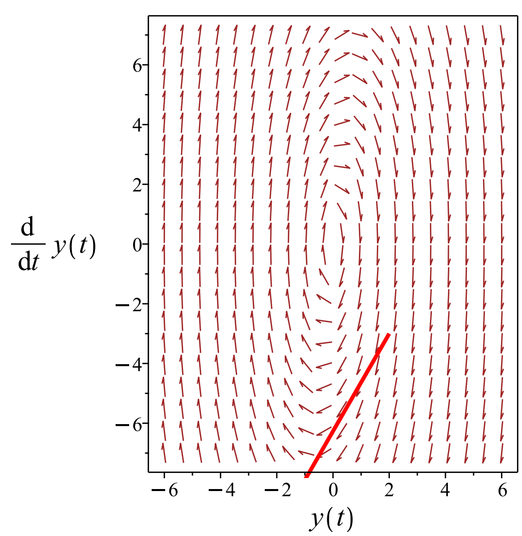
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{2\,e^{\frac{t}{2}}\left(3\cos\left(3t\right)-2\sin\left(3t\right)\right)}{3} \tag{1}$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2\,e^{\frac{t}{2}}\left(3\cos\left(3t\right)-2\sin\left(3t\right)\right)}{3}$$

Verified OK.

### 3.6.2 Maple step by step solution

Let's solve

$$\left[4y'' - 4y' + 37y = 0, y(0) = 2, y'\Big|_{\{t=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = y' - \frac{37y}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{37y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - r + \frac{37}{4} = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{1 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - 3\,\mathrm{I}, \frac{1}{2} + 3\,\mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{t}{2}} \cos(3t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\frac{t}{2}} \cos(3t) + c_2 \mathrm{e}^{\frac{t}{2}} \sin(3t)$

□ Check validity of solution $y = c_1 \mathrm{e}^{\frac{t}{2}} \cos(3t) + c_2 \mathrm{e}^{\frac{t}{2}} \sin(3t)$

  - Use initial condition $y(0) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $y' = \frac{c_1 \mathrm{e}^{\frac{t}{2}} \cos(3t)}{2} - 3c_1 \mathrm{e}^{\frac{t}{2}} \sin(3t) + \frac{c_2 \mathrm{e}^{\frac{t}{2}} \sin(3t)}{2} + 3c_2 \mathrm{e}^{\frac{t}{2}} \cos(3t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = -3$

  $-3 = \frac{c_1}{2} + 3c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 2, c_2 = -\frac{4}{3}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{2\,e^{\frac{t}{2}}(3\cos(3t)-2\sin(3t))}{3}$

- Solution to the IVP

  $y = \frac{2\,e^{\frac{t}{2}}(3\cos(3t)-2\sin(3t))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.359 (sec). Leaf size: 23

```
dsolve([4*diff(y(t),t$2)-4*diff(y(t),t)+37*y(t)=0,y(0) = 2, D(y)(0) = -3],y(t), singsol=all)
```

$$y(t) = \frac{2\,e^{\frac{t}{2}}(3\cos(3t) - 2\sin(3t))}{3}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 29

```
DSolve[{4*y''[t]-4*y'[t]+37*y[t]==0,{y[0]==2,y'[0]==-3}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \frac{2}{3}e^{t/2}(3\cos(3t) - 2\sin(3t))$$

## 3.7 problem Problem 8



Internal problem ID [12289]
Internal file name [`OUTPUT/10941_Saturday_September_30_2023_08_26_33_PM_50385180/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 8.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' + 2y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 3]$$

### 3.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = 0$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 2\\ y'(0) &= 3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 9 - 2s + 3sY(s) + 2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s + 9}{s^2 + 3s + 2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{5}{s+2} + \frac{7}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{5}{s+2}\right) &= -5\,\mathrm{e}^{-2t}\\ \mathcal{L}^{-1}\left(\frac{7}{s+1}\right) &= 7\,\mathrm{e}^{-t}\end{aligned}$$

Adding the above results and simplifying gives

$$y = -5\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$$

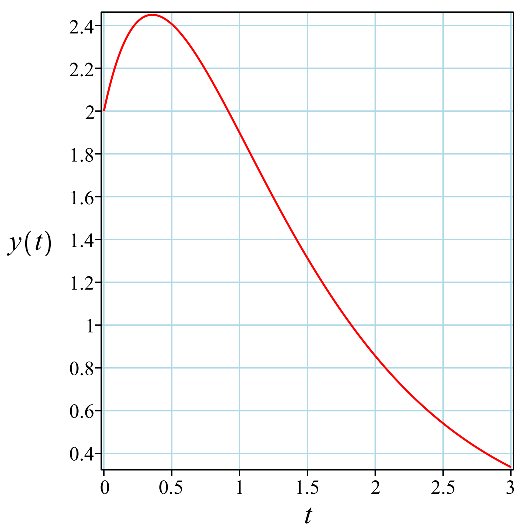
Simplifying the solution gives

$$y = -5\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$$

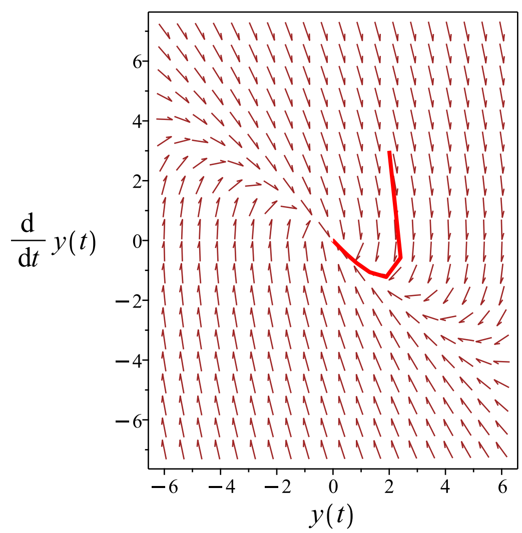
Summary

The solution(s) found are the following

$$y = -5\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = -5\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$$

Verified OK.

### 3.7.2 Maple step by step solution

Let's solve

$\left[y''+3y'+2y=0, y(0)=2, y'\Big|_{\{t=0\}}=3\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2+3r+2=0$

- Factor the characteristic polynomial

  $(r+2)(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-2,-1)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{-t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{-t}$

□ Check validity of solution $y=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{-t}$

  - Use initial condition $y(0)=2$

    $2=c_1+c_2$

  - Compute derivative of the solution

    $y'=-2c_1\mathrm{e}^{-2t}-c_2\mathrm{e}^{-t}$

  - Use the initial condition $y'\Big|_{\{t=0\}}=3$

    $3=-2c_1-c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1=-5, c_2=7\}$

- Substitute constant values into general solution and simplify

  $y = -5\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$

- Solution to the IVP

  $y = -5\,\mathrm{e}^{-2t} + 7\,\mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.562 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=0,y(0) = 2, D(y)(0) = 3],y(t), singsol=all)
```

$$y(t) = 7\,\mathrm{e}^{-t} - 5\,\mathrm{e}^{-2t}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==0,{y[0]==2,y'[0]==3}},y[t],t,IncludeSingularSolutions -> True
```

$$y(t) \to e^{-2t}(7e^t - 5)$$

## 3.8 problem Problem 9



Internal problem ID [12290]
Internal file name [`OUTPUT/10942_Saturday_September_30_2023_08_26_33_PM_26773818/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 9.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

### 3.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 0$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 2sY(s) + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s+1}{s^2 + 2s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2s + 2 - 4i} + \frac{1}{2s + 2 + 4i}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{1}{2s + 2 - 4i}\right) &= \frac{e^{(-1+2i)t}}{2}\\ \mathcal{L}^{-1}\left(\frac{1}{2s + 2 + 4i}\right) &= \frac{e^{(-1-2i)t}}{2}\end{aligned}$$

Adding the above results and simplifying gives

$$y = e^{-t} \cos(2t)$$

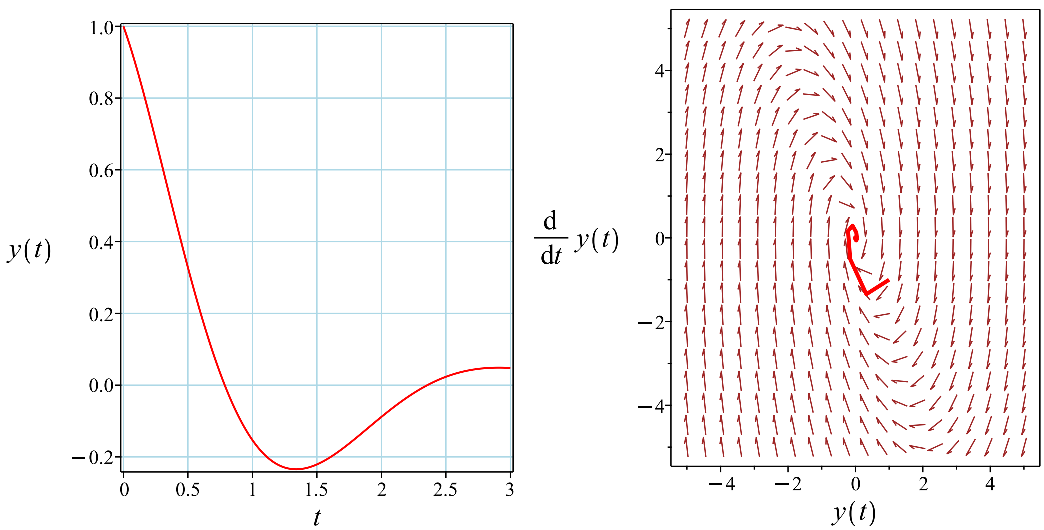
Simplifying the solution gives

$$y = e^{-t} \cos(2t)$$

Summary

The solution(s) found are the following

$$y = e^{-t} \cos(2t) \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-t} \cos(2t)$$

Verified OK.

### 3.8.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 5y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t} \cos(2t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-t} \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t)$

□ Check validity of solution $y = c_1 \mathrm{e}^{-t} \cos(2t) + c_2 \mathrm{e}^{-t} \sin(2t)$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = -c_1 \mathrm{e}^{-t} \cos(2t) - 2c_1 \mathrm{e}^{-t} \sin(2t) - c_2 \mathrm{e}^{-t} \sin(2t) + 2c_2 \mathrm{e}^{-t} \cos(2t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -c_1 + 2c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $$y = e^{-t} \cos(2t)$$

- Solution to the IVP

  $$y = e^{-t} \cos(2t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.125 (sec). Leaf size: 13

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+5*y(t)=0,y(0) = 1, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = e^{-t} \cos(2t)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 15

```
DSolve[{y''[t]+2*y'[t]+5*y[t]==0,{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-t} \cos(2t)$$

## 3.9 problem Problem 10



Internal problem ID [12291]
Internal file name [`OUTPUT/10943_Saturday_September_30_2023_08_26_33_PM_60338816/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 10.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 12y' + 13y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 3]$$

### 3.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= \frac{13}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 3y' + \frac{13y}{4} = 0$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{13}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) - 12sY(s) + 12y(0) + 13Y(s) = 0 \tag{1}$$

But the initial conditions are

$$y(0) = 2$$
$$y'(0) = 3$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) + 12 - 8s - 12sY(s) + 13Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{8s - 12}{4s^2 - 12s + 13}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s - \frac{3}{2} - i} + \frac{1}{s - \frac{3}{2} + i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s-\frac{3}{2}-i}\right) = \mathrm{e}^{\left(\frac{3}{2}+i\right)t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s-\frac{3}{2}+i}\right) = \mathrm{e}^{\left(\frac{3}{2}-i\right)t}$$

Adding the above results and simplifying gives

$$y = 2\,\mathrm{e}^{\frac{3t}{2}}\cos(t)$$

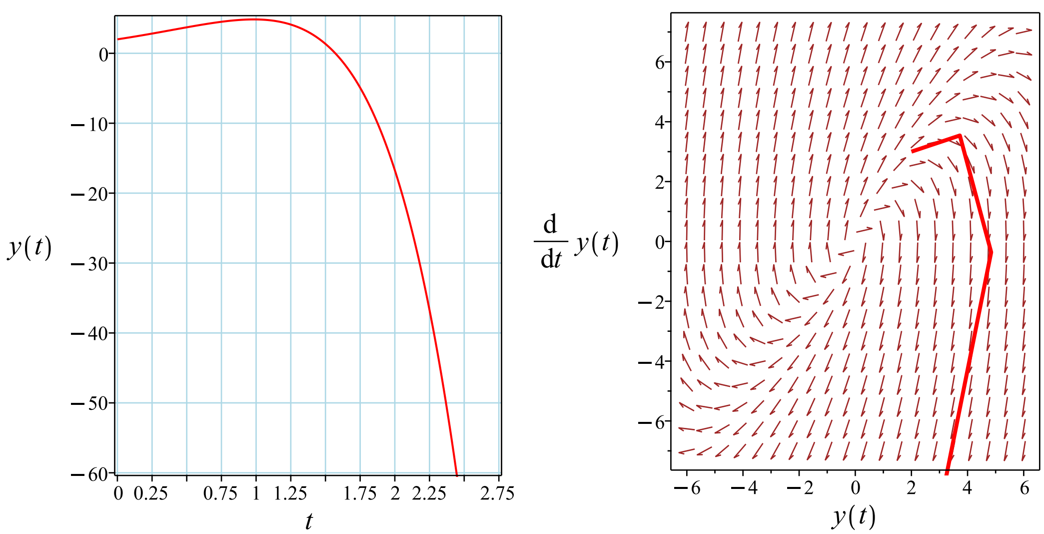
Simplifying the solution gives

$$y = 2\,\mathrm{e}^{\frac{3t}{2}}\cos(t)$$

Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{\frac{3t}{2}}\cos(t) \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{\frac{3t}{2}}\cos(t)$$

Verified OK.

### 3.9.2 Maple step by step solution

Let's solve

$\left[4y'' - 12y' + 13y = 0, y(0) = 2, y'\Big|_{\{t=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = 3y' - \frac{13y}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 3y' + \frac{13y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - 3r + \frac{13}{4} = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{3 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{3}{2} - \mathrm{I}, \frac{3}{2} + \mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{3t}{2}} \cos(t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{3t}{2}} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\frac{3t}{2}} \cos(t) + c_2 \mathrm{e}^{\frac{3t}{2}} \sin(t)$

□ Check validity of solution $y = c_1 \mathrm{e}^{\frac{3t}{2}} \cos(t) + c_2 \mathrm{e}^{\frac{3t}{2}} \sin(t)$

  - Use initial condition $y(0) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $y' = \frac{3c_1 \mathrm{e}^{\frac{3t}{2}} \cos(t)}{2} - c_1 \mathrm{e}^{\frac{3t}{2}} \sin(t) + \frac{3c_2 \mathrm{e}^{\frac{3t}{2}} \sin(t)}{2} + c_2 \mathrm{e}^{\frac{3t}{2}} \cos(t)$

- Use the initial condition $y'\Big|_{\{t=0\}} = 3$

  $3 = \frac{3c_1}{2} + c_2$
- Solve for $c_1$ and $c_2$

  $\{c_1 = 2, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $y = 2\,\mathrm{e}^{\frac{3t}{2}} \cos(t)$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{\frac{3t}{2}} \cos(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.609 (sec). Leaf size: 12

```
dsolve([4*diff(y(t),t$2)-12*diff(y(t),t)+13*y(t)=0,y(0) = 2, D(y)(0) = 3],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{\frac{3t}{2}} \cos(t)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 16

```
DSolve[{4*y''[t]-12*y'[t]+13*y[t]==0,{y[0]==2,y'[0]==3}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to 2e^{3t/2} \cos(t)$$

## 3.10 problem Problem 11



Internal problem ID [12292]
Internal file name [`OUTPUT/10944_Saturday_September_30_2023_08_26_34_PM_54095803/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 11.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 13y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -6]$$

### 3.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 13 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 13y = 0$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 13Y(s) = 0 \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = -6$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s + 4sY(s) + 13Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s-2}{s^2+4s+13}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{2}+\frac{2i}{3}}{s+2-3i} + \frac{\frac{1}{2}-\frac{2i}{3}}{s+2+3i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}+\frac{2i}{3}}{s+2-3i}\right) = \left(\frac{1}{2}+\frac{2i}{3}\right)e^{(-2+3i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2}-\frac{2i}{3}}{s+2+3i}\right) = \left(\frac{1}{2}-\frac{2i}{3}\right)e^{(-2-3i)t}$$

Adding the above results and simplifying gives

$$y = \frac{e^{-2t}(3\cos(3t) - 4\sin(3t))}{3}$$

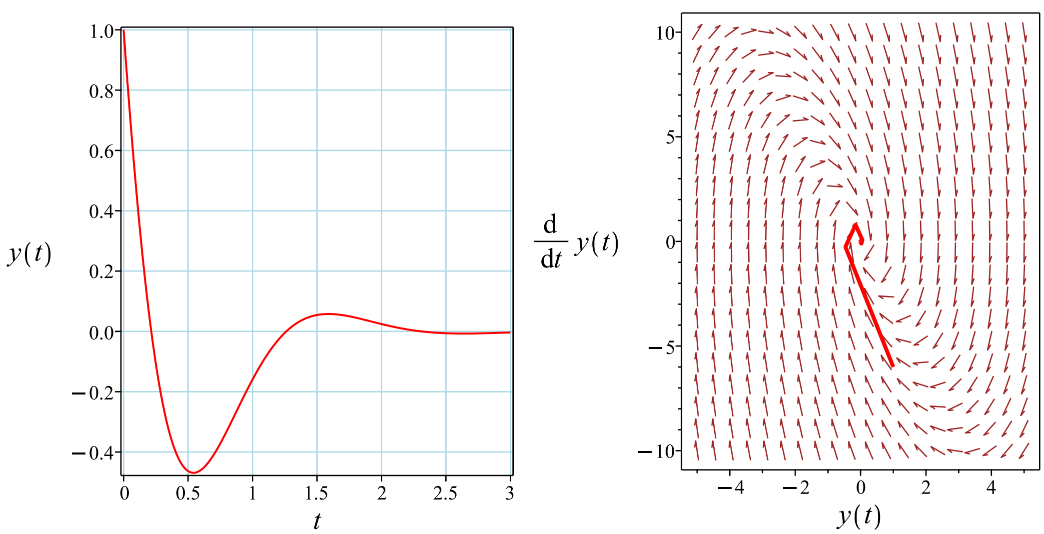
Simplifying the solution gives

$$y = \frac{e^{-2t}(3\cos(3t) - 4\sin(3t))}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2t}(3\cos(3t) - 4\sin(3t))}{3} \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}(3\cos(3t) - 4\sin(3t))}{3}$$

Verified OK.

### 3.10.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 13y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = -6\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(3t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-2t} \cos(3t) + c_2 \mathrm{e}^{-2t} \sin(3t)$

□ Check validity of solution $y = c_1 \mathrm{e}^{-2t} \cos(3t) + c_2 \mathrm{e}^{-2t} \sin(3t)$

  ◦ Use initial condition $y(0) = 1$

    $1 = c_1$

  ◦ Compute derivative of the solution

    $y' = -2c_1 \mathrm{e}^{-2t} \cos(3t) - 3c_1 \mathrm{e}^{-2t} \sin(3t) - 2c_2 \mathrm{e}^{-2t} \sin(3t) + 3c_2 \mathrm{e}^{-2t} \cos(3t)$

  ◦ Use the initial condition $y'\Big|_{\{t=0\}} = -6$

    $-6 = -2c_1 + 3c_2$

  ◦ Solve for $c_1$ and $c_2$

    $\left\{c_1 = 1, c_2 = -\frac{4}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{-2t}(3\cos(3t)-4\sin(3t))}{3}$

- Solution to the IVP

  $y = \frac{e^{-2t}(3\cos(3t)-4\sin(3t))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.843 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=0,y(0) = 1, D(y)(0) = -6],y(t), singsol=all)
```

$$y(t) = \frac{e^{-2t}(3\cos(3t) - 4\sin(3t))}{3}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 27

```
DSolve[{y''[t]+4*y'[t]+13*y[t]==0,{y[0]==1,y'[0]==-6}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{3}e^{-2t}(3\cos(3t) - 4\sin(3t))$$

## 3.11 problem Problem 12



Internal problem ID [12293]
Internal file name [`OUTPUT/10945_Saturday_September_30_2023_08_26_34_PM_81295945/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 12.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 9y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -3]$$

### 3.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 9y = 0$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 6sY(s) - 6y(0) + 9Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - s + 6sY(s) + 9Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s+3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{1}{s+3}\right)\\ &= \mathrm{e}^{-3t}\end{aligned}$$

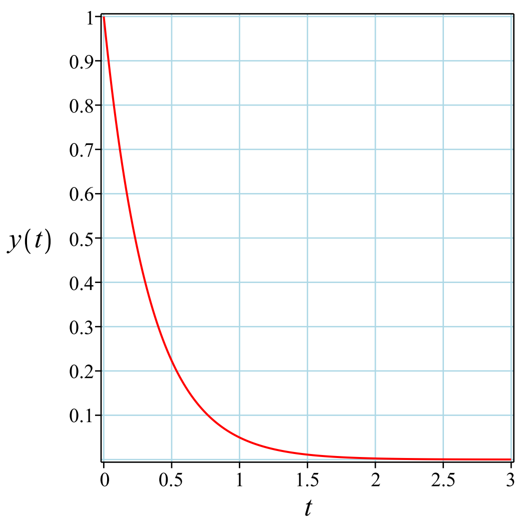
Simplifying the solution gives

$$y = e^{-3t}$$

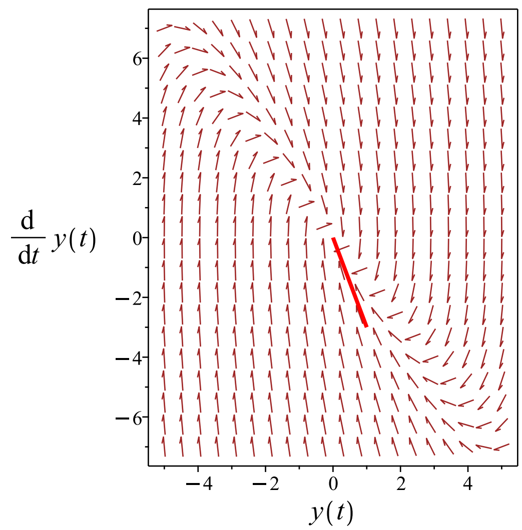
Summary

The solution(s) found are the following

$$y = e^{-3t} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = e^{-3t}$$

Verified OK.

### 3.11.2 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 9y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$
- Root of the characteristic polynomial

  $r = -3$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-3t}$
- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-3t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y = \mathrm{e}^{-3t} c_1 + c_2 t\,\mathrm{e}^{-3t}$

□ Check validity of solution $y = \mathrm{e}^{-3t} c_1 + c_2 t \mathrm{e}^{-3t}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$
  - Compute derivative of the solution

    $y' = -3\,\mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-3t} - 3c_2 t\,\mathrm{e}^{-3t}$
  - Use the initial condition $y'\Big|_{\{t=0\}} = -3$

    $-3 = -3c_1 + c_2$
  - Solve for $c_1$ and $c_2$

    $\{c_1 = 1, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{-3t}$

- Solution to the IVP

  $y = \mathrm{e}^{-3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.578 (sec). Leaf size: 8

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+9*y(t)=0,y(0) = 1, D(y)(0) = -3],y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-3t}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 10

```
DSolve[{y''[t]+6*y'[t]+9*y[t]==0,{y[0]==1,y'[0]==-3}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-3t}$$

## 3.12 problem Problem 13



Internal problem ID [12294]
Internal file name [`OUTPUT/10946_Saturday_September_30_2023_08_26_34_PM_40367264/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 13.
**ODE order**: 4.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y = 0}$$

With initial conditions

$$\left[y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = \frac{\sqrt{2}}{2}\right]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 1 \\
y'(0) &= 0 \\
y''(0) &= 0 \\
y'''(0) &= \frac{\sqrt{2}}{2}
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4 Y(s) - \frac{\sqrt{2}}{2} - s^3 + Y(s) = 0
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{2s^3 + \sqrt{2}}{2s^4 + 2}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{\frac{1}{4} - \frac{\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)}{8}}{s - \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}} + \frac{\frac{1}{4} - \frac{\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)}{8}}{s + \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}} + \frac{\frac{1}{4} - \frac{\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)\sqrt{2}}{8}}{s + \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}} + \frac{\frac{1}{4} - \frac{\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)}{8}}{s - \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\frac{1}{4} - \frac{\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)}{8}}{s - \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}}\right) &= \left(\frac{1}{8} - \frac{i}{8}\right) e^{\left(\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\, t} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{4} - \frac{\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)}{8}}{s + \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}}\right) &= \left(\frac{3}{8} - \frac{i}{8}\right) e^{\left(-\frac{1}{2} + \frac{i}{2}\right)\sqrt{2}\, t} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{4} - \frac{\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)\sqrt{2}}{8}}{s + \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}}\right) &= \left(\frac{3}{8} + \frac{i}{8}\right) e^{\left(-\frac{1}{2} - \frac{i}{2}\right)\sqrt{2}\, t} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{4} - \frac{\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)}{8}}{s - \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}}\right) &= \left(\frac{1}{8} + \frac{i}{8}\right) e^{\left(\frac{1}{2} - \frac{i}{2}\right)\sqrt{2}\, t}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = -\frac{\cos\left(\frac{\sqrt{2}\, t}{2}\right) \sinh\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + \frac{\cosh\left(\frac{\sqrt{2}\, t}{2}\right)\left(2\cos\left(\frac{\sqrt{2}\, t}{2}\right) + \sin\left(\frac{\sqrt{2}\, t}{2}\right)\right)}{2}
$$

Summary
The solution(s) found are the following

$$y = -\frac{\cos\left(\frac{\sqrt{2}\, t}{2}\right) \sinh\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + \frac{\cosh\left(\frac{\sqrt{2}\, t}{2}\right) \left(2\cos\left(\frac{\sqrt{2}\, t}{2}\right) + \sin\left(\frac{\sqrt{2}\, t}{2}\right)\right)}{2} \tag{1}$$

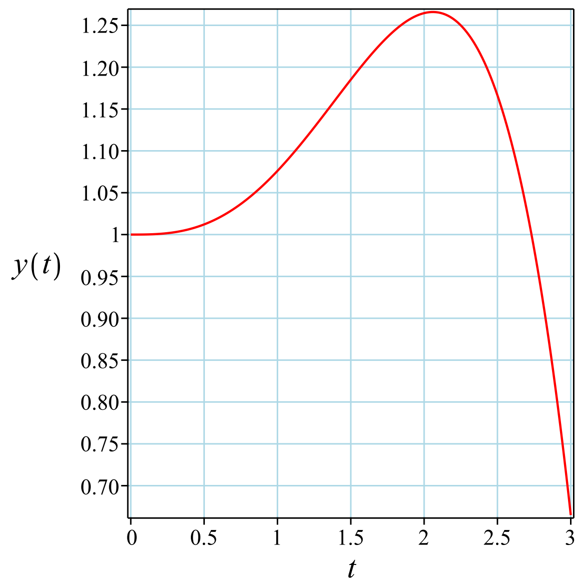



Figure 47: Solution plot

Verification of solutions

$$y = -\frac{\cos\left(\frac{\sqrt{2}\, t}{2}\right) \sinh\left(\frac{\sqrt{2}\, t}{2}\right)}{2} + \frac{\cosh\left(\frac{\sqrt{2}\, t}{2}\right) \left(2\cos\left(\frac{\sqrt{2}\, t}{2}\right) + \sin\left(\frac{\sqrt{2}\, t}{2}\right)\right)}{2}$$

Verified OK.

### 3.12.1 Maple step by step solution

Let's solve

$$\left[y'''' + y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 0, y'''\Big|_{\{t=0\}} = \frac{\sqrt{2}}{2}\right]$$

- Highest derivative means the order of the ODE is 4

$y''''$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(t)$

$y_1(t) = y$

- Define new variable $y_2(t)$

  $y_2(t) = y'$

- Define new variable $y_3(t)$

  $y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = -y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = -y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$

- Eigenpairs of $A$

$$\left[\left[-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1\end{array}\right]\right], \left[-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}} \\ 1\end{array}\right]\right], \left[\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}, \left[\begin{array}{c}\frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1\end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)t} \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{\sqrt{2}\,t}{2}} \cdot \left(\cos\left(\frac{\sqrt{2}\,t}{2}\right)-I\sin\left(\frac{\sqrt{2}\,t}{2}\right)\right) \cdot \left[\begin{array}{c}\frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1\end{array}\right]$$

- Simplify expression

$$e^{-\frac{\sqrt{2}\,t}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{\cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{-\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ \cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(t) = e^{-\frac{\sqrt{2}\,t}{2}} \cdot \begin{bmatrix} \frac{\sqrt{2}\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{2}\,t}{2}\right) \\ -\frac{\sqrt{2}\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{2}\,t}{2}\right) \end{bmatrix}, \vec{y}_2(t) = e^{-\frac{\sqrt{2}\,t}{2}} \cdot \begin{bmatrix} \frac{\sqrt{2}\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}-\frac{\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\cos\left(\frac{\sqrt{2}\,t}{2}\right) \\ \frac{\sqrt{2}\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{2}\,t}{2}\right) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\frac{\sqrt{2}\,t}{2}} \cdot \left(\cos\left(\frac{\sqrt{2}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{\sqrt{2}\,t}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{2}\,t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ \cos\left(\frac{\sqrt{2}\,t}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{2}\,t}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(t) = e^{\frac{\sqrt{2}\,t}{2}} \cdot \left[\begin{array}{c} -\frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ \sin\left(\frac{\sqrt{2}\,t}{2}\right) \\ \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{2}\,t}{2}\right) \end{array}\right], \vec{y}_4(t) = e^{\frac{\sqrt{2}\,t}{2}} \cdot \left[\begin{array}{c} \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{2}\,t}{2}\right) \\ \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}-\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{2}\,t}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + c_4\vec{y}_4(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{\sqrt{2}\,t}{2}} \cdot \left[\begin{array}{c} \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{2}\,t}{2}\right) \\ -\frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{2}\,t}{2}\right) \end{array}\right] + c_2 e^{-\frac{\sqrt{2}\,t}{2}} \cdot \left[\begin{array}{c} \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}-\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\cos\left(\frac{\sqrt{2}\,t}{2}\right) \\ \frac{\sqrt{2}\,\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2}+\frac{\sqrt{2}\,\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{2}\,t}{2}\right) \end{array}\right] + c_3 e^{\frac{\sqrt{2}\,t}{2}}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left((c_1+c_2)\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} - \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)\right)\cos\left(\frac{\sqrt{2}\,t}{2}\right) + \sin\left(\frac{\sqrt{2}\,t}{2}\right)\left((c_1-c_2)\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} + \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3+c_4)\right)\right)\sqrt{2}}{2}$$

- Use the initial condition $y(0) = 1$

$$1 = \frac{(c_1+c_2-c_3+c_4)\sqrt{2}}{2}$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(\left(-\frac{(c_1+c_2)\sqrt{2}\,\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}}}{2} - \frac{\sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)}{2}\right)\cos\left(\frac{\sqrt{2}\,t}{2}\right) - \frac{\left((c_1+c_2)\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} - \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)\right)\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} + \frac{\sqrt{2}\cos\left(\frac{\sqrt{2}\,t}{2}\right)\left((c_1-c\right.}{}\right.}{2}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = \frac{\left(-\frac{(c_1+c_2)\sqrt{2}}{2} - \frac{\sqrt{2}(c_3-c_4)}{2} + \frac{\sqrt{2}(c_1-c_2+c_3+c_4)}{2}\right)\sqrt{2}}{2}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(\left(\frac{(c_1+c_2)\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}}}{2} - \frac{\mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)}{2}\right)\cos\left(\frac{\sqrt{2}\,t}{2}\right) - \left(-\frac{(c_1+c_2)\sqrt{2}\,\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}}}{2} - \frac{\sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)}{2}\right)\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right) - \frac{\left((c_1+c_2)\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} - \right.}{}\right.}{}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 0$

$$0 = -\frac{(c_1-c_2)\sqrt{2}}{2} + \frac{\sqrt{2}(c_3+c_4)}{2}$$

- Calculate the 3rd derivative of the solution

$$y''' = \frac{\left(\left(-\frac{(c_1+c_2)\sqrt{2}\,\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}}}{4} - \frac{\sqrt{2}\,\mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)}{4}\right)\cos\left(\frac{\sqrt{2}\,t}{2}\right) - \frac{3\left(\frac{(c_1+c_2)\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}}}{2} - \frac{\mathrm{e}^{\frac{\sqrt{2}\,t}{2}}(c_3-c_4)}{2}\right)\sqrt{2}\sin\left(\frac{\sqrt{2}\,t}{2}\right)}{2} - \frac{3\left(-\frac{(c_1+c_2)\sqrt{2}\,\mathrm{e}^{-}}{2}\right.}{}\right.}{}$$

- Use the initial condition $y'''\Big|_{\{t=0\}} = \frac{\sqrt{2}}{2}$

$$\frac{\sqrt{2}}{2} = \frac{\left(\frac{(c_1+c_2)\sqrt{2}}{2} + \frac{\sqrt{2}(c_3-c_4)}{2} - \frac{\sqrt{2}(c_1-c_2+c_3+c_4)}{4} + \frac{3\sqrt{2}\left(\frac{c_1}{2} - \frac{c_2}{2} + \frac{c_3}{2} + \frac{c_4}{2}\right)}{2}\right)\sqrt{2}}{2}$$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{\sqrt{2}}{2}, c_2 = \frac{\sqrt{2}}{4}, c_3 = 0, c_4 = \frac{\sqrt{2}}{4}\right\}$$

- Solution to the IVP

$$y = \frac{\left(3\,\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} + \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}\right)\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{4} + \frac{\sin\left(\frac{\sqrt{2}\,t}{2}\right)\left(\mathrm{e}^{-\frac{\sqrt{2}\,t}{2}} + \mathrm{e}^{\frac{\sqrt{2}\,t}{2}}\right)}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.985 (sec). Leaf size: 47

```
dsolve([diff(y(t),t$4)+y(t)=0,y(0) = 1, D(y)(0) = 0, (D@@2)(y)(0) = 0, (D@@3)(y)(0) = 1/sqrt
```

$$y(t) = -\frac{\sinh\left(\frac{\sqrt{2}\,t}{2}\right)\cos\left(\frac{\sqrt{2}\,t}{2}\right)}{2} + \frac{\cosh\left(\frac{\sqrt{2}\,t}{2}\right)\left(2\cos\left(\frac{\sqrt{2}\,t}{2}\right) + \sin\left(\frac{\sqrt{2}\,t}{2}\right)\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 61

```
DSolve[{y''''[t]+y[t]==0,{y[0]==0,y'[0]==0,y''[0]==0,y'''[0]==1/Sqrt[2]}},y[t],t,IncludeSing
```

$$y(t) \to \frac{1}{4}e^{-\frac{t}{\sqrt{2}}}\left(\left(e^{\sqrt{2}t} + 1\right)\sin\left(\frac{t}{\sqrt{2}}\right) - \left(e^{\sqrt{2}t} - 1\right)\cos\left(\frac{t}{\sqrt{2}}\right)\right)$$

## 3.13 problem Problem 14



Internal problem ID [12295]
Internal file name [`OUTPUT/10947_Saturday_September_30_2023_08_26_34_PM_7363715/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 14.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1]$$

### 3.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = 0$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - 2sY(s) + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{1}{s^2 - 2s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{i}{4s - 4 - 8i} - \frac{i}{4(s - 1 + 2i)}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{i}{4s - 4 - 8i}\right) &= \frac{i e^{(1+2i)t}}{4}\\ \mathcal{L}^{-1}\left(-\frac{i}{4(s - 1 + 2i)}\right) &= -\frac{i e^{(1-2i)t}}{4}\end{aligned}$$

Adding the above results and simplifying gives

$$y = -\frac{e^t \sin(2t)}{2}$$

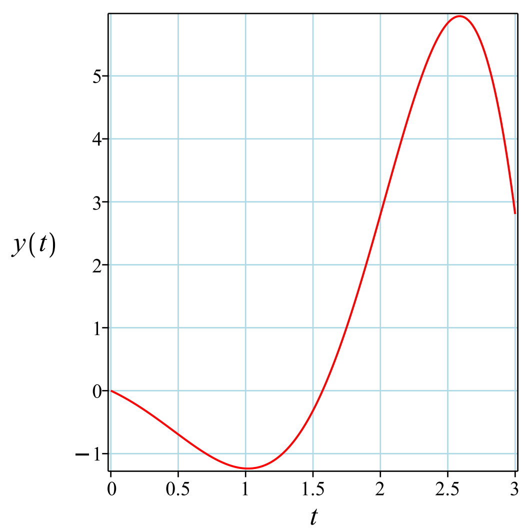
Simplifying the solution gives

$$y = -\frac{e^t \sin(2t)}{2}$$

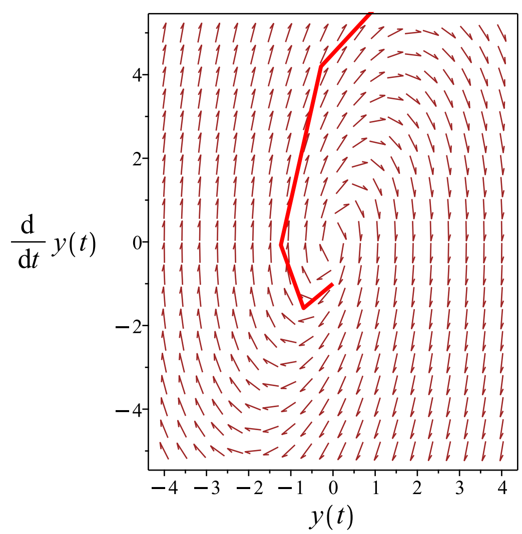
Summary

The solution(s) found are the following

$$y = -\frac{e^t \sin(2t)}{2} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = -\frac{e^t \sin(2t)}{2}$$

Verified OK.

### 3.13.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 5y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t \cos(2t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^t \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^t \cos(2t) + c_2 \mathrm{e}^t \sin(2t)$

□ Check validity of solution $y = c_1 \mathrm{e}^t \cos(2t) + c_2 \mathrm{e}^t \sin(2t)$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1$

- ○ Compute derivative of the solution

  $y' = c_1 \mathrm{e}^t \cos(2t) - 2c_1 \mathrm{e}^t \sin(2t) + c_2 \mathrm{e}^t \sin(2t) + 2c_2 \mathrm{e}^t \cos(2t)$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = 2c_2 + c_1$

- ○ Solve for $c_1$ and $c_2$

  $\left\{c_1 = 0, c_2 = -\frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{e^t \sin(2t)}{2}$

- Solution to the IVP

  $y = -\frac{e^t \sin(2t)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.484 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+5*y(t)=0,y(0) = 0, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = -\frac{\sin(2t) e^t}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 14

```
DSolve[{y''[t]-2*y'[t]+5*y[t]==0,{y[0]==0,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to -e^t \sin(t) \cos(t)$$

## 3.14 problem Problem 15



Internal problem ID [12296]
Internal file name [`OUTPUT/10948_Saturday_September_30_2023_08_26_35_PM_65893225/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 15.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 20y' + 51y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = -14]$$

### 3.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -20 \\ q(t) &= 51 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 20y' + 51y = 0$$

The domain of $p(t) = -20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 51$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 20sY(s) + 20y(0) + 51Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -14\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 14 - 20sY(s) + 51Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{14}{s^2 - 20s + 51}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s - 17} + \frac{1}{s - 3}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{1}{s-17}\right) &= -e^{17t}\\ \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) &= e^{3t}\end{aligned}$$

Adding the above results and simplifying gives

$$y = -2\,\mathrm{e}^{10t} \sinh(7t)$$

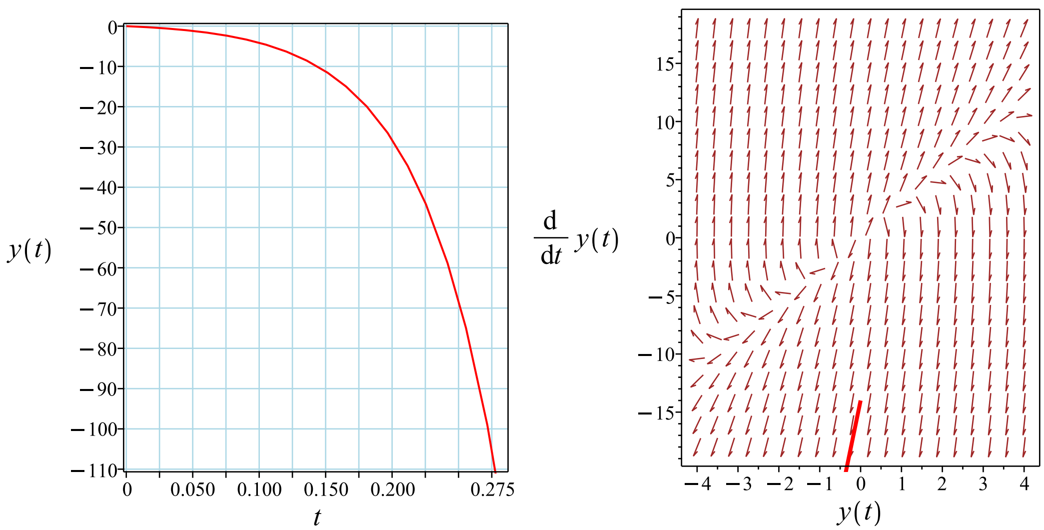
Simplifying the solution gives

$$y = -2\,\mathrm{e}^{10t} \sinh(7t)$$

Summary

The solution(s) found are the following

$$y = -2\,\mathrm{e}^{10t} \sinh(7t) \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2\,\mathrm{e}^{10t} \sinh(7t)$$

Verified OK.

### 3.14.2 Maple step by step solution

Let's solve

$\left[y'' - 20y' + 51y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = -14\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 - 20r + 51 = 0$

- Factor the characteristic polynomial

  $(r - 3)(r - 17) = 0$

- Roots of the characteristic polynomial

  $r = (3, 17)$

- 1st solution of the ODE

  $y_1(t) = e^{3t}$

- 2nd solution of the ODE

  $y_2(t) = e^{17t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 e^{3t} + c_2 e^{17t}$

□ Check validity of solution $y = c_1 e^{3t} + c_2 e^{17t}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = 3c_1 e^{3t} + 17c_2 e^{17t}$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -14$

    $-14 = 3c_1 + 17c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1 = 1, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $$y = e^{3t} - e^{17t}$$

- Solution to the IVP

  $$y = e^{3t} - e^{17t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.5 (sec). Leaf size: 14

```
dsolve([diff(y(t),t$2)-20*diff(y(t),t)+51*y(t)=0,y(0) = 0, D(y)(0) = -14],y(t), singsol=all)
```

$$y(t) = -2\,e^{10t}\sinh(7t)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[{y''[t]-20*y'[t]+51*y[t]==0,{y[0]==0,y'[0]==-14}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to e^{3t} - e^{17t}$$

## 3.15 problem Problem 16



Internal problem ID [12297]
Internal file name [OUTPUT/10949_Saturday_September_30_2023_08_26_35_PM_99942574/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 16.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + 3y' + y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -1]$$

### 3.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{3}{2} \\ q(t) &= \frac{1}{2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{3y'}{2} + \frac{y}{2} = 0$$

The domain of $p(t) = \frac{3}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2s^2Y(s) - 2y'(0) - 2sy(0) + 3sY(s) - 3y(0) + Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^2Y(s) - 7 - 6s + 3sY(s) + Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{6s + 7}{2s^2 + 3s + 1}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{4}{s + \frac{1}{2}} - \frac{1}{s + 1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{4}{s+\frac{1}{2}}\right) = 4\,\mathrm{e}^{-\frac{t}{2}}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+1}\right) = -\mathrm{e}^{-t}$$

Adding the above results and simplifying gives

$$y = 4\,\mathrm{e}^{-\frac{t}{2}} - \mathrm{e}^{-t}$$

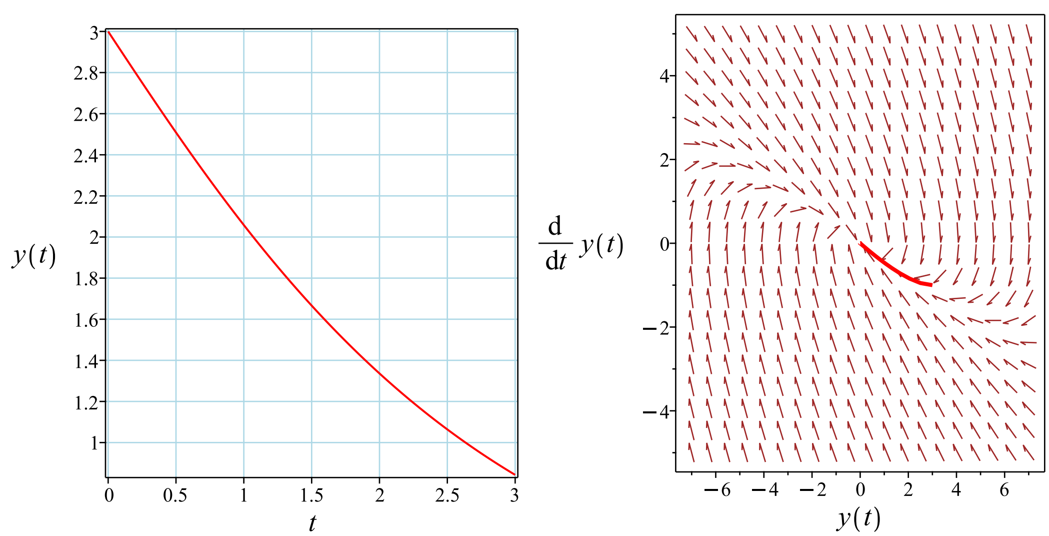
Simplifying the solution gives

$$y = 4\,\mathrm{e}^{-\frac{t}{2}} - \mathrm{e}^{-t}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 4\,\mathrm{e}^{-\frac{t}{2}} - \mathrm{e}^{-t} \tag{1}$$




(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 4\,\mathrm{e}^{-\frac{t}{2}} - \mathrm{e}^{-t}$$

Verified OK.

### 3.15.2 Maple step by step solution

Let's solve

$$\left[2y''+3y'+y=0, y(0)=3, y'\Big|_{\{t=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=-\frac{3y'}{2}-\frac{y}{2}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{2}+\frac{y}{2}=0$

- Characteristic polynomial of ODE

  $r^2+\frac{3}{2}r+\frac{1}{2}=0$

- Factor the characteristic polynomial

  $\frac{(r+1)(2r+1)}{2}=0$

- Roots of the characteristic polynomial

  $r=\left(-1,-\frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{-\frac{t}{2}}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{-\frac{t}{2}}$

□ Check validity of solution $y=c_1\mathrm{e}^{-t}+c_2\mathrm{e}^{-\frac{t}{2}}$

  - Use initial condition $y(0)=3$

    $3=c_1+c_2$

  - Compute derivative of the solution

    $y'=-c_1\mathrm{e}^{-t}-\frac{c_2\mathrm{e}^{-\frac{t}{2}}}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -c_1 - \frac{c_2}{2}$
- Solve for $c_1$ and $c_2$

  $\{c_1 = -1, c_2 = 4\}$
- Substitute constant values into general solution and simplify

  $y = 4\,\mathrm{e}^{-\frac{t}{2}} - \mathrm{e}^{-t}$

- Solution to the IVP

  $y = 4\,\mathrm{e}^{-\frac{t}{2}} - \mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.594 (sec). Leaf size: 17

```
dsolve([2*diff(y(t),t$2)+3*diff(y(t),t)+y(t)=0,y(0) = 3, D(y)(0) = -1],y(t), singsol=all)
```

$$y(t) = -\mathrm{e}^{-t} + 4\,\mathrm{e}^{-\frac{t}{2}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 22

```
DSolve[{2*y''[t]+3*y'[t]+y[t]==0,{y[0]==3,y'[0]==-1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \to e^{-t}\left(4e^{t/2} - 1\right)$$

## 3.16 problem Problem 17



Internal problem ID [12298]
Internal file name [OUTPUT/10950_Saturday_September_30_2023_08_26_35_PM_64669449/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 17.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{3y'' + 8y' - 3y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -4]$$

### 3.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{8}{3} \\ q(t) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{8y'}{3} - y = 0$$

The domain of $p(t) = \frac{8}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$3s^2Y(s) - 3y'(0) - 3sy(0) + 8sY(s) - 8y(0) - 3Y(s) = 0 \tag{1}$$

But the initial conditions are

$$y(0) = 3$$
$$y'(0) = -4$$

Substituting these initial conditions in above in Eq (1) gives

$$3s^2Y(s) - 12 - 9s + 8sY(s) - 3Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{9s + 12}{3s^2 + 8s - 3}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3}{2(s+3)} + \frac{3}{2\left(s - \frac{1}{3}\right)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{3}{2(s+3)}\right) = \frac{3\,\mathrm{e}^{-3t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{3}{2\left(s - \frac{1}{3}\right)}\right) = \frac{3\,\mathrm{e}^{\frac{t}{3}}}{2}$$

Adding the above results and simplifying gives

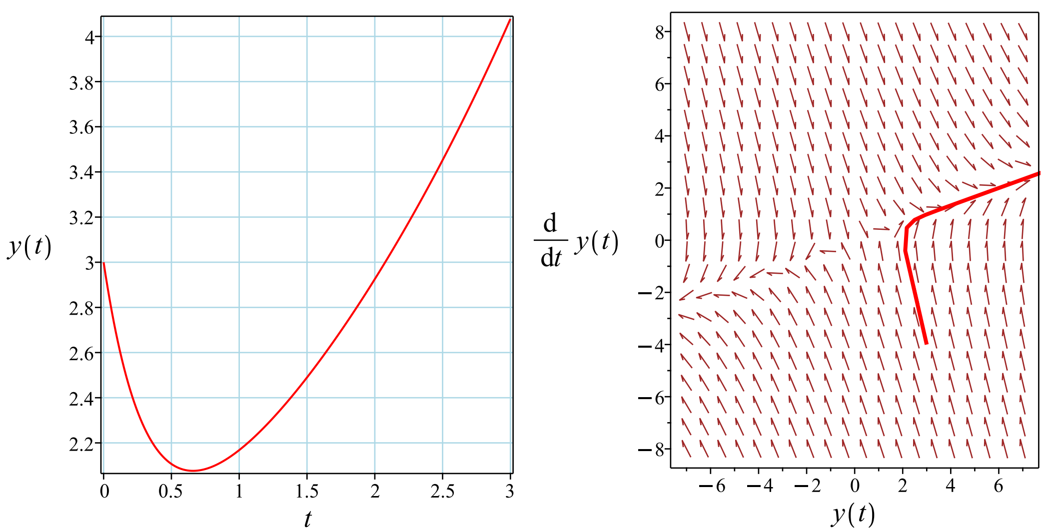
$$y = 3\,\mathrm{e}^{-\frac{4t}{3}} \cosh\left(\frac{5t}{3}\right)$$

Simplifying the solution gives

$$y = 3\,\mathrm{e}^{-\frac{4t}{3}} \cosh\left(\frac{5t}{3}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-\frac{4t}{3}} \cosh\left(\frac{5t}{3}\right) \tag{1}$$




(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 3\,\mathrm{e}^{-\frac{4t}{3}} \cosh\left(\frac{5t}{3}\right)$$

Verified OK.

### 3.16.2 Maple step by step solution

Let's solve

$\left[3y''+8y'-3y=0, y(0)=3, y'\Big|_{\{t=0\}}=-4\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=-\frac{8y'}{3}+y$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{8y'}{3}-y=0$

- Characteristic polynomial of ODE

  $r^2+\frac{8}{3}r-1=0$

- Factor the characteristic polynomial

  $\frac{(r+3)(3r-1)}{3}=0$

- Roots of the characteristic polynomial

  $r=\left(-3,\frac{1}{3}\right)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-3t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{\frac{t}{3}}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=\mathrm{e}^{-3t}c_1+c_2\mathrm{e}^{\frac{t}{3}}$

□ Check validity of solution $y=\mathrm{e}^{-3t}c_1+c_2\mathrm{e}^{\frac{t}{3}}$

  ◦ Use initial condition $y(0)=3$

    $3=c_1+c_2$

  ◦ Compute derivative of the solution

    $y'=-3\,\mathrm{e}^{-3t}c_1+\frac{c_2\mathrm{e}^{\frac{t}{3}}}{3}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -4$

  $-4 = -3c_1 + \frac{c_2}{3}$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = \frac{3}{2}, c_2 = \frac{3}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{3\left(e^{\frac{10t}{3}}+1\right)e^{-3t}}{2}$

- Solution to the IVP

  $y = \frac{3\left(e^{\frac{10t}{3}}+1\right)e^{-3t}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.625 (sec). Leaf size: 14

```
dsolve([3*diff(y(t),t$2)+8*diff(y(t),t)-3*y(t)=0,y(0) = 3, D(y)(0) = -4],y(t), singsol=all)
```

$$y(t) = 3\,e^{-\frac{4t}{3}} \cosh\left(\frac{5t}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 23

```
DSolve[{3*y''[t]+8*y'[t]-3*y[t]==0,{y[0]==3,y'[0]==-4}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to \frac{3}{2}e^{-3t}\left(e^{10t/3}+1\right)$$

## 3.17 problem Problem 18



Internal problem ID [12299]
Internal file name [`OUTPUT/10951_Saturday_September_30_2023_08_26_35_PM_90907946/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 18.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + 20y' + 51y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -5]$$

### 3.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 10 \\ q(t) &= \frac{51}{2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 10y' + \frac{51y}{2} = 0$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{51}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2s^2Y(s) - 2y'(0) - 2sy(0) + 20sY(s) - 20y(0) + 51Y(s) = 0 \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = -5$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^2Y(s) - 10 - 2s + 20sY(s) + 51Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s + 10}{2s^2 + 20s + 51}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2s + 10 - i\sqrt{2}} + \frac{1}{2s + 10 + i\sqrt{2}}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2s+10-i\sqrt{2}}\right) = \frac{e^{-\frac{\left(-i\sqrt{2}+10\right)t}{2}}}{2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2s+10+i\sqrt{2}}\right) = \frac{e^{-\frac{\left(i\sqrt{2}+10\right)t}{2}}}{2}$$

Adding the above results and simplifying gives

$$y = e^{-5t}\cos\left(\frac{\sqrt{2}\,t}{2}\right)$$

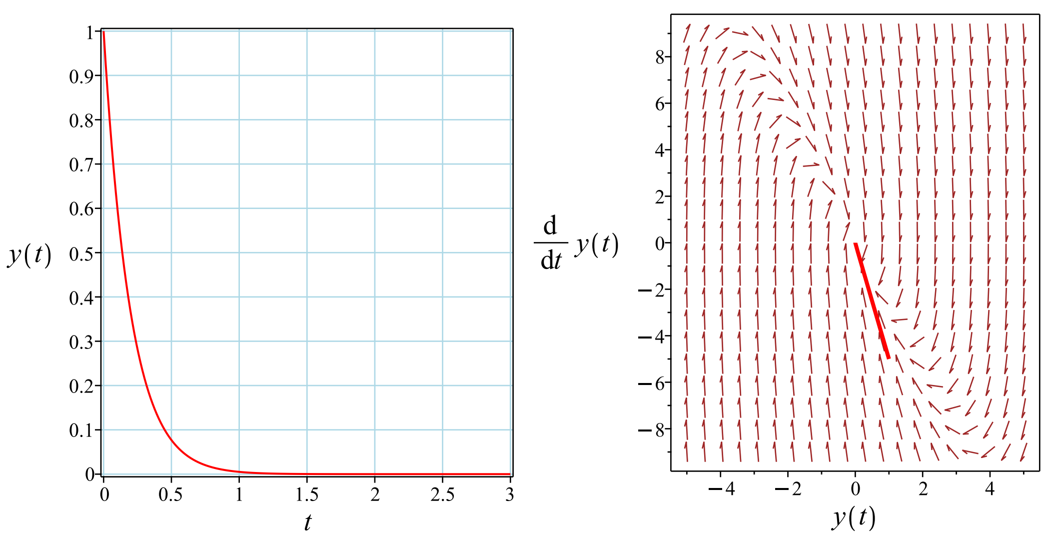
Simplifying the solution gives

$$y = e^{-5t}\cos\left(\frac{\sqrt{2}\,t}{2}\right)$$

Summary

The solution(s) found are the following

$$y = e^{-5t}\cos\left(\frac{\sqrt{2}\,t}{2}\right) \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right)$$

Verified OK.

### 3.17.2 Maple step by step solution

Let's solve

$\left[2y'' + 20y' + 51y = 0, y(0) = 1, y'\Big|_{\{t=0\}} = -5\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -10y' - \frac{51y}{2}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 10y' + \frac{51y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + 10r + \frac{51}{2} = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-10)\pm(\sqrt{-2})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-5 - \frac{I\sqrt{2}}{2}, -5 + \frac{I\sqrt{2}}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right)$

- 2nd solution of the ODE

  $y_2(t) = e^{-5t} \sin\left(\frac{\sqrt{2}\, t}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

$$y = c_1 e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right) + c_2 e^{-5t} \sin\left(\frac{\sqrt{2}\, t}{2}\right)$$

- ☐ Check validity of solution $y = c_1 e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right) + c_2 e^{-5t} \sin\left(\frac{\sqrt{2}\, t}{2}\right)$
  - Use initial condition $y(0) = 1$

    $1 = c_1$
  - Compute derivative of the solution

    $$y' = -5c_1 e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right) - \frac{c_1 e^{-5t} \sqrt{2}\, \sin\left(\frac{\sqrt{2}\, t}{2}\right)}{2} - 5c_2 e^{-5t} \sin\left(\frac{\sqrt{2}\, t}{2}\right) + \frac{c_2 e^{-5t} \sqrt{2}\, \cos\left(\frac{\sqrt{2}\, t}{2}\right)}{2}$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = -5$

    $-5 = -5c_1 + \frac{\sqrt{2}\, c_2}{2}$
  - Solve for $c_1$ and $c_2$

    $\{c_1 = 1, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $y = e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right)$
- Solution to the IVP

  $y = e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.61 (sec). Leaf size: 16

```
dsolve([2*diff(y(t),t$2)+20*diff(y(t),t)+51*y(t)=0,y(0) = 1, D(y)(0) = -5],y(t), singsol=all
```

$$y(t) = e^{-5t} \cos\left(\frac{\sqrt{2}\, t}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 19

```
DSolve[{2*y''[t]+20*y'[t]+51*y[t]==0,{y[0]==1,y'[0]==-5}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to e^{-5t} \cos\left(\frac{t}{\sqrt{2}}\right)$$

## 3.18 problem Problem 19



Internal problem ID [12300]
Internal file name [OUTPUT/10952_Saturday_September_30_2023_08_26_35_PM_54658325/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 19.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 40y' + 101y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -5]$$

### 3.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 10 \\ q(t) &= \frac{101}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 10y' + \frac{101y}{4} = 0$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{101}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) + 40sY(s) - 40y(0) + 101Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1 \\ y'(0) &= -5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) - 20 - 4s + 40sY(s) + 101Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s + 20}{4s^2 + 40s + 101}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2s + 10 - i} + \frac{1}{2s + 10 + i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2s+10-i}\right) = \frac{e^{\left(-5+\frac{i}{2}\right)t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{2s+10+i}\right) = \frac{e^{\left(-5-\frac{i}{2}\right)t}}{2}$$

Adding the above results and simplifying gives

$$y = e^{-5t}\cos\left(\frac{t}{2}\right)$$

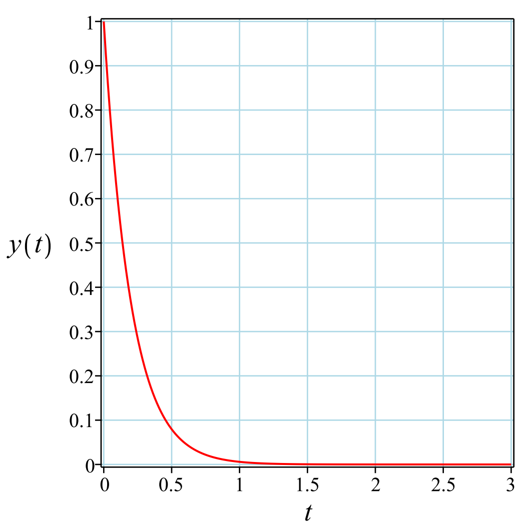
Simplifying the solution gives

$$y = e^{-5t}\cos\left(\frac{t}{2}\right)$$

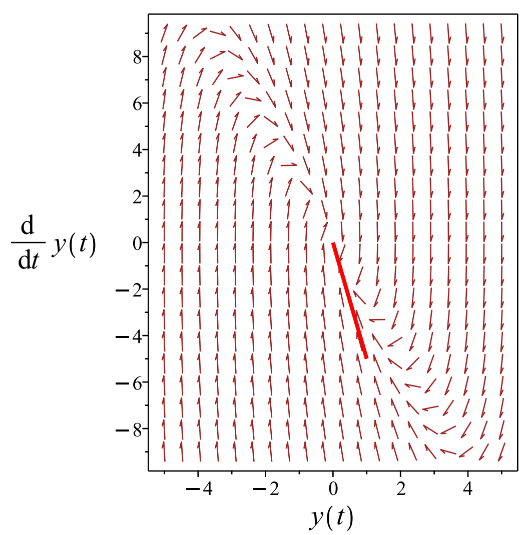
Summary

The solution(s) found are the following

$$y = e^{-5t}\cos\left(\frac{t}{2}\right) \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = e^{-5t}\cos\left(\frac{t}{2}\right)$$

Verified OK.

### 3.18.2 Maple step by step solution

Let's solve

$$\left[4y''+40y'+101y=0, y(0)=1, y'\Big|_{\{t=0\}}=-5\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=-10y'-\frac{101y}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+10y'+\frac{101y}{4}=0$

- Characteristic polynomial of ODE

  $r^2+10r+\frac{101}{4}=0$

- Use quadratic formula to solve for $r$

  $r=\frac{(-10)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-5-\frac{I}{2}, -5+\frac{I}{2}\right)$

- 1st solution of the ODE

  $y_1(t)=e^{-5t}\cos\left(\frac{t}{2}\right)$

- 2nd solution of the ODE

  $y_2(t)=e^{-5t}\sin\left(\frac{t}{2}\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y=c_1e^{-5t}\cos\left(\frac{t}{2}\right)+c_2e^{-5t}\sin\left(\frac{t}{2}\right)$

□ Check validity of solution $y=c_1e^{-5t}\cos\left(\frac{t}{2}\right)+c_2e^{-5t}\sin\left(\frac{t}{2}\right)$

  ◦ Use initial condition $y(0)=1$

    $1=c_1$

  ◦ Compute derivative of the solution

    $y'=-5c_1e^{-5t}\cos\left(\frac{t}{2}\right)-\frac{c_1e^{-5t}\sin\left(\frac{t}{2}\right)}{2}-5c_2e^{-5t}\sin\left(\frac{t}{2}\right)+\frac{c_2e^{-5t}\cos\left(\frac{t}{2}\right)}{2}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -5$

  $-5 = -5c_1 + \frac{c_2}{2}$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{-5t} \cos\left(\frac{t}{2}\right)$

- Solution to the IVP

  $y = \mathrm{e}^{-5t} \cos\left(\frac{t}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.64 (sec). Leaf size: 13

```
dsolve([4*diff(y(t),t$2)+40*diff(y(t),t)+101*y(t)=0,y(0) = 1, D(y)(0) = -5],y(t), singsol=al
```

$$y(t) = \mathrm{e}^{-5t} \cos\left(\frac{t}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 17

```
DSolve[{4*y''[t]+40*y'[t]+101*y[t]==0,{y[0]==1,y'[0]==-5}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to e^{-5t} \cos\left(\frac{t}{2}\right)$$

## 3.19 problem Problem 20



Internal problem ID [12301]
Internal file name [`OUTPUT/10953_Saturday_September_30_2023_08_26_36_PM_29950073/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 20.
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 34y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

### 3.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 34 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 34y = 0$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 34$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 6sY(s) - 6y(0) + 34Y(s) = 0 \tag{1}$$

But the initial conditions are

$$y(0) = 3$$
$$y'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 19 - 3s + 6sY(s) + 34Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3s + 19}{s^2 + 6s + 34}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{3}{2} - i}{s + 3 - 5i} + \frac{\frac{3}{2} + i}{s + 3 + 5i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{3}{2} - i}{s + 3 - 5i}\right) = \left(\frac{3}{2} - i\right) e^{(-3+5i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{3}{2} + i}{s + 3 + 5i}\right) = \left(\frac{3}{2} + i\right) e^{(-3-5i)t}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-3t}(3\cos(5t) + 2\sin(5t))$$

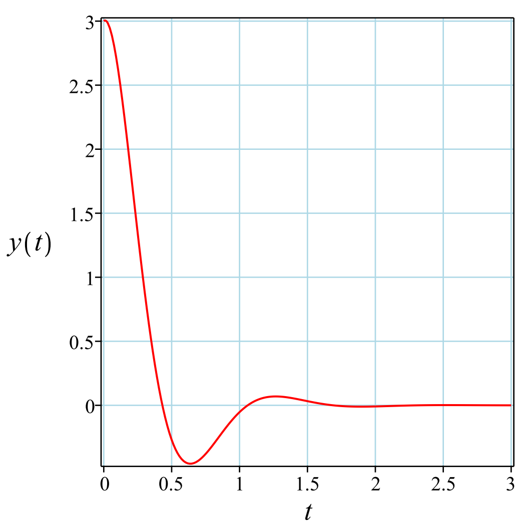
Simplifying the solution gives

$$y = \mathrm{e}^{-3t}(3\cos(5t) + 2\sin(5t))$$

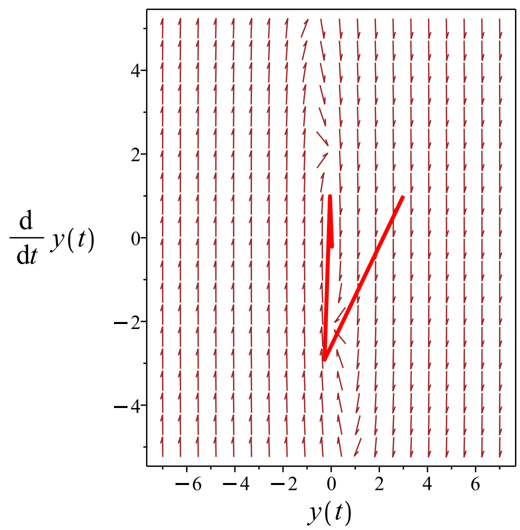
<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{-3t}(3\cos(5t) + 2\sin(5t)) \tag{1}$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-3t}(3\cos(5t) + 2\sin(5t))$$

Verified OK.

### 3.19.2 Maple step by step solution

Let's solve

$\left[y'' + 6y' + 34y = 0, y(0) = 3, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of ODE

  $r^2 + 6r + 34 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-6)\pm(\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 5\,\mathrm{I}, -3 + 5\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-3t} \cos(5t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-3t} \sin(5t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3t} \cos(5t) + c_2 \mathrm{e}^{-3t} \sin(5t)$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-3t} \cos(5t) + c_2 \mathrm{e}^{-3t} \sin(5t)$

- ○ Use initial condition $y(0) = 3$

  $3 = c_1$

- ○ Compute derivative of the solution

  $y' = -3c_1 \mathrm{e}^{-3t} \cos(5t) - 5c_1 \mathrm{e}^{-3t} \sin(5t) - 3c_2 \mathrm{e}^{-3t} \sin(5t) + 5c_2 \mathrm{e}^{-3t} \cos(5t)$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -3c_1 + 5c_2$

- ○ Solve for $c_1$ and $c_2$

  $\{c_1 = 3, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = e^{-3t}(3\cos(5t) + 2\sin(5t))$

- Solution to the IVP

  $y = e^{-3t}(3\cos(5t) + 2\sin(5t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.438 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$2)+6*diff(y(t),t)+34*y(t)=0,y(0) = 3, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = e^{-3t}(3\cos(5t) + 2\sin(5t))$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 24

```
DSolve[{y''[t]+6*y'[t]+34*y[t]==0,{y[0]==3,y'[0]==1}},y[t],t,IncludeSingularSolutions -> Tru
```

$$y(t) \rightarrow e^{-3t}(2\sin(5t) + 3\cos(5t))$$

## 3.20 problem Problem 21



Internal problem ID [12302]
Internal file name [`OUTPUT/10954_Saturday_September_30_2023_08_26_36_PM_50568081/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 21.
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_x]]`

$$\boxed{y''' + 8y'' + 16y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1, y''(0) = -8]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + 8s^2Y(s) - 8y'(0) - 8sy(0) + 16sY(s) - 16y(0) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 1 \\
y''(0) &= -8
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 16 - 9s - s^2 + 8s^2Y(s) + 16sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 9s + 16}{s\,(s^2 + 8s + 16)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s} + \frac{1}{(s+4)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$
$$\mathcal{L}^{-1}\left(\frac{1}{(s+4)^2}\right) = t\,e^{-4t}$$

Adding the above results and simplifying gives

$$y = t\,e^{-4t} + 1$$

<u>Summary</u>
The solution(s) found are the following

$$y = t\,e^{-4t} + 1 \tag{1}$$

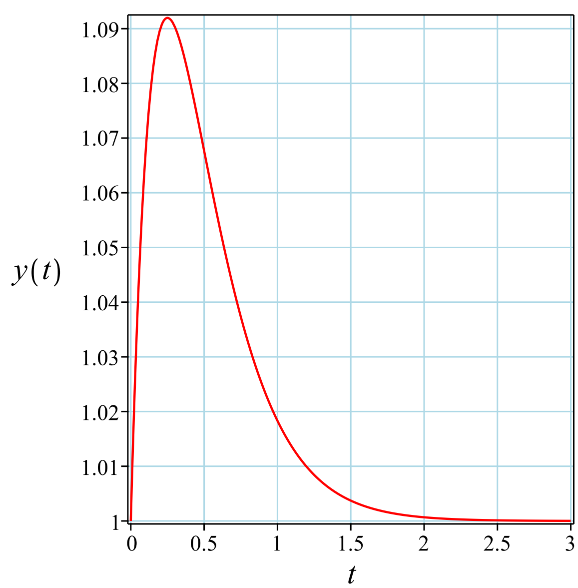



Figure 55: Solution plot

Verification of solutions

$$y = t\,\mathrm{e}^{-4t} + 1$$

Verified OK.

### 3.20.1 Maple step by step solution

Let's solve

$$\left[y''' + 8y'' + 16y' = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 1, y''\Big|_{\{t=0\}} = -8\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y$

  - Define new variable $y_2(t)$

    $y_2(t) = y'$

  - Define new variable $y_3(t)$

    $y_3(t) = y''$

  - Isolate for $y_3'(t)$ using original ODE

    $y_3'(t) = -8y_3(t) - 16y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = -8y_3(t) - 16y_2(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -16 & -8 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -16 & -8 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(t) = A \cdot \overrightarrow{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[-4, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-4, \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $-4$

$$\overrightarrow{y}_1(t) = e^{-4t} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -4$ is the eigenvalue, a

$$\overrightarrow{y}_2(t) = e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the $t$ multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\overrightarrow{y}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = \left(e^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of $A$

$$\lambda e^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda t}\overrightarrow{v} = e^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$

- Make use of the identity matrix I

$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(t)$ to be a solution to the homogeneous system

$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $-4$

$$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -16 & -8\end{bmatrix}-(-4)\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}\frac{1}{16}\\ -\frac{1}{4}\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}\frac{1}{64}\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue $-4$

$$\vec{y}_2(t)=\mathrm{e}^{-4t}\cdot\left(t\cdot\begin{bmatrix}\frac{1}{16}\\ -\frac{1}{4}\\ 1\end{bmatrix}+\begin{bmatrix}\frac{1}{64}\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

$$\left[0,\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

- General solution to the system of ODEs

$\vec{y}=c_1\vec{y}_1(t)+c_2\vec{y}_2(t)+c_3\vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-4t} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix} + c_2 e^{-4t} \cdot \left( t \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{64} \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_3 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{((4t+1)c_2+4c_1)e^{-4t}}{64} + c_3$
- Use the initial condition $y(0) = 1$

  $1 = \frac{c_1}{16} + \frac{c_2}{64} + c_3$
- Calculate the 1st derivative of the solution

  $y' = \frac{c_2 e^{-4t}}{16} - \frac{((4t+1)c_2+4c_1)e^{-4t}}{16}$
- Use the initial condition $y'\big|_{\{t=0\}} = 1$

  $1 = -\frac{c_1}{4}$
- Calculate the 2nd derivative of the solution

  $y'' = -\frac{c_2 e^{-4t}}{2} + \frac{((4t+1)c_2+4c_1)e^{-4t}}{4}$
- Use the initial condition $y''\big|_{\{t=0\}} = -8$

  $-8 = -\frac{c_2}{4} + c_1$
- Solve for the unknown coefficients

  $\{c_1 = -4, c_2 = 16, c_3 = 1\}$
- Solution to the IVP

  $y = t\,e^{-4t} + 1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.594 (sec). Leaf size: 12

```
dsolve([diff(y(t),t$3)+8*diff(y(t),t$2)+16*diff(y(t),t)=0,y(0) = 1, D(y)(0) = 1, (D@@2)(y)(0
```

$$y(t) = t\,\mathrm{e}^{-4t} + 1$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 14

```
DSolve[{y'''[t]+8*y''[t]+16*y'[t]==0,{y[0]==1,y'[0]==1,y''[0]==-8}},y[t],t,IncludeSingularSo
```

$$y(t) \to e^{-4t}t + 1$$

## 3.21 problem Problem 22



Internal problem ID [12303]
Internal file name [`OUTPUT/10955_Saturday_September_30_2023_08_26_36_PM_28153444/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 22.
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 6y'' + 13y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1, y''(0) = -6]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\\ \mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + 6s^2Y(s) - 6y'(0) - 6sy(0) + 13sY(s) - 13y(0) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 1\\ y''(0) &= -6\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 13 - 7s - s^2 + 6s^2Y(s) + 13sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 7s + 13}{s\,(s^2 + 6s + 13)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{4\,(s + 3 - 2i)} + \frac{i}{4s + 12 + 8i} + \frac{1}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{4\,(s + 3 - 2i)}\right) = -\frac{i\mathrm{e}^{(-3+2i)t}}{4}$$
$$\mathcal{L}^{-1}\left(\frac{i}{4s + 12 + 8i}\right) = \frac{i\mathrm{e}^{(-3-2i)t}}{4}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

Adding the above results and simplifying gives

$$y = \frac{\mathrm{e}^{-3t}\sin(2t)}{2} + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-3t}\sin(2t)}{2} + 1 \tag{1}$$

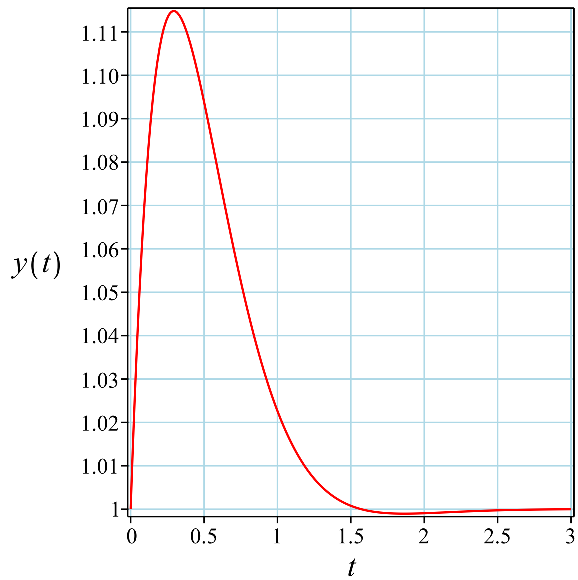



Figure 56: Solution plot

Verification of solutions

$$y = \frac{e^{-3t} \sin(2t)}{2} + 1$$

Verified OK.

### 3.21.1 Maple step by step solution

Let's solve

$$\left[y''' + 6y'' + 13y' = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 1, y''\Big|_{\{t=0\}} = -6\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(t)$

    $y_1(t) = y$

  - ◦ Define new variable $y_2(t)$

    $y_2(t) = y'$

  - ◦ Define new variable $y_3(t)$

$y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = -6y_3(t) - 13y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = -6y_3(t) - 13y_2(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -13 & -6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -13 & -6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-3 - 2\,\mathrm{I}, \begin{bmatrix} \frac{5}{169} - \frac{12\,\mathrm{I}}{169} \\ -\frac{3}{13} + \frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right], \left[-3 + 2\,\mathrm{I}, \begin{bmatrix} \frac{5}{169} + \frac{12\,\mathrm{I}}{169} \\ -\frac{3}{13} - \frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3-2\mathrm{I}, \begin{bmatrix} \frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{3}{13}+\frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-3-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{3}{13}+\frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{-3t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \frac{5}{169}-\frac{12\,\mathrm{I}}{169} \\ -\frac{3}{13}+\frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-3t} \cdot \begin{bmatrix} \left(\frac{5}{169}-\frac{12\,\mathrm{I}}{169}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \left(-\frac{3}{13}+\frac{2\,\mathrm{I}}{13}\right)(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = \mathrm{e}^{-3t} \cdot \begin{bmatrix} \frac{5\cos(2t)}{169} - \frac{12\sin(2t)}{169} \\ -\frac{3\cos(2t)}{13} + \frac{2\sin(2t)}{13} \\ \cos(2t) \end{bmatrix}, \vec{y}_3(t) = \mathrm{e}^{-3t} \cdot \begin{bmatrix} -\frac{5\sin(2t)}{169} - \frac{12\cos(2t)}{169} \\ \frac{3\sin(2t)}{13} + \frac{2\cos(2t)}{13} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{-3t} \cdot \begin{bmatrix} \frac{5\cos(2t)}{169} - \frac{12\sin(2t)}{169} \\ -\frac{3\cos(2t)}{13} + \frac{2\sin(2t)}{13} \\ \cos(2t) \end{bmatrix} + c_3 e^{-3t} \cdot \begin{bmatrix} -\frac{5\sin(2t)}{169} - \frac{12\cos(2t)}{169} \\ \frac{3\sin(2t)}{13} + \frac{2\cos(2t)}{13} \\ -\sin(2t) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((5c_2 - 12c_3)\cos(2t) - 12\left(c_2 + \frac{5c_3}{12}\right)\sin(2t)\right)e^{-3t}}{169} + c_1$$

- Use the initial condition $y(0) = 1$

$$1 = \frac{5c_2}{169} - \frac{12c_3}{169} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(-2(5c_2 - 12c_3)\sin(2t) - 24\left(c_2 + \frac{5c_3}{12}\right)\cos(2t)\right)e^{-3t}}{169} - \frac{3\left((5c_2 - 12c_3)\cos(2t) - 12\left(c_2 + \frac{5c_3}{12}\right)\sin(2t)\right)e^{-3t}}{169}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

$$1 = -\frac{3c_2}{13} + \frac{2c_3}{13}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(-4(5c_2 - 12c_3)\cos(2t) + 48\left(c_2 + \frac{5c_3}{12}\right)\sin(2t)\right)e^{-3t}}{169} - \frac{6\left(-2(5c_2 - 12c_3)\sin(2t) - 24\left(c_2 + \frac{5c_3}{12}\right)\cos(2t)\right)e^{-3t}}{169} + \frac{9\left((5c_2 - 12c_3\right.}{}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = -6$

$$-6 = c_2$$

- Solve for the unknown coefficients

$$\left\{c_1 = 1, c_2 = -6, c_3 = -\frac{5}{2}\right\}$$

- Solution to the IVP

$$y = \frac{e^{-3t}\sin(2t)}{2} + 1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.594 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$3)+6*diff(y(t),t$2)+13*diff(y(t),t)=0,y(0) = 1, D(y)(0) = 1, (D@@2)(y)(0
```

$$y(t) = \frac{e^{-3t} \sin(2t)}{2} + 1$$

✓ Solution by Mathematica

Time used: 0.456 (sec). Leaf size: 17

```
DSolve[{y'''[t]+6*y''[t]+13*y'[t]==0,{y[0]==1,y'[0]==1,y''[0]==-6}},y[t],t,IncludeSingularSo
```

$$y(t) \to e^{-3t} \sin(t) \cos(t) + 1$$

## 3.22 problem Problem 23



Internal problem ID [12304]
Internal file name [`OUTPUT/10956_Saturday_September_30_2023_08_26_36_PM_79997059/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 23.
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 13y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1, y''(0) = 6]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) - 6s^2Y(s) + 6y'(0) + 6sy(0) + 13sY(s) - 13y(0) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 1 \\
y''(0) &= 6
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 13 + 5s - s^2 - 6s^2Y(s) + 13sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 - 5s + 13}{s\,(s^2 - 6s + 13)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{4\,(s - 3 - 2i)} + \frac{i}{4s - 12 + 8i} + \frac{1}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{i}{4\,(s - 3 - 2i)}\right) = -\frac{i\mathrm{e}^{(3+2i)t}}{4}$$

$$\mathcal{L}^{-1}\left(\frac{i}{4s - 12 + 8i}\right) = \frac{i\mathrm{e}^{(3-2i)t}}{4}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

Adding the above results and simplifying gives

$$y = \frac{\mathrm{e}^{3t}\sin(2t)}{2} + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{3t}\sin(2t)}{2} + 1 \tag{1}$$

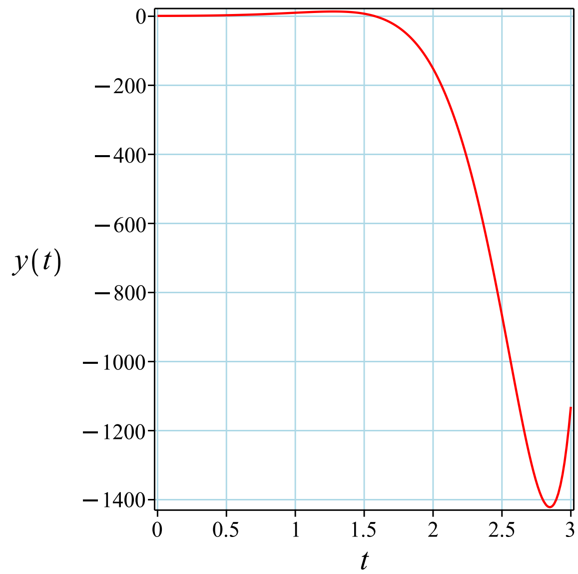



Figure 57: Solution plot

Verification of solutions

$$y = \frac{e^{3t} \sin(2t)}{2} + 1$$

Verified OK.

### 3.22.1 Maple step by step solution

Let's solve

$$\left[y''' - 6y'' + 13y' = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 1, y''\Big|_{\{t=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(t)$

    $y_1(t) = y$
  - ∘ Define new variable $y_2(t)$

    $y_2(t) = y'$
  - ∘ Define new variable $y_3(t)$

    $y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = 6y_3(t) - 13y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = 6y_3(t) - 13y_2(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -13 & 6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -13 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3-2\,\mathrm{I}, \begin{bmatrix} \frac{5}{169} + \frac{12\,\mathrm{I}}{169} \\ \frac{3}{13} + \frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right], \left[3+2\,\mathrm{I}, \begin{bmatrix} \frac{5}{169} - \frac{12\,\mathrm{I}}{169} \\ \frac{3}{13} - \frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3-2\,\mathrm{I}, \begin{bmatrix} \frac{5}{169}+\frac{12\,\mathrm{I}}{169} \\ \frac{3}{13}+\frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-2\,\mathrm{I})t}\cdot \begin{bmatrix} \frac{5}{169}+\frac{12\,\mathrm{I}}{169} \\ \frac{3}{13}+\frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{3t}\cdot\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right)\cdot \begin{bmatrix} \frac{5}{169}+\frac{12\,\mathrm{I}}{169} \\ \frac{3}{13}+\frac{2\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{3t}\cdot \begin{bmatrix} \left(\frac{5}{169}+\frac{12\,\mathrm{I}}{169}\right)\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right) \\ \left(\frac{3}{13}+\frac{2\,\mathrm{I}}{13}\right)\left(\cos\left(2t\right)-\mathrm{I}\sin\left(2t\right)\right) \\ \cos\left(2t\right)-\mathrm{I}\sin\left(2t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t)=\mathrm{e}^{3t}\cdot \begin{bmatrix} \frac{5\cos(2t)}{169}+\frac{12\sin(2t)}{169} \\ \frac{3\cos(2t)}{13}+\frac{2\sin(2t)}{13} \\ \cos\left(2t\right) \end{bmatrix}, \vec{y}_3(t)=\mathrm{e}^{3t}\cdot \begin{bmatrix} -\frac{5\sin(2t)}{169}+\frac{12\cos(2t)}{169} \\ -\frac{3\sin(2t)}{13}+\frac{2\cos(2t)}{13} \\ -\sin\left(2t\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(t)+c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{3t} \cdot \begin{bmatrix} \frac{5\cos(2t)}{169} + \frac{12\sin(2t)}{169} \\ \frac{3\cos(2t)}{13} + \frac{2\sin(2t)}{13} \\ \cos(2t) \end{bmatrix} + e^{3t} c_3 \cdot \begin{bmatrix} -\frac{5\sin(2t)}{169} + \frac{12\cos(2t)}{169} \\ -\frac{3\sin(2t)}{13} + \frac{2\cos(2t)}{13} \\ -\sin(2t) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $$y = \frac{\left((5c_2+12c_3)\cos(2t)+12\left(c_2-\frac{5c_3}{12}\right)\sin(2t)\right)e^{3t}}{169} + c_1$$

- Use the initial condition $y(0) = 1$

  $$1 = \frac{5c_2}{169} + \frac{12c_3}{169} + c_1$$

- Calculate the 1st derivative of the solution

  $$y' = \frac{\left(-2(5c_2+12c_3)\sin(2t)+24\left(c_2-\frac{5c_3}{12}\right)\cos(2t)\right)e^{3t}}{169} + \frac{3\left((5c_2+12c_3)\cos(2t)+12\left(c_2-\frac{5c_3}{12}\right)\sin(2t)\right)e^{3t}}{169}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $$1 = \frac{3c_2}{13} + \frac{2c_3}{13}$$

- Calculate the 2nd derivative of the solution

  $$y'' = \frac{\left(-4(5c_2+12c_3)\cos(2t)-48\left(c_2-\frac{5c_3}{12}\right)\sin(2t)\right)e^{3t}}{169} + \frac{6\left(-2(5c_2+12c_3)\sin(2t)+24\left(c_2-\frac{5c_3}{12}\right)\cos(2t)\right)e^{3t}}{169} + \frac{9\left((5c_2+12c_3)\,\mathrm{c}\right.}{}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 6$

  $$6 = c_2$$

- Solve for the unknown coefficients

  $$\left\{c_1 = 1, c_2 = 6, c_3 = -\frac{5}{2}\right\}$$

- Solution to the IVP

  $$y = \frac{e^{3t}\sin(2t)}{2} + 1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.532 (sec). Leaf size: 16

```
dsolve([diff(y(t),t$3)-6*diff(y(t),t$2)+13*diff(y(t),t)=0,y(0) = 1, D(y)(0) = 1, (D@@2)(y)(0
```

$$y(t) = \frac{e^{3t} \sin(2t)}{2} + 1$$

✓ Solution by Mathematica

Time used: 0.443 (sec). Leaf size: 17

```
DSolve[{y'''[t]-6*y''[t]+13*y'[t]==0,{y[0]==1,y'[0]==1,y''[0]==6}},y[t],t,IncludeSingularSol
```

$$y(t) \to e^{3t} \sin(t) \cos(t) + 1$$

## 3.23 problem Problem 24



Internal problem ID [12305]
Internal file name [`OUTPUT/10957_Saturday_September_30_2023_08_26_36_PM_73489955/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 24.
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_x]]`

$$\boxed{y''' + 4y'' + 29y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 5, y''(0) = -20]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + 4s^2Y(s) - 4y'(0) - 4sy(0) + 29sY(s) - 29y(0) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 5 \\
y''(0) &= -20
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 29 - 9s - s^2 + 4s^2Y(s) + 29sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 9s + 29}{s(s^2 + 4s + 29)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s} - \frac{i}{2(s + 2 - 5i)} + \frac{i}{2s + 4 + 10i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathcal{L}^{-1}\left(-\frac{i}{2(s + 2 - 5i)}\right) = -\frac{i\mathrm{e}^{(-2+5i)t}}{2}$$

$$\mathcal{L}^{-1}\left(\frac{i}{2s + 4 + 10i}\right) = \frac{i\mathrm{e}^{(-2-5i)t}}{2}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-2t}\sin(5t) + 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2t}\sin(5t) + 1 \tag{1}$$

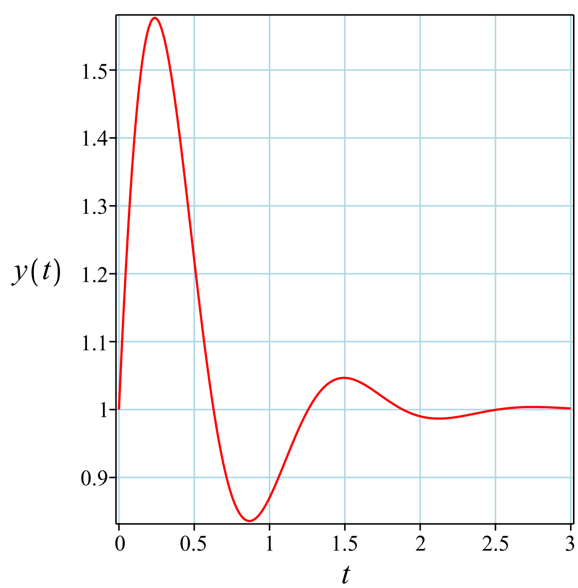



Figure 58: Solution plot

Verification of solutions

$$y = e^{-2t} \sin(5t) + 1$$

Verified OK.

### 3.23.1 Maple step by step solution

Let's solve

$\left[y''' + 4y'' + 29y' = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 5, y''\Big|_{\{t=0\}} = -20\right]$

- Highest derivative means the order of the ODE is 3

  $y'''$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$
  - Define new variable $y_3(t)$

    $y_3(t) = y''$
  - Isolate for $y_3'(t)$ using original ODE

    $y_3'(t) = -4y_3(t) - 29y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = -4y_3(t) - 29y_2(t)]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

  $$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -29 & -4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -29 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-2-5\,\mathrm{I}, \begin{bmatrix} -\frac{21}{841} - \frac{20\,\mathrm{I}}{841} \\ -\frac{2}{29} + \frac{5\,\mathrm{I}}{29} \\ 1 \end{bmatrix}\right], \left[-2+5\,\mathrm{I}, \begin{bmatrix} -\frac{21}{841} + \frac{20\,\mathrm{I}}{841} \\ -\frac{2}{29} - \frac{5\,\mathrm{I}}{29} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2-5\,\mathrm{I}, \begin{bmatrix} -\frac{21}{841} - \frac{20\,\mathrm{I}}{841} \\ -\frac{2}{29} + \frac{5\,\mathrm{I}}{29} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-2-5\,\mathrm{I})t} \cdot \begin{bmatrix} -\frac{21}{841} - \frac{20\,\mathrm{I}}{841} \\ -\frac{2}{29} + \frac{5\,\mathrm{I}}{29} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-2t} \cdot (\cos(5t) - I\sin(5t)) \cdot \begin{bmatrix} -\frac{21}{841} - \frac{20\,I}{841} \\ -\frac{2}{29} + \frac{5\,I}{29} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-2t} \cdot \begin{bmatrix} \left(-\frac{21}{841} - \frac{20\,I}{841}\right)(\cos(5t) - I\sin(5t)) \\ \left(-\frac{2}{29} + \frac{5\,I}{29}\right)(\cos(5t) - I\sin(5t)) \\ \cos(5t) - I\sin(5t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = e^{-2t} \cdot \begin{bmatrix} -\frac{21\cos(5t)}{841} - \frac{20\sin(5t)}{841} \\ -\frac{2\cos(5t)}{29} + \frac{5\sin(5t)}{29} \\ \cos(5t) \end{bmatrix}, \vec{y}_3(t) = e^{-2t} \cdot \begin{bmatrix} \frac{21\sin(5t)}{841} - \frac{20\cos(5t)}{841} \\ \frac{2\sin(5t)}{29} + \frac{5\cos(5t)}{29} \\ -\sin(5t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{-2t} \cdot \begin{bmatrix} -\frac{21\cos(5t)}{841} - \frac{20\sin(5t)}{841} \\ -\frac{2\cos(5t)}{29} + \frac{5\sin(5t)}{29} \\ \cos(5t) \end{bmatrix} + c_3 e^{-2t} \cdot \begin{bmatrix} \frac{21\sin(5t)}{841} - \frac{20\cos(5t)}{841} \\ \frac{2\sin(5t)}{29} + \frac{5\cos(5t)}{29} \\ -\sin(5t) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((-21c_2 - 20c_3)\cos(5t) - 20\left(c_2 - \frac{21c_3}{20}\right)\sin(5t)\right)e^{-2t}}{841} + c_1$$

- Use the initial condition $y(0) = 1$

$$1 = -\frac{21c_2}{841} - \frac{20c_3}{841} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(-5(-21c_2 - 20c_3)\sin(5t) - 100\left(c_2 - \frac{21c_3}{20}\right)\cos(5t)\right)e^{-2t}}{841} - \frac{2\left((-21c_2 - 20c_3)\cos(5t) - 20\left(c_2 - \frac{21c_3}{20}\right)\sin(5t)\right)e^{-2t}}{841}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 5$

$$5 = -\frac{2c_2}{29} + \frac{5c_3}{29}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(-25(-21c_2-20c_3)\cos(5t)+500\left(c_2-\frac{21c_3}{20}\right)\sin(5t)\right)e^{-2t}}{841} - \frac{4\left(-5(-21c_2-20c_3)\sin(5t)-100\left(c_2-\frac{21c_3}{20}\right)\cos(5t)\right)e^{-2t}}{841} + \frac{4}{}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = -20$

  $-20 = c_2$

- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = -20, c_3 = 21\}$

- Solution to the IVP

  $y = e^{-2t}\sin(5t) + 1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.625 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$3)+4*diff(y(t),t$2)+29*diff(y(t),t)=0,y(0) = 1, D(y)(0) = 5, (D@@2)(y)(0
```

$$y(t) = 1 + e^{-2t}\sin(5t)$$

✓ Solution by Mathematica

Time used: 0.58 (sec). Leaf size: 49

```
DSolve[{y'''[t]+4*y''[t]-20*y'[t]==0,{y[0]==1,y'[0]==5,y''[0]==-20}},y[t],t,IncludeSingularS
```

$$y(t) \to \frac{5e^{2\left(\sqrt{6}-1\right)t}}{4\sqrt{6}} - \frac{5e^{-2\left(1+\sqrt{6}\right)t}}{4\sqrt{6}} + 1$$

## 3.24 problem Problem 25



Internal problem ID [12306]
Internal file name [`OUTPUT/10958_Saturday_September_30_2023_08_26_37_PM_81202127/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 25.
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 6y'' + 25y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 4, y''(0) = -24]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + 6s^2Y(s) - 6y'(0) - 6sy(0) + 25sY(s) - 25y(0) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 4 \\
y''(0) &= -24
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 25 - 10s - s^2 + 6s^2Y(s) + 25sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 10s + 25}{s\,(s^2 + 6s + 25)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s} - \frac{i}{2\,(s + 3 - 4i)} + \frac{i}{2s + 6 + 8i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$$

$$\mathcal{L}^{-1}\left(-\frac{i}{2\,(s + 3 - 4i)}\right) = -\frac{i\,\mathrm{e}^{(-3+4i)t}}{2}$$

$$\mathcal{L}^{-1}\left(\frac{i}{2s + 6 + 8i}\right) = \frac{i\,\mathrm{e}^{(-3-4i)t}}{2}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-3t}\sin(4t) + 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3t}\sin(4t) + 1 \tag{1}$$

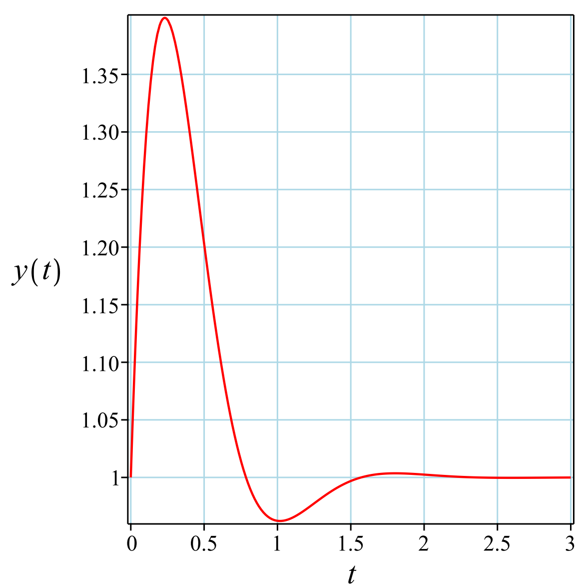



Figure 59: Solution plot

Verification of solutions

$$y = e^{-3t} \sin(4t) + 1$$

Verified OK.

### 3.24.1 Maple step by step solution

Let's solve

$$\left[y''' + 6y'' + 25y' = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 4, y''\Big|_{\{t=0\}} = -24\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(t)$

    $y_1(t) = y$

  ◦ Define new variable $y_2(t)$

    $y_2(t) = y'$

  ◦ Define new variable $y_3(t)$

    $y_3(t) = y''$

  ◦ Isolate for $y_3'(t)$ using original ODE

    $y_3'(t) = -6y_3(t) - 25y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = -6y_3(t) - 25y_2(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & -6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -25 & -6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-3-4\mathrm{I}, \begin{bmatrix} -\frac{7}{625}-\frac{24\,\mathrm{I}}{625} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right], \left[-3+4\mathrm{I}, \begin{bmatrix} -\frac{7}{625}+\frac{24\,\mathrm{I}}{625} \\ -\frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3-4\mathrm{I}, \begin{bmatrix} -\frac{7}{625}-\frac{24\,\mathrm{I}}{625} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-3-4\,\mathrm{I})t} \cdot \begin{bmatrix} -\frac{7}{625}-\frac{24\,\mathrm{I}}{625} \\ -\frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-3t} \cdot (\cos(4t) - \mathrm{I}\sin(4t)) \cdot \begin{bmatrix} -\frac{7}{625} - \frac{24\,\mathrm{I}}{625} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-3t} \cdot \begin{bmatrix} \left(-\frac{7}{625} - \frac{24\,\mathrm{I}}{625}\right)(\cos(4t) - \mathrm{I}\sin(4t)) \\ \left(-\frac{3}{25} + \frac{4\,\mathrm{I}}{25}\right)(\cos(4t) - \mathrm{I}\sin(4t)) \\ \cos(4t) - \mathrm{I}\sin(4t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = e^{-3t} \cdot \begin{bmatrix} -\frac{7\cos(4t)}{625} - \frac{24\sin(4t)}{625} \\ -\frac{3\cos(4t)}{25} + \frac{4\sin(4t)}{25} \\ \cos(4t) \end{bmatrix}, \vec{y}_3(t) = e^{-3t} \cdot \begin{bmatrix} \frac{7\sin(4t)}{625} - \frac{24\cos(4t)}{625} \\ \frac{3\sin(4t)}{25} + \frac{4\cos(4t)}{25} \\ -\sin(4t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{-3t} \cdot \begin{bmatrix} -\frac{7\cos(4t)}{625} - \frac{24\sin(4t)}{625} \\ -\frac{3\cos(4t)}{25} + \frac{4\sin(4t)}{25} \\ \cos(4t) \end{bmatrix} + c_3 e^{-3t} \cdot \begin{bmatrix} \frac{7\sin(4t)}{625} - \frac{24\cos(4t)}{625} \\ \frac{3\sin(4t)}{25} + \frac{4\cos(4t)}{25} \\ -\sin(4t) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((-7c_2 - 24c_3)\cos(4t) - 24\left(c_2 - \frac{7c_3}{24}\right)\sin(4t)\right)e^{-3t}}{625} + c_1$$

- Use the initial condition $y(0) = 1$

$$1 = -\frac{7c_2}{625} - \frac{24c_3}{625} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(-4(-7c_2 - 24c_3)\sin(4t) - 96\left(c_2 - \frac{7c_3}{24}\right)\cos(4t)\right)e^{-3t}}{625} - \frac{3\left((-7c_2 - 24c_3)\cos(4t) - 24\left(c_2 - \frac{7c_3}{24}\right)\sin(4t)\right)e^{-3t}}{625}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 4$

$$4 = -\frac{3c_2}{25} + \frac{4c_3}{25}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(-16(-7c_2-24c_3)\cos(4t)+384\left(c_2-\frac{7c_3}{24}\right)\sin(4t)\right)e^{-3t}}{625} - \frac{6\left(-4(-7c_2-24c_3)\sin(4t)-96\left(c_2-\frac{7c_3}{24}\right)\cos(4t)\right)e^{-3t}}{625} + \frac{9\left((-7$$

- Use the initial condition $y''\Big|_{\{t=0\}} = -24$

  $-24 = c_2$

- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = -24, c_3 = 7\}$

- Solution to the IVP

  $y = e^{-3t}\sin(4t) + 1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.656 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$3)+6*diff(y(t),t$2)+25*diff(y(t),t)=0,y(0) = 1, D(y)(0) = 4, (D@@2)(y)(0
```

$$y(t) = e^{-3t}\sin(4t) + 1$$

✓ Solution by Mathematica

Time used: 0.467 (sec). Leaf size: 17

```
DSolve[{y'''[t]+6*y''[t]+25*y'[t]==0,{y[0]==1,y'[0]==4,y''[0]==-24}},y[t],t,IncludeSingularS
```

$$y(t) \to e^{-3t}\sin(4t) + 1$$

## 3.25 problem Problem 26



Internal problem ID [12307]
Internal file name [`OUTPUT/10959_Saturday_September_30_2023_08_26_37_PM_89131153/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 26.
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 10y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3, y''(0) = 8]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) - 6s^2Y(s) + 6y'(0) + 6sy(0) + 10sY(s) - 10y(0) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= 3 \\
y''(0) &= 8
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) + 3s - s^2 - 6s^2Y(s) + 10sY(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s-3}{s^2 - 6s + 10}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2s-6-2i} + \frac{1}{2s-6+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2s-6-2i}\right) = \frac{e^{(3+i)t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{2s-6+2i}\right) = \frac{e^{(3-i)t}}{2}$$

Adding the above results and simplifying gives

$$y = e^{3t}\cos(t)$$

Summary

The solution(s) found are the following

$$y = e^{3t}\cos(t) \tag{1}$$

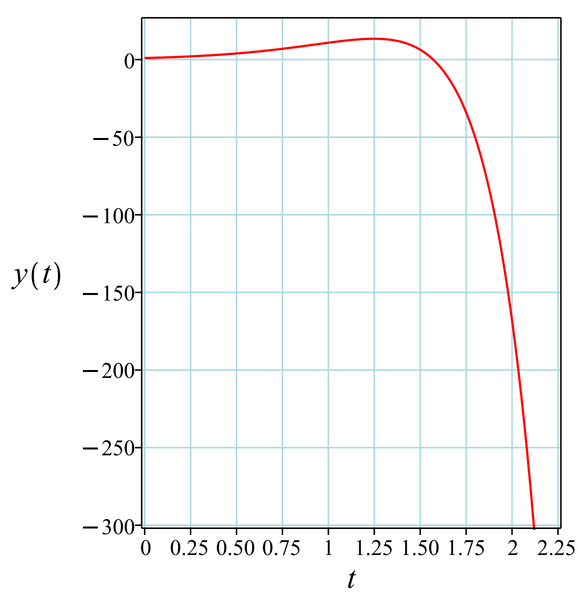



Figure 60: Solution plot

Verification of solutions

$$y = e^{3t} \cos(t)$$

Verified OK.

### 3.25.1 Maple step by step solution

Let's solve

$$\left[y''' - 6y'' + 10y' = 0, y(0) = 1, y'\Big|_{\{t=0\}} = 3, y''\Big|_{\{t=0\}} = 8\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$
  - Define new variable $y_3(t)$

    $y_3(t) = y''$
  - Isolate for $y_3'(t)$ using original ODE

    $y_3'(t) = 6y_3(t) - 10y_2(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = 6y_3(t) - 10y_2(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & 6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3-\mathrm{I}, \begin{bmatrix} \frac{2}{25}+\frac{3\,\mathrm{I}}{50} \\ \frac{3}{10}+\frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}\right], \left[3+\mathrm{I}, \begin{bmatrix} \frac{2}{25}-\frac{3\,\mathrm{I}}{50} \\ \frac{3}{10}-\frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3-\mathrm{I}, \begin{bmatrix} \frac{2}{25}+\frac{3\,\mathrm{I}}{50} \\ \frac{3}{10}+\frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(3-\mathrm{I})t} \cdot \begin{bmatrix} \frac{2}{25}+\frac{3\,\mathrm{I}}{50} \\ \frac{3}{10}+\frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{3t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} \frac{2}{25} + \frac{3\,\mathrm{I}}{50} \\ \frac{3}{10} + \frac{\mathrm{I}}{10} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{3t} \cdot \begin{bmatrix} \left(\frac{2}{25} + \frac{3\,\mathrm{I}}{50}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \left(\frac{3}{10} + \frac{\mathrm{I}}{10}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = e^{3t} \cdot \begin{bmatrix} \frac{2\cos(t)}{25} + \frac{3\sin(t)}{50} \\ \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \\ \cos(t) \end{bmatrix}, \vec{y}_3(t) = e^{3t} \cdot \begin{bmatrix} -\frac{2\sin(t)}{25} + \frac{3\cos(t)}{50} \\ -\frac{3\sin(t)}{10} + \frac{\cos(t)}{10} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{3t} \cdot \begin{bmatrix} \frac{2\cos(t)}{25} + \frac{3\sin(t)}{50} \\ \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \\ \cos(t) \end{bmatrix} + e^{3t} c_3 \cdot \begin{bmatrix} -\frac{2\sin(t)}{25} + \frac{3\cos(t)}{50} \\ -\frac{3\sin(t)}{10} + \frac{\cos(t)}{10} \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((4c_2+3c_3)\cos(t)+3\sin(t)\left(c_2-\frac{4c_3}{3}\right)\right)e^{3t}}{50} + c_1$$

- Use the initial condition $y(0) = 1$

$$1 = \frac{2c_2}{25} + \frac{3c_3}{50} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(-(4c_2+3c_3)\sin(t)+3\cos(t)\left(c_2-\frac{4c_3}{3}\right)\right)e^{3t}}{50} + \frac{3\left((4c_2+3c_3)\cos(t)+3\sin(t)\left(c_2-\frac{4c_3}{3}\right)\right)e^{3t}}{50}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 3$

$$3 = \frac{3c_2}{10} + \frac{c_3}{10}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{\left(-(4c_2+3c_3)\cos(t)-3\sin(t)\left(c_2-\frac{4c_3}{3}\right)\right)e^{3t}}{50} + \frac{3\left(-(4c_2+3c_3)\sin(t)+3\cos(t)\left(c_2-\frac{4c_3}{3}\right)\right)e^{3t}}{25} + \frac{9\left((4c_2+3c_3)\cos(t)+3\sin(t\right.}{50}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 8$

  $8 = c_2$

- Solve for the unknown coefficients

  $\{c_1 = 0, c_2 = 8, c_3 = 6\}$

- Solution to the IVP

  $y = e^{3t}\cos(t)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.531 (sec). Leaf size: 11

```
dsolve([diff(y(t),t$3)-6*diff(y(t),t$2)+10*diff(y(t),t)=0,y(0) = 1, D(y)(0) = 3, (D@@2)(y)(0
```

$$y(t) = e^{3t}\cos(t)$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 13

```
DSolve[{y'''[t]-6*y''[t]+10*y'[t]==0,{y[0]==1,y'[0]==3,y''[0]==8}},y[t],t,IncludeSingularSol
```

$$y(t) \to e^{3t}\cos(t)$$

## 3.26 problem Problem 27



Internal problem ID [12308]
Internal file name [`OUTPUT/10960_Saturday_September_30_2023_08_26_37_PM_5883988/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.5 Laplace transform. Homogeneous equations. Problems page 357
**Problem number**: Problem 27.
**ODE order**: 4.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 13y'' + 36y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1, y''(0) = 5, y'''(0) = 19]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) + 13s^2Y(s) - 13y'(0) - 13sy(0) + 36Y(s) = 0 \quad (1)$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= -1 \\
y''(0) &= 5 \\
y'''(0) &= 19
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4 Y(s) - 6 - 5s + s^2 + 13s^2 Y(s) + 36Y(s) = 0
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = -\frac{s^2 - 5s - 6}{s^4 + 13s^2 + 36}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{\frac{1}{2} - \frac{i}{2}}{s - 2i} + \frac{\frac{1}{2} + \frac{i}{2}}{s + 2i} + \frac{-\frac{1}{2} + \frac{i}{2}}{s - 3i} + \frac{-\frac{1}{2} - \frac{i}{2}}{s + 3i}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\frac{1}{2} - \frac{i}{2}}{s - 2i}\right) &= \left(\frac{1}{2} - \frac{i}{2}\right) \mathrm{e}^{2it} \\
\mathcal{L}^{-1}\left(\frac{\frac{1}{2} + \frac{i}{2}}{s + 2i}\right) &= \left(\frac{1}{2} + \frac{i}{2}\right) \mathrm{e}^{-2it} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{2} + \frac{i}{2}}{s - 3i}\right) &= \left(-\frac{1}{2} + \frac{i}{2}\right) \mathrm{e}^{3it} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{2} - \frac{i}{2}}{s + 3i}\right) &= \left(-\frac{1}{2} - \frac{i}{2}\right) \mathrm{e}^{-3it}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = -\cos(3t) - \sin(3t) + \cos(2t) + \sin(2t)
$$

Summary

The solution(s) found are the following

$$
y = -\cos(3t) - \sin(3t) + \cos(2t) + \sin(2t) \tag{1}
$$

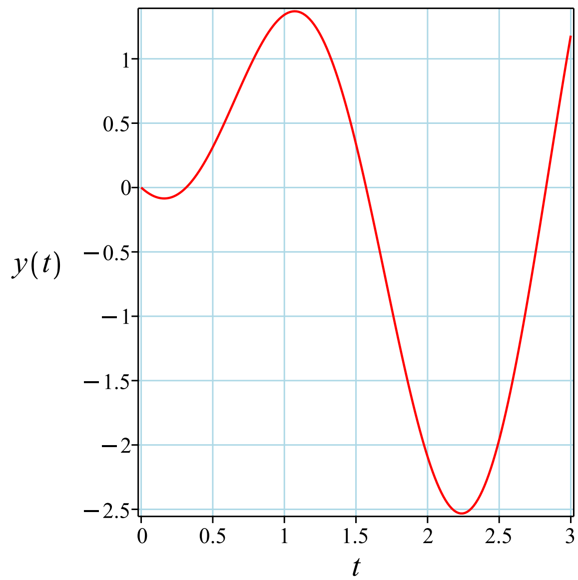



Figure 61: Solution plot

Verification of solutions

$$y = -\cos(3t) - \sin(3t) + \cos(2t) + \sin(2t)$$

Verified OK.

### 3.26.1 Maple step by step solution

Let's solve

$$\left[y'''' + 13y'' + 36y = 0, y(0) = 0, y'\Big|_{\{t=0\}} = -1, y''\Big|_{\{t=0\}} = 5, y'''\Big|_{\{t=0\}} = 19\right]$$

- Highest derivative means the order of the ODE is 4

  $y''''$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y$

  - Define new variable $y_2(t)$

    $y_2(t) = y'$

  - Define new variable $y_3(t)$

    $y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = -13y_3(t) - 36y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t)\,, y_3(t) = y_2'(t)\,, y_4(t) = y_3'(t)\,, y_4'(t) = -13y_3(t) - 36y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & -13 & 0 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 0 & -13 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-3 I, \left[\begin{array}{c} -\frac{I}{27} \\ -\frac{1}{9} \\ \frac{I}{3} \\ 1 \end{array}\right]\right], \left[-2 I, \left[\begin{array}{c} -\frac{I}{8} \\ -\frac{1}{4} \\ \frac{I}{2} \\ 1 \end{array}\right]\right], \left[2 I, \left[\begin{array}{c} \frac{I}{8} \\ -\frac{1}{4} \\ -\frac{I}{2} \\ 1 \end{array}\right]\right], \left[3 I, \left[\begin{array}{c} \frac{I}{27} \\ -\frac{1}{9} \\ -\frac{I}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3 I, \left[\begin{array}{c} -\frac{I}{27} \\ -\frac{1}{9} \\ \frac{I}{3} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-3 It} \cdot \left[\begin{array}{c} -\frac{I}{27} \\ -\frac{1}{9} \\ \frac{I}{3} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(3t) - I\sin(3t)) \cdot \left[\begin{array}{c} -\frac{I}{27} \\ -\frac{1}{9} \\ \frac{I}{3} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{I}{27}(\cos(3t) - I\sin(3t)) \\ -\frac{\cos(3t)}{9} + \frac{I\sin(3t)}{9} \\ \frac{I}{3}(\cos(3t) - I\sin(3t)) \\ \cos(3t) - I\sin(3t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(t) = \begin{bmatrix} -\frac{\sin(3t)}{27} \\ -\frac{\cos(3t)}{9} \\ \frac{\sin(3t)}{3} \\ \cos(3t) \end{bmatrix}, \vec{y}_2(t) = \begin{bmatrix} -\frac{\cos(3t)}{27} \\ \frac{\sin(3t)}{9} \\ \frac{\cos(3t)}{3} \\ -\sin(3t) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2t) - \mathrm{I}\sin(2t)) \\ -\frac{\cos(2t)}{4} + \frac{\mathrm{I}\sin(2t)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(t) = \left[\begin{array}{c} -\frac{\sin(2t)}{8} \\ -\frac{\cos(2t)}{4} \\ \frac{\sin(2t)}{2} \\ \cos(2t) \end{array}\right], \vec{y}_4(t) = \left[\begin{array}{c} -\frac{\cos(2t)}{8} \\ \frac{\sin(2t)}{4} \\ \frac{\cos(2t)}{2} \\ -\sin(2t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(t) + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + c_4\vec{y}_4(t)$$

- Substitute solutions into the general solution

$$\vec{y} = \left[\begin{array}{c} -\frac{c_4\cos(2t)}{8} - \frac{c_3\sin(2t)}{8} - \frac{c_2\cos(3t)}{27} - \frac{c_1\sin(3t)}{27} \\ \frac{c_4\sin(2t)}{4} - \frac{c_3\cos(2t)}{4} + \frac{c_2\sin(3t)}{9} - \frac{c_1\cos(3t)}{9} \\ \frac{c_4\cos(2t)}{2} + \frac{c_3\sin(2t)}{2} + \frac{c_2\cos(3t)}{3} + \frac{c_1\sin(3t)}{3} \\ -c_4\sin(2t) + c_3\cos(2t) - c_2\sin(3t) + c_1\cos(3t) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_4\cos(2t)}{8} - \frac{c_3\sin(2t)}{8} - \frac{c_2\cos(3t)}{27} - \frac{c_1\sin(3t)}{27}$$

- Use the initial condition $y(0) = 0$

$$0 = -\frac{c_4}{8} - \frac{c_2}{27}$$

- Calculate the 1st derivative of the solution

$$y' = \frac{c_4\sin(2t)}{4} - \frac{c_3\cos(2t)}{4} + \frac{c_2\sin(3t)}{9} - \frac{c_1\cos(3t)}{9}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

$$-1 = -\frac{c_3}{4} - \frac{c_1}{9}$$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{c_4\cos(2t)}{2} + \frac{c_3\sin(2t)}{2} + \frac{c_2\cos(3t)}{3} + \frac{c_1\sin(3t)}{3}$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 5$

$$5 = \frac{c_4}{2} + \frac{c_2}{3}$$

- Calculate the 3rd derivative of the solution

$$y''' = -c_4\sin(2t) + c_3\cos(2t) - c_2\sin(3t) + c_1\cos(3t)$$

- Use the initial condition $y'''\Big|_{\{t=0\}} = 19$

$$19 = c_3 + c_1$$

- Solve for the unknown coefficients

  $\{c_1 = 27, c_2 = 27, c_3 = -8, c_4 = -8\}$

- Solution to the IVP

  $y = -\cos(3t) - \sin(3t) + \cos(2t) + \sin(2t)$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.953 (sec). Leaf size: 25

```
dsolve([diff(y(t),t$4)+13*diff(y(t),t$2)+36*y(t)=0,y(0) = 0, D(y)(0) = -1, (D@@2)(y)(0) = 5,
```

$$y(t) = \cos(2t) + \sin(2t) - \cos(3t) - \sin(3t)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 26

```
DSolve[{y''''[t]+13*y''[t]+36*y[t]==0,{y[0]==0,y'[0]==-1,y''[0]==5,y'''[0]==19}},y[t],t,Incl
```

$$y(t) \rightarrow \sin(2t) - \sin(3t) + \cos(2t) - \cos(3t)$$

# 4 Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368

## 4.1 problem Problem 2(a)



Internal problem ID [12309]
Internal file name [`OUTPUT/10961_Saturday_September_30_2023_08_26_37_PM_37522487/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + 3y = 9t}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

### 4.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 3 \\ F &= 9t \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 3y = 9t$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 9t$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 3Y(s) = \frac{9}{s^2} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 2sY(s) + 3Y(s) = \frac{9}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 9}{s^2 (s^2 + 2s + 3)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s - i\sqrt{2} + 1} + \frac{1}{s + 1 + i\sqrt{2}} - \frac{2}{s} + \frac{3}{s^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s-i\sqrt{2}+1}\right) = \mathrm{e}^{-\left(1-i\sqrt{2}\right)t}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s+1+i\sqrt{2}}\right) = \mathrm{e}^{-\left(1+i\sqrt{2}\right)t}$$
$$\mathcal{L}^{-1}\left(-\frac{2}{s}\right) = -2$$
$$\mathcal{L}^{-1}\left(\frac{3}{s^2}\right) = 3t$$

Adding the above results and simplifying gives

$$y = 3t - 2 + 2\cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$$

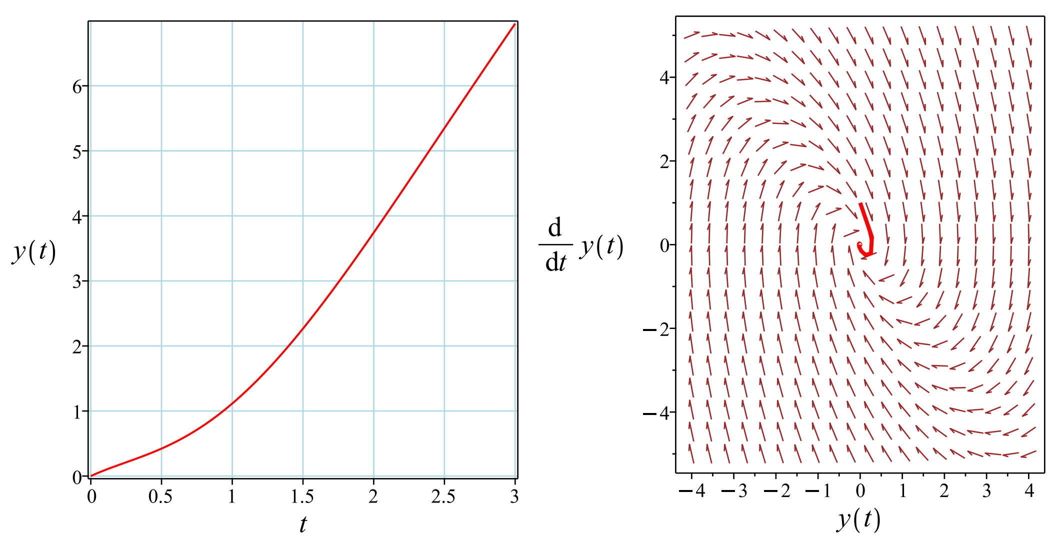
Simplifying the solution gives

$$y = 3t - 2 + 2\cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$$

Summary

The solution(s) found are the following

$$y = 3t - 2 + 2\cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t} \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3t - 2 + 2\cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$$

Verified OK.

### 4.1.2 Maple step by step solution

Let's solve

$\left[y'' + 2y' + 3y = 9t, y(0) = 0, y'\Big|_{\{t=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 3 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-2)\pm(\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-1 - \mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2} - 1\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin\left(\sqrt{2}\,t\right)\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\sqrt{2}\,t\right)\mathrm{e}^{-t} + c_2 \sin\left(\sqrt{2}\,t\right)\mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 9t\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} & \sin(\sqrt{2}\,t)\,\mathrm{e}^{-t} \\ -\sin(\sqrt{2}\,t)\,\sqrt{2}\,\mathrm{e}^{-t} - \cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} & \sqrt{2}\,\mathrm{e}^{-t}\cos(\sqrt{2}\,t) - \sin(\sqrt{2}\,t)\,\mathrm{e}^{-t} \end{bmatrix}$$

- ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{2}\,\mathrm{e}^{-2t}$

- ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{9\sqrt{2}\,\mathrm{e}^{-t}\left(-\cos(\sqrt{2}\,t)\left(\int t\,\mathrm{e}^{t}\sin(\sqrt{2}\,t)dt\right)+\sin(\sqrt{2}\,t)\left(\int t\,\mathrm{e}^{t}\cos(\sqrt{2}\,t)dt\right)\right)}{2}$

- ◦ Compute integrals

  $y_p(t) = -2 + 3t$

- • Substitute particular solution into general solution to ODE

  $y = c_1\cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} + c_2\sin(\sqrt{2}\,t)\,\mathrm{e}^{-t} - 2 + 3t$

- □ Check validity of solution $y = c_1\cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} + c_2\sin(\sqrt{2}\,t)\,\mathrm{e}^{-t} - 2 + 3t$

- ◦ Use initial condition $y(0) = 0$

  $0 = -2 + c_1$

- ◦ Compute derivative of the solution

  $y' = -c_1\sqrt{2}\,\sin(\sqrt{2}\,t)\,\mathrm{e}^{-t} - c_1\cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} + c_2\sqrt{2}\,\cos(\sqrt{2}\,t)\,\mathrm{e}^{-t} - c_2\sin(\sqrt{2}\,t)\,\mathrm{e}^{-t} + 3$

- ◦ Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = 3 - c_1 + \sqrt{2}\,c_2$

- ◦ Solve for $c_1$ and $c_2$

  $\{c_1 = 2, c_2 = 0\}$

- ◦ Substitute constant values into general solution and simplify

  $y = 3t - 2 + 2\cos(\sqrt{2}\,t)\,\mathrm{e}^{-t}$

- • Solution to the IVP

  $y = 3t - 2 + 2\cos(\sqrt{2}\,t)\,\mathrm{e}^{-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.219 (sec). Leaf size: 21

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+3*y(t)=9*t,y(0) = 0, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = 2\,\mathrm{e}^{-t} \cos\left(\sqrt{2}\,t\right) + 3t - 2$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 13

```
DSolve[{y''[t]+2*y''[t]+3*y[t]==9*t,{y[0]==0,y'[0]==1}},y[t],t,IncludeSingularSolutions -> T
```

$$y(t) \to 3t - 2\sin(t)$$

## 4.2 problem Problem 2(b)



Internal problem ID [12310]
Internal file name [`OUTPUT/10962_Saturday_September_30_2023_08_26_37_PM_33751077/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4y'' + 16y' + 17y = 17t - 1}$$

With initial conditions

$$[y(0) = -1, y'(0) = 2]$$

### 4.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= \frac{17}{4} \\ F &= \frac{17t}{4} - \frac{1}{4} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + \frac{17y}{4} = \frac{17t}{4} - \frac{1}{4}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{17}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{17t}{4} - \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) + 16sY(s) - 16y(0) + 17Y(s) = \frac{17}{s^2} - \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= -1 \\
y'(0) &= 2
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) + 8 + 4s + 16sY(s) + 17Y(s) = \frac{17}{s^2} - \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{4s^3 + 8s^2 + s - 17}{s^2 (4s^2 + 16s + 17)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{i}{s+2-\frac{i}{2}} + \frac{i}{s+2+\frac{i}{2}} - \frac{1}{s} + \frac{1}{s^2}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{i}{s+2-\frac{i}{2}}\right) &= -i\mathrm{e}^{\left(-2+\frac{i}{2}\right)t} \\
\mathcal{L}^{-1}\left(\frac{i}{s+2+\frac{i}{2}}\right) &= i\mathrm{e}^{\left(-2-\frac{i}{2}\right)t} \\
\mathcal{L}^{-1}\left(-\frac{1}{s}\right) &= -1 \\
\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) &= t
\end{aligned}$$

Adding the above results and simplifying gives

$$y = 2\,\mathrm{e}^{-2t} \sin\left(\frac{t}{2}\right) + t - 1$$

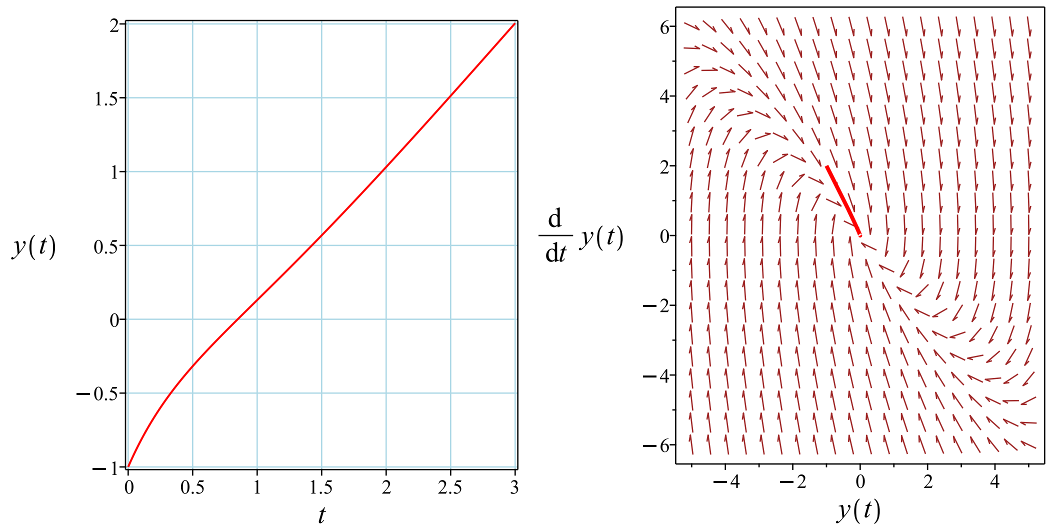
Simplifying the solution gives

$$y = 2\,\mathrm{e}^{-2t} \sin\left(\frac{t}{2}\right) + t - 1$$

<u>Summary</u>

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{-2t} \sin\left(\frac{t}{2}\right) + t - 1 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,\mathrm{e}^{-2t} \sin\left(\frac{t}{2}\right) + t - 1$$

Verified OK.

### 4.2.2 Maple step by step solution

Let's solve

$$\left[4y'' + 16y' + 17y = 17t - 1, y(0) = -1, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -4y' - \frac{17y}{4} + \frac{17t}{4} - \frac{1}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y' + \frac{17y}{4} = \frac{17t}{4} - \frac{1}{4}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + \frac{17}{4} = 0$

- Use quadratic formula to solve for $r$

$r = \frac{(-4)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2-\frac{\mathrm{I}}{2}, -2+\frac{\mathrm{I}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right) + c_2\mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \frac{17t}{4} - \frac{1}{4}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right) & \mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right) \\ -2\,\mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right) - \frac{\mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right)}{2} & -2\,\mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right) + \frac{\mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right)}{2} \end{array}\right]$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \frac{\mathrm{e}^{-4t}}{2}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\mathrm{e}^{-2t}\left(\cos\left(\frac{t}{2}\right)\left(\int (17t-1)\sin\left(\frac{t}{2}\right)\mathrm{e}^{2t}dt\right) - \sin\left(\frac{t}{2}\right)\left(\int (17t-1)\cos\left(\frac{t}{2}\right)\mathrm{e}^{2t}dt\right)\right)}{2}$

- ○ Compute integrals

  $y_p(t) = t - 1$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right) + c_2\mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right) + t - 1$

□ Check validity of solution $y = c_1\mathrm{e}^{-2t}\cos\left(\frac{t}{2}\right) + c_2\mathrm{e}^{-2t}\sin\left(\frac{t}{2}\right) + t - 1$

- ○ Use initial condition $y(0) = -1$

  $-1 = -1 + c_1$

- Compute derivative of the solution

  $y' = -2c_1 e^{-2t} \cos\left(\frac{t}{2}\right) - \frac{c_1 e^{-2t} \sin\left(\frac{t}{2}\right)}{2} - 2c_2 e^{-2t} \sin\left(\frac{t}{2}\right) + \frac{c_2 e^{-2t} \cos\left(\frac{t}{2}\right)}{2} + 1$

- Use the initial condition $y'\big|_{\{t=0\}} = 2$

  $2 = -2c_1 + 1 + \frac{c_2}{2}$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 0, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = 2\,e^{-2t} \sin\left(\frac{t}{2}\right) + t - 1$

- Solution to the IVP

  $y = 2\,e^{-2t} \sin\left(\frac{t}{2}\right) + t - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.171 (sec). Leaf size: 17

```
dsolve([4*diff(y(t),t$2)+16*diff(y(t),t)+17*y(t)=17*t-1,y(0) = -1, D(y)(0) = 2],y(t), singso
```

$$y(t) = 2\,e^{-2t} \sin\left(\frac{t}{2}\right) + t - 1$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 21

```
DSolve[{4*y''[t]+16*y'[t]+17*y[t]==17*t-1,{y[0]==-1,y'[0]==2}},y[t],t,IncludeSingularSolutio
```

$$y(t) \to t + 2e^{-2t} \sin\left(\frac{t}{2}\right) - 1$$

## 4.3 problem Problem 2(c)



Internal problem ID [12311]
Internal file name [`OUTPUT/10963_Saturday_September_30_2023_08_26_38_PM_1620668/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4y'' + 5y' + 4y = 3\,\mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = -1, y'(0) = 1]$$

### 4.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{5}{4} \\ q(t) &= 1 \\ F &= \frac{3\,\mathrm{e}^{-t}}{4} \end{aligned}$$

Hence the ode is

$$y'' + \frac{5y'}{4} + y = \frac{3\,\mathrm{e}^{-t}}{4}$$

The domain of $p(t) = \frac{5}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{3\,\mathrm{e}^{-t}}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) + 5sY(s) - 5y(0) + 4Y(s) = \frac{3}{s+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= -1 \\
y'(0) &= 1
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) + 1 + 4s + 5sY(s) + 4Y(s) = \frac{3}{s+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{4s^2 + 5s - 2}{(s+1)\,(4s^2 + 5s + 4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-1-\frac{i\sqrt{39}}{13}}{s+\frac{5}{8}-\frac{i\sqrt{39}}{8}} + \frac{-1+\frac{i\sqrt{39}}{13}}{s+\frac{5}{8}+\frac{i\sqrt{39}}{8}} + \frac{1}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-1-\frac{i\sqrt{39}}{13}}{s+\frac{5}{8}-\frac{i\sqrt{39}}{8}}\right) = -\frac{\left(i\sqrt{39}+13\right)\mathrm{e}^{-\frac{\left(-i\sqrt{39}+5\right)\left(-\frac{5t}{8\left(-\frac{5}{8}+\frac{i\sqrt{39}}{8}\right)}+\frac{i\sqrt{39}\,t}{-5+i\sqrt{39}}\right)}{8}}}{13}$$

$$\mathcal{L}^{-1}\left(\frac{-1+\frac{i\sqrt{39}}{13}}{s+\frac{5}{8}+\frac{i\sqrt{39}}{8}}\right) = -\frac{\left(13-i\sqrt{39}\right)\mathrm{e}^{-\frac{\left(i\sqrt{39}+5\right)\left(-\frac{5t}{8\left(-\frac{5}{8}-\frac{i\sqrt{39}}{8}\right)}-\frac{i\sqrt{39}\,t}{8\left(-\frac{5}{8}-\frac{i\sqrt{39}}{8}\right)}\right)}{8}}}{13}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = \mathrm{e}^{-t}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{-t} + \frac{2\,\mathrm{e}^{-\frac{5t}{8}}\left(\sqrt{39}\,\sin\left(\frac{\sqrt{39}\,t}{8}\right) - 13\cos\left(\frac{\sqrt{39}\,t}{8}\right)\right)}{13}$$

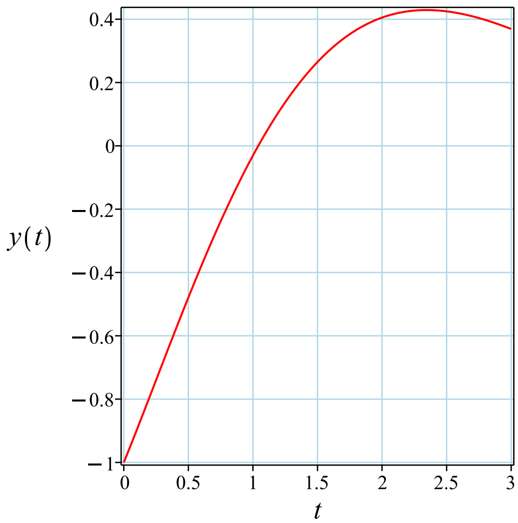
Simplifying the solution gives

$$y = \frac{2\,\mathrm{e}^{-\frac{5t}{8}}\sin\left(\frac{\sqrt{39}\,t}{8}\right)\sqrt{39}}{13} - 2\,\mathrm{e}^{-\frac{5t}{8}}\cos\left(\frac{\sqrt{39}\,t}{8}\right) + \mathrm{e}^{-t}$$

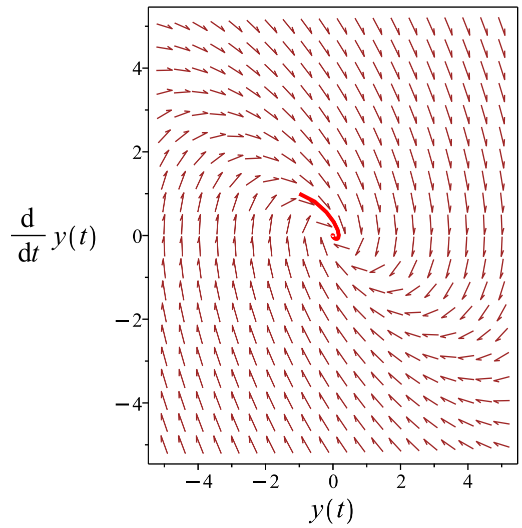
Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{-\frac{5t}{8}}\sin\left(\frac{\sqrt{39}\,t}{8}\right)\sqrt{39}}{13} - 2\,\mathrm{e}^{-\frac{5t}{8}}\cos\left(\frac{\sqrt{39}\,t}{8}\right) + \mathrm{e}^{-t} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = \frac{2\,\mathrm{e}^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\,t}{8}\right)\sqrt{39}}{13} - 2\,\mathrm{e}^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\,t}{8}\right) + \mathrm{e}^{-t}$$

Verified OK.

### 4.3.2 Maple step by step solution

Let's solve

$$\left[4y'' + 5y' + 4y = 3\,\mathrm{e}^{-t}, y(0) = -1, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{4} - y + \frac{3\,\mathrm{e}^{-t}}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{4} + y = \frac{3\,\mathrm{e}^{-t}}{4}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{5}{4}r + 1 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{\left(-\frac{5}{4}\right) \pm \left(\sqrt{-\frac{39}{16}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{5}{8} - \frac{I\sqrt{39}}{8}, -\frac{5}{8} + \frac{I\sqrt{39}}{8}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) + c_2 e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right) + y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \frac{3\, e^{-t}}{4}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) & e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right) \\ -\frac{5\, e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right)}{8} - \frac{e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right)\sqrt{39}}{8} & -\frac{5\, e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right)}{8} + \frac{e^{-\frac{5t}{8}} \sqrt{39} \cos\left(\frac{\sqrt{39}\, t}{8}\right)}{8} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \frac{\sqrt{39}\, e^{-\frac{5t}{4}}}{8}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{2\sqrt{39}\, e^{-\frac{5t}{8}} \left(-\cos\left(\frac{\sqrt{39}\, t}{8}\right)\left(\int e^{-\frac{3t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right) dt\right) + \sin\left(\frac{\sqrt{39}\, t}{8}\right)\left(\int e^{-\frac{3t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) dt\right)\right)}{13}$

  - Compute integrals

    $y_p(t) = e^{-t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) + c_2 e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right) + e^{-t}$

- □ Check validity of solution $y = c_1 e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) + c_2 e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right) + e^{-t}$
  - Use initial condition $y(0) = -1$

    $-1 = 1 + c_1$
  - Compute derivative of the solution

    $$y' = -\frac{5c_1 e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right)}{8} - \frac{c_1 e^{-\frac{5t}{8}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\, t}{8}\right)}{8} - \frac{5c_2 e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right)}{8} + \frac{c_2 e^{-\frac{5t}{8}} \sqrt{39}\, \cos\left(\frac{\sqrt{39}\, t}{8}\right)}{8} - e^{-t}$$
  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = -\frac{5c_1}{8} - 1 + \frac{c_2\sqrt{39}}{8}$
  - Solve for $c_1$ and $c_2$

    $\left\{c_1 = -2, c_2 = \frac{2\sqrt{39}}{13}\right\}$
  - Substitute constant values into general solution and simplify

    $$y = \frac{2\, e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right)\sqrt{39}}{13} - 2\, e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) + e^{-t}$$
- Solution to the IVP

  $$y = \frac{2\, e^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\, t}{8}\right)\sqrt{39}}{13} - 2\, e^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\, t}{8}\right) + e^{-t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.547 (sec). Leaf size: 36

```
dsolve([4*diff(y(t),t$2)+5*diff(y(t),t)+4*y(t)=3*exp(-t),y(0) = -1, D(y)(0) = 1],y(t), sings
```

$$y(t) = \frac{2\sqrt{39}\,\mathrm{e}^{-\frac{5t}{8}} \sin\left(\frac{\sqrt{39}\,t}{8}\right)}{13} - 2\,\mathrm{e}^{-\frac{5t}{8}} \cos\left(\frac{\sqrt{39}\,t}{8}\right) + \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 58

```
DSolve[{4*y''[t]+5*y'[t]+4*y[t]==3*Exp[-t],{y[0]==-1,y'[0]==1}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to e^{-t} + 2\sqrt{\frac{3}{13}}e^{-5t/8} \sin\left(\frac{\sqrt{39}t}{8}\right) - 2e^{-5t/8} \cos\left(\frac{\sqrt{39}t}{8}\right)$$

## 4.4 problem Problem 2(d)



Internal problem ID [12312]
Internal file name [`OUTPUT/10964_Monday_October_02_2023_02_47_37_AM_64152595/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = \mathrm{e}^{2t} t^2}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

### 4.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -4 \\ q(t) &= 4 \\ F &= \mathrm{e}^{2t} t^2 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 4y = e^{2t}t^2$$

The domain of $p(t) = -4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{2t}t^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 4Y(s) = \frac{2}{(s-2)^3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - 4sY(s) + 4Y(s) = \frac{2}{(s-2)^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^4 - 8s^3 + 24s^2 - 32s + 18}{(s-2)^3(s^2 - 4s + 4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-2} + \frac{2}{(s-2)^5}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{(s-2)^5}\right) = \frac{t^4 e^{2t}}{12}$$

Adding the above results and simplifying gives

$$y = \frac{e^{2t}(t^4+12)}{12}$$

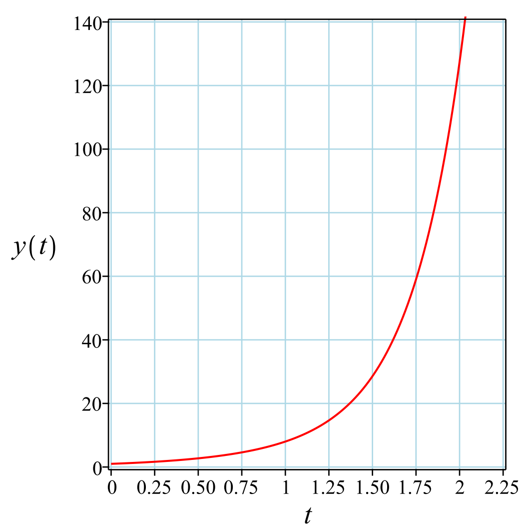
Simplifying the solution gives

$$y = \frac{e^{2t}(t^4+12)}{12}$$

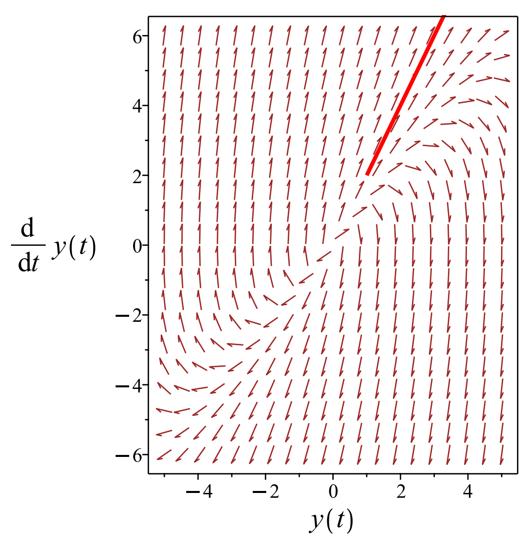
Summary

The solution(s) found are the following

$$y = \frac{e^{2t}(t^4+12)}{12} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = \frac{e^{2t}(t^4 + 12)}{12}$$

Verified OK.

### 4.4.2 Maple step by step solution

Let's solve

$\left[y'' - 4y' + 4y = e^{2t}t^2, y(0) = 1, y'\Big|_{\{t=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{2t}$

- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t) = e^{2t}t$

- General solution of the ODE

  $y = c_1y_1(t) + c_2y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1e^{2t} + c_2e^{2t}t + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = e^{2t}t^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{2t} & e^{2t}t \\ 2e^{2t} & 2e^{2t}t + e^{2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{4t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = e^{2t}\left(-\left(\int t^3 dt\right) + \left(\int t^2 dt\right) t\right)$

- Compute integrals

  $y_p(t) = \frac{t^4 e^{2t}}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2t} + c_2 e^{2t} t + \frac{t^4 e^{2t}}{12}$

☐ Check validity of solution $y = c_1 e^{2t} + c_2 e^{2t} t + \frac{t^4 e^{2t}}{12}$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = 2c_1 e^{2t} + 2c_2 e^{2t} t + c_2 e^{2t} + \frac{e^{2t} t^3}{3} + \frac{t^4 e^{2t}}{6}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = 2c_1 + c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = e^{2t}\left(1 + \frac{t^4}{12}\right)$

- Solution to the IVP

  $y = e^{2t}\left(1 + \frac{t^4}{12}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.875 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)-4*diff(y(t),t)+4*y(t)=t^2*exp(2*t),y(0) = 1, D(y)(0) = 2],y(t), sings
```

$$y(t) = \frac{e^{2t}(t^4 + 12)}{12}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 19

```
DSolve[{y''[t]-4*y'[t]+4*y[t]==t^2*Exp[2*t],{y[0]==1,y'[0]==2}},y[t],t,IncludeSingularSoluti
```

$$y(t) \to \frac{1}{12} e^{2t} (t^4 + 12)$$

## 4.5 problem Problem 2(e)



Internal problem ID [12313]
Internal file name [`OUTPUT/10965_Monday_October_02_2023_02_47_37_AM_10656506/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = e^{-2t}}$$

With initial conditions

$$\left[y(0) = -\frac{2}{13}, y'(0) = \frac{1}{13}\right]$$

### 4.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 9 \\ F &= e^{-2t} \end{aligned}$$

Hence the ode is

$$y'' + 9y = e^{-2t}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{1}{s+2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= -\frac{2}{13} \\
y'(0) &= \frac{1}{13}
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - \frac{1}{13} + \frac{2s}{13} + 9Y(s) = \frac{1}{s+2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2s^2 + 3s - 15}{13\,(s+2)\,(s^2+9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{3}{26}-\frac{i}{26}}{s-3i}+\frac{-\frac{3}{26}+\frac{i}{26}}{s+3i}+\frac{1}{13s+26}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{3}{26}-\frac{i}{26}}{s-3i}\right) = \left(-\frac{3}{26}-\frac{i}{26}\right)e^{3it}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{3}{26}+\frac{i}{26}}{s+3i}\right) = \left(-\frac{3}{26}+\frac{i}{26}\right)e^{-3it}$$

$$\mathcal{L}^{-1}\left(\frac{1}{13s+26}\right) = \frac{e^{-2t}}{13}$$

Adding the above results and simplifying gives

$$y = \frac{e^{-2t}}{13}-\frac{3\cos(3t)}{13}+\frac{\sin(3t)}{13}$$

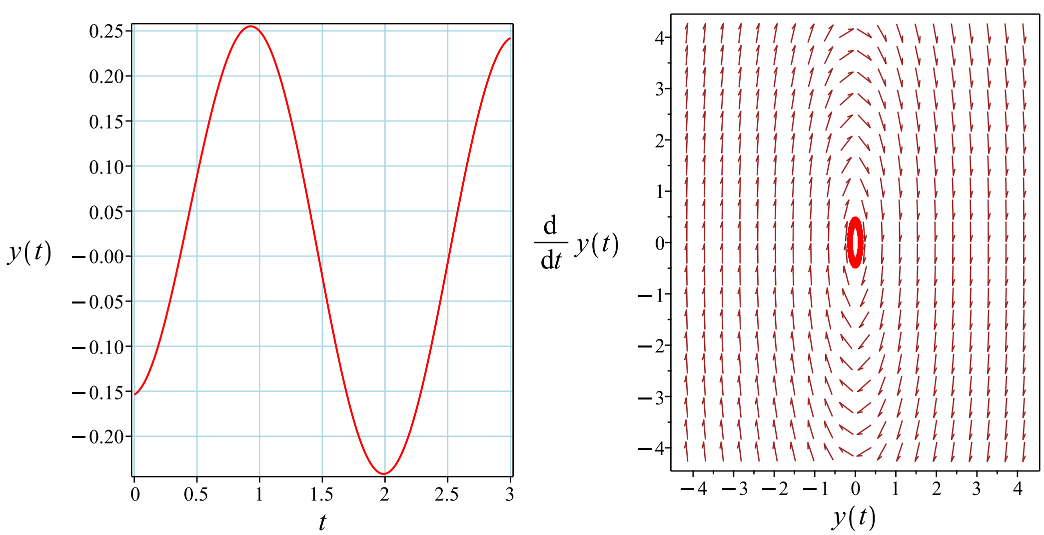
Simplifying the solution gives

$$y = \frac{e^{-2t}}{13}-\frac{3\cos(3t)}{13}+\frac{\sin(3t)}{13}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2t}}{13}-\frac{3\cos(3t)}{13}+\frac{\sin(3t)}{13} \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{-2t}}{13} - \frac{3\cos(3t)}{13} + \frac{\sin(3t)}{13}$$

Verified OK.

### 4.5.2 Maple step by step solution

Let's solve

$\left[y'' + 9y = e^{-2t}, y(0) = -\frac{2}{13}, y'\Big|_{\{t=0\}} = \frac{1}{13}\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3t) + c_2 \sin(3t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ◦ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = e^{-2t}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(3t)\left(\int e^{-2t}\sin(3t)dt\right)}{3} + \frac{\sin(3t)\left(\int e^{-2t}\cos(3t)dt\right)}{3}$

- ○ Compute integrals

  $y_p(t) = \frac{e^{-2t}}{13}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1\cos(3t) + c_2\sin(3t) + \frac{e^{-2t}}{13}$

- □ Check validity of solution $y = c_1\cos(3t) + c_2\sin(3t) + \frac{e^{-2t}}{13}$

- ○ Use initial condition $y(0) = -\frac{2}{13}$

  $-\frac{2}{13} = c_1 + \frac{1}{13}$

- ○ Compute derivative of the solution

  $y' = -3c_1\sin(3t) + 3c_2\cos(3t) - \frac{2e^{-2t}}{13}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = \frac{1}{13}$

  $\frac{1}{13} = -\frac{2}{13} + 3c_2$

- ○ Solve for $c_1$ and $c_2$

  $\left\{c_1 = -\frac{3}{13}, c_2 = \frac{1}{13}\right\}$

- ○ Substitute constant values into general solution and simplify

  $y = \frac{e^{-2t}}{13} - \frac{3\cos(3t)}{13} + \frac{\sin(3t)}{13}$

- ● Solution to the IVP

  $y = \frac{e^{-2t}}{13} - \frac{3\cos(3t)}{13} + \frac{\sin(3t)}{13}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.0 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)+9*y(t)=exp(-2*t),y(0) = -2/13, D(y)(0) = 1/13],y(t), singsol=all)
```

$$y(t) = -\frac{3\cos(3t)}{13} + \frac{\sin(3t)}{13} + \frac{e^{-2t}}{13}$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 25

```
DSolve[{y''[t]+9*y[t]==Exp[-2*t],{y[0]==-2/13,y'[0]==1/13}},y[t],t,IncludeSingularSolutions
```

$$y(t) \to \frac{1}{13}\left(e^{-2t} + \sin(3t) - 3\cos(3t)\right)$$

## 4.6 problem Problem 2(f)



Internal problem ID [12314]
Internal file name [`OUTPUT/10966_Monday_October_02_2023_02_47_38_AM_95537058/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y'' - 3y' + 17y = 17t - 1}$$

With initial conditions

$$[y(0) = -1, y'(0) = 2]$$

### 4.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -\frac{3}{2} \\ q(t) &= \frac{17}{2} \\ F &= \frac{17t}{2} - \frac{1}{2} \end{aligned}$$

Hence the ode is

$$y'' - \frac{3y'}{2} + \frac{17y}{2} = \frac{17t}{2} - \frac{1}{2}$$

The domain of $p(t) = -\frac{3}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{17}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{17t}{2} - \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2s^2Y(s) - 2y'(0) - 2sy(0) - 3sY(s) + 3y(0) + 17Y(s) = \frac{17}{s^2} - \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -1 \\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^2Y(s) - 7 + 2s - 3sY(s) + 17Y(s) = \frac{17}{s^2} - \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2s^3 - 7s^2 + s - 17}{s^2\,(2s^2 - 3s + 17)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{19}{34} - \frac{125 i \sqrt{127}}{4318}}{s - \frac{3}{4} - \frac{i \sqrt{127}}{4}} + \frac{-\frac{19}{34} + \frac{125 i \sqrt{127}}{4318}}{s - \frac{3}{4} + \frac{i \sqrt{127}}{4}} + \frac{1}{s^2} + \frac{2}{17 s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{19}{34} - \frac{125 i \sqrt{127}}{4318}}{s - \frac{3}{4} - \frac{i \sqrt{127}}{4}}\right) = -\frac{\left(125 i \sqrt{127} + 2413\right) \mathrm{e}^{\frac{\left(3+i \sqrt{127}\right)\left(\frac{3t}{4\left(\frac{3}{4}+\frac{i \sqrt{127}}{4}\right)} + \frac{i \sqrt{127}\, t}{3+i \sqrt{127}}\right)}{4}}}{4318}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{19}{34} + \frac{125 i \sqrt{127}}{4318}}{s - \frac{3}{4} + \frac{i \sqrt{127}}{4}}\right) = -\frac{\left(2413 - 125 i \sqrt{127}\right) \mathrm{e}^{\frac{\left(3-i \sqrt{127}\right)\left(\frac{3t}{4\left(\frac{3}{4}-\frac{i \sqrt{127}}{4}\right)} - \frac{i \sqrt{127}\, t}{4\left(\frac{3}{4}-\frac{i \sqrt{127}}{4}\right)}\right)}{4}}}{4318}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

$$\mathcal{L}^{-1}\left(\frac{2}{17 s}\right) = \frac{2}{17}$$

Adding the above results and simplifying gives

$$y = \frac{2}{17} + t + \frac{\left(125 \sqrt{127}\, \sin\left(\frac{\sqrt{127}\, t}{4}\right) - 2413 \cos\left(\frac{\sqrt{127}\, t}{4}\right)\right) \mathrm{e}^{\frac{3t}{4}}}{2159}$$

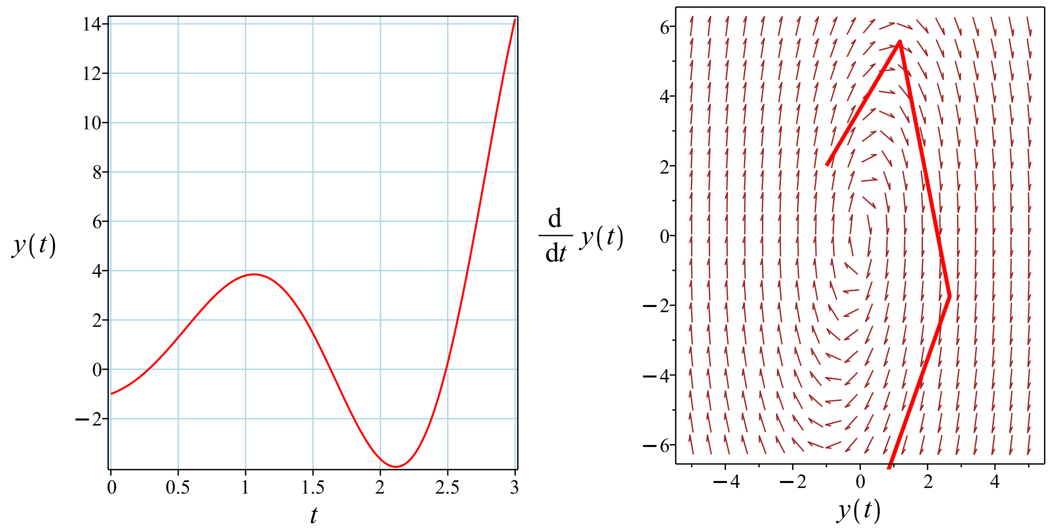
Simplifying the solution gives

$$y = \frac{125\, \mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\, t}{4}\right) \sqrt{127}}{2159} - \frac{19\, \mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\, t}{4}\right)}{17} + t + \frac{2}{17}$$

Summary

The solution(s) found are the following

$$y = \frac{125\, \mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\, t}{4}\right) \sqrt{127}}{2159} - \frac{19\, \mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\, t}{4}\right)}{17} + t + \frac{2}{17} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{125\,\mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right)\sqrt{127}}{2159} - \frac{19\,\mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right)}{17} + t + \frac{2}{17}$$

Verified OK.

### 4.6.2 Maple step by step solution

Let's solve

$$\left[2y'' - 3y' + 17y = 17t - 1, y(0) = -1, y'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2} - \frac{17y}{2} + \frac{17t}{2} - \frac{1}{2}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2} + \frac{17y}{2} = \frac{17t}{2} - \frac{1}{2}$

- Characteristic polynomial of homogeneous ODE

  $r^2 - \frac{3}{2}r + \frac{17}{2} = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{\left(\frac{3}{2}\right) \pm \left(\sqrt{-\frac{127}{4}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{3}{4} - \frac{\mathrm{I}\sqrt{127}}{4}, \frac{3}{4} + \frac{\mathrm{I}\sqrt{127}}{4}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right) + c_2 \mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right) + y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \frac{17t}{2} - \frac{1}{2}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right) & \mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right) \\ \frac{3\,\mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right)}{4} - \frac{\mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right)\sqrt{127}}{4} & \frac{3\,\mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right)}{4} + \frac{\mathrm{e}^{\frac{3t}{4}}\sqrt{127} \cos\left(\frac{\sqrt{127}\,t}{4}\right)}{4} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \frac{\sqrt{127}\,\mathrm{e}^{\frac{3t}{2}}}{4}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{2\sqrt{127}\,\mathrm{e}^{\frac{3t}{4}}\left(-\cos\left(\frac{\sqrt{127}\,t}{4}\right)\left(\int (17t-1)\mathrm{e}^{-\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right) dt\right) + \sin\left(\frac{\sqrt{127}\,t}{4}\right)\left(\int (17t-1)\mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right) dt\right)\right)}{127}$

  - Compute integrals

    $y_p(t) = t + \frac{2}{17}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right) + c_2 \mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right) + t + \frac{2}{17}$

□ Check validity of solution $y = c_1 e^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\, t}{4}\right) + c_2 e^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\, t}{4}\right) + t + \frac{2}{17}$

- Use initial condition $y(0) = -1$

  $-1 = c_1 + \frac{2}{17}$

- Compute derivative of the solution

  $$y' = \frac{3c_1 e^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\, t}{4}\right)}{4} - \frac{c_1 e^{\frac{3t}{4}} \sqrt{127}\, \sin\left(\frac{\sqrt{127}\, t}{4}\right)}{4} + \frac{3c_2 e^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\, t}{4}\right)}{4} + \frac{c_2 e^{\frac{3t}{4}} \sqrt{127}\, \cos\left(\frac{\sqrt{127}\, t}{4}\right)}{4} + 1$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = \frac{3c_1}{4} + 1 + \frac{c_2\sqrt{127}}{4}$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = -\frac{19}{17}, c_2 = \frac{125\sqrt{127}}{2159}\right\}$

- Substitute constant values into general solution and simplify

  $$y = \frac{125\, e^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\, t}{4}\right)\sqrt{127}}{2159} - \frac{19\, e^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\, t}{4}\right)}{17} + t + \frac{2}{17}$$

• Solution to the IVP

  $$y = \frac{125\, e^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\, t}{4}\right)\sqrt{127}}{2159} - \frac{19\, e^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\, t}{4}\right)}{17} + t + \frac{2}{17}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.359 (sec). Leaf size: 35

```
dsolve([2*diff(y(t),t$2)-3*diff(y(t),t)+17*y(t)=17*t-1,y(0) = -1, D(y)(0) = 2],y(t), singsol
```

$$y(t) = \frac{125\sqrt{127}\,\mathrm{e}^{\frac{3t}{4}} \sin\left(\frac{\sqrt{127}\,t}{4}\right)}{2159} - \frac{19\,\mathrm{e}^{\frac{3t}{4}} \cos\left(\frac{\sqrt{127}\,t}{4}\right)}{17} + t + \frac{2}{17}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 59

```
DSolve[{2*y''[t]-3*y'[t]+17*y[t]==17*t-1,{y[0]==-1,y'[0]==2}},y[t],t,IncludeSingularSolution
```

$$y(t) \to t + \frac{125e^{3t/4} \sin\left(\frac{\sqrt{127}t}{4}\right)}{17\sqrt{127}} - \frac{19}{17}e^{3t/4} \cos\left(\frac{\sqrt{127}t}{4}\right) + \frac{2}{17}$$

## 4.7 problem Problem 2(g)



Internal problem ID [12315]
Internal file name [`OUTPUT/10967_Monday_October_02_2023_02_47_38_AM_84918379/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(g).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + y = \mathrm{e}^{-t}}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

### 4.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= \mathrm{e}^{-t} \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = e^{-t}$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = \frac{1}{s+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 2sY(s) + Y(s) = \frac{1}{s+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2 + 2s + 2}{(s+1)(s^2 + 2s + 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{(s+1)^3} + \frac{1}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^3}\right) = \frac{t^2 \mathrm{e}^{-t}}{2}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = \mathrm{e}^{-t}$$

Adding the above results and simplifying gives

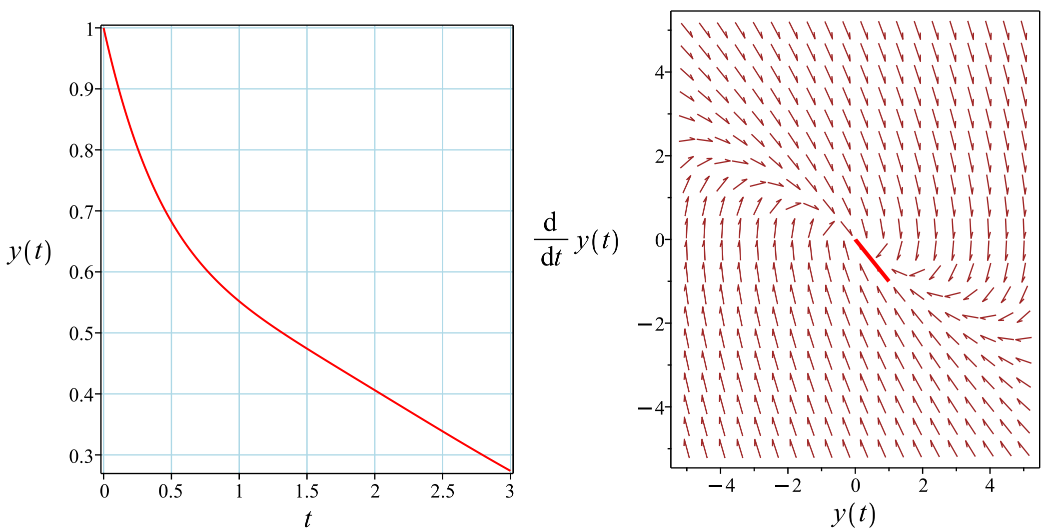
$$y = \frac{\mathrm{e}^{-t}(t^2+2)}{2}$$

Simplifying the solution gives

$$y = \frac{\mathrm{e}^{-t}(t^2+2)}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-t}(t^2+2)}{2} \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{e^{-t}(t^2+2)}{2}$$

Verified OK.

### 4.7.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+y=e^{-t}, y(0)=1, y'\Big|_{\{t=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+1=0$

- Factor the characteristic polynomial

  $(r+1)^2=0$

- Root of the characteristic polynomial

  $r=-1$

- 1st solution of the homogeneous ODE

  $y_1(t)=e^{-t}$

- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t)=t\,e^{-t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-t}+c_2t\,e^{-t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=e^{-t}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t)\,,y_2(t)) = \left[\begin{array}{cc} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t}-t\,\mathrm{e}^{-t} \end{array}\right]$$

- ○ Compute Wronskian

  $W(y_1(t)\,,y_2(t)) = \mathrm{e}^{-2t}$

- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \mathrm{e}^{-t}\left(-\left(\int t dt\right) + \left(\int 1 dt\right) t\right)$

- ○ Compute integrals

  $y_p(t) = \frac{t^2 \mathrm{e}^{-t}}{2}$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} + \frac{t^2 \mathrm{e}^{-t}}{2}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-t} + c_2 t \mathrm{e}^{-t} + \frac{t^2 \mathrm{e}^{-t}}{2}$

- ○ Use initial condition $y(0) = 1$

  $1 = c_1$

- ○ Compute derivative of the solution

  $y' = -c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-t} - c_2 t\,\mathrm{e}^{-t} + t\,\mathrm{e}^{-t} - \frac{t^2 \mathrm{e}^{-t}}{2}$

- ○ Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -c_1 + c_2$

- ○ Solve for $c_1$ and $c_2$

  $\{c_1 = 1, c_2 = 0\}$

- ○ Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{-t}\left(\frac{t^2}{2} + 1\right)$

- ● Solution to the IVP

  $y = \mathrm{e}^{-t}\left(\frac{t^2}{2} + 1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.672 (sec). Leaf size: 15

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=exp(-t),y(0) = 1, D(y)(0) = -1],y(t), singsol=all
```

$$y(t) = \frac{e^{-t}(t^2 + 2)}{2}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 19

```
DSolve[{y''[t]+2*y'[t]+y[t]==Exp[-t],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \frac{1}{2}e^{-t}(t^2 + 2)$$

## 4.8 problem Problem 2(h)



Internal problem ID [12316]
Internal file name [`OUTPUT/10968_Monday_October_02_2023_02_47_38_AM_16016325/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(h).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' + 5y = t + 2}$$

With initial conditions

$$[y(0) = 4, y'(0) = 1]$$

### 4.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -2 \\ q(t) &= 5 \\ F &= t + 2 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = t + 2$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = t + 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{2s+1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 4\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 7 - 4s - 2sY(s) + 5Y(s) = \frac{2s+1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^3 - 7s^2 + 2s + 1}{s^2\left(s^2 - 2s + 5\right)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{5s^2} + \frac{12}{25s} + \frac{\frac{44}{25} + \frac{17i}{25}}{s - 1 - 2i} + \frac{\frac{44}{25} - \frac{17i}{25}}{s - 1 + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{5s^2}\right) = \frac{t}{5}$$
$$\mathcal{L}^{-1}\left(\frac{12}{25s}\right) = \frac{12}{25}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{44}{25}+\frac{17i}{25}}{s-1-2i}\right) = \left(\frac{44}{25}+\frac{17i}{25}\right) e^{(1+2i)t}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{44}{25}-\frac{17i}{25}}{s-1+2i}\right) = \left(\frac{44}{25}-\frac{17i}{25}\right) e^{(1-2i)t}$$

Adding the above results and simplifying gives

$$y = \frac{12}{25} + \frac{t}{5} + \frac{2\,e^{t}(44\cos(2t) - 17\sin(2t))}{25}$$

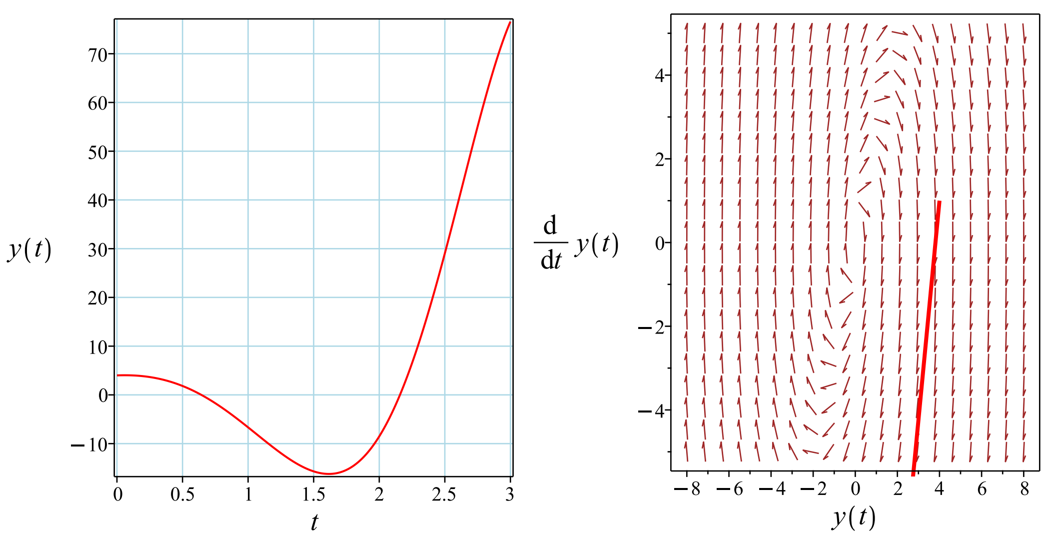
Simplifying the solution gives

$$y = \frac{88\,e^{t}\cos(2t)}{25} - \frac{34\,e^{t}\sin(2t)}{25} + \frac{t}{5} + \frac{12}{25}$$

Summary

The solution(s) found are the following

$$y = \frac{88\,e^{t}\cos(2t)}{25} - \frac{34\,e^{t}\sin(2t)}{25} + \frac{t}{5} + \frac{12}{25} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{88\,\mathrm{e}^{t}\cos\left(2t\right)}{25}-\frac{34\,\mathrm{e}^{t}\sin\left(2t\right)}{25}+\frac{t}{5}+\frac{12}{25}$$

Verified OK.

### 4.8.2 Maple step by step solution

Let's solve

$\left[y''-2y'+5y=t+2, y(0)=4, y'\Big|_{\{t=0\}}=1\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2-2r+5=0$

- Use quadratic formula to solve for $r$

  $r=\frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(1-2\,\mathrm{I},1+2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{t}\cos\left(2t\right)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{t}\sin\left(2t\right)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{t}\cos\left(2t\right)+c_2\mathrm{e}^{t}\sin\left(2t\right)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=t+2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^t \cos(2t) & e^t \sin(2t) \\ e^t \cos(2t) - 2\,e^t \sin(2t) & e^t \sin(2t) + 2\,e^t \cos(2t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{2t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^t\left(\cos(2t)\left(\int (t+2)\sin(2t)e^{-t}dt\right) - \sin(2t)\left(\int (t+2)\cos(2t)e^{-t}dt\right)\right)}{2}$

- Compute integrals

  $y_p(t) = \frac{12}{25} + \frac{t}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{12}{25} + \frac{t}{5}$

☐ Check validity of solution $y = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + \frac{12}{25} + \frac{t}{5}$

- Use initial condition $y(0) = 4$

  $4 = c_1 + \frac{12}{25}$

- Compute derivative of the solution

  $y' = c_1 e^t \cos(2t) - 2c_1 e^t \sin(2t) + c_2 e^t \sin(2t) + 2c_2 e^t \cos(2t) + \frac{1}{5}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = c_1 + \frac{1}{5} + 2c_2$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = \frac{88}{25}, c_2 = -\frac{34}{25}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{88\,e^t \cos(2t)}{25} - \frac{34\,e^t \sin(2t)}{25} + \frac{t}{5} + \frac{12}{25}$

- Solution to the IVP

  $y = \frac{88\,e^t \cos(2t)}{25} - \frac{34\,e^t \sin(2t)}{25} + \frac{t}{5} + \frac{12}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.688 (sec). Leaf size: 26

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)+5*y(t)=2+t,y(0) = 4, D(y)(0) = 1],y(t), singsol=all)
```

$$y(t) = \frac{88\cos(2t)\,e^{t}}{25} - \frac{34\sin(2t)\,e^{t}}{25} + \frac{t}{5} + \frac{12}{25}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 32

```
DSolve[{y''[t]-2*y'[t]+5*y[t]==2+t,{y[0]==4,y'[0]==1}},y[t],t,IncludeSingularSolutions -> Tr
```

$$y(t) \to \frac{1}{25}\left(5t - 34e^{t}\sin(2t) + 88e^{t}\cos(2t) + 12\right)$$

## 4.9 problem Problem 2(i)



Internal problem ID [12317]
Internal file name [OUTPUT/10969_Monday_October_02_2023_02_47_39_AM_27159909/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(i).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{2y' + y = e^{-\frac{t}{2}}}$$

With initial conditions

$$[y(0) = -1]$$

### 4.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{2}$$
$$q(t) = \frac{e^{-\frac{t}{2}}}{2}$$

Hence the ode is

$$y' + \frac{y}{2} = \frac{e^{-\frac{t}{2}}}{2}$$

The domain of $p(t) = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{e^{-\frac{t}{2}}}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

### 4.9.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2sY(s) - 2y(0) + Y(s) = \frac{2}{2s+1} \tag{1}$$

Replacing initial condition gives

$$2sY(s) + 2 + Y(s) = \frac{2}{2s+1}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{4s}{(2s+1)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2\left(s+\frac{1}{2}\right)^2} - \frac{1}{s+\frac{1}{2}}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2\left(s+\frac{1}{2}\right)^2}\right) = \frac{e^{-\frac{t}{2}} t}{2}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{s+\frac{1}{2}}\right) = -e^{-\frac{t}{2}}$$

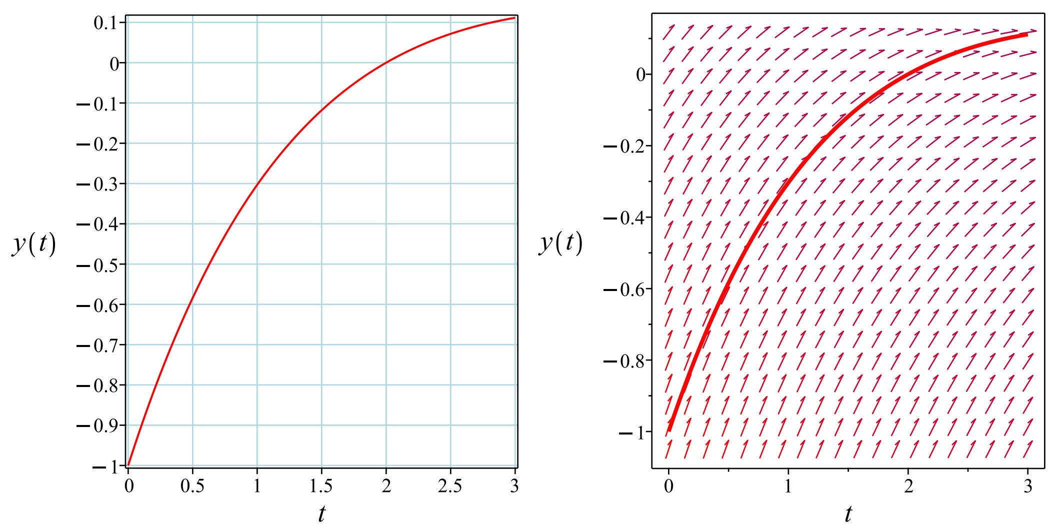
Adding the above results and simplifying gives

$$y = \frac{e^{-\frac{t}{2}}(-2+t)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{t}{2}}(-2+t)}{2} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{t}{2}}(-2+t)}{2}$$

Verified OK.

### 4.9.3 Maple step by step solution

Let's solve

$\left[2y' + y = \mathrm{e}^{-\frac{t}{2}}, y(0) = -1\right]$

- Highest derivative means the order of the ODE is 1

  $y'$

- Isolate the derivative

  $y' = -\frac{y}{2} + \frac{\mathrm{e}^{-\frac{t}{2}}}{2}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2} = \frac{\mathrm{e}^{-\frac{t}{2}}}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \frac{\mu(t)\mathrm{e}^{-\frac{t}{2}}}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y' + \frac{y}{2}\right) = \mu'(t)\, y + \mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)}{2}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{\frac{t}{2}}$

- Integrate both sides with respect to $t$

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \frac{\mu(t)\mathrm{e}^{-\frac{t}{2}}}{2} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \frac{\mu(t)\mathrm{e}^{-\frac{t}{2}}}{2} dt + c_1$

- Solve for $y$

  $y = \frac{\int \frac{\mu(t)\mathrm{e}^{-\frac{t}{2}}}{2} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{\frac{t}{2}}$

  $y = \frac{\int \frac{\mathrm{e}^{\frac{t}{2}}\mathrm{e}^{-\frac{t}{2}}}{2} dt + c_1}{\mathrm{e}^{\frac{t}{2}}}$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{t}{2}+c_1}{e^{\frac{t}{2}}}$$

- Simplify

$$y = \frac{e^{-\frac{t}{2}}(t+2c_1)}{2}$$

- Use initial condition $y(0) = -1$

$$-1 = c_1$$

- Solve for $c_1$

$$c_1 = -1$$

- Substitute $c_1 = -1$ into general solution and simplify

$$y = \frac{e^{-\frac{t}{2}}(-2+t)}{2}$$

- Solution to the IVP

$$y = \frac{e^{-\frac{t}{2}}(-2+t)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 4.578 (sec). Leaf size: 13

```
dsolve([2*diff(y(t),t)+y(t)=exp(-t/2),y(0) = -1],y(t), singsol=all)
```

$$y(t) = \frac{e^{-\frac{t}{2}}(t-2)}{2}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 19

```
DSolve[{2*y'[t]+y[t]==Exp[-t/2],{y[0]==-1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{2}e^{-t/2}(t-2)$$

## 4.10 problem Problem 2(i)[j]



Internal problem ID [12318]
Internal file name [`OUTPUT/10970_Monday_October_02_2023_02_47_39_AM_81295945/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(i)[j].
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 8y' + 20y = \sin(2t)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -4]$$

### 4.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 8 \\ q(t) &= 20 \\ F &= \sin(2t) \end{aligned}$$

Hence the ode is

$$y'' + 8y' + 20y = \sin(2t)$$

The domain of $p(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 8sY(s) - 8y(0) + 20Y(s) = \frac{2}{s^2+4} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 - s + 8sY(s) + 20Y(s) = \frac{2}{s^2+4}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3 + 4s^2 + 4s + 18}{(s^2+4)(s^2+8s+20)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{64} - \frac{i}{64}}{s-2i} + \frac{-\frac{1}{64} + \frac{i}{64}}{s+2i} + \frac{\frac{33}{64} - \frac{i}{64}}{s+4-2i} + \frac{\frac{33}{64} + \frac{i}{64}}{s+4+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{64}-\frac{i}{64}}{s-2i}\right) = \left(-\frac{1}{64}-\frac{i}{64}\right)\mathrm{e}^{2it}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{64}+\frac{i}{64}}{s+2i}\right) = \left(-\frac{1}{64}+\frac{i}{64}\right)\mathrm{e}^{-2it}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{33}{64}-\frac{i}{64}}{s+4-2i}\right) = \left(\frac{33}{64}-\frac{i}{64}\right)\mathrm{e}^{(-4+2i)t}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{33}{64}+\frac{i}{64}}{s+4+2i}\right) = \left(\frac{33}{64}+\frac{i}{64}\right)\mathrm{e}^{(-4-2i)t}$$

Adding the above results and simplifying gives

$$y = -\frac{\cos(2t)\left(1-33\,\mathrm{e}^{-4t}\right)}{32} + \frac{\sin(2t)\left(1+\mathrm{e}^{-4t}\right)}{32}$$

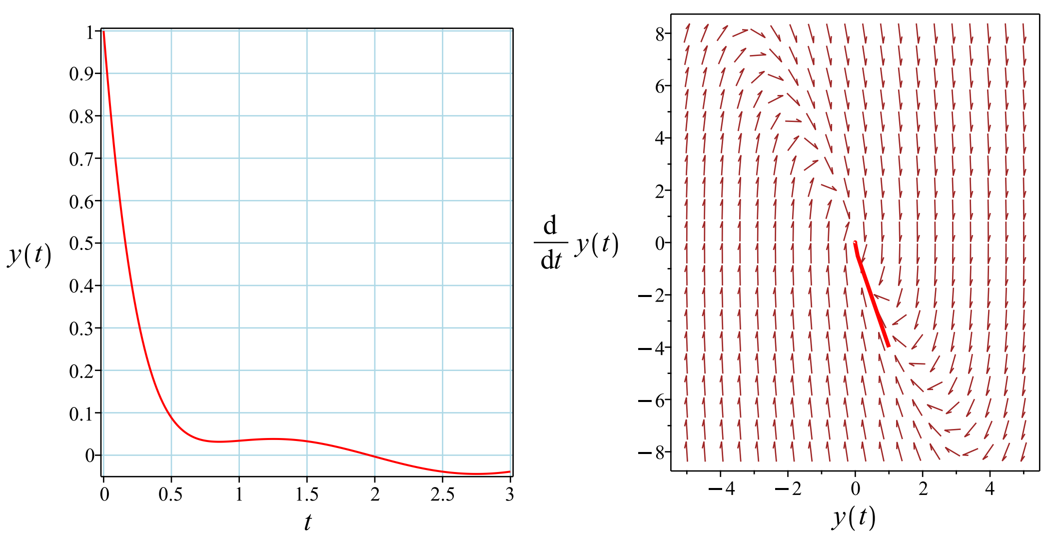
Simplifying the solution gives

$$y = \frac{\left(-1+33\,\mathrm{e}^{-4t}\right)\cos(2t)}{32} + \frac{\sin(2t)\left(1+\mathrm{e}^{-4t}\right)}{32}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-1+33\,\mathrm{e}^{-4t}\right)\cos(2t)}{32} + \frac{\sin(2t)\left(1+\mathrm{e}^{-4t}\right)}{32} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{(-1+33\,e^{-4t})\cos(2t)}{32} + \frac{\sin(2t)\,(1+e^{-4t})}{32}$$

Verified OK.

### 4.10.2 Maple step by step solution

Let's solve

$$\left[y''+8y'+20y=\sin(2t)\,, y(0)=1, y'\Big|_{\{t=0\}}=-4\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2+8r+20=0$

- Use quadratic formula to solve for $r$

  $r=\frac{(-8)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r=(-4-2\,\mathrm{I},-4+2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(2t)\,e^{-4t}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(2t)\,e^{-4t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(2t)\,e^{-4t}+c_2\sin(2t)\,e^{-4t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\sin(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t)\,e^{-4t} & \sin(2t)\,e^{-4t} \\ -2\sin(2t)\,e^{-4t} - 4\cos(2t)\,e^{-4t} & 2\cos(2t)\,e^{-4t} - 4\sin(2t)\,e^{-4t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-8t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{e^{-4t}\left(\sin(2t)\left(\int \sin(4t)e^{4t}dt\right) - 2\cos(2t)\left(\int e^{4t}\sin(2t)^2 dt\right)\right)}{4}$

- Compute integrals

  $y_p(t) = -\frac{\cos(2t)}{32} + \frac{\sin(2t)}{32}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2t)\,e^{-4t} + c_2\sin(2t)\,e^{-4t} - \frac{\cos(2t)}{32} + \frac{\sin(2t)}{32}$

□ Check validity of solution $y = c_1\cos(2t)\,e^{-4t} + c_2\sin(2t)\,e^{-4t} - \frac{\cos(2t)}{32} + \frac{\sin(2t)}{32}$

- Use initial condition $y(0) = 1$

  $1 = c_1 - \frac{1}{32}$

- Compute derivative of the solution

  $y' = -2c_1\sin(2t)\,e^{-4t} - 4c_1\cos(2t)\,e^{-4t} + 2c_2\cos(2t)\,e^{-4t} - 4c_2\sin(2t)\,e^{-4t} + \frac{\sin(2t)}{16} + \frac{\cos(2t)}{16}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -4$

  $-4 = \frac{1}{16} - 4c_1 + 2c_2$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = \frac{33}{32}, c_2 = \frac{1}{32}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(-1+33\,e^{-4t})\cos(2t)}{32} + \frac{\sin(2t)(1+e^{-4t})}{32}$

- Solution to the IVP

  $y = \frac{(-1+33\,e^{-4t})\cos(2t)}{32} + \frac{\sin(2t)(1+e^{-4t})}{32}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.531 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$2)+8*diff(y(t),t)+20*y(t)=sin(2*t),y(0) = 1, D(y)(0) = -4],y(t), singsol
```

$$y(t) = \frac{(-1 + 33\,\mathrm{e}^{-4t})\cos(2t)}{32} + \frac{\sin(2t)\,(1 + \mathrm{e}^{-4t})}{32}$$

✓ Solution by Mathematica

Time used: 0.292 (sec). Leaf size: 40

```
DSolve[{y''[t]+8*y'[t]+20*y[t]==Sin[2*t],{y[0]==1,y'[0]==-4}},y[t],t,IncludeSingularSolution
```

$$y(t) \to \frac{1}{32}e^{-4t}\left(\left(e^{4t}+1\right)\sin(2t) - \left(e^{4t}-33\right)\cos(2t)\right)$$

## 4.11 problem Problem 2(j)[k]



Internal problem ID [12319]
Internal file name [`OUTPUT/10971_Monday_October_02_2023_02_47_39_AM_43924496/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(j)[k].
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{4y'' - 4y' + y = t^2}$$

With initial conditions

$$[y(0) = -12, y'(0) = 7]$$

### 4.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \frac{1}{4} \\ F &= \frac{t^2}{4} \end{aligned}$$

Hence the ode is

$$y'' - y' + \frac{y}{4} = \frac{t^2}{4}$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{t^2}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) - 4sY(s) + 4y(0) + Y(s) = \frac{2}{s^3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -12\\ y'(0) &= 7\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) - 76 + 48s - 4sY(s) + Y(s) = \frac{2}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2(24s^4 - 38s^3 - 1)}{s^3(4s^2 - 4s + 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s^3} + \frac{8}{s^2} + \frac{24}{s} + \frac{17}{\left(s - \frac{1}{2}\right)^2} - \frac{36}{s - \frac{1}{2}}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) &= t^2 \\
\mathcal{L}^{-1}\left(\frac{8}{s^2}\right) &= 8t \\
\mathcal{L}^{-1}\left(\frac{24}{s}\right) &= 24 \\
\mathcal{L}^{-1}\left(\frac{17}{\left(s - \frac{1}{2}\right)^2}\right) &= 17t\, e^{\frac{t}{2}} \\
\mathcal{L}^{-1}\left(-\frac{36}{s - \frac{1}{2}}\right) &= -36\, e^{\frac{t}{2}}
\end{aligned}$$

Adding the above results and simplifying gives

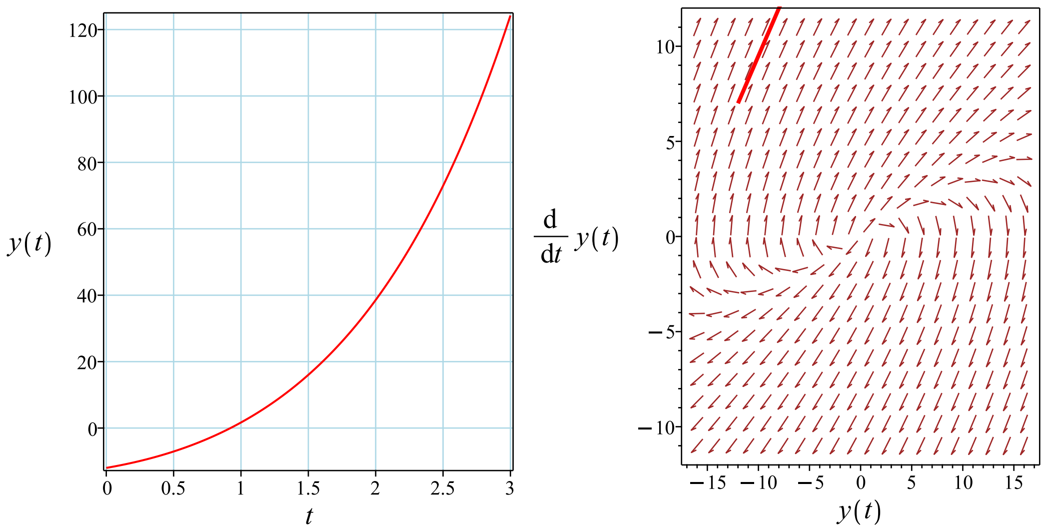
$$y = t^2 + 8t + 24 + e^{\frac{t}{2}}(17t - 36)$$

Simplifying the solution gives

$$y = t^2 + 8t + 24 + e^{\frac{t}{2}}(17t - 36)$$

<u>Summary</u>
The solution(s) found are the following

$$y = t^2 + 8t + 24 + e^{\frac{t}{2}}(17t - 36) \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = t^2 + 8t + 24 + e^{\frac{t}{2}}(17t - 36)$$

Verified OK.

### 4.11.2 Maple step by step solution

Let's solve

$$\left[4y'' - 4y' + y = t^2, y(0) = -12, y'\Big|_{\{t=0\}} = 7\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = y' - \frac{y}{4} + \frac{t^2}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y' + \frac{y}{4} = \frac{t^2}{4}$

- Characteristic polynomial of homogeneous ODE

  $r^2 - r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

$\frac{(2r-1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{1}{2}$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{\frac{t}{2}}$

- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t) = t\,e^{\frac{t}{2}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\frac{t}{2}} + c_2 t\,e^{\frac{t}{2}} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \frac{t^2}{4}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \left[\begin{array}{cc} e^{\frac{t}{2}} & t\,e^{\frac{t}{2}} \\ \frac{e^{\frac{t}{2}}}{2} & e^{\frac{t}{2}} + \frac{t\,e^{\frac{t}{2}}}{2} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = e^{t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{e^{\frac{t}{2}}\left(\int t^3 e^{-\frac{t}{2}}dt - \left(\int e^{-\frac{t}{2}} t^2 dt\right)t\right)}{4}$

  - Compute integrals

    $y_p(t) = t^2 + 8t + 24$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{\frac{t}{2}} + c_2 t\,e^{\frac{t}{2}} + t^2 + 8t + 24$

□ Check validity of solution $y = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}} + t^2 + 8t + 24$

  - Use initial condition $y(0) = -12$

    $-12 = c_1 + 24$

- Compute derivative of the solution

  $$y' = \frac{c_1 e^{\frac{t}{2}}}{2} + c_2 e^{\frac{t}{2}} + \frac{c_2 t\, e^{\frac{t}{2}}}{2} + 2t + 8$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 7$

  $$7 = \frac{c_1}{2} + c_2 + 8$$

- Solve for $c_1$ and $c_2$

  $$\{c_1 = -36, c_2 = 17\}$$

- Substitute constant values into general solution and simplify

  $$y = t^2 + 8t + 24 + e^{\frac{t}{2}}(17t - 36)$$

- Solution to the IVP

  $$y = t^2 + 8t + 24 + e^{\frac{t}{2}}(17t - 36)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.219 (sec). Leaf size: 22

```
dsolve([4*diff(y(t),t$2)-4*diff(y(t),t)+y(t)=t^2,y(0) = -12, D(y)(0) = 7],y(t), singsol=all)
```

$$y(t) = t^2 + 8t + 24 + e^{\frac{t}{2}}(-36 + 17t)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 26

```
DSolve[{4*y''[t]-4*y'[t]+y[t]==t^2,{y[0]==-12,y'[0]==7}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to t^2 + 8t + e^{t/2}(17t - 36) + 24$$

## 4.12 problem Problem 2(k)[l]



Internal problem ID [12320]
Internal file name [`OUTPUT/10972_Monday_October_02_2023_02_47_39_AM_45017633/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(k)[l].
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2y'' + y' - y = 4\sin(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = -4]$$

### 4.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{2} \\ q(t) &= -\frac{1}{2} \\ F &= 2\sin(t) \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{2} - \frac{y}{2} = 2\sin(t)$$

The domain of $p(t) = \frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -\frac{1}{2}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 2\sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$2s^2Y(s) - 2y'(0) - 2sy(0) + sY(s) - y(0) - Y(s) = \frac{4}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -4\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$2s^2Y(s) + 8 + sY(s) - Y(s) = \frac{4}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{4(2s^2+1)}{(s^2+1)(2s^2+s-1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{8}{5\left(s-\frac{1}{2}\right)}+\frac{-\frac{1}{5}+\frac{3i}{5}}{s-i}+\frac{-\frac{1}{5}-\frac{3i}{5}}{s+i}+\frac{2}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{8}{5\left(s-\frac{1}{2}\right)}\right) &= -\frac{8\,\mathrm{e}^{\frac{t}{2}}}{5} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{5}+\frac{3i}{5}}{s-i}\right) &= \left(-\frac{1}{5}+\frac{3i}{5}\right)\mathrm{e}^{it} \\
\mathcal{L}^{-1}\left(\frac{-\frac{1}{5}-\frac{3i}{5}}{s+i}\right) &= \left(-\frac{1}{5}-\frac{3i}{5}\right)\mathrm{e}^{-it} \\
\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) &= 2\,\mathrm{e}^{-t}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = -\frac{2\cos\left(t\right)}{5}-\frac{6\sin\left(t\right)}{5}+2\,\mathrm{e}^{-t}-\frac{8\,\mathrm{e}^{\frac{t}{2}}}{5}$$

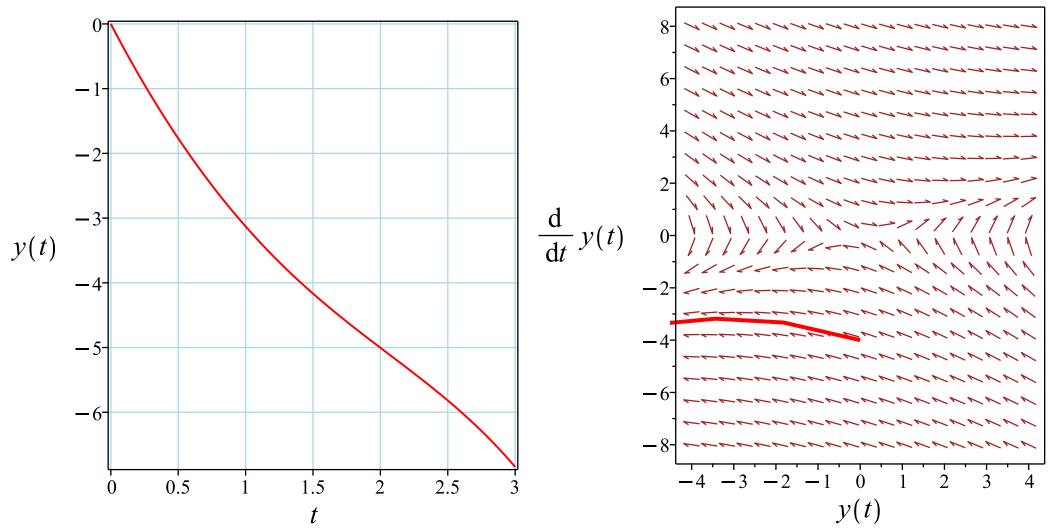
Simplifying the solution gives

$$y = -\frac{2\left(4\,\mathrm{e}^{\frac{3t}{2}}-5+\left(\cos\left(t\right)+3\sin\left(t\right)\right)\mathrm{e}^{t}\right)\mathrm{e}^{-t}}{5}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\left(4\,\mathrm{e}^{\frac{3t}{2}}-5+\left(\cos\left(t\right)+3\sin\left(t\right)\right)\mathrm{e}^{t}\right)\mathrm{e}^{-t}}{5} \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2\left(4\,e^{\frac{3t}{2}} - 5 + (\cos(t) + 3\sin(t))\,e^{t}\right)e^{-t}}{5}$$

Verified OK.

### 4.12.2 Maple step by step solution

Let's solve

$$\left[2y'' + y' - y = 4\sin(t)\,, y(0) = 0, y'\Big|_{\{t=0\}} = -4\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2} + \frac{y}{2} + 2\sin(t)$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2} - \frac{y}{2} = 2\sin(t)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{1}{2}r - \frac{1}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(r+1)(2r-1)}{2} = 0$
- Roots of the characteristic polynomial

  $r = \left(-1, \frac{1}{2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{\frac{t}{2}} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 2\sin(t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & \mathrm{e}^{\frac{t}{2}} \\ -\mathrm{e}^{-t} & \frac{\mathrm{e}^{\frac{t}{2}}}{2} \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \frac{3\,\mathrm{e}^{-\frac{t}{2}}}{2}$
- ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{4\left(\mathrm{e}^{\frac{3t}{2}}\left(\int \sin(t)\mathrm{e}^{-\frac{t}{2}}dt\right) - \left(\int \mathrm{e}^{t}\sin(t)dt\right)\right)\mathrm{e}^{-t}}{3}$
- ○ Compute integrals

  $y_p(t) = -\frac{2\cos(t)}{5} - \frac{6\sin(t)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{\frac{t}{2}} - \frac{2\cos(t)}{5} - \frac{6\sin(t)}{5}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{\frac{t}{2}} - \frac{2\cos(t)}{5} - \frac{6\sin(t)}{5}$

- ○ Use initial condition $y(0) = 0$

$0 = c_1 + c_2 - \frac{2}{5}$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} + \frac{c_2 e^{\frac{t}{2}}}{2} + \frac{2\sin(t)}{5} - \frac{6\cos(t)}{5}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -4$

  $-4 = -c_1 + \frac{c_2}{2} - \frac{6}{5}$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = 2, c_2 = -\frac{8}{5}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{2\left(4\,e^{\frac{3t}{2}} - 5 + (\cos(t) + 3\sin(t))e^{t}\right)e^{-t}}{5}$

- Solution to the IVP

  $y = -\frac{2\left(4\,e^{\frac{3t}{2}} - 5 + (\cos(t) + 3\sin(t))e^{t}\right)e^{-t}}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.516 (sec). Leaf size: 25

```
dsolve([2*diff(y(t),t$2)+diff(y(t),t)-y(t)=4*sin(t),y(0) = 0, D(y)(0) = -4],y(t), singsol=al
```

$$y(t) = -\frac{2\,e^{-t}\left(4\,e^{\frac{3t}{2}} - 5 + (\cos(t) + 3\sin(t))\,e^{t}\right)}{5}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 34

```
DSolve[{2*y''[t]+y'[t]-y[t]==4*Sin[t],{y[0]==0,y'[0]==-4}},y[t],t,IncludeSingularSolutions -
```

$$y(t) \to \frac{2}{5}\left(5e^{-t} - 4e^{t/2} - 3\sin(t) - \cos(t)\right)$$

## 4.13 problem Problem 2(m)



Internal problem ID [12321]
Internal file name [`OUTPUT/10973_Monday_October_02_2023_02_47_39_AM_19903849/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(m).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = e^{2t}}$$

With initial conditions

$$[y(0) = 1]$$

### 4.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -1$$
$$q(t) = e^{2t}$$

Hence the ode is

$$y' - y = e^{2t}$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = e^{2t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

### 4.13.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{1}{s-2} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 - Y(s) = \frac{1}{s-2}$$

Solving for $Y(s)$ gives

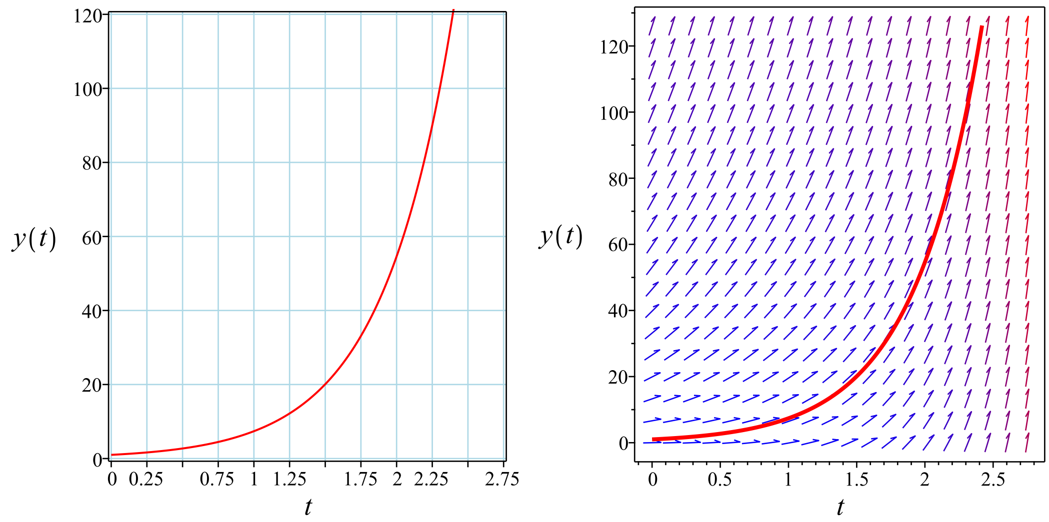
$$Y(s) = \frac{1}{s-2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) \\ &= e^{2t} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{2t} \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{2t}$$

Verified OK.

### 4.13.3 Maple step by step solution

Let's solve

$[y' - y = e^{2t}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  $y'$

- Isolate the derivative

  $y' = y + e^{2t}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = e^{2t}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(y' - y) = \mu(t) e^{2t}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) y)$

  $\mu(t)(y' - y) = \mu'(t) y + \mu(t) y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$
- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$
- Integrate both sides with respect to $t$

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right) dt = \int \mu(t)\, \mathrm{e}^{2t} dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t)\, y = \int \mu(t)\, \mathrm{e}^{2t} dt + c_1$
- Solve for $y$

  $y = \frac{\int \mu(t) \mathrm{e}^{2t} dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $y = \frac{\int \mathrm{e}^{-t} \mathrm{e}^{2t} dt + c_1}{\mathrm{e}^{-t}}$
- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{t} + c_1}{\mathrm{e}^{-t}}$
- Simplify

  $y = \mathrm{e}^{t}(\mathrm{e}^{t} + c_1)$
- Use initial condition $y(0) = 1$

  $1 = 1 + c_1$
- Solve for $c_1$

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \mathrm{e}^{2t}$
- Solution to the IVP

  $y = \mathrm{e}^{2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 4.391 (sec). Leaf size: 8

```
dsolve([diff(y(t),t)-y(t)=exp(2*t),y(0) = 1],y(t), singsol=all)
```

$$y(t) = e^{2t}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 10

```
DSolve[{y'[t]-y[t]==Exp[2*t],{y[0]==1}},y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{2t}$$

## 4.14 problem Problem 2(l)[n]



Internal problem ID [12322]
Internal file name [`OUTPUT/10974_Monday_October_02_2023_02_47_40_AM_13174011/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 2(l)[n].
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3y'' + 5y' - 2y = 7\,\mathrm{e}^{-2t}}$$

With initial conditions

$$[y(0) = 3, y'(0) = 0]$$

### 4.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= \frac{5}{3} \\ q(t) &= -\frac{2}{3} \\ F &= \frac{7\,\mathrm{e}^{-2t}}{3} \end{aligned}$$

Hence the ode is

$$y'' + \frac{5y'}{3} - \frac{2y}{3} = \frac{7\,\mathrm{e}^{-2t}}{3}$$

The domain of $p(t) = \frac{5}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -\frac{2}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{7\,\mathrm{e}^{-2t}}{3}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$3s^2Y(s) - 3y'(0) - 3sy(0) + 5sY(s) - 5y(0) - 2Y(s) = \frac{7}{s+2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$3s^2Y(s) - 15 - 9s + 5sY(s) - 2Y(s) = \frac{7}{s+2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{9s^2 + 33s + 37}{(s+2)\,(3s^2 + 5s - 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3}{s - \frac{1}{3}} - \frac{1}{(s + 2)^2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{3}{s - \frac{1}{3}}\right) = 3\,\mathrm{e}^{\frac{t}{3}}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{(s + 2)^2}\right) = -t\,\mathrm{e}^{-2t}$$

Adding the above results and simplifying gives

$$y = 3\,\mathrm{e}^{\frac{t}{3}} - t\,\mathrm{e}^{-2t}$$

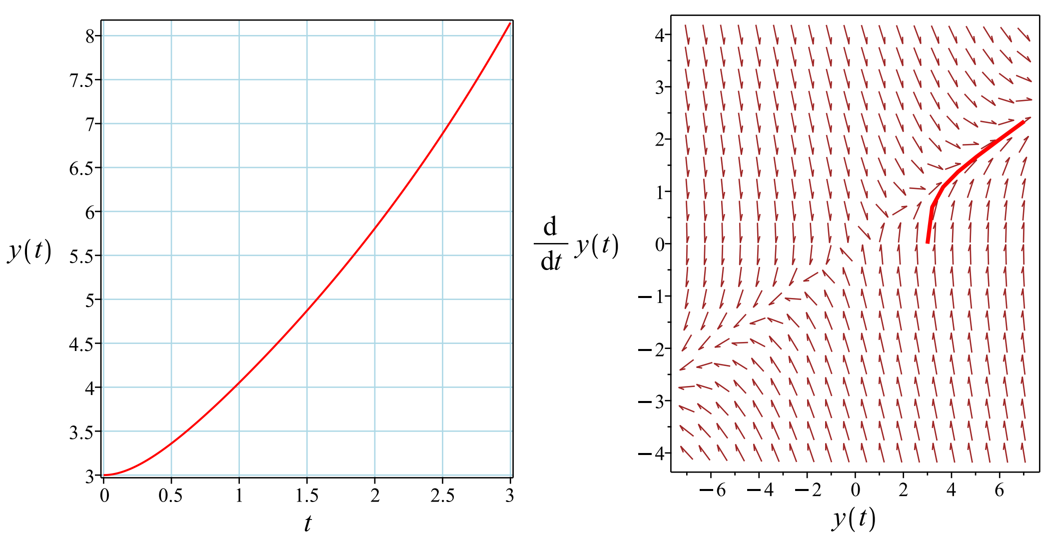
Simplifying the solution gives

$$y = -\left(-3\,\mathrm{e}^{\frac{7t}{3}} + t\right)\mathrm{e}^{-2t}$$

Summary

The solution(s) found are the following

$$y = -\left(-3\,\mathrm{e}^{\frac{7t}{3}} + t\right)\mathrm{e}^{-2t} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\left(-3\,\mathrm{e}^{\frac{7t}{3}} + t\right)\mathrm{e}^{-2t}$$

Verified OK.

### 4.14.2 Maple step by step solution

Let's solve

$\left[3y'' + 5y' - 2y = 7\,\mathrm{e}^{-2t}, y(0) = 3, y'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{3} + \frac{2y}{3} + \frac{7\,\mathrm{e}^{-2t}}{3}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{3} - \frac{2y}{3} = \frac{7\,\mathrm{e}^{-2t}}{3}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{5}{3}r - \frac{2}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(r+2)(3r-1)}{3} = 0$

- Roots of the characteristic polynomial

  $r = \left(-2, \frac{1}{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{3}}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{\frac{t}{3}} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \frac{7\,\mathrm{e}^{-2t}}{3}\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{\frac{t}{3}} \\ -2\,\mathrm{e}^{-2t} & \frac{\mathrm{e}^{\frac{t}{3}}}{3} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \frac{7\,\mathrm{e}^{-\frac{5t}{3}}}{3}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \left(\mathrm{e}^{\frac{7t}{3}}\left(\int \mathrm{e}^{-\frac{7t}{3}}dt\right) - \left(\int 1dt\right)\right)\mathrm{e}^{-2t}$

- Compute integrals

  $y_p(t) = -\frac{(3+7t)\mathrm{e}^{-2t}}{7}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{\frac{t}{3}} - \frac{(3+7t)\mathrm{e}^{-2t}}{7}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{\frac{t}{3}} - \frac{(3+7t)\mathrm{e}^{-2t}}{7}$

- Use initial condition $y(0) = 3$

  $3 = c_1 + c_2 - \frac{3}{7}$

- Compute derivative of the solution

  $y' = -2c_1\mathrm{e}^{-2t} + \frac{c_2\mathrm{e}^{\frac{t}{3}}}{3} - \mathrm{e}^{-2t} + \frac{2(3+7t)\mathrm{e}^{-2t}}{7}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 + \frac{c_2}{3} - \frac{1}{7}$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = \frac{3}{7}, c_2 = 3\right\}$

- Substitute constant values into general solution and simplify

  $y = -\left(-3\,\mathrm{e}^{\frac{7t}{3}} + t\right)\mathrm{e}^{-2t}$

- Solution to the IVP

  $y = -\left(-3\,\mathrm{e}^{\frac{7t}{3}} + t\right)\mathrm{e}^{-2t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.922 (sec). Leaf size: 18

```
dsolve([3*diff(y(t),t$2)+5*diff(y(t),t)-2*y(t)=7*exp(-2*t),y(0) = 3, D(y)(0) = 0],y(t), sing
```

$$y(t) = -\left(-3\,e^{\frac{7t}{3}} + t\right)e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 23

```
DSolve[{3*y''[t]+5*y'[t]-2*y[t]==7*Exp[-2*t],{y[0]==3,y'[0]==0}},y[t],t,IncludeSingularSolut
```

$$y(t) \to 3e^{t/3} - e^{-2t}t$$

## 4.15 problem Problem 3(a)



Internal problem ID [12323]
Internal file name [`OUTPUT/10975_Monday_October_02_2023_02_47_40_AM_58456927/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \text{Heaviside}\,(t) - \text{Heaviside}\,(-2 + t)}$$

With initial conditions

$$[y(0) = 1]$$

### 4.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \text{Heaviside}\,(t) - \text{Heaviside}\,(-2 + t) \end{aligned}$$

Hence the ode is

$$y' + y = \text{Heaviside}\,(t) - \text{Heaviside}\,(-2 + t)$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \text{Heaviside}(t) - \text{Heaviside}(-2+t)$ is

$$\{0 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

### 4.15.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{1 - \mathrm{e}^{-2s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + Y(s) = \frac{1 - \mathrm{e}^{-2s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{-1 + \mathrm{e}^{-2s} - s}{s(s+1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-1 + \mathrm{e}^{-2s} - s}{s(s+1)}\right) \\ &= \text{Heaviside}(-t+2) + \text{Heaviside}(-2+t)\,\mathrm{e}^{-t+2} \end{aligned}$$

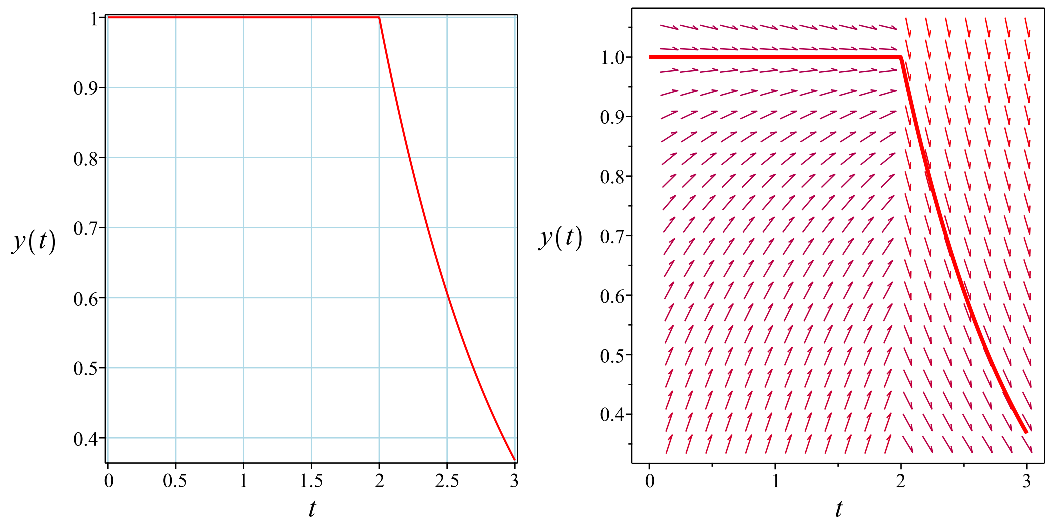
Hence the final solution is

$$y = \text{Heaviside}(-t+2) + \text{Heaviside}(-2+t)\,\mathrm{e}^{-t+2}$$

Summary

The solution(s) found are the following

$$y = \text{Heaviside}\left(-t+2\right)+\text{Heaviside}\left(-2+t\right) \mathrm{e}^{-t+2} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\left(-t+2\right)+\text{Heaviside}\left(-2+t\right) \mathrm{e}^{-t+2}$$

Verified OK.

### 4.15.3 Maple step by step solution

Let's solve

$\left[y'+y=\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t), y(0)=1\right]$

- Highest derivative means the order of the ODE is 1

  $y'$

- Isolate the derivative

  $y'=-y+\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t)$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+y=\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t)\left(y'+y\right)=\mu(t)\left(\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t)\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,y)$

$$\mu(t)\left(y'+y\right)=\mu'(t)\,y+\mu(t)\,y'$$

- Isolate $\mu'(t)$

$$\mu'(t)=\mu(t)$$

- Solve to find the integrating factor

$$\mu(t)=\mathrm{e}^{t}$$

- Integrate both sides with respect to $t$

$$\int\left(\frac{d}{dt}(\mu(t)\,y)\right)dt=\int\mu(t)\left(\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t)\right)dt+c_1$$

- Evaluate the integral on the lhs

$$\mu(t)\,y=\int\mu(t)\left(\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t)\right)dt+c_1$$

- Solve for $y$

$$y=\frac{\int\mu(t)(\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t))dt+c_1}{\mu(t)}$$

- Substitute $\mu(t)=\mathrm{e}^{t}$

$$y=\frac{\int\mathrm{e}^{t}(\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t))dt+c_1}{\mathrm{e}^{t}}$$

- Evaluate the integrals on the rhs

$$y=\frac{-\mathrm{e}^{t}\mathit{Heaviside}(-2+t)+\mathit{Heaviside}(-2+t)\mathrm{e}^{2}+\mathrm{e}^{t}\mathit{Heaviside}(t)-\mathit{Heaviside}(t)+c_1}{\mathrm{e}^{t}}$$

- Simplify

$$y=\mathit{Heaviside}(-2+t)\,\mathrm{e}^{-t+2}-\mathit{Heaviside}(-2+t)+\left(-\mathit{Heaviside}(t)+c_1\right)\mathrm{e}^{-t}+\mathit{Heaviside}(t)$$

- Use initial condition $y(0)=1$

$$1=\mathit{undefined}+c_1$$

- Solve for $c_1$

$$c_1=\mathit{undefined}$$

- Substitute $c_1=\mathit{undefined}$ into general solution and simplify

$$y=\mathit{Heaviside}(-2+t)\,\mathrm{e}^{-t+2}-\mathit{Heaviside}(-2+t)+\left(-\mathit{Heaviside}(t)+\mathit{undefined}\right)\mathrm{e}^{-t}+\mathit{Heaviside}($$

- Solution to the IVP

$$y=\mathit{Heaviside}(-2+t)\,\mathrm{e}^{-t+2}-\mathit{Heaviside}(-2+t)+\left(-\mathit{Heaviside}(t)+\mathit{undefined}\right)\mathrm{e}^{-t}+\mathit{Heaviside}($$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.203 (sec). Leaf size: 22

```
dsolve([diff(y(t),t)+y(t)=Heaviside(t)-Heaviside(t-2),y(0) = 1],y(t), singsol=all)
```

$$y(t) = 1 - \text{Heaviside}\,(t-2) + \text{Heaviside}\,(t-2)\,\mathrm{e}^{2-t}$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 31

```
DSolve[{y'[t]+y[t]==UnitStep[t]-UnitStep[t-2],{y[0]==1}},y[t],t,IncludeSingularSolutions ->
```

$$y(t) \to \left\{ \begin{array}{cc} 1 & 0 \leq t \leq 2 \\ e^{2-t} & t > 2 \\ e^{-t} & \text{True} \end{array} \right.$$

## 4.16 problem Problem 3(b)



Internal problem ID [12324]
Internal file name [`OUTPUT/10976_Monday_October_02_2023_02_47_40_AM_31494725/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = 4t(\text{Heaviside}(t) - \text{Heaviside}(-2+t))}$$

With initial conditions

$$[y(0) = 1]$$

### 4.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = -2$$
$$q(t) = -4t(-\text{Heaviside}(t) + \text{Heaviside}(-2+t))$$

Hence the ode is

$$y' - 2y = -4t(-\text{Heaviside}(t) + \text{Heaviside}(-2+t))$$

The domain of $p(t) = -2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = -4t(-\text{Heaviside}(t) + \text{Heaviside}(-2+t))$ is

$$\{0 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

### 4.16.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 2Y(s) = \frac{4 - 4\,\mathrm{e}^{-2s}(2s+1)}{s^2} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 - 2Y(s) = \frac{4 - 4\,\mathrm{e}^{-2s}(2s+1)}{s^2}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{8\,\mathrm{e}^{-2s}s - s^2 + 4\,\mathrm{e}^{-2s} - 4}{s^2\,(s-2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{8\,\mathrm{e}^{-2s}s - s^2 + 4\,\mathrm{e}^{-2s} - 4}{s^2\,(s-2)}\right) \\ &= -\left(1 + 2t - 5\,\mathrm{e}^{-4+2t}\right)\text{Heaviside}\,(-t+2) + 2\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{-4+2t} \end{aligned}$$

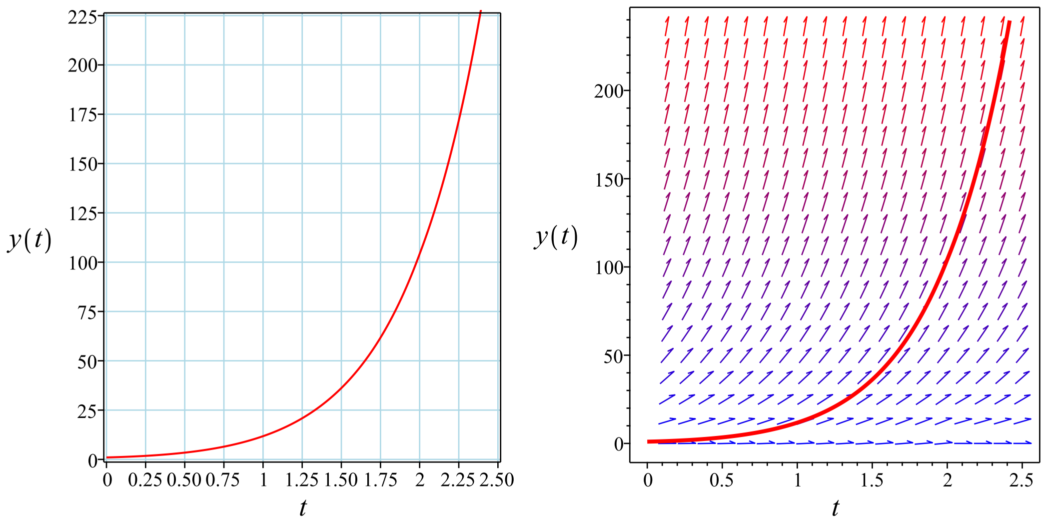
Hence the final solution is

$$y = -\left(1 + 2t - 5\,\mathrm{e}^{-4+2t}\right)\text{Heaviside}\,(-t+2) + 2\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{-4+2t}$$

Summary

The solution(s) found are the following

$$y = -\left(1+2t-5\,\mathrm{e}^{-4+2t}\right) \mathrm{Heaviside}\left(-t+2\right)+2\,\mathrm{e}^{2t}-5\,\mathrm{e}^{-4+2t} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\left(1+2t-5\,\mathrm{e}^{-4+2t}\right) \mathrm{Heaviside}\left(-t+2\right)+2\,\mathrm{e}^{2t}-5\,\mathrm{e}^{-4+2t}$$

Verified OK.

### 4.16.3 Maple step by step solution

Let's solve

$[y'-2y=4t(\mathit{Heaviside}(t)-\mathit{Heaviside}(-2+t))\,, y(0)=1]$

- Highest derivative means the order of the ODE is 1

  $y'$

- Isolate the derivative

  $y'=2y-4t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-2y=-4t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$\mu(t)\left(y'-2y\right) = -4\mu(t)\, t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, y)$

  $\mu(t)\left(y'-2y\right) = \mu'(t)\, y+\mu(t)\, y'$

- Isolate $\mu'(t)$

  $\mu'(t) = -2\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-2t}$

- Integrate both sides with respect to $t$

  $\int \left(\frac{d}{dt}(\mu(t)\, y)\right)dt = \int -4\mu(t)\, t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, y = \int -4\mu(t)\, t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))\, dt + c_1$

- Solve for $y$

  $y = \frac{\int -4\mu(t)t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))dt+c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-2t}$

  $y = \frac{\int -4\,\mathrm{e}^{-2t}t(-\mathit{Heaviside}(t)+\mathit{Heaviside}(-2+t))dt+c_1}{\mathrm{e}^{-2t}}$

- Evaluate the integrals on the rhs

  $y = \frac{-(2t+1)\mathrm{e}^{-2t}\mathit{Heaviside}(t)+\mathit{Heaviside}(t)+(2t+1)\mathrm{e}^{-2t}\mathit{Heaviside}(-2+t)-5\mathit{Heaviside}(-2+t)\mathrm{e}^{-4}+c_1}{\mathrm{e}^{-2t}}$

- Simplify

  $y = -5\,\mathrm{e}^{-4+2t}\mathit{Heaviside}(-2+t)+\mathit{Heaviside}(-2+t)\,(2t+1)+(c_1+\mathit{Heaviside}(t))\,\mathrm{e}^{2t}+(-2t-1$

- Use initial condition $y(0) = 1$

  $1 = \mathit{undefined}+c_1$

- Solve for $c_1$

  $c_1 = \mathit{undefined}$

- Substitute $c_1 = \mathit{undefined}$ into general solution and simplify

  $y = -5\,\mathrm{e}^{-4+2t}\mathit{Heaviside}(-2+t)+\mathit{Heaviside}(-2+t)\,(2t+1)+(\mathit{undefined}+\mathit{Heaviside}(t))\,\mathrm{e}^{2t}+($

- Solution to the IVP

  $y = -5\,\mathrm{e}^{-4+2t}\mathit{Heaviside}(-2+t)+\mathit{Heaviside}(-2+t)\,(2t+1)+(\mathit{undefined}+\mathit{Heaviside}(t))\,\mathrm{e}^{2t}+($

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.0 (sec). Leaf size: 40

```
dsolve([diff(y(t),t)-2*y(t)=4*t*(Heaviside(t)-Heaviside(t-2)),y(0) = 1],y(t), singsol=all)
```

$$y(t) = -5\,\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{2t-4} + 2t\,\mathrm{Heaviside}\,(t-2) - 2t + 2\,\mathrm{e}^{2t} - 1 + \mathrm{Heaviside}\,(t-2)$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 47

```
DSolve[{y'[t]-2*y[t]==4*t*(UnitStep[t]-UnitStep[t-2]),{y[0]==1}},y[t],t,IncludeSingularSolut
```

$$y(t) \to \left\{ \begin{array}{ll} e^{2t} & t < 0 \\ e^{2t-4}(-5+2e^4) & t > 2 \\ -2t + 2e^{2t} - 1 & \text{True} \end{array} \right.$$

## 4.17 problem Problem 3(c)



Internal problem ID [12325]
Internal file name [OUTPUT/10977_Monday_October_02_2023_02_47_40_AM_57838993/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 24\sin(t)(\text{Heaviside}(t) + \text{Heaviside}(t-\pi))}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}p(t) &= 0\\ q(t) &= 9\\ F &= 24\sin(t)(\text{Heaviside}(t) + \text{Heaviside}(t-\pi))\end{aligned}$$

Hence the ode is

$$y'' + 9y = 24\sin(t)(\text{Heaviside}(t) + \text{Heaviside}(t-\pi))$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 24\sin(t)\,(\text{Heaviside}(t) + \text{Heaviside}(t -$
is

$$\{0 \leq t \leq \pi, \pi \leq t \leq \infty, -\infty \leq t \leq 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{24 - 24\,\mathrm{e}^{-s\pi}}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \frac{24 - 24\,\mathrm{e}^{-s\pi}}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{24(-1 + \mathrm{e}^{-s\pi})}{(s^2 + 1)(s^2 + 9)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{24(-1+e^{-s\pi})}{(s^2+1)(s^2+9)}\right) \\
&= 4(1+\text{Heaviside}(t-\pi))\sin(t)^3
\end{aligned}
$$

Hence the final solution is

$$
y = 4(1+\text{Heaviside}(t-\pi))\sin(t)^3
$$

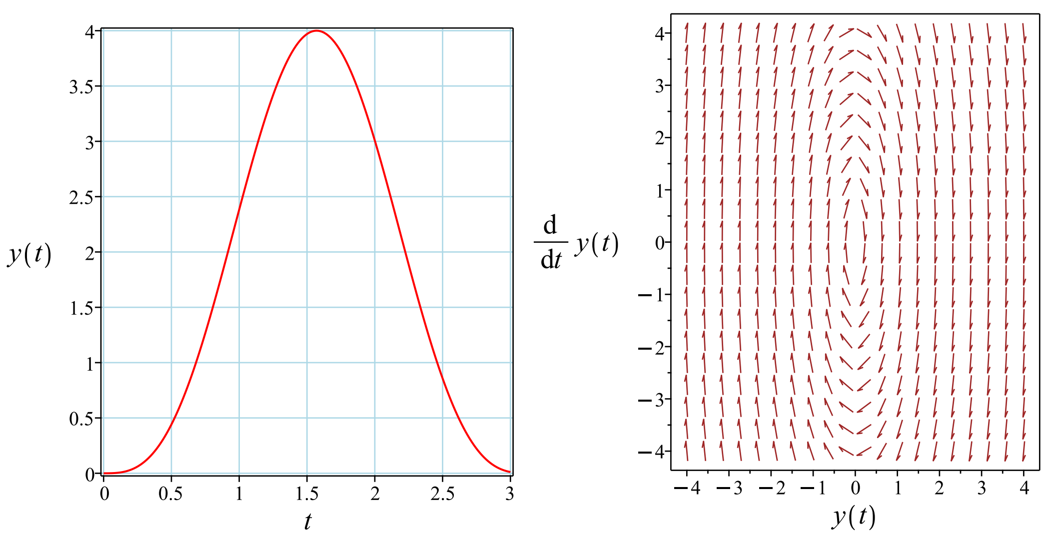
Simplifying the solution gives

$$
y = 4(1+\text{Heaviside}(t-\pi))\sin(t)^3
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = 4(1+\text{Heaviside}(t-\pi))\sin(t)^3 \tag{1}
$$




(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$
y = 4(1+\text{Heaviside}(t-\pi))\sin(t)^3
$$

Verified OK.

### 4.17.2 Maple step by step solution

Let's solve

$$\left[y''+9y=\sin(t)\left(24\mathit{Heaviside}(t)+24\mathit{Heaviside}(t-\pi)\right), y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=-9y+24\sin(t)\left(\mathit{Heaviside}(t)+\mathit{Heaviside}(t-\pi)\right)$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+9y=24\sin(t)\left(\mathit{Heaviside}(t)+\mathit{Heaviside}(t-\pi)\right)$

- Characteristic polynomial of homogeneous ODE

  $r^2+9=0$

- Use quadratic formula to solve for $r$

  $r=\frac{0\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r=(-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(3t)$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(3t)+c_2\sin(3t)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=24\sin(t)\left(\mathit{Heaviside}(t)+\mathit{H}\right.\right.$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 3$
- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -8\cos(3t) \left(\int \sin(3t)\sin(t)\left(\mathit{Heaviside}(t) + \mathit{Heaviside}(t-\pi)\right)dt\right) + 8\sin(3t)\left(\int \cos(3t)\right.$
- Compute integrals

  $y_p(t) = 4\sin(t)^3\left(\mathit{Heaviside}(t) + \mathit{Heaviside}(t-\pi)\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(3t) + c_2\sin(3t) + 4\sin(t)^3\left(\mathit{Heaviside}(t) + \mathit{Heaviside}(t-\pi)\right)$

□ Check validity of solution $y = c_1\cos(3t) + c_2\sin(3t) + 4\sin(t)^3\left(\mathit{Heaviside}(t) + \mathit{Heaviside}(t-\right.$

- Use initial condition $y(0) = 0$

  $0 = \mathit{undefined} + c_1$
- Compute derivative of the solution

  $y' = -3c_1\sin(3t) + 3c_2\cos(3t) + 12\sin(t)^2\left(\mathit{Heaviside}(t) + \mathit{Heaviside}(t-\pi)\right)\cos(t) + 4\sin(t)^3$
- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \mathit{undefined} + 3c_2$
- Solve for $c_1$ and $c_2$

  $\{c_1 = \mathit{undefined}, c_2 = \mathit{undefined}\}$
- Substitute constant values into general solution and simplify

  $y = 4\sin(t)^3\,\mathit{Heaviside}(t) + 4\sin(t)^3\,\mathit{Heaviside}(t-\pi) + \mathit{undefined}\sin(3t) + \mathit{undefined}\cos(3t)$

- Solution to the IVP

  $y = 4\sin(t)^3\,\mathit{Heaviside}(t) + 4\sin(t)^3\,\mathit{Heaviside}(t-\pi) + \mathit{undefined}\sin(3t) + \mathit{undefined}\cos(3t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.484 (sec). Leaf size: 18

```
dsolve([diff(y(t),t$2)+9*y(t)=24*sin(t)*(Heaviside(t)+Heaviside(t-Pi)),y(0) = 0, D(y)(0) = 0
```

$$y(t) = 4(1 + \text{Heaviside}\,(t-\pi))\sin\,(t)^3$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 24

```
DSolve[{y''[t]+9*y[t]==24*Sin[t]*(UnitStep[t]+UnitStep[t-Pi]),{y[0]==0,y'[0]==0}},y[t],t,Inc
```

$$y(t) \to 4(\theta(\pi - t)(\theta(t) - 2) + 2)\sin^3(t)$$

## 4.18 problem Problem 3(d)



Internal problem ID [12326]
Internal file name [`OUTPUT/10978_Monday_October_02_2023_02_47_40_AM_43400438/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = \text{Heaviside}\left(t\right) - \text{Heaviside}\left(t-1\right)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

### 4.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 1 \\ F &= \text{Heaviside}\left(t\right) - \text{Heaviside}\left(t-1\right) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = \text{Heaviside}(t) - \text{Heaviside}(t-1)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}(t) - \text{Heaviside}(t-1)$ is

$$\{0 \le t \le 1, 1 \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + Y(s) = \frac{-\mathrm{e}^{-s} + 1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 2sY(s) + Y(s) = \frac{-\mathrm{e}^{-s} + 1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-s^2 + \mathrm{e}^{-s} - s - 1}{s\,(s^2 + 2s + 1)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-s^2 + \mathrm{e}^{-s} - s - 1}{s\,(s^2 + 2s + 1)}\right) \\
&= \text{Heaviside}\,(1-t) + \left(\text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} - \mathrm{e}^{-t}\right) t
\end{aligned}
$$

Hence the final solution is

$$
y = \text{Heaviside}\,(1-t) + \left(\text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} - \mathrm{e}^{-t}\right) t
$$

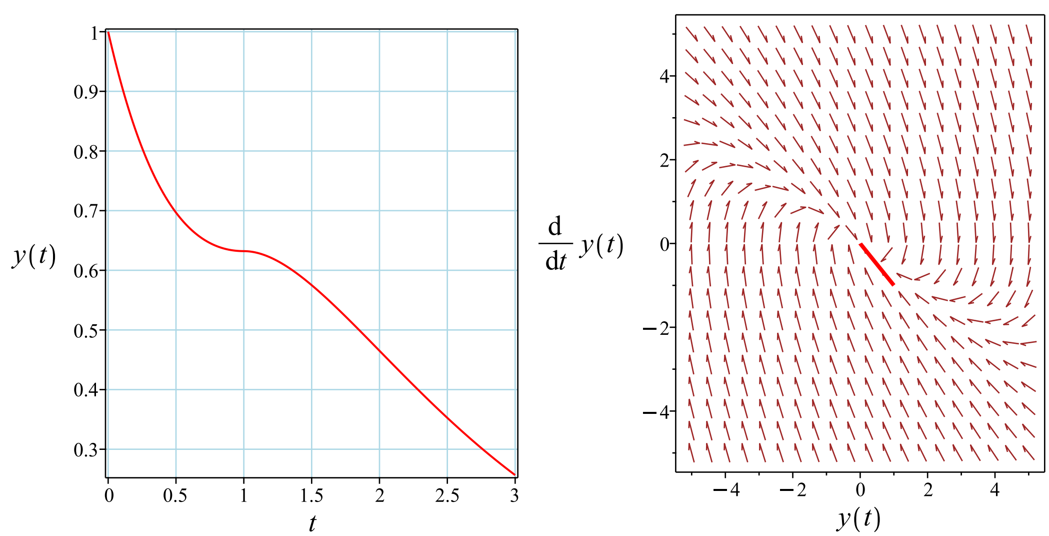
Simplifying the solution gives

$$
y = \text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} t - t\,\mathrm{e}^{-t} + 1 - \text{Heaviside}\,(t-1)
$$

Summary

The solution(s) found are the following

$$
y = \text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} t - t\,\mathrm{e}^{-t} + 1 - \text{Heaviside}\,(t-1) \tag{1}
$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \text{Heaviside}\,(t-1)\,\mathrm{e}^{1-t} t - t\,\mathrm{e}^{-t} + 1 - \text{Heaviside}\,(t-1)
$$

Verified OK.

### 4.18.2 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + y = Heaviside(t) - Heaviside(t-1)\,, y(0) = 1, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = Heaviside(t) - Heaviside(t\right.$$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t)\,, y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(t)\,, y_2(t)) = \mathrm{e}^{-2t}$

  ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = e^{-t}\left(-\left(\int (Heaviside(t) - Heaviside(t-1))\, t\, e^t dt\right) + \left(\int (Heaviside(t) - Heaviside(t-1)\right.$$

- Compute integrals

  $$y_p(t) = Heaviside(t-1)\, e^{1-t} t - Heaviside(t-1) + (-1-t)\, Heaviside(t)\, e^{-t} + Heaviside(t)$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-t} + c_2 t\, e^{-t} + Heaviside(t-1)\, e^{1-t} t - Heaviside(t-1) + (-1-t)\, Heaviside(t)\, e^{-t} + He$$

□ Check validity of solution $y = c_1 e^{-t} + c_2 t e^{-t} + Heaviside(t-1)\, e^{1-t} t - Heaviside(t-1) + (-1$

- Use initial condition $y(0) = 1$

  $$1 = undefined + c_1$$

- Compute derivative of the solution

  $$y' = -c_1 e^{-t} + c_2 e^{-t} - c_2 t\, e^{-t} + Dirac(t-1)\, e^{1-t} t - Heaviside(t-1)\, e^{1-t} t + Heaviside(t-1)\, e^{1-}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $$-1 = -c_1 + c_2 + undefined$$

- Solve for $c_1$ and $c_2$

  $$\{c_1 = undefined, c_2 = undefined\}$$

- Substitute constant values into general solution and simplify

  $$y = Heaviside(t-1)\, e^{1-t} t + (-Heaviside(t) + undefined)\,(1+t)\, e^{-t} + Heaviside(t) - Heaviside$$

- Solution to the IVP

  $$y = Heaviside(t-1)\, e^{1-t} t + (-Heaviside(t) + undefined)\,(1+t)\, e^{-t} + Heaviside(t) - Heaviside$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.985 (sec). Leaf size: 31

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+y(t)=Heaviside(t)-Heaviside(t-1),y(0) = 1, D(y)(0) = -
```

$$y(t) = t \,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1} - t\,\mathrm{e}^{-t} + 1 - \mathrm{Heaviside}\,(t-1)$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 43

```
DSolve[{y''[t]+2*y'[t]+y[t]==UnitStep[t]-UnitStep[t-1],{y[0]==1,y'[0]==-1}},y[t],t,IncludeSi
```

$$y(t) \to \{ \begin{array}{cc} e^{-t} & t<0 \\ 1-e^{-t}t & 0\leq t\leq 1 \\ (-1+e)e^{-t}t & \text{True} \end{array}$$

## 4.19 problem Problem 3(e)



Internal problem ID [12327]
Internal file name [`OUTPUT/10979_Monday_October_02_2023_02_47_41_AM_384554/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = 5\cos(t)\left(\text{Heaviside}(t) - \text{Heaviside}\left(t - \frac{\pi}{2}\right)\right)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

### 4.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= 5\cos(t)\left(\text{Heaviside}(t) - \text{Heaviside}\left(t - \frac{\pi}{2}\right)\right) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = 5\cos(t)\left(\text{Heaviside}(t) - \text{Heaviside}\left(t - \frac{\pi}{2}\right)\right)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5\cos(t)\,(\text{Heaviside}(t) - \text{Heaviside}\,(t -$
is

$$\left\{0 \leq t \leq \frac{\pi}{2}, \frac{\pi}{2} \leq t \leq \infty, -\infty \leq t \leq 0\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = \frac{5\,\mathrm{e}^{-\frac{s\pi}{2}} + 5s}{s^2 + 1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= -1
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 2sY(s) + 2Y(s) = \frac{5\,\mathrm{e}^{-\frac{s\pi}{2}} + 5s}{s^2 + 1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^3+s^2+5\,\mathrm{e}^{-\frac{s\pi}{2}}+6s+1}{(s^2+1)(s^2+2s+2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{s^3+s^2+5\,\mathrm{e}^{-\frac{s\pi}{2}}+6s+1}{(s^2+1)(s^2+2s+2)}\right)\\ &= \frac{7\cos(t)}{5}+\frac{9\sin(t)}{5}-2\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}\left(2\sin(t)\sinh\left(\frac{t}{2}-\frac{\pi}{4}\right)+\cos(t)\cosh\left(\frac{t}{2}-\frac{\pi}{4}\right)\right) -\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= \frac{7\cos(t)}{5}+\frac{9\sin(t)}{5}\\ &\quad -2\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}\left(2\sin(t)\sinh\left(\frac{t}{2}-\frac{\pi}{4}\right)+\cos(t)\cosh\left(\frac{t}{2}-\frac{\pi}{4}\right)\right)\\ &\quad -\frac{2\,\mathrm{e}^{-t}(\cos(t)+8\sin(t))}{5}+\frac{2\,\mathrm{e}^{-\frac{t}{2}}\left(\sin(t)\cosh\left(\frac{t}{2}\right)-2\cos(t)\sinh\left(\frac{t}{2}\right)\right)}{5}\end{aligned}$$

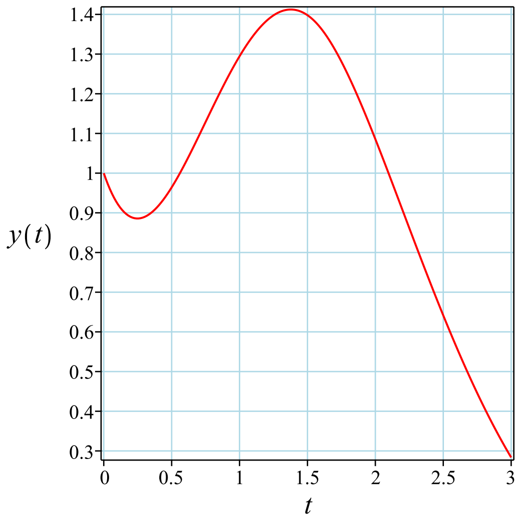
Simplifying the solution gives

$$\begin{aligned}y &= -\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)(\cos(t)-2\sin(t))\,\mathrm{e}^{-t+\frac{\pi}{2}}\\ &\quad +(-\cos(t)-2\sin(t))\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)-3\,\mathrm{e}^{-t}\sin(t)+\cos(t)+2\sin(t)\end{aligned}$$

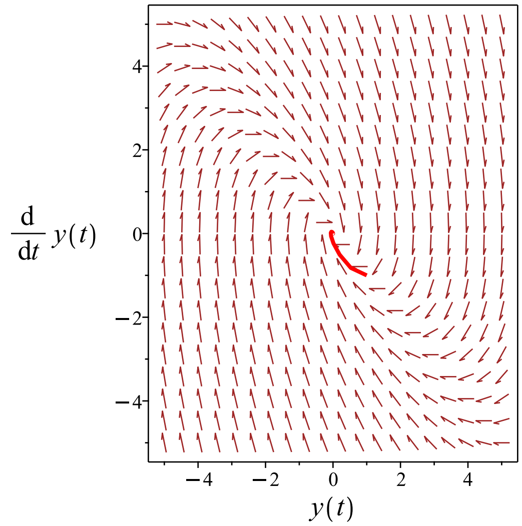
Summary

The solution(s) found are the following

$$\begin{aligned}y &= -\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)(\cos(t)-2\sin(t))\,\mathrm{e}^{-t+\frac{\pi}{2}}\\ &\quad +(-\cos(t)-2\sin(t))\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)-3\,\mathrm{e}^{-t}\sin(t)+\cos(t)+2\sin(t)\end{aligned} \tag{1}$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\operatorname{Heaviside}\left(t-\frac{\pi}{2}\right)(\cos(t)-2\sin(t))\,e^{-t+\frac{\pi}{2}} + (-\cos(t)-2\sin(t))\operatorname{Heaviside}\left(t-\frac{\pi}{2}\right) - 3\,e^{-t}\sin(t)+\cos(t)+2\sin(t)$$

Verified OK.

### 4.19.2 Maple step by step solution

Let's solve

$$\left[y''+2y'+2y=\cos(t)\left(5\mathit{Heaviside}(t)-5\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\right), y(0)=1, y'\Big|_{\{t=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=-5\cos(t)\,\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)+5\cos(t)\,\mathit{Heaviside}(t)-2y-2y'$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+2y'+2y=5\cos(t)\left(\mathit{Heaviside}(t)-\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\right)$

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+2=0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-1-\mathrm{I}, -1+\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}\cos(t)$
- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(t)$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 5\cos(t)\left(\mathit{Heaviside}(t) - \mathit{He}\right.\right.$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$
  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{5\,\mathrm{e}^{-t}\cos(t)\left(\int\left(\mathit{Heaviside}(t) - \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)\right)\sin(2t)\mathrm{e}^t dt\right)}{2} + 5\,\mathrm{e}^{-t}\sin(t)\left(\int \cos(t)^2\,\mathrm{e}^t\left(\mathit{Heaviside}(t) - \right.\right.$
  - Compute integrals

    $y_p(t) = -\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)(\cos(t) - 2\sin(t))\,\mathrm{e}^{-t+\frac{\pi}{2}} + (-\cos(t) - 2\sin(t))\,\mathit{Heaviside}\left(t-\frac{\pi}{2}\right) -$
- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) - \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)(\cos(t) - 2\sin(t))\,\mathrm{e}^{-t+\frac{\pi}{2}} + (-\cos(t) - 2\sin($

□ Check validity of solution $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) - \mathit{Heaviside}\left(t-\frac{\pi}{2}\right)(\cos(t) - 2\sin(t))$

  - Use initial condition $y(0) = 1$

    $1 = \mathit{undefined} + c_1$

- Compute derivative of the solution

  $y' = -c_1 e^{-t} \cos(t) - c_1 e^{-t} \sin(t) - c_2 e^{-t} \sin(t) + c_2 e^{-t} \cos(t) - Dirac\left(t - \frac{\pi}{2}\right) (\cos(t) - 2\sin(t))$

- Use the initial condition $y'\Big|_{\{t=0\}} = -1$

  $-1 = -c_1 + c_2 + undefined$

- Solve for $c_1$ and $c_2$

  $\{c_1 = undefined, c_2 = undefined\}$

- Substitute constant values into general solution and simplify

  $y = -Heaviside\left(t - \frac{\pi}{2}\right) (\cos(t) - 2\sin(t)) e^{-t+\frac{\pi}{2}} + (-\cos(t) - 2\sin(t)) Heaviside\left(t - \frac{\pi}{2}\right) + ((-$

- Solution to the IVP

  $y = -Heaviside\left(t - \frac{\pi}{2}\right) (\cos(t) - 2\sin(t)) e^{-t+\frac{\pi}{2}} + (-\cos(t) - 2\sin(t)) Heaviside\left(t - \frac{\pi}{2}\right) + ((-$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.734 (sec). Leaf size: 88

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=5*cos(t)*(Heaviside(t)-Heaviside(t-Pi/2)),y(0)
```

$$y(t) = -\text{Heaviside}\left(t - \frac{\pi}{2}\right) (\cos(t) - 2\sin(t)) e^{-t+\frac{\pi}{2}} + \text{Heaviside}\left(t - \frac{\pi}{2}\right) (-\cos(t) - 2\sin(t)) - 3\sin(t) e^{-t} + \cos(t) + 2\sin(t)$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 72

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==5*Cos[t]*(UnitStep[t]-UnitStep[t-Pi/2]),{y[0]==1,y'[0]==-1}},
```

$$y(t) \to \begin{cases} e^{-t}\cos(t) & t < 0 \\ e^{-t}\left(\left(-3+2e^{\pi/2}\right)\sin(t) - e^{\pi/2}\cos(t)\right) & 2t > \pi \\ \cos(t) + \left(2-3e^{-t}\right)\sin(t) & \text{True} \end{cases}$$

## 4.20 problem Problem 3(f)



Internal problem ID [12328]
Internal file name [`OUTPUT/10980_Monday_October_02_2023_02_47_41_AM_74959009/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 6y = 36t(\text{Heaviside}(t) - \text{Heaviside}(t-1))}$$

With initial conditions

$$[y(0) = -1, y'(0) = -2]$$

### 4.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= 36t(\text{Heaviside}(t) - \text{Heaviside}(t-1)) \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = 36t(\text{Heaviside}(t) - \text{Heaviside}(t-1))$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 36t(\text{Heaviside}\,(t) - \text{Heaviside}\,(t-1))$ is

$$\{0 \leq t \leq 1, 1 \leq t \leq \infty, -\infty \leq t \leq 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 5sY(s) - 5y(0) + 6Y(s) = \frac{36 - 36\,\mathrm{e}^{-s}(s+1)}{s^2} \tag{1}$$

But the initial conditions are

$$y(0) = -1$$
$$y'(0) = -2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 7 + s + 5sY(s) + 6Y(s) = \frac{36 - 36\,\mathrm{e}^{-s}(s+1)}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^3 + 36\,\mathrm{e}^{-s}s + 7s^2 + 36\,\mathrm{e}^{-s} - 36}{s^2\,(s^2 + 5s + 6)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{s^3 + 36\,\mathrm{e}^{-s} s + 7s^2 + 36\,\mathrm{e}^{-s} - 36}{s^2\,(s^2 + 5s + 6)}\right) \\
&= (-5 + 6t)\,\mathrm{Heaviside}\,(1 - t) + \left(-8\,\mathrm{e}^{-3t+3} + 9\,\mathrm{e}^{-2t+2}\right)\mathrm{Heaviside}\,(t - 1) + 4\,\mathrm{e}^{-2t}
\end{aligned}
$$

Hence the final solution is

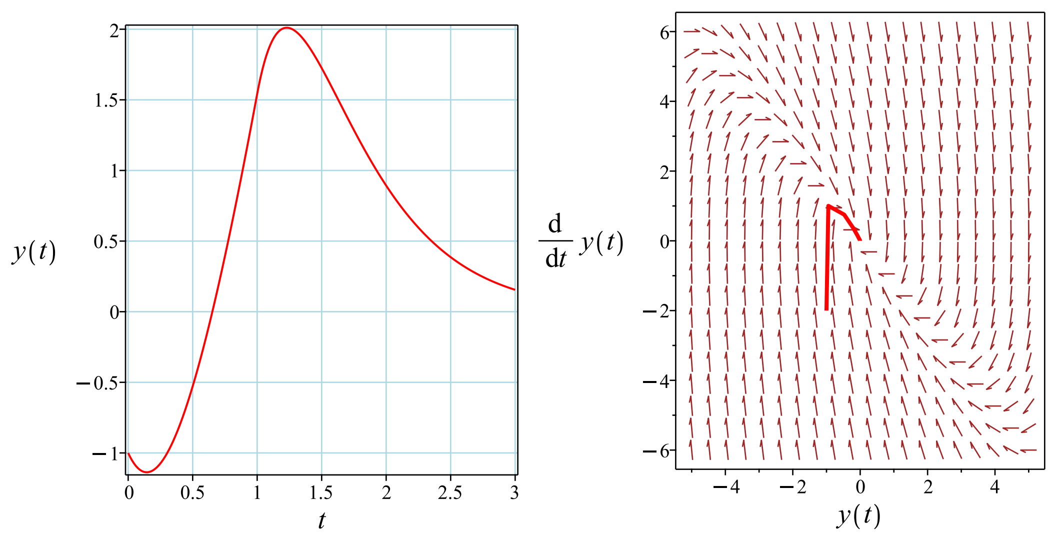
$$
y = (-5 + 6t)\,\mathrm{Heaviside}\,(1 - t) + \left(-8\,\mathrm{e}^{-3t+3} + 9\,\mathrm{e}^{-2t+2}\right)\mathrm{Heaviside}\,(t - 1) + 4\,\mathrm{e}^{-2t}
$$

Simplifying the solution gives

$$
\begin{aligned}
y = {} & -8\,\mathrm{Heaviside}\,(t - 1)\,\mathrm{e}^{-3t+3} + 9\,\mathrm{Heaviside}\,(t - 1)\,\mathrm{e}^{-2t+2} \\
& + (-6t + 5)\,\mathrm{Heaviside}\,(t - 1) + 6t + 4\,\mathrm{e}^{-2t} - 5
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {} & -8\,\mathrm{Heaviside}\,(t - 1)\,\mathrm{e}^{-3t+3} + 9\,\mathrm{Heaviside}\,(t - 1)\,\mathrm{e}^{-2t+2} \\
& + (-6t + 5)\,\mathrm{Heaviside}\,(t - 1) + 6t + 4\,\mathrm{e}^{-2t} - 5
\end{aligned} \tag{1}
$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = -8\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-3t+3} + 9\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} + (-6t+5)\,\mathrm{Heaviside}\,(t-1) + 6t + 4\,\mathrm{e}^{-2t} - 5
$$

Verified OK.

### 4.20.2 Maple step by step solution

Let's solve

$$
\left[y'' + 5y' + 6y = 36t(\mathit{Heaviside}(t) - \mathit{Heaviside}(t-1)), y(0) = -1, y'\Big|_{\{t=0\}} = -2\right]
$$

- Highest derivative means the order of the ODE is 2
  $y''$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 5r + 6 = 0$
- Factor the characteristic polynomial
  $(r+3)(r+2) = 0$
- Roots of the characteristic polynomial
  $r = (-3, -2)$
- 1st solution of the homogeneous ODE
  $y_1(t) = \mathrm{e}^{-3t}$
- 2nd solution of the homogeneous ODE
  $y_2(t) = \mathrm{e}^{-2t}$
- General solution of the ODE
  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y = \mathrm{e}^{-3t} c_1 + c_2 \mathrm{e}^{-2t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function
  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 36t(\mathit{Heaviside}(t) - \mathit{Heavisi}\right.$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{-2t} \\ -3\,e^{-3t} & -2\,e^{-2t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-5t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -36\,e^{-3t}\left(\int t(\mathit{Heaviside}(t) - \mathit{Heaviside}(t-1))\,e^{3t}dt\right) + 36\,e^{-2t}\left(\int t(\mathit{Heaviside}(t) - \mathit{Heavis}\right.$

- Compute integrals

  $y_p(t) = -8\mathit{Heaviside}(t-1)\,e^{-3t+3} + 9\mathit{Heaviside}(t-1)\,e^{-2t+2} + (-6t+5)\,\mathit{Heaviside}(t-1) + 6\Big($

- Substitute particular solution into general solution to ODE

  $y = e^{-3t}c_1 + c_2e^{-2t} - 8\mathit{Heaviside}(t-1)\,e^{-3t+3} + 9\mathit{Heaviside}(t-1)\,e^{-2t+2} + (-6t+5)\,\mathit{Heaviside}$

☐ Check validity of solution $y = e^{-3t}c_1 + c_2e^{-2t} - 8\mathit{Heaviside}(t-1)\,e^{-3t+3} + 9\mathit{Heaviside}(t-1)\,e^{-}$

- Use initial condition $y(0) = -1$

  $-1 = c_1 + c_2 + \mathit{undefined}$

- Compute derivative of the solution

  $y' = -3\,e^{-3t}c_1 - 2c_2e^{-2t} - 8\mathit{Dirac}(t-1)\,e^{-3t+3} + 24\mathit{Heaviside}(t-1)\,e^{-3t+3} + 9\mathit{Dirac}(t-1)\,e^{-2t}$

- Use the initial condition $y'\big|_{\{t=0\}} = -2$

  $-2 = -3c_1 - 2c_2 + \mathit{undefined}$

- Solve for $c_1$ and $c_2$

  $\{c_1 = \mathit{undefined}, c_2 = \mathit{undefined}\}$

- Substitute constant values into general solution and simplify

  $y = -8\mathit{Heaviside}(t-1)\,e^{-3t+3} + 9\mathit{Heaviside}(t-1)\,e^{-2t+2} + (-6t+5)\,\mathit{Heaviside}(t-1) + (\mathit{undef}$

- Solution to the IVP

  $y = -8\mathit{Heaviside}(t-1)\,e^{-3t+3} + 9\mathit{Heaviside}(t-1)\,e^{-2t+2} + (-6t+5)\,\mathit{Heaviside}(t-1) + (\mathit{undef}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 5.203 (sec). Leaf size: 45

```
dsolve([diff(y(t),t$2)+5*diff(y(t),t)+6*y(t)=36*t*(Heaviside(t)-Heaviside(t-1)),y(0) = -1, D
```

$$y(t) = -8\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-3t+3} + 9\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} + (-6t+5)\,\mathrm{Heaviside}\,(t-1) + 6t + 4\,\mathrm{e}^{-2t} - 5$$

✓ Solution by Mathematica
Time used: 0.071 (sec). Leaf size: 64

```
DSolve[{y''[t]+5*y'[t]+6*y[t]==36*t*(UnitStep[t]-UnitStep[t-1]),{y[0]==-1,y'[0]==-2}},y[t],t
```

$$y(t) \to \begin{cases} e^{-3t}(4-5e^t) & t<0 \\ e^{-3t}(-8e^3+4e^t+9e^{t+2}) & t>1 \\ 6t+4e^{-2t}-5 & \text{True} \end{cases}$$

## 4.21 problem Problem 3(g)



Internal problem ID [12329]
Internal file name [`OUTPUT/10981_Monday_October_02_2023_02_47_41_AM_74918722/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(g).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 13y = 39 \operatorname{Heaviside}(t) - 507(-2+t) \operatorname{Heaviside}(-2+t)}$$

With initial conditions

$$[y(0) = 3, y'(0) = 1]$$

### 4.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 13 \\ F &= (-507t + 1014) \operatorname{Heaviside}(-2+t) + 39 \operatorname{Heaviside}(t) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 13y = (-507t + 1014) \operatorname{Heaviside}(-2+t) + 39 \operatorname{Heaviside}(t)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = (-507t + 1014)\,\text{Heaviside}\,(-2 + t) + 39\,\text{Heaviside}\,(t)$ is

$$\{0 \le t \le 2, 2 \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 13Y(s) = -\frac{507\,\mathrm{e}^{-2s}}{s^2} + \frac{39}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 3\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 13 - 3s + 4sY(s) + 13Y(s) = -\frac{507\,\mathrm{e}^{-2s}}{s^2} + \frac{39}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-3s^3 - 13s^2 + 507\,\mathrm{e}^{-2s} - 39s}{s^2\,(s^2 + 4s + 13)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-3s^3 - 13s^2 + 507\,\mathrm{e}^{-2s} - 39s}{s^2\,(s^2 + 4s + 13)}\right) \\
&= 3 + \frac{\mathrm{e}^{-2t}\sin(3t)}{3} + \left(90 - 39t + (-12\cos(-6+3t) + 5\sin(-6+3t))\,\mathrm{e}^{4-2t}\right)\mathrm{Heaviside}(-2+t)
\end{aligned}
$$

Hence the final solution is

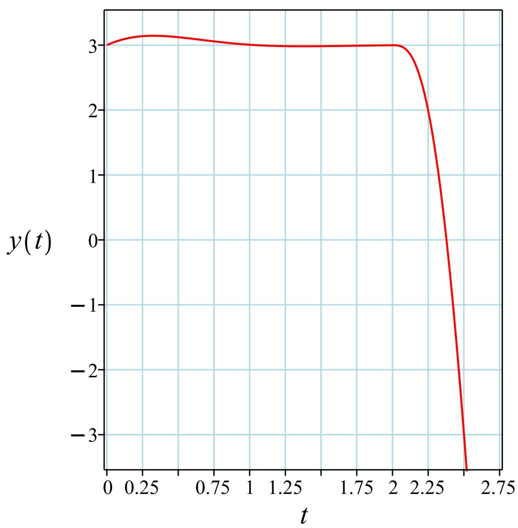
$$
\begin{aligned}
y = 3 &+ \frac{\mathrm{e}^{-2t}\sin(3t)}{3} \\
&+ \left(90 - 39t + (-12\cos(-6+3t) + 5\sin(-6+3t))\,\mathrm{e}^{4-2t}\right)\mathrm{Heaviside}(-2+t)
\end{aligned}
$$

Simplifying the solution gives

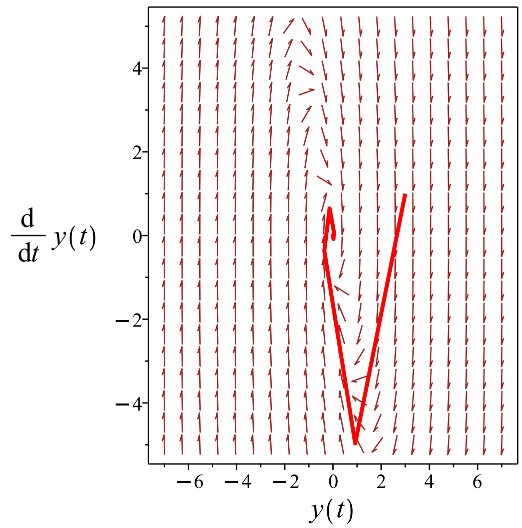
$$
\begin{aligned}
y = 3 - 12\left(\left(-\frac{5\cos(6)}{12} + \sin(6)\right)\sin(3t) + \cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right)\right)\mathrm{Heaviside}(-2 \\
+ t)\,\mathrm{e}^{4-2t} + 3(30 - 13t)\,\mathrm{Heaviside}(-2+t) + \frac{\mathrm{e}^{-2t}\sin(3t)}{3}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = 3 - 12&\left(\left(-\frac{5\cos(6)}{12} + \sin(6)\right)\sin(3t)\right. \\
&\left. + \cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right)\right)\mathrm{Heaviside}(-2+t)\,\mathrm{e}^{4-2t} \\
&+ 3(30 - 13t)\,\mathrm{Heaviside}(-2+t) + \frac{\mathrm{e}^{-2t}\sin(3t)}{3}
\end{aligned}
\tag{1}
$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = 3 - 12\left(\left(-\frac{5\cos(6)}{12} + \sin(6)\right)\sin(3t) + \cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right)\right)\text{Heaviside}(-2 + t)\,e^{4-2t} + 3(30 - 13t)\,\text{Heaviside}(-2 + t) + \frac{e^{-2t}\sin(3t)}{3}$$

Verified OK.

### 4.21.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 13y = (-507t + 1014)\,\mathit{Heaviside}(-2 + t) + 39\mathit{Heaviside}(t)\,, y(0) = 3, y'\Big|_{\{t=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -507t\mathit{Heaviside}(-2 + t) + 1014\mathit{Heaviside}(-2 + t) + 39\mathit{Heaviside}(t) - 13y - 4y'$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y' + 13y = -507t\mathit{Heaviside}(-2 + t) + 1014\mathit{Heaviside}(-2 + t) + 39\mathit{Heaviside}(t)$

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 13 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-4) \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 3\,\mathrm{I}, -2 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(3t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(3t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} \cos(3t) + c_2 \mathrm{e}^{-2t} \sin(3t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -507t\mathit{Heaviside}(-2+t) +\right.$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} \cos(3t) & \mathrm{e}^{-2t} \sin(3t) \\ -2\,\mathrm{e}^{-2t} \cos(3t) - 3\,\mathrm{e}^{-2t} \sin(3t) & -2\,\mathrm{e}^{-2t} \sin(3t) + 3\,\mathrm{e}^{-2t} \cos(3t) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 3\,\mathrm{e}^{-4t}$

  ○ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -13\,\mathrm{e}^{-2t} (\sin(3t) (\int \mathrm{e}^{2t} \cos(3t) (13t\mathit{Heaviside}(-2+t) - 26\mathit{Heaviside}(-2+t) - \mathit{Heavisid}$

  ○ Compute integrals

  $y_p(t) = -12\left(\cos(3t) \left(\cos(6) + \frac{5\sin(6)}{12}\right) - \frac{5\sin(3t)\left(\cos(6) - \frac{12\sin(6)}{5}\right)}{12}\right) \mathit{Heaviside}(-2+t)\,\mathrm{e}^{4-2t} + (90 -$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2t} \cos(3t) + c_2 \mathrm{e}^{-2t} \sin(3t) - 12\left(\cos(3t) \left(\cos(6) + \frac{5\sin(6)}{12}\right) - \frac{5\sin(3t)\left(\cos(6) - \frac{12\sin(6)}{5}\right)}{12}\right) \mathit{Hea}$

☐ Check validity of solution $y = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) - 12\left(\cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right) - \right.$

- Use initial condition $y(0) = 3$

  $3 = \mathit{undefined} + c_1$

- Compute derivative of the solution

  $y' = -2c_1 e^{-2t} \cos(3t) - 3c_1 e^{-2t} \sin(3t) - 2c_2 e^{-2t} \sin(3t) + 3c_2 e^{-2t} \cos(3t) - 12\left(-3\sin(3t)\left(\cos\right.\right.$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $1 = -2c_1 + \mathit{undefined} + 3c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = \mathit{undefined}, c_2 = \mathit{undefined}\}$

- Substitute constant values into general solution and simplify

  $y = -12\left(\cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right) - \frac{5\sin(3t)\left(\cos(6) - \frac{12\sin(6)}{5}\right)}{12}\right) \mathit{Heaviside}(-2+t)\, e^{4-2t} + (90 - 39$

- Solution to the IVP

  $y = -12\left(\cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right) - \frac{5\sin(3t)\left(\cos(6) - \frac{12\sin(6)}{5}\right)}{12}\right) \mathit{Heaviside}(-2+t)\, e^{4-2t} + (90 - 39$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.844 (sec). Leaf size: 50

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+13*y(t)=39*Heaviside(t)-507*(t-2)*Heaviside(t-2),y(0)
```

$$y(t) = 3 - 12\left(\left(-\frac{5\cos(6)}{12} + \sin(6)\right)\sin(3t) + \cos(3t)\left(\cos(6) + \frac{5\sin(6)}{12}\right)\right)\text{Heaviside}(t-2)\,\mathrm{e}^{-2t+4} + 3(30-13t)\,\text{Heaviside}(t-2) + \frac{\mathrm{e}^{-2t}\sin(3t)}{3}$$

✓ Solution by Mathematica

Time used: 0.097 (sec). Leaf size: 103

```
DSolve[{y''[t]+4*y'[t]+13*y[t]==39*UnitStep[t]-507*(t-2)*UnitStep[t-2],{y[0]==3,y'[0]==1}},y
```

$$y(t) \to \begin{cases} -39t - 12e^{4-2t}\cos(6-3t) - 5e^{4-2t}\sin(6-3t) + \frac{1}{3}e^{-2t}\sin(3t) + 93 & t > 2 \\ \frac{1}{3}e^{-2t}\sin(3t) + 3 & 0 \leq t \leq 2 \\ \frac{1}{3}e^{-2t}(9\cos(3t) + 7\sin(3t)) & \text{True} \end{cases}$$

## 4.22 problem Problem 3(h)



Internal problem ID [12330]
Internal file name [`OUTPUT/10982_Monday_October_02_2023_02_47_42_AM_54582995/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(h).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 3\,\text{Heaviside}\,(t) - 3\,\text{Heaviside}\,(t-4) + (2t-5)\,\text{Heaviside}\,(t-4)}$$

With initial conditions

$$\left[y(0) = \frac{3}{4}, y'(0) = 2\right]$$

### 4.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= (2t-8)\,\text{Heaviside}\,(t-4) + 3\,\text{Heaviside}\,(t)
\end{aligned}$$

Hence the ode is

$$y'' + 4y = (2t - 8)\,\text{Heaviside}\,(t - 4) + 3\,\text{Heaviside}\,(t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = (2t - 8)\,\text{Heaviside}\,(t - 4) + 3\,\text{Heaviside}\,(t)$ is

$$\{0 \le t \le 4, 4 \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{2\,\mathrm{e}^{-4s}}{s^2} + \frac{3}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= \frac{3}{4}\\ y'(0) &= 2\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 2 - \frac{3s}{4} + 4Y(s) = \frac{2\,\mathrm{e}^{-4s}}{s^2} + \frac{3}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3s^3 + 8s^2 + 8\,\mathrm{e}^{-4s} + 12s}{4s^2\,(s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3s^3+8s^2+8\,\mathrm{e}^{-4s}+12s}{4s^2\,(s^2+4)}\right) \\
&= \sin(2t)+\frac{\mathrm{Heaviside}(t-4)\,(2t-8-\sin(2t-8))}{4}+\frac{3}{4}
\end{aligned}
$$

Hence the final solution is

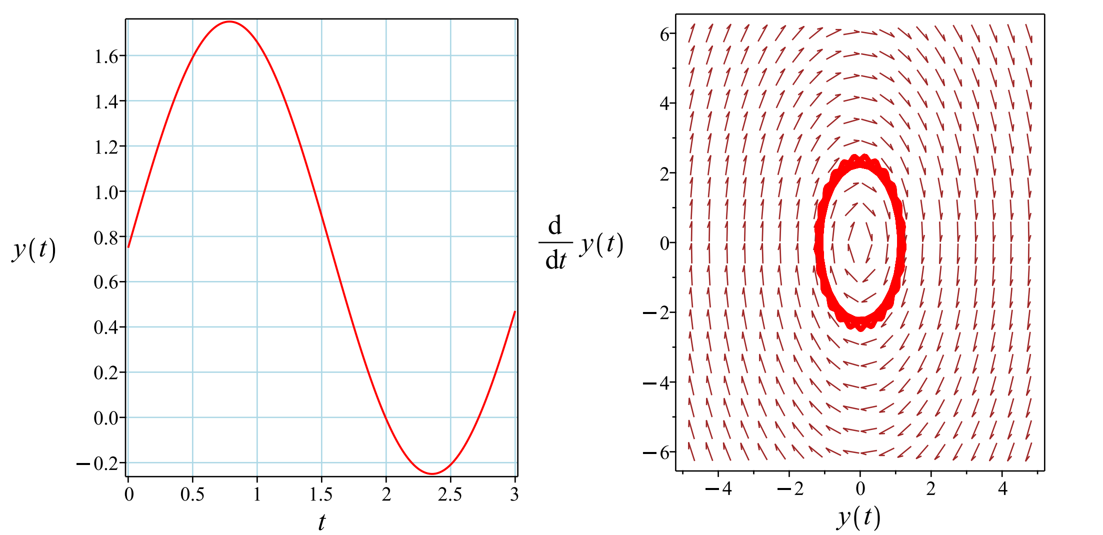
$$
y = \sin(2t)+\frac{\mathrm{Heaviside}(t-4)\,(2t-8-\sin(2t-8))}{4}+\frac{3}{4}
$$

Simplifying the solution gives

$$
y = -\frac{\mathrm{Heaviside}(t-4)\sin(2t-8)}{4}+\frac{\mathrm{Heaviside}(t-4)\,t}{2}-2\,\mathrm{Heaviside}(t-4)+\sin(2t)+\frac{3}{4}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {} & -\frac{\mathrm{Heaviside}(t-4)\sin(2t-8)}{4}+\frac{\mathrm{Heaviside}(t-4)\,t}{2} \\
& -2\,\mathrm{Heaviside}(t-4)+\sin(2t)+\frac{3}{4}
\end{aligned} \tag{1}
$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\text{Heaviside}\,(t-4)\sin(2t-8)}{4} + \frac{\text{Heaviside}\,(t-4)\,t}{2} - 2\,\text{Heaviside}\,(t-4) + \sin(2t) + \frac{3}{4}$$

Verified OK.

### 4.22.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = (2t-8)\,\mathit{Heaviside}(t-4) + 3\mathit{Heaviside}(t)\,, y(0) = \frac{3}{4}, y'\Big|_{\{t=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = 2\mathit{Heaviside}(t-4)\,t - 8\mathit{Heaviside}(t-4) + 3\mathit{Heaviside}(t) - 4y$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y = 2\mathit{Heaviside}(t-4)\,t - 8\mathit{Heaviside}(t-4) + 3\mathit{Heaviside}(t)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 2\mathit{Heaviside}(t-4)\, t - 8\mathit{Hea}\right.$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{\cos(2t)\left(\int \sin(2t)(2\mathit{Heaviside}(t-4)t - 8\mathit{Heaviside}(t-4) + 3\mathit{Heaviside}(t))dt\right)}{2} + \frac{\sin(2t)\left(\int \cos(2t)(2\mathit{Heaviside}(t-4)t - 8\mathit{He}\right.}{2}$

- Compute integrals

  $y_p(t) = \frac{\mathit{Heaviside}(t-4)(2t - 8 - \sin(2t)\cos(8) + \cos(2t)\sin(8))}{4} - \frac{3\mathit{Heaviside}(t)(-1 + \cos(2t))}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\mathit{Heaviside}(t-4)(2t - 8 - \sin(2t)\cos(8) + \cos(2t)\sin(8))}{4} - \frac{3\mathit{Heaviside}(t)(-1 + \cos(2t))}{4}$

□ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\mathit{Heaviside}(t-4)(2t - 8 - \sin(2t)\cos(8) + \cos(2t)\sin(8))}{4} -$

- Use initial condition $y(0) = \frac{3}{4}$

  $\frac{3}{4} = \mathit{undefined} + c_1$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \frac{\mathit{Dirac}(t-4)(2t - 8 - \sin(2t)\cos(8) + \cos(2t)\sin(8))}{4} + \frac{\mathit{Heaviside}(t-4)(2 - 2\cos(2t)\cos(}{4}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = \mathit{undefined} + 2c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = \mathit{undefined}, c_2 = \mathit{undefined}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\mathit{Heaviside}(t-4)(2t - 8 - \sin(2t)\cos(8) + \cos(2t)\sin(8))}{4} + \frac{(\mathit{undefined} - 3\mathit{Heaviside}(t))\cos(2t)}{4} + \mathit{undefined}\sin(2t) + \frac{3H}{}$

- Solution to the IVP

  $y = \frac{\mathit{Heaviside}(t-4)(2t - 8 - \sin(2t)\cos(8) + \cos(2t)\sin(8))}{4} + \frac{(\mathit{undefined} - 3\mathit{Heaviside}(t))\cos(2t)}{4} + \mathit{undefined}\sin(2t) + \frac{3H}{}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.5 (sec). Leaf size: 29

```
dsolve([diff(y(t),t$2)+4*y(t)=3*(Heaviside(t)-Heaviside(t-4))+(2*t-5)*Heaviside(t-4),y(0) =
```

$$y(t) = -\frac{\text{Heaviside}(t-4)\sin(2t-8)}{4} + \frac{\text{Heaviside}(t-4)\,t}{2} + \sin(2t) - 2\,\text{Heaviside}(t-4) + \frac{3}{4}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 60

```
DSolve[{y''[t]+4*y[t]==3*(UnitStep[t]-UnitStep[t-4])+(2*t-5)*UnitStep[t-4],{y[0]==3/4,y'[0]=
```

$$y(t) \to \begin{cases} \sin(2t) + \frac{3}{4} & 0 \leq t \leq 4 \\ \frac{3}{4}\cos(2t) + \sin(2t) & t < 0 \\ \frac{1}{4}(2t + \sin(8-2t) + 4\sin(2t) - 5) & \text{True} \end{cases}$$

## 4.23 problem Problem 3(i)



Internal problem ID [12331]
Internal file name [`OUTPUT/10983_Monday_October_02_2023_02_47_42_AM_32227872/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(i).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'' + 4y' + 5y = 25t\left(\text{Heaviside}\left(t\right) - \text{Heaviside}\left(t - \frac{\pi}{2}\right)\right)}$$

With initial conditions

$$[y(0) = 2, y'(0) = 2]$$

### 4.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \frac{5}{4} \\ F &= -\frac{25t\left(\text{Heaviside}\left(t-\frac{\pi}{2}\right) - \text{Heaviside}\left(t\right)\right)}{4} \end{aligned}$$

Hence the ode is

$$y'' + y' + \frac{5y}{4} = -\frac{25t\left(\text{Heaviside}\left(t - \frac{\pi}{2}\right) - \text{Heaviside}\left(t\right)\right)}{4}$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{5}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = -\frac{25t\left(\text{Heaviside}\left(t - \frac{\pi}{2}\right) - \text{Heaviside}(t)\right)}{4}$ is

$$\left\{0 \leq t \leq \frac{\pi}{2}, \frac{\pi}{2} \leq t \leq \infty, -\infty \leq t \leq 0\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) + 4sY(s) - 4y(0) + 5Y(s) = \frac{25 - \frac{25\,\mathrm{e}^{-\frac{s\pi}{2}}(s\pi+2)}{2}}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 2 \\
y'(0) &= 2
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) - 16 - 8s + 4sY(s) + 5Y(s) = \frac{25 - \frac{25\,\mathrm{e}^{-\frac{s\pi}{2}}(s\pi+2)}{2}}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{25\pi\,\mathrm{e}^{-\frac{s\pi}{2}} s - 16s^3 - 32s^2 + 50\,\mathrm{e}^{-\frac{s\pi}{2}} - 50}{2s^2\left(4s^2 + 4s + 5\right)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{25\pi\,\mathrm{e}^{-\frac{s\pi}{2}} s - 16s^3 - 32s^2 + 50\,\mathrm{e}^{-\frac{s\pi}{2}} - 50}{2s^2\left(4s^2 + 4s + 5\right)}\right) \\
&= 6\cos(t)\,\mathrm{e}^{-\frac{t}{2}} + (-4 + 5t)\,\text{Heaviside}\left(-t + \frac{\pi}{2}\right) + \left(\frac{1}{20} - \frac{i}{40}\right)\left(25\pi\,\mathrm{e}^{\left(-\frac{1}{4}+\frac{i}{2}\right)(2t-\pi)} + (15 + 20i)\,\mathrm{e}^{\left(-\frac{1}{4}-\frac{i}{2}\right)(}\right.
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = 6\cos(t)\,\mathrm{e}^{-\frac{t}{2}} &+ (-4 + 5t)\,\text{Heaviside}\left(-t + \frac{\pi}{2}\right) + \left(\frac{1}{20} - \frac{i}{40}\right)\left(25\pi\,\mathrm{e}^{\left(-\frac{1}{4}+\frac{i}{2}\right)(2t-\pi)}\right. \\
&\left.+ (15 + 20i)\,\mathrm{e}^{\left(-\frac{1}{4}-\frac{i}{2}\right)(2t-\pi)}\pi + (-16 - 8i)\,\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}(3\cos(t) + 4\sin(t))\right)\text{Heaviside}\left(t - \frac{\pi}{2}\right)
\end{aligned}$$

Simplifying the solution gives

$$\begin{aligned}
y = -4 &+ \left(\frac{5}{4} - \frac{5i}{8}\right)\pi\,\text{Heaviside}\left(t - \frac{\pi}{2}\right)\mathrm{e}^{\left(\frac{1}{4}-\frac{i}{2}\right)(-2t+\pi)} \\
&+ \left(\frac{5}{4} + \frac{5i}{8}\right)\pi\,\text{Heaviside}\left(t - \frac{\pi}{2}\right)\mathrm{e}^{\left(\frac{1}{4}+\frac{i}{2}\right)(-2t+\pi)} \\
&- 3\left(\cos(t) + \frac{4\sin(t)}{3}\right)\text{Heaviside}\left(t - \frac{\pi}{2}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}} \\
&+ (4 - 5t)\,\text{Heaviside}\left(t - \frac{\pi}{2}\right) + 6\cos(t)\,\mathrm{e}^{-\frac{t}{2}} + 5t
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = -4 &+ \left(\frac{5}{4} - \frac{5i}{8}\right)\pi\,\text{Heaviside}\left(t - \frac{\pi}{2}\right)\mathrm{e}^{\left(\frac{1}{4}-\frac{i}{2}\right)(-2t+\pi)} \\
&+ \left(\frac{5}{4} + \frac{5i}{8}\right)\pi\,\text{Heaviside}\left(t - \frac{\pi}{2}\right)\mathrm{e}^{\left(\frac{1}{4}+\frac{i}{2}\right)(-2t+\pi)} \\
&- 3\left(\cos(t) + \frac{4\sin(t)}{3}\right)\text{Heaviside}\left(t - \frac{\pi}{2}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}} \\
&+ (4 - 5t)\,\text{Heaviside}\left(t - \frac{\pi}{2}\right) + 6\cos(t)\,\mathrm{e}^{-\frac{t}{2}} + 5t
\end{aligned} \tag{1}$$

Verification of solutions

$$y = -4+\left(\frac{5}{4}-\frac{5i}{8}\right)\pi \,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right) \mathrm{e}^{\left(\frac{1}{4}-\frac{i}{2}\right)(-2t+\pi)} + \left(\frac{5}{4}+\frac{5i}{8}\right)\pi \,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right) \mathrm{e}^{\left(\frac{1}{4}+\frac{i}{2}\right)(-2t+\pi)} - 3\left(\cos(t)+\frac{4\sin(t)}{3}\right)\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right) \mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}} + (4-5t)\,\mathrm{Heaviside}\left(t-\frac{\pi}{2}\right)+6\cos(t)\,\mathrm{e}^{-\frac{t}{2}}+5t$$

Verified OK.

### 4.23.2 Maple step by step solution

Let's solve

$\left[4y''+4y'+5y=-25t\left(\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)-\mathit{Heaviside}(t)\right), y(0)=2, y'\Big|_{\{t=0\}}=2\right]$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y''=-\frac{5y}{4}-y'-\frac{25t\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)}{4}+\frac{25t\mathit{Heaviside}(t)}{4}$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+\frac{5y}{4}=-\frac{25t\left(\mathit{Heaviside}\left(t-\frac{\pi}{2}\right)-\mathit{Heaviside}(t)\right)}{4}$

- Characteristic polynomial of homogeneous ODE

  $r^2+r+\frac{5}{4}=0$

- Use quadratic formula to solve for $r$

  $r=\frac{(-1)\pm\left(\sqrt{-4}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\mathrm{I},-\frac{1}{2}+\mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\cos(t)\,\mathrm{e}^{-\frac{t}{2}}$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin(t)\,\mathrm{e}^{-\frac{t}{2}}$

- General solution of the ODE

$y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t)\, \mathrm{e}^{-\frac{t}{2}} + c_2 \sin(t)\, \mathrm{e}^{-\frac{t}{2}} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = -\frac{25t\left(Heaviside\left(t-\frac{\pi}{2}\right)-Heaviside(t\right.}{4}\right.$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t)\, \mathrm{e}^{-\frac{t}{2}} & \sin(t)\, \mathrm{e}^{-\frac{t}{2}} \\ -\sin(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{\cos(t)\mathrm{e}^{-\frac{t}{2}}}{2} & \cos(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{\sin(t)\mathrm{e}^{-\frac{t}{2}}}{2} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = \mathrm{e}^{-t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{25\, \mathrm{e}^{-\frac{t}{2}}\left(\cos(t)\left(\int \left(Heaviside\left(t-\frac{\pi}{2}\right)-Heaviside(t)\right) \sin(t) t\, \mathrm{e}^{\frac{t}{2}} dt\right)-\sin(t)\left(\int \left(Heaviside\left(t-\frac{\pi}{2}\right)-Heaviside(t)\right) \cos(t) t\, \mathrm{e}^{\frac{t}{2}} dt\right)\right.}{4}$

- Compute integrals

  $y_p(t) = -\frac{5\left(\left(\frac{12}{5}+\pi\right)\cos(t)-2\left(-\frac{8}{5}+\pi\right)\sin(t)\right) Heaviside\left(t-\frac{\pi}{2}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}}{4} + (4-5t)\, Heaviside\left(t-\frac{\pi}{2}\right) + 5 Heaviside($

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(t)\, \mathrm{e}^{-\frac{t}{2}} + c_2 \sin(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{5\left(\left(\frac{12}{5}+\pi\right)\cos(t)-2\left(-\frac{8}{5}+\pi\right)\sin(t)\right) Heaviside\left(t-\frac{\pi}{2}\right)\mathrm{e}^{-\frac{t}{2}+\frac{\pi}{4}}}{4} + (4-5t)\, Heavi$

☐ Check validity of solution $y = c_1 \cos(t)\, \mathrm{e}^{-\frac{t}{2}} + c_2 \sin(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{5\left(\left(\frac{12}{5}+\pi\right)\cos(t)-2\left(-\frac{8}{5}+\pi\right)\sin(t)\right) Heaviside(t}{4}$

- Use initial condition $y(0) = 2$

  $2 = undefined + c_1$

- Compute derivative of the solution

  $y' = -c_1 \sin(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{c_1 \cos(t)\mathrm{e}^{-\frac{t}{2}}}{2} + c_2 \cos(t)\, \mathrm{e}^{-\frac{t}{2}} - \frac{c_2 \sin(t)\mathrm{e}^{-\frac{t}{2}}}{2} - \frac{5\left(-\left(\frac{12}{5}+\pi\right)\sin(t)-2\left(-\frac{8}{5}+\pi\right)\cos(t)\right) Heavisid}{4}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 2$

  $2 = undefined - \frac{c_1}{2} + c_2$

- Solve for $c_1$ and $c_2$

$\{c_1 = undefined, c_2 = undefined\}$

- Substitute constant values into general solution and simplify

$$y = -\frac{5\left(\left(\frac{12}{5}+\pi\right)\cos(t)-2\left(-\frac{8}{5}+\pi\right)\sin(t)\right) Heaviside\left(t-\frac{\pi}{2}\right) e^{-\frac{t}{2}+\frac{\pi}{4}}}{4} + (4-5t)\, Heaviside\left(t-\frac{\pi}{2}\right) + ((4\cos(t) - 3\,\mathrm{s}$$

- Solution to the IVP

$$y = -\frac{5\left(\left(\frac{12}{5}+\pi\right)\cos(t)-2\left(-\frac{8}{5}+\pi\right)\sin(t)\right) Heaviside\left(t-\frac{\pi}{2}\right) e^{-\frac{t}{2}+\frac{\pi}{4}}}{4} + (4-5t)\, Heaviside\left(t-\frac{\pi}{2}\right) + ((4\cos(t) - 3\,\mathrm{s}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.875 (sec). Leaf size: 91

```
dsolve([4*diff(y(t),t$2)+4*diff(y(t),t)+5*y(t)=25*t*(Heaviside(t)-Heaviside(t-Pi/2)),y(0) =
```

$$\begin{aligned} y(t) = {} & -4 + \left(\frac{5}{4} - \frac{5i}{8}\right)\pi \operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) e^{\left(\frac{1}{4}-\frac{i}{2}\right)(-2t+\pi)} \\ & + \left(\frac{5}{4} + \frac{5i}{8}\right)\pi \operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) e^{\left(\frac{1}{4}+\frac{i}{2}\right)(-2t+\pi)} \\ & - 3\left(\cos(t) + \frac{4\sin(t)}{3}\right) \operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) e^{-\frac{t}{2}+\frac{\pi}{4}} \\ & + (4-5t) \operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) + 6\cos(t)\, e^{-\frac{t}{2}} + 5t \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 101

```
DSolve[{4*y''[t]+4*y'[t]+5*y[t]==25*t*(UnitStep[t]-UnitStep[t-Pi/2]),{y[0]==2,y'[0]==2}},y[t
```

$$y(t) \to \begin{cases} 5t + 6e^{-t/2}\cos(t) - 4 & t \geq 0 \land 2t \leq \pi \\ e^{-t/2}(2\cos(t) + 3\sin(t)) & t < 0 \\ \frac{1}{4}e^{-t/2}\left(\left(24 - e^{\pi/4}(12 + 5\pi)\right)\cos(t) + 2e^{\pi/4}(-8 + 5\pi)\sin(t)\right) & \text{True} \end{cases}$$

## 4.24 problem Problem 3(j)



Internal problem ID [12332]
Internal file name [`OUTPUT/10984_Monday_October_02_2023_02_47_43_AM_68847940/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 3(j).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 3y = \text{Heaviside}(t) - \text{Heaviside}(t-1) + \text{Heaviside}(-2+t) - \text{Heaviside}(-3+t)}$$

With initial conditions

$$\left[y(0) = -\frac{2}{3}, y'(0) = 1\right]$$

### 4.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= \text{Heaviside}(t) - \text{Heaviside}(t-1) + \text{Heaviside}(-2+t) - \text{Heaviside}(-3+t) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 3y = \text{Heaviside}(t) - \text{Heaviside}(t-1) + \text{Heaviside}(-2+t) - \text{Heaviside}(-3+t)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \text{Heaviside}(t) - \text{Heaviside}(t-1) + \text{Heaviside}(-2+t) - \text{Heaviside}(-3+t)$ is

$$\{0 \le t \le 1, 1 \le t \le 2, 2 \le t \le 3, 3 \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 3Y(s) = \frac{1 - e^{-s} + e^{-2s} - e^{-3s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -\frac{2}{3}\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{5}{3} + \frac{2s}{3} + 4sY(s) + 3Y(s) = \frac{1 - e^{-s} + e^{-2s} - e^{-3s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2s^2 + 3\,e^{-s} - 3\,e^{-2s} + 3\,e^{-3s} + 5s - 3}{3s\,(s^2 + 4s + 3)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{2s^2 + 3\,e^{-s} - 3\,e^{-2s} + 3\,e^{-3s} + 5s - 3}{3s\,(s^2+4s+3)}\right) \\
&= \frac{\text{Heaviside}\,(3-t)}{3} - e^{-t} + \frac{(-e^{9-3t} + 3\,e^{3-t})\,\text{Heaviside}\,(-3+t)}{6} + \frac{(2 + e^{6-3t} - 3\,e^{-t+2})\,\text{Heaviside}\,(-2
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y = {} & \frac{\text{Heaviside}\,(3-t)}{3} - e^{-t} + \frac{(-e^{9-3t} + 3\,e^{3-t})\,\text{Heaviside}\,(-3+t)}{6} \\
& + \frac{(2 + e^{6-3t} - 3\,e^{-t+2})\,\text{Heaviside}\,(-2+t)}{6} \\
& + \frac{(-2 - e^{-3t+3} + 3\,e^{1-t})\,\text{Heaviside}\,(t-1)}{6}
\end{aligned}
$$

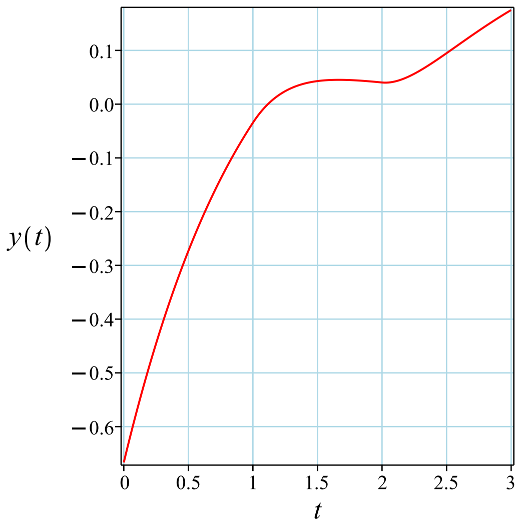
Simplifying the solution gives

$$
\begin{aligned}
y = {} & \frac{1}{3} - \frac{\text{Heaviside}\,(-3+t)}{3} - e^{-t} - \frac{\text{Heaviside}\,(-3+t)\,e^{9-3t}}{6} + \frac{\text{Heaviside}\,(-3+t)\,e^{3-t}}{2} \\
& + \frac{\text{Heaviside}\,(-2+t)\,e^{6-3t}}{6} - \frac{\text{Heaviside}\,(-2+t)\,e^{-t+2}}{2} + \frac{\text{Heaviside}\,(-2+t)}{3} \\
& - \frac{\text{Heaviside}\,(t-1)\,e^{-3t+3}}{6} + \frac{\text{Heaviside}\,(t-1)\,e^{1-t}}{2} - \frac{\text{Heaviside}\,(t-1)}{3}
\end{aligned}
$$

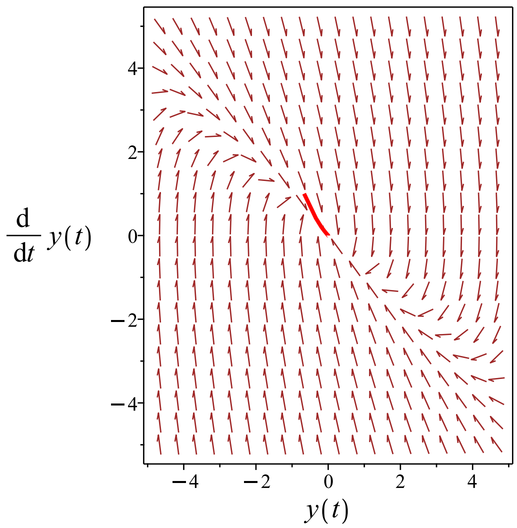
Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \frac{1}{3} - \frac{\text{Heaviside}\,(-3+t)}{3} - e^{-t} - \frac{\text{Heaviside}\,(-3+t)\,e^{9-3t}}{6} \\
& + \frac{\text{Heaviside}\,(-3+t)\,e^{3-t}}{2} + \frac{\text{Heaviside}\,(-2+t)\,e^{6-3t}}{6} \\
& - \frac{\text{Heaviside}\,(-2+t)\,e^{-t+2}}{2} + \frac{\text{Heaviside}\,(-2+t)}{3} \\
& - \frac{\text{Heaviside}\,(t-1)\,e^{-3t+3}}{6} + \frac{\text{Heaviside}\,(t-1)\,e^{1-t}}{2} - \frac{\text{Heaviside}\,(t-1)}{3}
\end{aligned} \tag{1}
$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{3} - \frac{\text{Heaviside}(-3+t)}{3} - e^{-t} - \frac{\text{Heaviside}(-3+t)\,e^{9-3t}}{6} + \frac{\text{Heaviside}(-3+t)\,e^{3-t}}{2} + \frac{\text{Heaviside}(-2+t)\,e^{6-3t}}{6} - \frac{\text{Heaviside}(-2+t)\,e^{-t+2}}{2} + \frac{\text{Heaviside}(-2+t)}{3} - \frac{\text{Heaviside}(t-1)\,e^{-3t+3}}{6} + \frac{\text{Heaviside}(t-1)\,e^{1-t}}{2} - \frac{\text{Heaviside}(t-1)}{3}$$

Verified OK.

### 4.24.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 3y = \mathit{Heaviside}(t) - \mathit{Heaviside}(t-1) + \mathit{Heaviside}(-2+t) - \mathit{Heaviside}(-3+t), y(0\right.$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-3t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-3t} c_1 + c_2 e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \mathit{Heaviside}(t) - \mathit{Heaviside}(t\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-3t} & e^{-t} \\ -3\,e^{-3t} & -e^{-t} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\,e^{-4t}$

  ◦ Substitute functions into equation for $y_p(t)$

  $y_p(t) = -\frac{e^{-3t}\left(\int (\mathit{Heaviside}(t) - \mathit{Heaviside}(t-1) + \mathit{Heaviside}(-2+t) - \mathit{Heaviside}(-3+t))e^{3t}dt\right)}{2} + \frac{e^{-t}\left(\int (\mathit{Heaviside}(t) - \mathit{Heaviside}\right.}{}$

  ◦ Compute integrals

  $y_p(t) = \frac{\mathit{Heaviside}(t)}{3} + \frac{e^{-3t}\mathit{Heaviside}(t)}{6} - \frac{\mathit{Heaviside}(t-1)}{3} - \frac{\mathit{Heaviside}(t-1)e^{-3t+3}}{6} + \frac{\mathit{Heaviside}(-2+t)}{3} + \frac{\mathit{Heaviside}(-2}{6}$

- Substitute particular solution into general solution to ODE

  $y = e^{-3t} c_1 + c_2 e^{-t} + \frac{\mathit{Heaviside}(t)}{3} + \frac{e^{-3t}\mathit{Heaviside}(t)}{6} - \frac{\mathit{Heaviside}(t-1)}{3} - \frac{\mathit{Heaviside}(t-1)e^{-3t+3}}{6} + \frac{\mathit{Heaviside}(-2+t)}{3}$

☐ Check validity of solution $y = e^{-3t} c_1 + c_2 e^{-t} + \frac{\mathit{Heaviside}(t)}{3} + \frac{e^{-3t}\mathit{Heaviside}(t)}{6} - \frac{\mathit{Heaviside}(t-1)}{3} - \mathit{Heavisi}$

  ◦ Use initial condition $y(0) = -\frac{2}{3}$

  $-\frac{2}{3} = c_1 + c_2 + \mathit{undefined}$

  ◦ Compute derivative of the solution

$$y' = -3\,\mathrm{e}^{-3t} c_1 - c_2 \mathrm{e}^{-t} + \frac{\mathit{Dirac}(t)}{3} - \frac{\mathrm{e}^{-3t}\mathit{Heaviside}(t)}{2} + \frac{\mathrm{e}^{-3t}\mathit{Dirac}(t)}{6} - \frac{\mathit{Dirac}(t-1)}{3} - \frac{\mathit{Dirac}(t-1)\mathrm{e}^{-3t+3}}{6} + \frac{\mathit{Heavi}}{}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 1$

  $$1 = -3c_1 - c_2 + \mathit{undefined}$$

- Solve for $c_1$ and $c_2$

  $$\{c_1 = \mathit{undefined}, c_2 = \mathit{undefined}\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{1-t}}{2} + \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}}{2} + \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} - \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{9-3t}}{6} - \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{-3t}}{6}$$

- Solution to the IVP

  $$y = \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{1-t}}{2} + \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{3-t}}{2} + \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{6-3t}}{6} - \frac{\mathit{Heaviside}(-3+t)\mathrm{e}^{9-3t}}{6} - \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{-3t}}{6}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.922 (sec). Leaf size: 88

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+3*y(t)=Heaviside(t)-Heaviside(t-1)+Heaviside(t-2)-Heav
```

$$y(t) = \frac{1}{3} - \frac{\mathrm{Heaviside}\,(t-3)}{3} - \mathrm{e}^{-t} - \frac{\mathrm{Heaviside}\,(t-3)\,\mathrm{e}^{-3t+9}}{6} + \frac{\mathrm{Heaviside}\,(t-3)\,\mathrm{e}^{-t+3}}{2} + \frac{\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{6-3t}}{6} - \frac{\mathrm{Heaviside}\,(t-2)\,\mathrm{e}^{2-t}}{2} + \frac{\mathrm{Heaviside}\,(t-2)}{3} - \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-3t+3}}{6} + \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1}}{2} - \frac{\mathrm{Heaviside}\,(t-1)}{3}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 199

```
DSolve[{y''[t]+4*y'[t]+3*y[t]==UnitStep[t]-UnitStep[t-1]+UnitStep[t-2]-UnitStep[t-3],{y[0]==
```

$$y(t) \to \begin{cases} \frac{1}{3} - e^{-t} & 0 \leq t \leq 1 \\ -\frac{1}{6}e^{-3t}(1 + 3e^{2t}) & t < 0 \\ \frac{1}{6}e^{-3t}(-e^3 - 6e^{2t} + 3e^{2t+1}) & 1 < t \leq 2 \\ \frac{1}{6}e^{-3t}(-e^3 + e^6 - 6e^{2t} + 2e^{3t} + 3e^{2t+1} - 3e^{2t+2}) & 2 < t \leq 3 \\ \frac{1}{6}e^{-3t}(-e^3 + e^6 - e^9 - 6e^{2t} + 3e^{2t+1} - 3e^{2t+2} + 3e^{2t+3}) & \text{True} \end{cases}$$

## 4.25 problem Problem 4(a)



Internal problem ID [12333]
Internal file name [`OUTPUT/10985_Monday_October_02_2023_02_47_43_AM_94070824/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 4(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_y]]`

$$y'' - 2y' = \begin{cases} 4 & 0 \leq t < 1 \\ 6 & 1 \leq t \end{cases}$$

With initial conditions

$$[y(0) = -6, y'(0) = 1]$$

### 4.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= -2 \\
q(t) &= 0 \\
F &= \begin{cases} 0 & t<0 \\ 4 & t<1 \\ 6 & 1\le t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y''-2y' = \begin{cases} 0 & t<0 \\ 4 & t<1 \\ 6 & 1\le t \end{cases}
$$

The domain of $p(t)=-2$ is

$$
\{-\infty<t<\infty\}
$$

And the point $t_0=0$ is inside this domain. The domain of $F=\begin{cases} 0 & t<0 \\ 4 & t<1 \\ 6 & 1\le t \end{cases}$ is

$$
\{0\le t\le 1, 1\le t\le \infty, -\infty\le t\le 0\}
$$

And the point $t_0=0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y)=Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s)-y(0) \\
\mathcal{L}(y'') &= s^2Y(s)-y'(0)-sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$
s^2Y(s)-y'(0)-sy(0)-2sY(s)+2y(0)=\frac{4+2\,\mathrm{e}^{-s}}{s} \tag{1}
$$

But the initial conditions are

$$\begin{aligned}y(0) &= -6\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) - 13 + 6s - 2sY(s) = \frac{4 + 2\,\mathrm{e}^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{-6s^2 + 2\,\mathrm{e}^{-s} + 13s + 4}{s^2\,(s-2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{-6s^2 + 2\,\mathrm{e}^{-s} + 13s + 4}{s^2\,(s-2)}\right)\\ &= -\frac{15}{2} - 2t + \frac{3\,\mathrm{e}^{2t}}{2} + \frac{(1 - \mathrm{Heaviside}\,(1-t))\,\mathrm{e}^{2t-2}}{2} - \frac{\mathrm{Heaviside}\,(t-1)\,(2t-1)}{2}\end{aligned}$$

Hence the final solution is

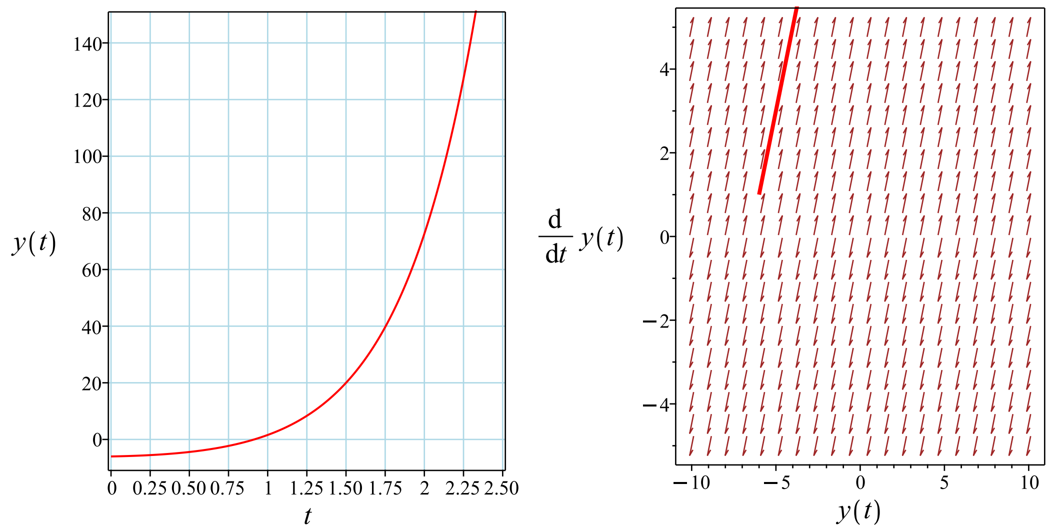
$$y = -\frac{15}{2} - 2t + \frac{3\,\mathrm{e}^{2t}}{2} + \frac{(1 - \mathrm{Heaviside}\,(1-t))\,\mathrm{e}^{2t-2}}{2} - \frac{\mathrm{Heaviside}\,(t-1)\,(2t-1)}{2}$$

Simplifying the solution gives

$$y = \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{2t-2}}{2} + \frac{(1-2t)\,\mathrm{Heaviside}\,(t-1)}{2} - 2t + \frac{3\,\mathrm{e}^{2t}}{2} - \frac{15}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{2t-2}}{2} + \frac{(1-2t)\,\mathrm{Heaviside}\,(t-1)}{2} - 2t + \frac{3\,\mathrm{e}^{2t}}{2} - \frac{15}{2} \tag{1}$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\left(t-1\right) e^{2t-2}}{2}+\frac{\left(1-2t\right) \text{Heaviside}\left(t-1\right)}{2}-2t+\frac{3\,e^{2t}}{2}-\frac{15}{2}$$

Verified OK.

### 4.25.2 Maple step by step solution

Let's solve

$$\left[y''-2y'=\left\{\begin{array}{cc}0 & t<0\\ 4 & t<1\\ 6 & 1\le t\end{array}\right., y(0)=-6, y'\Big|_{\{t=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2-2r=0$

- Factor the characteristic polynomial

  $r(r-2)=0$

- Roots of the characteristic polynomial

$r=(0,2)$

- 1st solution of the homogeneous ODE

  $y_1(t)=1$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{2t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1+c_2\mathrm{e}^{2t}+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

$$\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\begin{cases}0 & t<0\\ 4 & t<1\\ 6 & 1\leq t\end{cases}\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t),y_2(t))=\begin{bmatrix}1 & \mathrm{e}^{2t}\\ 0 & 2\,\mathrm{e}^{2t}\end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t),y_2(t))=2\,\mathrm{e}^{2t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t)=-\left(\int\left(\begin{cases}0 & t<0\\ 2 & t<1\\ 3 & 1\leq t\end{cases}\right)dt\right)+\frac{\mathrm{e}^{2t}\left(\int\left(\begin{cases}0 & t<0\\ 4 & t<1\\ 6 & 1\leq t\end{cases}\right)\mathrm{e}^{-2t}dt\right)}{2}$$

- ○ Compute integrals

$$y_p(t)=\begin{cases}0 & t\leq 0\\ -1+\mathrm{e}^{2t}-2t & t\leq 1\\ -\frac{1}{2}-3t+\frac{\mathrm{e}^{2t-2}}{2}+\mathrm{e}^{2t} & 1<t\end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 + c_2 e^{2t} + \begin{cases} 0 & t \leq 0 \\ -1 + e^{2t} - 2t & t \leq 1 \\ -\frac{1}{2} - 3t + \frac{e^{2t-2}}{2} + e^{2t} & 1 < t \end{cases}$$

☐ Check validity of solution $y = c_1 + c_2 e^{2t} + \begin{cases} 0 & t \leq 0 \\ -1 + e^{2t} - 2t & t \leq 1 \\ -\frac{1}{2} - 3t + \frac{e^{2t-2}}{2} + e^{2t} & 1 < t \end{cases}$

  - Use initial condition $y(0) = -6$

    $-6 = c_1 + c_2$

  - Compute derivative of the solution

    $$y' = 2c_2 e^{2t} + \begin{cases} 0 & t \leq 0 \\ 2\,e^{2t} - 2 & t \leq 1 \\ -3 + e^{2t-2} + 2\,e^{2t} & 1 < t \end{cases}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 1$

    $1 = 2c_2$

  - Solve for $c_1$ and $c_2$

    $\left\{c_1 = -\frac{13}{2}, c_2 = \frac{1}{2}\right\}$

  - Substitute constant values into general solution and simplify

    $$y = -\frac{13}{2} + \frac{e^{2t}}{2} - \left(\begin{cases} 0 & t \leq 0 \\ 1 - e^{2t} + 2t & t \leq 1 \\ \frac{1}{2} + 3t - \frac{e^{2t-2}}{2} - e^{2t} & 1 < t \end{cases}\right)$$

- Solution to the IVP

$$y = -\frac{13}{2} + \frac{e^{2t}}{2} - \left(\begin{cases} 0 & t \leq 0 \\ 1 - e^{2t} + 2t & t \leq 1 \\ \frac{1}{2} + 3t - \frac{e^{2t-2}}{2} - e^{2t} & 1 < t \end{cases}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 2*_b(_a)+4*Heaviside(_a)+2*Heaviside(_a
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 7.703 (sec). Leaf size: 50

```
dsolve([diff(y(t),t$2)-2*diff(y(t),t)=piecewise(0<=t and t<1,4,t>=1,6),y(0) = -6, D(y)(0) =
```

$$y(t) = -\frac{\left(\begin{cases} 15+4t-3\,\mathrm{e}^{2t} & t<1 \\ 20-3\,\mathrm{e}^{2} & t=1 \\ 14+6t-3\,\mathrm{e}^{2t}-\mathrm{e}^{2t-2} & 1<t \end{cases}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 68

```
DSolve[{y''[t]-2*y'[t]==Piecewise[{{4,0<=t<1},{6,t>=1}}],{y[0]==-6,y'[0]==1}},y[t],t,Include
```

$$y(t) \rightarrow \begin{cases} \frac{1}{2}(-13+e^{2t}) & t\leq 0 \\ \frac{1}{2}(-4t+3e^{2t}-15) & 0<t\leq 1 \\ \frac{1}{2}(-6t+3e^{2t}+e^{2t-2}-14) & \text{True} \end{cases}$$

## 4.26 problem Problem 4(b)



Internal problem ID [12334]
Internal file name [`OUTPUT/10986_Monday_October_02_2023_02_47_43_AM_35630610/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 4(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ -1 & 2 \le t \end{cases}}$$

With initial conditions

$$[y(0) = 3, y'(0) = -1]$$

### 4.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$
\begin{aligned}
p(t) &= -3 \\
q(t) &= 2 \\
F &= \begin{cases} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\le t \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y''-3y'+2y = \begin{cases} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\le t \end{cases}
$$

The domain of $p(t)=-3$ is

$$
\{-\infty<t<\infty\}
$$

And the point $t_0=0$ is inside this domain. The domain of $q(t)=2$ is

$$
\{-\infty<t<\infty\}
$$

And the point $t_0=0$ is also inside this domain. The domain of $F=\begin{cases} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\le t \end{cases}$ is

$$
\{1\le t\le 2, 2\le t\le\infty, -\infty\le t\le 1\}
$$

And the point $t_0=0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$
\mathcal{L}(y)=Y(s)
$$

Taking the Laplace transform of the ode and using the relations that

$$
\begin{aligned}
\mathcal{L}(y') &= sY(s)-y(0) \\
\mathcal{L}(y'') &= s^2Y(s)-y'(0)-sy(0)
\end{aligned}
$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 3sY(s) + 3y(0) + 2Y(s) = \frac{-2\,e^{-2s} + e^{-s}}{s} \tag{1}$$

But the initial conditions are

$$y(0) = 3$$
$$y'(0) = -1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 10 - 3s - 3sY(s) + 2Y(s) = \frac{-2\,e^{-2s} + e^{-s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-3s^2 + 2\,e^{-2s} - e^{-s} + 10s}{s\,(s^2 - 3s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-3s^2 + 2\,e^{-2s} - e^{-s} + 10s}{s\,(s^2 - 3s + 2)}\right) \\
&= \frac{\text{Heaviside}\,(t-1)}{2} - \text{Heaviside}\,(-2+t) + 7\,e^{t} - 4\,e^{2t} + 2\,e^{-2+t} - e^{t-1} - e^{-4+2t} + \frac{e^{2t-2}}{2} + \frac{(2\,e^{t-1} - e^{2t}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = {} & \frac{\text{Heaviside}\,(t-1)}{2} - \text{Heaviside}\,(-2+t) + 7\,e^{t} - 4\,e^{2t} + 2\,e^{-2+t} - e^{t-1} - e^{-4+2t} \\
& + \frac{e^{2t-2}}{2} + \frac{(2\,e^{t-1} - e^{2t-2})\,\text{Heaviside}\,(1-t)}{2} + \left(-2\,e^{-2+t} + e^{-4+2t}\right)\text{Heaviside}\,(-t+2)
\end{aligned}$$

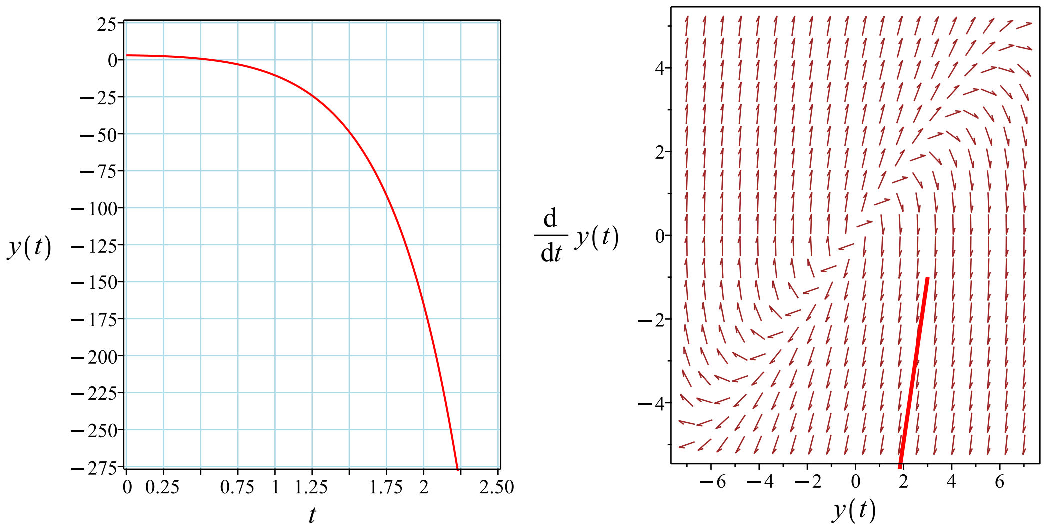
Simplifying the solution gives

$$\begin{aligned}
y = {} & -e^{-4+2t}\,\text{Heaviside}\,(-2+t) + \frac{\text{Heaviside}\,(t-1)\,e^{2t-2}}{2} \\
& + \left(-1 + 2\,e^{-2+t}\right)\text{Heaviside}\,(-2+t) + \frac{(-2\,e^{t-1} + 1)\,\text{Heaviside}\,(t-1)}{2} + 7\,e^{t} - 4\,e^{2t}
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = {} & -e^{-4+2t}\,\text{Heaviside}\,(-2+t) + \frac{\text{Heaviside}\,(t-1)\,e^{2t-2}}{2} \\
& + \left(-1 + 2\,e^{-2+t}\right)\text{Heaviside}\,(-2+t) \\
& + \frac{(-2\,e^{t-1} + 1)\,\text{Heaviside}\,(t-1)}{2} + 7\,e^{t} - 4\,e^{2t}
\end{aligned} \tag{1}$$




(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\mathrm{e}^{-4+2t}\,\mathrm{Heaviside}\,(-2+t) + \frac{\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{2t-2}}{2} + \left(-1+2\,\mathrm{e}^{-2+t}\right)\mathrm{Heaviside}\,(-2+t) + \frac{\left(-2\,\mathrm{e}^{t-1}+1\right)\mathrm{Heaviside}\,(t-1)}{2} + 7\,\mathrm{e}^{t} - 4\,\mathrm{e}^{2t}$$

Verified OK.

### 4.26.2 Maple step by step solution

Let's solve

$$\left[y'' - 3y' + 2y = \left\{\begin{array}{cc} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\leq t \end{array}\right., y(0)=3, y'\Big|_{\{t=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)\,(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$
- 1st solution of the homogeneous ODE

  $y_1(t) = e^t$
- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{2t}$
- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^t + c_2 e^{2t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = \begin{cases} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\leq t \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(t), y_2(t)) = e^{3t}$$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = -e^t\left(\int \left(\begin{cases} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\leq t \end{cases}\right) e^{-t}dt\right) + e^{2t}\left(\int \left(\begin{cases} 0 & t<1 \\ 1 & t<2 \\ -1 & 2\leq t \end{cases}\right) e^{-2t}dt\right)$$

- Compute integrals

$$y_p(t) = \frac{\left(\begin{cases} 0 & t\leq 1 \\ 1-2e^{t-1}+e^{2t-2} & t\leq 2 \\ -2e^{t-1}+4e^{-2+t}-1+e^{2t-2}-2e^{-4+2t} & 2<t \end{cases}\right)}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + \frac{\left(\begin{cases} 0 & t \leq 1 \\ 1 - 2\,\mathrm{e}^{t-1} + \mathrm{e}^{2t-2} & t \leq 2 \\ -2\,\mathrm{e}^{t-1} + 4\,\mathrm{e}^{-2+t} - 1 + \mathrm{e}^{2t-2} - 2\,\mathrm{e}^{-4+2t} & 2 < t \end{cases}\right)}{2}$$

□ Check validity of solution $y = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + \frac{\left(\begin{cases} 0 & t \leq 1 \\ 1 - 2\mathrm{e}^{t-1} + \mathrm{e}^{2t-2} & t \leq 2 \\ -2\mathrm{e}^{t-1} + 4\mathrm{e}^{-2+t} - 1 + \mathrm{e}^{2t-2} - 2\mathrm{e}^{-4+2t} & 2 < t \end{cases}\right)}{2}$

  - Use initial condition $y(0) = 3$

    $3 = c_1 + c_2$

  - Compute derivative of the solution

$$y' = c_1 \mathrm{e}^t + 2c_2 \mathrm{e}^{2t} + \frac{\left(\begin{cases} 0 & t \leq 1 \\ -2\,\mathrm{e}^{t-1} + 2\,\mathrm{e}^{2t-2} & t \leq 2 \\ -2\,\mathrm{e}^{t-1} + 4\,\mathrm{e}^{-2+t} + 2\,\mathrm{e}^{2t-2} - 4\,\mathrm{e}^{-4+2t} & 2 < t \end{cases}\right)}{2}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = -1$

    $-1 = 2c_2 + c_1$

  - Solve for $c_1$ and $c_2$

    $\{c_1 = 7, c_2 = -4\}$

  - Substitute constant values into general solution and simplify

$$y = 7\,\mathrm{e}^t - 4\,\mathrm{e}^{2t} - \frac{\left(\begin{cases} 0 & t \leq 1 \\ -1 + 2\,\mathrm{e}^{t-1} - \mathrm{e}^{2t-2} & t \leq 2 \\ 2\,\mathrm{e}^{t-1} - 4\,\mathrm{e}^{-2+t} + 1 - \mathrm{e}^{2t-2} + 2\,\mathrm{e}^{-4+2t} & 2 < t \end{cases}\right)}{2}$$

- Solution to the IVP

$$y = 7\,\mathrm{e}^t - 4\,\mathrm{e}^{2t} - \frac{\left(\begin{cases} 0 & t \leq 1 \\ -1 + 2\,\mathrm{e}^{t-1} - \mathrm{e}^{2t-2} & t \leq 2 \\ 2\,\mathrm{e}^{t-1} - 4\,\mathrm{e}^{-2+t} + 1 - \mathrm{e}^{2t-2} + 2\,\mathrm{e}^{-4+2t} & 2 < t \end{cases}\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 7.172 (sec). Leaf size: 121

```
dsolve([diff(y(t),t$2)-3*diff(y(t),t)+2*y(t)=piecewise(0<=t and t<1,0,t>=1 and t<2,1,t>=2,-1
```

$$y(t) = \begin{cases} 7\,\mathrm{e}^{t}-4\,\mathrm{e}^{2t} & t<1 \\ 7\,\mathrm{e}-4\,\mathrm{e}^{2}+\frac{1}{2} & t=1 \\ 7\,\mathrm{e}^{t}-4\,\mathrm{e}^{2t}-\mathrm{e}^{t-1}+\frac{\mathrm{e}^{2t-2}}{2}+\frac{1}{2} & t<2 \\ \frac{15\,\mathrm{e}^{2}}{2}-4\,\mathrm{e}^{4}-\mathrm{e}-\frac{1}{2} & t=2 \\ 7\,\mathrm{e}^{t}-4\,\mathrm{e}^{2t}+2\,\mathrm{e}^{t-2}-\mathrm{e}^{t-1}-\mathrm{e}^{2t-4}+\frac{\mathrm{e}^{2t-2}}{2}-\frac{1}{2} & 2<t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 109

```
DSolve[{y''[t]-3*y'[t]+2*y[t]==Piecewise[{{0,0<=t<1},{1,1<=t<2},{-1,t>=2}}],{y[0]==3,y'[0]==
```

$$y(t) \rightarrow \{ \begin{array}{cc} e^{t}(7-4e^{t}) & t\leq 1 \\ \frac{1}{2}(1-2e^{t-1}+14e^{t}-8e^{2t}+e^{2t-2}) & 1<t\leq 2 \\ \frac{1}{2}(-1+4e^{t-2}-2e^{t-1}+14e^{t}-8e^{2t}-2e^{2t-4}+e^{2t-2}) & \text{True} \end{array}$$

## 4.27 problem Problem 4(c)



Internal problem ID [12335]
Internal file name [`OUTPUT/10987_Monday_October_02_2023_02_47_43_AM_97693164/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 4(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \begin{cases} 1 & 0 \le t < 2 \\ -1 & 2 \le t \end{cases}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \le t \end{cases} \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \leq t \end{cases}$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \leq t \end{cases}$ is

$$\{0 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique. Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{-2\,\mathrm{e}^{-2s} + 1}{s} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{-2\,e^{-2s} + 1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2\,e^{-2s} - 1}{s\,(s^2 + 3s + 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{2\,e^{-2s} - 1}{s\,(s^2 + 3s + 2)}\right) \\ &= \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t} + \left(-1 - e^{4-2t} + 2\,e^{-t+2}\right) \text{Heaviside}\,(-2+t) \end{aligned}$$

Hence the final solution is

$$y = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t} + \left(-1 - e^{4-2t} + 2\,e^{-t+2}\right) \text{Heaviside}\,(-2+t)$$

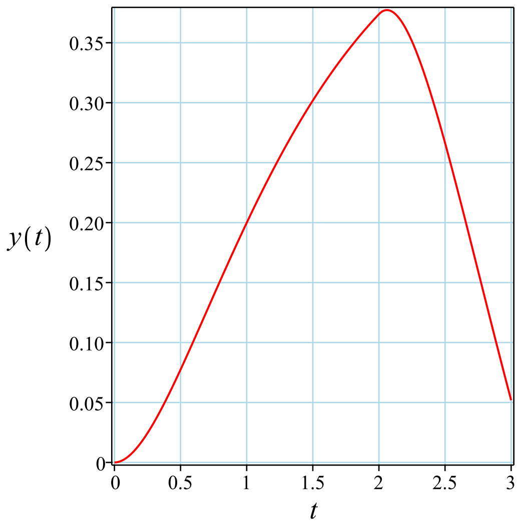
Simplifying the solution gives

$$\begin{aligned} y = {} & \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t} - \text{Heaviside}\,(-2+t)\,e^{4-2t} \\ & + 2\,\text{Heaviside}\,(-2+t)\,e^{-t+2} - \text{Heaviside}\,(-2+t) \end{aligned}$$

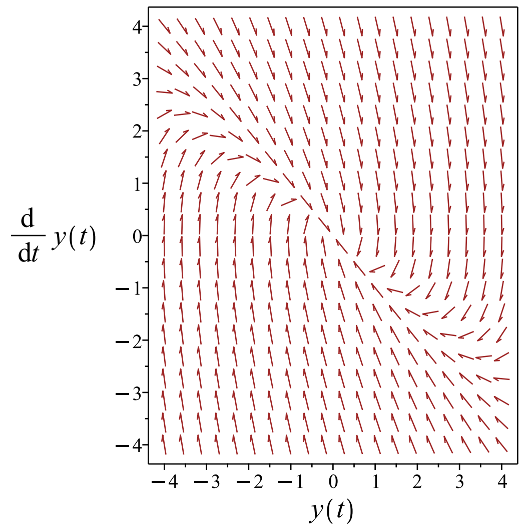
Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t} - \text{Heaviside}\,(-2+t)\,e^{4-2t} \\ & + 2\,\text{Heaviside}\,(-2+t)\,e^{-t+2} - \text{Heaviside}\,(-2+t) \end{aligned} \tag{1}$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t} - \text{Heaviside}(-2+t)\, e^{4-2t} + 2\,\text{Heaviside}(-2+t)\, e^{-t+2} - \text{Heaviside}(-2+t)$$

Verified OK.

### 4.27.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \leq t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$
- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$
- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

$$\left[ y_p(t) = -y_1(t) \left( \int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt \right) + y_2(t) \left( \int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt \right), f(t) = \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \leq t \end{cases} \right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

- ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -e^{-2t} \left( \int \left( \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \leq t \end{cases} \right) e^{2t} dt \right) + e^{-t} \left( \int \left( \begin{cases} 0 & t < 0 \\ 1 & t < 2 \\ -1 & 2 \leq t \end{cases} \right) e^{t} dt \right)$$

- ○ Compute integrals

$$y_p(t) = -\frac{\left( \begin{cases} 0 & t \leq 0 \\ -1 - e^{-2t} + 2\,e^{-t} & t \leq 2 \\ 1 - e^{-2t} + 2\,e^{-t} + 2\,e^{4-2t} - 4\,e^{-t+2} & 2 < t \end{cases} \right)}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2t} + c_2 e^{-t} - \frac{\left(\begin{cases} 0 & t \le 0 \\ -1 - e^{-2t} + 2\,e^{-t} & t \le 2 \\ 1 - e^{-2t} + 2\,e^{-t} + 2\,e^{4-2t} - 4\,e^{-t+2} & 2 < t \end{cases}\right)}{2}$$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{-t} - \frac{\left(\begin{cases} 0 & t \le 0 \\ -1 - e^{-2t} + 2e^{-t} & t \le 2 \\ 1 - e^{-2t} + 2e^{-t} + 2e^{4-2t} - 4e^{-t+2} & 2 < t \end{cases}\right)}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $$y' = -2c_1 e^{-2t} - c_2 e^{-t} - \frac{\left(\begin{cases} 0 & t \le 0 \\ -2\,e^{-t} + 2\,e^{-2t} & t \le 2 \\ 2\,e^{-2t} - 2\,e^{-t} - 4\,e^{4-2t} + 4\,e^{-t+2} & 2 < t \end{cases}\right)}{2}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 - c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $$y = -\frac{\left(\begin{cases} 0 & t \le 0 \\ -1 - e^{-2t} + 2\,e^{-t} & t \le 2 \\ 1 - e^{-2t} + 2\,e^{-t} + 2\,e^{4-2t} - 4\,e^{-t+2} & 2 < t \end{cases}\right)}{2}$$

- Solution to the IVP

$$y = -\frac{\left(\begin{cases} 0 & t \le 0 \\ -1 - e^{-2t} + 2\,e^{-t} & t \le 2 \\ 1 - e^{-2t} + 2\,e^{-t} + 2\,e^{4-2t} - 4\,e^{-t+2} & 2 < t \end{cases}\right)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 6.079 (sec). Leaf size: 55

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=piecewise(0<=t and t<2,1,t>=2,-1),y(0) = 0, D(y
```

$$y(t) = -\mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{2} - \frac{\left(\begin{cases} -1 & t < 2 \\ 1 - 4\,\mathrm{e}^{2-t} + 2\,\mathrm{e}^{-2t+4} & 2 \leq t \end{cases}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 68

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==Piecewise[{{1,0<=t<2},{-1,t>=2}}],{y[0]==0,y'[0]==0}},y[t],t,
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ \frac{1}{2}e^{-2t}(-1+e^t)^2 & 0 < t \leq 2 \\ -\frac{1}{2}e^{-2t}(-1+2e^4+2e^t+e^{2t}-4e^{t+2}) & \text{True} \end{cases}$$

## 4.28 problem Problem 4(d)



Internal problem ID [12336]
Internal file name [`OUTPUT/10988_Monday_October_02_2023_02_47_44_AM_15034228/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 4(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + y = \begin{cases} t & 0 \leq t < \pi \\ -t & \pi \leq t \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$p(t) = 0$$
$$q(t) = 1$$
$$F = \begin{cases} 0 & t < 0 \\ t & t < \pi \\ -t & \pi \leq t \end{cases}$$

Hence the ode is

$$y'' + y = \begin{cases} 0 & t < 0 \\ t & t < \pi \\ -t & \pi \le t \end{cases}$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & t < 0 \\ t & t < \pi \\ -t & \pi \le t \end{cases}$ is

$$\{0 \le t \le \pi, \pi \le t \le \infty, -\infty \le t \le 0\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + Y(s) = \frac{-2(s\pi + 1)\,\mathrm{e}^{-s\pi} + 1}{s^2} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + Y(s) = \frac{-2(s\pi + 1)\,\mathrm{e}^{-s\pi} + 1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2\pi\,\mathrm{e}^{-s\pi} s + 2\,\mathrm{e}^{-s\pi} - 1}{s^2\,(s^2 + 1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{2\pi\,\mathrm{e}^{-s\pi} s + 2\,\mathrm{e}^{-s\pi} - 1}{s^2\,(s^2 + 1)}\right) \\
&= -\sin(t) + t + 2\left(-2\pi\cos\left(\frac{t}{2}\right)^2 + \pi - \sin(t) - t\right)\mathrm{Heaviside}(t - \pi)
\end{aligned}$$

Hence the final solution is

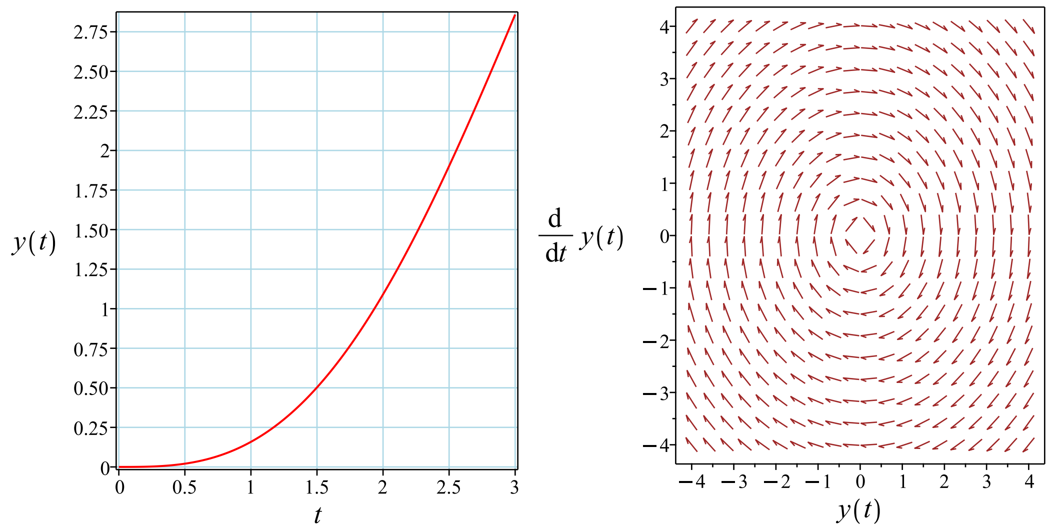
$$y = -\sin(t) + t + 2\left(-2\pi\cos\left(\frac{t}{2}\right)^2 + \pi - \sin(t) - t\right)\mathrm{Heaviside}(t - \pi)$$

Simplifying the solution gives

$$y = (-2\pi\cos(t) - 2t - 2\sin(t))\,\mathrm{Heaviside}(t - \pi) + t - \sin(t)$$

Summary
The solution(s) found are the following

$$y = (-2\pi\cos(t) - 2t - 2\sin(t))\,\mathrm{Heaviside}(t - \pi) + t - \sin(t) \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (-2\pi \cos(t) - 2t - 2\sin(t))\,\text{Heaviside}(t - \pi) + t - \sin(t)$$

Verified OK.

### 4.28.2 Maple step by step solution

Let's solve

$$\left[y'' + y = \begin{cases} 0 & t < 0 \\ t & t < \pi \\ -t & \pi \le t \end{cases}, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  $y''$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for $r$
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(t) + c_2 \sin(t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

$$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \begin{cases} 0 & t < 0 \\ t & t < \pi \\ -t & \pi \leq t \end{cases}\right]$$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

  ○ Compute Wronskian

$$W(y_1(t), y_2(t)) = 1$$

  ○ Substitute functions into equation for $y_p(t)$

$$y_p(t) = -\cos(t)\left(\int \sin(t)\, t \left(\begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ -1 & \pi \leq t \end{cases}\right) dt\right) + \sin(t)\left(\int \cos(t)\, t \left(\begin{cases} 0 & t < 0 \\ 1 & t < \pi \\ -1 & \pi \leq t \end{cases}\right) dt\right)$$

  ○ Compute integrals

$$y_p(t) = \begin{cases} 0 & t \leq 0 \\ -\sin(t) + t & t \leq \pi \\ -2\pi \cos(t) - 3\sin(t) - t & \pi < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(t) + c_2 \sin(t) + \begin{cases} 0 & t \le 0 \\ -\sin(t) + t & t \le \pi \\ -2\pi\cos(t) - 3\sin(t) - t & \pi < t \end{cases}$$

- ☐ Check validity of solution $y = c_1 \cos(t) + c_2 \sin(t) + \begin{cases} 0 & t \le 0 \\ -\sin(t) + t & t \le \pi \\ -2\pi\cos(t) - 3\sin(t) - t & \pi < t \end{cases}$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $$y' = -c_1 \sin(t) + c_2 \cos(t) + \begin{cases} 0 & t \le 0 \\ -\cos(t) + 1 & t \le \pi \\ 2\pi\sin(t) - 3\cos(t) - 1 & \pi < t \end{cases}$$

  - Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = c_2$

  - Solve for $c_1$ and $c_2$

    $\{c_1 = 0, c_2 = 0\}$

  - Substitute constant values into general solution and simplify

    $$y = \begin{cases} 0 & t \le 0 \\ -\sin(t) + t & t \le \pi \\ -2\pi\cos(t) - 3\sin(t) - t & \pi < t \end{cases}$$

- Solution to the IVP

$$y = \begin{cases} 0 & t \le 0 \\ -\sin(t) + t & t \le \pi \\ -2\pi\cos(t) - 3\sin(t) - t & \pi < t \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.281 (sec). Leaf size: 37

```
dsolve([diff(y(t),t$2)+y(t)=piecewise(0<=t and t<Pi,t,t>=Pi,-t),y(0) = 0, D(y)(0) = 0],y(t),
```

$$y(t) = \begin{cases} t - \sin(t) & t < \pi \\ -2\cos(t)\pi - 3\sin(t) - t & \pi \leq t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 38

```
DSolve[{y''[t]+y[t]==Piecewise[{{t,0<=t<Pi},{-t,t>=Pi}}],{y[0]==0,y'[0]==0}},y[t],t,IncludeS
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ t - \sin(t) & 0 < t \leq \pi \\ -t - 2\pi\cos(t) - 3\sin(t) & \text{True} \end{cases}$$

## 4.29 problem Problem 4(e)



Internal problem ID [12337]
Internal file name [`OUTPUT/10989_Monday_October_02_2023_02_47_44_AM_6249229/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 4(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second__order__laplace", "second__order__linear__constant__coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$
\boxed{y'' + 4y = \begin{cases} 8t & 0 \le t < \frac{\pi}{2} \\ 8\pi & \frac{\pi}{2} \le t \end{cases}}
$$

With initial conditions

$$
[y(0) = 0, y'(0) = 0]
$$

### 4.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$
y'' + p(t)y' + q(t)y = F
$$

Where here

$$
\begin{aligned}
p(t) &= 0 \\
q(t) &= 4 \\
F &= 8\left(\begin{cases} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \le t \end{cases}\right)
\end{aligned}
$$

Hence the ode is

$$y'' + 4y = 8\left(\left\{\begin{array}{cc} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \le t \end{array}\right.\right)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 8\left(\left\{\begin{array}{cc} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \le t \end{array}\right.\right)$ is

$$\left\{0 \le t \le \frac{\pi}{2}, \frac{\pi}{2} \le t \le \infty, -\infty \le t \le 0\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \frac{8 + 4\,\mathrm{e}^{-\frac{s\pi}{2}}(s\pi - 2)}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) + 4Y(s) = \frac{8 + 4\,\mathrm{e}^{-\frac{s\pi}{2}} (s\pi - 2)}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4\pi\,\mathrm{e}^{-\frac{s\pi}{2}} s - 8\,\mathrm{e}^{-\frac{s\pi}{2}} + 8}{s^2 (s^2 + 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{4\pi\,\mathrm{e}^{-\frac{s\pi}{2}} s - 8\,\mathrm{e}^{-\frac{s\pi}{2}} + 8}{s^2 (s^2 + 4)}\right) \\ &= -\sin(2t) + 2t + \text{Heaviside}\left(t - \frac{\pi}{2}\right)\left(2\pi \cos(t)^2 + \pi - \sin(2t) - 2t\right) \end{aligned}$$

Hence the final solution is

$$y = -\sin(2t) + 2t + \text{Heaviside}\left(t - \frac{\pi}{2}\right)\left(2\pi \cos(t)^2 + \pi - \sin(2t) - 2t\right)$$

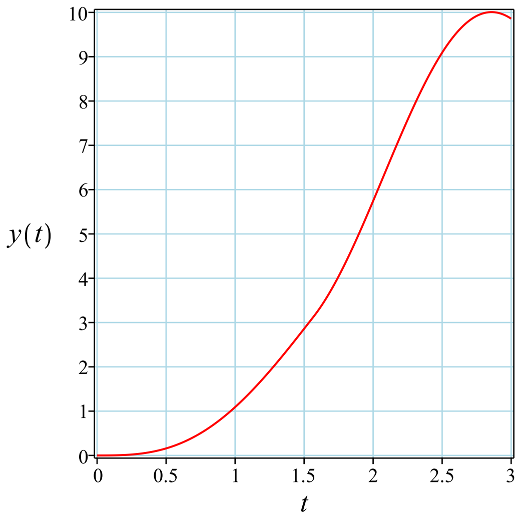
Simplifying the solution gives

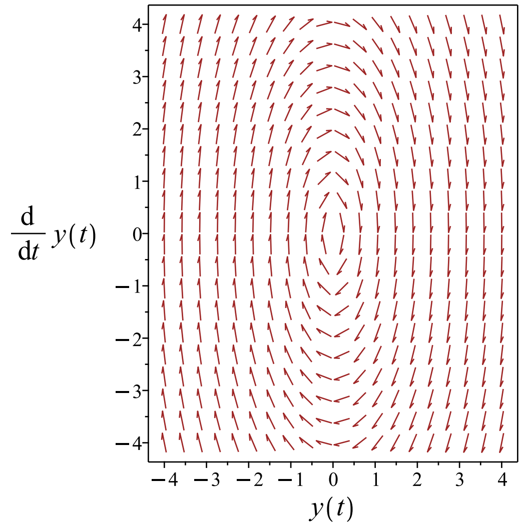
$$y = (\pi \cos(2t) - 2t + 2\pi - \sin(2t))\,\text{Heaviside}\left(t - \frac{\pi}{2}\right) + 2t - \sin(2t)$$

Summary

The solution(s) found are the following

$$y = (\pi \cos(2t) - 2t + 2\pi - \sin(2t))\,\text{Heaviside}\left(t - \frac{\pi}{2}\right) + 2t - \sin(2t) \tag{1}$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (\pi \cos(2t) - 2t + 2\pi - \sin(2t)) \operatorname{Heaviside}\left(t - \frac{\pi}{2}\right) + 2t - \sin(2t)$$

Verified OK.

### 4.29.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = 8\left(\begin{cases} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \leq t \end{cases}\right), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$
- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$
- Use quadratic formula to solve for $r$

  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2t) + c_2 \sin(2t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 8\left(\begin{cases} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \leq t \end{cases}\right)\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = 2$

- ○ Substitute functions into equation for $y_p(t)$

  $$y_p(t) = -4\cos(2t)\left(\int \sin(2t)\left(\left(\begin{cases} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \leq t \end{cases}\right)\right) dt\right) + 4\sin(2t)\left(\int \cos(2t)\left(\left(\begin{cases} 0 & t < 0 \\ t & t < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} \leq t \end{cases}\right)\right) dt\right)$$

- ○ Compute integrals

  $$y_p(t) = \begin{cases} 0 & t \leq 0 \\ -\sin(2t) + 2t & t \leq \frac{\pi}{2} \\ \pi\cos(2t) + 2\pi - 2\sin(2t) & \frac{\pi}{2} < t \end{cases}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \begin{cases} 0 & t \leq 0 \\ -\sin(2t) + 2t & t \leq \frac{\pi}{2} \\ \pi \cos(2t) + 2\pi - 2\sin(2t) & \frac{\pi}{2} < t \end{cases}$$

☐ Check validity of solution $y = c_1 \cos(2t) + c_2 \sin(2t) + \begin{cases} 0 & t \leq 0 \\ -\sin(2t) + 2t & t \leq \frac{\pi}{2} \\ \pi \cos(2t) + 2\pi - 2\sin(2t) & \frac{\pi}{2} < t \end{cases}$

- Use initial condition $y(0) = 0$

  $0 = c_1$
- Compute derivative of the solution

  $$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \begin{cases} 0 & t \leq 0 \\ -2\cos(2t) + 2 & t \leq \frac{\pi}{2} \\ -2\pi \sin(2t) - 4\cos(2t) & \frac{\pi}{2} < t \end{cases}$$
- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2$
- Solve for $c_1$ and $c_2$

  $\{c_1 = 0, c_2 = 0\}$
- Substitute constant values into general solution and simplify

  $$y = \begin{cases} 0 & t \leq 0 \\ -\sin(2t) + 2t & t \leq \frac{\pi}{2} \\ \pi \cos(2t) + 2\pi - 2\sin(2t) & \frac{\pi}{2} < t \end{cases}$$

- Solution to the IVP

  $$y = \begin{cases} 0 & t \leq 0 \\ -\sin(2t) + 2t & t \leq \frac{\pi}{2} \\ \pi \cos(2t) + 2\pi - 2\sin(2t) & \frac{\pi}{2} < t \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.25 (sec). Leaf size: 40

```
dsolve([diff(y(t),t$2)+4*y(t)=piecewise(0<=t and t<Pi/2,8*t,t>=Pi/2,8*Pi),y(0) = 0, D(y)(0)
```

$$y(t) = \begin{cases} -\sin(2t) + 2t & t < \frac{\pi}{2} \\ -2\sin(2t) + 2\cos(t)^2 \pi + \pi & \frac{\pi}{2} \leq t \end{cases}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 48

```
DSolve[{y''[t]+4*y[t]==Piecewise[{{8*t,0<=t<Pi/2},{8*Pi,t>=Pi/2}}],{y[0]==0,y'[0]==0}},y[t],
```

$$y(t) \to \begin{cases} 0 & t \leq 0 \\ 2t - \sin(2t) & t > 0 \land 2t \leq \pi \\ \pi\cos(2t) - 2\sin(2t) + 2\pi & \text{True} \end{cases}$$

## 4.30 problem Problem 5(a)



Internal problem ID [12338]
Internal file name [`OUTPUT/10990_Monday_October_02_2023_02_47_44_AM_46001342/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 5(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4\pi^2 y = 3\delta\left(t - \frac{1}{3}\right) - \delta(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4\pi^2 \\ F &= 3\delta\left(t - \frac{1}{3}\right) - \delta(t-1) \end{aligned}$$

Hence the ode is

$$y'' + 4\pi^2 y = 3\delta\left(t - \frac{1}{3}\right) - \delta(t-1)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4\pi^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\delta\left(t - \frac{1}{3}\right) - \delta(t-1)$ is

$$\left\{1 \leq t \leq \frac{1}{3}, \frac{1}{3} \leq t \leq \infty, -\infty \leq t \leq 1\right\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) + 4\pi^2 Y(s) = 3\,\mathrm{e}^{-\frac{s}{3}} - \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) + 4\pi^2 Y(s) = 3\,\mathrm{e}^{-\frac{s}{3}} - \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3\,\mathrm{e}^{-\frac{s}{3}} - \mathrm{e}^{-s}}{4\pi^2 + s^2}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{3\,\mathrm{e}^{-\frac{s}{3}}-\mathrm{e}^{-s}}{4\pi^2+s^2}\right) \\
&= \frac{-\operatorname{Heaviside}(t-1)\sin(2\pi t)+3\sin\left(\frac{2\pi(-1+3t)}{3}\right)\operatorname{Heaviside}\left(t-\frac{1}{3}\right)}{2\pi}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{-\operatorname{Heaviside}(t-1)\sin(2\pi t)+3\sin\left(\frac{2\pi(-1+3t)}{3}\right)\operatorname{Heaviside}\left(t-\frac{1}{3}\right)}{2\pi}
$$

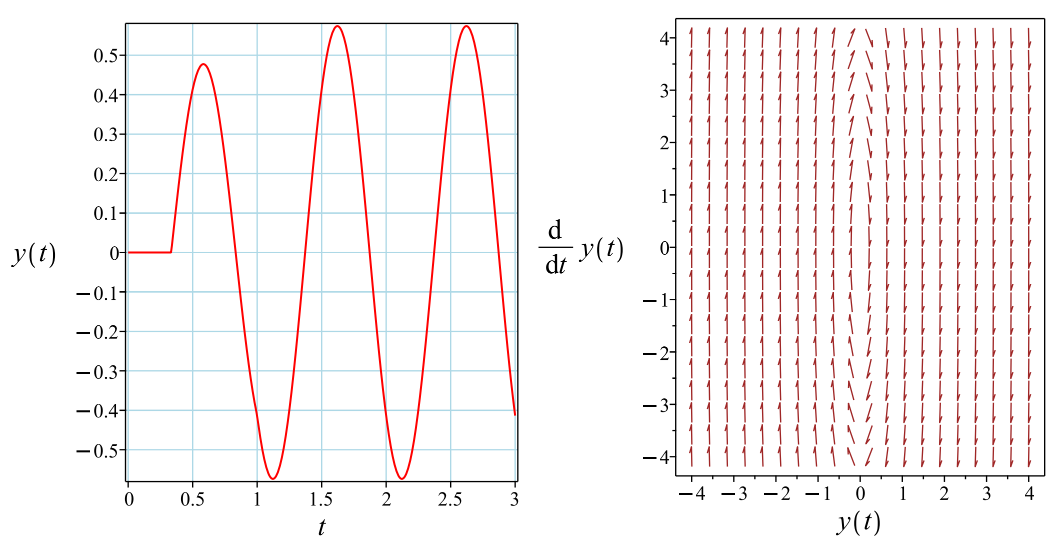
Simplifying the solution gives

$$
y = \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)\operatorname{Heaviside}\left(t-\frac{1}{3}\right)-2\operatorname{Heaviside}(t-1)\sin(2\pi t)}{4\pi}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)\operatorname{Heaviside}\left(t-\frac{1}{3}\right)-2\operatorname{Heaviside}(t-1)\sin(2\pi t)}{4\pi} \tag{1}
$$




(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\left(-3\cos\left(2\pi t\right)\sqrt{3}-3\sin\left(2\pi t\right)\right)\text{Heaviside}\left(t-\frac{1}{3}\right)-2\,\text{Heaviside}\left(t-1\right)\sin\left(2\pi t\right)}{4\pi}$$

Verified OK.

### 4.30.2 Maple step by step solution

Let's solve

$$\left[y''+4\pi^2 y = 3Dirac\left(t-\frac{1}{3}\right)-Dirac(t-1)\,, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$
- Characteristic polynomial of homogeneous ODE

  $4\pi^2+r^2=0$
- Use quadratic formula to solve for $r$

  $r=\frac{0\pm\left(\sqrt{-16\pi^2}\right)}{2}$
- Roots of the characteristic polynomial

  $r=(-2\,\mathrm{I}\pi, 2\,\mathrm{I}\pi)$
- 1st solution of the homogeneous ODE

  $y_1(t)=\cos\left(2\pi t\right)$
- 2nd solution of the homogeneous ODE

  $y_2(t)=\sin\left(2\pi t\right)$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos\left(2\pi t\right)+c_2\sin\left(2\pi t\right)+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int\frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int\frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=3Dirac\left(t-\frac{1}{3}\right)-Dirac(t-\right.$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -2\pi\sin(2\pi t) & 2\pi\cos(2\pi t) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 2\pi$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{-3\cos(2\pi t)\sqrt{3}\left(\int Dirac\left(t-\frac{1}{3}\right)dt\right) - \sin(2\pi t)\left(\int\left(2Dirac(t-1)+3Dirac\left(t-\frac{1}{3}\right)\right)dt\right)}{4\pi}$

- Compute integrals

  $y_p(t) = \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)Heaviside\left(t-\frac{1}{3}\right)-2Heaviside(t-1)\sin(2\pi t)}{4\pi}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(2\pi t) + c_2\sin(2\pi t) + \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)Heaviside\left(t-\frac{1}{3}\right)-2Heaviside(t-1)\sin(2\pi t)}{4\pi}$

☐ Check validity of solution $y = c_1\cos(2\pi t) + c_2\sin(2\pi t) + \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)Heaviside\left(t-\frac{1}{3}\right)-2He}{4\pi}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -2c_1\pi\sin(2\pi t) + 2c_2\pi\cos(2\pi t) + \frac{\left(6\pi\sin(2\pi t)\sqrt{3}-6\pi\cos(2\pi t)\right)Heaviside\left(t-\frac{1}{3}\right)+\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)}{4\pi}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = 2c_2\pi$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)Heaviside\left(t-\frac{1}{3}\right)-2Heaviside(t-1)\sin(2\pi t)}{4\pi}$

- Solution to the IVP

  $y = \frac{\left(-3\cos(2\pi t)\sqrt{3}-3\sin(2\pi t)\right)Heaviside\left(t-\frac{1}{3}\right)-2Heaviside(t-1)\sin(2\pi t)}{4\pi}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.843 (sec). Leaf size: 36

```
dsolve([diff(y(t),t$2)+(2*Pi)^2*y(t)=3*Dirac(t-1/3)-Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t),
```

$$y(t) = \frac{\left(-3\sqrt{3}\, \cos\left(2\pi t\right) - 3\sin\left(2\pi t\right)\right) \text{Heaviside}\left(t - \frac{1}{3}\right) - 2\sin\left(2\pi t\right) \text{Heaviside}\left(t - 1\right)}{4\pi}$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 49

```
DSolve[{y''[t]+(2*Pi)^2*y[t]==3*DiracDelta[t-1/3]-DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t
```

$$y(t) \to -\frac{2\theta(t-1)\sin(2\pi t) + 3\theta(3t-1)\left(\sin(2\pi t) + \sqrt{3}\cos(2\pi t)\right)}{4\pi}$$

## 4.31 problem Problem 5(b)



Internal problem ID [12339]
Internal file name [`OUTPUT/10991_Monday_October_02_2023_02_47_45_AM_79997059/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 5(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = 3\delta(t - 1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 2 \\ F &= 3\delta(t - 1) \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 2y = 3\delta(t - 1)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 3\delta(t-1)$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 2sY(s) - 2y(0) + 2Y(s) = 3\,\mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2sY(s) + 2Y(s) = 3\,\mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3\,\mathrm{e}^{-s}}{s^2 + 2s + 2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{3\,\mathrm{e}^{-s}}{s^2 + 2s + 2}\right)\\ &= 3\,\mathrm{Heaviside}\,(t-1)\sin\,(t-1)\,\mathrm{e}^{1-t}\end{aligned}$$

Hence the final solution is

$$y = 3\,\mathrm{Heaviside}\left(t-1\right)\sin\left(t-1\right)\mathrm{e}^{1-t}$$

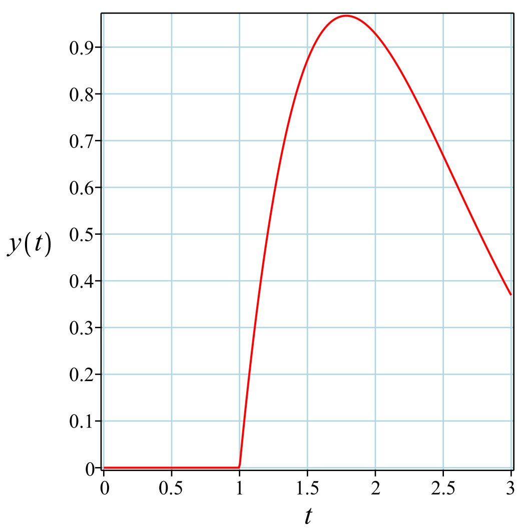
Simplifying the solution gives

$$y = 3\,\mathrm{Heaviside}\left(t-1\right)\sin\left(t-1\right)\mathrm{e}^{1-t}$$

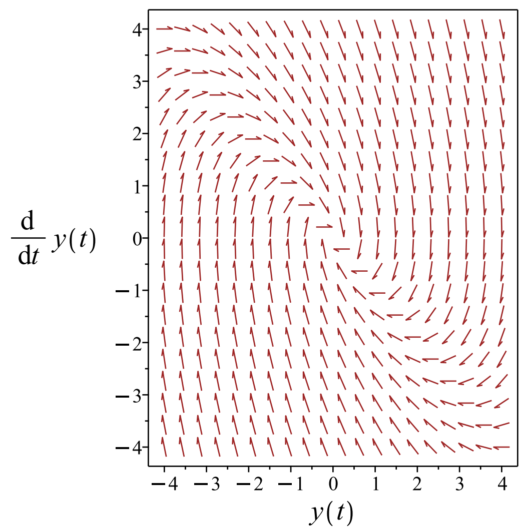
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{Heaviside}\left(t-1\right)\sin\left(t-1\right)\mathrm{e}^{1-t} \tag{1}$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{Heaviside}\left(t-1\right)\sin\left(t-1\right)\mathrm{e}^{1-t}$$

Verified OK.

### 4.31.2 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 2y = 3Dirac(t-1)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1-\mathrm{I}, -1+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}\cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}\sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t}\cos(t) + c_2 \mathrm{e}^{-t}\sin(t) + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 3Dirac(t-1)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t)\,, y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ -\mathrm{e}^{-t}\cos(t) - \mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\sin(t) + \mathrm{e}^{-t}\cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t)\,, y_2(t)) = \mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

$y_p(t) = 3\left(\int Dirac(t-1)\, dt\right) (\sin(t)\cos(1) - \sin(1)\cos(t))\, \mathrm{e}^{1-t}$

- ○ Compute integrals

  $y_p(t) = 3Heaviside(t-1) (\sin(t)\cos(1) - \sin(1)\cos(t))\, \mathrm{e}^{1-t}$

- • Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) + 3Heaviside(t-1)(\sin(t)\cos(1) - \sin(1)\cos(t))\, \mathrm{e}^{1-t}$

- □ Check validity of solution $y = c_1\mathrm{e}^{-t}\cos(t) + c_2\mathrm{e}^{-t}\sin(t) + 3Heaviside(t-1)(\sin(t)\cos(1) - \mathrm{si}$

  - ○ Use initial condition $y(0) = 0$

    $0 = c_1$

  - ○ Compute derivative of the solution

    $y' = -c_1\mathrm{e}^{-t}\cos(t) - c_1\mathrm{e}^{-t}\sin(t) - c_2\mathrm{e}^{-t}\sin(t) + c_2\mathrm{e}^{-t}\cos(t) + 3Dirac(t-1)(\sin(t)\cos(1) - \mathrm{s}$

  - ○ Use the initial condition $y'\Big|_{\{t=0\}} = 0$

    $0 = -c_1 + c_2$

  - ○ Solve for $c_1$ and $c_2$

    $\{c_1 = 0, c_2 = 0\}$

  - ○ Substitute constant values into general solution and simplify

    $y = 3Heaviside(t-1)(\sin(t)\cos(1) - \sin(1)\cos(t))\, \mathrm{e}^{1-t}$

- • Solution to the IVP

  $y = 3Heaviside(t-1)(\sin(t)\cos(1) - \sin(1)\cos(t))\, \mathrm{e}^{1-t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.734 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=3*Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), sings
```

$$y(t) = 3\,\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1}\sin\left(t-1\right)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 24

```
DSolve[{y''[t]+2*y'[t]+2*y[t]==3*DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularS
```

$$y(t) \to -3e^{1-t}\theta(t-1)\sin(1-t)$$

## 4.32 problem Problem 5(c)



Internal problem ID [12340]
Internal file name [`OUTPUT/10992_Monday_October_02_2023_02_47_45_AM_90181001/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 5(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 29y = 5\delta(t-\pi) - 5\delta(-2\pi + t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 29 \\ F &= 5\delta(t-\pi) - 5\delta(-2\pi + t) \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 29y = 5\delta(t-\pi) - 5\delta(-2\pi + t)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 29$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5\delta(t-\pi) - 5\delta(-2\pi + t)$ is

$$\{\pi \leq t \leq 2\pi, 2\pi \leq t \leq \infty, -\infty \leq t \leq \pi\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 29Y(s) = 5\,\mathrm{e}^{-s\pi} - 5\,\mathrm{e}^{-2s\pi} \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4sY(s) + 29Y(s) = 5\,\mathrm{e}^{-s\pi} - 5\,\mathrm{e}^{-2s\pi}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{5\,\mathrm{e}^{-s\pi} - 5\,\mathrm{e}^{-2s\pi}}{s^2 + 4s + 29}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{5\,\mathrm{e}^{-s\pi} - 5\,\mathrm{e}^{-2s\pi}}{s^2 + 4s + 29}\right) \\
&= -\sin(5t)\left(\mathrm{Heaviside}(t-\pi)\,\mathrm{e}^{-2t+2\pi} + \mathrm{Heaviside}(-2\pi+t)\,\mathrm{e}^{4\pi-2t}\right)
\end{aligned}
$$

Hence the final solution is

$$
y = -\sin(5t)\left(\mathrm{Heaviside}(t-\pi)\,\mathrm{e}^{-2t+2\pi} + \mathrm{Heaviside}(-2\pi+t)\,\mathrm{e}^{4\pi-2t}\right)
$$

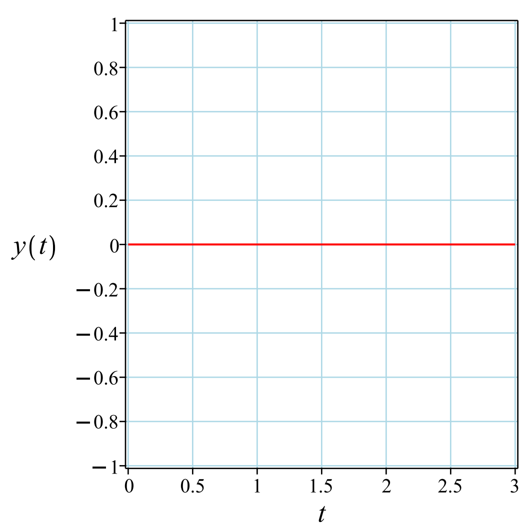
Simplifying the solution gives

$$
y = -\sin(5t)\left(\mathrm{Heaviside}(t-\pi)\,\mathrm{e}^{-2t+2\pi} + \mathrm{Heaviside}(-2\pi+t)\,\mathrm{e}^{4\pi-2t}\right)
$$

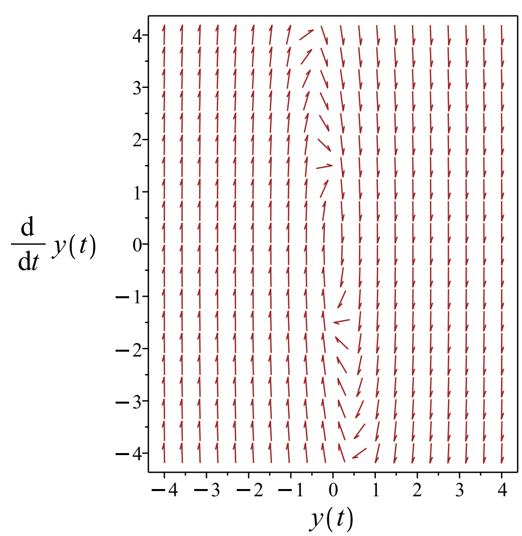
Summary

The solution(s) found are the following

$$
y = -\sin(5t)\left(\mathrm{Heaviside}(t-\pi)\,\mathrm{e}^{-2t+2\pi} + \mathrm{Heaviside}(-2\pi+t)\,\mathrm{e}^{4\pi-2t}\right) \tag{1}
$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$
y = -\sin(5t)\left(\mathrm{Heaviside}(t-\pi)\,\mathrm{e}^{-2t+2\pi} + \mathrm{Heaviside}(-2\pi+t)\,\mathrm{e}^{4\pi-2t}\right)
$$

Verified OK.

### 4.32.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 29y = 5Dirac(t-\pi) - 5Dirac(-2\pi + t), y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 29 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{(-4)\pm(\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 5\,\mathrm{I}, -2 + 5\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}\cos(5t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t}\sin(5t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{-2t}\cos(5t) + c_2\mathrm{e}^{-2t}\sin(5t) + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  ◦ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 5Dirac(t-\pi) - 5Dirac(-2\right.$$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t}\cos(5t) & \mathrm{e}^{-2t}\sin(5t) \\ -2\,\mathrm{e}^{-2t}\cos(5t) - 5\,\mathrm{e}^{-2t}\sin(5t) & -2\,\mathrm{e}^{-2t}\sin(5t) + 5\,\mathrm{e}^{-2t}\cos(5t) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(t), y_2(t)) = 5\,\mathrm{e}^{-4t}$

  ◦ Substitute functions into equation for $y_p(t)$

$$y_p(t) = \mathrm{e}^{-2t} \sin(5t) \left(\int \left(-\mathrm{e}^{4\pi} \mathit{Dirac}(-2\pi + t) - \mathrm{e}^{2\pi} \mathit{Dirac}(t - \pi)\right) dt\right)$$

- Compute integrals

$$y_p(t) = \mathrm{e}^{-2t} \sin(5t) \left(-\mathrm{e}^{2\pi} \mathit{Heaviside}(t - \pi) - \mathrm{e}^{4\pi} \mathit{Heaviside}(-2\pi + t)\right)$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-2t} \cos(5t) + c_2 \mathrm{e}^{-2t} \sin(5t) + \mathrm{e}^{-2t} \sin(5t) \left(-\mathrm{e}^{2\pi} \mathit{Heaviside}(t - \pi) - \mathrm{e}^{4\pi} \mathit{Heaviside}(-2\pi + t\right.$$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-2t} \cos(5t) + c_2 \mathrm{e}^{-2t} \sin(5t) + \mathrm{e}^{-2t} \sin(5t) \left(-\mathrm{e}^{2\pi} \mathit{Heaviside}(t -\right.$

- Use initial condition $y(0) = 0$

$$0 = c_1$$

- Compute derivative of the solution

$$y' = -2c_1 \mathrm{e}^{-2t} \cos(5t) - 5c_1 \mathrm{e}^{-2t} \sin(5t) - 2c_2 \mathrm{e}^{-2t} \sin(5t) + 5c_2 \mathrm{e}^{-2t} \cos(5t) - 2\,\mathrm{e}^{-2t} \sin(5t) \left(-\mathrm{e}^{2\pi}\right.$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

$$0 = -2c_1 + 5c_2$$

- Solve for $c_1$ and $c_2$

$$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$y = \mathrm{e}^{-2t} \sin(5t) \left(-\mathrm{e}^{2\pi} \mathit{Heaviside}(t - \pi) - \mathrm{e}^{4\pi} \mathit{Heaviside}(-2\pi + t)\right)$$

- Solution to the IVP

$$y = \mathrm{e}^{-2t} \sin(5t) \left(-\mathrm{e}^{2\pi} \mathit{Heaviside}(t - \pi) - \mathrm{e}^{4\pi} \mathit{Heaviside}(-2\pi + t)\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.234 (sec). Leaf size: 41

```
dsolve([diff(y(t),t$2)+4*diff(y(t),t)+29*y(t)=5*Dirac(t-Pi)-5*Dirac(t-2*Pi),y(0) = 0, D(y)(0
```

$$y(t) = -\sin(5t)\left(\mathrm{e}^{-2t+2\pi}\,\mathrm{Heaviside}(t-\pi) + \mathrm{Heaviside}(-2\pi+t)\,\mathrm{e}^{4\pi-2t}\right)$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 39

```
DSolve[{y''[t]+4*y'[t]+29*y[t]==5*DiracDelta[t-Pi]-5*DiracDelta[t-2*Pi],{y[0]==0,y'[0]==0}},
```

$$y(t) \to -e^{2\pi-2t}\left(e^{2\pi}\theta(t-2\pi) + \theta(t-\pi)\right)\sin(5t)$$

## 4.33 problem Problem 5(d)



Internal problem ID [12341]
Internal file name [`OUTPUT/10993_Monday_October_02_2023_02_47_46_AM_46263500/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 5(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = 1 - \delta(t - 1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= 2 \\ F &= 1 - \delta(t - 1) \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = 1 - \delta(t - 1)$$

The domain of $p(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 1 - \delta(t-1)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) + 2Y(s) = \frac{1}{s} - \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s} - \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{\mathrm{e}^{-s}s - 1}{s\,(s^2 + 3s + 2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{\mathrm{e}^{-s}s-1}{s\,(s^2+3s+2)}\right) \\
&= \left(\mathrm{e}^{-2t+2}-\mathrm{e}^{1-t}\right)\mathrm{Heaviside}\,(t-1)+\frac{\mathrm{e}^{-2t}}{2}-\mathrm{e}^{-t}+\frac{1}{2}
\end{aligned}
$$

Hence the final solution is

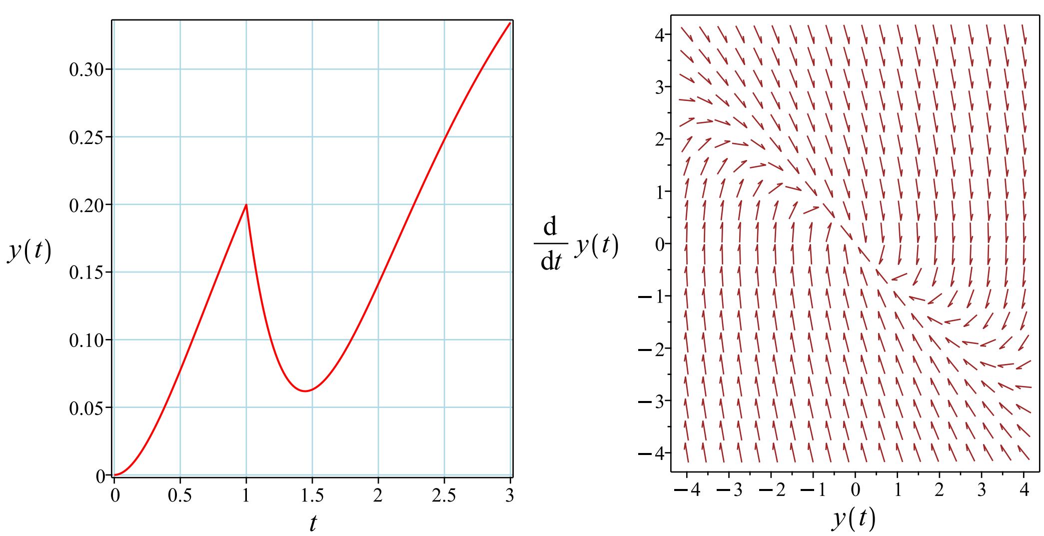
$$
y = \left(\mathrm{e}^{-2t+2}-\mathrm{e}^{1-t}\right)\mathrm{Heaviside}\,(t-1)+\frac{\mathrm{e}^{-2t}}{2}-\mathrm{e}^{-t}+\frac{1}{2}
$$

Simplifying the solution gives

$$
y = \mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}-\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{1-t}+\frac{\mathrm{e}^{-2t}}{2}-\mathrm{e}^{-t}+\frac{1}{2}
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2}-\mathrm{Heaviside}\,(t-1)\,\mathrm{e}^{1-t}+\frac{\mathrm{e}^{-2t}}{2}-\mathrm{e}^{-t}+\frac{1}{2} \tag{1}
$$




(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\left(t-1\right) \mathrm{e}^{-2t+2} - \text{Heaviside}\left(t-1\right) \mathrm{e}^{1-t} + \frac{\mathrm{e}^{-2t}}{2} - \mathrm{e}^{-t} + \frac{1}{2}$$

Verified OK.

### 4.33.2 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = 1 - \mathit{Dirac}(t-1)\,, y(0) = 0, y'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 1 - \mathit{Dirac}(t-1)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{-t} \\ -2\,e^{-2t} & -e^{-t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = e^{-3t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = -e^{-2t}\left(\int (-Dirac(t-1)\,e^2 + e^{2t})\,dt\right) - e^{-t}\left(\int (-1 + Dirac(t-1))\,e^t dt\right)$

- Compute integrals

  $y_p(t) = Heaviside(t-1)\,e^{-2t+2} + \frac{1}{2} - Heaviside(t-1)\,e^{1-t}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2t} + c_2 e^{-t} + Heaviside(t-1)\,e^{-2t+2} + \frac{1}{2} - Heaviside(t-1)\,e^{1-t}$

☐ Check validity of solution $y = c_1 e^{-2t} + c_2 e^{-t} + Heaviside(t-1)\,e^{-2t+2} + \frac{1}{2} - Heaviside(t-1)\,e$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{2}$

- Compute derivative of the solution

  $y' = -2c_1 e^{-2t} - c_2 e^{-t} + Dirac(t-1)\,e^{-2t+2} - 2Heaviside(t-1)\,e^{-2t+2} - Dirac(t-1)\,e^{1-t} + He$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = -2c_1 - c_2$

- Solve for $c_1$ and $c_2$

  $\left\{c_1 = \frac{1}{2}, c_2 = -1\right\}$

- Substitute constant values into general solution and simplify

  $y = Heaviside(t-1)\,e^{-2t+2} - Heaviside(t-1)\,e^{1-t} + \frac{e^{-2t}}{2} - e^{-t} + \frac{1}{2}$

- Solution to the IVP

  $y = Heaviside(t-1)\,e^{-2t+2} - Heaviside(t-1)\,e^{1-t} + \frac{e^{-2t}}{2} - e^{-t} + \frac{1}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 4.985 (sec). Leaf size: 38

```
dsolve([diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=1-Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), sings
```

$$y(t) = \text{Heaviside}\,(t-1)\,\mathrm{e}^{-2t+2} - \text{Heaviside}\,(t-1)\,\mathrm{e}^{-t+1} - \mathrm{e}^{-t} + \frac{\mathrm{e}^{-2t}}{2} + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 36

```
DSolve[{y''[t]+3*y'[t]+2*y[t]==1-DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularS
```

$$y(t) \to \frac{1}{2}e^{-2t}\left(\left(e^t-1\right)^2 - 2e\left(e^t-e\right)\theta(t-1)\right)$$

## 4.34 problem Problem 5(e)



Internal problem ID [12342]
Internal file name [`OUTPUT/10994_Monday_October_02_2023_02_47_46_AM_73489955/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 5(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'' + 4y' + y = e^{-\frac{t}{2}}\delta(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= 1 \\ q(t) &= \frac{1}{4} \\ F &= \frac{e^{-\frac{1}{2}}\delta(t-1)}{4} \end{aligned}$$

Hence the ode is

$$y'' + y' + \frac{y}{4} = \frac{e^{-\frac{1}{2}}\delta(t-1)}{4}$$

The domain of $p(t) = 1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{1}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \frac{e^{-\frac{1}{2}}\delta(t-1)}{4}$ is

$$\{t < 1 \lor 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)
\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$4s^2Y(s) - 4y'(0) - 4sy(0) + 4sY(s) - 4y(0) + Y(s) = e^{-\frac{1}{2}-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$4s^2Y(s) + 4sY(s) + Y(s) = e^{-\frac{1}{2}-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{e^{-\frac{1}{2}-s}}{4s^2 + 4s + 1}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-\frac{1}{2}-s}}{4s^2+4s+1}\right) \\
&= \frac{\text{Heaviside}\,(t-1)\,(t-1)\,e^{-\frac{t}{2}}}{4}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\text{Heaviside}\,(t-1)\,(t-1)\,e^{-\frac{t}{2}}}{4}
$$

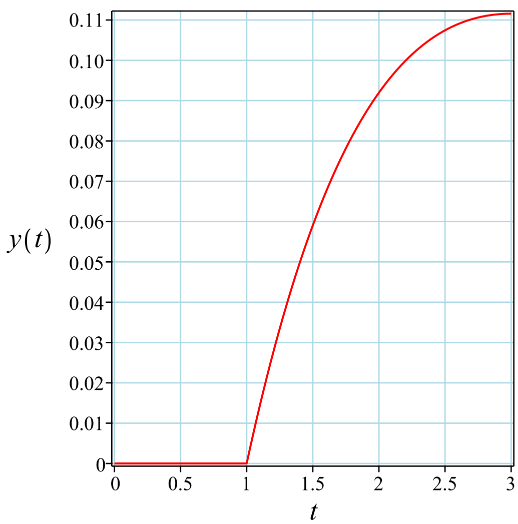
Simplifying the solution gives

$$
y = \frac{\text{Heaviside}\,(t-1)\,(t-1)\,e^{-\frac{t}{2}}}{4}
$$

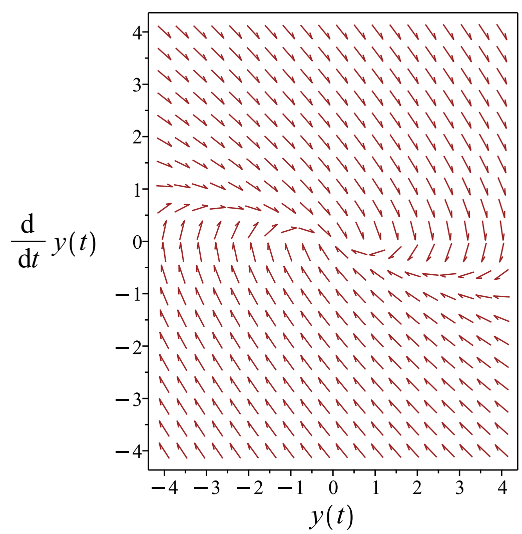
Summary

The solution(s) found are the following

$$
y = \frac{\text{Heaviside}\,(t-1)\,(t-1)\,e^{-\frac{t}{2}}}{4} \tag{1}
$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}(t-1)(t-1)\,\mathrm{e}^{-\frac{t}{2}}}{4}$$

Verified OK.

### 4.34.2 Maple step by step solution

Let's solve

$\left[4y''+4y'+y=\mathrm{e}^{-\frac{1}{2}}\mathit{Dirac}(t-1), y(0)=0, y'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  $y''$
- Isolate 2nd derivative

  $y''=-y'-\frac{y}{4}+\frac{\mathrm{e}^{-\frac{1}{2}}\mathit{Dirac}(t-1)}{4}$
- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y'+\frac{y}{4}=\frac{\mathrm{e}^{-\frac{1}{2}}\mathit{Dirac}(t-1)}{4}$
- Characteristic polynomial of homogeneous ODE

  $r^2+r+\frac{1}{4}=0$
- Factor the characteristic polynomial

  $\frac{(2r+1)^2}{4}=0$
- Root of the characteristic polynomial

  $r=-\frac{1}{2}$
- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^{-\frac{t}{2}}$
- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t)=\mathrm{e}^{-\frac{t}{2}}t$
- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-\frac{t}{2}}+c_2\mathrm{e}^{-\frac{t}{2}}t+y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $$\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \frac{e^{-\frac{1}{2}} Dirac(t-1)}{4}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} & e^{-\frac{t}{2}} t \\ -\frac{e^{-\frac{t}{2}}}{2} & -\frac{e^{-\frac{t}{2}} t}{2} + e^{-\frac{t}{2}} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(t), y_2(t)) = e^{-t}$$

- Substitute functions into equation for $y_p(t)$

  $$y_p(t) = \frac{e^{-\frac{t}{2}} \left(\int Dirac(t-1) dt\right)(t-1)}{4}$$

- Compute integrals

  $$y_p(t) = \frac{Heaviside(t-1)(t-1) e^{-\frac{t}{2}}}{4}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-\frac{t}{2}} + c_2 e^{-\frac{t}{2}} t + \frac{Heaviside(t-1)(t-1) e^{-\frac{t}{2}}}{4}$$

□ Check validity of solution $y = c_1 e^{-\frac{t}{2}} + c_2 e^{-\frac{t}{2}} t + \frac{Heaviside(t-1)(t-1) e^{-\frac{t}{2}}}{4}$

- Use initial condition $y(0) = 0$

  $$0 = c_1$$

- Compute derivative of the solution

  $$y' = -\frac{c_1 e^{-\frac{t}{2}}}{2} - \frac{c_2 e^{-\frac{t}{2}} t}{2} + c_2 e^{-\frac{t}{2}} + \frac{Dirac(t-1)(t-1) e^{-\frac{t}{2}}}{4} + \frac{Heaviside(t-1) e^{-\frac{t}{2}}}{4} - \frac{Heaviside(t-1)(t-1) e^{-\frac{t}{2}}}{8}$$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $$0 = -\frac{c_1}{2} + c_2$$

- Solve for $c_1$ and $c_2$

  $$\{c_1 = 0, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

  $$y = \frac{Heaviside(t-1)(t-1) e^{-\frac{t}{2}}}{4}$$

- Solution to the IVP

  $$y = \frac{Heaviside(t-1)(t-1) e^{-\frac{t}{2}}}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.875 (sec). Leaf size: 17

```
dsolve([4*diff(y(t),t$2)+4*diff(y(t),t)+y(t)=exp(-t/2)*Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t
```

$$y(t) = \frac{\text{Heaviside}\,(t-1)\,(t-1)\,\mathrm{e}^{-\frac{t}{2}}}{4}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 23

```
DSolve[{4*y''[t]+4*y'[t]+y[t]==Exp[-t/2]*DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeS
```

$$y(t) \to \frac{1}{4} e^{-t/2}(t-1)\theta(t-1)$$

## 4.35 problem Problem 5(f)



Internal problem ID [12343]
Internal file name [`OUTPUT/10995_Monday_October_02_2023_02_47_46_AM_81202127/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 5(f).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 7y' + 6y = \delta(t - 1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

### 4.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(t)y' + q(t)y = F$$

Where here

$$\begin{aligned} p(t) &= -7 \\ q(t) &= 6 \\ F &= \delta(t - 1) \end{aligned}$$

Hence the ode is

$$y'' - 7y' + 6y = \delta(t - 1)$$

The domain of $p(t) = -7$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \delta(t-1)$ is

$$\{t < 1 \vee 1 < t\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 7sY(s) + 7y(0) + 6Y(s) = \mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 7sY(s) + 6Y(s) = \mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s}}{s^2 - 7s + 6}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2 - 7s + 6}\right) \\
&= \frac{\text{Heaviside}\,(t-1)\left(e^{6t-6} - e^{t-1}\right)}{5}
\end{aligned}
$$

Hence the final solution is

$$
y = \frac{\text{Heaviside}\,(t-1)\left(e^{6t-6} - e^{t-1}\right)}{5}
$$

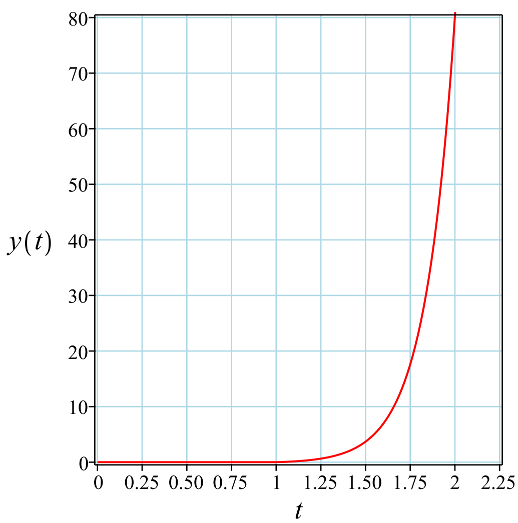
Simplifying the solution gives

$$
y = \frac{\text{Heaviside}\,(t-1)\left(e^{6t-6} - e^{t-1}\right)}{5}
$$

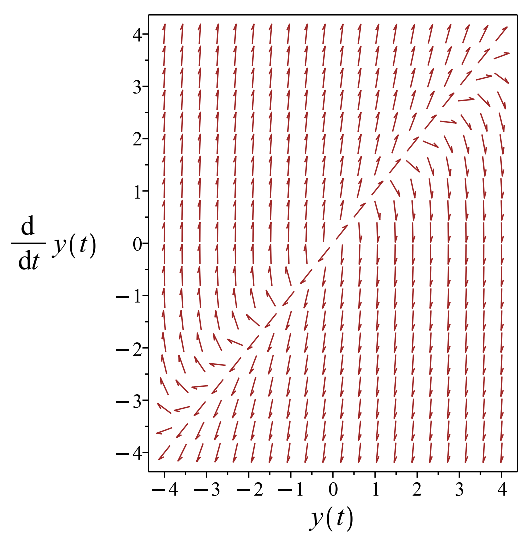
Summary

The solution(s) found are the following

$$
y = \frac{\text{Heaviside}\,(t-1)\left(e^{6t-6} - e^{t-1}\right)}{5} \tag{1}
$$




(a) Solution plot




(b) Slope field plot

Verification of solutions

$$y = \frac{\text{Heaviside}\,(t-1)\left(\mathrm{e}^{6t-6}-\mathrm{e}^{t-1}\right)}{5}$$

Verified OK.

### 4.35.2 Maple step by step solution

Let's solve

$$\left[y''-7y'+6y = \mathit{Dirac}(t-1)\,, y(0)=0, y'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2-7r+6=0$

- Factor the characteristic polynomial

  $(r-1)(r-6)=0$

- Roots of the characteristic polynomial

  $r=(1,6)$

- 1st solution of the homogeneous ODE

  $y_1(t)=\mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $y_2(t)=\mathrm{e}^{6t}$

- General solution of the ODE

  $y=c_1y_1(t)+c_2y_2(t)+y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^t+c_2\mathrm{e}^{6t}+y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t)=-y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right)+y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t)=\mathit{Dirac}(t-1)\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(t), y_2(t)) = \begin{bmatrix} e^t & e^{6t} \\ e^t & 6\,e^{6t} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(t), y_2(t)) = 5\,e^{7t}$

- Substitute functions into equation for $y_p(t)$

  $y_p(t) = \frac{\left(\int Dirac(t-1)dt\right)\left(e^{6t-6}-e^{t-1}\right)}{5}$

- Compute integrals

  $y_p(t) = \frac{Heaviside(t-1)\left(e^{6t-6}-e^{t-1}\right)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^t + c_2 e^{6t} + \frac{Heaviside(t-1)\left(e^{6t-6}-e^{t-1}\right)}{5}$

□ Check validity of solution $y = c_1 e^t + c_2 e^{6t} + \frac{Heaviside(t-1)\left(e^{6t-6}-e^{t-1}\right)}{5}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- Compute derivative of the solution

  $y' = c_1 e^t + 6c_2 e^{6t} + \frac{Dirac(t-1)\left(e^{6t-6}-e^{t-1}\right)}{5} + \frac{Heaviside(t-1)\left(6\,e^{6t-6}-e^{t-1}\right)}{5}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = c_1 + 6c_2$

- Solve for $c_1$ and $c_2$

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = \frac{Heaviside(t-1)\left(e^{6t-6}-e^{t-1}\right)}{5}$

- Solution to the IVP

  $y = \frac{Heaviside(t-1)\left(e^{6t-6}-e^{t-1}\right)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.781 (sec). Leaf size: 23

```
dsolve([diff(y(t),t$2)-7*diff(y(t),t)+6*y(t)=Dirac(t-1),y(0) = 0, D(y)(0) = 0],y(t), singsol
```

$$y(t) = \frac{\text{Heaviside}\,(t-1)\,(\mathrm{e}^{-6+6t} - \mathrm{e}^{t-1})}{5}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 29

```
DSolve[{y''[t]-7*y'[t]+6*y[t]==DiracDelta[t-1],{y[0]==0,y'[0]==0}},y[t],t,IncludeSingularSol
```

$$y(t) \to \frac{1}{5}e^{t-6}\left(e^{5t} - e^5\right)\theta(t-1)$$

## 4.36 problem Problem 6(a)



Internal problem ID [12344]
Internal file name [`OUTPUT/10996_Monday_October_02_2023_02_47_46_AM_33879010/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 6(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{10Q' + 100Q = \text{Heaviside}\,(t-1) - \text{Heaviside}\,(-2+t)}$$

With initial conditions

$$[Q(0) = 0]$$

### 4.36.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$Q' + p(t)Q = q(t)$$

Where here

$$\begin{aligned} p(t) &= 10 \\ q(t) &= \frac{\text{Heaviside}\,(t-1)}{10} - \frac{\text{Heaviside}\,(-2+t)}{10} \end{aligned}$$

Hence the ode is

$$Q' + 10Q = \frac{\text{Heaviside}(t-1)}{10} - \frac{\text{Heaviside}(-2+t)}{10}$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{\text{Heaviside}(t-1)}{10} - \frac{\text{Heaviside}(-2+t)}{10}$ is

$$\{1 \leq t \leq 2, 2 \leq t \leq \infty, -\infty \leq t \leq 1\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

### 4.36.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(Q) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(Q') = sY(s) - Q(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$10sY(s) - 10Q(0) + 100Y(s) = \frac{e^{-s} - e^{-2s}}{s} \tag{1}$$

Replacing initial condition gives

$$10sY(s) + 100Y(s) = \frac{e^{-s} - e^{-2s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-s} - e^{-2s}}{10s(s+10)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} Q &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{e^{-s} - e^{-2s}}{10s(s+10)}\right) \\ &= \frac{\text{Heaviside}(t-1)(1 - e^{-10t+10})}{100} - \frac{\text{Heaviside}(-2+t)(1 - e^{-10t+20})}{100} \end{aligned}$$

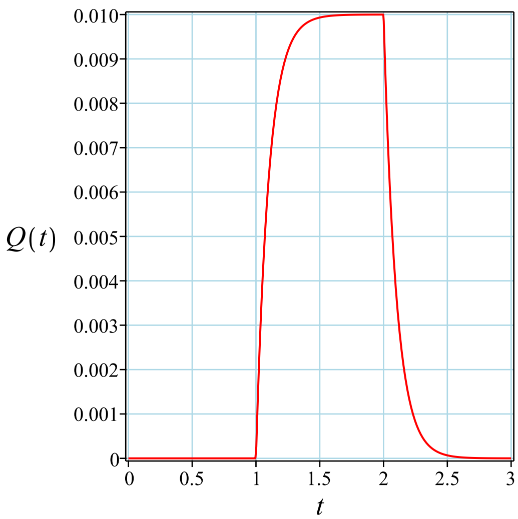
Hence the final solution is

$$Q = \frac{\text{Heaviside}\left(t-1\right)\left(1-\mathrm{e}^{-10t+10}\right)}{100}-\frac{\text{Heaviside}\left(-2+t\right)\left(1-\mathrm{e}^{-10t+20}\right)}{100}$$

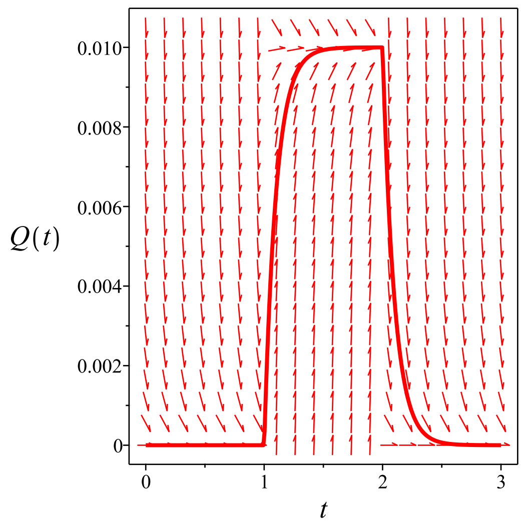
<u>Summary</u>

The solution(s) found are the following

$$Q = \frac{\text{Heaviside}\left(t-1\right)\left(1-\mathrm{e}^{-10t+10}\right)}{100}-\frac{\text{Heaviside}\left(-2+t\right)\left(1-\mathrm{e}^{-10t+20}\right)}{100} \tag{1}$$




(a) Solution plot




(b) Slope field plot

<u>Verification of solutions</u>

$$Q = \frac{\text{Heaviside}\left(t-1\right)\left(1-\mathrm{e}^{-10t+10}\right)}{100}-\frac{\text{Heaviside}\left(-2+t\right)\left(1-\mathrm{e}^{-10t+20}\right)}{100}$$

Verified OK.

### 4.36.3 Maple step by step solution

Let's solve

$[10Q'+100Q = \mathit{Heaviside}(t-1)-\mathit{Heaviside}(-2+t), Q(0)=0]$

- Highest derivative means the order of the ODE is 1

  $Q'$

- Isolate the derivative

  $Q' = -10Q + \frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}$

- Group terms with $Q$ on the lhs of the ODE and the rest on the rhs of the ODE

  $Q' + 10Q = \frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(Q' + 10Q\right) = \mu(t)\left(\frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, Q)$

  $\mu(t)\left(Q' + 10Q\right) = \mu'(t)\, Q + \mu(t)\, Q'$

- Isolate $\mu'(t)$

  $\mu'(t) = 10\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{10t}$

- Integrate both sides with respect to $t$

  $\int \left(\frac{d}{dt}(\mu(t)\, Q)\right) dt = \int \mu(t)\left(\frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}\right) dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, Q = \int \mu(t)\left(\frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}\right) dt + c_1$

- Solve for $Q$

  $Q = \frac{\int \mu(t)\left(\frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}\right) dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{10t}$

  $Q = \frac{\int \mathrm{e}^{10t}\left(\frac{\mathit{Heaviside}(t-1)}{10} - \frac{\mathit{Heaviside}(-2+t)}{10}\right) dt + c_1}{\mathrm{e}^{10t}}$

- Evaluate the integrals on the rhs

  $Q = \frac{\frac{\mathrm{e}^{10t}\mathit{Heaviside}(t-1)}{100} - \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{10}}{100} - \frac{\mathrm{e}^{10t}\mathit{Heaviside}(-2+t)}{100} + \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{20}}{100} + c_1}{\mathrm{e}^{10t}}$

- Simplify

  $Q = \frac{\mathit{Heaviside}(-2+t)\mathrm{e}^{-10t+20}}{100} - \frac{\mathit{Heaviside}(t-1)\mathrm{e}^{-10t+10}}{100} + \frac{\mathit{Heaviside}(t-1)}{100} - \frac{\mathit{Heaviside}(-2+t)}{100} + \mathrm{e}^{-10t}c_1$

- Use initial condition $Q(0) = 0$

  $0 = c_1$

- Solve for $c_1$

$c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

$$Q = \frac{Heaviside(-2+t)\mathrm{e}^{-10t+20}}{100} - \frac{Heaviside(t-1)\mathrm{e}^{-10t+10}}{100} + \frac{Heaviside(t-1)}{100} - \frac{Heaviside(-2+t)}{100}$$

- Solution to the IVP

$$Q = \frac{Heaviside(-2+t)\mathrm{e}^{-10t+20}}{100} - \frac{Heaviside(t-1)\mathrm{e}^{-10t+10}}{100} + \frac{Heaviside(t-1)}{100} - \frac{Heaviside(-2+t)}{100}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.094 (sec). Leaf size: 37

```
dsolve([10*diff(Q(t),t)+100*Q(t)=Heaviside(t-1)-Heaviside(t-2),Q(0) = 0],Q(t), singsol=all)
```

$$Q(t) = \frac{\mathrm{Heaviside}(t-2)\,\mathrm{e}^{-10t+20}}{100} - \frac{\mathrm{Heaviside}(t-2)}{100} - \frac{\mathrm{Heaviside}(t-1)\,\mathrm{e}^{-10t+10}}{100} + \frac{\mathrm{Heaviside}(t-1)}{100}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 50

```
DSolve[{10*q'[t]+100*q[t]==UnitStep[t-1]-UnitStep[t-2],{q[0]==0}},q[t],t,IncludeSingularSolu
```

$$q(t) \to \begin{cases} \frac{1}{100}e^{10-10t}(-1+e^{10}) & t > 2 \\ \frac{1}{100}(1-e^{10-10t}) & 1 < t \leq 2 \end{cases}$$

## 4.37 problem Problem 13(a)



Internal problem ID [12345]
Internal file name [`OUTPUT/10997_Monday_October_02_2023_02_47_46_AM_86939990/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 13(a).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' + 4y' + 4y = 8}$$

With initial conditions

$$[y(0) = 4, y'(0) = -3, y''(0) = -3]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + s^2Y(s) - y'(0) - sy(0) + 4sY(s) - 4y(0) + 4Y(s) = \frac{8}{s} \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 4 \\
y'(0) &= -3 \\
y''(0) &= -3
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 10 - s - 4s^2 + s^2Y(s) + 4sY(s) + 4Y(s) = \frac{8}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{4s^3 + s^2 + 10s + 8}{s\left(s^3 + s^2 + 4s + 4\right)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{1}{2} + \frac{i}{2}}{s - 2i} + \frac{\frac{1}{2} - \frac{i}{2}}{s + 2i} + \frac{1}{s + 1} + \frac{2}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2} + \frac{i}{2}}{s - 2i}\right) = \left(\frac{1}{2} + \frac{i}{2}\right) \mathrm{e}^{2it}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{2} - \frac{i}{2}}{s + 2i}\right) = \left(\frac{1}{2} - \frac{i}{2}\right) \mathrm{e}^{-2it}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s + 1}\right) = \mathrm{e}^{-t}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s}\right) = 2$$

Adding the above results and simplifying gives

$$y = \cos(2t) - \sin(2t) + \mathrm{e}^{-t} + 2$$

Summary
The solution(s) found are the following

$$y = \cos(2t) - \sin(2t) + \mathrm{e}^{-t} + 2 \tag{1}$$

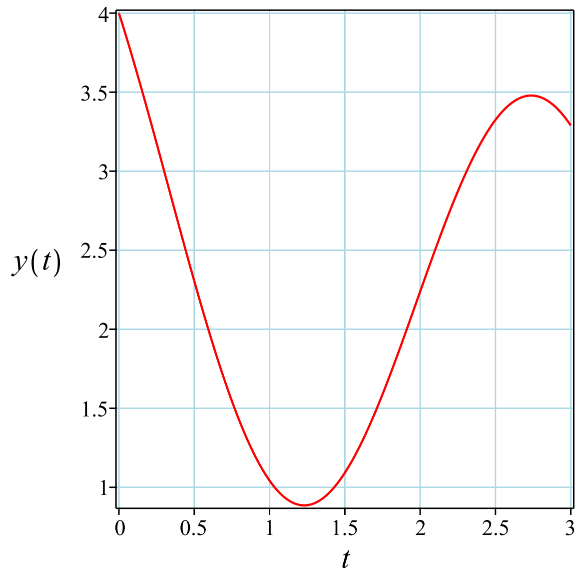


Figure 97: Solution plot

Verification of solutions

$$y = \cos(2t) - \sin(2t) + e^{-t} + 2$$

Verified OK.

### 4.37.1 Maple step by step solution

Let's solve

$$\left[y''' + y'' + 4y' + 4y = 8, y(0) = 4, y'\Big|_{\{t=0\}} = -3, y''\Big|_{\{t=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(t)$

    $y_1(t) = y$

  ◦ Define new variable $y_2(t)$

    $y_2(t) = y'$

  ◦ Define new variable $y_3(t)$

    $y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = 8 - y_3(t) - 4y_2(t) - 4y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = 8 - y_3(t) - 4y_2(t) - 4y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -1 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}t} \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(2t\right) - \mathrm{I}\sin\left(2t\right)\right) \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos(2t)}{4} + \frac{\mathrm{I}\sin(2t)}{4} \\ \frac{\mathrm{I}}{2}\left(\cos\left(2t\right) - \mathrm{I}\sin\left(2t\right)\right) \\ \cos\left(2t\right) - \mathrm{I}\sin\left(2t\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = \left[\begin{array}{c} -\frac{\cos(2t)}{4} \\ \frac{\sin(2t)}{2} \\ \cos\left(2t\right) \end{array}\right], \vec{y}_3(t) = \left[\begin{array}{c} \frac{\sin(2t)}{4} \\ \frac{\cos(2t)}{2} \\ -\sin\left(2t\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(t) = c_1 \vec{y}_1 + c_2 \vec{y}_2(t) + c_3 \vec{y}_3(t) + \vec{y}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^{-t} & -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} \\ -e^{-t} & \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} \\ e^{-t} & \cos(2t) & -\sin(2t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-t} & -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} \\ -e^{-t} & \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} \\ e^{-t} & \cos(2t) & -\sin(2t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ -1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{4e^{-t}}{5} + \frac{\cos(2t)}{5} + \frac{2\sin(2t)}{5} & \frac{\sin(2t)}{2} & \frac{e^{-t}}{5} - \frac{\cos(2t)}{5} + \frac{\sin(2t)}{10} \\ -\frac{4e^{-t}}{5} - \frac{2\sin(2t)}{5} + \frac{4\cos(2t)}{5} & \cos(2t) & -\frac{e^{-t}}{5} + \frac{2\sin(2t)}{5} + \frac{\cos(2t)}{5} \\ \frac{4e^{-t}}{5} - \frac{4\cos(2t)}{5} - \frac{8\sin(2t)}{5} & -2\sin(2t) & \frac{e^{-t}}{5} + \frac{4\cos(2t)}{5} - \frac{2\sin(2t)}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{y}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{y}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(t) = \begin{bmatrix} \frac{8}{5} - \frac{8\cos(t)\sin(t)}{5} + \frac{4\sin(t)^2}{5} - \frac{8\,\mathrm{e}^{-t}}{5} \\ -\frac{8}{5} + \frac{8\cos(t)\sin(t)}{5} + \frac{16\sin(t)^2}{5} + \frac{8\,\mathrm{e}^{-t}}{5} \\ \frac{8}{5} + \frac{32\cos(t)\sin(t)}{5} - \frac{16\sin(t)^2}{5} - \frac{8\,\mathrm{e}^{-t}}{5} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + \begin{bmatrix} \frac{8}{5} - \frac{8\cos(t)\sin(t)}{5} + \frac{4\sin(t)^2}{5} - \frac{8\,\mathrm{e}^{-t}}{5} \\ -\frac{8}{5} + \frac{8\cos(t)\sin(t)}{5} + \frac{16\sin(t)^2}{5} + \frac{8\,\mathrm{e}^{-t}}{5} \\ \frac{8}{5} + \frac{32\cos(t)\sin(t)}{5} - \frac{16\sin(t)^2}{5} - \frac{8\,\mathrm{e}^{-t}}{5} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-8-5c_2)\cos(2t)}{20} + \frac{(20c_1-32)\mathrm{e}^{-t}}{20} + 2 + \frac{(5c_3-16)\sin(2t)}{20}$

- Use the initial condition $y(0) = 4$

  $4 = -\frac{c_2}{4} + c_1$

- Calculate the 1st derivative of the solution

  $y' = -\frac{(-8-5c_2)\sin(2t)}{10} - \frac{(20c_1-32)\mathrm{e}^{-t}}{20} + \frac{(5c_3-16)\cos(2t)}{10}$

- Use the initial condition $y'\Big|_{\{t=0\}} = -3$

  $-3 = -c_1 + \frac{c_3}{2}$

- Calculate the 2nd derivative of the solution

  $y'' = -\frac{(-8-5c_2)\cos(2t)}{5} + \frac{(20c_1-32)\mathrm{e}^{-t}}{20} - \frac{(5c_3-16)\sin(2t)}{5}$

- Use the initial condition $y''\Big|_{\{t=0\}} = -3$

  $-3 = c_1 + c_2$
- Solve for the unknown coefficients

  $\left\{c_1 = \frac{13}{5}, c_2 = -\frac{28}{5}, c_3 = -\frac{4}{5}\right\}$
- Solution to the IVP

  $y = \cos(2t) - \sin(2t) + e^{-t} + 2$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.031 (sec). Leaf size: 20

```
dsolve([diff(y(t),t$3)+diff(y(t),t$2)+4*diff(y(t),t)+4*y(t)=8,y(0) = 4, D(y)(0) = -3, (D@@2)
```

$$y(t) = \cos(2t) - \sin(2t) + e^{-t} + 2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 22

```
DSolve[{y'''[t]+y''[t]+4*y'[t]+4*y[t]==8,{y[0]==4,y'[0]==-3,y''[0]==-3}},y[t],t,IncludeSingu
```

$$y(t) \to e^{-t} - \sin(2t) + \cos(2t) + 2$$

## 4.38 problem Problem 13(b)



Internal problem ID [12346]
Internal file name [`OUTPUT/10998_Monday_October_02_2023_02_47_47_AM_18318295/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 13(b).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

$$\boxed{y''' - 2y'' - y' + 2y = 4t}$$

With initial conditions

$$[y(0) = 2, y'(0) = -2, y''(0) = 4]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) - 2s^2Y(s) + 2y'(0) + 2sy(0) - sY(s) + y(0) + 2Y(s) = \frac{4}{s^2} \quad (1)$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 2 \\
y'(0) &= -2 \\
y''(0) &= 4
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 6 + 6s - 2s^2 - 2s^2Y(s) - sY(s) + 2Y(s) = \frac{4}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^4 - 6s^3 + 6s^2 + 4}{s^2 (s^3 - 2s^2 - s + 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-2} + \frac{2}{s^2} - \frac{3}{s-1} + \frac{3}{s+1} + \frac{1}{s}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) &= e^{2t} \\
\mathcal{L}^{-1}\left(\frac{2}{s^2}\right) &= 2t \\
\mathcal{L}^{-1}\left(-\frac{3}{s-1}\right) &= -3\,e^{t} \\
\mathcal{L}^{-1}\left(\frac{3}{s+1}\right) &= 3\,e^{-t} \\
\mathcal{L}^{-1}\left(\frac{1}{s}\right) &= 1
\end{aligned}$$

Adding the above results and simplifying gives

$$y = e^{2t} - 6\sinh(t) + 2t + 1$$

Summary

The solution(s) found are the following

$$y = e^{2t} - 6\sinh(t) + 2t + 1 \tag{1}$$

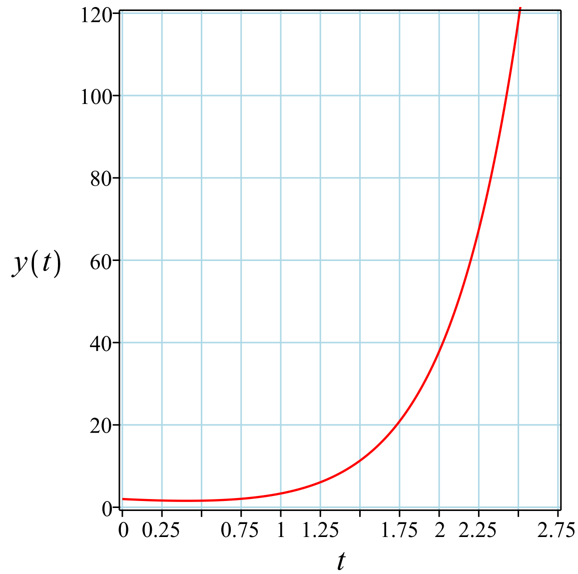



Figure 98: Solution plot

<u>Verification of solutions</u>

$$y = e^{2t} - 6\sinh(t) + 2t + 1$$

Verified OK.

### 4.38.1 Maple step by step solution

Let's solve

$$\left[y''' - 2y'' - y' + 2y = 4t, y(0) = 2, y'\Big|_{\{t=0\}} = -2, y''\Big|_{\{t=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(t)$

    $y_1(t) = y$

  ◦ Define new variable $y_2(t)$

    $y_2(t) = y'$

  ◦ Define new variable $y_3(t)$

    $y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = 4t + 2y_3(t) + y_2(t) - 2y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = 4t + 2y_3(t) + y_2(t) - 2y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 4t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 4t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(t)$$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\begin{bmatrix} e^{-t} & e^{t} & \frac{e^{2t}}{4} \\ -e^{-t} & e^{t} & \frac{e^{2t}}{2} \\ e^{-t} & e^{t} & e^{2t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where $I$ is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t)=\begin{bmatrix} e^{-t} & e^{t} & \frac{e^{2t}}{4} \\ -e^{-t} & e^{t} & \frac{e^{2t}}{2} \\ e^{-t} & e^{t} & e^{2t} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t)=\begin{bmatrix} \frac{e^{-t}}{3}+e^{t}-\frac{e^{2t}}{3} & \frac{e^{t}}{2}-\frac{e^{-t}}{2} & \frac{e^{-t}}{6}-\frac{e^{t}}{2}+\frac{e^{2t}}{3} \\ -\frac{e^{-t}}{3}+e^{t}-\frac{2e^{2t}}{3} & \frac{e^{-t}}{2}+\frac{e^{t}}{2} & -\frac{e^{-t}}{6}-\frac{e^{t}}{2}+\frac{2e^{2t}}{3} \\ \frac{e^{-t}}{3}+e^{t}-\frac{4e^{2t}}{3} & \frac{e^{t}}{2}-\frac{e^{-t}}{2} & \frac{e^{-t}}{6}-\frac{e^{t}}{2}+\frac{4e^{2t}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{y}_p(t)=\Phi(t)\cdot\vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{y}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(t) = \begin{bmatrix} 2t+1+\frac{2\,\mathrm{e}^{-t}}{3} - 2\,\mathrm{e}^{t} + \frac{\mathrm{e}^{2t}}{3} \\ 2 - \frac{2\,\mathrm{e}^{-t}}{3} - 2\,\mathrm{e}^{t} + \frac{2\,\mathrm{e}^{2t}}{3} \\ \frac{2\,\mathrm{e}^{-t}}{3} - 2\,\mathrm{e}^{t} + \frac{4\,\mathrm{e}^{2t}}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} 2t+1+\frac{2\,\mathrm{e}^{-t}}{3} - 2\,\mathrm{e}^{t} + \frac{\mathrm{e}^{2t}}{3} \\ 2 - \frac{2\,\mathrm{e}^{-t}}{3} - 2\,\mathrm{e}^{t} + \frac{2\,\mathrm{e}^{2t}}{3} \\ \frac{2\,\mathrm{e}^{-t}}{3} - 2\,\mathrm{e}^{t} + \frac{4\,\mathrm{e}^{2t}}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = 1 + \frac{(2+3c_1)\mathrm{e}^{-t}}{3} + \frac{(3c_3+4)\mathrm{e}^{2t}}{12} + (c_2-2)\,\mathrm{e}^{t} + 2t$
- Use the initial condition $y(0) = 2$

  $2 = c_1 + \frac{c_3}{4} + c_2$
- Calculate the 1st derivative of the solution

  $y' = -\frac{(2+3c_1)\mathrm{e}^{-t}}{3} + \frac{(3c_3+4)\mathrm{e}^{2t}}{6} + (c_2-2)\,\mathrm{e}^{t} + 2$
- Use the initial condition $y'\Big|_{\{t=0\}} = -2$

  $-2 = -c_1 + \frac{c_3}{2} + c_2$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{(2+3c_1)\mathrm{e}^{-t}}{3} + \frac{(3c_3+4)\mathrm{e}^{2t}}{3} + (c_2-2)\,\mathrm{e}^{t}$
- Use the initial condition $y''\Big|_{\{t=0\}} = 4$

  $4 = c_1 + c_2 + c_3$
- Solve for the unknown coefficients

  $\left\{c_1 = \frac{7}{3}, c_2 = -1, c_3 = \frac{8}{3}\right\}$
- Solution to the IVP

$$y = 1 + 3\,\mathrm{e}^{-t} + \mathrm{e}^{2t} - 3\,\mathrm{e}^{t} + 2t$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 3.86 (sec). Leaf size: 17

```
dsolve([diff(y(t),t$3)-2*diff(y(t),t$2)-diff(y(t),t)+2*y(t)=4*t,y(0) = 2, D(y)(0) = -2, (D@@
```

$$y(t) = -6\sinh\left(t\right) + 2t + \mathrm{e}^{2t} + 1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 27

```
DSolve[{y'''[t]-2*y''[t]-y'[t]+2*y[t]==4*t,{y[0]==2,y'[0]==-2,y''[0]==4}},y[t],t,IncludeSing
```

$$y(t) \to 2t + 3e^{-t} - 3e^{t} + e^{2t} + 1$$

## 4.39 problem Problem 13(c)



Internal problem ID [12347]
Internal file name [`OUTPUT/10999_Monday_October_02_2023_02_47_47_AM_5883988/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 13(c).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' + 4y' - 4y = 8\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{t}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0, y''(0) = 3]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) - s^2Y(s) + y'(0) + sy(0) + 4sY(s) - 4y(0) - 4Y(s) = \frac{3s+2}{(s-2)(s-1)} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 2 \\
y'(0) &= 0 \\
y''(0) &= 3
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^3 Y(s) - 11 - 2s^2 - s^2 Y(s) + 2s + 4sY(s) - 4Y(s) = \frac{3s+2}{(s-2)(s-1)}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = \frac{2s^4 - 8s^3 + 21s^2 - 34s + 24}{(s-2)(s-1)(s^3 - s^2 + 4s - 4)}
$$

Applying partial fractions decomposition results in

$$
Y(s) = \frac{i}{2s-4i} - \frac{i}{2(s+2i)} + \frac{1}{s-2} - \frac{1}{(s-1)^2} + \frac{1}{s-1}
$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{i}{2s-4i}\right) &= \frac{i\mathrm{e}^{2it}}{2} \\
\mathcal{L}^{-1}\left(-\frac{i}{2(s+2i)}\right) &= -\frac{i\mathrm{e}^{-2it}}{2} \\
\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) &= \mathrm{e}^{2t} \\
\mathcal{L}^{-1}\left(-\frac{1}{(s-1)^2}\right) &= -t\,\mathrm{e}^{t} \\
\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) &= \mathrm{e}^{t}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = \mathrm{e}^{2t} - \sin(2t) - \mathrm{e}^{t}(t-1)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{2t} - \sin(2t) - \mathrm{e}^{t}(t-1) \tag{1}
$$

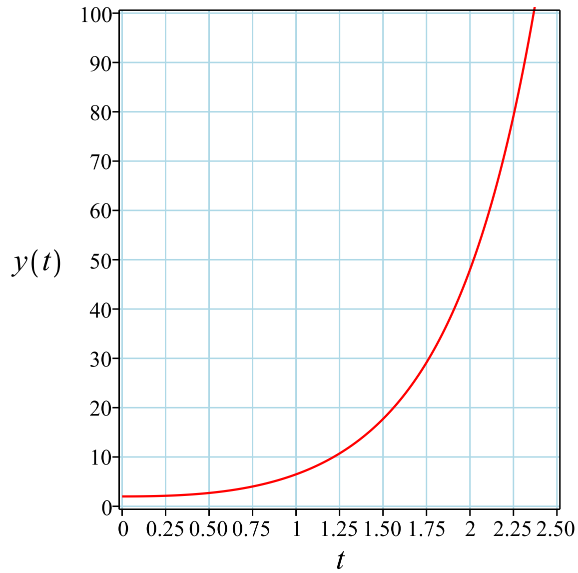



Figure 99: Solution plot

<u>Verification of solutions</u>

$$y = e^{2t} - \sin(2t) - e^{t}(t-1)$$

Verified OK.

### 4.39.1 Maple step by step solution

Let's solve

$$\left[y''' - y'' + 4y' - 4y = 8\,e^{2t} - 5\,e^{t}, y(0) = 2, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 3\right]$$

- Highest derivative means the order of the ODE is 3

  $y'''$

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(t)$

    $y_1(t) = y$
  - ◦ Define new variable $y_2(t)$

    $y_2(t) = y'$
  - ◦ Define new variable $y_3(t)$

    $y_3(t) = y''$

- Isolate for $y_3'(t)$ using original ODE

  $y_3'(t) = 8\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{t} + y_3(t) - 4y_2(t) + 4y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = 8\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{t} + y_3(t) - 4y_2(t) + 4y_1(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 8\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 8\,\mathrm{e}^{2t} - 5\,\mathrm{e}^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(2t)}{4} + \frac{\mathrm{I}\sin(2t)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = \begin{bmatrix} -\frac{\cos(2t)}{4} \\ \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \vec{y}_3(t) = \begin{bmatrix} \frac{\sin(2t)}{4} \\ \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(t) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(t) + c_3\overrightarrow{y}_3(t) + \overrightarrow{y}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^t & -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} \\ e^t & \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} \\ e^t & \cos(2t) & -\sin(2t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^t & -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} \\ e^t & \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} \\ e^t & \cos(2t) & -\sin(2t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 1 & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{4e^t}{5} + \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5} & \frac{\sin(2t)}{2} & \frac{e^t}{5} - \frac{\cos(2t)}{5} - \frac{\sin(2t)}{10} \\ \frac{4e^t}{5} - \frac{2\sin(2t)}{5} - \frac{4\cos(2t)}{5} & \cos(2t) & \frac{e^t}{5} + \frac{2\sin(2t)}{5} - \frac{\cos(2t)}{5} \\ \frac{4e^t}{5} - \frac{4\cos(2t)}{5} + \frac{8\sin(2t)}{5} & -2\sin(2t) & \frac{e^t}{5} + \frac{4\cos(2t)}{5} + \frac{2\sin(2t)}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{y}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(t) = \left[\begin{array}{c} \frac{\sin(2t)}{10} + \frac{\cos(2t)}{5} + \mathrm{e}^{2t} - \frac{6\,\mathrm{e}^{t}}{5} - t\,\mathrm{e}^{t} \\ \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5} + 2\,\mathrm{e}^{2t} - \frac{11\,\mathrm{e}^{t}}{5} - t\,\mathrm{e}^{t} \\ -\frac{2\sin(2t)}{5} - \frac{4\cos(2t)}{5} + 4\,\mathrm{e}^{2t} - \frac{16\,\mathrm{e}^{t}}{5} - t\,\mathrm{e}^{t} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + \left[\begin{array}{c} \frac{\sin(2t)}{10} + \frac{\cos(2t)}{5} + \mathrm{e}^{2t} - \frac{6\,\mathrm{e}^{t}}{5} - t\,\mathrm{e}^{t} \\ \frac{\cos(2t)}{5} - \frac{2\sin(2t)}{5} + 2\,\mathrm{e}^{2t} - \frac{11\,\mathrm{e}^{t}}{5} - t\,\mathrm{e}^{t} \\ -\frac{2\sin(2t)}{5} - \frac{4\cos(2t)}{5} + 4\,\mathrm{e}^{2t} - \frac{16\,\mathrm{e}^{t}}{5} - t\,\mathrm{e}^{t} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-5c_2+4)\cos(2t)}{20} + \frac{(5c_3+2)\sin(2t)}{20} + \mathrm{e}^{2t} + \frac{(-6-5t+5c_1)\mathrm{e}^{t}}{5}$

- Use the initial condition $y(0) = 2$

  $2 = -\frac{c_2}{4} + c_1$

- Calculate the 1st derivative of the solution

  $y' = -\frac{(-5c_2+4)\sin(2t)}{10} + \frac{(5c_3+2)\cos(2t)}{10} + 2\,\mathrm{e}^{2t} - \mathrm{e}^{t} + \frac{(-6-5t+5c_1)\mathrm{e}^{t}}{5}$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \frac{c_3}{2} + c_1$

- Calculate the 2nd derivative of the solution

  $y'' = -\frac{(-5c_2+4)\cos(2t)}{5} - \frac{(5c_3+2)\sin(2t)}{5} + 4\,\mathrm{e}^{2t} - 2\,\mathrm{e}^{t} + \frac{(-6-5t+5c_1)\mathrm{e}^{t}}{5}$

- Use the initial condition $y''\Big|_{\{t=0\}} = 3$

  $3 = c_1 + c_2$
- Solve for the unknown coefficients

  $\left\{c_1 = \frac{11}{5}, c_2 = \frac{4}{5}, c_3 = -\frac{22}{5}\right\}$
- Solution to the IVP

  $y = -t\,\mathrm{e}^t - \sin(2t) + \mathrm{e}^{2t} + \mathrm{e}^t$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.563 (sec). Leaf size: 22

```
dsolve([diff(y(t),t$3)-diff(y(t),t$2)+4*diff(y(t),t)-4*y(t)=8*exp(2*t)-5*exp(t),y(0) = 2, D(
```

$$y(t) = -t\,\mathrm{e}^t + \mathrm{e}^{2t} + \mathrm{e}^t - \sin(2t)$$

✓ Solution by Mathematica

Time used: 0.561 (sec). Leaf size: 24

```
DSolve[{y'''[t]-y''[t]+4*y'[t]-4*y[t]==8*Exp[2*t]-5*Exp[t],{y[0]==2,y'[0]==0,y''[0]==3}},y[t
```

$$y(t) \to e^t\left(-t + e^t + 1\right) - \sin(2t)$$

## 4.40 problem Problem 13(d)

Internal problem ID [12348]
Internal file name [OUTPUT/11000_Monday_October_02_2023_02_47_47_AM_80377253/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 13(d).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' - 5y'' + y' - y = -t^2 + 2t - 10}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0, y''(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s)-y''(0)-sy'(0)-s^2y(0)-5s^2Y(s)+5y'(0)+5sy(0)+sY(s)-y(0)-Y(s) = -\frac{2}{s^3}+\frac{2}{s^2}-\frac{10}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 2 \\
y'(0) &= 0 \\
y''(0) &= 0
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 2 - 2s^2 - 5s^2Y(s) + 10s + sY(s) - Y(s) = -\frac{2}{s^3} + \frac{2}{s^2} - \frac{10}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2s^5 - 10s^4 + 2s^3 - 10s^2 + 2s - 2}{s^3\left(s^3 - 5s^2 + s - 1\right)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}}{26} - \frac{11\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{78} - \frac{121}{39\left(116+6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{29}{78}}{s - \frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3} - \frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}} - \frac{5}{3}} + \frac{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6} - \frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}} + \frac{5}{3}\right.}{}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}+\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}\right)^{2}}{26}-\frac{11\left(\cdots\right.}{\cdots}\cdots}{s-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\cdots}\right.$$

$$\mathcal{L}^{-1}\left(\frac{\frac{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}+\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)^{2}}{26}+\frac{11\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{156}+\frac{121}{78\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{29}{78}-\cdots}{s+\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}+\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}-\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\cdots}\right.$$

$$\mathcal{L}^{-1}\left(\frac{\frac{\left(-\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}-\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{5}{3}-\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\right)^{2}}{26}+\frac{11\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{156}+\frac{121}{78\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}+\frac{29}{78}+\cdots}{s+\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{6}+\frac{11}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}-\frac{5}{3}+\frac{i\sqrt{3}\left(\frac{\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}{3}-\frac{22}{3\left(116+6\sqrt{78}\right)^{\frac{1}{3}}}\right)}{2}\cdots}\right.$$

Adding the above results and simplifying gives

$$y=t^{2}+\frac{\left(\sum_{\_\alpha=\mathrm{RootOf}\left(\_Z^{3}-5\_Z^{2}+\_Z-1\right)}\left(\_\alpha^{2}-11\_\alpha+28\right)\mathrm{e}^{\_\alpha t}\right)}{26}$$

Summary

The solution(s) found are the following

$$y = t^2 + \frac{\left(\sum_{\_\alpha = \mathrm{RootOf}\left(\_Z^3 - 5\_Z^2 + \_Z - 1\right)} \left(\_\alpha^2 - 11\_\alpha + 28\right) \mathrm{e}^{\_\alpha t}\right)}{26} \tag{1}$$

Verification of solutions

$$y = t^2 + \frac{\left(\sum_{\_\alpha = \mathrm{RootOf}\left(\_Z^3 - 5\_Z^2 + \_Z - 1\right)} \left(\_\alpha^2 - 11\_\alpha + 28\right) \mathrm{e}^{\_\alpha t}\right)}{26}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 4.391 (sec). Leaf size: 38

```
dsolve([diff(y(t),t$3)-5*diff(y(t),t$2)+diff(y(t),t)-y(t)=2*t-10-t^2,y(0) = 2, D(y)(0) = 0,
```

$$y(t) = \frac{\left(\sum_{\_\alpha = \mathrm{RootOf}\left(\_Z^3 - 5\_Z^2 + \_Z - 1\right)} \left(\_\alpha - 4\right)\left(\_\alpha - 7\right) \mathrm{e}^{\_\alpha t}\right)}{26} + t^2$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 1009

```
DSolve[{y'''[t]-5*y''[t]+y'[t]-y[t]==2*t-10-t^2,{y[0]==2,y'[0]==0,y''[0]==0}},y[t],t,Include
```

$y(t)$

$$\rightarrow \frac{-\text{Root}\left[\#1^3-5\#1^2+\#1-1\&,2\right]\text{Root}\left[\#1^3-5\#1^2+\#1-1\&,3\right]^2t^2+\text{Root}\left[\#1^3-5\#1^2+\#1-\right.}{}$$

## 4.41 problem Problem 14(a)



Internal problem ID [12349]
Internal file name [`OUTPUT/11001_Monday_October_02_2023_02_47_48_AM_24196050/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 14(a).
**ODE order**: 4.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y'' + 4y = 12\,\text{Heaviside}\,(t) - 12\,\text{Heaviside}\,(t-1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) - 5s^2Y(s) + 5y'(0) + 5sy(0) + 4Y(s) = \frac{-12\,\mathrm{e}^{-s} + 12}{s} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0 \\
y''(0) &= 0 \\
y'''(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4 Y(s) - 5s^2 Y(s) + 4Y(s) = \frac{-12\,\mathrm{e}^{-s} + 12}{s}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = -\frac{12(\mathrm{e}^{-s} - 1)}{s\left(s^4 - 5s^2 + 4\right)}
$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{12(\mathrm{e}^{-s} - 1)}{s\left(s^4 - 5s^2 + 4\right)}\right) \\
&= -4\cosh(t) + \cosh(2t) + 2\,\mathrm{e}^{t-1} - \frac{\mathrm{e}^{2t-2}}{2} + \frac{\text{Heaviside}(1-t)\left(6 + \mathrm{e}^{2t-2} - 4\,\mathrm{e}^{t-1}\right)}{2} + \frac{\left(-\mathrm{e}^{-2t+2} + 4\,\mathrm{e}^{1-t}\right) \mathrm{I}}{2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y = {} & -4\cosh(t) + \cosh(2t) + 2\,\mathrm{e}^{t-1} - \frac{\mathrm{e}^{2t-2}}{2} \\
& + \frac{\text{Heaviside}(1-t)\left(6 + \mathrm{e}^{2t-2} - 4\,\mathrm{e}^{t-1}\right)}{2} + \frac{\left(-\mathrm{e}^{-2t+2} + 4\,\mathrm{e}^{1-t}\right)\text{Heaviside}(t-1)}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & -4\cosh(t) + \cosh(2t) + 2\,\mathrm{e}^{t-1} - \frac{\mathrm{e}^{2t-2}}{2} \\
& + \frac{\text{Heaviside}(1-t)\left(6 + \mathrm{e}^{2t-2} - 4\,\mathrm{e}^{t-1}\right)}{2} + \frac{\left(-\mathrm{e}^{-2t+2} + 4\,\mathrm{e}^{1-t}\right)\text{Heaviside}(t-1)}{2}
\end{aligned} \tag{1}
$$

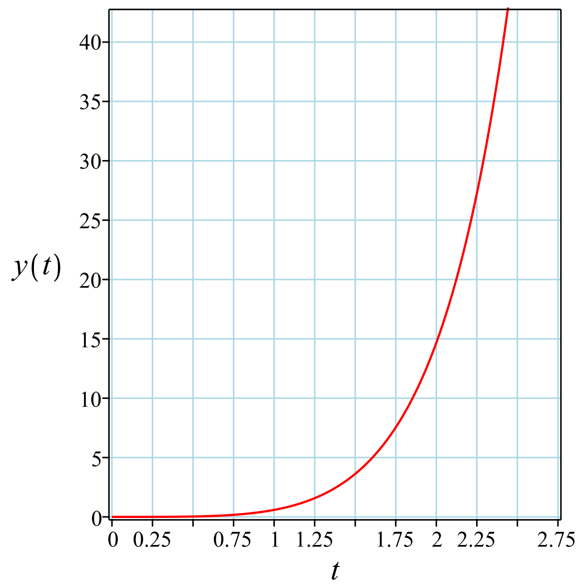



Figure 100: Solution plot

Verification of solutions

$$y = -4\cosh(t) + \cosh(2t) + 2\,\mathrm{e}^{t-1} - \frac{\mathrm{e}^{2t-2}}{2} + \frac{\mathrm{Heaviside}(1-t)\left(6+\mathrm{e}^{2t-2}-4\,\mathrm{e}^{t-1}\right)}{2} + \frac{\left(-\mathrm{e}^{-2t+2}+4\,\mathrm{e}^{1-t}\right)\mathrm{Heaviside}(t-1)}{2}$$

Verified OK.

### 4.41.1 Maple step by step solution

Let's solve

$$\left[y'''' - 5y'' + 4y = 12\mathit{Heaviside}(t) - 12\mathit{Heaviside}(t-1), y(0) = 0, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 0, y'''\Big|\right.$$

- Highest derivative means the order of the ODE is 4

  $y''''$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$

- Define new variable $y_3(t)$

  $y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = 12 Heaviside(t) - 12 Heaviside(t-1) + 5y_3(t) - 4y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = 12 Heaviside(t) - 12 Heaviside(t-1) + 5y_3(t)$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 12 Heaviside(t) - 12 Heaviside(t-1) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 12 Heaviside(t) - 12 Heaviside(t-1) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2t} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4 + \vec{y}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{8} & -e^{-t} & e^t & \frac{e^{2t}}{8} \\ \frac{e^{-2t}}{4} & e^{-t} & e^t & \frac{e^{2t}}{4} \\ -\frac{e^{-2t}}{2} & -e^{-t} & e^t & \frac{e^{2t}}{2} \\ e^{-2t} & e^{-t} & e^t & e^{2t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{8} & -e^{-t} & e^{t} & \frac{e^{2t}}{8} \\ \frac{e^{-2t}}{4} & e^{-t} & e^{t} & \frac{e^{2t}}{4} \\ -\frac{e^{-2t}}{2} & -e^{-t} & e^{t} & \frac{e^{2t}}{2} \\ e^{-2t} & e^{-t} & e^{t} & e^{2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & -1 & 1 & \frac{1}{8} \\ \frac{1}{4} & 1 & 1 & \frac{1}{4} \\ -\frac{1}{2} & -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} -\frac{(e^{4t}-4e^{3t}-4e^{t}+1)e^{-2t}}{6} & -\frac{(e^{4t}-8e^{3t}+8e^{t}-1)e^{-2t}}{12} & -\frac{(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{6} & \frac{(e^{4t}-2e^{3t}+2e^{t}-1)e^{-2t}}{12} \\ -\frac{(e^{4t}-2e^{3t}+2e^{t}-1)e^{-2t}}{3} & -\frac{(e^{4t}-4e^{3t}-4e^{t}+1)e^{-2t}}{6} & \frac{(2e^{4t}-e^{3t}+e^{t}-2)e^{-2t}}{6} & -\frac{(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{6} \\ \frac{2(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{3} & -\frac{(e^{4t}-2e^{3t}+2e^{t}-1)e^{-2t}}{3} & -\frac{(-4e^{4t}+e^{3t}+e^{t}-4)e^{-2t}}{6} & \frac{(2e^{4t}-e^{3t}+e^{t}-2)e^{-2t}}{6} \\ -\frac{2(2e^{4t}-e^{3t}+e^{t}-2)e^{-2t}}{3} & \frac{2(-e^{4t}+e^{3t}+e^{t}-1)e^{-2t}}{3} & \frac{(8e^{4t}-e^{3t}+e^{t}-8)e^{-2t}}{6} & -\frac{(-4e^{4t}+e^{3t}+e^{t}-4)e^{-2t}}{6}
\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{y}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{y}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(t) = \begin{bmatrix} 2\left(\mathrm{e}^{-1+3t} Heaviside(t-1) - \frac{\mathrm{e}^{4t-2} Heaviside(t-1)}{4} + \left(\mathrm{e}^{1+t} - \frac{\mathrm{e}^2}{4} - \frac{3\,\mathrm{e}^{2t}}{2}\right) Heaviside(t-1) - \\ -\left(-2\,\mathrm{e}^{-1+3t} Heaviside(t-1) + \mathrm{e}^{4t-2} Heaviside(t-1) + \left(-\mathrm{e}^2 + 2\,\mathrm{e}^{1+t}\right) Heaviside(t - \\ -2\left(-\mathrm{e}^{-1+3t} Heaviside(t-1) + \mathrm{e}^{4t-2} Heaviside(t-1) + \left(\mathrm{e}^2 - \mathrm{e}^{1+t}\right) Heaviside(t-1\right. \\ -4\left(-\frac{\mathrm{e}^{-1+3t} Heaviside(t-1)}{2} + \mathrm{e}^{4t-2} Heaviside(t-1) + \left(-\mathrm{e}^2 + \frac{\mathrm{e}^{1+t}}{2}\right) Heaviside(t\right. \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(t) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + \begin{bmatrix} 2\left(\mathrm{e}^{-1+3t} Heaviside(t-1) - \frac{\mathrm{e}^{4t-2} Heaviside(t-1)}{4} + \left(\mathrm{e}^{1+t} - \right.\right. \\ -\left(-2\,\mathrm{e}^{-1+3t} Heaviside(t-1) + \mathrm{e}^{4t-2} Heaviside(t-1) + \right. \\ -2\left(-\mathrm{e}^{-1+3t} Heaviside(t-1) + \mathrm{e}^{4t-2} Heaviside(t-1)\right. \\ -4\left(-\frac{\mathrm{e}^{-1+3t} Heaviside(t-1)}{2} + \mathrm{e}^{4t-2} Heaviside(t-1)\right. \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = -2\left(-\mathrm{e}^{-1+3t} Heaviside(t-1) + \frac{\mathrm{e}^{4t-2} Heaviside(t-1)}{4} + \left(-\mathrm{e}^{1+t} + \frac{\mathrm{e}^2}{4} + \frac{3\,\mathrm{e}^{2t}}{2}\right) Heaviside(t-1) - \right.$

- Use the initial condition $y(0) = 0$

  $0 = undefined + c_3 + \frac{c_4}{8} - c_2 - \frac{c_1}{8}$

- Calculate the 1st derivative of the solution

  $y' = -2\left(-3\,\mathrm{e}^{-1+3t} Heaviside(t-1) - \mathrm{e}^{-1+3t} Dirac(t-1) + \mathrm{e}^{4t-2} Heaviside(t-1) + \frac{\mathrm{e}^{4t-2} Dirac(t-1)}{4}\right.$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = undefined + c_3 + \frac{c_4}{4} + c_2 + \frac{c_1}{4}$

- Calculate the 2nd derivative of the solution

  $y'' = -2\left(9\left(Heaviside(t) - \frac{c_3}{2}\right) \mathrm{e}^{3t} + 16\left(-\frac{Heaviside(t)}{4} - \frac{c_4}{16}\right) \mathrm{e}^{4t} - 2 Dirac(t)\, \mathrm{e}^{4t} - \mathrm{e}^{-1+3t} Dirac(1, t\right.$

- Use the initial condition $y''\Big|_{\{t=0\}} = 0$

$$0 = undefined + c_3 + \frac{c_4}{2} - c_2 - \frac{c_1}{2}$$

- Calculate the 3rd derivative of the solution

$$y''' = -2\left(27\left(Heaviside(t) - \frac{c_3}{2}\right) e^{3t} + 64\left(-\frac{Heaviside(t)}{4} - \frac{c_4}{16}\right) e^{4t} - 12 Dirac(t) e^{4t} - 9 e^{-1+3t} Dirac($$

- Use the initial condition $y'''\Big|_{\{t=0\}} = 0$

$$0 = undefined + c_3 + c_4 + c_2 + c_1$$

- Solve for the unknown coefficients

$$\{c_1 = undefined, c_2 = 0, c_3 = 0, c_4 = 0\}$$

- Solution to the IVP

$$y = -\frac{(-4 e^{3t+1} Heaviside(t-1) + 4 e^{3t+2} Heaviside(t) - e^{4t+2} Heaviside(t) + 6(-Heaviside(t) + Heaviside(t-1)) e^{2t+2} + (e^4 - 4 e^{3+t} + e}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 5.031 (sec). Leaf size: 72

```
dsolve([diff(y(t),t$4)-5*diff(y(t),t$2)+4*y(t)=12*(Heaviside(t)-Heaviside(t-1)),y(0) = 0, D(
```

$$y(t) = 2(-1 + \cosh(t))^2 - \frac{\text{Heaviside}(t-1)\left(e^{-2t+2} - 4 e^{-t+1} + e^{2t-2} + 6 - 4 e^{t-1}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 88

```
DSolve[{y''''[t]-5*y''[t]+4*y[t]==12*(UnitStep[t]-UnitStep[t-1]),{y[0]==0,y'[0]==0,y''[0]==0
```

$$y(t) \to \begin{cases} \frac{1}{2}e^{-2t}(-1+e^t)^4 & 0 \leq t \leq 1 \\ \frac{1}{2}(-1+e)e^{-2(t+1)}(-e^2-e^3+e^{4t}+4e^{t+2}-4e^{3t+1}+e^{4t+1}) & t > 1 \end{cases}$$

## 4.42 problem Problem 14(b)



Internal problem ID [12350]
Internal file name [OUTPUT/11002_Monday_October_02_2023_02_47_49_AM_8359695/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 5.6 Laplace transform. Nonhomogeneous equations. Problems page 368
**Problem number**: Problem 14(b).
**ODE order**: 4.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 16y = -32\,\text{Heaviside}\,(t-\pi) + 32\,\text{Heaviside}\,(t)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) - 16Y(s) = \frac{-32\,\mathrm{e}^{-s\pi} + 32}{s} \tag{1}$$

But the initial conditions are

$$
\begin{aligned}
y(0) &= 0 \\
y'(0) &= 0 \\
y''(0) &= 0 \\
y'''(0) &= 0
\end{aligned}
$$

Substituting these initial conditions in above in Eq (1) gives

$$
s^4 Y(s) - 16Y(s) = \frac{-32\,\mathrm{e}^{-s\pi} + 32}{s}
$$

Solving the above equation for $Y(s)$ results in

$$
Y(s) = -\frac{32(\mathrm{e}^{-s\pi} - 1)}{s\,(s^4 - 16)}
$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{32(\mathrm{e}^{-s\pi} - 1)}{s\,(s^4 - 16)}\right) \\
&= \left(2\cos(t)^2 - 3\right) \mathrm{Heaviside}\,(\pi - t) - \mathrm{Heaviside}\,(t - \pi) \cosh(2t - 2\pi) + \cosh(2t)
\end{aligned}
$$

Hence the final solution is

$$
y = \left(2\cos(t)^2 - 3\right) \mathrm{Heaviside}\,(\pi - t) - \mathrm{Heaviside}\,(t - \pi) \cosh(2t - 2\pi) + \cosh(2t)
$$

Summary

The solution(s) found are the following

$$
y = \left(2\cos(t)^2 - 3\right) \mathrm{Heaviside}\,(\pi - t) - \mathrm{Heaviside}\,(t - \pi) \cosh(2t - 2\pi) + \cosh(2t) \tag{1}
$$

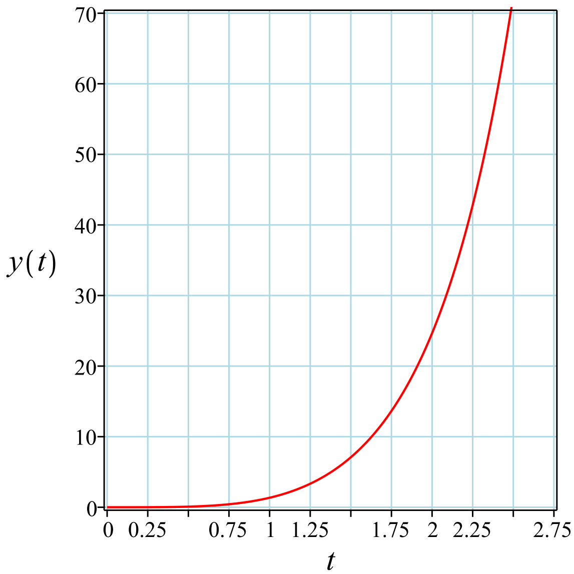



Figure 101: Solution plot

Verification of solutions

$$y = \left(2\cos(t)^2 - 3\right)\text{Heaviside}(\pi - t) - \text{Heaviside}(t - \pi)\cosh(2t - 2\pi) + \cosh(2t)$$

Verified OK.

### 4.42.1 Maple step by step solution

Let's solve

$$\left[y'''' - 16y = -32 Heaviside(t-\pi) + 32 Heaviside(t), y(0) = 0, y'\Big|_{\{t=0\}} = 0, y''\Big|_{\{t=0\}} = 0, y'''\Big|_{\{t=0\}}\right.$$

- Highest derivative means the order of the ODE is 4

  $y''''$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$
  - Define new variable $y_3(t)$

    $y_3(t) = y''$

- Define new variable $y_4(t)$

  $y_4(t) = y'''$

- Isolate for $y_4'(t)$ using original ODE

  $y_4'(t) = -32 Heaviside(t - \pi) + 32 Heaviside(t) + 16 y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_4(t) = y_3'(t), y_4'(t) = -32 Heaviside(t - \pi) + 32 Heaviside(t) + 16 y_1$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -32 Heaviside(t - \pi) + 32 Heaviside(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -32 Heaviside(t - \pi) + 32 Heaviside(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2t) - \mathrm{I}\sin(2t)) \\ -\frac{\cos(2t)}{4} + \frac{\mathrm{I}\sin(2t)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(t) = \begin{bmatrix} -\frac{\sin(2t)}{8} \\ -\frac{\cos(2t)}{4} \\ \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \vec{y}_4(t) = \begin{bmatrix} -\frac{\cos(2t)}{8} \\ \frac{\sin(2t)}{4} \\ \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(t) + c_4\vec{y}_4(t) + \vec{y}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{8} & \frac{e^{2t}}{8} & -\frac{\sin(2t)}{8} & -\frac{\cos(2t)}{8} \\ \frac{e^{-2t}}{4} & \frac{e^{2t}}{4} & -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} \\ -\frac{e^{-2t}}{2} & \frac{e^{2t}}{2} & \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} \\ e^{-2t} & e^{2t} & \cos(2t) & -\sin(2t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{8} & \frac{e^{2t}}{8} & -\frac{\sin(2t)}{8} & -\frac{\cos(2t)}{8} \\ \frac{e^{-2t}}{4} & \frac{e^{2t}}{4} & -\frac{\cos(2t)}{4} & \frac{\sin(2t)}{4} \\ -\frac{e^{-2t}}{2} & \frac{e^{2t}}{2} & \frac{\sin(2t)}{2} & \frac{\cos(2t)}{2} \\ e^{-2t} & e^{2t} & \cos(2t) & -\sin(2t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & \frac{1}{8} & 0 & -\frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{e^{-2t}}{4} + \frac{e^{2t}}{4} + \frac{\cos(2t)}{2} & -\frac{e^{-2t}}{8} + \frac{e^{2t}}{8} + \frac{\sin(2t)}{4} & \frac{e^{-2t}}{16} + \frac{e^{2t}}{16} - \frac{\cos(2t)}{8} & -\frac{e^{-2t}}{32} + \frac{e^{2t}}{32} \\ -\frac{e^{-2t}}{2} + \frac{e^{2t}}{2} - \sin(2t) & \frac{e^{-2t}}{4} + \frac{e^{2t}}{4} + \frac{\cos(2t)}{2} & -\frac{e^{-2t}}{8} + \frac{e^{2t}}{8} + \frac{\sin(2t)}{4} & \frac{e^{-2t}}{16} + \frac{e^{2t}}{16} \\ e^{-2t} + e^{2t} - 2\cos(2t) & -\frac{e^{-2t}}{2} + \frac{e^{2t}}{2} - \sin(2t) & \frac{e^{-2t}}{4} + \frac{e^{2t}}{4} + \frac{\cos(2t)}{2} & -\frac{e^{-2t}}{8} + \frac{e^{2t}}{8} \\ -2e^{-2t} + 2e^{2t} + 4\sin(2t) & e^{-2t} + e^{2t} - 2\cos(2t) & -\frac{e^{-2t}}{2} + \frac{e^{2t}}{2} - \sin(2t) & \frac{e^{-2t}}{4} + \frac{e^{2t}}{4} + \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{y}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{y}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(t) = \left[\begin{array}{c} \frac{(-2(\cos(2t)-2)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t))\mathrm{e}^{2t+2\pi}+\mathrm{e}^{4t+2\pi}\mathit{Heaviside}(t)+(-\mathrm{e}^{4t}-\mathrm{e}^{4\pi})\mathit{H}}{2} \\ \mathrm{e}^{-2t-2\pi}(2\sin(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t))\,\mathrm{e}^{2t+2\pi}+\mathrm{e}^{4t+2\pi}\mathit{Heaviside}(t)+(- \\ 2(2\cos(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t))\,\mathrm{e}^{2t+2\pi}+\mathrm{e}^{4t+2\pi}\mathit{Heaviside}(t)+(-\mathrm{e}^{4t}-\mathrm{e} \\ 4\,\mathrm{e}^{-2t-2\pi}(-2\sin(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t))\,\mathrm{e}^{2t+2\pi}+\mathrm{e}^{4t+2\pi}\mathit{Heaviside}(t)+( \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(t) + c_4\vec{y}_4(t) + \left[\begin{array}{c} \frac{(-2(\cos(2t)-2)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(}{} \\ \mathrm{e}^{-2t-2\pi}(2\sin(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t \\ 2(2\cos(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside}(t))\,\mathrm{e}^{2t+2} \\ 4\,\mathrm{e}^{-2t-2\pi}(-2\sin(2t)(\mathit{Heaviside}(t-\pi)-\mathit{Heaviside} \end{array}\right.$$

- First component of the vector is the solution to the ODE

  $y = -\frac{\mathit{Heaviside}(t-\pi)\mathrm{e}^{-2t+2\pi}}{2} - \frac{\mathit{Heaviside}(t-\pi)\mathrm{e}^{2t-2\pi}}{2} + (-\cos(2t)+2)\,\mathit{Heaviside}(t-\pi) + \frac{(8\mathit{Heaviside}(t)-c_4}{8}$

- Use the initial condition $y(0) = 0$

  $0 = \mathit{undefined} - \frac{c_4}{8} - \frac{c_1}{8} + \frac{c_2}{8}$

- Calculate the 1st derivative of the solution

  $y' = -\frac{\mathit{Dirac}(t-\pi)\mathrm{e}^{-2t+2\pi}}{2} + \mathit{Heaviside}(t-\pi)\,\mathrm{e}^{-2t+2\pi} - \frac{\mathit{Dirac}(t-\pi)\mathrm{e}^{2t-2\pi}}{2} - \mathit{Heaviside}(t-\pi)\,\mathrm{e}^{2t-2\pi} + 2$

- Use the initial condition $y'\Big|_{\{t=0\}} = 0$

  $0 = \mathit{undefined} + \frac{c_1}{4} + \frac{c_2}{4} - \frac{c_3}{4}$

- Calculate the 2nd derivative of the solution

$$y'' = \frac{Dirac(1,t)\,e^{2t}}{2} + \frac{c_3 \sin(2t)}{2} - \frac{(8Heaviside(t)-c_4)\cos(2t)}{2} + \frac{(4Heaviside(t)-c_1)\,e^{-2t}}{2} + \frac{(4Heaviside(t)+c_2)\,e^{2t}}{2} + 4Dir$$

- Use the initial condition $y''\Big|_{\{t=0\}} = 0$

$$0 = undefined + \frac{c_4}{2} - \frac{c_1}{2} + \frac{c_2}{2}$$

- Calculate the 3rd derivative of the solution

$$y''' = 3Dirac(1,t)\,e^{2t} + \frac{Dirac(2,t)\,e^{2t}}{2} + c_3\cos(2t) - (4Heaviside(t) - c_1)\,e^{-2t} + (4Heaviside(t) + c_2$$

- Use the initial condition $y'''\Big|_{\{t=0\}} = 0$

$$0 = undefined + c_3 + c_1 + c_2$$

- Solve for the unknown coefficients

$$\{c_1 = undefined, c_2 = undefined, c_3 = undefined, c_4 = undefined\}$$

- Solution to the IVP

$$y = -\frac{Heaviside(t-\pi)\,e^{-2t+2\pi}}{2} - \frac{Heaviside(t-\pi)\,e^{2t-2\pi}}{2} + (-\cos(2t)+2)\,Heaviside(t-\pi) + Heaviside(t)\,($$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 5.016 (sec). Leaf size: 40

```
dsolve([diff(y(t),t$4)-16*y(t)=32*(Heaviside(t)-Heaviside(t-Pi)),y(0) = 0, D(y)(0) = 0, (D@@
```

$$y(t) = -\operatorname{Heaviside}(t-\pi)\cosh(2t-2\pi) + (-\cos(2t)+2)\operatorname{Heaviside}(t-\pi) + \cos(2t) + \cosh(2t) - 2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 72

```
DSolve[{y''''[t]-16*y[t]==32*(UnitStep[t]-UnitStep[t-Pi]),{y[0]==0,y'[0]==0,y''[0]==0,y'''[0
```

$$y(t) \to \begin{cases} \frac{1}{2}e^{-2(t+\pi)}(-1+e^{2\pi})(-e^{2\pi}+e^{4t}) & t > \pi \\ \frac{1}{2}(2\cos(2t)+e^{-2t}+e^{2t}-4) & 0 \leq t \leq \pi \end{cases}$$

# 5 Chapter 6. Introduction to Systems of ODEs. Problems page 408

## 5.1 problem Problem 1(a)



Internal problem ID [12351]
Internal file name [`OUTPUT/11003_Monday_October_02_2023_02_47_49_AM_782895/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 1(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{t^2y'' + 3y't + y = t^7}$$

### 5.1.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 3t, C = 1, f(t) = t^7$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for $y_h$ from

$$t^2y'' + 3y't + y = 0$$

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} + t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by $t^r$ gives

$$r(r-1) + 3r + 1 = 0$$

Or

$$r^2 + 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= -1 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = t^r$ and $y_2 = t^r \ln(t)$. Hence

$$y = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

Next, we find the particular solution to the ODE

$$t^2 y'' + 3y't + y = t^7$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $t$ as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{t}$$
$$y_2 = \frac{\ln(t)}{t}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & \frac{\ln(t)}{t} \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}\left(\frac{\ln(t)}{t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & \frac{\ln(t)}{t} \\ -\frac{1}{t^2} & -\frac{\ln(t)}{t^2} + \frac{1}{t^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)\left(-\frac{\ln(t)}{t^2} + \frac{1}{t^2}\right) - \left(\frac{\ln(t)}{t}\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = \frac{1}{t^3}$$

Which simplifies to

$$W = \frac{1}{t^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{t^6 \ln(t)}{\frac{1}{t}}\,dt$$

Which simplifies to

$$u_1 = -\int t^7 \ln(t)\,dt$$

Hence

$$u_1 = -\frac{t^8 \ln(t)}{8} + \frac{t^8}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^6}{\frac{1}{t}}\,dt$$

Which simplifies to

$$u_2 = \int t^7 dt$$

Hence

$$u_2 = \frac{t^8}{8}$$

Which simplifies to

$$u_1 = -\frac{t^8(-1 + 8\ln(t))}{64}$$

$$u_2 = \frac{t^8}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{t^7(-1+8\ln(t))}{64} + \frac{t^7\ln(t)}{8}$$

Which simplifies to

$$y_p(t) = \frac{t^7}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{t^7}{64} + \frac{c_1}{t} + \frac{c_2\ln(t)}{t} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{t^7}{64} + \frac{c_1}{t} + \frac{c_2\ln(t)}{t} \tag{1}$$

Verification of solutions

$$y = \frac{t^7}{64} + \frac{c_1}{t} + \frac{c_2\ln(t)}{t}$$

Verified OK.

### 5.1.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$t^2y'' + 3y't + y = 0$$

In normal form the ode

$$t^2y'' + 3y't + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{1}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where $\tau$ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\, \tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(t) dt\right)} dt \\ &= \int \mathrm{e}^{-\left(\int \frac{3}{t} dt\right)} dt \\ &= \int e^{-3\ln(t)}\, dt \\ &= \int \frac{1}{t^3} dt \\ &= -\frac{1}{2t^2} \end{aligned} \tag{6}$$

Using (6) to evaluate $q_1$ from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{1}{t^2}}{\frac{1}{t^6}} \\ &= t^4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + t^4 y(\tau) = 0$$

But in terms of $\tau$

$$t^4 = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by $\tau^r$ gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to $y$ using (6) which results in

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{t^2}\right)\right)}{2}$$

Therefore the homogeneous solution $y_h$ is

$$y_h = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\left(c_1 - c_2\ln(2) + c_2\ln\left(-\frac{1}{t^2}\right)\right)}{2}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $t$ as well. Let

$$y_p(t) = u_1y_1 + u_2y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{-\frac{1}{t^2}}$$

$$y_2 = -\frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{t^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2} \\ \frac{d}{dt}\left(\sqrt{-\frac{1}{t^2}}\right) & \frac{d}{dt}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{t^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{t^2}}\,t^3} & -\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{t^2}}\,t^3} + \frac{\sqrt{2}\,\ln\left(-\frac{1}{t^2}\right)}{2\sqrt{-\frac{1}{t^2}}\,t^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}}{t} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{-\frac{1}{t^2}}\right)\left(-\frac{\sqrt{2}\,\ln(2)}{2\sqrt{-\frac{1}{t^2}}\,t^3} + \frac{\sqrt{2}\,\ln\left(-\frac{1}{t^2}\right)}{2\sqrt{-\frac{1}{t^2}}\,t^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}}{t}\right) \\ & - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{t^2}}\,t^3}\right) \end{aligned}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{t^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{t^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2}\right)t^7}{\frac{\sqrt{2}}{t}}\,dt$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{-\frac{1}{t^2}}\,\left(\ln(2) - \ln\left(-\frac{1}{t^2}\right)\right) t^8}{2} dt$$

Hence

$$u_1 = -\frac{\sqrt{-\frac{1}{t^2}}\, t^9 \ln\left(-\frac{1}{t^2}\right)}{16} + \frac{\sqrt{-\frac{1}{t^2}}\, t^9 (4\ln(2) - 1)}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{t^2}}\, t^7}{\frac{\sqrt{2}}{t}} dt$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{t^2}}\, t^8 \sqrt{2}}{2} dt$$

Hence

$$u_2 = \frac{t^9 \sqrt{-\frac{1}{t^2}}\, \sqrt{2}}{16}$$

Which simplifies to

$$u_1 = \frac{\sqrt{-\frac{1}{t^2}}\, t^9 \left(-4\ln\left(-\frac{1}{t^2}\right) + 4\ln(2) - 1\right)}{64}$$

$$u_2 = \frac{t^9 \sqrt{-\frac{1}{t^2}}\, \sqrt{2}}{16}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{t^7 \left(-4\ln\left(-\frac{1}{t^2}\right) + 4\ln(2) - 1\right)}{64} + \frac{t^9 \sqrt{-\frac{1}{t^2}}\, \sqrt{2} \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\, \ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\, \ln\left(-\frac{1}{t^2}\right)}{2}\right)}{16}$$

Which simplifies to

$$y_p(t) = \frac{t^7}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2}\right) + \left(\frac{t^7}{64}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2} + \frac{t^7}{64} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\left(c_1 - c_2 \ln(2) + c_2 \ln\left(-\frac{1}{t^2}\right)\right)}{2} + \frac{t^7}{64}$$

Verified OK.

### 5.1.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 3t, C = 1, f(t) = t^7$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for $y_h$ from

$$t^2y'' + 3y't + y = 0$$

In normal form the ode

$$t^2y'' + 3y't + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= \frac{3}{t} \\
q(t) &= \frac{1}{t^2}
\end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left( \frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where $\tau$ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where $c$ is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{\frac{1}{t^2}}}{c} \\
\tau'' &= -\frac{1}{c\sqrt{\frac{1}{t^2}}\, t^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{t^2}}\, t^3} + \frac{3}{t}\frac{\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{\frac{1}{t^2}} dt}{c} \\
&= \frac{\sqrt{\frac{1}{t^2}}\, t \ln(t)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1}{t}
$$

Now the particular solution to this ODE is found

$$
t^2 y'' + 3y't + y = t^7
$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $t$ as well. Let

$$
y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{-\frac{1}{t^2}} \\
y_2 &= -\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\, \ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\, \ln\left(-\frac{1}{t^2}\right)}{2}
\end{aligned}
$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{-\frac{1}{t^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln\left(-\frac{1}{t^2}\right)}{2} \\ \frac{d}{dt}\left(\sqrt{-\frac{1}{t^2}}\right) & \frac{d}{dt}\left(-\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln\left(-\frac{1}{t^2}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{-\frac{1}{t^2}} & -\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln\left(-\frac{1}{t^2}\right)}{2} \\ \frac{1}{\sqrt{-\frac{1}{t^2}}\,t^3} & -\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{t^2}}\,t^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{t^2}\right)}{2\sqrt{-\frac{1}{t^2}}\,t^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}}{t} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{-\frac{1}{t^2}}\right)\left(-\frac{\sqrt{2}\ln(2)}{2\sqrt{-\frac{1}{t^2}}\,t^3} + \frac{\sqrt{2}\ln\left(-\frac{1}{t^2}\right)}{2\sqrt{-\frac{1}{t^2}}\,t^3} - \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}}{t}\right) - \left(-\frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln(2)}{2} + \frac{\sqrt{2}\sqrt{-\frac{1}{t^2}}\ln\left(-\frac{1}{t^2}\right)}{2}\right)\left(\frac{1}{\sqrt{-\frac{1}{t^2}}\,t^3}\right)$$

Which simplifies to

$$W = \frac{\sqrt{2}}{t^3}$$

Which simplifies to

$$W = \frac{\sqrt{2}}{t^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2} + \frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2}\right) t^7}{\frac{\sqrt{2}}{t}}\,dt$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{-\frac{1}{t^2}}\,\left(\ln\left(2\right) - \ln\left(-\frac{1}{t^2}\right)\right) t^8}{2} dt$$

Hence

$$u_1 = -\frac{\sqrt{-\frac{1}{t^2}}\,t^9 \ln\left(-\frac{1}{t^2}\right)}{16} + \frac{\sqrt{-\frac{1}{t^2}}\,t^9 (4\ln\left(2\right) - 1)}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{-\frac{1}{t^2}}\,t^7}{\frac{\sqrt{2}}{t}}\,dt$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{-\frac{1}{t^2}}\,t^8\sqrt{2}}{2} dt$$

Hence

$$u_2 = \frac{t^9\sqrt{-\frac{1}{t^2}}\,\sqrt{2}}{16}$$

Which simplifies to

$$u_1 = \frac{\sqrt{-\frac{1}{t^2}}\,t^9 \left(-4\ln\left(-\frac{1}{t^2}\right) + 4\ln\left(2\right) - 1\right)}{64}$$

$$u_2 = \frac{t^9\sqrt{-\frac{1}{t^2}}\,\sqrt{2}}{16}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(t) \\
&= -\frac{t^7\left(-4\ln\left(-\frac{1}{t^2}\right)+4\ln(2)-1\right)}{64}+\frac{t^9\sqrt{-\frac{1}{t^2}}\,\sqrt{2}\left(-\frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\ln(2)}{2}+\frac{\sqrt{2}\,\sqrt{-\frac{1}{t^2}}\,\ln\left(-\frac{1}{t^2}\right)}{2}\right)}{16}
\end{aligned}
$$

Which simplifies to

$$
y_p(t) = \frac{t^7}{64}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{t}\right) + \left(\frac{t^7}{64}\right) \\
&= \frac{t^7}{64} + \frac{c_1}{t}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{t^7}{64} + \frac{c_1}{t}
$$

Summary

The solution(s) found are the following

$$
y = \frac{t^7}{64} + \frac{c_1}{t} \tag{1}
$$

Verification of solutions

$$
y = \frac{t^7}{64} + \frac{c_1}{t}
$$

Verified OK.

### 5.1.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = t^2, B = 3t, C = 1, f(t) = t^7$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. Solving for $y_h$ from

$$t^2y'' + 3y't + y = 0$$

In normal form the ode

$$t^2y'' + 3y't + y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{3}{t} \\ q(t) &= \frac{1}{t^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not $y$.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} + \frac{3n}{t^2} + \frac{1}{t^2} = 0 \tag{5}$$

Solving (5) for $n$ gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}
v''(t) + \frac{v'(t)}{t} &= 0 \\
v''(t) + \frac{v'(t)}{t} &= 0 \qquad (7)
\end{aligned}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(t, u) \\
&= f(t)g(u) \\
&= -\frac{u}{t}
\end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{t}\,dt \\
\int \frac{1}{u}\,du &= \int -\frac{1}{t}\,dt \\
\ln(u) &= -\ln(t) + c_1 \\
u &= e^{-\ln(t)+c_1} \\
&= \frac{c_1}{t}
\end{aligned}$$

Now that $u(t)$ is known, then

$$\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= c_1 \ln(t) + c_2
\end{aligned}$$

Hence

$$\begin{aligned} y &= v(t)\, t^n \\ &= \frac{c_1 \ln(t) + c_2}{t} \\ &= \frac{c_1 \ln(t) + c_2}{t} \end{aligned}$$

Now the particular solution to this ODE is found

$$t^2 y'' + 3y't + y = t^7$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $t$ as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{t} \\ y_2 &= \frac{\ln(t)}{t} \end{aligned}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & \frac{\ln(t)}{t} \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}\left(\frac{\ln(t)}{t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & \frac{\ln(t)}{t} \\ -\frac{1}{t^2} & -\frac{\ln(t)}{t^2} + \frac{1}{t^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)\left(-\frac{\ln(t)}{t^2} + \frac{1}{t^2}\right) - \left(\frac{\ln(t)}{t}\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = \frac{1}{t^3}$$

Which simplifies to

$$W = \frac{1}{t^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{t^6 \ln(t)}{\frac{1}{t}} \, dt$$

Which simplifies to

$$u_1 = -\int t^7 \ln(t) \, dt$$

Hence

$$u_1 = -\frac{t^8 \ln(t)}{8} + \frac{t^8}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^6}{\frac{1}{t}} \, dt$$

Which simplifies to

$$u_2 = \int t^7 dt$$

Hence

$$u_2 = \frac{t^8}{8}$$

Which simplifies to

$$u_1 = -\frac{t^8(-1 + 8\ln(t))}{64}$$

$$u_2 = \frac{t^8}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{t^7(-1 + 8\ln(t))}{64} + \frac{t^7\ln(t)}{8}$$

Which simplifies to

$$y_p(t) = \frac{t^7}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\ln(t) + c_2}{t}\right) + \left(\frac{t^7}{64}\right) \\ &= \frac{t^7}{64} + \frac{c_1\ln(t) + c_2}{t} \end{aligned}$$

Which simplifies to

$$y = \frac{t^7}{64} + \frac{c_1\ln(t) + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{t^7}{64} + \frac{c_1\ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{t^7}{64} + \frac{c_1\ln(t) + c_2}{t}$$

Verified OK.

### 5.1.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t $t$ gives

$$\int \left(t^2 y'' + 3y't + y\right) dt = \int t^7 dt$$
$$t^2 y' + ty = \frac{t^8}{8} + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{t^8 + 8c_1}{8t^2}$$

Hence the ode is

$$y' + \frac{y}{t} = \frac{t^8 + 8c_1}{8t^2}$$

The integrating factor $\mu$ is

$$\mu = e^{\int \frac{1}{t} dt}$$
$$= t$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu)\left(\frac{t^8 + 8c_1}{8t^2}\right)$$
$$\frac{d}{dt}(ty) = (t)\left(\frac{t^8 + 8c_1}{8t^2}\right)$$
$$d(ty) = \left(\frac{t^8 + 8c_1}{8t}\right) dt$$

Integrating gives

$$ty = \int \frac{t^8 + 8c_1}{8t} dt$$
$$ty = \frac{t^8}{64} + c_1 \ln(t) + c_2$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t)}{t} + \frac{c_2}{t}$$

which simplifies to

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t}$$

Verified OK.

### 5.1.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$t^2y'' + 3y't + y = t^7$$

Integrating both sides of the ODE w.r.t $t$ gives

$$\int (t^2y'' + 3y't + y)\, dt = \int t^7 dt$$
$$t^2y' + ty = \frac{t^8}{8} + c_1$$

Which is now solved for $y$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(t)y = q(t)$$

Where here

$$p(t) = \frac{1}{t}$$
$$q(t) = \frac{t^8 + 8c_1}{8t^2}$$

Hence the ode is

$$y' + \frac{y}{t} = \frac{t^8 + 8c_1}{8t^2}$$

The integrating factor $\mu$ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{t}dt} \\ &= t\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{t^8 + 8c_1}{8t^2}\right) \\ \frac{d}{dt}(ty) &= (t)\left(\frac{t^8 + 8c_1}{8t^2}\right) \\ d(ty) &= \left(\frac{t^8 + 8c_1}{8t}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}ty &= \int \frac{t^8 + 8c_1}{8t}\,dt \\ ty &= \frac{t^8}{64} + c_1 \ln(t) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t)}{t} + \frac{c_2}{t}$$

which simplifies to

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t}$$

Verified OK.

### 5.1.7 Solving using Kovacic algorithm

Writing the ode as

$$t^2y'' + 3y't + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= 3t \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then $y$ is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 130: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of $r$ in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at $\infty$ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of $r$ is

$$
r = -\frac{1}{4t^2}
$$

For the pole at $t = 0$ let $b$ be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of $r$ given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of $r$ at $\infty$ is 2 then $[\sqrt{r}]_\infty = 0$. Let $b$ be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of $r$ at $\infty$. which can be found by dividing the leading coefficient of $s$ by the leading coefficient of $t$ from

$$
r = \frac{s}{t} = -\frac{1}{4t^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of $r$ at $\infty$ where $r$ is

$$
r = -\frac{1}{4t^2}
$$

| pole $c$ location | pole order | $[\sqrt{r}]_c$ | $\alpha_c^+$ | $\alpha_c^-$ |
|---|---|---|---|---|
| 0 | 2 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |

| Order of $r$ at $\infty$ | $[\sqrt{r}]_\infty$ | $\alpha_\infty^+$ | $\alpha_\infty^-$ |
|---|---|---|---|
| 2 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set $\Gamma$ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer $d$ from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such $d$ is found to work in finding candidate $\omega$.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since $d$ an integer and $d \geq 0$ then it can be used to find $\omega$ using

$$\omega = \sum_{c \in \Gamma} \left( s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left( (-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2t} + (-)\,(0) \\ &= \frac{1}{2t} \\ &= \frac{1}{2t} \end{aligned}$$

Now that $\omega$ is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= \mathrm{e}^{\int \frac{1}{2t} dt} \\ &= \sqrt{t} \end{aligned}$$

The first solution to the original ode in $y$ is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3t}{t^2}\,dt} \\
&= z_1 e^{-\frac{3\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{t}$$

The second solution $y_2$ to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1(\ln(t))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{t}\right) + c_2\left(\frac{1}{t}(\ln(t))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$t^2 y'' + 3y't + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{t} + \frac{c_2 \ln(t)}{t}$$

The particular solution $y_p$ can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on $t$ as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where $u_1, u_2$ to be determined, and $y_1, y_2$ are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{t}$$
$$y_2 = \frac{\ln(t)}{t}$$

In the Variation of parameters $u_1, u_2$ are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and $a$ is the coefficient in front of $y''$ in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t} & \frac{\ln(t)}{t} \\ \frac{d}{dt}\left(\frac{1}{t}\right) & \frac{d}{dt}\left(\frac{\ln(t)}{t}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t} & \frac{\ln(t)}{t} \\ -\frac{1}{t^2} & -\frac{\ln(t)}{t^2} + \frac{1}{t^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t}\right)\left(-\frac{\ln(t)}{t^2} + \frac{1}{t^2}\right) - \left(\frac{\ln(t)}{t}\right)\left(-\frac{1}{t^2}\right)$$

Which simplifies to

$$W = \frac{1}{t^3}$$

Which simplifies to

$$W = \frac{1}{t^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{t^6 \ln(t)}{\frac{1}{t}} dt$$

Which simplifies to

$$u_1 = -\int t^7 \ln(t)\, dt$$

Hence

$$u_1 = -\frac{t^8 \ln(t)}{8} + \frac{t^8}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{t^6}{\frac{1}{t}} dt$$

Which simplifies to

$$u_2 = \int t^7 dt$$

Hence

$$u_2 = \frac{t^8}{8}$$

Which simplifies to

$$u_1 = -\frac{t^8(-1 + 8\ln(t))}{64}$$

$$u_2 = \frac{t^8}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\frac{t^7(-1+8\ln(t))}{64} + \frac{t^7\ln(t)}{8}$$

Which simplifies to

$$y_p(t) = \frac{t^7}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{t} + \frac{c_2\ln(t)}{t}\right) + \left(\frac{t^7}{64}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_2\ln(t) + c_1}{t} + \frac{t^7}{64}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2\ln(t) + c_1}{t} + \frac{t^7}{64} \tag{1}$$

Verification of solutions

$$y = \frac{c_2\ln(t) + c_1}{t} + \frac{t^7}{64}$$

Verified OK.

### 5.1.8 Solving as exact linear second order ode ode

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= t^2 \\
q(x) &= 3t \\
r(x) &= 1 \\
s(x) &= t^7
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 3
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (3) + (1) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)
$$

Integrating gives

$$
p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\; dt
$$

Substituting the above values for $p, q, r, s$ gives

$$
t^2 y' + ty = \int t^7\, dt
$$

We now have a first order ode to solve which is

$$
t^2 y' + ty = \frac{t^8}{8} + c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(t)y = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= \frac{1}{t} \\
q(t) &= \frac{t^8 + 8c_1}{8t^2}
\end{aligned}
$$

Hence the ode is

$$y' + \frac{y}{t} = \frac{t^8 + 8c_1}{8t^2}$$

The integrating factor $\mu$ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{t}dt} \\ &= t\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{t^8 + 8c_1}{8t^2}\right) \\ \frac{d}{dt}(ty) &= (t)\left(\frac{t^8 + 8c_1}{8t^2}\right) \\ d(ty) &= \left(\frac{t^8 + 8c_1}{8t}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}ty &= \int \frac{t^8 + 8c_1}{8t} dt \\ ty &= \frac{t^8}{64} + c_1 \ln(t) + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t$ results in

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t)}{t} + \frac{c_2}{t}$$

which simplifies to

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t}$$

Summary

The solution(s) found are the following

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t} \tag{1}$$

Verification of solutions

$$y = \frac{\frac{t^8}{64} + c_1 \ln(t) + c_2}{t}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(t^2*diff(y(t),t$2)+3*t*diff(y(t),t)+y(t)=t^7,y(t), singsol=all)
```

$$y(t) = \frac{c_2}{t} + \frac{t^7}{64} + \frac{c_1 \ln(t)}{t}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 26

```
DSolve[t^2*y''[t]+3*t*y'[t]+y[t]==t^7,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{t^8 + 64c_2 \log(t) + 64c_1}{64t}$$

## 5.2 problem Problem 1(b)

Internal problem ID [12352]
Internal file name [`OUTPUT/11004_Monday_October_02_2023_02_47_52_AM_91889946/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 1(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{t^2y'' - 6y't + \sin(2t)\, y = \ln(t)}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         -> trying with_periodic_functions in the coefficients
            --- Trying Lie symmetry methods, 2nd order ---
            `, `-> Computing symmetries using: way = 5
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         -> trying with_periodic_functions in the coefficients
            --- Trying Lie symmetry methods, 2nd order ---
            `, `-> Computing symmetries using: way = 5
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
```

✗ Solution by Maple

```
dsolve(t^2*diff(y(t),t$2)-6*t*diff(y(t),t)+sin(2*t)*y(t)=ln(t),y(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[t^2*y''[t]-6*t*y'[t]+Sin[2*t]*y[t]==Log[t],y[t],t,IncludeSingularSolutions -> True]
```

Not solved

## 5.3 problem Problem 1(c)

Internal problem ID [12353]
Internal file name [`OUTPUT/11005_Monday_October_02_2023_02_47_52_AM_16697868/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 1(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + 3y' + \frac{y}{t} = t}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 36

```
dsolve(diff(y(t),t$2)+3*diff(y(t),t)+y(t)/t=t,y(t), singsol=all)
```

$$y(t) = \frac{\left(7\,\mathrm{e}^{-3t}\,\mathrm{KummerU}\left(\frac{2}{3}, 2, 3t\right) c_1 + 7\,\mathrm{e}^{-3t}\,\mathrm{KummerM}\left(\frac{2}{3}, 2, 3t\right) c_2 + t - \frac{1}{2}\right) t}{7}$$

✓ Solution by Mathematica

Time used: 23.552 (sec). Leaf size: 253

```
DSolve[y''[t]+3*y'[t]+y[t]/t==t,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to G_{1,2}^{2,0}\left(3t \middle| \begin{array}{c} \frac{2}{3} \\ 0,1 \end{array}\right)\left(\int_1^t -\frac{3\,\text{Hypergeometric1F1}\left(\frac{4}{3},2,-3\ldots\right.}{3\,\text{Hypergeometric1F1}\left(\frac{4}{3},2,-3K[2]\right)G_{1,2}^{2,0}\left(3K[2] \middle| \begin{array}{c} \frac{2}{3} \\ 0,1 \end{array}\right)+3\,\text{Hypergeometric1F1}\left(\frac{4}{3},2,-3K[2]\right)G\ldots} + c_2\right) - 3t\,\text{Hypergeometric1F1}\left(\frac{4}{3},2,-3t\right)\left(\int_1^t \frac{G_{1,2}^{2,0}\left(3K\ldots\right.}{-9\,\text{Hypergeometric1F1}\left(\frac{4}{3},2,-3K[1]\right)G_{1,2}^{2,0}\left(3K[1] \middle| \begin{array}{c} \frac{2}{3} \\ 0,1 \end{array}\right)-9\,\text{Hypergeometric1F1}\left(\frac{4}{3},2,\ldots\right.} + c_1\right)$$

## 5.4 problem Problem 1(d)

Internal problem ID [12354]
Internal file name [`OUTPUT/11006_Monday_October_02_2023_02_47_52_AM_14110659/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 1(d).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{y'' + y't - y\ln(t) = \cos(2t)}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
         trying Riccati sub-methods:
```

✗ Solution by Maple

```
dsolve(diff(y(t),t$2)+t*diff(y(t),t)-y(t)*ln(t)=cos(2*t),y(t), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y''[t]+t*y'[t]-y[t]*Log[t]==Cos[2*t],y[t],t,IncludeSingularSolutions -> True]
```

Not solved

## 5.5 problem Problem 1(e)

Internal problem ID [12355]
Internal file name [`OUTPUT/11007_Monday_October_02_2023_02_47_52_AM_52665797/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 1(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

Unable to solve or complete the solution.

$$\boxed{t^3y'' - 2y't + y = t^4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 114

```
dsolve(t^3*diff(y(t),t$2)-2*t*diff(y(t),t)+y(t)=t^4,y(t), singsol=all)
```

$$y(t) = -\left(\left(-\operatorname{BesselI}\left(0, \frac{1}{t}\right) - \operatorname{BesselI}\left(1, \frac{1}{t}\right)\right)\left(\int t\left(\operatorname{BesselK}\left(0, \frac{1}{t}\right) - \operatorname{BesselK}\left(1, \frac{1}{t}\right)\right) e^{\frac{1}{t}} dt\right) + \left(\int t\left(\operatorname{BesselI}\left(0, \frac{1}{t}\right) + \operatorname{BesselI}\left(1, \frac{1}{t}\right)\right) e^{\frac{1}{t}} dt\right)\left(\operatorname{BesselK}\left(0, \frac{1}{t}\right) - \operatorname{BesselK}\left(1, \frac{1}{t}\right)\right) - \operatorname{BesselK}\left(0, \frac{1}{t}\right) c_1 + \operatorname{BesselK}\left(1, \frac{1}{t}\right) c_1 - \operatorname{BesselI}\left(0, \frac{1}{t}\right) c_2 - \operatorname{BesselI}\left(1, \frac{1}{t}\right) c_2\right) e^{-\frac{1}{t}}$$

✓ Solution by Mathematica

Time used: 27.071 (sec). Leaf size: 272

```
DSolve[t^3*y''[t]-2*t*y'[t]+y[t]==t^4,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-1/t}\left(\text{BesselI}\left(0,\frac{1}{t}\right)\right.$$

$$\left.+\text{BesselI}\left(1,\frac{1}{t}\right)\right)\left(\int_1^t \frac{2e^{\frac{2}{K[1]}}\sqrt{\pi}K[1]^3G_{1,2}^{2,0}\left(\frac{2}{K[1]}\middle| \begin{array}{c} \frac{1}{2} \\ -1,0 \end{array}\right)}{e^{\frac{1}{K[1]}}\sqrt{\pi}\left(\text{BesselI}\left(0,\frac{1}{K[1]}\right)-\text{BesselI}\left(2,\frac{1}{K[1]}\right)\right)G_{1,2}^{2,0}\left(\frac{2}{K[1]}\middle| \begin{array}{c} \frac{1}{2} \\ -1,0 \end{array}\right)-2\left(\text{Bes}\right.}\right.$$

## 5.6 problem Problem 2(a)



Internal problem ID [12356]
Internal file name [`OUTPUT/11008_Monday_October_02_2023_02_47_53_AM_83656395/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 2(a).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 1}$$

### 5.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 2, C = 1, f(t) = 1$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-t} + c_2 t\, e^{-t} \tag{1}$$

Therefore the homogeneous solution $y_h$ is

$$y_h = c_1 e^{-t} + c_2 t\, e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, e^{-t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t}\right) + (1) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-t}(c_2 t + c_1) + 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-t}(c_2 t + c_1) + 1 \tag{1}$$

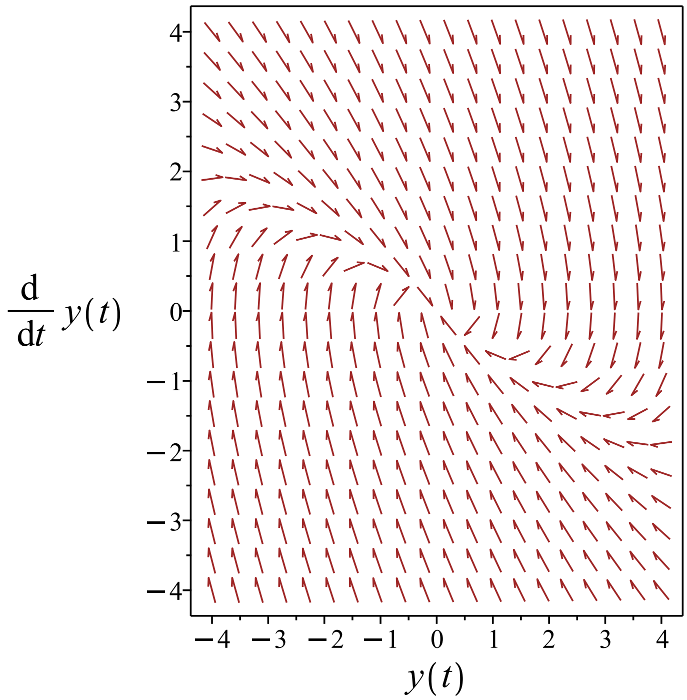


Figure 102: Slope field plot

Verification of solutions

$$y = e^{-t}(c_2 t + c_1) + 1$$

Verified OK.

### 5.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(t)\, y' + \frac{(p(t)^2 + p'(t))\, y}{2} = f(t)$$

Where $p(t) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^t$$
$$(e^t y)'' = e^t$$

Integrating once gives

$$(e^t y)' = e^t + c_1$$

Integrating again gives

$$(e^t y) = c_1 t + e^t + c_2$$

Hence the solution is

$$y = \frac{c_1 t + e^t + c_2}{e^t}$$

Or

$$y = t\,e^{-t} c_1 + c_2 e^{-t} + 1$$

<u>Summary</u>

The solution(s) found are the following

$$y = t\,e^{-t} c_1 + c_2 e^{-t} + 1 \tag{1}$$

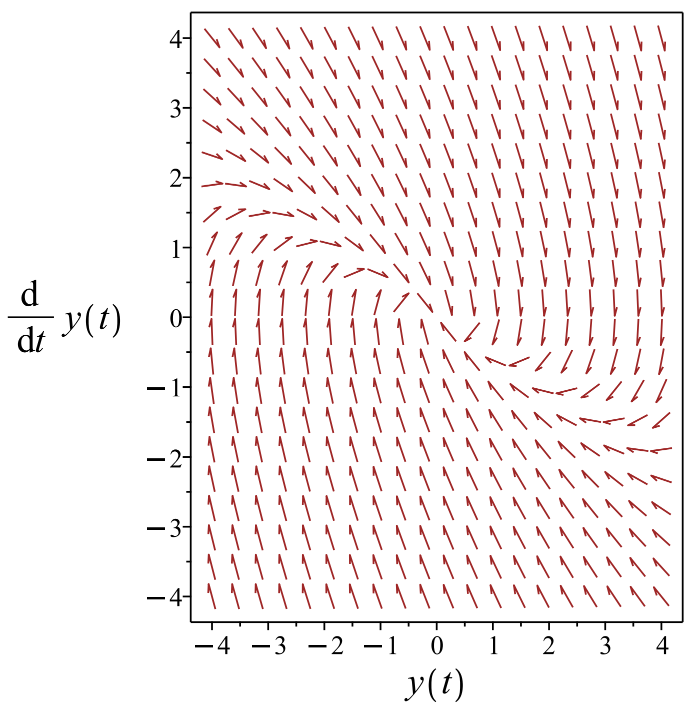



Figure 103: Slope field plot

Verification of solutions

$$y = t\,\mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-t} + 1$$

Verified OK.

### 5.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then $y$ is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 131: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of $t$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1\left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-t}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-2t}}{(y_1)^2}\,dt \\
&= y_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-t}\right) + c_2\left(\mathrm{e}^{-t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-t} + c_2 t \, e^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t \, e^{-t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-t} + c_2 t \, e^{-t}\right) + (1) \end{aligned}$$

Which simplifies to

$$y = e^{-t}(c_2 t + c_1) + 1$$

Summary

The solution(s) found are the following

$$y = e^{-t}(c_2 t + c_1) + 1 \tag{1}$$

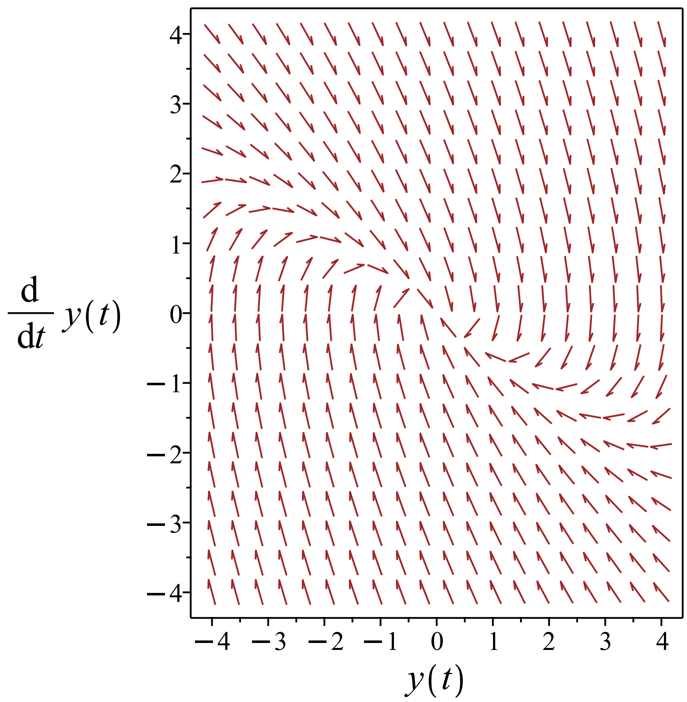



Figure 104: Slope field plot

Verification of solutions

$$y = e^{-t}(c_2 t + c_1) + 1$$

Verified OK.

### 5.6.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 1$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-t}$

- Repeated root, multiply $y_1(t)$ by $t$ to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-t} & t\,\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & \mathrm{e}^{-t} - t\,\mathrm{e}^{-t} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \mathrm{e}^{-t}\left(-\left(\int t\,\mathrm{e}^{t} dt\right) + \left(\int \mathrm{e}^{t} dt\right) t\right)$

- Compute integrals

  $y_p(t) = 1$

- Substitute particular solution into general solution to ODE

  $y = c_2 t\,e^{-t} + c_1 e^{-t} + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(t),t$2)+2*diff(y(t),t)+y(t)=1,y(t), singsol=all)
```

$$y(t) = 1 + (c_1 t + c_2)\,e^{-t}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 21

```
DSolve[y''[t]+2*y'[t]+y[t]==1,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}\left(e^t + c_2 t + c_1\right)$$

## 5.7 problem Problem 2(b)



Internal problem ID [12357]
Internal file name [`OUTPUT/11009_Monday_October_02_2023_02_47_54_AM_55377506/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 2(b).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' + 5y = \mathrm{e}^{t}}$$

### 5.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -2, C = 5, f(t) = \mathrm{e}^{t}$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$y'' - 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 2\lambda\, e^{\lambda t} + 5\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)}\\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Therefore the homogeneous solution $y_h$ is

$$y_h = e^{t}(c_1 \cos(2t) + c_2 \sin(2t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^t$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^t\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t \cos(2t), e^t \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^t$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^t = e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = \frac{e^t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^t (c_1 \cos(2t) + c_2 \sin(2t))\right) + \left(\frac{e^t}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^t (c_1 \cos(2t) + c_2 \sin(2t)) + \frac{e^t}{4} \tag{1}$$

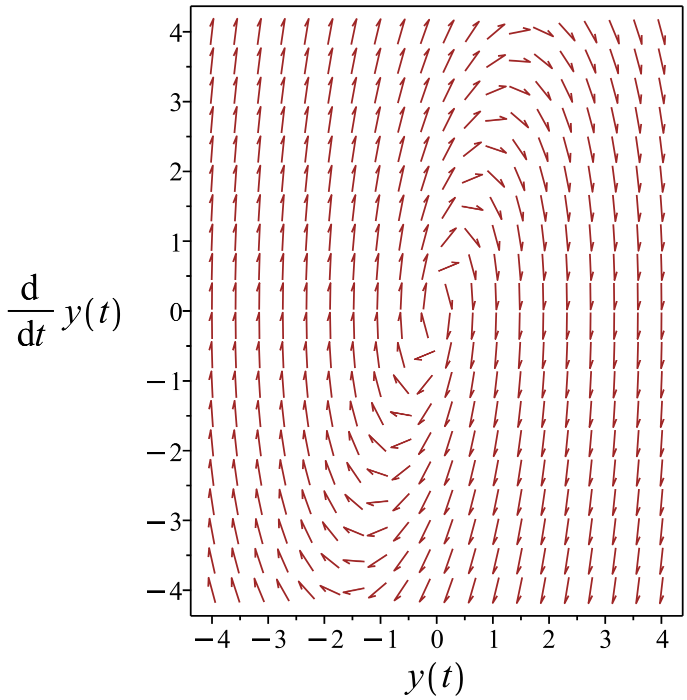



Figure 105: Slope field plot

Verification of solutions

$$y = e^t(c_1 \cos(2t) + c_2 \sin(2t)) + \frac{e^t}{4}$$

Verified OK.

### 5.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then $y$ is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 133: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of $t$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(2t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dt} \\
&= z_1 e^{t} \\
&= z_1\left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{t}\cos(2t)
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{2t}}{(y_1)^2}\,dt \\
&= y_1\left(\frac{\tan(2t)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^t \cos(2t)\right) + c_2 \left(e^t \cos(2t) \left(\frac{\tan(2t)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$
y'' - 2y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = e^t \cos(2t) c_1 + \frac{e^t \sin(2t) c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^t
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^t\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{e^t \cos(2t), \frac{e^t \sin(2t)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^t
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^t = e^t$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = \frac{e^t}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^t \cos(2t)\, c_1 + \frac{e^t \sin(2t)\, c_2}{2}\right) + \left(\frac{e^t}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^t \cos(2t)\, c_1 + \frac{e^t \sin(2t)\, c_2}{2} + \frac{e^t}{4} \tag{1}$$

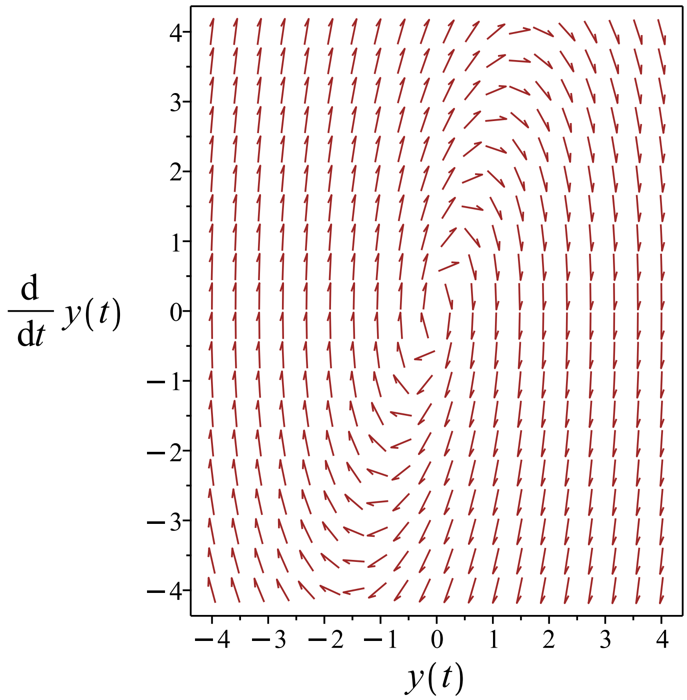



Figure 106: Slope field plot

Verification of solutions

$$y = e^{t} \cos(2t) c_1 + \frac{e^{t} \sin(2t) c_2}{2} + \frac{e^{t}}{4}$$

Verified OK.

### 5.7.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = e^{t}$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^t \cos(2t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^t \sin(2t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^t \cos(2t)\, c_1 + \mathrm{e}^t \sin(2t)\, c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t)\left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = \mathrm{e}^t\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^t \cos(2t) & \mathrm{e}^t \sin(2t) \\ \mathrm{e}^t \cos(2t) - 2\,\mathrm{e}^t \sin(2t) & \mathrm{e}^t \sin(2t) + 2\,\mathrm{e}^t \cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 2\,\mathrm{e}^{2t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\frac{\mathrm{e}^t \left(\cos(2t)\left(\int \sin(2t) dt\right) - \sin(2t)\left(\int \cos(2t) dt\right)\right)}{2}$

  - Compute integrals

    $y_p(t) = \frac{\mathrm{e}^t}{4}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^t \cos(2t)\, c_1 + \mathrm{e}^t \sin(2t)\, c_2 + \frac{\mathrm{e}^t}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(t),t$2)-2*diff(y(t),t)+5*y(t)=exp(t),y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{t}\left(\frac{1}{4}+\sin\left(2t\right)c_2+\cos\left(2t\right)c_1\right)$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 33

```
DSolve[y''[t]-2*y'[t]+5*y[t]==Exp[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{4}e^{t}((1+4c_2)\cos(2t)+4c_1\sin(2t)+1)$$

## 5.8 problem Problem 2(c)



Internal problem ID [12358]
Internal file name [`OUTPUT/11010_Monday_October_02_2023_02_47_57_AM_27011075/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 2(c).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' - 7y = 4}$$

### 5.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = -3, C = -7, f(t) = 4$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$y'' - 3y' - 7y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = -3, C = -7$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} - 3\lambda\, e^{\lambda t} - 7\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda - 7 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = -7$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(-7)} \\
&= \frac{3}{2} \pm \frac{\sqrt{37}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{\sqrt{37}}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{\sqrt{37}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{\sqrt{37}}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{\sqrt{37}}{2}
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
y &= c_1 e^{\left(\frac{3}{2}+\frac{\sqrt{37}}{2}\right)t} + c_2 e^{\left(\frac{3}{2}-\frac{\sqrt{37}}{2}\right)t}
\end{aligned}$$

Or

$$y = c_1 e^{\left(\frac{3}{2}+\frac{\sqrt{37}}{2}\right)t} + c_2 e^{\left(\frac{3}{2}-\frac{\sqrt{37}}{2}\right)t}$$

Therefore the homogeneous solution $y_h$ is

$$y_h = c_1 e^{\left(\frac{3}{2}+\frac{\sqrt{37}}{2}\right)t} + c_2 e^{\left(\frac{3}{2}-\frac{\sqrt{37}}{2}\right)t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\left(\frac{3}{2}-\frac{\sqrt{37}}{2}\right)t}, e^{\left(\frac{3}{2}+\frac{\sqrt{37}}{2}\right)t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{7}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -\frac{4}{7}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\left(\frac{3}{2}+\frac{\sqrt{37}}{2}\right)t} + c_2 e^{\left(\frac{3}{2}-\frac{\sqrt{37}}{2}\right)t}\right) + \left(-\frac{4}{7}\right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{\frac{\left(3+\sqrt{37}\right)t}{2}} + c_2 e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} - \frac{4}{7}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{\left(3+\sqrt{37}\right)t}{2}} + c_2 e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} - \frac{4}{7} \tag{1}$$

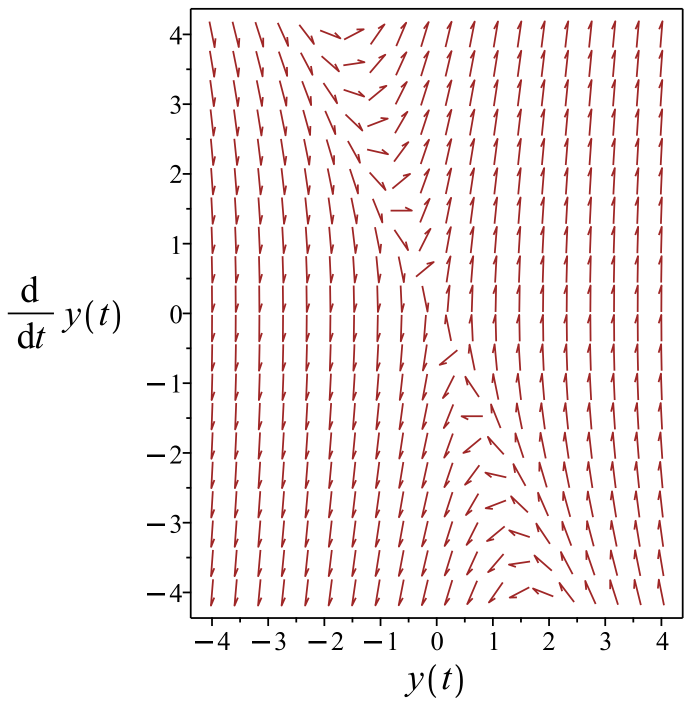



Figure 107: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{\left(3+\sqrt{37}\right)t}{2}} + c_2 e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} - \frac{4}{7}$$

Verified OK.

### 5.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' - 7y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= -7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{37}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 37 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{37z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then $y$ is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 135: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{37}{4}$ is not a function of $t$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t\sqrt{37}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\, dt} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{\left(-3+\sqrt{37}\right)t}{2}}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dt}}{\left(y_1\right)^2}\, dt \\
&= y_1 \int \frac{e^{3t}}{\left(y_1\right)^2}\, dt \\
&= y_1 \left(\frac{\sqrt{37}\, \mathrm{e}^{t\sqrt{37}}}{37}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{\left(-3+\sqrt{37}\right)t}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} \left(\frac{\sqrt{37}\, \mathrm{e}^{t\sqrt{37}}}{37}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$
y'' - 3y' - 7y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} + \frac{c_2 \sqrt{37}\, e^{\frac{\left(3+\sqrt{37}\right)t}{2}}}{37}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{37}\, e^{\frac{\left(3+\sqrt{37}\right)t}{2}}}{37}, e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{7}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -\frac{4}{7}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} + \frac{c_2 \sqrt{37}\, \mathrm{e}^{\frac{\left(3+\sqrt{37}\right)t}{2}}}{37}\right) + \left(-\frac{4}{7}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} + \frac{c_2 \sqrt{37}\, \mathrm{e}^{\frac{\left(3+\sqrt{37}\right)t}{2}}}{37} - \frac{4}{7} \tag{1}
$$

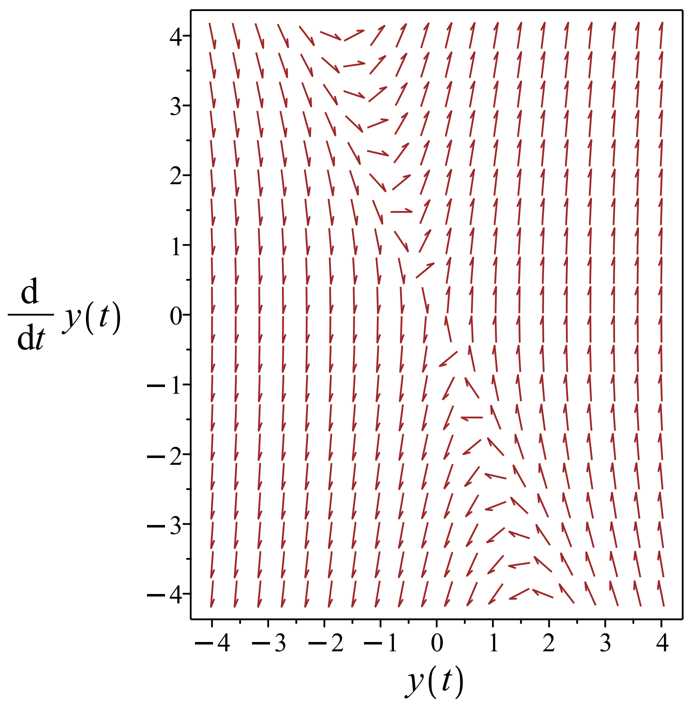



Figure 108: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} + \frac{c_2 \sqrt{37}\, \mathrm{e}^{\frac{\left(3+\sqrt{37}\right)t}{2}}}{37} - \frac{4}{7}
$$

Verified OK.

### 5.8.3 Maple step by step solution

Let's solve

$y'' - 3y' - 7y = 4$

- Highest derivative means the order of the ODE is 2

  $y''$

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r - 7 = 0$

- Use quadratic formula to solve for $r$

  $r = \frac{3 \pm (\sqrt{37})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{3}{2} - \frac{\sqrt{37}}{2}, \frac{3}{2} + \frac{\sqrt{37}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{37}}{2}\right)t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{37}}{2}\right)t}$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{37}}{2}\right)t} + c_2 \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{37}}{2}\right)t} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- ○ Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

  $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 4\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{37}}{2}\right)t} & \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{37}}{2}\right)t} \\ \left(\frac{3}{2} - \frac{\sqrt{37}}{2}\right) \mathrm{e}^{\left(\frac{3}{2} - \frac{\sqrt{37}}{2}\right)t} & \left(\frac{3}{2} + \frac{\sqrt{37}}{2}\right) \mathrm{e}^{\left(\frac{3}{2} + \frac{\sqrt{37}}{2}\right)t} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(t), y_2(t)) = \sqrt{37}\, \mathrm{e}^{3t}$

- Substitute functions into equation for $y_p(t)$

$$y_p(t) = \frac{4\sqrt{37}\left(-e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}}\left(\int e^{\frac{\left(-3+\sqrt{37}\right)t}{2}}dt\right)+e^{\frac{\left(3+\sqrt{37}\right)t}{2}}\left(\int e^{-\frac{\left(3+\sqrt{37}\right)t}{2}}dt\right)\right)}{37}$$

- Compute integrals

$$y_p(t) = -\frac{4}{7}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{\left(\frac{3}{2}-\frac{\sqrt{37}}{2}\right)t} + c_2 e^{\left(\frac{3}{2}+\frac{\sqrt{37}}{2}\right)t} - \frac{4}{7}$$

## Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

## ✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(t),t$2)-3*diff(y(t),t)-7*y(t)=4,y(t), singsol=all)
```

$$y(t) = e^{\frac{\left(3+\sqrt{37}\right)t}{2}} c_2 + e^{-\frac{\left(-3+\sqrt{37}\right)t}{2}} c_1 - \frac{4}{7}$$

## ✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 43

```
DSolve[y''[t]-3*y'[t]-7*y[t]==4,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to c_1 e^{-\frac{1}{2}\left(\sqrt{37}-3\right)t} + c_2 e^{\frac{1}{2}\left(3+\sqrt{37}\right)t} - \frac{4}{7}$$

## 5.9 problem Problem 2(d)

Internal problem ID [12359]
Internal file name [`OUTPUT/11011_Monday_October_02_2023_02_48_01_AM_59068219/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 2(d).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' + 3y' + y = 5}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE And $y_p$ is a particular solution to the nonhomogeneous ODE. $y_h$ is the solution to

$$y''' + 3y'' + 3y' + y = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(t) = c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t} + t^2 \mathrm{e}^{-t} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-t} \\
y_2 &= t\,\mathrm{e}^{-t} \\
y_3 &= t^2 \mathrm{e}^{-t}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + 3y'' + 3y' + y = 5
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t\,\mathrm{e}^{-t}, t^2\mathrm{e}^{-t}, \mathrm{e}^{-t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
A_1 = 5
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = 5]
$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$
y_p = 5
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-t} + c_2 t\, \mathrm{e}^{-t} + t^2 \mathrm{e}^{-t} c_3\right) + (5)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-t}\left(c_3 t^2 + c_2 t + c_1\right) + 5
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-t}\left(c_3 t^2 + c_2 t + c_1\right) + 5 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-t}\left(c_3 t^2 + c_2 t + c_1\right) + 5
$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(t),t$3)+3*diff(y(t),t$2)+3*diff(y(t),t)+y(t)=5,y(t), singsol=all)
```

$$
y(t) = 5 + \left(c_3 t^2 + c_2 t + c_1\right) \mathrm{e}^{-t}
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 28

```
DSolve[y'''[t]+3*y''[t]+3*y'[t]+y[t]==5,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to e^{-t}\left(5e^t + t(c_3 t + c_2) + c_1\right)$$

## 5.10 problem Problem 2(e)



Internal problem ID [12360]
Internal file name [`OUTPUT/11012_Monday_October_02_2023_02_48_01_AM_54245314/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 2(e).
**ODE order**: 2.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3y'' + 5y' - 2y = 3t^2}$$

### 5.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 3, B = 5, C = -2, f(t) = 3t^2$. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$3y'' + 5y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 3, B = 5, C = -2$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$3\lambda^2 \mathrm{e}^{\lambda t} + 5\lambda\, \mathrm{e}^{\lambda t} - 2\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$3\lambda^2 + 5\lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = 5, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(3)} \pm \frac{1}{(2)\,(3)}\sqrt{5^2 - (4)\,(3)\,(-2)}\\ &= -\frac{5}{6} \pm \frac{7}{6}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{5}{6} + \frac{7}{6}$$

$$\lambda_2 = -\frac{5}{6} - \frac{7}{6}$$

Which simplifies to

$$\lambda_1 = \frac{1}{3}$$

$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$y = c_1 e^{\left(\frac{1}{3}\right)t} + c_2 e^{(-2)t}$$

Or

$$y = c_1 \mathrm{e}^{\frac{t}{3}} + c_2 \mathrm{e}^{-2t}$$

Therefore the homogeneous solution $y_h$ is

$$y_h = c_1 \mathrm{e}^{\frac{t}{3}} + c_2 \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$t^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, t, t^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2t}, e^{\frac{t}{3}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 t^2 + A_2 t + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3 t^2 - 2A_2 t + 10tA_3 - 2A_1 + 5A_2 + 6A_3 = 3t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{93}{4}, A_2 = -\frac{15}{2}, A_3 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -\frac{3}{2}t^2 - \frac{15}{2}t - \frac{93}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{t}{3}} + c_2 e^{-2t}\right) + \left(-\frac{3}{2}t^2 - \frac{15}{2}t - \frac{93}{4}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{\frac{t}{3}} + c_2 e^{-2t} - \frac{3t^2}{2} - \frac{15t}{2} - \frac{93}{4} \tag{1}$$

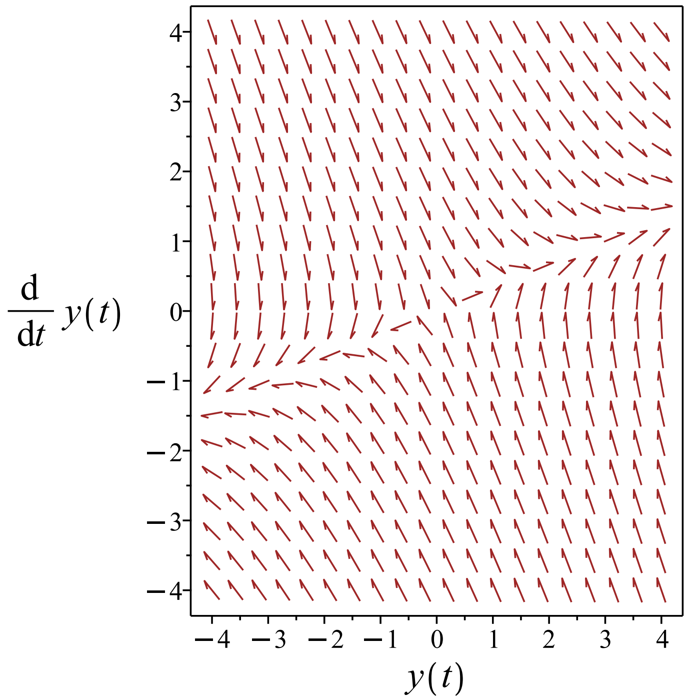



Figure 109: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{t}{3}} + c_2 \mathrm{e}^{-2t} - \frac{3t^2}{2} - \frac{15t}{2} - \frac{93}{4}$$

Verified OK.

### 5.10.2 Solving using Kovacic algorithm

Writing the ode as

$$3y'' + 5y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= 5 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where $r$ is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of $A, B, C$ from (3) in the above and simplifying gives

$$r = \frac{49}{36} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 36 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{49z(t)}{36} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then $y$ is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of $r$ and the order of $r$ at $\infty$. The following table summarizes these cases.

| Case | Allowed pole order for $r$ | Allowed value for $\mathcal{O}(\infty)$ |
|---|---|---|
| 1 | $\{0, 1, 2, 4, 6, 8, \cdots\}$ | $\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$ |
| 2 | Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$. | no condition |
| 3 | $\{1, 2\}$ | $\{2, 3, 4, 5, 6, 7, \cdots\}$ |

Table 137: Necessary conditions for each Kovacic case

The order of $r$ at $\infty$ is the degree of $t$ minus the degree of $s$. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in $r$. Therefore the set of poles $\Gamma$ is empty. Since there is no odd order pole larger than 2 and the order at $\infty$ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{49}{36}$ is not a function of $t$, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{7t}{6}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in $y$ is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{3}\,dt} \\
&= z_1 e^{-\frac{5t}{6}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5t}{6}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2t}
$$

The second solution $y_2$ to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{3}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{-\frac{5t}{3}}}{(y_1)^2}\,dt \\
&= y_1 \left(\frac{3\,\mathrm{e}^{\frac{7t}{3}}}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t} \left(\frac{3\,\mathrm{e}^{\frac{7t}{3}}}{7}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where $y_h$ is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and $y_p$ is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. $y_h$ is the solution to

$$
3y'' + 5y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2t} + \frac{3c_2 \mathrm{e}^{\frac{t}{3}}}{7}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
t^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, t, t^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{3\,\mathrm{e}^{\frac{t}{3}}}{7}, \mathrm{e}^{-2t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 t^2 + A_2 t + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3t^2 - 2A_2t + 10tA_3 - 2A_1 + 5A_2 + 6A_3 = 3t^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{93}{4}, A_2 = -\frac{15}{2}, A_3 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -\frac{3}{2}t^2 - \frac{15}{2}t - \frac{93}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2t} + \frac{3c_2 e^{\frac{t}{3}}}{7}\right) + \left(-\frac{3}{2}t^2 - \frac{15}{2}t - \frac{93}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2t} + \frac{3c_2 e^{\frac{t}{3}}}{7} - \frac{3t^2}{2} - \frac{15t}{2} - \frac{93}{4} \tag{1}$$

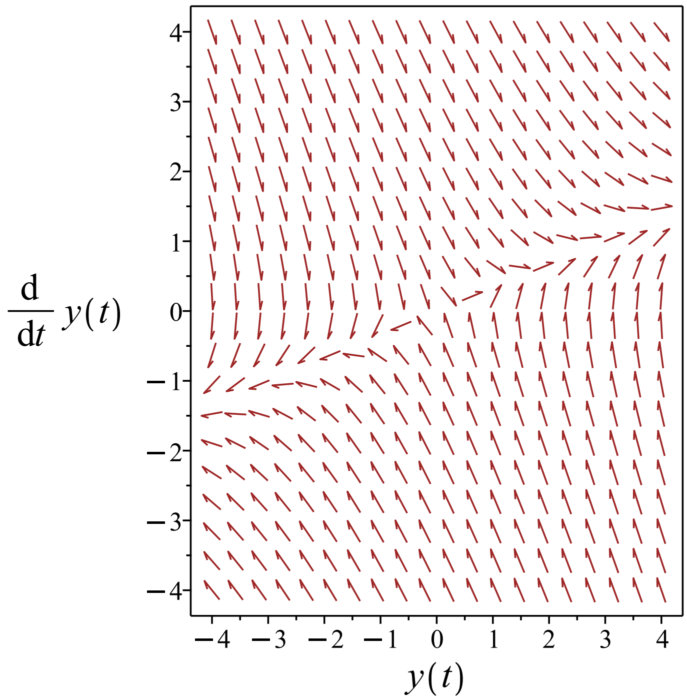



Figure 110: Slope field plot

Verification of solutions

$$y = c_1 e^{-2t} + \frac{3c_2 e^{\frac{t}{3}}}{7} - \frac{3t^2}{2} - \frac{15t}{2} - \frac{93}{4}$$

Verified OK.

### 5.10.3 Maple step by step solution

Let's solve

$3y'' + 5y' - 2y = 3t^2$

- Highest derivative means the order of the ODE is 2

  $y''$

- Isolate 2nd derivative

  $y'' = -\frac{5y'}{3} + \frac{2y}{3} + t^2$

- Group terms with $y$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{5y'}{3} - \frac{2y}{3} = t^2$

- Characteristic polynomial of homogeneous ODE

$r^2 + \frac{5}{3}r - \frac{2}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(r+2)(3r-1)}{3} = 0$

- Roots of the characteristic polynomial

  $r = \left(-2, \frac{1}{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the homogeneous ODE

  $y_2(t) = e^{\frac{t}{3}}$

- General solution of the ODE

  $y = c_1y_1(t) + c_2y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1e^{-2t} + c_2e^{\frac{t}{3}} + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find $y_p$ here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = t^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} e^{-2t} & e^{\frac{t}{3}} \\ -2e^{-2t} & \frac{e^{\frac{t}{3}}}{3} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \frac{7e^{-\frac{5t}{3}}}{3}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = \frac{3\left(e^{\frac{7t}{3}}\left(\int t^2e^{-\frac{t}{3}}dt\right) - \left(\int e^{2t}t^2dt\right)\right)e^{-2t}}{7}$

  - Compute integrals

    $y_p(t) = -\frac{3}{2}t^2 - \frac{15}{2}t - \frac{93}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-2t} + c_2e^{\frac{t}{3}} - \frac{3t^2}{2} - \frac{15t}{2} - \frac{93}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(3*diff(y(t),t$2)+5*diff(y(t),t)-2*y(t)=3*t^2,y(t), singsol=all)
```

$$y(t) = -\frac{3\,\mathrm{e}^{-2t}\left(-\frac{2\,\mathrm{e}^{\frac{7t}{3}}c_1}{3}+\left(t^2+5t+\frac{31}{2}\right)\mathrm{e}^{2t}-\frac{2c_2}{3}\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 38

```
DSolve[3*y''[t]+5*y'[t]-2*y[t]==3*t^2,y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to -\frac{3}{4}\left(2t^2+10t+31\right)+c_1e^{t/3}+c_2e^{-2t}$$

## 5.11 problem Problem 2(f)



Internal problem ID [12361]
Internal file name [`OUTPUT/11013_Monday_October_02_2023_02_48_03_AM_40015569/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 2(f).
**ODE order**: 3.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 2y'' + 4y' = \sin(t)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where $y_h$ is the solution to the homogeneous ODE And $y_p$ is a particular solution to the nonhomogeneous ODE. $y_h$ is the solution to

$$y''' - 2y'' + 4y' = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1 + i\sqrt{3}$$
$$\lambda_3 = 1 - i\sqrt{3}$$

Therefore the homogeneous solution is

$$y_h(t) = c_1 + e^{\left(1+i\sqrt{3}\right)t} c_2 + e^{\left(1-i\sqrt{3}\right)t} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = e^{\left(1+i\sqrt{3}\right)t}$$

$$y_3 = e^{\left(1-i\sqrt{3}\right)t}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' + 4y' = \sin(t)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, e^{\left(1-i\sqrt{3}\right)t}, e^{\left(1+i\sqrt{3}\right)t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution $y_p$ into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \sin(t) + 3A_2 \cos(t) + 2A_1 \cos(t) + 2A_2 \sin(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{13}, A_2 = \frac{2}{13}\right]$$

Substituting the above back in the above trial solution $y_p$, gives the particular solution

$$y_p = -\frac{3\cos(t)}{13} + \frac{2\sin(t)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \mathrm{e}^{\left(1+i\sqrt{3}\right)t} c_2 + \mathrm{e}^{\left(1-i\sqrt{3}\right)t} c_3\right) + \left(-\frac{3\cos(t)}{13} + \frac{2\sin(t)}{13}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \mathrm{e}^{\left(1+i\sqrt{3}\right)t} c_2 + \mathrm{e}^{\left(1-i\sqrt{3}\right)t} c_3 - \frac{3\cos(t)}{13} + \frac{2\sin(t)}{13} \tag{1}$$

Verification of solutions

$$y = c_1 + \mathrm{e}^{\left(1+i\sqrt{3}\right)t} c_2 + \mathrm{e}^{\left(1-i\sqrt{3}\right)t} c_3 - \frac{3\cos(t)}{13} + \frac{2\sin(t)}{13}$$

Verified OK.

### 5.11.1 Maple step by step solution

Let's solve

$y''' - 2y'' + 4y' = \sin(t)$

- Highest derivative means the order of the ODE is 3

  $y'''$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(t)$

    $y_1(t) = y$
  - Define new variable $y_2(t)$

    $y_2(t) = y'$
  - Define new variable $y_3(t)$

    $y_3(t) = y''$
  - Isolate for $y_3'(t)$ using original ODE

$y_3'(t) = \sin(t) + 2y_3(t) - 4y_2(t)$

Convert linear ODE into a system of first order ODEs

$[y_2(t) = y_1'(t), y_3(t) = y_2'(t), y_3'(t) = \sin(t) + 2y_3(t) - 4y_2(t)]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 2 \end{bmatrix} \cdot \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ \sin(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \sin(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(t) = A \cdot \vec{y}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - I\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-I\sqrt{3}\right)^2} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(1-I\sqrt{3}\right)t} \cdot \begin{bmatrix} \frac{1}{\left(1-I\sqrt{3}\right)^2} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^t \cdot \left(\cos\left(\sqrt{3}\,t\right) - I\sin\left(\sqrt{3}\,t\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(1-I\sqrt{3}\right)^2} \\ \frac{1}{1-I\sqrt{3}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^t \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{3}\,t\right) - I\sin\left(\sqrt{3}\,t\right)}{\left(1-I\sqrt{3}\right)^2} \\ \frac{\cos\left(\sqrt{3}\,t\right) - I\sin\left(\sqrt{3}\,t\right)}{1-I\sqrt{3}} \\ \cos\left(\sqrt{3}\,t\right) - I\sin\left(\sqrt{3}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(t) = e^t \cdot \begin{bmatrix} -\frac{\cos(\sqrt{3}\,t)}{8} + \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{8} \\ \frac{\cos(\sqrt{3}\,t)}{4} + \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{4} \\ \cos(\sqrt{3}\,t) \end{bmatrix}, \vec{y}_3(t) = e^t \cdot \begin{bmatrix} \frac{\cos(\sqrt{3}\,t)\sqrt{3}}{8} + \frac{\sin(\sqrt{3}\,t)}{8} \\ \frac{\cos(\sqrt{3}\,t)\sqrt{3}}{4} - \frac{\sin(\sqrt{3}\,t)}{4} \\ -\sin(\sqrt{3}\,t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($
  $\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + \vec{y}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} 1 & e^t\left(-\frac{\cos(\sqrt{3}\,t)}{8} + \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{8}\right) & e^t\left(\frac{\cos(\sqrt{3}\,t)\sqrt{3}}{8} + \frac{\sin(\sqrt{3}\,t)}{8}\right) \\ 0 & e^t\left(\frac{\cos(\sqrt{3}\,t)}{4} + \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{4}\right) & e^t\left(\frac{\cos(\sqrt{3}\,t)\sqrt{3}}{4} - \frac{\sin(\sqrt{3}\,t)}{4}\right) \\ 0 & e^t\cos(\sqrt{3}\,t) & -e^t\sin(\sqrt{3}\,t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th
  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 1 & e^t\left(-\frac{\cos(\sqrt{3}\,t)}{8} + \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{8}\right) & e^t\left(\frac{\cos(\sqrt{3}\,t)\sqrt{3}}{8} + \frac{\sin(\sqrt{3}\,t)}{8}\right) \\ 0 & e^t\left(\frac{\cos(\sqrt{3}\,t)}{4} + \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{4}\right) & e^t\left(\frac{\cos(\sqrt{3}\,t)\sqrt{3}}{4} - \frac{\sin(\sqrt{3}\,t)}{4}\right) \\ 0 & e^t\cos(\sqrt{3}\,t) & -e^t\sin(\sqrt{3}\,t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{1}{8} & \frac{\sqrt{3}}{8} \\ 0 & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} 1 & \frac{e^t\cos(\sqrt{3}\,t)}{2} + \frac{e^t\sin(\sqrt{3}\,t)\sqrt{3}}{6} - \frac{1}{2} & -\frac{e^t\cos(\sqrt{3}\,t)}{4} + \frac{e^t\sin(\sqrt{3}\,t)\sqrt{3}}{12} + \frac{1}{4} \\ 0 & \frac{e^t\left(\cos(\sqrt{3}\,t)\sqrt{3} - \sin(\sqrt{3}\,t)\right)\sqrt{3}}{3} & \frac{e^t\sin(\sqrt{3}\,t)\sqrt{3}}{3} \\ 0 & -\frac{4\,e^t\sin(\sqrt{3}\,t)\sqrt{3}}{3} & \frac{e^t\left(\sin(\sqrt{3}\,t)\sqrt{3} + 3\cos(\sqrt{3}\,t)\right)}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{y}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{y}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{y}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(t) = \begin{bmatrix} \frac{2\sin(t)}{13} - \frac{3\cos(t)}{13} - \frac{7\,e^t \sin\left(\sqrt{3}\,t\right)\sqrt{3}}{156} - \frac{e^t \cos\left(\sqrt{3}\,t\right)}{52} + \frac{1}{4} \\ -\frac{2\,e^t \cos\left(\sqrt{3}\,t\right)}{13} + \frac{3\sin(t)}{13} - \frac{e^t \sin\left(\sqrt{3}\,t\right)\sqrt{3}}{39} + \frac{2\cos(t)}{13} \\ \frac{5\,e^t \sin\left(\sqrt{3}\,t\right)\sqrt{3}}{39} - \frac{3\,e^t \cos\left(\sqrt{3}\,t\right)}{13} - \frac{2\sin(t)}{13} + \frac{3\cos(t)}{13} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2(t) + c_3\vec{y}_3(t) + \begin{bmatrix} \frac{2\sin(t)}{13} - \frac{3\cos(t)}{13} - \frac{7\,e^t \sin\left(\sqrt{3}\,t\right)\sqrt{3}}{156} - \frac{e^t \cos\left(\sqrt{3}\,t\right)}{52} + \frac{1}{4} \\ -\frac{2\,e^t \cos\left(\sqrt{3}\,t\right)}{13} + \frac{3\sin(t)}{13} - \frac{e^t \sin\left(\sqrt{3}\,t\right)\sqrt{3}}{39} + \frac{2\cos(t)}{13} \\ \frac{5\,e^t \sin\left(\sqrt{3}\,t\right)\sqrt{3}}{39} - \frac{3\,e^t \cos\left(\sqrt{3}\,t\right)}{13} - \frac{2\sin(t)}{13} + \frac{3\cos(t)}{13} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^t\left(-c_3\sqrt{3}+c_2+\frac{2}{13}\right)\cos\left(\sqrt{3}\,t\right)}{8} + \frac{\left(\left(c_2-\frac{14}{39}\right)\sqrt{3}+c_3\right)e^t \sin\left(\sqrt{3}\,t\right)}{8} + c_1 - \frac{3\cos(t)}{13} + \frac{2\sin(t)}{13} + \frac{1}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 2*(diff(_b(_a), _a))-4*_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(diff(y(t),t$3)=2*diff(y(t),t$2)-4*diff(y(t),t)+sin(t),y(t), singsol=all)
```

$$y(t) = \frac{e^{t}\left(-c_2\sqrt{3}+c_1\right)\cos\left(\sqrt{3}\,t\right)}{4}+\frac{e^{t}\left(\sqrt{3}\,c_1+c_2\right)\sin\left(\sqrt{3}\,t\right)}{4}+c_3-\frac{3\cos\left(t\right)}{13}+\frac{2\sin\left(t\right)}{13}$$

✓ Solution by Mathematica

Time used: 1.636 (sec). Leaf size: 82

```
DSolve[y'''[t]==2*y''[t]-4*y'[t]+Sin[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \frac{1}{52}\left(8\sin(t)-12\cos(t)-13\left(\sqrt{3}c_1-c_2\right)e^{t}\cos\left(\sqrt{3}t\right)+13c_1e^{t}\sin\left(\sqrt{3}t\right)+13\sqrt{3}c_2e^{t}\sin\left(\sqrt{3}t\right)\right)+c_3$$

## 5.12 problem Problem 3(a)



Internal problem ID [12362]
Internal file name [`OUTPUT/11014_Monday_October_02_2023_02_48_03_AM_63640428/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - 2y \\
y' &= 3x(t) - 4y
\end{aligned}
$$

### 5.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix $A$, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} & -2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t} & -2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t} \\ 3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t} & 3\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{-t}\right) c_1 + \left(-2\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t}\right) c_2 \\ \left(3\,\mathrm{e}^{-t} - 3\,\mathrm{e}^{-2t}\right) c_1 + \left(3\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-t}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-2c_1 + 2c_2)\,\mathrm{e}^{-2t} + 3\left(c_1 - \frac{2c_2}{3}\right)\mathrm{e}^{-t} \\ (-3c_1 + 3c_2)\,\mathrm{e}^{-2t} + 3\left(c_1 - \frac{2c_2}{3}\right)\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-1$ | 1 | real eigenvalue |
| $-2$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \implies \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-1$ | 1 | 1 | No | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ |
| $-2$ | 1 | 1 | No | $\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-1$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue $-2$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\
&= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-t} + \frac{2 c_2 \mathrm{e}^{-2t}}{3} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{-2t} \end{bmatrix}
$$

The following is the phase plot of the system.

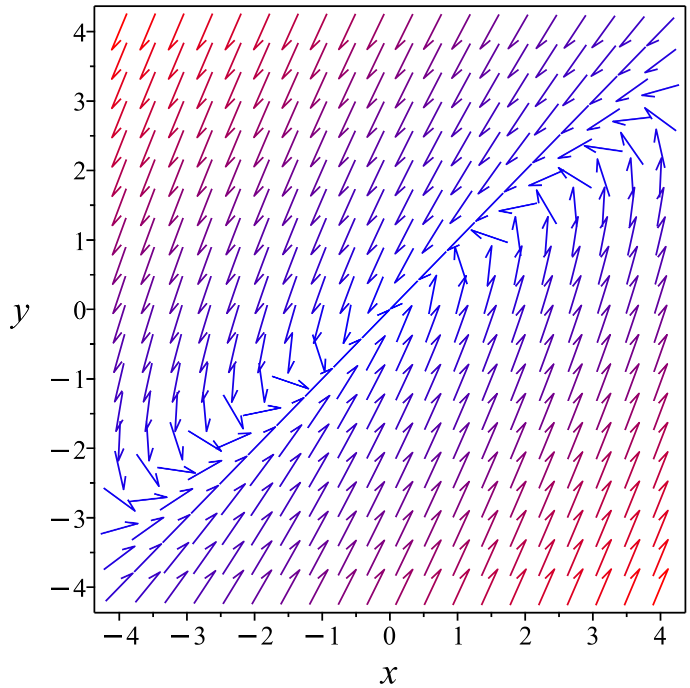



Figure 111: Phase plot

### 5.12.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - 2y, y' = 3x(t) - 4y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

  $$\left[\left[-2, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = c_1 \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{2c_1 e^{-2t}}{3} + c_2 e^{-t} \\ c_1 e^{-2t} + c_2 e^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{2c_1 e^{-2t}}{3} + c_2 e^{-t}, y = c_1 e^{-2t} + c_2 e^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=x(t)-2*y(t),diff(y(t),t)=3*x(t)-4*y(t)],singsol=all)
```

$$x(t) = e^{-t} c_1 + c_2 e^{-2t}$$
$$y(t) = e^{-t} c_1 + \frac{3c_2 e^{-2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 60

```
DSolve[{x'[t]==x[t]-2*y[t],y'[t]==3*x[t]-4*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^{-2t}(c_1(3e^t - 2) - 2c_2(e^t - 1))$$
$$y(t) \to e^{-2t}(3c_1(e^t - 1) + c_2(3 - 2e^t))$$

## 5.13 problem Problem 3(b)



Internal problem ID [12363]
Internal file name [`OUTPUT/11015_Monday_October_02_2023_11_46_12_PM_64152595/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{5x(t)}{4} + \frac{3y}{4} \\
y' &= \frac{x(t)}{2} - \frac{3y}{2}
\end{aligned}
$$

### 5.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(11\sqrt{145}+145\right)e^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290} + \frac{\left(-11\sqrt{145}+145\right)e^{-\frac{\left(1+\sqrt{145}\right)t}{8}}}{290} & -\frac{3\left(-e^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+e^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\right)\sqrt{145}}{145} \\ -\frac{2\left(-e^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+e^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\right)\sqrt{145}}{145} & \frac{\left(-11\sqrt{145}+145\right)e^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290} + \frac{e^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\left(11\sqrt{145}+14\right.}{290} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cc}
\frac{\left(11\sqrt{145}+145\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290}+\frac{\left(-11\sqrt{145}+145\right)\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}}{290} & -\frac{3\left(-\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\right)\sqrt{145}}{145} \\
-\frac{2\left(-\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\right)\sqrt{145}}{145} & \frac{\left(-11\sqrt{145}+145\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290}+\frac{\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\left(11\sqrt{145}+\right.}{290}
\end{array}\right. \\
&= \left[\begin{array}{c}
\left(\frac{\left(11\sqrt{145}+145\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290}+\frac{\left(-11\sqrt{145}+145\right)\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}}{290}\right)c_1-\frac{3\left(-\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\right)\sqrt{145}\,c_2}{145} \\
-\frac{2\left(-\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\right)\sqrt{145}\,c_1}{145}+\left(\frac{\left(-11\sqrt{145}+145\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290}+\frac{\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\left(11\sqrt{145}+145\right)}{290}\right)c_2
\end{array}\right] \\
&= \left[\begin{array}{c}
\frac{\left((11c_1+6c_2)\sqrt{145}+145c_1\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290}-\frac{11\,\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\left(\left(c_1+\frac{6c_2}{11}\right)\sqrt{145}-\frac{145c_1}{11}\right)}{290} \\
\frac{\left((4c_1-11c_2)\sqrt{145}+145c_2\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{290}-\frac{2\,\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\left(\left(c_1-\frac{11c_2}{4}\right)\sqrt{145}-\frac{145c_2}{4}\right)}{145}
\end{array}\right]
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{5}{4} - \lambda & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \frac{1}{4}\lambda - \frac{9}{4} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{8} + \frac{\sqrt{145}}{8}$$
$$\lambda_2 = -\frac{1}{8} - \frac{\sqrt{145}}{8}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-\frac{1}{8} - \frac{\sqrt{145}}{8}$ | 1 | real eigenvalue |
| $-\frac{1}{8} + \frac{\sqrt{145}}{8}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{8} - \frac{\sqrt{145}}{8}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} - \left(-\frac{1}{8} - \frac{\sqrt{145}}{8}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{11}{8} + \frac{\sqrt{145}}{8} & \frac{3}{4} \\ \frac{1}{2} & -\frac{11}{8} + \frac{\sqrt{145}}{8} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{11}{8} + \frac{\sqrt{145}}{8} & \frac{3}{4} & 0 \\ \frac{1}{2} & -\frac{11}{8} + \frac{\sqrt{145}}{8} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2\left(\frac{11}{8}+\frac{\sqrt{145}}{8}\right)} \Longrightarrow \left[\begin{array}{cc|c} \frac{11}{8}+\frac{\sqrt{145}}{8} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{11}{8}+\frac{\sqrt{145}}{8} & \frac{3}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{6t}{11+\sqrt{145}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{6t}{11+\sqrt{145}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6t}{11+\sqrt{145}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{6t}{11+\sqrt{145}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{6}{11+\sqrt{145}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{6t}{11+\sqrt{145}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{11+\sqrt{145}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{6t}{11+\sqrt{145}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{11+\sqrt{145}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{8} + \frac{\sqrt{145}}{8}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} - \left(-\frac{1}{8}+\frac{\sqrt{145}}{8}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{11}{8}-\frac{\sqrt{145}}{8} & \frac{3}{4} \\ \frac{1}{2} & -\frac{\sqrt{145}}{8}-\frac{11}{8} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{11}{8}-\frac{\sqrt{145}}{8} & \frac{3}{4} & 0 \\ \frac{1}{2} & -\frac{\sqrt{145}}{8}-\frac{11}{8} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2\left(\frac{11}{8}-\frac{\sqrt{145}}{8}\right)} \Longrightarrow \left[\begin{array}{cc|c} \frac{11}{8}-\frac{\sqrt{145}}{8} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{11}{8}-\frac{\sqrt{145}}{8} & \frac{3}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{6t}{-11+\sqrt{145}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{6t}{-11+\sqrt{145}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6t}{-11+\sqrt{145}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{6t}{-11+\sqrt{145}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{6}{-11+\sqrt{145}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{6t}{-11+\sqrt{145}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{-11+\sqrt{145}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{6t}{-11+\sqrt{145}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{-11+\sqrt{145}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-\frac{1}{8}+\frac{\sqrt{145}}{8}$ | 1 | 1 | No | $\left[\begin{array}{c}\frac{3}{-\frac{11}{2}+\frac{\sqrt{145}}{2}} \\ 1\end{array}\right]$ |
| $-\frac{1}{8}-\frac{\sqrt{145}}{8}$ | 1 | 1 | No | $\left[\begin{array}{c}\frac{3}{-\frac{11}{2}-\frac{\sqrt{145}}{2}} \\ 1\end{array}\right]$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{8}+\frac{\sqrt{145}}{8}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{1}{8}+\frac{\sqrt{145}}{8}\right)t} \\ &= \left[\begin{array}{c}\frac{3}{-\frac{11}{2}+\frac{\sqrt{145}}{2}} \\ 1\end{array}\right] e^{\left(-\frac{1}{8}+\frac{\sqrt{145}}{8}\right)t}\end{aligned}$$

Since eigenvalue $-\frac{1}{8}-\frac{\sqrt{145}}{8}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{1}{8}-\frac{\sqrt{145}}{8}\right)t} \\ &= \left[\begin{array}{c}\frac{3}{-\frac{11}{2}-\frac{\sqrt{145}}{2}} \\ 1\end{array}\right] e^{\left(-\frac{1}{8}-\frac{\sqrt{145}}{8}\right)t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1 \left[\begin{array}{c}\frac{3\,\mathrm{e}^{\left(-\frac{1}{8}+\frac{\sqrt{145}}{8}\right)t}}{-\frac{11}{2}+\frac{\sqrt{145}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{8}+\frac{\sqrt{145}}{8}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}\frac{3\,\mathrm{e}^{\left(-\frac{1}{8}-\frac{\sqrt{145}}{8}\right)t}}{-\frac{11}{2}-\frac{\sqrt{145}}{2}} \\ \mathrm{e}^{\left(-\frac{1}{8}-\frac{\sqrt{145}}{8}\right)t}\end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right]=\left[\begin{array}{c} \frac{c_1\left(11+\sqrt{145}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{4}-\frac{c_2\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\left(-11+\sqrt{145}\right)}{4} \\ c_1\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}+c_2\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}} \end{array}\right]$$

The following is the phase plot of the system.

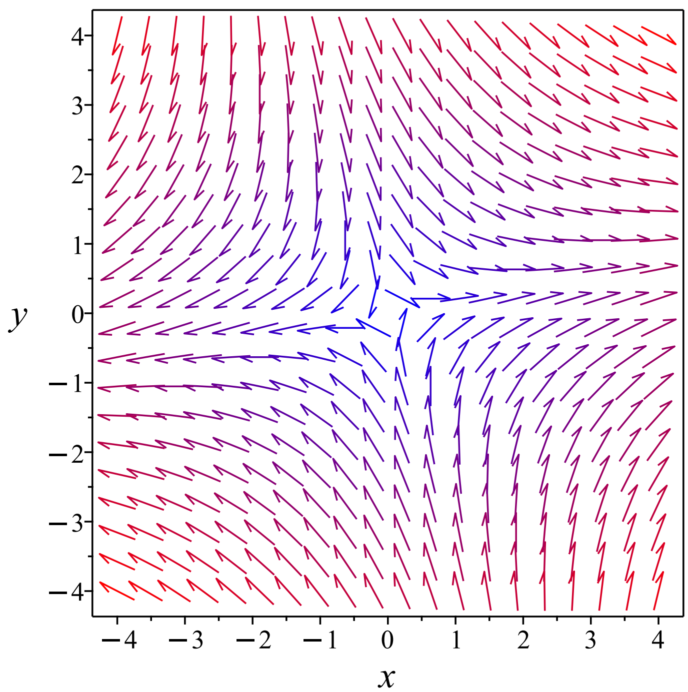



Figure 112: Phase plot

### 5.13.3 Maple step by step solution

Let's solve

$\left[x'(t)=\frac{5x(t)}{4}+\frac{3y}{4}, y'=\frac{x(t)}{2}-\frac{3y}{2}\right]$

- Define vector

$\overrightarrow{x}(t)=\left[\begin{array}{c} x(t) \\ y \end{array}\right]$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-\frac{1}{8} - \frac{\sqrt{145}}{8}, \left[\begin{array}{c} \frac{3}{-\frac{11}{2} - \frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]\right], \left[-\frac{1}{8} + \frac{\sqrt{145}}{8}, \left[\begin{array}{c} \frac{3}{-\frac{11}{2} + \frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{8} - \frac{\sqrt{145}}{8}, \left[\begin{array}{c} \frac{3}{-\frac{11}{2} - \frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(-\frac{1}{8} - \frac{\sqrt{145}}{8}\right)t} \cdot \left[\begin{array}{c} \frac{3}{-\frac{11}{2} - \frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{1}{8} + \frac{\sqrt{145}}{8}, \left[\begin{array}{c} \frac{3}{-\frac{11}{2} + \frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(-\frac{1}{8} + \frac{\sqrt{145}}{8}\right)t} \cdot \left[\begin{array}{c} \frac{3}{-\frac{11}{2} + \frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(-\frac{1}{8}-\frac{\sqrt{145}}{8}\right)t} \cdot \left[\begin{array}{c} \frac{3}{-\frac{11}{2}-\frac{\sqrt{145}}{2}} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\left(-\frac{1}{8}+\frac{\sqrt{145}}{8}\right)t} \cdot \left[\begin{array}{c} \frac{3}{-\frac{11}{2}+\frac{\sqrt{145}}{2}} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{c_2\left(11+\sqrt{145}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{4} - \frac{\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}} c_1\left(-11+\sqrt{145}\right)}{4} \\ c_1 \mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}} + c_2 \mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_2\left(11+\sqrt{145}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{4} - \frac{\mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}} c_1\left(-11+\sqrt{145}\right)}{4}, y = c_1 \mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}} + c_2 \mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=5/4*x(t)+3/4*y(t),diff(y(t),t)=1/2*x(t)-3/2*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}} + c_2 \mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}$$

$$y(t) = \frac{c_1 \mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}\sqrt{145}}{6} - \frac{c_2 \mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}\sqrt{145}}{6} - \frac{11 c_1 \mathrm{e}^{\frac{\left(-1+\sqrt{145}\right)t}{8}}}{6} - \frac{11 c_2 \mathrm{e}^{-\frac{\left(1+\sqrt{145}\right)t}{8}}}{6}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 161

```
DSolve[{x'[t]==5/4*x[t]+3/4*y[t],y'[t]==1/2*x[t]-3/2*y[t]},{x[t],y[t]},t,IncludeSingularSolu
```

$$x(t) \to \frac{1}{290}e^{-\frac{1}{8}\left(1+\sqrt{145}\right)t}\left(c_1\left(\left(145+11\sqrt{145}\right)e^{\frac{\sqrt{145}t}{4}}+145-11\sqrt{145}\right)\right.$$
$$\left.+6\sqrt{145}c_2\left(e^{\frac{\sqrt{145}t}{4}}-1\right)\right)$$
$$y(t) \to \frac{1}{290}e^{-\frac{1}{8}\left(1+\sqrt{145}\right)t}\left(4\sqrt{145}c_1\left(e^{\frac{\sqrt{145}t}{4}}-1\right)\right.$$
$$\left.-c_2\left(\left(11\sqrt{145}-145\right)e^{\frac{\sqrt{145}t}{4}}-145-11\sqrt{145}\right)\right)$$

## 5.14 problem Problem 3(c)



Internal problem ID [12364]
Internal file name [`OUTPUT/11016_Monday_October_02_2023_11_46_13_PM_64086885/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = x(t) - 2y$$
$$y' = x(t) - y$$

### 5.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + \sin(t) & -2\sin(t) \\ \sin(t) & \cos(t) - \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t)+\sin(t) & -2\sin(t) \\ \sin(t) & \cos(t)-\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t)+\sin(t))\,c_1 - 2\sin(t)\,c_2 \\ \sin(t)\,c_1 + (\cos(t)-\sin(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1-2c_2)\sin(t) + c_1\cos(t) \\ (c_1-c_2)\sin(t) + c_2\cos(t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left( \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0
$$

Therefore

$$
\det\left( \begin{bmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix} \right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $i$ | 1 | complex eigenvalue |
| $-i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1-i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1-i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1-\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \implies \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $i$ | 1 | 1 | No | $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ |
| $-i$ | 1 | 1 | No | $\begin{bmatrix} 1-i \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} (1+i)\,e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} (1-i)\,e^{-it} \\ e^{-it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} (1+i)\,c_1 e^{it} + (1-i)\,c_2 e^{-it} \\ c_1 e^{it} + c_2 e^{-it} \end{bmatrix}$$

The following is the phase plot of the system.

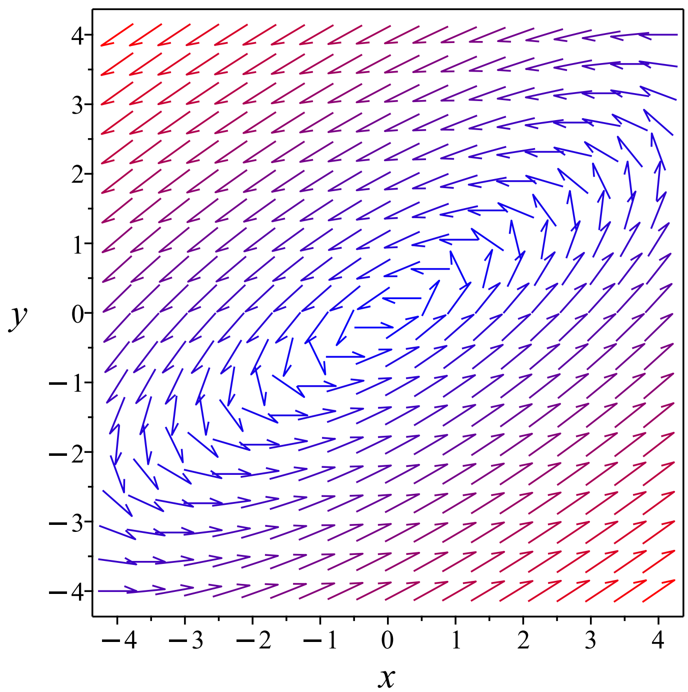



Figure 113: Phase plot

### 5.14.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - 2y, y' = x(t) - y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-\mathrm{I}, \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} 1+\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} 1-\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\left[\begin{array}{c}(1-\mathrm{I})\left(\cos\left(t\right)-\mathrm{I}\sin\left(t\right)\right)\\ \cos\left(t\right)-\mathrm{I}\sin\left(t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{1}(t)=\left[\begin{array}{c}\cos\left(t\right)-\sin\left(t\right)\\ \cos\left(t\right)\end{array}\right],\vec{x}_{2}(t)=\left[\begin{array}{c}-\cos\left(t\right)-\sin\left(t\right)\\ -\sin\left(t\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x}=c_{1}\vec{x}_{1}(t)+c_{2}\vec{x}_{2}(t)$$

- Substitute solutions into the general solution

$$\vec{x}=\left[\begin{array}{c}c_{2}(-\cos\left(t\right)-\sin\left(t\right))+c_{1}(\cos\left(t\right)-\sin\left(t\right))\\ c_{1}\cos\left(t\right)-c_{2}\sin\left(t\right)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y\end{array}\right]=\left[\begin{array}{c}(c_{1}-c_{2})\cos\left(t\right)-\sin\left(t\right)(c_{1}+c_{2})\\ c_{1}\cos\left(t\right)-c_{2}\sin\left(t\right)\end{array}\right]$$

- Solution to the system of ODEs

$$\{x(t)=(c_{1}-c_{2})\cos\left(t\right)-\sin\left(t\right)(c_{1}+c_{2}),y=c_{1}\cos\left(t\right)-c_{2}\sin\left(t\right)\}$$

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve([diff(x(t),t)-x(t)+2*y(t)=0,diff(y(t),t)+y(t)-x(t)=0],singsol=all)
```

$$x(t)=c_{1}\sin\left(t\right)+c_{2}\cos\left(t\right)$$
$$y(t)=-\frac{c_{1}\cos\left(t\right)}{2}+\frac{c_{2}\sin\left(t\right)}{2}+\frac{c_{1}\sin\left(t\right)}{2}+\frac{c_{2}\cos\left(t\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 39

```
DSolve[{x'[t]-x[t]+2*y[t]==0,y'[t]+y[t]-x[t]==0},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t)\to c_{1}(\sin(t)+\cos(t))-2c_{2}\sin(t)$$
$$y(t)\to c_{2}\cos(t)+(c_{1}-c_{2})\sin(t)$$

## 5.15 problem Problem 3(d)



Internal problem ID [12365]
Internal file name [`OUTPUT/11017_Monday_October_02_2023_11_46_13_PM_22296256/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -5x(t) + 2y \\
y' &= -2x(t) + y
\end{aligned}
$$

### 5.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix $A$, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{(3\sqrt{5}+5)\,e^{-(\sqrt{5}+2)t}}{10} + \frac{(-3\sqrt{5}+5)\,e^{(-2+\sqrt{5})t}}{10} & -\frac{\left(-e^{(-2+\sqrt{5})t}+e^{-(\sqrt{5}+2)t}\right)\sqrt{5}}{5} \\ \frac{\left(-e^{(-2+\sqrt{5})t}+e^{-(\sqrt{5}+2)t}\right)\sqrt{5}}{5} & \frac{(-3\sqrt{5}+5)\,e^{-(\sqrt{5}+2)t}}{10} + \frac{e^{(-2+\sqrt{5})t}(3\sqrt{5}+5)}{10} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{\left(3\sqrt{5}+5\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{10}+\frac{\left(-3\sqrt{5}+5\right)\mathrm{e}^{\left(-2+\sqrt{5}\right)t}}{10} & -\frac{\left(-\mathrm{e}^{\left(-2+\sqrt{5}\right)t}+\mathrm{e}^{-\left(\sqrt{5}+2\right)t}\right)\sqrt{5}}{5} \\ \frac{\left(-\mathrm{e}^{\left(-2+\sqrt{5}\right)t}+\mathrm{e}^{-\left(\sqrt{5}+2\right)t}\right)\sqrt{5}}{5} & \frac{\left(-3\sqrt{5}+5\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{10}+\frac{\mathrm{e}^{\left(-2+\sqrt{5}\right)t}\left(3\sqrt{5}+5\right)}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{\left(3\sqrt{5}+5\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{10}+\frac{\left(-3\sqrt{5}+5\right)\mathrm{e}^{\left(-2+\sqrt{5}\right)t}}{10}\right)c_1-\frac{\left(-\mathrm{e}^{\left(-2+\sqrt{5}\right)t}+\mathrm{e}^{-\left(\sqrt{5}+2\right)t}\right)\sqrt{5}\,c_2}{5} \\ \frac{\left(-\mathrm{e}^{\left(-2+\sqrt{5}\right)t}+\mathrm{e}^{-\left(\sqrt{5}+2\right)t}\right)\sqrt{5}\,c_1}{5}+\left(\frac{\left(-3\sqrt{5}+5\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{10}+\frac{\mathrm{e}^{\left(-2+\sqrt{5}\right)t}\left(3\sqrt{5}+5\right)}{10}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left((3c_1-2c_2)\sqrt{5}+5c_1\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{10}-\frac{3\left(\left(c_1-\frac{2c_2}{3}\right)\sqrt{5}-\frac{5c_1}{3}\right)\mathrm{e}^{\left(-2+\sqrt{5}\right)t}}{10} \\ \frac{\left((2c_1-3c_2)\sqrt{5}+5c_2\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{10}-\frac{\mathrm{e}^{\left(-2+\sqrt{5}\right)t}\left(\left(c_1-\frac{3c_2}{2}\right)\sqrt{5}-\frac{5c_2}{2}\right)}{5} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2 + \sqrt{5}$$
$$\lambda_2 = -2 - \sqrt{5}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-2-\sqrt{5}$ | 1 | real eigenvalue |
| $-2+\sqrt{5}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - \sqrt{5}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} - \left(-2-\sqrt{5}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{5}-3 & 2 \\ -2 & 3+\sqrt{5} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{5}-3 & 2 & 0 \\ -2 & 3+\sqrt{5} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{\sqrt{5}-3} \Longrightarrow \left[\begin{array}{cc|c} \sqrt{5}-3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{5}-3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{5}-3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{5}-3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{5}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}-3} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 + \sqrt{5}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} - \left(-2+\sqrt{5}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-\sqrt{5} & 2 \\ -2 & 3-\sqrt{5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-\sqrt{5} & 2 & 0 \\ -2 & 3-\sqrt{5} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{-3-\sqrt{5}} \Longrightarrow \left[\begin{array}{cc|c} -3-\sqrt{5} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-\sqrt{5} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3+\sqrt{5}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3+\sqrt{5}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| eigenvalue | multiplicity | | defective? | eigenvectors |
|---|---|---|---|---|
| | algebraic $m$ | geometric $k$ | | |
| $-2+\sqrt{5}$ | 1 | 1 | No | $\left[\begin{array}{c}\frac{2}{3+\sqrt{5}}\\ 1\end{array}\right]$ |
| $-2-\sqrt{5}$ | 1 | 1 | No | $\left[\begin{array}{c}\frac{2}{3-\sqrt{5}}\\ 1\end{array}\right]$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-2+\sqrt{5}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-2+\sqrt{5}\right)t}\\
&= \left[\begin{array}{c}\frac{2}{3+\sqrt{5}}\\ 1\end{array}\right] e^{\left(-2+\sqrt{5}\right)t}
\end{aligned}
$$

Since eigenvalue $-2-\sqrt{5}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(-2-\sqrt{5}\right)t}\\
&= \left[\begin{array}{c}\frac{2}{3-\sqrt{5}}\\ 1\end{array}\right] e^{\left(-2-\sqrt{5}\right)t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x(t)\\ y\end{array}\right] = c_1\left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(-2+\sqrt{5}\right)t}}{3+\sqrt{5}}\\ \mathrm{e}^{\left(-2+\sqrt{5}\right)t}\end{array}\right] + c_2\left[\begin{array}{c}\frac{2\,\mathrm{e}^{\left(-2-\sqrt{5}\right)t}}{3-\sqrt{5}}\\ \mathrm{e}^{\left(-2-\sqrt{5}\right)t}\end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c}x(t)\\ y\end{array}\right] = \left[\begin{array}{c}\frac{c_2\left(3+\sqrt{5}\right)\mathrm{e}^{-\left(\sqrt{5}+2\right)t}}{2} - \frac{\mathrm{e}^{\left(-2+\sqrt{5}\right)t}c_1\left(\sqrt{5}-3\right)}{2}\\ c_1\mathrm{e}^{\left(-2+\sqrt{5}\right)t} + c_2\mathrm{e}^{-\left(\sqrt{5}+2\right)t}\end{array}\right]
$$

The following is the phase plot of the system.

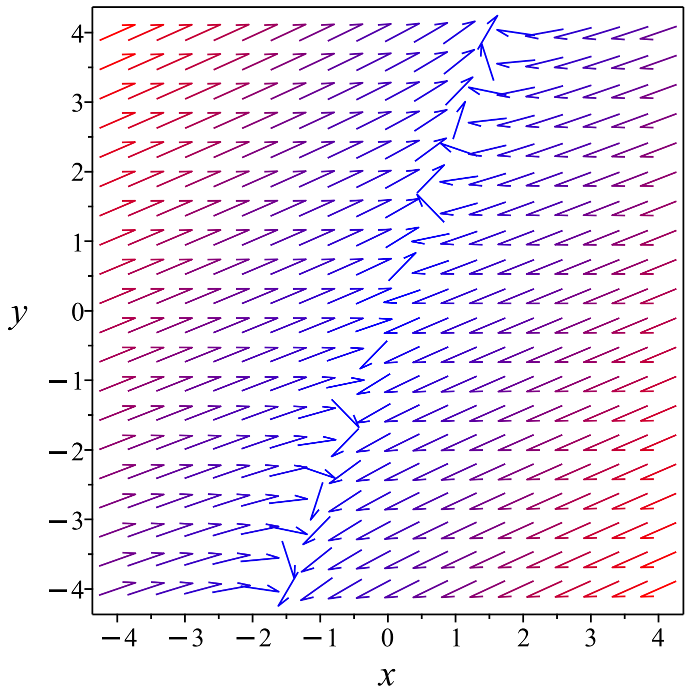



Figure 114: Phase plot

### 5.15.3 Maple step by step solution

Let's solve

$[x'(t) = -5x(t) + 2y, y' = -2x(t) + y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & 2 \\ -2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-2-\sqrt{5}, \begin{bmatrix} \frac{2}{3-\sqrt{5}} \\ 1 \end{bmatrix}\right], \left[-2+\sqrt{5}, \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2-\sqrt{5}, \begin{bmatrix} \frac{2}{3-\sqrt{5}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{\left(-2-\sqrt{5}\right)t} \cdot \begin{bmatrix} \frac{2}{3-\sqrt{5}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2+\sqrt{5}, \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{\left(-2+\sqrt{5}\right)t} \cdot \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{\left(-2-\sqrt{5}\right)t} \cdot \begin{bmatrix} \frac{2}{3-\sqrt{5}} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{\left(-2+\sqrt{5}\right)t} \cdot \begin{bmatrix} \frac{2}{3+\sqrt{5}} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1\left(3+\sqrt{5}\right)e^{-\left(\sqrt{5}+2\right)t}}{2} - \frac{c_2 e^{\left(-2+\sqrt{5}\right)t}\left(\sqrt{5}-3\right)}{2} \\ c_1 e^{-\left(\sqrt{5}+2\right)t} + c_2 e^{\left(-2+\sqrt{5}\right)t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{c_1\left(3+\sqrt{5}\right)e^{-\left(\sqrt{5}+2\right)t}}{2} - \frac{c_2 e^{\left(-2+\sqrt{5}\right)t}\left(\sqrt{5}-3\right)}{2}, y = c_1 e^{-\left(\sqrt{5}+2\right)t} + c_2 e^{\left(-2+\sqrt{5}\right)t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 83

```
dsolve([diff(x(t),t)+5*x(t)-2*y(t)=0,diff(y(t),t)+2*x(t)-y(t)=0],singsol=all)
```

$$x(t) = c_1 e^{\left(-2+\sqrt{5}\right)t} + c_2 e^{-\left(2+\sqrt{5}\right)t}$$

$$y(t) = \frac{c_1 e^{\left(-2+\sqrt{5}\right)t}\sqrt{5}}{2} - \frac{c_2 e^{-\left(2+\sqrt{5}\right)t}\sqrt{5}}{2} + \frac{3c_1 e^{\left(-2+\sqrt{5}\right)t}}{2} + \frac{3c_2 e^{-\left(2+\sqrt{5}\right)t}}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 148

```
DSolve[{x'[t]+5*x[t]-2*y[t]==0,y'[t]+2*x[t]-y[t]==0},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{10}e^{-\left(\left(2+\sqrt{5}\right)t\right)}\left(c_1\left(\left(5-3\sqrt{5}\right)e^{2\sqrt{5}t} + 5 + 3\sqrt{5}\right) + 2\sqrt{5}c_2\left(e^{2\sqrt{5}t} - 1\right)\right)$$

$$y(t) \to \frac{1}{10}e^{-\left(\left(2+\sqrt{5}\right)t\right)}\left(c_2\left(\left(5+3\sqrt{5}\right)e^{2\sqrt{5}t} + 5 - 3\sqrt{5}\right) - 2\sqrt{5}c_1\left(e^{2\sqrt{5}t} - 1\right)\right)$$

## 5.16 problem Problem 3(e)



Internal problem ID [12366]
Internal file name [`OUTPUT/11018_Monday_October_02_2023_11_46_13_PM_78487314/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(e).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) - 2y \\
y' &= x(t) - 3y
\end{aligned}
$$

### 5.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(-3\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{e^{\sqrt{7}\,t}\left(3\sqrt{7}+7\right)}{14} & -\frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{7} \\ \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(3\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left(-3\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(-3\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{e^{\sqrt{7}\,t}\left(3\sqrt{7}+7\right)}{14} & -\frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{7} \\ \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}}{14} & \frac{\left(3\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left(-3\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(-3\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{e^{\sqrt{7}\,t}\left(3\sqrt{7}+7\right)}{14}\right)c_1 - \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}\,c_2}{7} \\ \frac{\left(-e^{-\sqrt{7}\,t}+e^{\sqrt{7}\,t}\right)\sqrt{7}\,c_1}{14} + \left(\frac{\left(3\sqrt{7}+7\right)e^{-\sqrt{7}\,t}}{14} + \frac{\left(-3\sqrt{7}+7\right)e^{\sqrt{7}\,t}}{14}\right)c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((-3c_1+2c_2)\sqrt{7}+7c_1\right)e^{-\sqrt{7}\,t}}{14} + \frac{3\left(\left(c_1-\frac{2c_2}{3}\right)\sqrt{7}+\frac{7c_1}{3}\right)e^{\sqrt{7}\,t}}{14} \\ \frac{\left((-c_1+3c_2)\sqrt{7}+7c_2\right)e^{-\sqrt{7}\,t}}{14} + \frac{e^{\sqrt{7}\,t}\left((c_1-3c_2)\sqrt{7}+7c_2\right)}{14} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 7 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{7}$$
$$\lambda_2 = -\sqrt{7}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $\sqrt{7}$ | 1 | real eigenvalue |
| $-\sqrt{7}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} - \left(\sqrt{7}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3-\sqrt{7} & -2 \\ 1 & -3-\sqrt{7} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3-\sqrt{7} & -2 & 0 \\ 1 & -3-\sqrt{7} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3-\sqrt{7}} \Longrightarrow \left[\begin{array}{cc|c} 3-\sqrt{7} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-\sqrt{7} & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{-3+\sqrt{7}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{-3+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{-3+\sqrt{7}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{-3+\sqrt{7}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{-3+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\sqrt{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} - \left(-\sqrt{7}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3+\sqrt{7} & -2 \\ 1 & -3+\sqrt{7} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3+\sqrt{7} & -2 & 0 \\ 1 & -3+\sqrt{7} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3+\sqrt{7}} \Longrightarrow \left[\begin{array}{cc|c} 3+\sqrt{7} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+\sqrt{7} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3+\sqrt{7}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3+\sqrt{7}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3+\sqrt{7}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3+\sqrt{7}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{7}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3+\sqrt{7}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3+\sqrt{7}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $\sqrt{7}$ | 1 | 1 | No | $\begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix}$ |
| $-\sqrt{7}$ | 1 | 1 | No | $\begin{bmatrix} -\frac{2}{-3-\sqrt{7}} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{7}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{7}\,t} \\
&= \left[\begin{array}{c} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{array}\right] e^{\sqrt{7}\,t}
\end{aligned}
$$

Since eigenvalue $-\sqrt{7}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{7}\,t} \\
&= \left[\begin{array}{c} -\frac{2}{-3-\sqrt{7}} \\ 1 \end{array}\right] e^{-\sqrt{7}\,t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{\sqrt{7}\,t}}{-3+\sqrt{7}} \\ \mathrm{e}^{\sqrt{7}\,t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{-\sqrt{7}\,t}}{-3-\sqrt{7}} \\ \mathrm{e}^{-\sqrt{7}\,t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -c_2\left(-3+\sqrt{7}\right) \mathrm{e}^{-\sqrt{7}\,t} + \mathrm{e}^{\sqrt{7}\,t} c_1 \left(3+\sqrt{7}\right) \\ c_1 \mathrm{e}^{\sqrt{7}\,t} + c_2 \mathrm{e}^{-\sqrt{7}\,t} \end{array}\right]
$$

The following is the phase plot of the system.

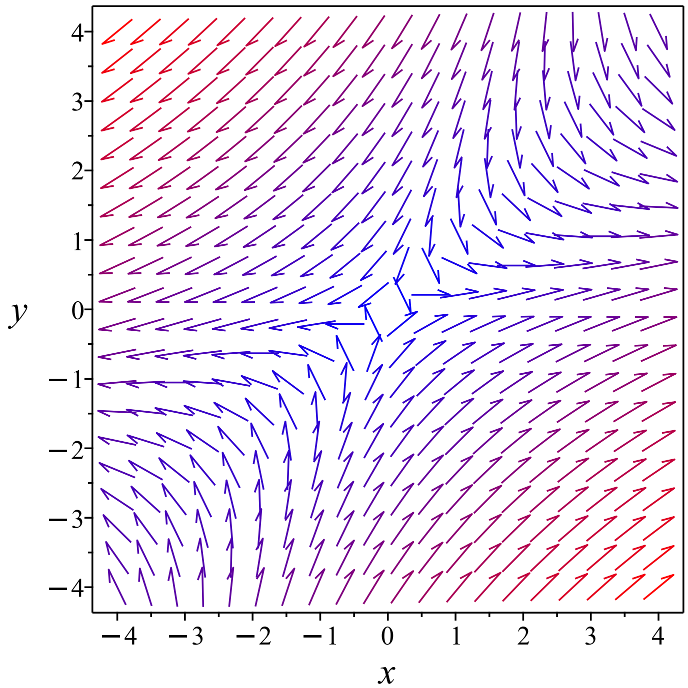



Figure 115: Phase plot

### 5.16.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - 2y, y' = x(t) - 3y]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix}$$

- Rewrite the system as

  $$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

  $$\left[\left[\sqrt{7}, \begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix}\right], \left[-\sqrt{7}, \begin{bmatrix} -\frac{2}{-3-\sqrt{7}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[\sqrt{7}, \begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = \mathrm{e}^{\sqrt{7}\,t} \cdot \begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[-\sqrt{7}, \begin{bmatrix} -\frac{2}{-3-\sqrt{7}} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = \mathrm{e}^{-\sqrt{7}\,t} \cdot \begin{bmatrix} -\frac{2}{-3-\sqrt{7}} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

- Substitute solutions into the general solution

  $$\vec{x} = c_1 \mathrm{e}^{\sqrt{7}\,t} \cdot \begin{bmatrix} -\frac{2}{-3+\sqrt{7}} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-\sqrt{7}\,t} \cdot \begin{bmatrix} -\frac{2}{-3-\sqrt{7}} \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_2(-3+\sqrt{7})\,\mathrm{e}^{-\sqrt{7}\,t} + \mathrm{e}^{\sqrt{7}\,t}c_1(3+\sqrt{7}) \\ c_1\mathrm{e}^{\sqrt{7}\,t} + c_2\mathrm{e}^{-\sqrt{7}\,t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -c_2(-3+\sqrt{7})\,\mathrm{e}^{-\sqrt{7}\,t} + \mathrm{e}^{\sqrt{7}\,t}c_1(3+\sqrt{7})\,, y = c_1\mathrm{e}^{\sqrt{7}\,t} + c_2\mathrm{e}^{-\sqrt{7}\,t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 71

```
dsolve([diff(x(t),t)-3*x(t)+2*y(t)=0,diff(y(t),t)-x(t)+3*y(t)=0],singsol=all)
```

$$x(t) = c_1\mathrm{e}^{\sqrt{7}\,t} + c_2\mathrm{e}^{-\sqrt{7}\,t}$$
$$y(t) = -\frac{c_1\sqrt{7}\,\mathrm{e}^{\sqrt{7}\,t}}{2} + \frac{c_2\sqrt{7}\,\mathrm{e}^{-\sqrt{7}\,t}}{2} + \frac{3c_1\mathrm{e}^{\sqrt{7}\,t}}{2} + \frac{3c_2\mathrm{e}^{-\sqrt{7}\,t}}{2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 144

```
DSolve[{x'[t]-3*x[t]+2*y[t]==0,y'[t]-x[t]+3*y[t]==0},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{14}e^{-\sqrt{7}t}\left(c_1\left(\left(7+3\sqrt{7}\right)e^{2\sqrt{7}t} + 7 - 3\sqrt{7}\right) - 2\sqrt{7}c_2\left(e^{2\sqrt{7}t} - 1\right)\right)$$
$$y(t) \to \frac{1}{14}e^{-\sqrt{7}t}\left(\sqrt{7}c_1\left(e^{2\sqrt{7}t} - 1\right) - c_2\left(\left(3\sqrt{7} - 7\right)e^{2\sqrt{7}t} - 7 - 3\sqrt{7}\right)\right)$$

## 5.17 problem Problem 3(f)



Internal problem ID [12367]
Internal file name [`OUTPUT/11019_Monday_October_02_2023_11_46_14_PM_79597876/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(f).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -x(t) + z(t) \\ y' &= -x(t) + y \\ z'(t) &= -x(t) - 2y + 3z(t) \end{aligned}$$

### 5.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1-t & -\frac{2\,e^{3t}}{9} + \frac{2}{9} + \frac{2t}{3} & \frac{2\,e^{3t}}{9} - \frac{2}{9} + \frac{t}{3} \\ -t & \frac{8}{9} + \frac{2t}{3} + \frac{e^{3t}}{9} & -\frac{e^{3t}}{9} + \frac{1}{9} + \frac{t}{3} \\ -t & \frac{2t}{3} - \frac{8\,e^{3t}}{9} + \frac{8}{9} & \frac{1}{9} + \frac{8\,e^{3t}}{9} + \frac{t}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1-t & -\frac{2\,e^{3t}}{9}+\frac{2}{9}+\frac{2t}{3} & \frac{2\,e^{3t}}{9}-\frac{2}{9}+\frac{t}{3} \\ -t & \frac{8}{9}+\frac{2t}{3}+\frac{e^{3t}}{9} & -\frac{e^{3t}}{9}+\frac{1}{9}+\frac{t}{3} \\ -t & \frac{2t}{3}-\frac{8\,e^{3t}}{9}+\frac{8}{9} & \frac{1}{9}+\frac{8\,e^{3t}}{9}+\frac{t}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (1-t)\,c_1+\left(-\frac{2\,e^{3t}}{9}+\frac{2}{9}+\frac{2t}{3}\right)c_2+\left(\frac{2\,e^{3t}}{9}-\frac{2}{9}+\frac{t}{3}\right)c_3 \\ -tc_1+\left(\frac{8}{9}+\frac{2t}{3}+\frac{e^{3t}}{9}\right)c_2+\left(-\frac{e^{3t}}{9}+\frac{1}{9}+\frac{t}{3}\right)c_3 \\ -tc_1+\left(\frac{2t}{3}-\frac{8\,e^{3t}}{9}+\frac{8}{9}\right)c_2+\left(\frac{1}{9}+\frac{8\,e^{3t}}{9}+\frac{t}{3}\right)c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 & 1 \\ -1 & 1-\lambda & 0 \\ -1 & -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 0$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 0 | 1 | real eigenvalue |
| 3 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 0 & 1 \\ -1 & -2 & 0 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 1 & 0 \\ 0 & -2 & -\frac{1}{4} & 0 \\ -1 & -2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 1 & 0 \\ 0 & -2 & -\frac{1}{4} & 0 \\ 0 & -2 & -\frac{1}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 0 & 1 & 0 \\ 0 & -2 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 0 & 1 \\ 0 & -2 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{4}, v_2 = -\frac{t}{8}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ -\frac{t}{8} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{8} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{8} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 3 | 1 | 1 | No | $\begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$ |
| 0 | 2 | 1 | Yes | $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

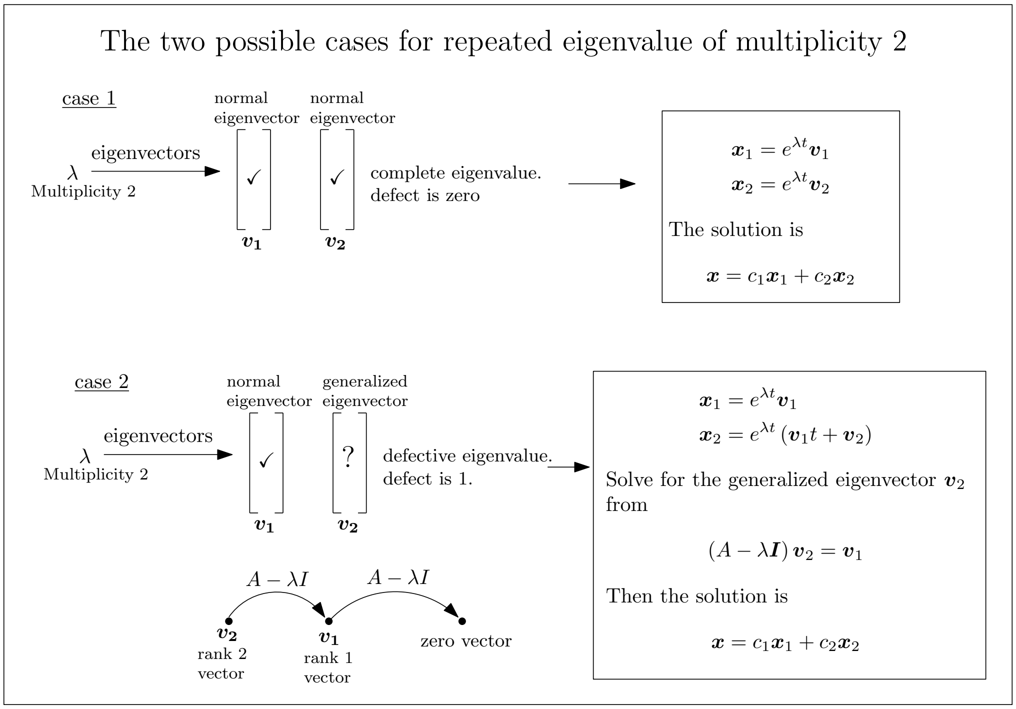



Figure 116: Possible case for repeated $\lambda$ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left( \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 0. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 1 \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} \vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) 1 \\ &= \begin{bmatrix} t \\ 1+t \\ 1+t \end{bmatrix} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{3t}}{4} \\ -\frac{e^{3t}}{8} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} t \\ 1+t \\ 1+t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{3t}}{4} + c_2 + c_3 t \\ -\frac{c_1 e^{3t}}{8} + c_2 + c_3 t + c_3 \\ c_1 e^{3t} + c_3 t + c_2 + c_3 \end{bmatrix}$$

### 5.17.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + z(t), y' = -x(t) + y, z'(t) = -x(t) - 2y + 3z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$

- Substitute solutions into the general solution

$$\vec{x} = e^{3t} c_3 \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_1 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{3t} c_3}{4} + c_1 \\ -\frac{e^{3t} c_3}{8} + c_1 \\ e^{3t} c_3 + c_1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{e^{3t} c_3}{4} + c_1, y = -\frac{e^{3t} c_3}{8} + c_1, z(t) = e^{3t} c_3 + c_1\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)+x(t)-z(t)=0,diff(y(t),t)-y(t)+x(t)=0,diff(z(t),t)+x(t)+2*y(t)-3*z(t)=0]
```

$$x(t) = c_1 + c_2 t + c_3 e^{3t}$$
$$y(t) = -\frac{c_3 e^{3t}}{2} + c_2 + c_1 + c_2 t$$
$$z(t) = c_2 + 4 c_3 e^{3t} + c_1 + c_2 t$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 132

```
DSolve[{x'[t]+x[t]-z[t]==0,y'[t]-y[t]+x[t]==0,z'[t]+x[t]+2*y[t]-3*z[t]==0},{x[t],y[t],z[t]},
```

$$x(t) \to \frac{1}{9}\left(-9c_1(t-1) + c_2\left(6t - 2e^{3t} + 2\right) + c_3\left(3t + 2e^{3t} - 2\right)\right)$$
$$y(t) \to \frac{1}{9}\left(-9c_1 t + c_2\left(6t + e^{3t} + 8\right) + c_3\left(3t - e^{3t} + 1\right)\right)$$
$$z(t) \to \frac{1}{9}\left(-9c_1 t - 2c_2\left(-3t + 4e^{3t} - 4\right) + c_3\left(3t + 8e^{3t} + 1\right)\right)$$

## 5.18 problem Problem 3(g)



Internal problem ID [12368]
Internal file name [`OUTPUT/11020_Monday_October_02_2023_11_46_14_PM_95537058/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6. Introduction to Systems of ODEs. Problems page 408
**Problem number**: Problem 3(g).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -\frac{x(t)}{2} + 2y - 3z(t) \\
y' &= y - \frac{z(t)}{2} \\
z'(t) &= -2x(t) + z(t)
\end{aligned}
$$

### 5.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(-2\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{33} + \frac{\left(2\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{33} + \frac{e^{3t}}{3} & \frac{\left(3\sqrt{33}-11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(-3\sqrt{33}-11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} + \cdots \\ \frac{\left(-5\sqrt{33}-11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left(5\sqrt{33}-11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} + \frac{e^{3t}}{12} & \frac{\left(23\sqrt{33}+121\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left(-23\sqrt{33}+121\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} + \cdots \\ \frac{\left(-3\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(3\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} - \frac{e^{3t}}{3} & \frac{\left(5\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(-5\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} - \cdots \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{\left(-2\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{33} + \frac{\left(2\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{33} + \frac{e^{3t}}{3} & \frac{\left(3\sqrt{33}-11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(-3\sqrt{33}-11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} + \cdots \\ \frac{\left(-5\sqrt{33}-11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left(5\sqrt{33}-11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} + \frac{e^{3t}}{12} & \frac{\left(23\sqrt{33}+121\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left(-23\sqrt{33}+121\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} \cdots \\ \frac{\left(-3\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(3\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} - \frac{e^{3t}}{3} & \frac{\left(5\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(-5\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} - \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\left(-2\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{33} + \frac{\left(2\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{33} + \frac{e^{3t}}{3}\right)c_1 + \left(\frac{\left(3\sqrt{33}-11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(-3\sqrt{33}-11\right)e^{-\cdots}}{66} \cdots\right. \\ \left(\frac{\left(-5\sqrt{33}-11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left(5\sqrt{33}-11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} + \frac{e^{3t}}{12}\right)c_1 + \left(\frac{\left(23\sqrt{33}+121\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left(-23\sqrt{33}+121\right)\cdots}{264} \cdots\right. \\ \left(\frac{\left(-3\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(3\sqrt{33}+11\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{66} - \frac{e^{3t}}{3}\right)c_1 + \left(\frac{\left(5\sqrt{33}+11\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{66} + \frac{\left(-5\sqrt{33}+11\right)e^{-\cdots}}{66} \cdots\right. \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left((-16c_1+12c_2-13c_3)\sqrt{33}+88c_1-44c_2+77c_3\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left((16c_1-12c_2+13c_3)\sqrt{33}+88c_1-44c_2+77c_3\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} + \frac{(c_1 \cdots}{\cdots} \\ \frac{\left((-20c_1+92c_2+3c_3)\sqrt{33}-44c_1+484c_2+77c_3\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{1056} + \frac{\left((20c_1-92c_2-3c_3)\sqrt{33}-44c_1+484c_2+77c_3\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{1056} + \frac{(c_1 \cdots}{\cdots} \\ \frac{\left((-12c_1+20c_2-7c_3)\sqrt{33}+44c_1+44c_2+55c_3\right)e^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{264} + \frac{\left((12c_1-20c_2+7c_3)\sqrt{33}+44c_1+44c_2+55c_3\right)e^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{264} - \frac{(c_1+ \cdots}{\cdots} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 5.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{1}{2} - \lambda & 2 & -3 \\ 0 & 1 - \lambda & -\frac{1}{2} \\ -2 & 0 & 1 - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - \frac{3}{2}\lambda^2 - 6\lambda + \frac{9}{2} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{3}{4} + \frac{\sqrt{33}}{4}$$
$$\lambda_2 = -\frac{3}{4} - \frac{\sqrt{33}}{4}$$
$$\lambda_3 = 3$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 3 | 1 | real eigenvalue |
| $-\frac{3}{4}+\frac{\sqrt{33}}{4}$ | 1 | real eigenvalue |
| $-\frac{3}{4}-\frac{\sqrt{33}}{4}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{7}{2} & 2 & -3 \\ 0 & -2 & -\frac{1}{2} \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\frac{7}{2} & 2 & -3 & 0 \\ 0 & -2 & -\frac{1}{2} & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{7} \implies \left[\begin{array}{ccc|c} -\frac{7}{2} & 2 & -3 & 0 \\ 0 & -2 & -\frac{1}{2} & 0 \\ 0 & -\frac{8}{7} & -\frac{2}{7} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_2}{7} \implies \left[\begin{array}{ccc|c} -\frac{7}{2} & 2 & -3 & 0 \\ 0 & -2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{7}{2} & 2 & -3 \\ 0 & -2 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -t \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{3}{4} - \frac{\sqrt{33}}{4}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} - \left(-\frac{3}{4} - \frac{\sqrt{33}}{4}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} + \frac{\sqrt{33}}{4} & 2 & -3 \\ 0 & \frac{7}{4} + \frac{\sqrt{33}}{4} & -\frac{1}{2} \\ -2 & 0 & \frac{7}{4} + \frac{\sqrt{33}}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{4}+\frac{\sqrt{33}}{4} & 2 & -3 & 0 \\ 0 & \frac{7}{4}+\frac{\sqrt{33}}{4} & -\frac{1}{2} & 0 \\ -2 & 0 & \frac{7}{4}+\frac{\sqrt{33}}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{\frac{1}{4}+\frac{\sqrt{33}}{4}} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{4}+\frac{\sqrt{33}}{4} & 2 & -3 & 0 \\ 0 & \frac{7}{4}+\frac{\sqrt{33}}{4} & -\frac{1}{2} & 0 \\ 0 & \frac{16}{1+\sqrt{33}} & \frac{-14+2\sqrt{33}}{1+\sqrt{33}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{16R_2}{\left(1+\sqrt{33}\right)\left(\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{4}+\frac{\sqrt{33}}{4} & 2 & -3 & 0 \\ 0 & \frac{7}{4}+\frac{\sqrt{33}}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{4}+\frac{\sqrt{33}}{4} & 2 & -3 \\ 0 & \frac{7}{4}+\frac{\sqrt{33}}{4} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{\left(7+\sqrt{33}\right)t}{8}, v_2 = -\frac{\left(-7+\sqrt{33}\right)t}{8}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{\left(7+\sqrt{33}\right)t}{8} \\ -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\left(7+\sqrt{33}\right)t}{8} \\ -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{\left(7+\sqrt{33}\right)t}{8} \\ -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{\sqrt{33}}{8}+\frac{7}{8} \\ \frac{7}{8}-\frac{\sqrt{33}}{8} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{\left(7+\sqrt{33}\right)t}{8} \\ -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{33}}{8} + \frac{7}{8} \\ \frac{7}{8} - \frac{\sqrt{33}}{8} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{\left(7+\sqrt{33}\right)t}{8} \\ -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = \begin{bmatrix} 7 + \sqrt{33} \\ 7 - \sqrt{33} \\ 8 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\frac{3}{4} + \frac{\sqrt{33}}{4}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} - \left(-\frac{3}{4} + \frac{\sqrt{33}}{4}\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} - \frac{\sqrt{33}}{4} & 2 & -3 \\ 0 & \frac{7}{4} - \frac{\sqrt{33}}{4} & -\frac{1}{2} \\ -2 & 0 & \frac{7}{4} - \frac{\sqrt{33}}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{4} - \frac{\sqrt{33}}{4} & 2 & -3 & 0 \\ 0 & \frac{7}{4} - \frac{\sqrt{33}}{4} & -\frac{1}{2} & 0 \\ -2 & 0 & \frac{7}{4} - \frac{\sqrt{33}}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{\frac{1}{4} - \frac{\sqrt{33}}{4}} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{4} - \frac{\sqrt{33}}{4} & 2 & -3 & 0 \\ 0 & \frac{7}{4} - \frac{\sqrt{33}}{4} & -\frac{1}{2} & 0 \\ 0 & -\frac{16}{-1+\sqrt{33}} & \frac{14+2\sqrt{33}}{-1+\sqrt{33}} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{16R_2}{\left(-1+\sqrt{33}\right)\left(\frac{7}{4}-\frac{\sqrt{33}}{4}\right)} \Longrightarrow \left[\begin{array}{ccc|c} \frac{1}{4}-\frac{\sqrt{33}}{4} & 2 & -3 & 0 \\ 0 & \frac{7}{4}-\frac{\sqrt{33}}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{4}-\frac{\sqrt{33}}{4} & 2 & -3 \\ 0 & \frac{7}{4}-\frac{\sqrt{33}}{4} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{\left(-7+\sqrt{33}\right)t}{8}, v_2 = \frac{\left(7+\sqrt{33}\right)t}{8}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ \frac{\left(7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ \frac{\left(7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ \frac{\left(7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{7}{8}-\frac{\sqrt{33}}{8} \\ \frac{\sqrt{33}}{8}+\frac{7}{8} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ \frac{\left(7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{7}{8}-\frac{\sqrt{33}}{8} \\ \frac{\sqrt{33}}{8}+\frac{7}{8} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{\left(-7+\sqrt{33}\right)t}{8} \\ \frac{\left(7+\sqrt{33}\right)t}{8} \\ t \end{bmatrix} = \begin{bmatrix} 7-\sqrt{33} \\ 7+\sqrt{33} \\ 8 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-\frac{3}{4}+\frac{\sqrt{33}}{4}$ | 1 | 1 | No | $\begin{bmatrix} -\frac{2\left(-\frac{17}{4}+\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix}$ |
| $-\frac{3}{4}-\frac{\sqrt{33}}{4}$ | 1 | 1 | No | $\begin{bmatrix} -\frac{2\left(-\frac{17}{4}-\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}-\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix}$ |
| 3 | 1 | 1 | No | $\begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{3}{4}+\frac{\sqrt{33}}{4}$ is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t} \\ &= \begin{bmatrix} -\frac{2\left(-\frac{17}{4}+\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix} e^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t}\end{aligned}$$

Since eigenvalue $-\frac{3}{4}-\frac{\sqrt{33}}{4}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\left(-\frac{3}{4}-\frac{\sqrt{33}}{4}\right)t} \\ &= \begin{bmatrix} -\frac{2\left(-\frac{17}{4}-\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}-\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix} e^{\left(-\frac{3}{4}-\frac{\sqrt{33}}{4}\right)t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{3t} \\ &= \begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,e^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t}\left(-\frac{17}{4}+\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{33}}{2}\right)} \\ -\frac{e^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t}}{2\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \\ e^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,e^{\left(-\frac{3}{4}-\frac{\sqrt{33}}{4}\right)t}\left(-\frac{17}{4}-\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}-\frac{\sqrt{33}}{2}\right)} \\ -\frac{e^{\left(-\frac{3}{4}-\frac{\sqrt{33}}{4}\right)t}}{2\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)} \\ e^{\left(-\frac{3}{4}-\frac{\sqrt{33}}{4}\right)t} \end{bmatrix} + c_3 \begin{bmatrix} -e^{3t} \\ -\frac{e^{3t}}{4} \\ e^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1\left(-7+\sqrt{33}\right)\mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{8} + \frac{c_2\left(7+\sqrt{33}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{8} - c_3\mathrm{e}^{3t} \\ \frac{c_1\left(7+\sqrt{33}\right)\mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{8} - \frac{c_2\left(-7+\sqrt{33}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{8} - \frac{c_3\mathrm{e}^{3t}}{4} \\ \mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}c_1 + \mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}c_2 + c_3\mathrm{e}^{3t} \end{bmatrix}$$

### 5.18.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -\frac{x(t)}{2} + 2y - 3z(t), y' = y - \frac{z(t)}{2}, z'(t) = -2x(t) + z(t)\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -\frac{1}{2} & 2 & -3 \\ 0 & 1 & -\frac{1}{2} \\ -2 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[3, \begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right], \left[-\frac{3}{4} - \frac{\sqrt{33}}{4}, \begin{bmatrix} -\frac{2\left(-\frac{17}{4} - \frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4} - \frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2} - \frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4} - \frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix}\right], \left[-\frac{3}{4} + \frac{\sqrt{33}}{4}, \begin{bmatrix} -\frac{2\left(-\frac{17}{4} + \frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4} + \frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2} + \frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4} + \frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{3t} \cdot \begin{bmatrix} -1 \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{3}{4} - \frac{\sqrt{33}}{4}, \begin{bmatrix} -\frac{2\left(-\frac{17}{4} - \frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4} - \frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2} - \frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4} - \frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(-\frac{3}{4} - \frac{\sqrt{33}}{4}\right)t} \cdot \begin{bmatrix} -\frac{2\left(-\frac{17}{4} - \frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4} - \frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2} - \frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4} - \frac{\sqrt{33}}{4}\right)} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\frac{3}{4}+\frac{\sqrt{33}}{4}, \left[\begin{array}{c} -\frac{2\left(-\frac{17}{4}+\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t} \cdot \left[\begin{array}{c} -\frac{2\left(-\frac{17}{4}+\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1\mathrm{e}^{3t} \cdot \left[\begin{array}{c} -1 \\ -\frac{1}{4} \\ 1 \end{array}\right] + c_2\mathrm{e}^{\left(-\frac{3}{4}-\frac{\sqrt{33}}{4}\right)t} \cdot \left[\begin{array}{c} -\frac{2\left(-\frac{17}{4}-\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}-\frac{\sqrt{33}}{2}\right)} \\ -\frac{1}{2\left(-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)} \\ 1 \end{array}\right] + c_3\mathrm{e}^{\left(-\frac{3}{4}+\frac{\sqrt{33}}{4}\right)t} \cdot \left[\begin{array}{c} -\frac{2\left(-\frac{17}{4}+\frac{3\sqrt{33}}{4}\right)}{\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)\left(-\frac{1}{2}+\right.} \\ -\frac{1}{2\left(-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)} \\ 1 \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_3\left(-7+\sqrt{33}\right)\mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{8} + \frac{c_2\left(7+\sqrt{33}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{8} - c_1\mathrm{e}^{3t} \\ \frac{c_3\left(7+\sqrt{33}\right)\mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{8} - \frac{c_2\left(-7+\sqrt{33}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{8} - \frac{c_1\mathrm{e}^{3t}}{4} \\ c_1\mathrm{e}^{3t} + \mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}c_2 + \mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}c_3 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{c_3\left(-7+\sqrt{33}\right)\mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{8} + \frac{c_2\left(7+\sqrt{33}\right)\mathrm{e}^{-\frac{\left(3+\sqrt{33}\right)t}{4}}}{8} - c_1\mathrm{e}^{3t}, y = \frac{c_3\left(7+\sqrt{33}\right)\mathrm{e}^{\frac{\left(-3+\sqrt{33}\right)t}{4}}}{8} - \frac{c_2\left(-7+\sqrt{33}\right)\mathrm{e}}{8}\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 164

```
dsolve([diff(x(t),t)=-1/2*x(t)+2*y(t)-3*z(t),diff(y(t),t)=y(t)-1/2*z(t),diff(z(t),t)=-2*x(t)
```

$$x(t) = -\frac{c_2 e^{\frac{(-3+\sqrt{33})t}{4}} \sqrt{33}}{8} + \frac{c_3 e^{-\frac{(3+\sqrt{33})t}{4}} \sqrt{33}}{8} + \frac{7c_2 e^{\frac{(-3+\sqrt{33})t}{4}}}{8} + \frac{7c_3 e^{-\frac{(3+\sqrt{33})t}{4}}}{8} - c_1 e^{3t}$$

$$y(t) = \frac{c_2 e^{\frac{(-3+\sqrt{33})t}{4}} \sqrt{33}}{8} - \frac{c_3 e^{-\frac{(3+\sqrt{33})t}{4}} \sqrt{33}}{8} + \frac{7c_2 e^{\frac{(-3+\sqrt{33})t}{4}}}{8} + \frac{7c_3 e^{-\frac{(3+\sqrt{33})t}{4}}}{8} - \frac{c_1 e^{3t}}{4}$$

$$z(t) = c_1 e^{3t} + c_2 e^{\frac{(-3+\sqrt{33})t}{4}} + c_3 e^{-\frac{(3+\sqrt{33})t}{4}}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 523

```
DSolve[{x'[t]==-1/2*x[t]+2*y[t]-3*z[t],y'[t]==y[t]-1/2*z[t],z'[t]==-2*x[t]+z[t]},{x[t],y[t],
```

$$\begin{aligned} x(t) \to \frac{1}{264} e^{-\frac{1}{4}(3+\sqrt{33})t} \Bigg( c_1 \left( \left(88 - 16\sqrt{33}\right) e^{\frac{\sqrt{33}t}{2}} + 88 e^{\frac{1}{4}(15+\sqrt{33})t} + 88 + 16\sqrt{33} \right) \\ + 4c_2 \left( \left(3\sqrt{33} - 11\right) e^{\frac{\sqrt{33}t}{2}} + 22 e^{\frac{1}{4}(15+\sqrt{33})t} - 11 - 3\sqrt{33} \right) \\ - c_3 \left( \left(13\sqrt{33} - 77\right) e^{\frac{\sqrt{33}t}{2}} + 154 e^{\frac{1}{4}(15+\sqrt{33})t} - 77 - 13\sqrt{33} \right) \Bigg) \end{aligned}$$

$$y(t) \to \frac{e^{-\frac{1}{4}(3+\sqrt{33})t} \left( -4c_1 \left( \left(11 + 5\sqrt{33}\right) e^{\frac{\sqrt{33}t}{2}} - 22 e^{\frac{1}{4}(15+\sqrt{33})t} + 11 - 5\sqrt{33} \right) + c_2 \left( \left(484 + 92\sqrt{33}\right) e^{\frac{\sqrt{33}t}{2}} + 88\ldots \right.\right.}{1056}$$

$$\begin{aligned} z(t) \to -\frac{1}{264} e^{-\frac{1}{4}(3+\sqrt{33})t} \Bigg( 4c_1 \left( \left(3\sqrt{33} - 11\right) e^{\frac{\sqrt{33}t}{2}} + 22 e^{\frac{1}{4}(15+\sqrt{33})t} - 11 - 3\sqrt{33} \right) \\ - 4c_2 \left( \left(11 + 5\sqrt{33}\right) e^{\frac{\sqrt{33}t}{2}} - 22 e^{\frac{1}{4}(15+\sqrt{33})t} + 11 - 5\sqrt{33} \right) \\ + c_3 \left( \left(7\sqrt{33} - 55\right) e^{\frac{\sqrt{33}t}{2}} - 154 e^{\frac{1}{4}(15+\sqrt{33})t} - 55 - 7\sqrt{33} \right) \Bigg) \end{aligned}$$

# 6 Chapter 6.4 Reduction to a single ODE. Problems page 415

## 6.1 problem Problem 4(a)



Internal problem ID [12369]
Internal file name [`OUTPUT/11021_Monday_October_02_2023_11_46_15_PM_23389732/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{x(t)}{2}+\frac{y}{2}\\
y' &= -\frac{x(t)}{2}+\frac{y}{2}
\end{aligned}
$$

### 6.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}
$$

For the above matrix $A$, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{t}{2}\right) & \mathrm{e}^{\frac{t}{2}} \sin\left(\frac{t}{2}\right) \\ -\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{t}{2}\right) & \mathrm{e}^{\frac{t}{2}} \cos\left(\frac{t}{2}\right) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\cos\left(\frac{t}{2}\right) & e^{\frac{t}{2}}\sin\left(\frac{t}{2}\right) \\ -e^{\frac{t}{2}}\sin\left(\frac{t}{2}\right) & e^{\frac{t}{2}}\cos\left(\frac{t}{2}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\cos\left(\frac{t}{2}\right) c_1 + e^{\frac{t}{2}}\sin\left(\frac{t}{2}\right) c_2 \\ -e^{\frac{t}{2}}\sin\left(\frac{t}{2}\right) c_1 + e^{\frac{t}{2}}\cos\left(\frac{t}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{\frac{t}{2}}\left(\cos\left(\frac{t}{2}\right) c_1 + \sin\left(\frac{t}{2}\right) c_2\right) \\ -e^{\frac{t}{2}}\left(\sin\left(\frac{t}{2}\right) c_1 - \cos\left(\frac{t}{2}\right) c_2\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

### 6.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda + \frac{1}{2} = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2} + \frac{i}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i}{2}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $\frac{1}{2} + \frac{i}{2}$ | 1 | complex eigenvalue |
| $\frac{1}{2} - \frac{i}{2}$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2} - \frac{i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} - \left(\frac{1}{2} - \frac{i}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{i}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{i}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{i}{2} & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{i}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} + \frac{i}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} - \left(\frac{1}{2} + \frac{i}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{i}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{i}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{i}{2} & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -\frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $\frac{1}{2} + \frac{i}{2}$ | 1 | 1 | No | $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ |
| $\frac{1}{2} - \frac{i}{2}$ | 1 | 1 | No | $\begin{bmatrix} i \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{\left(\frac{1}{2}+\frac{i}{2}\right)t} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{i}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{\left(\frac{1}{2}-\frac{i}{2}\right)t} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} i\left(c_2\mathrm{e}^{\left(\frac{1}{2}-\frac{i}{2}\right)t} - c_1\mathrm{e}^{\left(\frac{1}{2}+\frac{i}{2}\right)t}\right) \\ c_1\mathrm{e}^{\left(\frac{1}{2}+\frac{i}{2}\right)t} + c_2\mathrm{e}^{\left(\frac{1}{2}-\frac{i}{2}\right)t} \end{bmatrix}$$

The following is the phase plot of the system.

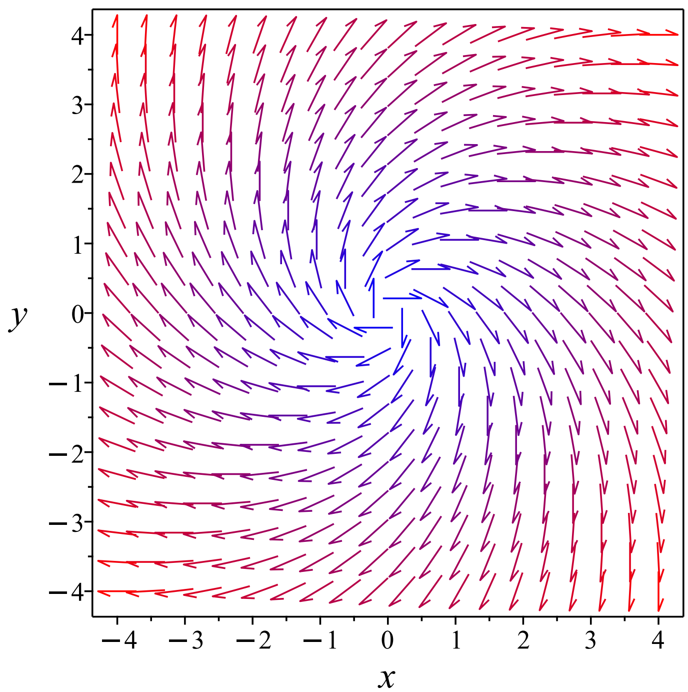



Figure 117: Phase plot

### 6.1.3 Maple step by step solution

Let's solve

$\left[x'(t) = \frac{x(t)}{2} + \frac{y}{2}, y' = -\frac{x(t)}{2} + \frac{y}{2}\right]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[\frac{1}{2} - \frac{\mathrm{I}}{2}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2} + \frac{\mathrm{I}}{2}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2} - \frac{\mathrm{I}}{2}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2} - \frac{\mathrm{I}}{2}\right)t} \cdot \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{t}{2}} \cdot \left(\cos\left(\frac{t}{2}\right) - \mathrm{I}\sin\left(\frac{t}{2}\right)\right) \cdot \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{\frac{t}{2}} \cdot \begin{bmatrix} I\left(\cos\left(\frac{t}{2}\right) - I\sin\left(\frac{t}{2}\right)\right) \\ \cos\left(\frac{t}{2}\right) - I\sin\left(\frac{t}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^{\frac{t}{2}} \cdot \begin{bmatrix} \sin\left(\frac{t}{2}\right) \\ \cos\left(\frac{t}{2}\right) \end{bmatrix}, \vec{x}_2(t) = e^{\frac{t}{2}} \cdot \begin{bmatrix} \cos\left(\frac{t}{2}\right) \\ -\sin\left(\frac{t}{2}\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{\frac{t}{2}} \cdot \begin{bmatrix} \sin\left(\frac{t}{2}\right) \\ \cos\left(\frac{t}{2}\right) \end{bmatrix} + c_2 e^{\frac{t}{2}} \cdot \begin{bmatrix} \cos\left(\frac{t}{2}\right) \\ -\sin\left(\frac{t}{2}\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{2}}\left(c_1\sin\left(\frac{t}{2}\right) + c_2\cos\left(\frac{t}{2}\right)\right) \\ e^{\frac{t}{2}}\left(c_1\cos\left(\frac{t}{2}\right) - c_2\sin\left(\frac{t}{2}\right)\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = e^{\frac{t}{2}}\left(c_1\sin\left(\frac{t}{2}\right) + c_2\cos\left(\frac{t}{2}\right)\right), y = e^{\frac{t}{2}}\left(c_1\cos\left(\frac{t}{2}\right) - c_2\sin\left(\frac{t}{2}\right)\right)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)+diff(y(t),t)=y(t),diff(x(t),t)-diff(y(t),t)=x(t)],singsol=all)
```

$$x(t) = e^{\frac{t}{2}}\left(c_2\cos\left(\frac{t}{2}\right) + c_1\sin\left(\frac{t}{2}\right)\right)$$
$$y(t) = e^{\frac{t}{2}}\left(\cos\left(\frac{t}{2}\right)c_1 - \sin\left(\frac{t}{2}\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 63

```
DSolve[{x'[t]+y'[t]==y[t],x'[t]-y'[t]==x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to e^{t/2}\left(c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right)\right)$$
$$y(t) \to e^{t/2}\left(c_2 \cos\left(\frac{t}{2}\right) - c_1 \sin\left(\frac{t}{2}\right)\right)$$

## 6.2 problem Problem 4(b)



Internal problem ID [12370]
Internal file name [`OUTPUT/11022_Monday_October_02_2023_11_46_15_PM_52810511/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{t}{3} + \frac{2x(t)}{3} + \frac{2y}{3} \\
y' &= \frac{t}{3} - \frac{x(t)}{3} - \frac{y}{3}
\end{aligned}
$$

### 6.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -1+2\,e^{\frac{t}{3}} & 2\,e^{\frac{t}{3}}-2 \\ -e^{\frac{t}{3}}+1 & 2-e^{\frac{t}{3}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -1+2\,e^{\frac{t}{3}} & 2\,e^{\frac{t}{3}}-2 \\ -e^{\frac{t}{3}}+1 & 2-e^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-1+2\,e^{\frac{t}{3}}\right)c_1 + \left(2\,e^{\frac{t}{3}}-2\right)c_2 \\ \left(-e^{\frac{t}{3}}+1\right)c_1 + \left(2-e^{\frac{t}{3}}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1+2c_2)\,e^{\frac{t}{3}} - c_1 - 2c_2 \\ (-c_1-c_2)\,e^{\frac{t}{3}} + c_1 + 2c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} 2\,e^{-\frac{t}{3}}-1 & -2+2\,e^{-\frac{t}{3}} \\ 1-e^{-\frac{t}{3}} & -e^{-\frac{t}{3}}+2 \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -1+2\,e^{\frac{t}{3}} & 2\,e^{\frac{t}{3}}-2 \\ -e^{\frac{t}{3}}+1 & 2-e^{\frac{t}{3}} \end{bmatrix} \int \begin{bmatrix} 2\,e^{-\frac{t}{3}}-1 & -2+2\,e^{-\frac{t}{3}} \\ 1-e^{-\frac{t}{3}} & -e^{-\frac{t}{3}}+2 \end{bmatrix} \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix} dt \\
&= \begin{bmatrix} -1+2\,e^{\frac{t}{3}} & 2\,e^{\frac{t}{3}}-2 \\ -e^{\frac{t}{3}}+1 & 2-e^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} -4t\,e^{-\frac{t}{3}} - 12\,e^{-\frac{t}{3}} - \frac{t^2}{2} \\ 2(3+t)\,e^{-\frac{t}{3}} + \frac{t^2}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2}t^2 - 4t - 12 \\ 2t + 6 + \frac{1}{2}t^2 \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} -12 + 2(c_1 + c_2)\,\mathrm{e}^{\frac{t}{3}} - \frac{t^2}{2} - 4t - c_1 - 2c_2 \\ (-c_1 - c_2)\,\mathrm{e}^{\frac{t}{3}} + c_1 + 2c_2 + 2t + 6 + \frac{t^2}{2} \end{bmatrix}\end{aligned}$$

### 6.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{2}{3} - \lambda & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \frac{1}{3}\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{1}{3}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 0 | 1 | real eigenvalue |
| $\frac{1}{3}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{2}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} - \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 0 | 1 | 1 | No | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ |
| $\frac{1}{3}$ | 1 | 1 | No | $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^0 \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0 \end{aligned}$$

Since eigenvalue $\frac{1}{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{\frac{t}{3}} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{\frac{t}{3}}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2\,\mathrm{e}^{\frac{t}{3}} \\ \mathrm{e}^{\frac{t}{3}} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -1 & -2\,\mathrm{e}^{\frac{t}{3}} \\ 1 & \mathrm{e}^{\frac{t}{3}} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 1 & 2 \\ -\mathrm{e}^{-\frac{t}{3}} & -\mathrm{e}^{-\frac{t}{3}} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -1 & -2\,e^{\frac{t}{3}} \\ 1 & e^{\frac{t}{3}} \end{bmatrix} \int \begin{bmatrix} 1 & 2 \\ -e^{-\frac{t}{3}} & -e^{-\frac{t}{3}} \end{bmatrix} \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix} dt$$

$$= \begin{bmatrix} -1 & -2\,e^{\frac{t}{3}} \\ 1 & e^{\frac{t}{3}} \end{bmatrix} \int \begin{bmatrix} t \\ -\frac{2t\,e^{-\frac{t}{3}}}{3} \end{bmatrix} dt$$

$$= \begin{bmatrix} -1 & -2\,e^{\frac{t}{3}} \\ 1 & e^{\frac{t}{3}} \end{bmatrix} \begin{bmatrix} \frac{t^2}{2} \\ 2(3+t)\,e^{-\frac{t}{3}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}t^2 - 4t - 12 \\ 2t + 6 + \frac{1}{2}t^2 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} -2c_2 e^{\frac{t}{3}} \\ c_2 e^{\frac{t}{3}} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t^2 - 4t - 12 \\ 2t + 6 + \frac{1}{2}t^2 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 - 2c_2 e^{\frac{t}{3}} - \frac{t^2}{2} - 4t - 12 \\ c_1 + c_2 e^{\frac{t}{3}} + 2t + 6 + \frac{t^2}{2} \end{bmatrix}$$

### 6.2.3 Maple step by step solution

Let's solve

$$\left[x'(t) = \frac{t}{3} + \frac{2x(t)}{3} + \frac{2y}{3}, y' = \frac{t}{3} - \frac{x(t)}{3} - \frac{y}{3}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[\frac{1}{3}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{1}{3}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$\vec{x}_2 = e^{\frac{t}{3}} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} -1 & -2\,e^{\frac{t}{3}} \\ 1 & e^{\frac{t}{3}} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} -1 & -2\,e^{\frac{t}{3}} \\ 1 & e^{\frac{t}{3}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} -1 + 2\,e^{\frac{t}{3}} & 2\,e^{\frac{t}{3}} - 2 \\ -e^{\frac{t}{3}} + 1 & 2 - e^{\frac{t}{3}} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t)\cdot \overrightarrow{v}'(t)=\overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t)=\frac{1}{\Phi(t)}\cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t)=\left[\begin{array}{c}-\frac{t^2}{2}+12\,e^{\frac{t}{3}}-4t-12\\ \frac{t^2}{2}-6\,e^{\frac{t}{3}}+2t+6\end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t)=c_1\overrightarrow{x}_1+c_2\overrightarrow{x}_2+\left[\begin{array}{c}-\frac{t^2}{2}+12\,e^{\frac{t}{3}}-4t-12\\ \frac{t^2}{2}-6\,e^{\frac{t}{3}}+2t+6\end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c}x(t)\\ y\end{array}\right]=\left[\begin{array}{c}-2c_2e^{\frac{t}{3}}-\frac{t^2}{2}+12\,e^{\frac{t}{3}}-4t-12-c_1\\ 6+(c_2-6)\,e^{\frac{t}{3}}+\frac{t^2}{2}+2t+c_1\end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t)=-2c_2e^{\frac{t}{3}}-\frac{t^2}{2}+12\,e^{\frac{t}{3}}-4t-12-c_1, y=6+(c_2-6)\,e^{\frac{t}{3}}+\frac{t^2}{2}+2t+c_1\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)+2*diff(y(t),t)=t,diff(x(t),t)-diff(y(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t)=3c_1e^{\frac{t}{3}}-\frac{t^2}{2}-4t+c_2$$
$$y(t)=-\frac{3c_1e^{\frac{t}{3}}}{2}+2t-6+\frac{t^2}{2}-c_2$$

✓ Solution by Mathematica

Time used: 0.178 (sec). Leaf size: 87

```
DSolve[{x'[t]+2*y'[t]==t,x'[t]-y'[t]==x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to -\frac{t^2}{2} - 4t + c_1\left(2e^{t/3} - 1\right) + 2c_2\left(e^{t/3} - 1\right) - 12$$
$$y(t) \to \frac{t^2}{2} + 2t - c_1e^{t/3} - c_2e^{t/3} + 6 + c_1 + 2c_2$$

## 6.3 problem Problem 4(c)



Internal problem ID [12371]
Internal file name [`OUTPUT/11023_Monday_October_02_2023_11_46_15_PM_51119002/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{6}{5} + \frac{3y}{5} - \frac{3t}{5} + x(t) \\
y' &= \frac{6}{5} - \frac{2y}{5} + \frac{2t}{5}
\end{aligned}
$$

### 6.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{6}{5} - \frac{3t}{5} \\ \frac{6}{5} + \frac{2t}{5} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & \frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{t} & \frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{t}c_1 + \frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-\frac{2t}{5}}c_2}{7} \\ e^{-\frac{2t}{5}}c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-\frac{2t}{5}}\left((7c_1+3c_2)e^{\frac{7t}{5}}-3c_2\right)}{7} \\ e^{-\frac{2t}{5}}c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{-t} & -\frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-t}}{7} \\ 0 & e^{\frac{2t}{5}} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{t} & \frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix} \int \begin{bmatrix} e^{-t} & -\frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-t}}{7} \\ 0 & e^{\frac{2t}{5}} \end{bmatrix} \begin{bmatrix} \frac{6}{5}-\frac{3t}{5} \\ \frac{6}{5}+\frac{2t}{5} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{t} & \frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix} \begin{bmatrix} -\frac{3\left(2e^{\frac{7t}{5}}t+e^{\frac{7t}{5}}-2t+6\right)e^{-t}}{14} \\ \frac{e^{\frac{2t}{5}}(2t+1)}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{2} \\ t+\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{e^{-\frac{2t}{5}}\left((7c_1+3c_2)e^{\frac{7t}{5}}-3c_2-\frac{21\,e^{\frac{2t}{5}}}{2}\right)}{7} \\ e^{-\frac{2t}{5}}c_2 + t + \frac{1}{2} \end{bmatrix}$$

### 6.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{6}{5} - \frac{3t}{5} \\ \frac{6}{5} + \frac{2t}{5} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & \frac{3}{5} \\ 0 & -\frac{2}{5}-\lambda \end{bmatrix}\right) = 0$$

Since the matrix $A$ is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(-\frac{2}{5}-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -\frac{2}{5}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 1 | 1 | real eigenvalue |
| $-\frac{2}{5}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & \frac{3}{5} \\ 0 & -\frac{7}{5} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & \frac{3}{5} & 0 \\ 0 & -\frac{7}{5} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{7R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 0 & \frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & \frac{3}{5} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{2}{5}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} - \left(-\frac{2}{5}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{7}{5} & \frac{3}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{7}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{7}{5} & \frac{3}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{7}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{7} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{7} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 1 | 1 | 1 | No | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ |
| $-\frac{2}{5}$ | 1 | 1 | No | $\begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t}$$

Since eigenvalue $-\frac{2}{5}$ is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-\frac{2t}{5}} = \begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix} e^{-\frac{2t}{5}}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} e^{t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3\,e^{-\frac{2t}{5}}}{7} \\ e^{-\frac{2t}{5}} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{t} & -\frac{3\,e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} e^{-t} & \frac{3\,e^{-t}}{7} \\ 0 & e^{\frac{2t}{5}} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^t & -\frac{3\,e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix} \int \begin{bmatrix} e^{-t} & \frac{3\,e^{-t}}{7} \\ 0 & e^{\frac{2t}{5}} \end{bmatrix} \begin{bmatrix} \frac{6}{5} - \frac{3t}{5} \\ \frac{6}{5} + \frac{2t}{5} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & -\frac{3\,e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix} \int \begin{bmatrix} -\frac{3\,e^{-t}(t-4)}{7} \\ \frac{2\,e^{\frac{2t}{5}}(3+t)}{5} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & -\frac{3\,e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix} \begin{bmatrix} \frac{3\,e^{-t}(-3+t)}{7} \\ \frac{e^{\frac{2t}{5}}(2t+1)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} \\ t + \frac{1}{2} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3c_2 e^{-\frac{2t}{5}}}{7} \\ c_2 e^{-\frac{2t}{5}} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ t + \frac{1}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{\left(14c_1 e^{\frac{7t}{5}} - 21\,e^{\frac{2t}{5}} - 6c_2\right) e^{-\frac{2t}{5}}}{14} \\ c_2 e^{-\frac{2t}{5}} + t + \frac{1}{2} \end{bmatrix}$$

### 6.3.3 Maple step by step solution

Let's solve

$\left[x'(t) = \frac{6}{5} + \frac{3y}{5} - \frac{3t}{5} + x(t), y' = \frac{6}{5} - \frac{2y}{5} + \frac{2t}{5}\right]$

- Define vector

  $\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{6}{5} - \frac{3t}{5} \\ \frac{6}{5} + \frac{2t}{5} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{6}{5} - \frac{3t}{5} \\ \frac{6}{5} + \frac{2t}{5} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \frac{6}{5} - \frac{3t}{5} \\ \frac{6}{5} + \frac{2t}{5} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & \frac{3}{5} \\ 0 & -\frac{2}{5} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-\frac{2}{5}, \begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{2}{5}, \begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-\frac{2t}{5}} \cdot \begin{bmatrix} -\frac{3}{7} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$\vec{x}_2 = e^t \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} -\frac{3\,e^{-\frac{2t}{5}}}{7} & e^t \\ e^{-\frac{2t}{5}} & 0 \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} -\frac{3\,e^{-\frac{2t}{5}}}{7} & e^t \\ e^{-\frac{2t}{5}} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{3}{7} & 1 \\ 1 & 0 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} e^t & \frac{3\left(e^{\frac{7t}{5}}-1\right)e^{-\frac{2t}{5}}}{7} \\ 0 & e^{-\frac{2t}{5}} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$$\Phi(t)\cdot \vec{v}'(t)=\vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot \left(\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t)=\left[\begin{array}{c}\frac{3\left(6\,e^{\frac{7t}{5}}-7\,e^{\frac{2t}{5}}+1\right)e^{-\frac{2t}{5}}}{14}\\ -\frac{e^{-\frac{2t}{5}}}{2}+t+\frac{1}{2}\end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\left[\begin{array}{c}\frac{3\left(6\,e^{\frac{7t}{5}}-7\,e^{\frac{2t}{5}}+1\right)e^{-\frac{2t}{5}}}{14}\\ -\frac{e^{-\frac{2t}{5}}}{2}+t+\frac{1}{2}\end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c}x(t)\\ y\end{array}\right]=\left[\begin{array}{c}\frac{\left(14c_2 e^{\frac{7t}{5}}+18\,e^{\frac{7t}{5}}-21\,e^{\frac{2t}{5}}-6c_1+3\right)e^{-\frac{2t}{5}}}{14}\\ c_1 e^{-\frac{2t}{5}}-\frac{e^{-\frac{2t}{5}}}{2}+t+\frac{1}{2}\end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t)=\frac{\left(14c_2 e^{\frac{7t}{5}}+18\,e^{\frac{7t}{5}}-21\,e^{\frac{2t}{5}}-6c_1+3\right)e^{-\frac{2t}{5}}}{14}, y=c_1 e^{-\frac{2t}{5}}-\frac{e^{-\frac{2t}{5}}}{2}+t+\frac{1}{2}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 30

```
dsolve([diff(x(t),t)-diff(y(t),t)=x(t)+y(t)-t,2*diff(x(t),t)+3*diff(y(t),t)=2*x(t)+6],singso
```

$$x(t)=c_2 e^t+e^{-\frac{2t}{5}}c_1-\frac{3}{2}$$
$$y(t)=-\frac{7\,e^{-\frac{2t}{5}}c_1}{3}+\frac{1}{2}+t$$

✓ Solution by Mathematica

Time used: 0.438 (sec). Leaf size: 53

```
DSolve[{x'[t]-y'[t]==x[t]+y[t]-t,2*x'[t]+3*y'[t]==2*x[t]+6},{x[t],y[t]},t,IncludeSingularSol
```

$$
\begin{aligned}
x(t) &\to \left(c_1 + \frac{3c_2}{7}\right)e^t - \frac{3}{7}c_2 e^{-2t/5} - \frac{3}{2} \\
y(t) &\to t + c_2 e^{-2t/5} + \frac{1}{2}
\end{aligned}
$$

## 6.4 problem Problem 4(d)



Internal problem ID [12372]
Internal file name [`OUTPUT/11024_Monday_October_02_2023_11_46_16_PM_7363715/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{2t}{7} + \frac{y}{7} \\
y' &= -\frac{3t}{7} + \frac{2y}{7}
\end{aligned}
$$

### 6.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} - \frac{1}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} - \frac{1}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} c_1 + \left(\frac{e^{\frac{2t}{7}}}{2} - \frac{1}{2}\right) c_2 \\ e^{\frac{2t}{7}} c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} 1 & -\frac{1}{2} + \frac{e^{-\frac{2t}{7}}}{2} \\ 0 & e^{-\frac{2t}{7}} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} - \frac{1}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \int \begin{bmatrix} 1 & -\frac{1}{2} + \frac{e^{-\frac{2t}{7}}}{2} \\ 0 & e^{-\frac{2t}{7}} \end{bmatrix} \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix} dt \\ &= \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} - \frac{1}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \begin{bmatrix} \frac{3(2t+7)e^{-\frac{2t}{7}}}{8} + \frac{t^2}{4} \\ \frac{3(2t+7)e^{-\frac{2t}{7}}}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4}t^2 + \frac{3}{4}t + \frac{21}{8} \\ \frac{3t}{2} + \frac{21}{4} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} c_1 + \frac{e^{\frac{2t}{7}} c_2}{2} - \frac{c_2}{2} + \frac{t^2}{4} + \frac{3t}{4} + \frac{21}{8} \\ e^{\frac{2t}{7}} c_2 + \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

### 6.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & \frac{1}{7} \\ 0 & \frac{2}{7} - \lambda \end{bmatrix}\right) = 0$$

Since the matrix $A$ is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(\frac{2}{7} - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{2}{7}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 0 | 1 | real eigenvalue |
| $\frac{2}{7}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & \frac{1}{7} & 0 \\ 0 & \frac{2}{7} & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & \frac{1}{7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{2}{7}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} - \left(\frac{2}{7}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 0 | 1 | 1 | No | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ |
| $\frac{2}{7}$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{0}$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0}$$

Since eigenvalue $\frac{2}{7}$ is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{\frac{2t}{7}}$$
$$= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{\frac{2t}{7}}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{\frac{2t}{7}}}{2} \\ e^{\frac{2t}{7}} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & e^{-\frac{2t}{7}} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \int \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & e^{-\frac{2t}{7}} \end{bmatrix} \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix} dt$$

$$= \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \int \begin{bmatrix} \frac{t}{2} \\ -\frac{3\,e^{-\frac{2t}{7}}t}{7} \end{bmatrix} dt$$

$$= \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \begin{bmatrix} \frac{t^2}{4} \\ \frac{3(2t+7)e^{-\frac{2t}{7}}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}t^2 + \frac{3}{4}t + \frac{21}{8} \\ \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{\frac{2t}{7}}}{2} \\ c_2 e^{\frac{2t}{7}} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}t^2 + \frac{3}{4}t + \frac{21}{8} \\ \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} c_1 + \frac{c_2 e^{\frac{2t}{7}}}{2} + \frac{t^2}{4} + \frac{3t}{4} + \frac{21}{8} \\ c_2 e^{\frac{2t}{7}} + \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

### 6.4.3 Maple step by step solution

Let's solve

$\left[x'(t) = \frac{2t}{7} + \frac{y}{7}, y' = -\frac{3t}{7} + \frac{2y}{7}\right]$

- Define vector

  $\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \frac{2t}{7} \\ -\frac{3t}{7} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & \frac{1}{7} \\ 0 & \frac{2}{7} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[\frac{2}{7}, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[\frac{2}{7}, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$\overrightarrow{x}_2 = e^{\frac{2t}{7}} \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} 1 & \frac{e^{\frac{2t}{7}}}{2} - \frac{1}{2} \\ 0 & e^{\frac{2t}{7}} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} \frac{t^2}{4} + \frac{21}{8} - \frac{21\,e^{\frac{2t}{7}}}{8} + \frac{3t}{4} \\ -\frac{21\,e^{\frac{2t}{7}}}{4} + \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} \frac{t^2}{4} + \frac{21}{8} - \frac{21\,e^{\frac{2t}{7}}}{8} + \frac{3t}{4} \\ -\frac{21\,e^{\frac{2t}{7}}}{4} + \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(4c_2-21)e^{\frac{2t}{7}}}{8} + \frac{t^2}{4} + \frac{3t}{4} + c_1 + \frac{21}{8} \\ c_2 e^{\frac{2t}{7}} - \frac{21\,e^{\frac{2t}{7}}}{4} + \frac{3t}{2} + \frac{21}{4} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{(4c_2-21)e^{\frac{2t}{7}}}{8} + \frac{t^2}{4} + \frac{3t}{4} + c_1 + \frac{21}{8}, y = c_2 e^{\frac{2t}{7}} - \frac{21\,e^{\frac{2t}{7}}}{4} + \frac{3t}{2} + \frac{21}{4}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([2*diff(x(t),t)-diff(y(t),t)=t,3*diff(x(t),t)+2*diff(y(t),t)=y(t)],singsol=all)
```

$$x(t) = \frac{t^2}{4} + \frac{7\,e^{\frac{2t}{7}} c_1}{2} + \frac{3t}{4} + c_2$$
$$y(t) = \frac{3t}{2} + 7\,e^{\frac{2t}{7}} c_1 + \frac{21}{4}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 60

```
DSolve[{2*x'[t]-y'[t]==t,3*x'[t]+2*y'[t]==y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tr
```

$$x(t) \to \frac{1}{8}\left(2t^2 + 6t + 4c_2e^{2t/7} + 21 + 8c_1 - 4c_2\right)$$
$$y(t) \to \frac{3t}{2} + c_2e^{2t/7} + \frac{21}{4}$$

## 6.5 problem Problem 4(e)



Internal problem ID [12373]
Internal file name [`OUTPUT/11025_Monday_October_02_2023_11_46_16_PM_84307621/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(e).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{3t}{4} - \frac{x(t)}{4} - \frac{y}{4} \\
y' &= \frac{5t}{4} - \frac{3x(t)}{4} - \frac{3y}{4}
\end{aligned}
$$

### 6.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{3t}{4} \\ \frac{5t}{4} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3}{4} & -\frac{1}{4} + \frac{e^{-t}}{4} \\ -\frac{3}{4} + \frac{3\,e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{1}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3}{4} & -\frac{1}{4} + \frac{e^{-t}}{4} \\ -\frac{3}{4} + \frac{3\,e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{1}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{-t}}{4} + \frac{3}{4}\right) c_1 + \left(-\frac{1}{4} + \frac{e^{-t}}{4}\right) c_2 \\ \left(-\frac{3}{4} + \frac{3\,e^{-t}}{4}\right) c_1 + \left(\frac{3\,e^{-t}}{4} + \frac{1}{4}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1+c_2)e^{-t}}{4} + \frac{3c_1}{4} - \frac{c_2}{4} \\ \frac{(3c_1+3c_2)e^{-t}}{4} - \frac{3c_1}{4} + \frac{c_2}{4} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{3}{4} + \frac{e^{t}}{4} & \frac{e^{t}}{4} - \frac{1}{4} \\ \frac{3\,e^{t}}{4} - \frac{3}{4} & \frac{1}{4} + \frac{3\,e^{t}}{4} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3}{4} & -\frac{1}{4} + \frac{e^{-t}}{4} \\ -\frac{3}{4} + \frac{3\,e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{1}{4} \end{bmatrix} \int \begin{bmatrix} \frac{3}{4} + \frac{e^{t}}{4} & \frac{e^{t}}{4} - \frac{1}{4} \\ \frac{3\,e^{t}}{4} - \frac{3}{4} & \frac{1}{4} + \frac{3\,e^{t}}{4} \end{bmatrix} \begin{bmatrix} \frac{3t}{4} \\ \frac{5t}{4} \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3}{4} & -\frac{1}{4} + \frac{e^{-t}}{4} \\ -\frac{3}{4} + \frac{3\,e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{(4t-4)e^{t}}{8} + \frac{t^2}{8} \\ \frac{3\,e^{t}(t-1)}{2} - \frac{t^2}{8} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}t - \frac{1}{2} + \frac{1}{8}t^2 \\ -\frac{1}{8}t^2 + \frac{3}{2}t - \frac{3}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(2c_1+2c_2)\mathrm{e}^{-t}}{8} + \frac{t^2}{8} + \frac{t}{2} + \frac{3c_1}{4} - \frac{c_2}{4} - \frac{1}{2} \\ \frac{(6c_1+6c_2)\mathrm{e}^{-t}}{8} - \frac{t^2}{8} + \frac{3t}{2} - \frac{3c_1}{4} + \frac{c_2}{4} - \frac{3}{2} \end{bmatrix}$$

### 6.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{3t}{4} \\ \frac{5t}{4} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{1}{4} - \lambda & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-1$ | 1 | real eigenvalue |
| 0 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{3}{4} & -\frac{3}{4} & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| eigenvalue | multiplicity | | defective? | eigenvectors |
|---|---|---|---|---|
| | algebraic $m$ | geometric $k$ | | |
| $-1$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$ |
| 0 | 1 | 1 | No | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-1$ is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-t} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{-t}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{0} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{0}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-t}}{3} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{e^{-t}}{3} & -1 \\ e^{-t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{3 e^{t}}{4} & \frac{3 e^{t}}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{-t}}{3} & -1 \\ e^{-t} & 1 \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{t}}{4} & \frac{3\,e^{t}}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{3t}{4} \\ \frac{5t}{4} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{e^{-t}}{3} & -1 \\ e^{-t} & 1 \end{bmatrix} \int \begin{bmatrix} \frac{3t\,e^{t}}{2} \\ -\frac{t}{4} \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{e^{-t}}{3} & -1 \\ e^{-t} & 1 \end{bmatrix} \begin{bmatrix} \frac{3\,e^{t}(t-1)}{2} \\ -\frac{t^2}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}t - \frac{1}{2} + \frac{1}{8}t^2 \\ -\frac{1}{8}t^2 + \frac{3}{2}t - \frac{3}{2} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-t}}{3} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t - \frac{1}{2} + \frac{1}{8}t^2 \\ -\frac{1}{8}t^2 + \frac{3}{2}t - \frac{3}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-t}}{3} - c_2 + \frac{t}{2} - \frac{1}{2} + \frac{t^2}{8} \\ c_1 e^{-t} + c_2 - \frac{t^2}{8} + \frac{3t}{2} - \frac{3}{2} \end{bmatrix}$$

### 6.5.3 Maple step by step solution

Let's solve

$$\left[x'(t) = \frac{3t}{4} - \frac{x(t)}{4} - \frac{y}{4}, y' = \frac{5t}{4} - \frac{3x(t)}{4} - \frac{3y}{4}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{3t}{4} \\ \frac{5t}{4} \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} \frac{3t}{4} \\ \frac{5t}{4} \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(t) = \left[\begin{array}{c} \frac{3t}{4} \\ \frac{5t}{4} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{3}{4} \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-1, \left[\begin{array}{c} \frac{1}{3} \\ 1 \end{array}\right]\right], \left[0, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} \frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($
  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst
  $$\phi(t) = \begin{bmatrix} \frac{e^{-t}}{3} & -1 \\ e^{-t} & 1 \end{bmatrix}$$
- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th
  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$
  $$\Phi(t) = \begin{bmatrix} \frac{e^{-t}}{3} & -1 \\ e^{-t} & 1 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{3} & -1 \\ 1 & 1 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix
  $$\Phi(t) = \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3}{4} & -\frac{1}{4} + \frac{e^{-t}}{4} \\ -\frac{3}{4} + \frac{3\,e^{-t}}{4} & \frac{3\,e^{-t}}{4} + \frac{1}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$
  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution
  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms
  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{-t}}{2} + \frac{t}{2} - \frac{1}{2} + \frac{t^2}{8} \\ -\frac{t^2}{8} - \frac{3}{2} + \frac{3\,e^{-t}}{2} + \frac{3t}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} \frac{e^{-t}}{2} + \frac{t}{2} - \frac{1}{2} + \frac{t^2}{8} \\ -\frac{t^2}{8} - \frac{3}{2} + \frac{3\,e^{-t}}{2} + \frac{3t}{2} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{(3+2c_1)e^{-t}}{6} + \frac{t^2}{8} + \frac{t}{2} - c_2 \\ c_1 e^{-t} - \frac{t^2}{8} - \frac{3}{2} + \frac{3\,e^{-t}}{2} + \frac{3t}{2} + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{1}{2} + \frac{(3+2c_1)e^{-t}}{6} + \frac{t^2}{8} + \frac{t}{2} - c_2, y = c_1 e^{-t} - \frac{t^2}{8} - \frac{3}{2} + \frac{3\,e^{-t}}{2} + \frac{3t}{2} + c_2\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve([5*diff(x(t),t)-3*diff(y(t),t)=x(t)+y(t),3*diff(x(t),t)-diff(y(t),t)=t],singsol=all)
```

$$x(t) = \frac{t^2}{8} - e^{-t}c_1 + \frac{t}{2} + c_2$$
$$y(t) = \frac{3t}{2} - 3\,e^{-t}c_1 - 2 - \frac{t^2}{8} - c_2$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 75

```
DSolve[{5*x'[t]-3*y'[t]==x[t]+y[t],3*x'[t]-y'[t]==t},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{8}\left(t^2 + 4t + 2(c_1 + c_2)e^{-t} - 4 + 6c_1 - 2c_2\right)$$
$$y(t) \to \frac{1}{8}\left(-t^2 + 12t + 2\left(3(c_1 + c_2)e^{-t} - 6 - 3c_1 + c_2\right)\right)$$

## 6.6 problem Problem 4(f)



Internal problem ID [12374]
Internal file name [`OUTPUT/11026_Monday_October_02_2023_11_46_16_PM_56013244/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(f).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{4y}{5} + \frac{4t}{5} \\
y' &= \frac{y}{5} + \frac{t}{5}
\end{aligned}
$$

### 6.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} - 4 \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} - 4 \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 + \left(4\,\mathrm{e}^{\frac{t}{5}} - 4\right) c_2 \\ \mathrm{e}^{\frac{t}{5}} c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} 1 & -4 + 4\,\mathrm{e}^{-\frac{t}{5}} \\ 0 & \mathrm{e}^{-\frac{t}{5}} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} - 4 \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix} \int \begin{bmatrix} 1 & -4 + 4\,\mathrm{e}^{-\frac{t}{5}} \\ 0 & \mathrm{e}^{-\frac{t}{5}} \end{bmatrix} \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix} dt \\
&= \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} - 4 \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix} \begin{bmatrix} -4(5+t)\,\mathrm{e}^{-\frac{t}{5}} \\ -(5+t)\,\mathrm{e}^{-\frac{t}{5}} \end{bmatrix} \\
&= \begin{bmatrix} -4t - 20 \\ -5 - t \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} 4\,\mathrm{e}^{\frac{t}{5}} c_2 - 4t + c_1 - 4c_2 - 20 \\ \mathrm{e}^{\frac{t}{5}} c_2 - 5 - t \end{bmatrix}
\end{aligned}$$

### 6.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & \frac{4}{5} \\ 0 & \frac{1}{5} - \lambda \end{bmatrix}\right) = 0$$

Since the matrix $A$ is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-\lambda)(\frac{1}{5} - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{5}$$
$$\lambda_2 = 0$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $0$ | $1$ | real eigenvalue |
| $\frac{1}{5}$ | $1$ | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & \frac{4}{5} & 0 \\ 0 & \frac{1}{5} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} 0 & \frac{4}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & \frac{4}{5} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{5}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} - \left(\frac{1}{5}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 4t\}$

Hence the solution is

$$\begin{bmatrix} 4t \\ t \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 4t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $\frac{1}{5}$ | 1 | 1 | No | $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ |
| 0 | 1 | 1 | No | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{1}{5}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\frac{t}{5}} \\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{\frac{t}{5}} \end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} 4\,\mathrm{e}^{\frac{t}{5}} \\ \mathrm{e}^{\frac{t}{5}} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 4\,\mathrm{e}^{\frac{t}{5}} & 1 \\ \mathrm{e}^{\frac{t}{5}} & 0 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 0 & \mathrm{e}^{-\frac{t}{5}} \\ 1 & -4 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 4\,\mathrm{e}^{\frac{t}{5}} & 1 \\ \mathrm{e}^{\frac{t}{5}} & 0 \end{bmatrix} \int \begin{bmatrix} 0 & \mathrm{e}^{-\frac{t}{5}} \\ 1 & -4 \end{bmatrix} \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix} dt \\
&= \begin{bmatrix} 4\,\mathrm{e}^{\frac{t}{5}} & 1 \\ \mathrm{e}^{\frac{t}{5}} & 0 \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{-\frac{t}{5}} t}{5} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} 4\,\mathrm{e}^{\frac{t}{5}} & 1 \\ \mathrm{e}^{\frac{t}{5}} & 0 \end{bmatrix} \begin{bmatrix} -(5+t)\,\mathrm{e}^{-\frac{t}{5}} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -4t - 20 \\ -5 - t \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 4c_1 \mathrm{e}^{\frac{t}{5}} \\ c_1 \mathrm{e}^{\frac{t}{5}} \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -4t - 20 \\ -5 - t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 4c_1 \mathrm{e}^{\frac{t}{5}} + c_2 - 4t - 20 \\ c_1 \mathrm{e}^{\frac{t}{5}} - 5 - t \end{bmatrix}$$

### 6.6.3 Maple step by step solution

Let's solve

$\left[x'(t) = \frac{4y}{5} + \frac{4t}{5}, y' = \frac{y}{5} + \frac{t}{5}\right]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix}$

- Define the forcing function

$\vec{f}(t) = \begin{bmatrix} \frac{4t}{5} \\ \frac{t}{5} \end{bmatrix}$

- Define the coefficient matrix

$A = \begin{bmatrix} 0 & \frac{4}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$\left[\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right], \left[\frac{1}{5}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right]\right]$

- Consider eigenpair

$\left[0, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]$

- Solution to homogeneous system from eigenpair

  $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- Consider eigenpair

  $\begin{bmatrix} \frac{1}{5}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{bmatrix}$

- Solution to homogeneous system from eigenpair

  $\vec{x}_2 = \mathrm{e}^{\frac{t}{5}} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} 1 & 4\,\mathrm{e}^{\frac{t}{5}} - 4 \\ 0 & \mathrm{e}^{\frac{t}{5}} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$

- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{x}_p(t) = \left[\begin{array}{c} 20\,\mathrm{e}^{\frac{t}{5}} - 4t - 20 \\ 5\,\mathrm{e}^{\frac{t}{5}} - 5 - t \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \left[\begin{array}{c} 20\,\mathrm{e}^{\frac{t}{5}} - 4t - 20 \\ 5\,\mathrm{e}^{\frac{t}{5}} - 5 - t \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} -20 + 4(5 + c_2)\,\mathrm{e}^{\frac{t}{5}} - 4t + c_1 \\ (5 + c_2)\,\mathrm{e}^{\frac{t}{5}} - 5 - t \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = -20 + 4(5 + c_2)\,\mathrm{e}^{\frac{t}{5}} - 4t + c_1, y = (5 + c_2)\,\mathrm{e}^{\frac{t}{5}} - 5 - t\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)-4*diff(y(t),t)=0,2*diff(x(t),t)-3*diff(y(t),t)=y(t)+t],singsol=all)
```

$$x(t) = 5\,\mathrm{e}^{\frac{t}{5}} c_1 - 4t + c_2$$
$$y(t) = \frac{5\,\mathrm{e}^{\frac{t}{5}} c_1}{4} - 5 - t$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 45

```
DSolve[{x'[t]-4*y'[t]==0,2*x'[t]-3*y'[t]==y[t]+t},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to -4t + 4c_2 e^{t/5} - 20 + c_1 - 4c_2$$
$$y(t) \to -t + c_2 e^{t/5} - 5$$

## 6.7 problem Problem 4(g)



Internal problem ID [12375]
Internal file name [`OUTPUT/11027_Monday_October_02_2023_11_46_17_PM_7488110/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 6.4 Reduction to a single ODE. Problems page 415
**Problem number**: Problem 4(g).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{\sin(t)}{4} + \frac{x(t)}{4} + \frac{y}{4} + \frac{t}{4} \\
y' &= \frac{\sin(t)}{8} - \frac{3x(t)}{8} - \frac{3y}{8} - \frac{3t}{8}
\end{aligned}
$$

### 6.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{\sin(t)}{4} + \frac{t}{4} \\ \frac{\sin(t)}{8} - \frac{3t}{8} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,e^{-\frac{t}{8}}+3 & 2-2\,e^{-\frac{t}{8}} \\ -3+3\,e^{-\frac{t}{8}} & 3\,e^{-\frac{t}{8}}-2 \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -2\,e^{-\frac{t}{8}}+3 & 2-2\,e^{-\frac{t}{8}} \\ -3+3\,e^{-\frac{t}{8}} & 3\,e^{-\frac{t}{8}}-2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(-2\,e^{-\frac{t}{8}}+3\right)c_1 + \left(2-2\,e^{-\frac{t}{8}}\right)c_2 \\ \left(-3+3\,e^{-\frac{t}{8}}\right)c_1 + \left(3\,e^{-\frac{t}{8}}-2\right)c_2 \end{bmatrix} \\ &= \begin{bmatrix} (-2c_1-2c_2)\,e^{-\frac{t}{8}}+3c_1+2c_2 \\ (3c_1+3c_2)\,e^{-\frac{t}{8}}-3c_1-2c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} 3-2\,e^{\frac{t}{8}} & -2\,e^{\frac{t}{8}}+2 \\ 3\,e^{\frac{t}{8}}-3 & -2+3\,e^{\frac{t}{8}} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -2\,e^{-\frac{t}{8}}+3 & 2-2\,e^{-\frac{t}{8}} \\ -3+3\,e^{-\frac{t}{8}} & 3\,e^{-\frac{t}{8}}-2 \end{bmatrix} \int \begin{bmatrix} 3-2\,e^{\frac{t}{8}} & -2\,e^{\frac{t}{8}}+2 \\ 3\,e^{\frac{t}{8}}-3 & -2+3\,e^{\frac{t}{8}} \end{bmatrix} \begin{bmatrix} \frac{\sin(t)}{4}+\frac{t}{4} \\ \frac{\sin(t)}{8}-\frac{3t}{8} \end{bmatrix} dt \\ &= \begin{bmatrix} -2\,e^{-\frac{t}{8}}+3 & 2-2\,e^{-\frac{t}{8}} \\ -3+3\,e^{-\frac{t}{8}} & 3\,e^{-\frac{t}{8}}-2 \end{bmatrix} \begin{bmatrix} \frac{2(-520+65t+24\cos(t)-3\sin(t))e^{\frac{t}{8}}}{65}-\cos(t) \\ \frac{3(520-65t-24\cos(t)+3\sin(t))e^{\frac{t}{8}}}{65}+\cos(t) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{17\cos(t)}{65}+2t-16-\frac{6\sin(t)}{65} \\ -\frac{7\cos(t)}{65}-3t+24+\frac{9\sin(t)}{65} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} -16 + 2(-c_1 - c_2)\,\mathrm{e}^{-\frac{t}{8}} + 3c_1 + 2c_2 - \frac{17\cos(t)}{65} + 2t - \frac{6\sin(t)}{65} \\ 24 + 3(c_1 + c_2)\,\mathrm{e}^{-\frac{t}{8}} - 3c_1 - 2c_2 - \frac{7\cos(t)}{65} - 3t + \frac{9\sin(t)}{65} \end{bmatrix}$$

### 6.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} \frac{\sin(t)}{4} + \frac{t}{4} \\ \frac{\sin(t)}{8} - \frac{3t}{8} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{1}{4} - \lambda & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \frac{1}{8}\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{8}$$
$$\lambda_2 = 0$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 0 | 1 | real eigenvalue |
| $-\frac{1}{8}$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{8} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{8}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} - \left(-\frac{1}{8}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{8} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{8} & \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{1}{4} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{8} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-\frac{1}{8}$ | 1 | 1 | No | $\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$ |
| 0 | 1 | 1 | No | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{1}{8}$ is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-\frac{t}{8}} \\ &= \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} e^{-\frac{t}{8}}\end{aligned}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{0} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{0}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{-\frac{t}{8}}}{3} \\ \mathrm{e}^{-\frac{t}{8}} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{-\frac{t}{8}}}{3} & -1 \\ \mathrm{e}^{-\frac{t}{8}} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 3\,\mathrm{e}^{\frac{t}{8}} & 3\,\mathrm{e}^{\frac{t}{8}} \\ -3 & -2 \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{2\,e^{-\frac{t}{8}}}{3} & -1 \\ e^{-\frac{t}{8}} & 1 \end{bmatrix} \int \begin{bmatrix} 3\,e^{\frac{t}{8}} & 3\,e^{\frac{t}{8}} \\ -3 & -2 \end{bmatrix} \begin{bmatrix} \frac{\sin(t)}{4} + \frac{t}{4} \\ \frac{\sin(t)}{8} - \frac{3t}{8} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{2\,e^{-\frac{t}{8}}}{3} & -1 \\ e^{-\frac{t}{8}} & 1 \end{bmatrix} \int \begin{bmatrix} -\frac{3\,e^{\frac{t}{8}}(-3\sin(t)+t)}{8} \\ -\sin(t) \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{2\,e^{-\frac{t}{8}}}{3} & -1 \\ e^{-\frac{t}{8}} & 1 \end{bmatrix} \begin{bmatrix} \frac{3(520-65t-24\cos(t)+3\sin(t))e^{\frac{t}{8}}}{65} \\ \cos(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{17\cos(t)}{65} + 2t - 16 - \frac{6\sin(t)}{65} \\ -\frac{7\cos(t)}{65} - 3t + 24 + \frac{9\sin(t)}{65} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-\frac{t}{8}}}{3} \\ c_1 e^{-\frac{t}{8}} \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{17\cos(t)}{65} + 2t - 16 - \frac{6\sin(t)}{65} \\ -\frac{7\cos(t)}{65} - 3t + 24 + \frac{9\sin(t)}{65} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{-\frac{t}{8}}}{3} - c_2 - \frac{17\cos(t)}{65} + 2t - 16 - \frac{6\sin(t)}{65} \\ c_1 e^{-\frac{t}{8}} + c_2 - \frac{7\cos(t)}{65} - 3t + 24 + \frac{9\sin(t)}{65} \end{bmatrix}$$

### 6.7.3 Maple step by step solution

Let's solve

$$\left[x'(t) = \frac{\sin(t)}{4} + \frac{x(t)}{4} + \frac{y}{4} + \frac{t}{4}, y' = \frac{\sin(t)}{8} - \frac{3x(t)}{8} - \frac{3y}{8} - \frac{3t}{8}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{\sin(t)}{4} + \frac{t}{4} \\ \frac{\sin(t)}{8} - \frac{3t}{8} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{\sin(t)}{4} + \frac{t}{4} \\ \frac{\sin(t)}{8} - \frac{3t}{8} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} \frac{\sin(t)}{4} + \frac{t}{4} \\ \frac{\sin(t)}{8} - \frac{3t}{8} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-\frac{1}{8}, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-\frac{1}{8}, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-\frac{t}{8}} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{-\frac{t}{8}}}{3} & -1 \\ \mathrm{e}^{-\frac{t}{8}} & 1 \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{-\frac{t}{8}}}{3} & -1 \\ \mathrm{e}^{-\frac{t}{8}} & 1 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} -2\,\mathrm{e}^{-\frac{t}{8}} + 3 & 2 - 2\,\mathrm{e}^{-\frac{t}{8}} \\ -3 + 3\,\mathrm{e}^{-\frac{t}{8}} & 3\,\mathrm{e}^{-\frac{t}{8}} - 2 \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t)\cdot \vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\vec{x}_p(t)=\left[\begin{array}{c} -\frac{17\cos(t)}{65}-15+\frac{992\,\mathrm{e}^{-\frac{t}{8}}}{65}-\frac{6\sin(t)}{65}+2t \\ -\frac{7\cos(t)}{65}+23-\frac{1488\,\mathrm{e}^{-\frac{t}{8}}}{65}+\frac{9\sin(t)}{65}-3t \end{array}\right]$

- Plug particular solution back into general solution

$$\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\left[\begin{array}{c} -\frac{17\cos(t)}{65}-15+\frac{992\,\mathrm{e}^{-\frac{t}{8}}}{65}-\frac{6\sin(t)}{65}+2t \\ -\frac{7\cos(t)}{65}+23-\frac{1488\,\mathrm{e}^{-\frac{t}{8}}}{65}+\frac{9\sin(t)}{65}-3t \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right]=\left[\begin{array}{c} -\frac{2c_1\mathrm{e}^{-\frac{t}{8}}}{3}-\frac{17\cos(t)}{65}-15+\frac{992\,\mathrm{e}^{-\frac{t}{8}}}{65}-\frac{6\sin(t)}{65}+2t-c_2 \\ c_1\mathrm{e}^{-\frac{t}{8}}-\frac{7\cos(t)}{65}+23-\frac{1488\,\mathrm{e}^{-\frac{t}{8}}}{65}+\frac{9\sin(t)}{65}-3t+c_2 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=-\frac{2c_1\mathrm{e}^{-\frac{t}{8}}}{3}-\frac{17\cos(t)}{65}-15+\frac{992\,\mathrm{e}^{-\frac{t}{8}}}{65}-\frac{6\sin(t)}{65}+2t-c_2, y=c_1\mathrm{e}^{-\frac{t}{8}}-\frac{7\cos(t)}{65}+23-\frac{1488\,\mathrm{e}^{-\frac{t}{8}}}{65}+\right.$$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 51

```
dsolve([3*diff(x(t),t)+2*diff(y(t),t)=sin(t),diff(x(t),t)-2*diff(y(t),t)=x(t)+y(t)+t],singso
```

$$x(t)=-8\,\mathrm{e}^{-\frac{t}{8}}c_1-\frac{6\sin(t)}{65}-\frac{17\cos(t)}{65}+2t+c_2$$
$$y(t)=12\,\mathrm{e}^{-\frac{t}{8}}c_1-\frac{7\cos(t)}{65}+\frac{9\sin(t)}{65}+8-3t-c_2$$

✓ <u>Solution by Mathematica</u>

Time used: 0.358 (sec). Leaf size: 98

```
DSolve[{x'[t]+2*y'[t]==Sin[t],x'[t]-2*y'[t]==x[t]+y[t]+t},{x[t],y[t]},t,IncludeSingularSolut
```

$$x(t) \to -2t - \frac{6\sin(t)}{17} - \frac{7\cos(t)}{17} + 2c_1e^{t/4} + 2c_2e^{t/4} - 8 - c_1 - 2c_2$$
$$y(t) \to t + \frac{3\sin(t)}{17} - \frac{5\cos(t)}{17} - c_1e^{t/4} - c_2e^{t/4} + 4 + c_1 + 2c_2$$

# 7 Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514

## 7.1 problem Problem 3(a)



Internal problem ID [12376]
Internal file name [`OUTPUT/11028_Wednesday_October_04_2023_01_27_05_AM_64152595/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 3(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -4x(t) + 9y + 12\,\mathrm{e}^{-t} \\
y' &= -5x(t) + 2y
\end{aligned}
$$

### 7.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 12\,\mathrm{e}^{-t} \\ 0 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t}\cos(6t) - \frac{e^{-t}\sin(6t)}{2} & \frac{3e^{-t}\sin(6t)}{2} \\ -\frac{5e^{-t}\sin(6t)}{6} & e^{-t}\cos(6t) + \frac{e^{-t}\sin(6t)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-t}(2\cos(6t)-\sin(6t))}{2} & \frac{3e^{-t}\sin(6t)}{2} \\ -\frac{5e^{-t}\sin(6t)}{6} & \frac{e^{-t}(2\cos(6t)+\sin(6t))}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{e^{-t}(2\cos(6t)-\sin(6t))}{2} & \frac{3e^{-t}\sin(6t)}{2} \\ -\frac{5e^{-t}\sin(6t)}{6} & \frac{e^{-t}(2\cos(6t)+\sin(6t))}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-t}(2\cos(6t)-\sin(6t))c_1}{2} + \frac{3e^{-t}\sin(6t)c_2}{2} \\ -\frac{5e^{-t}\sin(6t)c_1}{6} + \frac{e^{-t}(2\cos(6t)+\sin(6t))c_2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{-t}(-c_1+3c_2)\sin(6t)}{2} + e^{-t}\cos(6t)\,c_1 \\ -\frac{5e^{-t}\left(\left(c_1-\frac{3c_2}{5}\right)\sin(6t)-\frac{6c_2\cos(6t)}{5}\right)}{6} \end{bmatrix}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$e^{-At} = (e^{At})^{-1}$$

$$= \begin{bmatrix} \frac{(2\cos(6t)+\sin(6t))e^{t}}{2} & -\frac{3\sin(6t)e^{t}}{2} \\ \frac{5\sin(6t)e^{t}}{6} & \frac{(2\cos(6t)-\sin(6t))e^{t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{-t}(2\cos(6t)-\sin(6t))}{2} & \frac{3e^{-t}\sin(6t)}{2} \\ -\frac{5e^{-t}\sin(6t)}{6} & \frac{e^{-t}(2\cos(6t)+\sin(6t))}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(2\cos(6t)+\sin(6t))e^{t}}{2} & -\frac{3\sin(6t)e^{t}}{2} \\ \frac{5\sin(6t)e^{t}}{6} & \frac{(2\cos(6t)-\sin(6t))e^{t}}{2} \end{bmatrix} \begin{bmatrix} 12e^{-t} \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} \frac{e^{-t}(2\cos(6t)-\sin(6t))}{2} & \frac{3e^{-t}\sin(6t)}{2} \\ -\frac{5e^{-t}\sin(6t)}{6} & \frac{e^{-t}(2\cos(6t)+\sin(6t))}{2} \end{bmatrix} \begin{bmatrix} 2\sin(6t) - \cos(6t) \\ -\frac{5\cos(6t)}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} \\ -\frac{5e^{-t}}{3} \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} e^{-t}\left(-1 + \frac{(-c_1+3c_2)\sin(6t)}{2} + c_1\cos(6t)\right) \\ -\frac{5\left(\left(c_1-\frac{3c_2}{5}\right)\sin(6t)-\frac{6c_2\cos(6t)}{5}+2\right)e^{-t}}{6} \end{bmatrix}$$

### 7.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 12\,e^{-t} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -4-\lambda & 9 \\ -5 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 37 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + 6i$$
$$\lambda_2 = -1 - 6i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-1+6i$ | 1 | complex eigenvalue |
| $-1-6i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - 6i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} - (-1-6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3+6i & 9 \\ -5 & 3+6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3+6i & 9 & 0 \\ -5 & 3+6i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{3} - \frac{2i}{3}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -3+6i & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+6i & 9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{5} + \frac{6i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{6\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{5} + \frac{6i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{6\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} + \frac{6i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{6\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} + \frac{6i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{5} + \frac{6\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 3 + 6i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + 6i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} - (-1 + 6i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 - 6i & 9 \\ -5 & 3 - 6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 - 6i & 9 & 0 \\ -5 & 3 - 6i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{3} + \frac{2i}{3}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 - 6i & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-6i & 9 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{3}{5} - \frac{6i}{5}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{6\,\mathrm{I}}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{5} - \frac{6i}{5}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{6\,\mathrm{I}}{5}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{5} - \frac{6i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{6\,\mathrm{I}}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} - \frac{6i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{3}{5} - \frac{6\,\mathrm{I}}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 3-6i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| eigenvalue | multiplicity: algebraic $m$ | multiplicity: geometric $k$ | defective? | eigenvectors |
|---|---|---|---|---|
| $-1+6i$ | 1 | 1 | No | $\begin{bmatrix} \frac{3}{5}-\frac{6i}{5} \\ 1 \end{bmatrix}$ |
| $-1-6i$ | 1 | 1 | No | $\begin{bmatrix} \frac{3}{5}+\frac{6i}{5} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right) \mathrm{e}^{(-1+6i)t} \\ \mathrm{e}^{(-1+6i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{3}{5}+\frac{6i}{5}\right) \mathrm{e}^{(-1-6i)t} \\ \mathrm{e}^{(-1-6i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right) \mathrm{e}^{(-1+6i)t} & \left(\frac{3}{5}+\frac{6i}{5}\right) \mathrm{e}^{(-1-6i)t} \\ \mathrm{e}^{(-1+6i)t} & \mathrm{e}^{(-1-6i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{5i\mathrm{e}^{(1-6i)t}}{12} & \left(\frac{1}{2}-\frac{i}{4}\right) \mathrm{e}^{(1-6i)t} \\ -\frac{5i\mathrm{e}^{(1+6i)t}}{12} & \left(\frac{1}{2}+\frac{i}{4}\right) \mathrm{e}^{(1+6i)t} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right)e^{(-1+6i)t} & \left(\frac{3}{5}+\frac{6i}{5}\right)e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix} \int \begin{bmatrix} \frac{5i e^{(1-6i)t}}{12} & \left(\frac{1}{2}-\frac{i}{4}\right)e^{(1-6i)t} \\ -\frac{5i e^{(1+6i)t}}{12} & \left(\frac{1}{2}+\frac{i}{4}\right)e^{(1+6i)t} \end{bmatrix} \begin{bmatrix} 12\,e^{-t} \\ 0 \end{bmatrix} dt$$

$$= \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right)e^{(-1+6i)t} & \left(\frac{3}{5}+\frac{6i}{5}\right)e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix} \int \begin{bmatrix} 5ie^{-6it} \\ -5ie^{6it} \end{bmatrix} dt$$

$$= \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right)e^{(-1+6i)t} & \left(\frac{3}{5}+\frac{6i}{5}\right)e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix} \begin{bmatrix} -\frac{5\,e^{-6it}}{6} \\ -\frac{5\,e^{6it}}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} \\ -\frac{5\,e^{-t}}{3} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right)c_1e^{(-1+6i)t} \\ c_1e^{(-1+6i)t} \end{bmatrix} + \begin{bmatrix} \left(\frac{3}{5}+\frac{6i}{5}\right)c_2e^{(-1-6i)t} \\ c_2e^{(-1-6i)t} \end{bmatrix} + \begin{bmatrix} -e^{-t} \\ -\frac{5\,e^{-t}}{3} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{3}{5}-\frac{6i}{5}\right)c_1e^{(-1+6i)t} + \left(\frac{3}{5}+\frac{6i}{5}\right)c_2e^{(-1-6i)t} - e^{-t} \\ c_1e^{(-1+6i)t} + c_2e^{(-1-6i)t} - \frac{5\,e^{-t}}{3} \end{bmatrix}$$

### 7.1.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -4x(t) + 9y + \frac{12}{e^t}, y' = -5x(t) + 2y\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -\frac{4x(t)e^t - 9y\,e^t - 12}{e^t} + 4x(t) - 9y \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -4 & 9 \\ -5 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-1-6\,\mathrm{I}, \begin{bmatrix} \frac{3}{5}+\frac{6\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[-1+6\,\mathrm{I}, \begin{bmatrix} \frac{3}{5}-\frac{6\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-6\,\mathrm{I}, \begin{bmatrix} \frac{3}{5}+\frac{6\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-6\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{3}{5}+\frac{6\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(6t) - \mathrm{I}\sin(6t)) \cdot \begin{bmatrix} \frac{3}{5}+\frac{6\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{-t} \cdot \begin{bmatrix} \left(\frac{3}{5}+\frac{6\,\mathrm{I}}{5}\right)(\cos(6t) - \mathrm{I}\sin(6t)) \\ \cos(6t) - \mathrm{I}\sin(6t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{3\cos(6t)}{5} + \frac{6\sin(6t)}{5} \\ \cos(6t) \end{bmatrix}, \vec{x}_2(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{3\sin(6t)}{5} + \frac{6\cos(6t)}{5} \\ -\sin(6t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} \frac{3\cos(6t)}{5} + \frac{6\sin(6t)}{5} \\ \cos(6t) \end{array}\right] + c_2 \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -\frac{3\sin(6t)}{5} + \frac{6\cos(6t)}{5} \\ -\sin(6t) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{3\,\mathrm{e}^{-t}\left((2c_2+c_1)\cos(6t)+2\sin(6t)\left(c_1-\frac{c_2}{2}\right)\right)}{5} \\ \mathrm{e}^{-t}(c_1\cos(6t) - c_2\sin(6t)) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{3\,\mathrm{e}^{-t}\left((2c_2+c_1)\cos(6t)+2\sin(6t)\left(c_1-\frac{c_2}{2}\right)\right)}{5}, y = \mathrm{e}^{-t}(c_1\cos(6t) - c_2\sin(6t))\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 66

```
dsolve([diff(x(t),t)=-4*x(t)+9*y(t)+12*exp(-t),diff(y(t),t)=-5*x(t)+2*y(t)],singsol=all)
```

$$x(t) = \frac{\mathrm{e}^{-t}(6\sin(6t)\,c_1 + 3\sin(6t)\,c_2 + 3\cos(6t)\,c_1 - 6\cos(6t)\,c_2 - 5)}{5}$$
$$y(t) = \frac{\mathrm{e}^{-t}(-5 + 3\sin(6t)\,c_2 + 3\cos(6t)\,c_1)}{3}$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 73

```
DSolve[{x'[t]==-4*x[t]+9*y[t]+12*Exp[-t],y'[t]==-5*x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingula
```

$$x(t) \to \frac{1}{2}e^{-t}(2c_1\cos(6t) - (c_1 - 3c_2)\sin(6t) - 2)$$
$$y(t) \to \frac{1}{6}e^{-t}(6c_2\cos(6t) + (3c_2 - 5c_1)\sin(6t) - 10)$$

## 7.2 problem Problem 3(b)



Internal problem ID [12377]
Internal file name [`OUTPUT/11029_Wednesday_October_04_2023_01_27_08_AM_97534733/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 3(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -7x(t) + 6y + 6\,\mathrm{e}^{-t} \\
y' &= -12x(t) + 5y + 37
\end{aligned}
$$

### 7.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 6\,\mathrm{e}^{-t} \\ 37 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{-t}\cos(6t) - e^{-t}\sin(6t) & e^{-t}\sin(6t) \\ -2\,e^{-t}\sin(6t) & e^{-t}\cos(6t) + e^{-t}\sin(6t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(6t) - \sin(6t)) & e^{-t}\sin(6t) \\ -2\,e^{-t}\sin(6t) & e^{-t}(\cos(6t) + \sin(6t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}(\cos(6t) - \sin(6t)) & e^{-t}\sin(6t) \\ -2\,e^{-t}\sin(6t) & e^{-t}(\cos(6t) + \sin(6t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(6t) - \sin(6t))\, c_1 + e^{-t}\sin(6t)\, c_2 \\ -2\,e^{-t}\sin(6t)\, c_1 + e^{-t}(\cos(6t) + \sin(6t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((c_2 - c_1)\sin(6t) + c_1\cos(6t))\, e^{-t} \\ e^{-t}(c_2\cos(6t) - 2c_1\sin(6t) + \sin(6t)\, c_2) \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\, dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} (\cos(6t) + \sin(6t))\, e^{t} & -\sin(6t)\, e^{t} \\ 2\sin(6t)\, e^{t} & (\cos(6t) - \sin(6t))\, e^{t} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{-t}(\cos(6t) - \sin(6t)) & e^{-t}\sin(6t) \\ -2\,e^{-t}\sin(6t) & e^{-t}(\cos(6t) + \sin(6t)) \end{bmatrix} \int \begin{bmatrix} (\cos(6t) + \sin(6t))\, e^{t} & -\sin(6t)\, e^{t} \\ 2\sin(6t)\, e^{t} & (\cos(6t) - \sin(6 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(6t) - \sin(6t)) & e^{-t}\sin(6t) \\ -2\,e^{-t}\sin(6t) & e^{-t}(\cos(6t) + \sin(6t)) \end{bmatrix} \begin{bmatrix} (-1 + 6\,e^{t})\cos(6t) - \sin(6t)\,(e^{t} - 1) \\ (-2 + 7\,e^{t})\cos(6t) + 5\sin(6t)\, e^{t} \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t} + 6 \\ -2\,e^{-t} + 7 \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} 6 + ((c_2 - c_1)\sin(6t) + c_1\cos(6t) - 1)\,e^{-t} \\ 7 + ((-2c_1 + c_2)\sin(6t) + c_2\cos(6t) - 2)\,e^{-t} \end{bmatrix}$$

### 7.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 6\,e^{-t} \\ 37 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & 6 \\ -12 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 37 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + 6i$$
$$\lambda_2 = -1 - 6i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-1 + 6i$ | 1 | complex eigenvalue |
| $-1 - 6i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - 6i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} - (-1 - 6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -6 + 6i & 6 \\ -12 & 6 + 6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6 + 6i & 6 & 0 \\ -12 & 6 + 6i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} -6 + 6i & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 + 6i & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1+6i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} - (-1+6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6-6i & 6 \\ -12 & 6-6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -6-6i & 6 & 0 \\ -12 & 6-6i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1+i)R_1 \Longrightarrow \left[\begin{array}{cc|c} -6-6i & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6-6i & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-1 + 6i$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$ |
| $-1 - 6i$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+6i)t} \\ e^{(-1+6i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-6i)t} \\ e^{(-1-6i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+6i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} i e^{(1-6i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-6i)t} \\ -i e^{(1+6i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+6i)t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+6i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix} \int \begin{bmatrix} i e^{(1-6i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-6i)t} \\ -i e^{(1+6i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+6i)t} \end{bmatrix} \begin{bmatrix} 6\,e^{-t} \\ 37 \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+6i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix} \int \begin{bmatrix} \left(\frac{37}{2}-\frac{37i}{2}\right) e^{(1-6i)t} + 6i e^{-6it} \\ \left(\frac{37}{2}+\frac{37i}{2}\right) e^{(1+6i)t} - 6i e^{6it} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+6i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-6i)t} \\ e^{(-1+6i)t} & e^{(-1-6i)t} \end{bmatrix} \begin{bmatrix} \left(\frac{7}{2}+\frac{5i}{2}\right) e^{(1-6i)t} - e^{-6it} \\ \left(\frac{7}{2}-\frac{5i}{2}\right) e^{(1+6i)t} - e^{6it} \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t} + 6 \\ -2\,e^{-t} + 7 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(-1+6i)t} \\ c_1 \mathrm{e}^{(-1+6i)t} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(-1-6i)t} \\ c_2 \mathrm{e}^{(-1-6i)t} \end{bmatrix} + \begin{bmatrix} -\mathrm{e}^{-t}+6 \\ -2\,\mathrm{e}^{-t}+7 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_1 \mathrm{e}^{(-1+6i)t} + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 \mathrm{e}^{(-1-6i)t} - \mathrm{e}^{-t}+6 \\ c_1 \mathrm{e}^{(-1+6i)t} + c_2 \mathrm{e}^{(-1-6i)t} - 2\,\mathrm{e}^{-t}+7 \end{bmatrix}$$

### 7.2.3 Maple step by step solution

Let's solve

$\left[x'(t) = -7x(t) + 6y + \frac{6}{\mathrm{e}^t}, y' = -12x(t) + 5y + 37\right]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -\frac{7x(t)\mathrm{e}^t - 6y\,\mathrm{e}^t - 6}{\mathrm{e}^t} + 7x(t) - 6y \\ 37 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 37 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 37 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -7 & 6 \\ -12 & 5 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$
- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

  $$\left[\left[-1-6\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+6\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$
- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[-1-6\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$
- Solution from eigenpair

  $$\mathrm{e}^{(-1-6\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$
- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{-t} \cdot (\cos(6t) - \mathrm{I}\sin(6t)) \cdot \begin{bmatrix} \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$
- Simplify expression

  $$\mathrm{e}^{-t} \cdot \begin{bmatrix} \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(6t) - \mathrm{I}\sin(6t)) \\ \cos(6t) - \mathrm{I}\sin(6t) \end{bmatrix}$$
- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} \frac{\cos(6t)}{2} + \frac{\sin(6t)}{2} \\ \cos(6t) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{-t} \cdot \begin{bmatrix} -\frac{\sin(6t)}{2} + \frac{\cos(6t)}{2} \\ -\sin(6t) \end{bmatrix}\right]$$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} \mathrm{e}^{-t}\left(\frac{\cos(6t)}{2} + \frac{\sin(6t)}{2}\right) & \mathrm{e}^{-t}\left(-\frac{\sin(6t)}{2} + \frac{\cos(6t)}{2}\right) \\ \mathrm{e}^{-t}\cos(6t) & -\mathrm{e}^{-t}\sin(6t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where $I$ is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t)=\left[\begin{array}{cc} \mathrm{e}^{-t}\left(\frac{\cos(6t)}{2}+\frac{\sin(6t)}{2}\right) & \mathrm{e}^{-t}\left(-\frac{\sin(6t)}{2}+\frac{\cos(6t)}{2}\right) \\ \mathrm{e}^{-t}\cos(6t) & -\mathrm{e}^{-t}\sin(6t) \end{array}\right]\cdot\frac{1}{\left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{array}\right]}$$
- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t)=\left[\begin{array}{cc} \mathrm{e}^{-t}(\cos(6t)-\sin(6t)) & \mathrm{e}^{-t}\sin(6t) \\ -2\,\mathrm{e}^{-t}\sin(6t) & \mathrm{e}^{-t}(\cos(6t)+\sin(6t)) \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t)=\Phi(t)\cdot\vec{v}(t)$
- Take the derivative of the particular solution

  $\vec{x}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} 6 + (-6\cos(6t) - \sin(6t))\,\mathrm{e}^{-t} \\ 7 + (-7\cos(6t) + 5\sin(6t))\,\mathrm{e}^{-t} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \begin{bmatrix} 6 + (-6\cos(6t) - \sin(6t))\,\mathrm{e}^{-t} \\ 7 + (-7\cos(6t) + 5\sin(6t))\,\mathrm{e}^{-t} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} 6 + \frac{((c_1+c_2-12)\cos(6t)+\sin(6t)(c_1-c_2-2))\mathrm{e}^{-t}}{2} \\ 7 + ((c_1-7)\cos(6t) - \sin(6t)(c_2-5))\,\mathrm{e}^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = 6 + \frac{((c_1+c_2-12)\cos(6t)+\sin(6t)(c_1-c_2-2))\mathrm{e}^{-t}}{2}, y = 7 + ((c_1-7)\cos(6t) - \sin(6t)(c_2-5))\,\mathrm{e}^{-t}\right\}$$

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 82

```
dsolve([diff(x(t),t)=-7*x(t)+6*y(t)+6*exp(-t),diff(y(t),t)=-12*x(t)+5*y(t)+37],singsol=all)
```

$$x(t) = 6 + \frac{\mathrm{e}^{-t}(-2 + \sin(6t)\,c_1 + \sin(6t)\,c_2 + \cos(6t)\,c_1 - \cos(6t)\,c_2 - 2\sin(6t) - 2\cos(6t))}{2}$$

$$y(t) = 7 + \mathrm{e}^{-t}(-2 + \sin(6t)\,c_2 + \cos(6t)\,c_1 - 2\cos(6t))$$

✓ Solution by Mathematica

Time used: 0.387 (sec). Leaf size: 72

```
DSolve[{x'[t]==-7*x[t]+6*y[t]+6*Exp[-t],y'[t]==-12*x[t]+5*y[t]+37},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to e^{-t}\left(6e^t + c_1\cos(6t) + (c_2 - c_1)\sin(6t) - 1\right)$$
$$y(t) \to e^{-t}\left(7e^t + c_2\cos(6t) + (c_2 - 2c_1)\sin(6t) - 2\right)$$

## 7.3 problem Problem 3(c)



Internal problem ID [12378]
Internal file name [`OUTPUT/11030_Wednesday_October_04_2023_01_27_09_AM_7665776/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 3(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -7x(t) + 10y + 18\,\mathrm{e}^{t} \\
y' &= -10x(t) + 9y + 37
\end{aligned}
$$

### 7.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 18\,\mathrm{e}^{t} \\ 37 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(6t)\,\mathrm{e}^{t} - \frac{4\sin(6t)\mathrm{e}^{t}}{3} & \frac{5\sin(6t)\mathrm{e}^{t}}{3} \\ -\frac{5\sin(6t)\mathrm{e}^{t}}{3} & \cos(6t)\,\mathrm{e}^{t} + \frac{4\sin(6t)\mathrm{e}^{t}}{3} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{t}(3\cos(6t)-4\sin(6t))}{3} & \frac{5\sin(6t)\mathrm{e}^{t}}{3} \\ -\frac{5\sin(6t)\mathrm{e}^{t}}{3} & \frac{\mathrm{e}^{t}(3\cos(6t)+4\sin(6t))}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{t}(3\cos(6t)-4\sin(6t))}{3} & \frac{5\sin(6t)\mathrm{e}^{t}}{3} \\ -\frac{5\sin(6t)\mathrm{e}^{t}}{3} & \frac{\mathrm{e}^{t}(3\cos(6t)+4\sin(6t))}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{t}(3\cos(6t)-4\sin(6t))c_1}{3} + \frac{5\sin(6t)\mathrm{e}^{t}c_2}{3} \\ -\frac{5\sin(6t)\mathrm{e}^{t}c_1}{3} + \frac{\mathrm{e}^{t}(3\cos(6t)+4\sin(6t))c_2}{3} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{t}(-4c_1+5c_2)\sin(6t)}{3} + \mathrm{e}^{t}\cos(6t)\,c_1 \\ -\frac{5\left(\left(c_1-\frac{4c_2}{5}\right)\sin(6t)-\frac{3c_2\cos(6t)}{5}\right)\mathrm{e}^{t}}{3} \end{bmatrix}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$e^{-At} = (e^{At})^{-1}$$
$$= \begin{bmatrix} \frac{(3\cos(6t)+4\sin(6t))\mathrm{e}^{-t}}{3} & -\frac{5\,\mathrm{e}^{-t}\sin(6t)}{3} \\ \frac{5\,\mathrm{e}^{-t}\sin(6t)}{3} & \frac{(3\cos(6t)-4\sin(6t))\mathrm{e}^{-t}}{3} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{\mathrm{e}^{t}(3\cos(6t)-4\sin(6t))}{3} & \frac{5\sin(6t)\mathrm{e}^{t}}{3} \\ -\frac{5\sin(6t)\mathrm{e}^{t}}{3} & \frac{\mathrm{e}^{t}(3\cos(6t)+4\sin(6t))}{3} \end{bmatrix} \int \begin{bmatrix} \frac{(3\cos(6t)+4\sin(6t))\mathrm{e}^{-t}}{3} & -\frac{5\,\mathrm{e}^{-t}\sin(6t)}{3} \\ \frac{5\,\mathrm{e}^{-t}\sin(6t)}{3} & \frac{(3\cos(6t)-4\sin(6t))\mathrm{e}^{-t}}{3} \end{bmatrix} \begin{bmatrix} 18\,\mathrm{e}^{t} \\ 37 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{t}(3\cos(6t)-4\sin(6t))}{3} & \frac{5\sin(6t)\mathrm{e}^{t}}{3} \\ -\frac{5\sin(6t)\mathrm{e}^{t}}{3} & \frac{\mathrm{e}^{t}(3\cos(6t)+4\sin(6t))}{3} \end{bmatrix} \begin{bmatrix} \frac{(30\cos(6t)+5\sin(6t))\mathrm{e}^{-t}}{3} - 4\cos(6t) + 3\sin(6t) - 4 \\ -5 + (-5+7\,\mathrm{e}^{-t})\cos(6t) + \frac{22\,\mathrm{e}^{-t}\sin(6t)}{3} \end{bmatrix}$$
$$= \begin{bmatrix} -4\cos(6t)\,\mathrm{e}^{t} - 3\sin(6t)\,\mathrm{e}^{t} - 4\,\mathrm{e}^{t} + 10 \\ -5\cos(6t)\,\mathrm{e}^{t} - 5\,\mathrm{e}^{t} + 7 \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} -\frac{4\left(c_1 - \frac{5c_2}{4} + \frac{9}{4}\right)e^t \sin(6t)}{3} + e^t(c_1 - 4)\cos(6t) - 4\,e^t + 10 \\ e^t(c_2 - 5)\cos(6t) - \frac{5\left(c_1 - \frac{4c_2}{5}\right)e^t \sin(6t)}{3} - 5\,e^t + 7 \end{bmatrix}$$

### 7.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 18\,e^t \\ 37 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & 10 \\ -10 & 9-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 37 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 6i$$
$$\lambda_2 = 1 - 6i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $1 + 6i$ | 1 | complex eigenvalue |
| $1 - 6i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 6i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} - (1 - 6i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -8 + 6i & 10 \\ -10 & 8 + 6i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -8 + 6i & 10 & 0 \\ -10 & 8 + 6i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{4}{5} - \frac{3i}{5}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} -8 + 6i & 10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 + 6i & 10 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{4}{5} + \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{4}{5} + \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{4}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 4 + 3i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 6i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} - (1 + 6i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 - 6i & 10 \\ -10 & 8 - 6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -8 - 6i & 10 & 0 \\ -10 & 8 - 6i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{4}{5} + \frac{3i}{5}\right) R_1 \implies \left[\begin{array}{cc|c} -8 - 6i & 10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8-6i & 10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{4}{5} - \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{4}{5} - \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{5} - \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{5} - \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{4}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 4-3i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $1+6i$ | 1 | 1 | No | $\left[\begin{array}{c}\frac{4}{5}-\frac{3i}{5} \\ 1\end{array}\right]$ |
| $1-6i$ | 1 | 1 | No | $\left[\begin{array}{c}\frac{4}{5}+\frac{3i}{5} \\ 1\end{array}\right]$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t) \\ y\end{array}\right] = c_1\left[\begin{array}{c}\left(\frac{4}{5}-\frac{3i}{5}\right)\mathrm{e}^{(1+6i)t} \\ \mathrm{e}^{(1+6i)t}\end{array}\right] + c_2\left[\begin{array}{c}\left(\frac{4}{5}+\frac{3i}{5}\right)\mathrm{e}^{(1-6i)t} \\ \mathrm{e}^{(1-6i)t}\end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc}\left(\frac{4}{5}-\frac{3i}{5}\right)\mathrm{e}^{(1+6i)t} & \left(\frac{4}{5}+\frac{3i}{5}\right)\mathrm{e}^{(1-6i)t} \\ \mathrm{e}^{(1+6i)t} & \mathrm{e}^{(1-6i)t}\end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi\int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc}\frac{5i\,\mathrm{e}^{(-1-6i)t}}{6} & \left(\frac{1}{2}-\frac{2i}{3}\right)\mathrm{e}^{(-1-6i)t} \\ -\frac{5i\,\mathrm{e}^{(-1+6i)t}}{6} & \left(\frac{1}{2}+\frac{2i}{3}\right)\mathrm{e}^{(-1+6i)t}\end{array}\right]$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} \left(\frac{4}{5}-\frac{3i}{5}\right) e^{(1+6i)t} & \left(\frac{4}{5}+\frac{3i}{5}\right) e^{(1-6i)t} \\ e^{(1+6i)t} & e^{(1-6i)t} \end{array}\right] \int \left[\begin{array}{cc} \frac{5i e^{(-1-6i)t}}{6} & \left(\frac{1}{2}-\frac{2i}{3}\right) e^{(-1-6i)t} \\ -\frac{5i e^{(-1+6i)t}}{6} & \left(\frac{1}{2}+\frac{2i}{3}\right) e^{(-1+6i)t} \end{array}\right] \left[\begin{array}{c} 18 e^{t} \\ 37 \end{array}\right] dt$$

$$= \left[\begin{array}{cc} \left(\frac{4}{5}-\frac{3i}{5}\right) e^{(1+6i)t} & \left(\frac{4}{5}+\frac{3i}{5}\right) e^{(1-6i)t} \\ e^{(1+6i)t} & e^{(1-6i)t} \end{array}\right] \int \left[\begin{array}{c} \left(\frac{37}{2}-\frac{74i}{3}\right) e^{(-1-6i)t}+15i e^{-6it} \\ \left(\frac{37}{2}+\frac{74i}{3}\right) e^{(-1+6i)t}-15i e^{6it} \end{array}\right] dt$$

$$= \left[\begin{array}{cc} \left(\frac{4}{5}-\frac{3i}{5}\right) e^{(1+6i)t} & \left(\frac{4}{5}+\frac{3i}{5}\right) e^{(1-6i)t} \\ e^{(1+6i)t} & e^{(1-6i)t} \end{array}\right] \left[\begin{array}{c} \left(\frac{7}{2}+\frac{11i}{3}\right) e^{(-1-6i)t}-\frac{5 e^{-6it}}{2} \\ \left(\frac{7}{2}-\frac{11i}{3}\right) e^{(-1+6i)t}-\frac{5 e^{6it}}{2} \end{array}\right]$$

$$= \left[\begin{array}{c} 10-4 e^{t} \\ 7-5 e^{t} \end{array}\right]$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(\frac{4}{5}-\frac{3i}{5}\right) c_1 e^{(1+6i)t} \\ c_1 e^{(1+6i)t} \end{array}\right] + \left[\begin{array}{c} \left(\frac{4}{5}+\frac{3i}{5}\right) c_2 e^{(1-6i)t} \\ c_2 e^{(1-6i)t} \end{array}\right] + \left[\begin{array}{c} 10-4 e^{t} \\ 7-5 e^{t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \left(\frac{4}{5}-\frac{3i}{5}\right) c_1 e^{(1+6i)t} + \left(\frac{4}{5}+\frac{3i}{5}\right) c_2 e^{(1-6i)t} + 10 - 4 e^{t} \\ c_1 e^{(1+6i)t} + c_2 e^{(1-6i)t} + 7 - 5 e^{t} \end{array}\right]$$

### 7.3.3 Maple step by step solution

Let's solve

$[x'(t) = -7x(t) + 10y + 18 e^{t}, y' = -10x(t) + 9y + 37]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} -7 & 10 \\ -10 & 9 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 18 e^{t} \\ 37 \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 18\,e^{t} \\ 37 \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(t) = \begin{bmatrix} 18\,e^{t} \\ 37 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -7 & 10 \\ -10 & 9 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[1-6\,\mathrm{I}, \begin{bmatrix} \frac{4}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[1+6\,\mathrm{I}, \begin{bmatrix} \frac{4}{5}-\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-6\,\mathrm{I}, \begin{bmatrix} \frac{4}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-6\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{4}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{t} \cdot (\cos(6t) - \mathrm{I}\sin(6t)) \cdot \begin{bmatrix} \frac{4}{5}+\frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{t} \cdot \begin{bmatrix} \left(\frac{4}{5}+\frac{3\,\mathrm{I}}{5}\right)(\cos(6t) - \mathrm{I}\sin(6t)) \\ \cos(6t) - \mathrm{I}\sin(6t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \mathrm{e}^t \cdot \left[\begin{array}{c} \frac{4\cos(6t)}{5} + \frac{3\sin(6t)}{5} \\ \cos(6t) \end{array}\right], \vec{x}_2(t) = \mathrm{e}^t \cdot \left[\begin{array}{c} -\frac{4\sin(6t)}{5} + \frac{3\cos(6t)}{5} \\ -\sin(6t) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} \mathrm{e}^t\left(\frac{4\cos(6t)}{5} + \frac{3\sin(6t)}{5}\right) & \mathrm{e}^t\left(-\frac{4\sin(6t)}{5} + \frac{3\cos(6t)}{5}\right) \\ \cos(6t)\,\mathrm{e}^t & -\sin(6t)\,\mathrm{e}^t \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \mathrm{e}^t\left(\frac{4\cos(6t)}{5} + \frac{3\sin(6t)}{5}\right) & \mathrm{e}^t\left(-\frac{4\sin(6t)}{5} + \frac{3\cos(6t)}{5}\right) \\ \cos(6t)\,\mathrm{e}^t & -\sin(6t)\,\mathrm{e}^t \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} \frac{4}{5} & \frac{3}{5} \\ 1 & 0 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\mathrm{e}^t(3\cos(6t)-4\sin(6t))}{3} & \frac{5\sin(6t)\mathrm{e}^t}{3} \\ -\frac{5\sin(6t)\mathrm{e}^t}{3} & \frac{\mathrm{e}^t(3\cos(6t)+4\sin(6t))}{3} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot \vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t)=\left[\begin{array}{c}10-4\,\mathrm{e}^{t}+\frac{14\sin(6t)\mathrm{e}^{t}}{3}-6\cos(6t)\,\mathrm{e}^{t}\\ 7-5\,\mathrm{e}^{t}+\frac{22\sin(6t)\mathrm{e}^{t}}{3}-2\cos(6t)\,\mathrm{e}^{t}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+\left[\begin{array}{c}10-4\,\mathrm{e}^{t}+\frac{14\sin(6t)\mathrm{e}^{t}}{3}-6\cos(6t)\,\mathrm{e}^{t}\\ 7-5\,\mathrm{e}^{t}+\frac{22\sin(6t)\mathrm{e}^{t}}{3}-2\cos(6t)\,\mathrm{e}^{t}\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y\end{array}\right]=\left[\begin{array}{c}\frac{4\left(c_1+\frac{3c_2}{4}-\frac{15}{2}\right)\mathrm{e}^{t}\cos(6t)}{5}+\frac{3\,\mathrm{e}^{t}\left(c_1-\frac{4c_2}{3}+\frac{70}{9}\right)\sin(6t)}{5}-4\,\mathrm{e}^{t}+10\\ \mathrm{e}^{t}(-2+c_1)\cos(6t)+\frac{(-3c_2+22)\mathrm{e}^{t}\sin(6t)}{3}-5\,\mathrm{e}^{t}+7\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=\frac{4\left(c_1+\frac{3c_2}{4}-\frac{15}{2}\right)\mathrm{e}^{t}\cos(6t)}{5}+\frac{3\,\mathrm{e}^{t}\left(c_1-\frac{4c_2}{3}+\frac{70}{9}\right)\sin(6t)}{5}-4\,\mathrm{e}^{t}+10, y=\mathrm{e}^{t}(-2+c_1)\cos(6t)+\frac{(-3c_2+22}{3}\right.$$

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 81

```
dsolve([diff(x(t),t)=-7*x(t)+10*y(t)+18*exp(t),diff(y(t),t)=-10*x(t)+9*y(t)+37],singsol=all)
```

$$x(t)=10+\frac{\mathrm{e}^{t}(-20+3\sin(6t)\,c_1+4\sin(6t)\,c_2+4\cos(6t)\,c_1-3\cos(6t)\,c_2-15\sin(6t)-20\cos(6t))}{5}$$
$$y(t)=7+\mathrm{e}^{t}(-5+\sin(6t)\,c_2+\cos(6t)\,c_1-5\cos(6t))$$

✓ Solution by Mathematica

Time used: 0.622 (sec). Leaf size: 82

```
DSolve[{x'[t]==-7*x[t]+10*y[t]+18*Exp[t],y'[t]==-10*x[t]+9*y[t]+37},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to -4e^t + c_1e^t\cos(6t) - \frac{1}{3}(4c_1 - 5c_2)e^t\sin(6t) + 10$$
$$y(t) \to -5e^t + c_2e^t\cos(6t) - \frac{1}{3}(5c_1 - 4c_2)e^t\sin(6t) + 7$$

## 7.4 problem Problem 3(d)



Internal problem ID [12379]
Internal file name [`OUTPUT/11031_Wednesday_October_04_2023_01_27_10_AM_88058161/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 3(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -14x(t) + 39y + 78\sinh(t) \\
y' &= -6x(t) + 16y + 6\cosh(t)
\end{aligned}
$$

### 7.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 78\sinh(t) \\ 6\cosh(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{t}\cos(3t) - 5\,\mathrm{e}^{t}\sin(3t) & 13\,\mathrm{e}^{t}\sin(3t) \\ -2\,\mathrm{e}^{t}\sin(3t) & \mathrm{e}^{t}\cos(3t) + 5\,\mathrm{e}^{t}\sin(3t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(3t) - 5\sin(3t)) & 13\,\mathrm{e}^{t}\sin(3t) \\ -2\,\mathrm{e}^{t}\sin(3t) & \mathrm{e}^{t}(\cos(3t) + 5\sin(3t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(3t) - 5\sin(3t)) & 13\,\mathrm{e}^{t}\sin(3t) \\ -2\,\mathrm{e}^{t}\sin(3t) & \mathrm{e}^{t}(\cos(3t) + 5\sin(3t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(3t) - 5\sin(3t))\,c_1 + 13\,\mathrm{e}^{t}\sin(3t)\,c_2 \\ -2\,\mathrm{e}^{t}\sin(3t)\,c_1 + \mathrm{e}^{t}(\cos(3t) + 5\sin(3t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}((-5c_1 + 13c_2)\sin(3t) + c_1\cos(3t)) \\ \mathrm{e}^{t}(c_2\cos(3t) - 2c_1\sin(3t) + 5\sin(3t)\,c_2) \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} (\cos(3t) + 5\sin(3t))\,\mathrm{e}^{-t} & -13\,\mathrm{e}^{-t}\sin(3t) \\ 2\,\mathrm{e}^{-t}\sin(3t) & (\cos(3t) - 5\sin(3t))\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{t}(\cos(3t) - 5\sin(3t)) & 13\,\mathrm{e}^{t}\sin(3t) \\ -2\,\mathrm{e}^{t}\sin(3t) & \mathrm{e}^{t}(\cos(3t) + 5\sin(3t)) \end{bmatrix} \int \begin{bmatrix} (\cos(3t) + 5\sin(3t))\,\mathrm{e}^{-t} & -13\,\mathrm{e}^{-t}\,\mathrm{s} \\ 2\,\mathrm{e}^{-t}\sin(3t) & (\cos(3t) - 5 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\cos(3t) - 5\sin(3t)) & 13\,\mathrm{e}^{t}\sin(3t) \\ -2\,\mathrm{e}^{t}\sin(3t) & \mathrm{e}^{t}(\cos(3t) + 5\sin(3t)) \end{bmatrix} \begin{bmatrix} (-52 + 60\,\mathrm{e}^{-2t})\cos(3t) + 27\,\mathrm{e}^{-2t}\sin(3t) + \\ 21(-1 + \mathrm{e}^{-2t})\cos(3t) + 15\,\mathrm{e}^{-2t}\sin(3t) + \end{bmatrix} \\
&= \begin{bmatrix} -52\,\mathrm{e}^{t} + 60\,\mathrm{e}^{-t} \\ -21\,\mathrm{e}^{t} + 21\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t)\\ &= \begin{bmatrix} -5\left(c_1 - \frac{13c_2}{5}\right) e^t \sin(3t) + e^t \cos(3t) c_1 - 52 e^t + 60 e^{-t} \\ -2 e^t \left(c_1 - \frac{5c_2}{2}\right) \sin(3t) + e^t \cos(3t) c_2 - 21 e^t + 21 e^{-t} \end{bmatrix}\end{aligned}$$

### 7.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 78\sinh(t) \\ 6\cosh(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -14-\lambda & 39 \\ -6 & 16-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 3i$$
$$\lambda_2 = 1 - 3i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $1 + 3i$ | 1 | complex eigenvalue |
| $1 - 3i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} - (1 - 3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -15 + 3i & 39 \\ -6 & 15 + 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -15 + 3i & 39 & 0 \\ -6 & 15 + 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left( -\frac{5}{13} - \frac{i}{13} \right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -15 + 3i & 39 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -15 + 3i & 39 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{2}+\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{2}+\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} 5+i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} - (1+3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -15-3i & 39 \\ -6 & 15-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -15-3i & 39 & 0 \\ -6 & 15-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{5}{13} + \frac{i}{13}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -15-3i & 39 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -15-3i & 39 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{5}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{5}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{5}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{5}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{5}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{5}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 5 - i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $1 + 3i$ | 1 | 1 | No | $\begin{bmatrix} \frac{5}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$ |
| $1 - 3i$ | 1 | 1 | No | $\begin{bmatrix} \frac{5}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) \mathrm{e}^{(1+3i)t} \\ \mathrm{e}^{(1+3i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{5}{2} + \frac{i}{2}\right) \mathrm{e}^{(1-3i)t} \\ \mathrm{e}^{(1-3i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) \mathrm{e}^{(1+3i)t} & \left(\frac{5}{2} + \frac{i}{2}\right) \mathrm{e}^{(1-3i)t} \\ \mathrm{e}^{(1+3i)t} & \mathrm{e}^{(1-3i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} i\mathrm{e}^{(-1-3i)t} & \left(\frac{1}{2} - \frac{5i}{2}\right) \mathrm{e}^{(-1-3i)t} \\ -i\mathrm{e}^{(-1+3i)t} & \left(\frac{1}{2} + \frac{5i}{2}\right) \mathrm{e}^{(-1+3i)t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) \mathrm{e}^{(1+3i)t} & \left(\frac{5}{2} + \frac{i}{2}\right) \mathrm{e}^{(1-3i)t} \\ \mathrm{e}^{(1+3i)t} & \mathrm{e}^{(1-3i)t} \end{bmatrix} \int \begin{bmatrix} i\mathrm{e}^{(-1-3i)t} & \left(\frac{1}{2} - \frac{5i}{2}\right) \mathrm{e}^{(-1-3i)t} \\ -i\mathrm{e}^{(-1+3i)t} & \left(\frac{1}{2} + \frac{5i}{2}\right) \mathrm{e}^{(-1+3i)t} \end{bmatrix} \begin{bmatrix} 78 \sinh(t) \\ 6 \cosh(t) \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) \mathrm{e}^{(1+3i)t} & \left(\frac{5}{2} + \frac{i}{2}\right) \mathrm{e}^{(1-3i)t} \\ \mathrm{e}^{(1+3i)t} & \mathrm{e}^{(1-3i)t} \end{bmatrix} \int \begin{bmatrix} -15\, \mathrm{e}^{(-1-3i)t} \left(\left(-\frac{1}{5} + i\right) \cosh(t) - \frac{26i \sinh(t)}{5}\right) \\ 15\, \mathrm{e}^{(-1+3i)t} \left(\left(\frac{1}{5} + i\right) \cosh(t) - \frac{26i \sinh(t)}{5}\right) \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) \mathrm{e}^{(1+3i)t} & \left(\frac{5}{2} + \frac{i}{2}\right) \mathrm{e}^{(1-3i)t} \\ \mathrm{e}^{(1+3i)t} & \mathrm{e}^{(1-3i)t} \end{bmatrix} \begin{bmatrix} \left(\frac{21}{2} + \frac{15i}{2}\right) \cosh((2+3i)\, t) + \left(-\frac{21}{2} - \frac{15i}{2}\right) \sinh((2+3i)\, t) \\ \left(\frac{21}{2} - \frac{15i}{2}\right) \cosh((2-3i)\, t) + \left(-\frac{21}{2} + \frac{15i}{2}\right) \sinh((2-3i)\, t) \end{bmatrix} \\
&= \begin{bmatrix} 4\, \mathrm{e}^{t} (-13 + 15 \cosh(2t) - 15 \sinh(2t)) \\ 21\, \mathrm{e}^{t} (-1 + \cosh(2t) - \sinh(2t)) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{2} - \frac{i}{2}\right) c_1 \mathrm{e}^{(1+3i)t} \\ c_1 \mathrm{e}^{(1+3i)t} \end{bmatrix} + \begin{bmatrix} \left(\frac{5}{2} + \frac{i}{2}\right) c_2 \mathrm{e}^{(1-3i)t} \\ c_2 \mathrm{e}^{(1-3i)t} \end{bmatrix} + \begin{bmatrix} 4\,\mathrm{e}^t(-13 + 15\cosh(2t) - 15\sinh(2t)) \\ 21\,\mathrm{e}^t(-1 + \cosh(2t) - \sinh(2t)) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{2} + \frac{i}{2}\right) c_2 \mathrm{e}^{(1-3i)t} + \left(\frac{5}{2} - \frac{i}{2}\right) c_1 \mathrm{e}^{(1+3i)t} + 60\left(\cosh(2t) - \sinh(2t) - \frac{13}{15}\right) \mathrm{e}^t \\ c_1 \mathrm{e}^{(1+3i)t} + c_2 \mathrm{e}^{(1-3i)t} + 21\,\mathrm{e}^t(-1 + \cosh(2t) - \sinh(2t)) \end{bmatrix}$$

### 7.4.3 Maple step by step solution

Let's solve

$[x'(t) = -14x(t) + 39y + 78\sinh(t), y' = -6x(t) + 16y + 6\cosh(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 78\sinh(t) \\ 6\cosh(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 78\sinh(t) \\ 6\cosh(t) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 78\sinh(t) \\ 6\cosh(t) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

  $$\left[\left[1-3\,\mathrm{I}, \left[\begin{array}{c}\frac{5}{2}+\frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right], \left[1+3\,\mathrm{I}, \left[\begin{array}{c}\frac{5}{2}-\frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[1-3\,\mathrm{I}, \left[\begin{array}{c}\frac{5}{2}+\frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{(1-3\,\mathrm{I})t} \cdot \left[\begin{array}{c}\frac{5}{2}+\frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$\mathrm{e}^{t} \cdot (\cos(3t) - \mathrm{I}\sin(3t)) \cdot \left[\begin{array}{c}\frac{5}{2}+\frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Simplify expression

  $$\mathrm{e}^{t} \cdot \left[\begin{array}{c}\left(\frac{5}{2}+\frac{\mathrm{I}}{2}\right)(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c}\frac{5\cos(3t)}{2}+\frac{\sin(3t)}{2} \\ \cos(3t)\end{array}\right], \overrightarrow{x}_2(t) = \mathrm{e}^{t} \cdot \left[\begin{array}{c}-\frac{5\sin(3t)}{2}+\frac{\cos(3t)}{2} \\ -\sin(3t)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

  ○ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \left[\begin{array}{cc}\mathrm{e}^{t}\left(\frac{5\cos(3t)}{2}+\frac{\sin(3t)}{2}\right) & \mathrm{e}^{t}\left(-\frac{5\sin(3t)}{2}+\frac{\cos(3t)}{2}\right) \\ \mathrm{e}^{t}\cos(3t) & -\mathrm{e}^{t}\sin(3t)\end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th
  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$
  $\Phi(t) = \begin{bmatrix} \mathrm{e}^t\left(\frac{5\cos(3t)}{2} + \frac{\sin(3t)}{2}\right) & \mathrm{e}^t\left(-\frac{5\sin(3t)}{2} + \frac{\cos(3t)}{2}\right) \\ \mathrm{e}^t\cos(3t) & -\mathrm{e}^t\sin(3t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}}$
- Evaluate and simplify to get the fundamental matrix
  $\Phi(t) = \begin{bmatrix} \mathrm{e}^t(\cos(3t) - 5\sin(3t)) & 13\,\mathrm{e}^t\sin(3t) \\ -2\,\mathrm{e}^t\sin(3t) & \mathrm{e}^t(\cos(3t) + 5\sin(3t)) \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$
  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution
  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms
  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$
  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution
  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} 40\,\mathrm{e}^t \sin(3t) - 8\,\mathrm{e}^t \cos(3t) - 52\,\mathrm{e}^t + 60\,\mathrm{e}^{-t} \\ 16\,\mathrm{e}^t \sin(3t) - 21\,\mathrm{e}^t + 21\,\mathrm{e}^{-t} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \begin{bmatrix} 40\,\mathrm{e}^t \sin(3t) - 8\,\mathrm{e}^t \cos(3t) - 52\,\mathrm{e}^t + 60\,\mathrm{e}^{-t} \\ 16\,\mathrm{e}^t \sin(3t) - 21\,\mathrm{e}^t + 21\,\mathrm{e}^{-t} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{5\left(c_1+\frac{c_2}{5}-\frac{16}{5}\right)\mathrm{e}^t\cos(3t)}{2} + \frac{\mathrm{e}^t(c_1-5c_2+80)\sin(3t)}{2} - 52\,\mathrm{e}^t + 60\,\mathrm{e}^{-t} \\ -\mathrm{e}^t(c_2-16)\sin(3t) + c_1\mathrm{e}^t\cos(3t) - 21\,\mathrm{e}^t + 21\,\mathrm{e}^{-t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{5\left(c_1+\frac{c_2}{5}-\frac{16}{5}\right)\mathrm{e}^t\cos(3t)}{2} + \frac{\mathrm{e}^t(c_1-5c_2+80)\sin(3t)}{2} - 52\,\mathrm{e}^t + 60\,\mathrm{e}^{-t}, y = -\mathrm{e}^t(c_2-16)\sin(3t) + c_1\mathrm{e}^t \mathrm{co}\right.$$

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 86

```
dsolve([diff(x(t),t)=-14*x(t)+39*y(t)+78*sinh(t),diff(y(t),t)=-6*x(t)+16*y(t)+6*cosh(t)],sin
```

$$x(t) = \mathrm{e}^t \sin(3t)\, c_2 + \mathrm{e}^t \cos(3t)\, c_1 - 52\,\mathrm{e}^t + 60\,\mathrm{e}^{-t}$$
$$y(t) = \frac{5\,\mathrm{e}^t \sin(3t)\, c_2}{13} + \frac{\mathrm{e}^t \cos(3t)\, c_2}{13} + \frac{5\,\mathrm{e}^t \cos(3t)\, c_1}{13} - \frac{\mathrm{e}^t \sin(3t)\, c_1}{13} - 20\,\mathrm{e}^t + 20\,\mathrm{e}^{-t} - 2\sinh(t)$$

✓ Solution by Mathematica

Time used: 0.623 (sec). Leaf size: 90

```
DSolve[{x'[t]==-14*x[t]+39*y[t]+78*Sinh[t],y'[t]==-6*x[t]+16*y[t]+6*Cosh[t]},{x[t],y[t]},t,I
```

$$x(t) \to 60e^{-t} - 52e^t + c_1e^t\cos(3t) - (5c_1 - 13c_2)e^t\sin(3t)$$
$$y(t) \to 21e^{-t} - 21e^t + c_2e^t\cos(3t) - (2c_1 - 5c_2)e^t\sin(3t)$$

## 7.5 problem Problem 4(a)



Internal problem ID [12380]
Internal file name [`OUTPUT/11032_Wednesday_October_04_2023_01_27_11_AM_13432084/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 4(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 4y - 2z(t) - 2\sinh(t) \\
y' &= 4x(t) + 2y - 2z(t) + 10\cosh(t) \\
z'(t) &= -x(t) + 3y + z(t) + 5
\end{aligned}
$$

### 7.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} -2\sinh(t) \\ 10\cosh(t) \\ 5 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(12\,e^{7t}-7\,e^{4t}+9)\,e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}-27)\,e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(12\,e^{7t}-7\,e^{4t}-5)\,e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}+15)\,e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(3\,e^{7t}-7\,e^{4t}+4)\,e^{-2t}}{7} & \frac{(5\,e^{7t}+7\,e^{4t}-12)\,e^{-2t}}{21} & -\frac{e^{5t}}{3}+\frac{4\,e^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(12\,e^{7t}-7\,e^{4t}+9)\,e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}-27)\,e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(12\,e^{7t}-7\,e^{4t}-5)\,e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}+15)\,e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(3\,e^{7t}-7\,e^{4t}+4)\,e^{-2t}}{7} & \frac{(5\,e^{7t}+7\,e^{4t}-12)\,e^{-2t}}{21} & -\frac{e^{5t}}{3}+\frac{4\,e^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(12\,e^{7t}-7\,e^{4t}+9)\,e^{-2t}c_1}{14} + \frac{(20\,e^{7t}+7\,e^{4t}-27)\,e^{-2t}c_2}{42} + \left(-\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3}\right)c_3 \\ \frac{(12\,e^{7t}-7\,e^{4t}-5)\,e^{-2t}c_1}{14} + \frac{(20\,e^{7t}+7\,e^{4t}+15)\,e^{-2t}c_2}{42} + \left(-\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3}\right)c_3 \\ \frac{(3\,e^{7t}-7\,e^{4t}+4)\,e^{-2t}c_1}{7} + \frac{(5\,e^{7t}+7\,e^{4t}-12)\,e^{-2t}c_2}{21} + \left(-\frac{e^{5t}}{3}+\frac{4\,e^{2t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{e^{-2t}\left(\left(c_1-\frac{c_2}{3}-\frac{4c_3}{3}\right)e^{4t}+\left(-\frac{12c_1}{7}-\frac{20c_2}{21}+\frac{4c_3}{3}\right)e^{7t}-\frac{9c_1}{7}+\frac{9c_2}{7}\right)}{2} \\ -\frac{\left(\left(c_1-\frac{c_2}{3}-\frac{4c_3}{3}\right)e^{4t}+\left(-\frac{12c_1}{7}-\frac{20c_2}{21}+\frac{4c_3}{3}\right)e^{7t}+\frac{5c_1}{7}-\frac{5c_2}{7}\right)e^{-2t}}{2} \\ -\left(\left(c_1-\frac{c_2}{3}-\frac{4c_3}{3}\right)e^{4t}+\left(-\frac{3c_1}{7}-\frac{5c_2}{21}+\frac{c_3}{3}\right)e^{7t}-\frac{4c_1}{7}+\frac{4c_2}{7}\right)e^{-2t} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \frac{e^{-5t}(9\,e^{7t}-7\,e^{3t}+12)}{14} & -\frac{e^{-5t}(27\,e^{7t}-7\,e^{3t}-20)}{42} & \frac{2\,e^{-5t}(e^{3t}-1)}{3} \\ -\frac{e^{-5t}(5\,e^{7t}+7\,e^{3t}-12)}{14} & \frac{e^{-5t}(15\,e^{7t}+7\,e^{3t}+20)}{42} & \frac{2\,e^{-5t}(e^{3t}-1)}{3} \\ \frac{e^{-5t}(4\,e^{7t}-7\,e^{3t}+3)}{7} & -\frac{e^{-5t}(12\,e^{7t}-7\,e^{3t}-5)}{21} & \frac{e^{-5t}(4\,e^{3t}-1)}{3} \end{bmatrix}
\end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{(12\,e^{7t}-7\,e^{4t}+9)e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}-27)e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(12\,e^{7t}-7\,e^{4t}-5)e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}+15)e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(3\,e^{7t}-7\,e^{4t}+4)e^{-2t}}{7} & \frac{(5\,e^{7t}+7\,e^{4t}-12)e^{-2t}}{21} & -\frac{e^{5t}}{3}+\frac{4\,e^{2t}}{3} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-5t}(9\,e^{7t}-7\,e^{3t}+12)}{14} & -\frac{e^{-5t}(27\,e^{7t}-7\,e^{3t}-}{42} \\ -\frac{e^{-5t}(5\,e^{7t}+7\,e^{3t}-12)}{14} & \frac{e^{-5t}(15\,e^{7t}+7\,e^{3t}+20}{42} \\ \frac{e^{-5t}(4\,e^{7t}-7\,e^{3t}+3)}{7} & -\frac{e^{-5t}(12\,e^{7t}-7\,e^{3t}-}{21} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(12\,e^{7t}-7\,e^{4t}+9)e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}-27)e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(12\,e^{7t}-7\,e^{4t}-5)e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}+15)e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(3\,e^{7t}-7\,e^{4t}+4)e^{-2t}}{7} & \frac{(5\,e^{7t}+7\,e^{4t}-12)e^{-2t}}{21} & -\frac{e^{5t}}{3}+\frac{4\,e^{2t}}{3} \end{bmatrix} \begin{bmatrix} -\frac{5\,e^{-2t}}{3}+\frac{2\,e^{-5t}}{3}-\frac{26\sinh(t)}{21}-\frac{74\sinh(3t)}{63}+ \\ -\frac{5\,e^{-2t}}{3}+\frac{2\,e^{-5t}}{3}+\frac{52\sinh(3t)}{63}+\frac{8\sinh(4t)}{21}+ \\ \frac{e^{-5t}}{3}-\frac{10\,e^{-2t}}{3}+\frac{8\sinh(t)}{21}-\frac{58\sinh(3t)}{63}+\frac{4\,\mathrm{si}}{} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\,e^{-2t}\left(\frac{\left(-544\cosh(t)^6+544\cosh(t)^5\sinh(t)+720\cosh(t)^4-448\sinh(t)\cosh(t)^3-210\cosh(t)^2+54\cosh(t)\sinh(t)+5\right)e^{7t}}{7}-2\left(\cosh(t)^3-\sinh(t)\,\mathrm{co}\right.\right.}{9} \\ \frac{2\,e^{-2t}\left(\frac{\left(-544\cosh(t)^6+544\cosh(t)^5\sinh(t)+720\cosh(t)^4-448\sinh(t)\cosh(t)^3-210\cosh(t)^2+54\cosh(t)\sinh(t)+5\right)e^{7t}}{7}-2\left(\cosh(t)^3-\sinh(t)\right.\right.}{9} \\ -\frac{544\left(\left(\cosh(t)^6-\cosh(t)^5\sinh(t)-\frac{45\cosh(t)^4}{34}+\frac{14\sinh(t)\cosh(t)^3}{17}+\frac{105\cosh(t)^2}{272}-\frac{27\cosh(t)\sinh(t)}{272}-\frac{5}{544}\right)e^{7t}+\frac{7\left(\cosh(t)^3-\sinh(t)\,\mathrm{cos}\right.}{}\right.}{63} \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{6\,e^{-2t}\left(\left(-\frac{544\cosh(t)^6}{27}+\frac{544\cosh(t)^5\sinh(t)}{27}+\frac{80\cosh(t)^4}{3}-\frac{448\sinh(t)\cosh(t)^3}{27}-\frac{70\cosh(t)^2}{9}+2\cosh(t)\sinh(t)+c_1+\frac{5c_2}{9}-\frac{7c_3}{9}+\frac{5}{27}\right)e^{7t}+\left(\right.\right.}{} \\ \frac{6\,e^{-2t}\left(\left(-\frac{544\cosh(t)^6}{27}+\frac{544\cosh(t)^5\sinh(t)}{27}+\frac{80\cosh(t)^4}{3}-\frac{448\sinh(t)\cosh(t)^3}{27}-\frac{70\cosh(t)^2}{9}+2\cosh(t)\sinh(t)+c_1+\frac{5c_2}{9}-\frac{7c_3}{9}+\frac{5}{27}\right)e^{7t}+\left(\right.\right.}{} \\ \frac{3\,e^{-2t}\left(\left(-\frac{544\cosh(t)^6}{27}+\frac{544\cosh(t)^5\sinh(t)}{27}+\frac{80\cosh(t)^4}{3}-\frac{448\sinh(t)\cosh(t)^3}{27}-\frac{70\cosh(t)^2}{9}+2\cosh(t)\sinh(t)+c_1+\frac{5c_2}{9}-\frac{7c_3}{9}+\frac{5}{27}\right)e^{7t}\right.}{} \end{bmatrix}$$

### 7.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} -2\sinh(t) \\ 10\cosh(t) \\ 5 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 4 & -2 \\ 4 & 2-\lambda & -2 \\ -1 & 3 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 - 4\lambda + 20 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 5$$
$$\lambda_3 = 2$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-2$ | 1 | real eigenvalue |
| 2 | 1 | real eigenvalue |
| 5 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 4 & -2 & 0 \\ 4 & 4 & -2 & 0 \\ -1 & 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & \frac{5}{2} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 4 & 4 & -2 & 0 \\ 0 & 4 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 & -2 \\ 0 & 4 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{9t}{8}, v_2 = -\frac{5t}{8}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{9t}{8} \\ -\frac{5t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{9t}{8} \\ -\frac{5t}{8} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{9t}{8} \\ -\frac{5t}{8} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{9t}{8} \\ -\frac{5t}{8} \\ t \end{bmatrix} = \begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{9t}{8} \\ -\frac{5t}{8} \\ t \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 8 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & -2 \\ 4 & 0 & -2 \\ -1 & 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 4 & -2 & 0 \\ 4 & 0 & -2 & 0 \\ -1 & 3 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ -1 & 3 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{4} \implies \left[\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 3 & -\frac{3}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_2}{4} \implies \left[\begin{array}{ccc|c} 4 & 0 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 & -2 \\ 4 & -3 & -2 \\ -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 4 & -3 & -2 & 0 \\ -1 & 3 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{4R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & \frac{7}{3} & -\frac{14}{3} & 0 \\ -1 & 3 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & \frac{7}{3} & -\frac{14}{3} & 0 \\ 0 & \frac{5}{3} & -\frac{10}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_2}{7} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & \frac{7}{3} & -\frac{14}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 4 & -2 \\ 0 & \frac{7}{3} & -\frac{14}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-2$ | 1 | 1 | No | $\begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix}$ |
| 5 | 1 | 1 | No | $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ |
| 2 | 1 | 1 | No | $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-2$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix} e^{-2t}\end{aligned}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{5t} \\ &= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\ &= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{9\,\mathrm{e}^{-2t}}{8} \\ -\frac{5\,\mathrm{e}^{-2t}}{8} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{5t} \\ 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{2t}}{2} \\ \frac{\mathrm{e}^{2t}}{2} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{ccc} \frac{9\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ -\frac{5\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ {\mathrm e}^{-2 t} & {\mathrm e}^{5 t} & {\mathrm e}^{2 t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{ccc} \frac{4\,{\mathrm e}^{2 t}}{7} & -\frac{4\,{\mathrm e}^{2 t}}{7} & 0 \\ \frac{3\,{\mathrm e}^{-5 t}}{7} & \frac{5\,{\mathrm e}^{-5 t}}{21} & -\frac{{\mathrm e}^{-5 t}}{3} \\ -{\mathrm e}^{-2 t} & \frac{{\mathrm e}^{-2 t}}{3} & \frac{4\,{\mathrm e}^{-2 t}}{3} \end{array}\right]$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{ccc} \frac{9\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ -\frac{5\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ {\mathrm e}^{-2 t} & {\mathrm e}^{5 t} & {\mathrm e}^{2 t} \end{array}\right] \int \left[\begin{array}{ccc} \frac{4\,{\mathrm e}^{2 t}}{7} & -\frac{4\,{\mathrm e}^{2 t}}{7} & 0 \\ \frac{3\,{\mathrm e}^{-5 t}}{7} & \frac{5\,{\mathrm e}^{-5 t}}{21} & -\frac{{\mathrm e}^{-5 t}}{3} \\ -{\mathrm e}^{-2 t} & \frac{{\mathrm e}^{-2 t}}{3} & \frac{4\,{\mathrm e}^{-2 t}}{3} \end{array}\right] \left[\begin{array}{c} -2 \sinh\left(t\right) \\ 10 \cosh\left(t\right) \\ 5 \end{array}\right] dt \\
&= \left[\begin{array}{ccc} \frac{9\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ -\frac{5\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ {\mathrm e}^{-2 t} & {\mathrm e}^{5 t} & {\mathrm e}^{2 t} \end{array}\right] \int \left[\begin{array}{c} -\frac{8\,{\mathrm e}^{2 t} \left(\sinh\left(t\right)+5 \cosh\left(t\right)\right)}{7} \\ -\frac{{\mathrm e}^{-5 t} \left(18 \sinh\left(t\right)-50 \cosh\left(t\right)+35\right)}{21} \\ \frac{2\,{\mathrm e}^{-2 t} \left(3 \sinh\left(t\right)+5 \cosh\left(t\right)+10\right)}{3} \end{array}\right] dt \\
&= \left[\begin{array}{ccc} \frac{9\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ -\frac{5\,{\mathrm e}^{-2 t}}{8} & 2\,{\mathrm e}^{5 t} & \frac{{\mathrm e}^{2 t}}{2} \\ {\mathrm e}^{-2 t} & {\mathrm e}^{5 t} & {\mathrm e}^{2 t} \end{array}\right] \left[\begin{array}{c} -\frac{8 \sinh\left(3 t\right)}{7}-\frac{16 \sinh\left(t\right)}{7}-\frac{16 \cosh\left(t\right)}{7}-\frac{8 \cosh\left(3 t\right)}{7} \\ \frac{4 \sinh\left(4 t\right)}{21}+\frac{17 \sinh\left(6 t\right)}{63}-\frac{4 \cosh\left(4 t\right)}{21}-\frac{17 \cosh\left(6 t\right)}{63}+\frac{{\mathrm e}^{-5 t}}{3} \\ \frac{8 \sinh\left(t\right)}{3}+\frac{2 \sinh\left(3 t\right)}{9}-\frac{8 \cosh\left(t\right)}{3}-\frac{2 \cosh\left(3 t\right)}{9}-\frac{10\,{\mathrm e}^{-2 t}}{3} \end{array}\right] \\
&= \left[\begin{array}{c} \frac{2\,{\mathrm e}^{-2 t} \left(\frac{\left(-544 \cosh\left(t\right)^{6}+544 \cosh\left(t\right)^{5} \sinh\left(t\right)+720 \cosh\left(t\right)^{4}-448 \sinh\left(t\right) \cosh\left(t\right)^{3}-210 \cosh\left(t\right)^{2}+54 \cosh\left(t\right) \sinh\left(t\right)+5\right) {\mathrm e}^{7 t}}{7}-2 \left(\cosh\left(t\right)^{3}-\sinh\left(t\right) \cdots\right.\right.}{9} \\ \frac{2\,{\mathrm e}^{-2 t} \left(\frac{\left(-544 \cosh\left(t\right)^{6}+544 \cosh\left(t\right)^{5} \sinh\left(t\right)+720 \cosh\left(t\right)^{4}-448 \sinh\left(t\right) \cosh\left(t\right)^{3}-210 \cosh\left(t\right)^{2}+54 \cosh\left(t\right) \sinh\left(t\right)+5\right) {\mathrm e}^{7 t}}{7}-2 \left(\cosh\left(t\right)^{3}-\sinh\left(t\right) \cdots\right.\right.}{9} \\ -\frac{544 \left(\left(\cosh\left(t\right)^{6}-\cosh\left(t\right)^{5} \sinh\left(t\right)-\frac{45 \cosh\left(t\right)^{4}}{34}+\frac{14 \sinh\left(t\right) \cosh\left(t\right)^{3}}{17}+\frac{105 \cosh\left(t\right)^{2}}{272}-\frac{27 \cosh\left(t\right) \sinh\left(t\right)}{272}-\frac{5}{544}\right) {\mathrm e}^{7 t}+\frac{7 \left(\cosh\left(t\right)^{3}-\sinh\left(t\right) \cdots\right.}{\cdots}\right.}{63} \end{array}\right.
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{9c_1 e^{-2t}}{8} \\ -\frac{5c_1 e^{-2t}}{8} \\ c_1 e^{-2t} \end{bmatrix} + \begin{bmatrix} 2c_2 e^{5t} \\ 2c_2 e^{5t} \\ c_2 e^{5t} \end{bmatrix} + \begin{bmatrix} \frac{c_3 e^{2t}}{2} \\ \frac{c_3 e^{2t}}{2} \\ c_3 e^{2t} \end{bmatrix} + \begin{bmatrix} 2\,e^{-2t} \Big( \frac{(-544 \cosh(t)^6 + 544 \cosh(t)^5 \sinh(t) + 720 \cosh(t)^4 - 448 \sinh(t) \mathrm{co}}{7} \\ 2\,e^{-2t} \Big( \frac{(-544 \cosh(t)^6 + 544 \cosh(t)^5 \sinh(t) + 720 \cosh(t)^4 - 448 \sinh(t) \mathrm{c}}{7} \\ -\, 544 \Big( \big( \cosh(t)^6 - \cosh(t)^5 \sinh(t) - \frac{45 \cosh(t)^4}{34} + \frac{14 \sinh(t) \cosh(t)}{17} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} 2\Big(\Big(-\frac{544\cosh(t)^6}{63} + \frac{544\cosh(t)^5\sinh(t)}{63} + \frac{80\cosh(t)^4}{7} - \frac{64\sinh(t)\cosh(t)^3}{9} - \frac{10\cosh(t)^2}{3} + \frac{6\cosh(t)\sinh(t)}{7} + \\ 2\Big(\Big(-\frac{544\cosh(t)^6}{63} + \frac{544\cosh(t)^5\sinh(t)}{63} + \frac{80\cosh(t)^4}{7} - \frac{64\sinh(t)\cosh(t)^3}{9} - \frac{10\cosh(t)^2}{3} + \frac{6\cosh(t)\sinh(t)}{7} + \\ \Big(\Big(-\frac{544\cosh(t)^6}{63} + \frac{544\cosh(t)^5\sinh(t)}{63} + \frac{80\cosh(t)^4}{7} - \frac{64\sinh(t)\cosh(t)^3}{9} - \frac{10\cosh(t)^2}{3} + \frac{6\cosh(t)\sinh(t)}{7} + c_2 \end{bmatrix}$$

### 7.5.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) + 4y - 2z(t) - 2\sinh(t), y' = 4x(t) + 2y - 2z(t) + 10\cosh(t), z'(t) = -x(t) + 3y$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -2\sinh(t) \\ 10\cosh(t) \\ 5 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -2\sinh(t) \\ 10\cosh(t) \\ 5 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -2\sinh(t) \\ 10\cosh(t) \\ 5 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & -2 \\ -1 & 3 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-2, \begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{9}{8} \\ -\frac{5}{8} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[5, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{5t} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} \frac{9\,e^{-2t}}{8} & \frac{e^{2t}}{2} & 2\,e^{5t} \\ -\frac{5\,e^{-2t}}{8} & \frac{e^{2t}}{2} & 2\,e^{5t} \\ e^{-2t} & e^{2t} & e^{5t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} \frac{9\,e^{-2t}}{8} & \frac{e^{2t}}{2} & 2\,e^{5t} \\ -\frac{5\,e^{-2t}}{8} & \frac{e^{2t}}{2} & 2\,e^{5t} \\ e^{-2t} & e^{2t} & e^{5t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{9}{8} & \frac{1}{2} & 2 \\ -\frac{5}{8} & \frac{1}{2} & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{(12\,e^{7t}-7\,e^{4t}+9)e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}-27)e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(12\,e^{7t}-7\,e^{4t}-5)e^{-2t}}{14} & \frac{(20\,e^{7t}+7\,e^{4t}+15)e^{-2t}}{42} & -\frac{2\,e^{5t}}{3}+\frac{2\,e^{2t}}{3} \\ \frac{(3\,e^{7t}-7\,e^{4t}+4)e^{-2t}}{7} & \frac{(5\,e^{7t}+7\,e^{4t}-12)e^{-2t}}{21} & -\frac{e^{5t}}{3}+\frac{4\,e^{2t}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \left[\begin{array}{c} -\frac{4\left(\frac{2176\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{63}-\frac{2176\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{63}+\left(\frac{576\sinh\left(\frac{t}{2}\right)^{2}}{7}-\frac{544\,\mathrm{e}^{7t}}{7}\right)\cosh\left(\frac{t}{2}\right)^{7}+\left(\frac{544\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}}{9}+\frac{576\sinh\left(\frac{t}{2}\right)\cdots}{7}\right.\right.}{\cdots} \\ -\frac{4\left(\frac{2176\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{63}-\frac{2176\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{63}+\left(-\frac{320\sinh\left(\frac{t}{2}\right)^{2}}{7}-\frac{544\,\mathrm{e}^{7t}}{7}\right)\cosh\left(\frac{t}{2}\right)^{7}+\left(\frac{544\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}}{9}-\frac{320\sinh\left(\cdots\right)}{7}\right.\right.}{\cdots} \\ -\frac{12\,\mathrm{e}^{-2t}\left(\frac{1088\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{189}-\frac{1088\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{189}+\left(\frac{512\sinh\left(\frac{t}{2}\right)^{2}}{21}-\frac{272\,\mathrm{e}^{7t}}{21}\right)\cosh\left(\frac{t}{2}\right)^{7}+\left(\frac{272\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}}{27}+\frac{512\cdots}{}\right.\right.}{\cdots} \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \left[\begin{array}{c} -\frac{4\left(\frac{2176\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{63}-\frac{2176\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{63}+\left(\frac{576\sinh\left(\frac{t}{2}\right)^{2}}{7}-\frac{544\,\mathrm{e}^{7t}}{7}\right)\cosh\left(\frac{t}{2}\right)^{7}+\cdots\right.}{\cdots} \\ -\frac{4\left(\frac{2176\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{63}-\frac{2176\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{63}+\left(-\frac{320\sinh\left(\frac{t}{2}\right)^{2}}{7}-\frac{544\,\mathrm{e}^{7t}}{7}\right)\cosh\left(\frac{t}{2}\right)^{7}\cdots\right.}{\cdots} \\ -\frac{12\,\mathrm{e}^{-2t}\left(\frac{1088\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{189}-\frac{1088\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{189}+\left(\frac{512\sinh\left(\frac{t}{2}\right)^{2}}{21}-\frac{272\,\mathrm{e}^{7t}}{21}\right)\cos\cdots\right.}{\cdots} \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{2304\,\mathrm{e}^{-2t}\left(\frac{34\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{81}-\frac{34\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{81}+\left(\sinh\left(\frac{t}{2}\right)^{2}-\frac{17\,\mathrm{e}^{7t}}{18}\right)\cosh\left(\frac{t}{2}\right)^{7}+\sinh\left(\frac{t}{2}\right)\left(\sinh\left(\frac{t}{2}\right)^{2}+\frac{119\,\mathrm{e}^{7\cdots}}{162}\right.\right.}{\cdots} \\ -\frac{1280\,\mathrm{e}^{-2t}\left(-\frac{34\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{45}+\frac{34\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{45}+\left(\sinh\left(\frac{t}{2}\right)^{2}+\frac{17\,\mathrm{e}^{7t}}{10}\right)\cosh\left(\frac{t}{2}\right)^{7}+\sinh\left(\frac{t}{2}\right)\left(\sinh\left(\frac{t}{2}\right)^{2}-\frac{11\cdots}{}\right.\right.}{\cdots} \\ \frac{2048\,\mathrm{e}^{-2t}\left(\frac{17\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{72}-\frac{17\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{72}+\left(\sinh\left(\frac{t}{2}\right)^{2}-\frac{17\,\mathrm{e}^{7t}}{32}\right)\cosh\left(\frac{t}{2}\right)^{7}+\sinh\left(\frac{t}{2}\right)\left(\sinh\left(\frac{t}{2}\right)^{2}+\frac{119\,\mathrm{e}^{7t}}{288}\right.\right.}{\cdots} \end{array}\right.$$

- Solution to the system of ODEs

$$\begin{cases} x(t) = \dfrac{2304\,\mathrm{e}^{-2t}\left(\frac{34\,\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{9}}{81}-\frac{34\sinh\left(\frac{t}{2}\right)\mathrm{e}^{7t}\cosh\left(\frac{t}{2}\right)^{8}}{81}+\left(\sinh\left(\frac{t}{2}\right)^{2}-\frac{17\,\mathrm{e}^{7t}}{18}\right)\cosh\left(\frac{t}{2}\right)^{7}+\sinh\left(\frac{t}{2}\right)\left(\sinh\left(\frac{t}{2}\right)^{2}+\frac{119\,\mathrm{e}^{7t}}{162}\right)\cosh\left(\right.\right.}{} \end{cases}$$

✓ Solution by Maple

Time used: 0.282 (sec). Leaf size: 429

```
dsolve([diff(x(t),t)=2*x(t)+4*y(t)-2*z(t)-2*sinh(t),diff(y(t),t)=4*x(t)+2*y(t)-2*z(t)+10*cos
```

$$\begin{aligned}
x(t) &= -1+2c_3\mathrm{e}^{5t}+\frac{9c_1\mathrm{e}^{-2t}}{8}-\frac{45\cosh(t)}{16}-\frac{3\sinh(t)}{16}+\frac{c_2\mathrm{e}^{2t}}{2} \\
&\quad -\frac{275\,\mathrm{e}^{-2t}\sinh(t)}{224}-\frac{3\,\mathrm{e}^{-2t}\sinh(3t)}{14}-\frac{275\,\mathrm{e}^{-2t}\cosh(t)}{224}-\frac{3\,\mathrm{e}^{-2t}\cosh(3t)}{14} \\
&\quad +\frac{3\,\mathrm{e}^{2t}\sinh(t)}{2}+\frac{275\,\mathrm{e}^{2t}\sinh(3t)}{288}-\frac{3\,\mathrm{e}^{2t}\cosh(t)}{2}-\frac{275\,\mathrm{e}^{2t}\cosh(3t)}{288} \\
&\quad -\frac{3\,\mathrm{e}^{5t}\sinh(4t)}{14}-\frac{275\,\mathrm{e}^{5t}\sinh(6t)}{1008}+\frac{3\,\mathrm{e}^{5t}\cosh(4t)}{14}+\frac{275\,\mathrm{e}^{5t}\cosh(6t)}{1008} \\
y(t) &= -1+2c_3\mathrm{e}^{5t}-\frac{5c_1\mathrm{e}^{-2t}}{8}-\frac{15\cosh(t)}{16}-\frac{\sinh(t)}{16}+\frac{c_2\mathrm{e}^{2t}}{2} \\
&\quad +\frac{25\,\mathrm{e}^{-2t}\sinh(t)}{32}-\frac{\mathrm{e}^{-2t}\sinh(3t)}{14}+\frac{25\,\mathrm{e}^{-2t}\cosh(t)}{32}-\frac{\mathrm{e}^{-2t}\cosh(3t)}{14} \\
&\quad +\frac{\mathrm{e}^{2t}\sinh(t)}{2}-\frac{175\,\mathrm{e}^{2t}\sinh(3t)}{288}-\frac{\mathrm{e}^{2t}\cosh(t)}{2}+\frac{175\,\mathrm{e}^{2t}\cosh(3t)}{288} \\
&\quad -\frac{\mathrm{e}^{5t}\sinh(4t)}{14}+\frac{25\,\mathrm{e}^{5t}\sinh(6t)}{144}+\frac{\mathrm{e}^{5t}\cosh(4t)}{14}-\frac{25\,\mathrm{e}^{5t}\cosh(6t)}{144} \\
z(t) &= -\frac{25\,\mathrm{e}^{-2t}\sinh(t)}{14}-3-\frac{4\,\mathrm{e}^{-2t}\sinh(3t)}{7}-\frac{25\,\mathrm{e}^{-2t}\cosh(t)}{14}-\frac{4\,\mathrm{e}^{-2t}\cosh(3t)}{7} \\
&\quad +4\,\mathrm{e}^{2t}\sinh(t)+\frac{25\,\mathrm{e}^{2t}\sinh(3t)}{18}-4\,\mathrm{e}^{2t}\cosh(t)-\frac{25\,\mathrm{e}^{2t}\cosh(3t)}{18}-\frac{4\,\mathrm{e}^{5t}\sinh(4t)}{7} \\
&\quad -\frac{25\,\mathrm{e}^{5t}\sinh(6t)}{63}+\frac{4\,\mathrm{e}^{5t}\cosh(4t)}{7}+\frac{25\,\mathrm{e}^{5t}\cosh(6t)}{63}+c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{2t}+c_3\mathrm{e}^{5t}
\end{aligned}$$

✓ Solution by Mathematica

Time used: 0.292 (sec). Leaf size: 233

```
DSolve[{x'[t]==2*x[t]+4*y[t]-2*z[t]-2*Sinh[t],y'[t]==4*x[t]+2*y[t]-2*z[t]+10*Cosh[t],z'[t]==
```

$$x(t) \to -\frac{29e^{-t}}{9} - 3e^t + \frac{9}{14}(c_1 - c_2)e^{-2t} + \frac{2}{21}(9c_1 + 5c_2 - 7c_3)e^{5t} + \frac{1}{6}(-3c_1 + c_2 + 4c_3)e^{2t} - 1$$
$$y(t) \to \frac{7e^{-t}}{9} - e^t + \frac{5}{14}(c_2 - c_1)e^{-2t} + \frac{2}{21}(9c_1 + 5c_2 - 7c_3)e^{5t} + \frac{1}{6}(-3c_1 + c_2 + 4c_3)e^{2t} - 1$$
$$z(t) \to -\frac{25e^{-t}}{9} - 4e^t + \frac{4}{7}(c_1 - c_2)e^{-2t} + \frac{1}{21}(9c_1 + 5c_2 - 7c_3)e^{5t} + \frac{1}{3}(-3c_1 + c_2 + 4c_3)e^{2t} - 3$$

## 7.6 problem Problem 4(b)



Internal problem ID [12381]
Internal file name [`OUTPUT/11033_Wednesday_October_04_2023_01_27_12_AM_25873952/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 4(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + 6y - 2z(t) + 50\,\mathrm{e}^{t} \\
y' &= 6x(t) + 2y - 2z(t) + 21\,\mathrm{e}^{-t} \\
z'(t) &= -x(t) + 6y + z(t) + 9
\end{aligned}
$$

### 7.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 50\,\mathrm{e}^{t} \\ 21\,\mathrm{e}^{-t} \\ 9 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{ccc} \frac{\left(16\,\mathrm{e}^{10t}-10\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{2\left(8\,\mathrm{e}^{10t}-5\,\mathrm{e}^{7t}-3\right)\mathrm{e}^{-4t}}{15} & \frac{\left(3\,\mathrm{e}^{10t}+2\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{\left(16\,\mathrm{e}^{10t}-25\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{5\,\mathrm{e}^{3t}}{3} \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \left[\begin{array}{ccc} \frac{\left(16\,\mathrm{e}^{10t}-10\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{2\left(8\,\mathrm{e}^{10t}-5\,\mathrm{e}^{7t}-3\right)\mathrm{e}^{-4t}}{15} & \frac{\left(3\,\mathrm{e}^{10t}+2\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{\left(16\,\mathrm{e}^{10t}-25\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{5\,\mathrm{e}^{3t}}{3} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right] \\ &= \left[\begin{array}{c} \frac{\left(16\,\mathrm{e}^{10t}-10\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}c_1}{15}+\frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}c_2}{5}+\left(-\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3}\right)c_3 \\ \frac{2\left(8\,\mathrm{e}^{10t}-5\,\mathrm{e}^{7t}-3\right)\mathrm{e}^{-4t}c_1}{15}+\frac{\left(3\,\mathrm{e}^{10t}+2\right)\mathrm{e}^{-4t}c_2}{5}+\left(-\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3}\right)c_3 \\ \frac{\left(16\,\mathrm{e}^{10t}-25\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}c_1}{15}+\frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}c_2}{5}+\left(-\frac{2\,\mathrm{e}^{6t}}{3}+\frac{5\,\mathrm{e}^{3t}}{3}\right)c_3 \end{array}\right] \\ &= \left[\begin{array}{c} -\frac{2\left(\left(-\frac{8c_1}{5}-\frac{9c_2}{10}+c_3\right)\mathrm{e}^{10t}+\left(-c_3+c_1\right)\mathrm{e}^{7t}-\frac{9c_1}{10}+\frac{9c_2}{10}\right)\mathrm{e}^{-4t}}{3} \\ -\frac{2\left(\left(-\frac{8c_1}{5}-\frac{9c_2}{10}+c_3\right)\mathrm{e}^{10t}+\left(-c_3+c_1\right)\mathrm{e}^{7t}+\frac{3c_1}{5}-\frac{3c_2}{5}\right)\mathrm{e}^{-4t}}{3} \\ -\frac{5\left(\left(-\frac{16c_1}{25}-\frac{9c_2}{25}+\frac{2c_3}{5}\right)\mathrm{e}^{10t}+\left(-c_3+c_1\right)\mathrm{e}^{7t}-\frac{9c_1}{25}+\frac{9c_2}{25}\right)\mathrm{e}^{-4t}}{3} \end{array}\right] \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \left[\begin{array}{ccc} -\frac{\mathrm{e}^{-6t}\left(-9\,\mathrm{e}^{10t}+10\,\mathrm{e}^{3t}-16\right)}{15} & -\frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-6t}}{5} & \frac{2\,\mathrm{e}^{-6t}\left(\mathrm{e}^{3t}-1\right)}{3} \\ -\frac{2\,\mathrm{e}^{-6t}\left(3\,\mathrm{e}^{10t}+5\,\mathrm{e}^{3t}-8\right)}{15} & \frac{\mathrm{e}^{-6t}\left(2\,\mathrm{e}^{10t}+3\right)}{5} & \frac{2\,\mathrm{e}^{-6t}\left(\mathrm{e}^{3t}-1\right)}{3} \\ -\frac{\mathrm{e}^{-6t}\left(-9\,\mathrm{e}^{10t}+25\,\mathrm{e}^{3t}-16\right)}{15} & -\frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-6t}}{5} & \frac{\mathrm{e}^{-6t}\left(5\,\mathrm{e}^{3t}-2\right)}{3} \end{array}\right] \end{aligned}$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{ccc} \frac{\left(16\,\mathrm{e}^{10t}-10\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{2\left(8\,\mathrm{e}^{10t}-5\,\mathrm{e}^{7t}-3\right)\mathrm{e}^{-4t}}{15} & \frac{\left(3\,\mathrm{e}^{10t}+2\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{\left(16\,\mathrm{e}^{10t}-25\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{5\,\mathrm{e}^{3t}}{3} \end{array}\right] \int \left[\begin{array}{ccc} -\frac{\mathrm{e}^{-6t}\left(-9\,\mathrm{e}^{10t}+10\,\mathrm{e}^{3t}-16\right)}{15} & -\frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-6t}}{5} & 2\,\mathrm{e}\cdots \\ -\frac{2\,\mathrm{e}^{-6t}\left(3\,\mathrm{e}^{10t}+5\,\mathrm{e}^{3t}-8\right)}{15} & \frac{\mathrm{e}^{-6t}\left(2\,\mathrm{e}^{10t}+3\right)}{5} & 2\,\mathrm{e}\cdots \\ -\frac{\mathrm{e}^{-6t}\left(-9\,\mathrm{e}^{10t}+25\,\mathrm{e}^{3t}-16\right)}{15} & -\frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-6t}}{5} & \mathrm{e}^{-\cdots} \end{array}\right.
$$

$$
= \left[\begin{array}{ccc} \frac{\left(16\,\mathrm{e}^{10t}-10\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{2\left(8\,\mathrm{e}^{10t}-5\,\mathrm{e}^{7t}-3\right)\mathrm{e}^{-4t}}{15} & \frac{\left(3\,\mathrm{e}^{10t}+2\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{2\,\mathrm{e}^{3t}}{3} \\ \frac{\left(16\,\mathrm{e}^{10t}-25\,\mathrm{e}^{7t}+9\right)\mathrm{e}^{-4t}}{15} & \frac{3\left(\mathrm{e}^{10t}-1\right)\mathrm{e}^{-4t}}{5} & -\frac{2\,\mathrm{e}^{6t}}{3}+\frac{5\,\mathrm{e}^{3t}}{3} \end{array}\right] \left[\begin{array}{c} \frac{\left(90\,\mathrm{e}^{12t}-63\,\mathrm{e}^{10t}+250\,\mathrm{e}^{5t}-30\,\mathrm{e}^{4t}-160\,\mathrm{e}^{2t}+15\,\mathrm{e}^{t}-27\right)\mathrm{e}^{-7t}}{15} \\ \frac{\left(-60\,\mathrm{e}^{12t}+42\,\mathrm{e}^{10t}+250\,\mathrm{e}^{5t}-30\,\mathrm{e}^{4t}-160\,\mathrm{e}^{2t}+15\,\mathrm{e}^{t}-27\right)\mathrm{e}^{-7t}}{15} \\ \frac{\left(90\,\mathrm{e}^{12t}-63\,\mathrm{e}^{10t}+625\,\mathrm{e}^{5t}-75\,\mathrm{e}^{4t}-160\,\mathrm{e}^{2t}+15\,\mathrm{e}^{t}-27\right)\mathrm{e}^{-7t}}{15} \end{array}\right.
$$

$$
= \left[\begin{array}{c} -1+12\,\mathrm{e}^{t}-6\,\mathrm{e}^{-t} \\ -1+2\,\mathrm{e}^{t}+\mathrm{e}^{-t} \\ -4+37\,\mathrm{e}^{t}-6\,\mathrm{e}^{-t} \end{array}\right]
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
&= \left[\begin{array}{c} \mathrm{e}^{-4t}\left(\frac{(16c_1+9c_2-10c_3)\mathrm{e}^{10t}}{15}+\frac{2(c_3-c_1)\mathrm{e}^{7t}}{3}+\frac{3c_1}{5}-\frac{3c_2}{5}-6\,\mathrm{e}^{3t}-\mathrm{e}^{4t}+12\,\mathrm{e}^{5t}\right) \\ \mathrm{e}^{-4t}\left(\frac{(16c_1+9c_2-10c_3)\mathrm{e}^{10t}}{15}+\frac{2(c_3-c_1)\mathrm{e}^{7t}}{3}-\frac{2c_1}{5}+\frac{2c_2}{5}+\mathrm{e}^{3t}-\mathrm{e}^{4t}+2\,\mathrm{e}^{5t}\right) \\ \mathrm{e}^{-4t}\left(\frac{(16c_1+9c_2-10c_3)\mathrm{e}^{10t}}{15}+\frac{5(c_3-c_1)\mathrm{e}^{7t}}{3}+\frac{3c_1}{5}-\frac{3c_2}{5}-6\,\mathrm{e}^{3t}-4\,\mathrm{e}^{4t}+37\,\mathrm{e}^{5t}\right) \end{array}\right]
\end{aligned}
$$

### 7.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y' \\ z'(t) \end{array}\right] = \left[\begin{array}{ccc} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{array}\right] \left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] + \left[\begin{array}{c} 50\,\mathrm{e}^{t} \\ 21\,\mathrm{e}^{-t} \\ 9 \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 6 & -2 \\ 6 & 2-\lambda & -2 \\ -1 & 6 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 - 18\lambda + 72 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$
$$\lambda_2 = -4$$
$$\lambda_3 = 3$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-4$ | 1 | real eigenvalue |
| 3 | 1 | real eigenvalue |
| 6 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{bmatrix} - (-4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 & -2 \\ 6 & 6 & -2 \\ -1 & 6 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & 6 & -2 & 0 \\ 6 & 6 & -2 & 0 \\ -1 & 6 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 6 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} 6 & 6 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & \frac{14}{3} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 6 & 6 & -2 & 0 \\ 0 & 7 & \frac{14}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 6 & -2 \\ 0 & 7 & \frac{14}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -\frac{2t}{3}\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 6 & -2 \\ 6 & -1 & -2 \\ -1 & 6 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 6 & -2 & 0 \\ 6 & -1 & -2 & 0 \\ -1 & 6 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 6R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 6 & -2 & 0 \\ 0 & 35 & -14 & 0 \\ -1 & 6 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 6 & -2 & 0 \\ 0 & 35 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 6 & -2 \\ 0 & 35 & -14 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{5}, v_2 = \frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{5} \\ \frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 6 & -2 \\ 6 & -4 & -2 \\ -1 & 6 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 6 & -2 & 0 \\ 6 & -4 & -2 & 0 \\ -1 & 6 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} -4 & 6 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ -1 & 6 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{4} \implies \left[\begin{array}{ccc|c} -4 & 6 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & \frac{9}{2} & -\frac{9}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{9R_2}{10} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 6 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 6 & -2 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 6 | 1 | 1 | No | $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ |
| $-4$ | 1 | 1 | No | $\begin{bmatrix} 1 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$ |
| 3 | 1 | 1 | No | $\begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{6t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t} \end{aligned}$$

Since eigenvalue $-4$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{-4t} \\ &= \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 1 \end{bmatrix} e^{-4t} \end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{3t}$$
$$= \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} e^{3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{6t} \\ e^{6t} \\ e^{6t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-4t} \\ -\frac{2\,e^{-4t}}{3} \\ e^{-4t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{2\,e^{3t}}{5} \\ \frac{2\,e^{3t}}{5} \\ e^{3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{6t} & e^{-4t} & \frac{2\,e^{3t}}{5} \\ e^{6t} & -\frac{2\,e^{-4t}}{3} & \frac{2\,e^{3t}}{5} \\ e^{6t} & e^{-4t} & e^{3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{16\,e^{-6t}}{15} & \frac{3\,e^{-6t}}{5} & -\frac{2\,e^{-6t}}{3} \\ \frac{3\,e^{4t}}{5} & -\frac{3\,e^{4t}}{5} & 0 \\ -\frac{5\,e^{-3t}}{3} & 0 & \frac{5\,e^{-3t}}{3} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{6t} & e^{-4t} & \frac{2\,e^{3t}}{5} \\ e^{6t} & -\frac{2\,e^{-4t}}{3} & \frac{2\,e^{3t}}{5} \\ e^{6t} & e^{-4t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} \frac{16\,e^{-6t}}{15} & \frac{3\,e^{-6t}}{5} & -\frac{2\,e^{-6t}}{3} \\ \frac{3\,e^{4t}}{5} & -\frac{3\,e^{4t}}{5} & 0 \\ -\frac{5\,e^{-3t}}{3} & 0 & \frac{5\,e^{-3t}}{3} \end{bmatrix} \begin{bmatrix} 50\,e^{t} \\ 21\,e^{-t} \\ 9 \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{6t} & e^{-4t} & \frac{2\,e^{3t}}{5} \\ e^{6t} & -\frac{2\,e^{-4t}}{3} & \frac{2\,e^{3t}}{5} \\ e^{6t} & e^{-4t} & e^{3t} \end{bmatrix} \int \begin{bmatrix} \frac{160\,e^{-5t}}{3} + \frac{63\,e^{-7t}}{5} - 6\,e^{-6t} \\ 30\,e^{5t} - \frac{63\,e^{3t}}{5} \\ -\frac{250\,e^{-2t}}{3} + 15\,e^{-3t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{6t} & e^{-4t} & \frac{2\,e^{3t}}{5} \\ e^{6t} & -\frac{2\,e^{-4t}}{3} & \frac{2\,e^{3t}}{5} \\ e^{6t} & e^{-4t} & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{9\,e^{-7t}}{5} - \frac{32\,e^{-5t}}{3} + e^{-6t} \\ -\frac{21\,e^{3t}}{5} + 6\,e^{5t} \\ \frac{125\,e^{-2t}}{3} - 5\,e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} -1 + 12\,e^{t} - 6\,e^{-t} \\ -1 + 2\,e^{t} + e^{-t} \\ -4 + 37\,e^{t} - 6\,e^{-t} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{6t} \\ c_1 e^{6t} \\ c_1 e^{6t} \end{bmatrix} + \begin{bmatrix} c_2 e^{-4t} \\ -\frac{2c_2 e^{-4t}}{3} \\ c_2 e^{-4t} \end{bmatrix} + \begin{bmatrix} \frac{2c_3 e^{3t}}{5} \\ \frac{2c_3 e^{3t}}{5} \\ c_3 e^{3t} \end{bmatrix} + \begin{bmatrix} -1 + 12\,e^{t} - 6\,e^{-t} \\ -1 + 2\,e^{t} + e^{-t} \\ -4 + 37\,e^{t} - 6\,e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{(-5c_1 e^{10t} - 2c_3 e^{7t} - 60\,e^{5t} + 5\,e^{4t} + 30\,e^{3t} - 5c_2)\,e^{-4t}}{5} \\ \frac{(15c_1 e^{10t} + 6c_3 e^{7t} + 30\,e^{5t} - 15\,e^{4t} + 15\,e^{3t} - 10c_2)\,e^{-4t}}{15} \\ (c_1 e^{10t} + c_3 e^{7t} + 37\,e^{5t} - 4\,e^{4t} - 6\,e^{3t} + c_2)\,e^{-4t} \end{bmatrix}$$

### 7.6.3 Maple step by step solution

Let's solve

$\left[x'(t) = 2x(t) + 6y - 2z(t) + 50\,e^{t}, y' = 6x(t) + 2y - 2z(t) + \frac{21}{e^{t}}, z'(t) = -x(t) + 6y + z(t) + 9\right]$

- Define vector

$$\overrightarrow{x}(t) = \left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right]$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 50\,e^{t} \\ \frac{6x(t)e^{t}+2y\,e^{t}-2z(t)e^{t}+21}{e^{t}} - 6x(t) - 2y + 2z(t) \\ 9 \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{ccc} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 50\,e^{t} \\ 0 \\ 9 \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(t) = \left[\begin{array}{c} 50\,e^{t} \\ 0 \\ 9 \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 2 & 6 & -2 \\ 6 & 2 & -2 \\ -1 & 6 & 1 \end{array}\right]$$

- Rewrite the system as

$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-4, \left[\begin{array}{c} 1 \\ -\frac{2}{3} \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{array}\right]\right], \left[6, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-4, \left[\begin{array}{c} 1 \\ -\frac{2}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-4t} \cdot \left[\begin{array}{c} 1 \\ -\frac{2}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} \frac{2}{5} \\ \frac{2}{5} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[6, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{6t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($
  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^{-4t} & \frac{2e^{3t}}{5} & e^{6t} \\ -\frac{2e^{-4t}}{3} & \frac{2e^{3t}}{5} & e^{6t} \\ e^{-4t} & e^{3t} & e^{6t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-4t} & \frac{2e^{3t}}{5} & e^{6t} \\ -\frac{2e^{-4t}}{3} & \frac{2e^{3t}}{5} & e^{6t} \\ e^{-4t} & e^{3t} & e^{6t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{2}{5} & 1 \\ -\frac{2}{3} & \frac{2}{5} & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{(16e^{10t}-10e^{7t}+9)e^{-4t}}{15} & \frac{3(e^{10t}-1)e^{-4t}}{5} & -\frac{2e^{6t}}{3} + \frac{2e^{3t}}{3} \\ \frac{2(8e^{10t}-5e^{7t}-3)e^{-4t}}{15} & \frac{(3e^{10t}+2)e^{-4t}}{5} & -\frac{2e^{6t}}{3} + \frac{2e^{3t}}{3} \\ \frac{(16e^{10t}-25e^{7t}+9)e^{-4t}}{15} & \frac{3(e^{10t}-1)e^{-4t}}{5} & -\frac{2e^{6t}}{3} + \frac{5e^{3t}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t)=\left[\begin{array}{c}\frac{\left(29\,\mathrm{e}^{10t}-44\,\mathrm{e}^{7t}+36\,\mathrm{e}^{5t}-3\,\mathrm{e}^{4t}-18\right)\mathrm{e}^{-4t}}{3}\\ \frac{\left(29\,\mathrm{e}^{10t}-44\,\mathrm{e}^{7t}+6\,\mathrm{e}^{5t}-3\,\mathrm{e}^{4t}+12\right)\mathrm{e}^{-4t}}{3}\\ \frac{\left(29\,\mathrm{e}^{10t}-110\,\mathrm{e}^{7t}+111\,\mathrm{e}^{5t}-12\,\mathrm{e}^{4t}-18\right)\mathrm{e}^{-4t}}{3}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+c_3\vec{x}_3+\left[\begin{array}{c}\frac{\left(29\,\mathrm{e}^{10t}-44\,\mathrm{e}^{7t}+36\,\mathrm{e}^{5t}-3\,\mathrm{e}^{4t}-18\right)\mathrm{e}^{-4t}}{3}\\ \frac{\left(29\,\mathrm{e}^{10t}-44\,\mathrm{e}^{7t}+6\,\mathrm{e}^{5t}-3\,\mathrm{e}^{4t}+12\right)\mathrm{e}^{-4t}}{3}\\ \frac{\left(29\,\mathrm{e}^{10t}-110\,\mathrm{e}^{7t}+111\,\mathrm{e}^{5t}-12\,\mathrm{e}^{4t}-18\right)\mathrm{e}^{-4t}}{3}\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y\\ z(t)\end{array}\right]=\left[\begin{array}{c}\frac{\left(15c_3\mathrm{e}^{10t}+145\,\mathrm{e}^{10t}+6c_2\mathrm{e}^{7t}-220\,\mathrm{e}^{7t}+180\,\mathrm{e}^{5t}-15\,\mathrm{e}^{4t}+15c_1-90\right)\mathrm{e}^{-4t}}{15}\\ -\frac{2\left(\left(-\frac{3c_3}{2}-\frac{29}{2}\right)\mathrm{e}^{10t}+\left(-\frac{3c_2}{5}+22\right)\mathrm{e}^{7t}+c_1+\frac{3\,\mathrm{e}^{4t}}{2}-3\,\mathrm{e}^{5t}-6\right)\mathrm{e}^{-4t}}{3}\\ \frac{\left(3c_3\mathrm{e}^{10t}+29\,\mathrm{e}^{10t}+3c_2\mathrm{e}^{7t}-110\,\mathrm{e}^{7t}+111\,\mathrm{e}^{5t}-12\,\mathrm{e}^{4t}+3c_1-18\right)\mathrm{e}^{-4t}}{3}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=\frac{\left(15c_3\mathrm{e}^{10t}+145\,\mathrm{e}^{10t}+6c_2\mathrm{e}^{7t}-220\,\mathrm{e}^{7t}+180\,\mathrm{e}^{5t}-15\,\mathrm{e}^{4t}+15c_1-90\right)\mathrm{e}^{-4t}}{15}, y=-\frac{2\left(\left(-\frac{3c_3}{2}-\frac{29}{2}\right)\mathrm{e}^{10t}+\left(-\frac{3c_2}{5}+22\right)\mathrm{e}^{7t}+c_1\right.}{3}\right.$$

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 102

```
dsolve([diff(x(t),t)=2*x(t)+6*y(t)-2*z(t)+50*exp(t),diff(y(t),t)=6*x(t)+2*y(t)-2*z(t)+21*exp
```

$$x(t) = -6\,e^{-t} + c_3e^{6t} + e^{-4t}c_1 + \frac{2c_2e^{3t}}{5} + 12\,e^{t} - 1$$
$$y(t) = e^{-t} + c_3e^{6t} - \frac{2\,e^{-4t}c_1}{3} + \frac{2c_2e^{3t}}{5} + 2\,e^{t} - 1$$
$$z(t) = -6\,e^{-t} + c_3e^{6t} + e^{-4t}c_1 + c_2e^{3t} + 37\,e^{t} - 4$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 213

```
DSolve[{x'[t]==2*x[t]+6*y[t]-2*z[t]+50*Exp[t],y'[t]==6*x[t]+2*y[t]-2*z[t]+21*Exp[-t],z'[t]==
```

$$x(t) \to -6e^{-t} + 12e^{t} + \frac{3}{5}(c_1 - c_2)e^{-4t} + \frac{1}{15}(16c_1 + 9c_2 - 10c_3)e^{6t} - \frac{2}{3}(c_1 - c_3)e^{3t} - 1$$
$$y(t) \to e^{-t} + 2e^{t} - \frac{2}{5}(c_1 - c_2)e^{-4t} + \frac{1}{15}(16c_1 + 9c_2 - 10c_3)e^{6t} - \frac{2}{3}(c_1 - c_3)e^{3t} - 1$$
$$z(t) \to -6e^{-t} + 37e^{t} + \frac{3}{5}(c_1 - c_2)e^{-4t} + \frac{1}{15}(16c_1 + 9c_2 - 10c_3)e^{6t} - \frac{5}{3}(c_1 - c_3)e^{3t} - 4$$

## 7.7 problem Problem 4(c)



Internal problem ID [12382]
Internal file name [OUTPUT/11034_Wednesday_October_04_2023_01_27_13_AM_52232036/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 4(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) - 2y + 4z(t) \\
y' &= -2x(t) + y + 2z(t) \\
z'(t) &= -4x(t) - 2y + 6z(t) + \mathrm{e}^{2t}
\end{aligned}
$$

### 7.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -3\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 4\,e^{2t}-4\,e^{t} \\ -2\,e^{2t}+2\,e^{t} & e^{t} & 2\,e^{2t}-2\,e^{t} \\ -4\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 5\,e^{2t}-4\,e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -3\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 4\,e^{2t}-4\,e^{t} \\ -2\,e^{2t}+2\,e^{t} & e^{t} & 2\,e^{2t}-2\,e^{t} \\ -4\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 5\,e^{2t}-4\,e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(-3\,e^{2t}+4\,e^{t}\right)c_1 + \left(-2\,e^{2t}+2\,e^{t}\right)c_2 + \left(4\,e^{2t}-4\,e^{t}\right)c_3 \\ \left(-2\,e^{2t}+2\,e^{t}\right)c_1 + e^{t}c_2 + \left(2\,e^{2t}-2\,e^{t}\right)c_3 \\ \left(-4\,e^{2t}+4\,e^{t}\right)c_1 + \left(-2\,e^{2t}+2\,e^{t}\right)c_2 + \left(5\,e^{2t}-4\,e^{t}\right)c_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(-3c_1-2c_2+4c_3\right)e^{2t} + 4\left(c_1+\frac{c_2}{2}-c_3\right)e^{t} \\ \left(-2c_1+2c_3\right)e^{2t} + 2\left(c_1+\frac{c_2}{2}-c_3\right)e^{t} \\ \left(-4c_1-2c_2+5c_3\right)e^{2t} + 4\left(c_1+\frac{c_2}{2}-c_3\right)e^{t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} e^{-2t}(4\,e^{t}-3) & 2\,e^{-2t}(e^{t}-1) & -4\,e^{-2t}(e^{t}-1) \\ 2\,e^{-2t}(e^{t}-1) & e^{-t} & -2\,e^{-2t}(e^{t}-1) \\ 4\,e^{-2t}(e^{t}-1) & 2\,e^{-2t}(e^{t}-1) & (-4\,e^{t}+5)\,e^{-2t} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -3\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 4\,e^{2t}-4\,e^{t} \\ -2\,e^{2t}+2\,e^{t} & e^{t} & 2\,e^{2t}-2\,e^{t} \\ -4\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 5\,e^{2t}-4\,e^{t} \end{bmatrix} \int \begin{bmatrix} e^{-2t}(4\,e^{t}-3) & 2\,e^{-2t}(e^{t}-1) & -4\,e^{-2t}(e^{t}-1) \\ 2\,e^{-2t}(e^{t}-1) & e^{-t} & -2\,e^{-2t}(e^{t}-1) \\ 4\,e^{-2t}(e^{t}-1) & 2\,e^{-2t}(e^{t}-1) & (-4\,e^{t}+5)\,e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} -3\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 4\,e^{2t}-4\,e^{t} \\ -2\,e^{2t}+2\,e^{t} & e^{t} & 2\,e^{2t}-2\,e^{t} \\ -4\,e^{2t}+4\,e^{t} & -2\,e^{2t}+2\,e^{t} & 5\,e^{2t}-4\,e^{t} \end{bmatrix} \begin{bmatrix} 4t-4\,e^{t} \\ 2t-2\,e^{t} \\ 5t-4\,e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 4\,e^{2t}(t-1) \\ 2\,e^{2t}(t-1) \\ e^{2t}(-4+5t) \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} (4t-3c_1-2c_2+4c_3-4)\,e^{2t} + 4\left(c_1+\frac{c_2}{2}-c_3\right)e^{t} \\ (2t-2c_1+2c_3-2)\,e^{2t} + 2\left(c_1+\frac{c_2}{2}-c_3\right)e^{t} \\ (5t-4c_1-2c_2+5c_3-4)\,e^{2t} + 4\left(c_1+\frac{c_2}{2}-c_3\right)e^{t} \end{bmatrix} \end{aligned}$$

### 7.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & -2 & 4 \\ -2 & 1-\lambda & 2 \\ -4 & -2 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 1 | 1 | real eigenvalue |
| 2 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -2 & 4 \\ -2 & 0 & 2 \\ -4 & -2 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 0 \\ -2 & 0 & 2 & 0 \\ -4 & -2 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -2 & 4 & 0 \\ 0 & \frac{4}{3} & -\frac{2}{3} & 0 \\ -4 & -2 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -2 & 4 & 0 \\ 0 & \frac{4}{3} & -\frac{2}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -3 & -2 & 4 & 0 \\ 0 & \frac{4}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -2 & 4 \\ 0 & \frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ -4 & -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ -2 & -1 & 2 & 0 \\ -4 & -2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & -2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2} + s\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} + s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} + s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -\frac{t}{2} + s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -\frac{t}{2} + s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

Which are normalized to

$$\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 2 | 2 | 2 | No | $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| 1 | 1 | 1 | No | $\begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

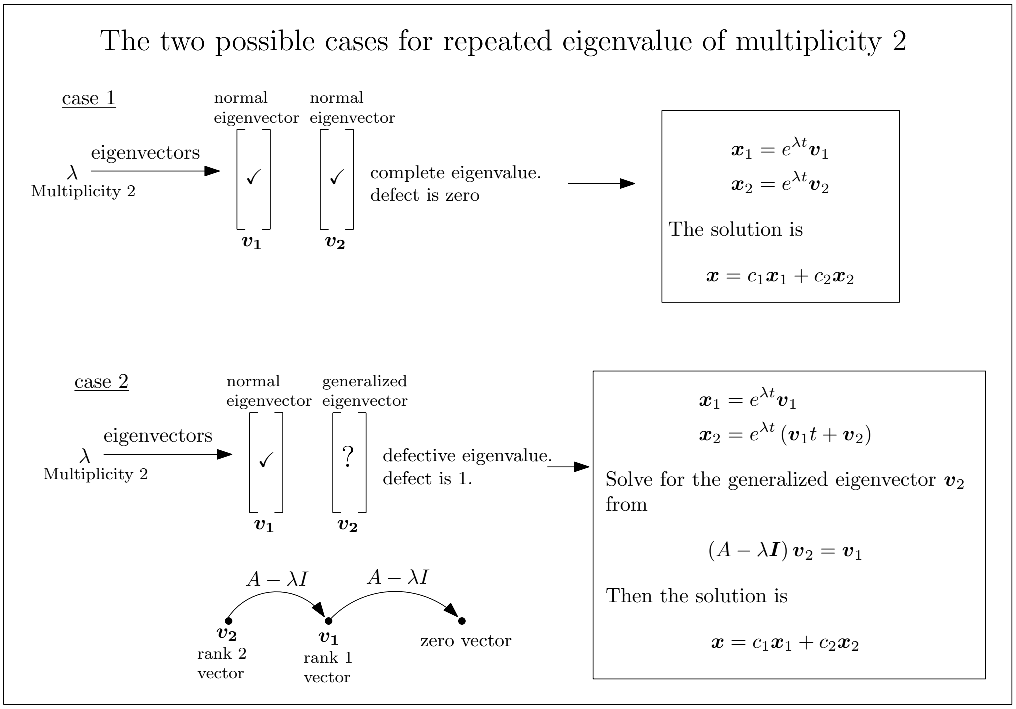



Figure 118: Possible case for repeated $\lambda$ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} e^{2t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^t$$
$$= \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} e^t$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{2} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t} \\ \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \mathrm{e}^{2t} & -\frac{\mathrm{e}^{2t}}{2} & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{2t} & \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{2t} & 0 & \mathrm{e}^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -4\,\mathrm{e}^{-2t} & -2\,\mathrm{e}^{-2t} & 5\,\mathrm{e}^{-2t} \\ -2\,\mathrm{e}^{-2t} & 0 & 2\,\mathrm{e}^{-2t} \\ 4\,\mathrm{e}^{-t} & 2\,\mathrm{e}^{-t} & -4\,\mathrm{e}^{-t} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t} & -\frac{e^{2t}}{2} & e^{t} \\ 0 & e^{2t} & \frac{e^{t}}{2} \\ e^{2t} & 0 & e^{t} \end{bmatrix} \int \begin{bmatrix} -4\,e^{-2t} & -2\,e^{-2t} & 5\,e^{-2t} \\ -2\,e^{-2t} & 0 & 2\,e^{-2t} \\ 4\,e^{-t} & 2\,e^{-t} & -4\,e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & -\frac{e^{2t}}{2} & e^{t} \\ 0 & e^{2t} & \frac{e^{t}}{2} \\ e^{2t} & 0 & e^{t} \end{bmatrix} \int \begin{bmatrix} 5 \\ 2 \\ -4\,e^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & -\frac{e^{2t}}{2} & e^{t} \\ 0 & e^{2t} & \frac{e^{t}}{2} \\ e^{2t} & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 5t \\ 2t \\ -4\,e^{t} \end{bmatrix} \\
&= \begin{bmatrix} 4\,e^{2t}(t-1) \\ 2\,e^{2t}(t-1) \\ e^{2t}(-4+5t) \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ 0 \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2 e^{2t}}{2} \\ c_2 e^{2t} \\ 0 \end{bmatrix} + \begin{bmatrix} c_3 e^{t} \\ \frac{c_3 e^{t}}{2} \\ c_3 e^{t} \end{bmatrix} + \begin{bmatrix} 4\,e^{2t}(t-1) \\ 2\,e^{2t}(t-1) \\ e^{2t}(-4+5t) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(8t+2c_1-c_2-8)e^{2t}}{2} + c_3 e^{t} \\ (c_2+2t-2)\,e^{2t} + \frac{c_3 e^{t}}{2} \\ (5t+c_1-4)\,e^{2t} + c_3 e^{t} \end{bmatrix}
$$

### 7.7.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -2x(t) - 2y + 4z(t), y' = -2x(t) + y + 2z(t), z'(t) = -4x(t) - 2y + 6z(t) + (e^t)^2\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ (e^t)^2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ (e^t)^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ (e^t)^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[1, \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} -\frac{1}{2} \\ 1 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^t \cdot \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 2

$$\overrightarrow{x}_2(t) = \mathrm{e}^{2t} \cdot \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\overrightarrow{x}_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the $t$ multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\overrightarrow{x}_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of $A$

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{x}_3(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} -2 & -2 & 4 \\ -2 & 1 & 2 \\ -4 & -2 & 6 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

  $$\vec{x}_3(t) = \mathrm{e}^{2t} \cdot \left(t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \vec{x}_p(t)$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \begin{bmatrix} \mathrm{e}^t & \mathrm{e}^{2t} & \mathrm{e}^{2t}\left(t - \frac{1}{4}\right) \\ \frac{\mathrm{e}^t}{2} & 0 & 0 \\ \mathrm{e}^t & \mathrm{e}^{2t} & \mathrm{e}^{2t}t \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{t} & e^{2t} & e^{2t}\left(t - \frac{1}{4}\right) \\ \frac{e^{t}}{2} & 0 & 0 \\ e^{t} & e^{2t} & e^{2t}t \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & -\frac{1}{4} \\ \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} (1-4t)\,e^{2t} & -2\,e^{2t} + 2\,e^{t} & 4\,e^{2t}t \\ 0 & e^{t} & 0 \\ -4\,e^{2t}t & -2\,e^{2t} + 2\,e^{t} & e^{2t}(4t+1) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} 2\,e^{2t}t^2 \\ 0 \\ e^{2t}(2t^2+t) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + \begin{bmatrix} 2\,e^{2t}t^2 \\ 0 \\ e^{2t}(2t^2+t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{((-1+4t)c_3+8t^2+4c_2)e^{2t}}{4} + c_1 e^t \\ \frac{c_1 e^t}{2} \\ (2t^2+(1+c_3)\,t+c_2)\,e^{2t}+c_1 e^t \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{((-1+4t)c_3+8t^2+4c_2)e^{2t}}{4} + c_1 e^t, y = \frac{c_1 e^t}{2}, z(t) = (2t^2+(1+c_3)\,t+c_2)\,e^{2t}+c_1 e^t\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 77

```
dsolve([diff(x(t),t)=-2*x(t)-2*y(t)+4*z(t),diff(y(t),t)=-2*x(t)+1*y(t)+2*z(t),diff(z(t),t)=-
```

$$x(t) = (e^t(4t+c_2-4)+c_3)\,e^t$$
$$y(t) = \left(\frac{c_2}{2}+2t-2+c_1\right)e^{2t}+\frac{c_3 e^t}{2}$$
$$z(t) = \frac{5c_2 e^{2t}}{4}+5\,e^{2t}t-4\,e^{2t}+c_3 e^t+\frac{c_1 e^{2t}}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 118

```
DSolve[{x'[t]==-2*x[t]-2*y[t]+4*z[t],y'[t]==-2*x[t]+y[t]+2*z[t],z'[t]==-4*x[t]-2*y[t]+6*z[t]
```

$$x(t) \to e^t\left(e^t(4t-4-3c_1-2c_2+4c_3)+2(2c_1+c_2-2c_3)\right)$$
$$y(t) \to e^t\left(2e^t(t-1-c_1+c_3)+2c_1+c_2-2c_3\right)$$
$$z(t) \to e^t\left(e^t(5t-4-4c_1-2c_2+5c_3)+2(2c_1+c_2-2c_3)\right)$$

## 7.8 problem Problem 4(d)



Internal problem ID [12383]
Internal file name [`OUTPUT/11035_Wednesday_October_04_2023_01_27_14_AM_59545152/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 4(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}
x'(t) &= 3x(t) - 2y + 3z(t) \\
y' &= x(t) - y + 2z(t) + 2\,\mathrm{e}^{-t} \\
z'(t) &= -2x(t) + 2y - 2z(t)
\end{aligned}$$

### 7.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2\,\mathrm{e}^{-t} \\ 0 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(6t+13)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & -\frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(6t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\ \frac{(-3t+4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{(\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{3} & \frac{(-3t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\ \frac{(-6t-4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} + \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{7\,\mathrm{e}^{-2t}}{9} + \frac{(-6t+2)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{(6t+13)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & -\frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(6t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\ \frac{(-3t+4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{(\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{3} & \frac{(-3t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\ \frac{(-6t-4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} + \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{7\,\mathrm{e}^{-2t}}{9} + \frac{(-6t+2)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{(6t+13)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9}\right)c_1 - \frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_2}{3} + \left(\frac{(6t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9}\right)c_3 \\ \left(\frac{(-3t+4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9}\right)c_1 + \frac{(\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}c_2}{3} + \left(\frac{(-3t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9}\right)c_3 \\ \left(\frac{(-6t-4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} + \frac{4\,\mathrm{e}^{-2t}}{9}\right)c_1 + \frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}c_2}{3} + \left(\frac{7\,\mathrm{e}^{-2t}}{9} + \frac{(-6t+2)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-2t}\left(\left(c_1\left(t+\frac{13}{6}\right)+\left(t+\frac{7}{6}\right)c_3-c_2\right)\mathrm{e}^{3t}-\frac{2c_1}{3}+c_2-\frac{7c_3}{6}\right)}{3} \\ -\frac{\left(\left(\left(t-\frac{4}{3}\right)c_1+\left(t-\frac{7}{3}\right)c_3-c_2\right)\mathrm{e}^{3t}+\frac{4c_1}{3}-2c_2+\frac{7c_3}{3}\right)\mathrm{e}^{-2t}}{3} \\ -\frac{2\left(\left(c_1\left(t+\frac{2}{3}\right)+\left(t-\frac{1}{3}\right)c_3-c_2\right)\mathrm{e}^{3t}-\frac{2c_1}{3}+c_2-\frac{7c_3}{6}\right)\mathrm{e}^{-2t}}{3} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \frac{(-6t+13)\mathrm{e}^{-t}}{9} - \frac{4\,\mathrm{e}^{2t}}{9} & \frac{2\,\mathrm{e}^{2t}}{3} - \frac{2\,\mathrm{e}^{-t}}{3} & \frac{(-6t+7)\mathrm{e}^{-t}}{9} - \frac{7\,\mathrm{e}^{2t}}{9} \\ \frac{(4+3t)\mathrm{e}^{-t}}{9} - \frac{4\,\mathrm{e}^{2t}}{9} & \frac{\mathrm{e}^{-t}}{3} + \frac{2\,\mathrm{e}^{2t}}{3} & \frac{(3t+7)\mathrm{e}^{-t}}{9} - \frac{7\,\mathrm{e}^{2t}}{9} \\ \frac{(6t-4)\mathrm{e}^{-t}}{9} + \frac{4\,\mathrm{e}^{2t}}{9} & -\frac{2\,\mathrm{e}^{2t}}{3} + \frac{2\,\mathrm{e}^{-t}}{3} & \frac{(6t+2)\mathrm{e}^{-t}}{9} + \frac{7\,\mathrm{e}^{2t}}{9} \end{bmatrix}
\end{aligned}$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{ccc}
\frac{(6t+13)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & -\frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(6t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\
\frac{(-3t+4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{(\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{3} & \frac{(-3t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\
\frac{(-6t-4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} + \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{7\,\mathrm{e}^{-2t}}{9} + \frac{(-6t+2)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9}
\end{array}\right] \int \left[\begin{array}{cc}
\frac{(-6t+13)\mathrm{e}^{-t}}{9} - \frac{4\,\mathrm{e}^{2t}}{9} & \frac{2\,\mathrm{e}^{2t}}{3} - \frac{2\,\mathrm{e}^{-}}{3} \\
\frac{(4+3t)\mathrm{e}^{-t}}{9} - \frac{4\,\mathrm{e}^{2t}}{9} & \frac{\mathrm{e}^{-t}}{3} + \frac{2\,\mathrm{e}^{2t}}{3} \\
\frac{(6t-4)\mathrm{e}^{-t}}{9} + \frac{4\,\mathrm{e}^{2t}}{9} & -\frac{2\,\mathrm{e}^{2t}}{3} + \frac{2\,\mathrm{e}}{3}
\end{array}\right.
$$

$$
= \left[\begin{array}{ccc}
\frac{(6t+13)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & -\frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{(6t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\
\frac{(-3t+4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{(\mathrm{e}^{3t}+2)\mathrm{e}^{-2t}}{3} & \frac{(-3t+7)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} - \frac{7\,\mathrm{e}^{-2t}}{9} \\
\frac{(-6t-4)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9} + \frac{4\,\mathrm{e}^{-2t}}{9} & \frac{2(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} & \frac{7\,\mathrm{e}^{-2t}}{9} + \frac{(-6t+2)\mathrm{e}^{-2t}\mathrm{e}^{3t}}{9}
\end{array}\right] \left[\begin{array}{c}
\frac{2(1+2\,\mathrm{e}^{3t})\mathrm{e}^{-2t}}{3} \\
\frac{(4\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{3} \\
-\frac{2(1+2\,\mathrm{e}^{3t})\mathrm{e}^{-2t}}{3}
\end{array}\right]
$$

$$
= \left[\begin{array}{c} 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-t} \end{array}\right]
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c}
\frac{2\,\mathrm{e}^{-2t}\left(\left(c_1\left(t+\frac{13}{6}\right)+\left(t+\frac{7}{6}\right)c_3-c_2\right)\mathrm{e}^{3t}-\frac{2c_1}{3}+c_2-\frac{7c_3}{6}+3\,\mathrm{e}^{t}\right)}{3} \\
-\frac{\mathrm{e}^{-2t}\left(\left(\left(t-\frac{4}{3}\right)c_1+\left(t-\frac{7}{3}\right)c_3-c_2\right)\mathrm{e}^{3t}+\frac{4c_1}{3}-2c_2+\frac{7c_3}{3}-3\,\mathrm{e}^{t}\right)}{3} \\
-\frac{2\left(\left(c_1\left(t+\frac{2}{3}\right)+\left(t-\frac{1}{3}\right)c_3-c_2\right)\mathrm{e}^{3t}-\frac{2c_1}{3}+c_2-\frac{7c_3}{6}+3\,\mathrm{e}^{t}\right)\mathrm{e}^{-2t}}{3}
\end{array}\right]
\end{aligned}
$$

### 7.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y' \\ z'(t) \end{array}\right] = \left[\begin{array}{ccc} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{array}\right] \left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ 2\,\mathrm{e}^{-t} \\ 0 \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 & 3 \\ 1 & -1-\lambda & 2 \\ -2 & 2 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= 1 \end{aligned}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-2$ | 1 | real eigenvalue |
| 1 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -2 & 3 \\ 1 & 1 & 2 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & -2 & 3 & 0 \\ 1 & 1 & 2 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & -2 & 3 & 0 \\ 0 & \frac{7}{5} & \frac{7}{5} & 0 \\ -2 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{5} \implies \left[\begin{array}{ccc|c} 5 & -2 & 3 & 0 \\ 0 & \frac{7}{5} & \frac{7}{5} & 0 \\ 0 & \frac{6}{5} & \frac{6}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{6R_2}{7} \implies \left[\begin{array}{ccc|c} 5 & -2 & 3 & 0 \\ 0 & \frac{7}{5} & \frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & \frac{7}{5} & \frac{7}{5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & -2 & 2 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 1 & -2 & 2 & 0 \\ -2 & 2 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ -2 & 2 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -t, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-2$ | 1 | 1 | No | $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ |
| 1 | 2 | 1 | Yes | $\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-2$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

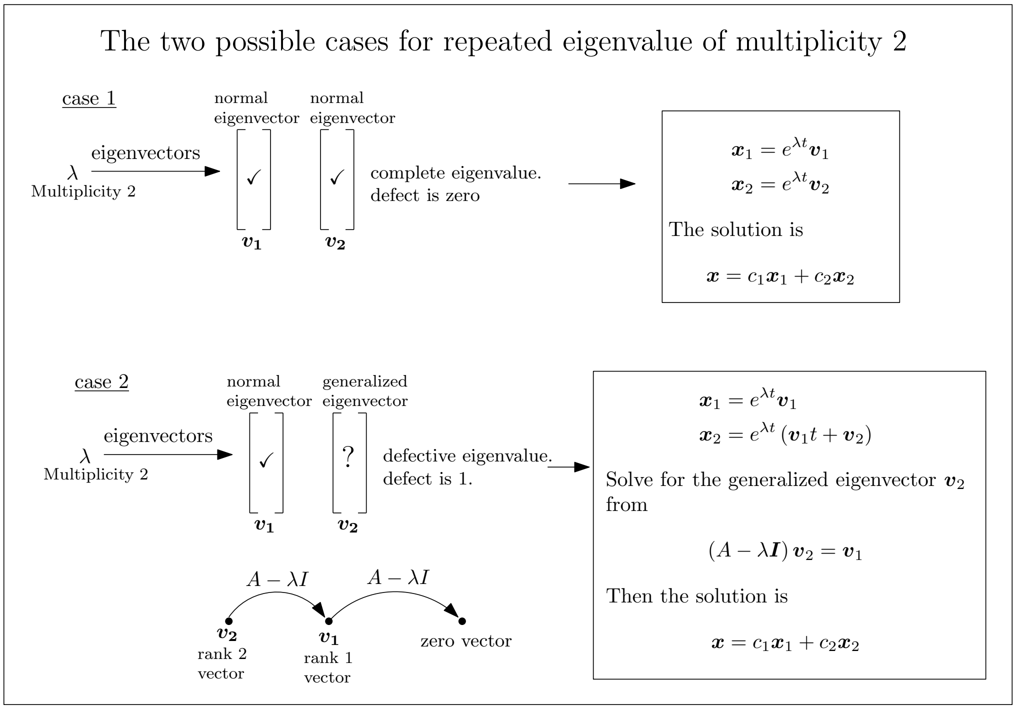



Figure 119: Possible case for repeated $\lambda$ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & -2 & 2 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{11}{2} \\ 1 \\ 4 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}
\vec{x}_2(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} -\mathrm{e}^{t} \\ \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t} \end{bmatrix}
\end{aligned}$$

And

$$\begin{aligned}
\vec{x}_3(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left( \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{11}{2} \\ 1 \\ 4 \end{bmatrix} \right) \mathrm{e}^{t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{t}(2t+11)}{2} \\ \frac{\mathrm{e}^{t}(t+2)}{2} \\ \mathrm{e}^{t}(t+4) \end{bmatrix}
\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{t} \\ \frac{\mathrm{e}^{t}}{2} \\ \mathrm{e}^{t} \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t}\left(-t - \frac{11}{2}\right) \\ \mathrm{e}^{t}\left(\frac{t}{2} + 1\right) \\ \mathrm{e}^{t}(t+4) \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-2t} & -\mathrm{e}^{t} & \mathrm{e}^{t}\left(-t-\frac{11}{2}\right) \\ -\mathrm{e}^{-2t} & \frac{\mathrm{e}^{t}}{2} & \mathrm{e}^{t}\left(\frac{t}{2}+1\right) \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} & \mathrm{e}^{t}(t+4) \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{4\,\mathrm{e}^{2t}}{9} & -\frac{2\,\mathrm{e}^{2t}}{3} & \frac{7\,\mathrm{e}^{2t}}{9} \\ \frac{2(3t+10)\mathrm{e}^{-t}}{9} & \frac{2\,\mathrm{e}^{-t}}{3} & \frac{2(3t+13)\mathrm{e}^{-t}}{9} \\ -\frac{2\,\mathrm{e}^{-t}}{3} & 0 & -\frac{2\,\mathrm{e}^{-t}}{3} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\mathrm{e}^{-2t} & -\mathrm{e}^{t} & \mathrm{e}^{t}\left(-t-\frac{11}{2}\right) \\ -\mathrm{e}^{-2t} & \frac{\mathrm{e}^{t}}{2} & \mathrm{e}^{t}\left(\frac{t}{2}+1\right) \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} & \mathrm{e}^{t}(t+4) \end{bmatrix} \int \begin{bmatrix} \frac{4\,\mathrm{e}^{2t}}{9} & -\frac{2\,\mathrm{e}^{2t}}{3} & \frac{7\,\mathrm{e}^{2t}}{9} \\ \frac{2(3t+10)\mathrm{e}^{-t}}{9} & \frac{2\,\mathrm{e}^{-t}}{3} & \frac{2(3t+13)\mathrm{e}^{-t}}{9} \\ -\frac{2\,\mathrm{e}^{-t}}{3} & 0 & -\frac{2\,\mathrm{e}^{-t}}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 2\,\mathrm{e}^{-t} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-2t} & -\mathrm{e}^{t} & \mathrm{e}^{t}\left(-t-\frac{11}{2}\right) \\ -\mathrm{e}^{-2t} & \frac{\mathrm{e}^{t}}{2} & \mathrm{e}^{t}\left(\frac{t}{2}+1\right) \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} & \mathrm{e}^{t}(t+4) \end{bmatrix} \int \begin{bmatrix} -\frac{4\,\mathrm{e}^{t}}{3} \\ \frac{4\,\mathrm{e}^{-2t}}{3} \\ 0 \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{-2t} & -\mathrm{e}^{t} & \mathrm{e}^{t}\left(-t-\frac{11}{2}\right) \\ -\mathrm{e}^{-2t} & \frac{\mathrm{e}^{t}}{2} & \mathrm{e}^{t}\left(\frac{t}{2}+1\right) \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} & \mathrm{e}^{t}(t+4) \end{bmatrix} \begin{bmatrix} -\frac{4\,\mathrm{e}^{t}}{3} \\ -\frac{2\,\mathrm{e}^{-2t}}{3} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{-2t} \\ -c_1 e^{-2t} \\ c_1 e^{-2t} \end{bmatrix} + \begin{bmatrix} -c_2 e^{t} \\ \frac{c_2 e^{t}}{2} \\ c_2 e^{t} \end{bmatrix} + \begin{bmatrix} c_3 e^{t}\left(-t - \frac{11}{2}\right) \\ c_3 e^{t}\left(\frac{t}{2} + 1\right) \\ c_3 e^{t}(t+4) \end{bmatrix} + \begin{bmatrix} 2\,e^{-t} \\ e^{-t} \\ -2\,e^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -e^{-2t}\left(\left(\left(t + \frac{11}{2}\right) c_3 + c_2\right) e^{3t} + c_1 - 2\,e^{t}\right) \\ \frac{e^{-2t}\left(((t+2)c_3 + c_2)e^{3t} - 2c_1 + 2\,e^{t}\right)}{2} \\ e^{-2t}\left(((t+4)\,c_3 + c_2)\,e^{3t} + c_1 - 2\,e^{t}\right) \end{bmatrix}$$

### 7.8.3 Maple step by step solution

Let's solve

$$\left[x'(t) = 3x(t) - 2y + 3z(t)\,, y' = x(t) - y + 2z(t) + \frac{2}{e^{t}}, z'(t) = -2x(t) + 2y - 2z(t)\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ \frac{x(t)e^{t} - y\,e^{t} + 2z(t)e^{t} + 2}{e^{t}} - x(t) + y - 2z(t) \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{array}\right]$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

  $$\left[\left[-2, \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} -1 \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

  $$\left[-2, \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = e^{-2t} \cdot \left[\begin{array}{c} -1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[1, \left[\begin{array}{c} -1 \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 1

  $$\overrightarrow{x}_2(t) = e^{t} \cdot \left[\begin{array}{c} -1 \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

$\overrightarrow{x}_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the $t$ multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\overrightarrow{x}_3(t)$ into the homogeneous system

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right)\cdot\left(t\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of $A$

  $\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$
- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$
- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{x}_3(t)$ to be a solution to the homogeneous system

  $(A - \lambda\cdot I)\cdot\overrightarrow{p} = \overrightarrow{v}$
- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

  $$\left(\left[\begin{array}{ccc} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -2 & 2 & -2 \end{array}\right] - 1\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p} = \left[\begin{array}{c} -1 \\ \frac{1}{2} \\ 1 \end{array}\right]$$
- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \left[\begin{array}{c} -\frac{1}{2} \\ 0 \\ 0 \end{array}\right]$$
- Second solution from eigenvalue 1

  $$\overrightarrow{x}_3(t) = \mathrm{e}^{t}\cdot\left(t\cdot\left[\begin{array}{c} -1 \\ \frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{2} \\ 0 \\ 0 \end{array}\right]\right)$$
- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2(t) + c_3\overrightarrow{x}_3(t)$
- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^{-2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{t} \cdot \left( t \cdot \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -\left(\left(\left(t + \frac{1}{2}\right) c_3 + c_2\right) e^{3t} + c_1\right) e^{-2t} \\ \frac{e^{-2t}\left((c_3 t + c_2) e^{3t} - 2c_1\right)}{2} \\ \left((c_3 t + c_2) e^{3t} + c_1\right) e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\left(\left(\left(t + \frac{1}{2}\right) c_3 + c_2\right) e^{3t} + c_1\right) e^{-2t}, y = \frac{e^{-2t}\left((c_3 t + c_2) e^{3t} - 2c_1\right)}{2}, z(t) = \left((c_3 t + c_2) e^{3t} + c_1\right) e^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 90

```
dsolve([diff(x(t),t)=3*x(t)-2*y(t)+3*z(t),diff(y(t),t)=x(t)-y(t)+2*z(t)+2*exp(-t),diff(z(t),
```

$$\begin{aligned} x(t) &= 2\,e^{-t} + c_1 e^{t} + c_2 e^{-2t} + c_3 e^{t} t \\ y(t) &= e^{-t} - \frac{c_1 e^{t}}{2} + c_2 e^{-2t} - \frac{c_3 e^{t} t}{2} + \frac{7 c_3 e^{t}}{4} \\ z(t) &= -2\,e^{-t} - c_1 e^{t} - c_2 e^{-2t} - c_3 e^{t} t + \frac{3 c_3 e^{t}}{2} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 174

```
DSolve[{x'[t]==3*x[t]-2*y[t]+3*z[t],y'[t]==x[t]-y[t]+2*z[t]+2*Exp[-t],z'[t]==-2*x[t]+2*y[t]-
```

$$\begin{aligned} x(t) &\to \frac{1}{9} e^{-2t} \left(18 e^{t} + e^{3t} (c_1(6t + 13) + c_3(6t + 7) - 6c_2) - 4c_1 + 6c_2 - 7c_3\right) \\ y(t) &\to \frac{1}{9} e^{-2t} \left(9 e^{t} + e^{3t} (c_1(4 - 3t) + c_3(7 - 3t) + 3c_2) - 4c_1 + 6c_2 - 7c_3\right) \\ z(t) &\to \frac{1}{9} e^{-2t} \left(-18 e^{t} + 2 e^{3t} (-(c_1(3t + 2)) - 3c_3 t + 3c_2 + c_3) + 4c_1 - 6c_2 + 7c_3\right) \end{aligned}$$

## 7.9 problem Problem 5(a)



Internal problem ID [12384]
Internal file name [`OUTPUT/11036_Wednesday_October_04_2023_01_27_15_AM_6134005/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 5(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 7x(t) + y - 1 - 6\,\mathrm{e}^{t} \\
y' &= -4x(t) + 3y + 4\,\mathrm{e}^{t} - 3
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

### 7.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} -1 - 6\,\mathrm{e}^{t} \\ 4\,\mathrm{e}^{t} - 3 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{5t}(2t+1) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} e^{5t}(2t+1) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(2t+1) - t\,e^{5t} \\ -4t\,e^{5t} - e^{5t}(1-2t) \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(1+t) \\ e^{5t}(-2t-1) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{-5t}(1-2t) & -t\,e^{-5t} \\ 4t\,e^{-5t} & e^{-5t}(2t+1) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{5t}(2t+1) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix} \int \begin{bmatrix} e^{-5t}(1-2t) & -t\,e^{-5t} \\ 4t\,e^{-5t} & e^{-5t}(2t+1) \end{bmatrix} \begin{bmatrix} -1-6\,e^{t} \\ 4\,e^{t}-3 \end{bmatrix} dt \\
&= \begin{bmatrix} e^{5t}(2t+1) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix} \begin{bmatrix} e^{-4t} - t\,e^{-5t} - 2t\,e^{-4t} \\ e^{-5t} + 2t\,e^{-5t} + 4t\,e^{-4t} \end{bmatrix} \\
&= \begin{bmatrix} e^{t} \\ 1 \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} t\,e^{5t} + e^t + e^{5t} \\ e^{5t}(-2t-1) + 1 \end{bmatrix}$$

### 7.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} -1-6\,e^t \\ 4\,e^t - 3 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 10\lambda + 25 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 5 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 5 | 2 | 1 | Yes | $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

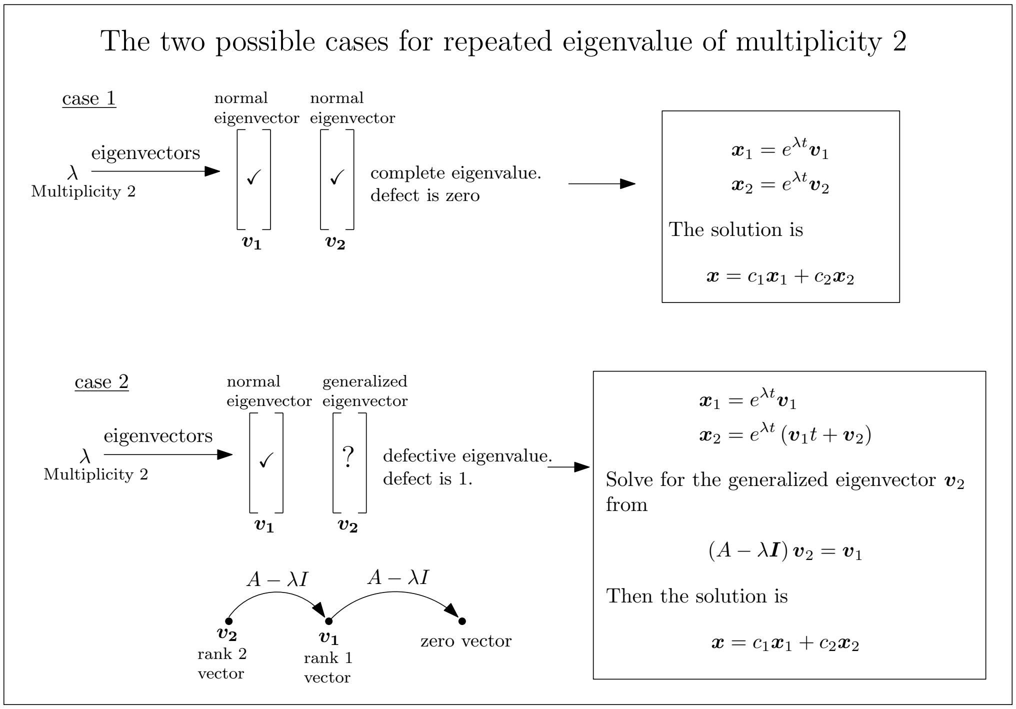



Figure 120: Possible case for repeated $\lambda$ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\, \vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left( \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{5t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left( \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix} \right) \mathrm{e}^{5t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{5t}(-2+t)}{2} \\ \frac{\mathrm{e}^{5t}(2t-5)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{5t}\left(-\frac{t}{2} + 1\right) \\ \mathrm{e}^{5t}\left(t - \frac{5}{2}\right) \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} & \mathrm{e}^{5t}\left(-\frac{t}{2} + 1\right) \\ \mathrm{e}^{5t} & \mathrm{e}^{5t}\left(t - \frac{5}{2}\right) \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} (4t-10)\,\mathrm{e}^{-5t} & 2(-2+t)\,\mathrm{e}^{-5t} \\ -4\,\mathrm{e}^{-5t} & -2\,\mathrm{e}^{-5t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} & \mathrm{e}^{5t}\left(-\frac{t}{2}+1\right) \\ \mathrm{e}^{5t} & \mathrm{e}^{5t}\left(t-\frac{5}{2}\right) \end{bmatrix} \int \begin{bmatrix} (4t-10)\,\mathrm{e}^{-5t} & 2(-2+t)\,\mathrm{e}^{-5t} \\ -4\,\mathrm{e}^{-5t} & -2\,\mathrm{e}^{-5t} \end{bmatrix} \begin{bmatrix} -1-6\,\mathrm{e}^{t} \\ 4\,\mathrm{e}^{t}-3 \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} & \mathrm{e}^{5t}\left(-\frac{t}{2}+1\right) \\ \mathrm{e}^{5t} & \mathrm{e}^{5t}\left(t-\frac{5}{2}\right) \end{bmatrix} \int \begin{bmatrix} (-10t+22)\,\mathrm{e}^{-5t}+(-16t+44)\,\mathrm{e}^{-4t} \\ 16\,\mathrm{e}^{-4t}+10\,\mathrm{e}^{-5t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} & \mathrm{e}^{5t}\left(-\frac{t}{2}+1\right) \\ \mathrm{e}^{5t} & \mathrm{e}^{5t}\left(t-\frac{5}{2}\right) \end{bmatrix} \begin{bmatrix} 2(-2+t)\,\mathrm{e}^{-5t}+2\,\mathrm{e}^{-4t}(2t-5) \\ -2\,\mathrm{e}^{-5t}-4\,\mathrm{e}^{-4t} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{t} \\ 1 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y \end{bmatrix} &= \begin{bmatrix} -\frac{c_1\mathrm{e}^{5t}}{2} \\ c_1\mathrm{e}^{5t} \end{bmatrix} + \begin{bmatrix} c_2\mathrm{e}^{5t}\left(-\frac{t}{2}+1\right) \\ c_2\mathrm{e}^{5t}\left(t-\frac{5}{2}\right) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t} \\ 1 \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{((-t+2)c_2-c_1)\mathrm{e}^{5t}}{2}+\mathrm{e}^{t} \\ 1+\frac{(c_2(2t-5)+2c_1)\mathrm{e}^{5t}}{2} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0)=1 \\ y(0)=-1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t=0$ gives

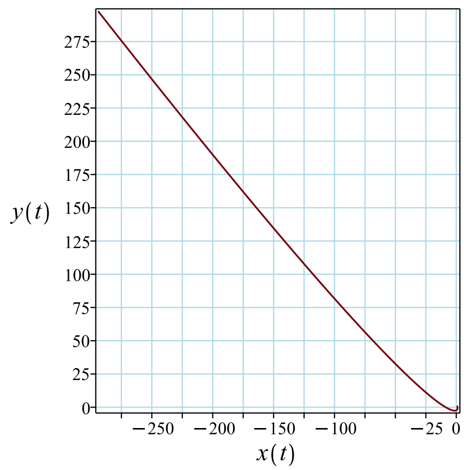
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_2-\frac{c_1}{2}+1 \\ 1-\frac{5c_2}{2}+c_1 \end{bmatrix}$$

Solving for the constants of integrations gives

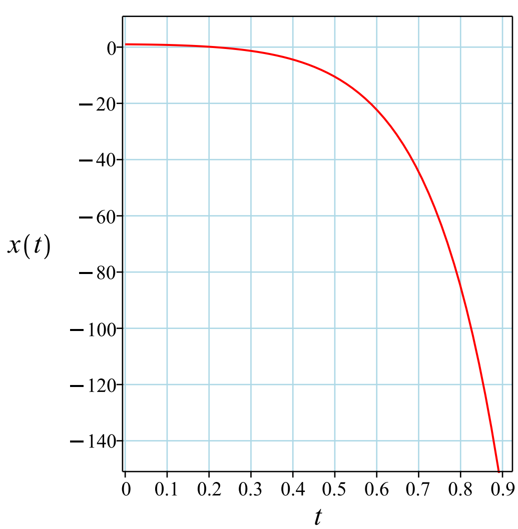
$$\begin{bmatrix} c_1 = 8 \\ c_2 = 4 \end{bmatrix}$$

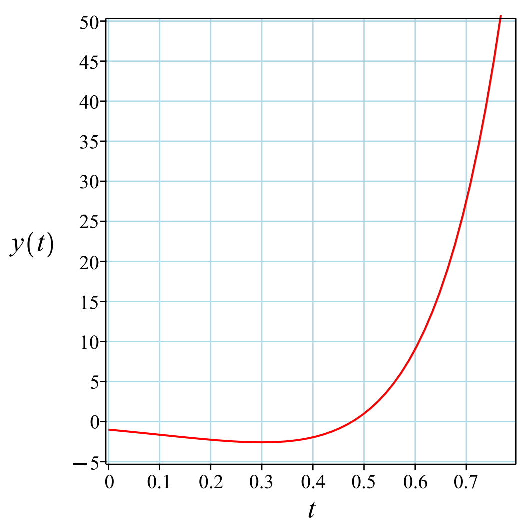
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -2t\,\mathrm{e}^{5t} + \mathrm{e}^{t} \\ 1 + \frac{(8t-4)\mathrm{e}^{5t}}{2} \end{bmatrix}$$




The following are plots of each solution.



✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve([diff(x(t),t) = 7*x(t)+y(t)-1-6*exp(t), diff(y(t),t) = -4*x(t)+3*y(t)+4*exp(t)-3, x(0
```

$$x(t) = -2t\,e^{5t} + e^{t}$$
$$y(t) = 1 - e^{5t}(-4t + 2)$$

✓ Solution by Mathematica

Time used: 0.325 (sec). Leaf size: 51

```
DSolve[{x'[t]==7*x[t]+y[t]-1-Exp[t],y'[t]==-4*x[t]+3*y[t]+4*Exp[t]-3},{x[0]==1,y[0]==-1},{x[
```

$$x(t) \to \frac{1}{8}e^{t}\left(e^{4t}(4t+5)+3\right)$$
$$y(t) \to \frac{1}{4}\left(-e^{5t}(4t+3)-5e^{t}+4\right)$$

## 7.10 problem Problem 5(b)



Internal problem ID [12385]
Internal file name [`OUTPUT/11037_Wednesday_October_04_2023_01_27_15_AM_19470867/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 5(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= 3x(t) - 2y + 24\sin(t) \\ y' &= 9x(t) - 3y + 12\cos(t) \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

### 7.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 24\sin(t) \\ 12\cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \sin(3t) + \cos(3t) & -\frac{2\sin(3t)}{3} \\ 3\sin(3t) & \cos(3t) - \sin(3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \sin(3t) + \cos(3t) & -\frac{2\sin(3t)}{3} \\ 3\sin(3t) & \cos(3t) - \sin(3t) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{5\sin(3t)}{3} + \cos(3t) \\ 4\sin(3t) - \cos(3t) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \cos(3t) - \sin(3t) & \frac{2\sin(3t)}{3} \\ -3\sin(3t) & \sin(3t) + \cos(3t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \sin(3t) + \cos(3t) & -\frac{2\sin(3t)}{3} \\ 3\sin(3t) & \cos(3t) - \sin(3t) \end{bmatrix} \int \begin{bmatrix} \cos(3t) - \sin(3t) & \frac{2\sin(3t)}{3} \\ -3\sin(3t) & \sin(3t) + \cos(3t) \end{bmatrix} \begin{bmatrix} 24\sin( \\ 12\cos( \end{bmatrix} \\
&= \begin{bmatrix} \sin(3t) + \cos(3t) & -\frac{2\sin(3t)}{3} \\ 3\sin(3t) & \cos(3t) - \sin(3t) \end{bmatrix} \begin{bmatrix} 4\cos(2t) - 4\cos(4t) - 6\sin(2t) + 3\sin(4t) \\ -15\sin(2t) + \frac{21\sin(4t)}{2} - 3\cos(2t) - \frac{3\cos(4t)}{2} \end{bmatrix} \\
&= \begin{bmatrix} 9\sin(t) \\ \frac{51\sin(t)}{2} - \frac{9\cos(t)}{2} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{5\sin(3t)}{3} + \cos(3t) + 9\sin(t) \\ 4\sin(3t) - \cos(3t) + \frac{51\sin(t)}{2} - \frac{9\cos(t)}{2} \end{bmatrix}$$

### 7.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 24\sin(t) \\ 12\cos(t) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 9 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $3i$ | 1 | complex eigenvalue |
| $-3i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3+3i & -2 \\ 9 & -3+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3+3i & -2 & 0 \\ 9 & -3+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{2} + \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3+3i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+3i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{3} - \frac{i}{3}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{3} - \frac{\mathrm{I}}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3} - \frac{i}{3}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{3} - \frac{\mathrm{I}}{3}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} - \frac{i}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{3} - \frac{\mathrm{I}}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} - \frac{i}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{3} - \frac{\mathrm{I}}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 3 & -2 \\ 9 & -3 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 - 3i & -2 \\ 9 & -3 - 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 - 3i & -2 & 0 \\ 9 & -3 - 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{2} - \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 - 3i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 - 3i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{3} + \frac{i}{3}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3} + \frac{i}{3}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} + \frac{i}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{i}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{3} + \frac{I}{3}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $3i$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{3} + \frac{i}{3} \\ 1 \end{bmatrix}$ |
| $-3i$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{3} - \frac{i}{3} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) e^{3it} \\ e^{3it} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{3}-\frac{i}{3}\right) e^{-3it} \\ e^{-3it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) e^{3it} & \left(\frac{1}{3}-\frac{i}{3}\right) e^{-3it} \\ e^{3it} & e^{-3it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{3i e^{-3it}}{2} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{-3it} \\ \frac{3i e^{3it}}{2} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{3it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) e^{3it} & \left(\frac{1}{3}-\frac{i}{3}\right) e^{-3it} \\ e^{3it} & e^{-3it} \end{bmatrix} \int \begin{bmatrix} -\frac{3i e^{-3it}}{2} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{-3it} \\ \frac{3i e^{3it}}{2} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{3it} \end{bmatrix} \begin{bmatrix} 24\sin(t) \\ 12\cos(t) \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) e^{3it} & \left(\frac{1}{3}-\frac{i}{3}\right) e^{-3it} \\ e^{3it} & e^{-3it} \end{bmatrix} \int \begin{bmatrix} 6\, e^{-3it}(-6i\sin(t) + (1+i)\cos(t)) \\ -6((-1+i)\cos(t) - 6i\sin(t))\, e^{3it} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) e^{3it} & \left(\frac{1}{3}-\frac{i}{3}\right) e^{-3it} \\ e^{3it} & e^{-3it} \end{bmatrix} \begin{bmatrix} -\frac{9\, e^{-3it}\left((1+i)\cos(t)+\left(-\frac{17}{3}+\frac{i}{3}\right)\sin(t)\right)}{4} \\ \frac{9\left((-1+i)\cos(t)+\left(\frac{17}{3}+\frac{i}{3}\right)\sin(t)\right)e^{3it}}{4} \end{bmatrix} \\
&= \begin{bmatrix} 9\sin(t) \\ \frac{51\sin(t)}{2} - \frac{9\cos(t)}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) c_1 e^{3it} \\ c_1 e^{3it} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{3}-\frac{i}{3}\right) c_2 e^{-3it} \\ c_2 e^{-3it} \end{bmatrix} + \begin{bmatrix} 9\sin(t) \\ \frac{51\sin(t)}{2} - \frac{9\cos(t)}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) c_1 e^{3it} + \left(\frac{1}{3}-\frac{i}{3}\right) c_2 e^{-3it} + 9\sin(t) \\ c_1 e^{3it} + c_2 e^{-3it} + \frac{51\sin(t)}{2} - \frac{9\cos(t)}{2} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

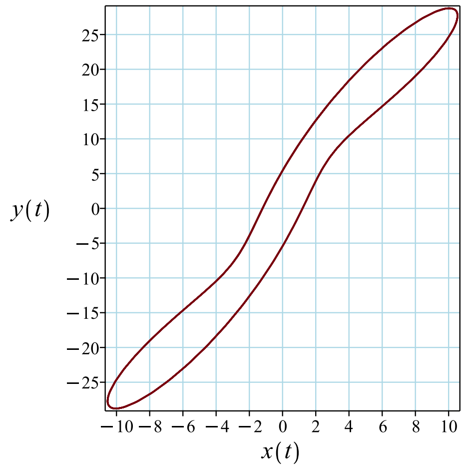
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{3}+\frac{i}{3}\right) c_1 + \left(\frac{1}{3}-\frac{i}{3}\right) c_2 \\ c_1 + c_2 - \frac{9}{2} \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = \frac{7}{4} + \frac{i}{4} \\ c_2 = \frac{7}{4} - \frac{i}{4} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{2i}{3}\right) e^{3it} + \left(\frac{1}{2}-\frac{2i}{3}\right) e^{-3it} + 9\sin(t) \\ \left(\frac{7}{4}+\frac{i}{4}\right) e^{3it} + \left(\frac{7}{4}-\frac{i}{4}\right) e^{-3it} + \frac{51\sin(t)}{2} - \frac{9\cos(t)}{2} \end{bmatrix}$$




The following are plots of each solution.

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 44

```
dsolve([diff(x(t),t) = 3*x(t)-2*y(t)+24*sin(t), diff(y(t),t) = 9*x(t)-3*y(t)+12*cos(t), x(0)
```

$$x(t) = -\frac{4\sin(3t)}{3} + \cos(3t) + 9\sin(t)$$

$$y(t) = \frac{7\cos(3t)}{2} - \frac{\sin(3t)}{2} - \frac{9\cos(t)}{2} + \frac{51\sin(t)}{2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 50

```
DSolve[{x'[t]==3*x[t]-2*y[t]+24*Sin[t],y'[t]==9*x[t]-3*y[t]+12*Cos[t]},{x[0]==1,y[0]==-1},{x
```

$$x(t) \to 9\sin(t) - \frac{4}{3}\sin(3t) + \cos(3t)$$

$$y(t) \to \frac{1}{2}(51\sin(t) - \sin(3t) - 9\cos(t) + 7\cos(3t))$$

## 7.11 problem Problem 5(c)



Internal problem ID [12386]
Internal file name [`OUTPUT/11038_Wednesday_October_04_2023_01_27_16_AM_14866388/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 5(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 7x(t) - 4y + 10\,\mathrm{e}^{t} \\
y' &= 3x(t) + 14y + 6\,\mathrm{e}^{2t}
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

### 7.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 3 & 14 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 10\,\mathrm{e}^{t} \\ 6\,\mathrm{e}^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 4\,\mathrm{e}^{10t} - 3\,\mathrm{e}^{11t} & -4\,\mathrm{e}^{11t} + 4\,\mathrm{e}^{10t} \\ 3\,\mathrm{e}^{11t} - 3\,\mathrm{e}^{10t} & -3\,\mathrm{e}^{10t} + 4\,\mathrm{e}^{11t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} 4\,\mathrm{e}^{10t} - 3\,\mathrm{e}^{11t} & -4\,\mathrm{e}^{11t} + 4\,\mathrm{e}^{10t} \\ 3\,\mathrm{e}^{11t} - 3\,\mathrm{e}^{10t} & -3\,\mathrm{e}^{10t} + 4\,\mathrm{e}^{11t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{e}^{11t} \\ -\mathrm{e}^{11t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \mathrm{e}^{-11t}(4\,\mathrm{e}^{t} - 3) & 4\,\mathrm{e}^{-11t}(\mathrm{e}^{t} - 1) \\ -3\,\mathrm{e}^{-11t}(\mathrm{e}^{t} - 1) & (-3\,\mathrm{e}^{t} + 4)\,\mathrm{e}^{-11t} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} 4\,\mathrm{e}^{10t} - 3\,\mathrm{e}^{11t} & -4\,\mathrm{e}^{11t} + 4\,\mathrm{e}^{10t} \\ 3\,\mathrm{e}^{11t} - 3\,\mathrm{e}^{10t} & -3\,\mathrm{e}^{10t} + 4\,\mathrm{e}^{11t} \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{-11t}(4\,\mathrm{e}^{t} - 3) & 4\,\mathrm{e}^{-11t}(\mathrm{e}^{t} - 1) \\ -3\,\mathrm{e}^{-11t}(\mathrm{e}^{t} - 1) & (-3\,\mathrm{e}^{t} + 4)\,\mathrm{e}^{-11t} \end{bmatrix} \begin{bmatrix} 10\,\mathrm{e}^{t} \\ 6\,\mathrm{e}^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} 4\,\mathrm{e}^{10t} - 3\,\mathrm{e}^{11t} & -4\,\mathrm{e}^{11t} + 4\,\mathrm{e}^{10t} \\ 3\,\mathrm{e}^{11t} - 3\,\mathrm{e}^{10t} & -3\,\mathrm{e}^{10t} + 4\,\mathrm{e}^{11t} \end{bmatrix} \begin{bmatrix} -3\,\mathrm{e}^{-8t} - \frac{16\,\mathrm{e}^{-9t}}{9} + 3\,\mathrm{e}^{-10t} \\ \frac{2\,\mathrm{e}^{-9t}}{3} + \frac{9\,\mathrm{e}^{-8t}}{4} - 3\,\mathrm{e}^{-10t} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{3} - \frac{13\,\mathrm{e}^{t}}{9} \\ -\frac{5\,\mathrm{e}^{2t}}{12} + \frac{\mathrm{e}^{t}}{3} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} e^{11t} - \frac{e^{2t}}{3} - \frac{13\,e^{t}}{9} \\ -e^{11t} - \frac{5\,e^{2t}}{12} + \frac{e^{t}}{3} \end{bmatrix}$$

### 7.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ 3 & 14 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 10\,e^{t} \\ 6\,e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 7 & -4 \\ 3 & 14 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 7-\lambda & -4 \\ 3 & 14-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 21\lambda + 110 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 10$$
$$\lambda_2 = 11$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 11 | 1 | real eigenvalue |
| 10 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 10$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & -4 \\ 3 & 14 \end{bmatrix} - (10)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -4 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{4t}{3}\}$

Hence the solution is

$$\begin{bmatrix} -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{4t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 11$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 7 & -4 \\ 3 & 14 \end{bmatrix} - (11) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 10 | 1 | 1 | No | $\begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}$ |
| 11 | 1 | 1 | No | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 10 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{10t} \\ &= \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix} e^{10t}\end{aligned}$$

Since eigenvalue 11 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{11t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{11t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{4\,\mathrm{e}^{10t}}{3} \\ \mathrm{e}^{10t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{11t} \\ \mathrm{e}^{11t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{4\,\mathrm{e}^{10t}}{3} & -\mathrm{e}^{11t} \\ \mathrm{e}^{10t} & \mathrm{e}^{11t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -3\,\mathrm{e}^{-10t} & -3\,\mathrm{e}^{-10t} \\ 3\,\mathrm{e}^{-11t} & 4\,\mathrm{e}^{-11t} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{4\,\mathrm{e}^{10t}}{3} & -\mathrm{e}^{11t} \\ \mathrm{e}^{10t} & \mathrm{e}^{11t} \end{bmatrix} \int \begin{bmatrix} -3\,\mathrm{e}^{-10t} & -3\,\mathrm{e}^{-10t} \\ 3\,\mathrm{e}^{-11t} & 4\,\mathrm{e}^{-11t} \end{bmatrix} \begin{bmatrix} 10\,\mathrm{e}^{t} \\ 6\,\mathrm{e}^{2t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{4\,\mathrm{e}^{10t}}{3} & -\mathrm{e}^{11t} \\ \mathrm{e}^{10t} & \mathrm{e}^{11t} \end{bmatrix} \int \begin{bmatrix} -30\,\mathrm{e}^{-9t} - 18\,\mathrm{e}^{-8t} \\ 30\,\mathrm{e}^{-10t} + 24\,\mathrm{e}^{-9t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{4\,\mathrm{e}^{10t}}{3} & -\mathrm{e}^{11t} \\ \mathrm{e}^{10t} & \mathrm{e}^{11t} \end{bmatrix} \begin{bmatrix} \frac{10\,\mathrm{e}^{-9t}}{3} + \frac{9\,\mathrm{e}^{-8t}}{4} \\ -3\,\mathrm{e}^{-10t} - \frac{8\,\mathrm{e}^{-9t}}{3} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{3} - \frac{13\,\mathrm{e}^{t}}{9} \\ -\frac{5\,\mathrm{e}^{2t}}{12} + \frac{\mathrm{e}^{t}}{3} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4c_1\mathrm{e}^{10t}}{3} \\ c_1\mathrm{e}^{10t} \end{bmatrix} + \begin{bmatrix} -c_2\mathrm{e}^{11t} \\ c_2\mathrm{e}^{11t} \end{bmatrix} + \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{3} - \frac{13\,\mathrm{e}^{t}}{9} \\ -\frac{5\,\mathrm{e}^{2t}}{12} + \frac{\mathrm{e}^{t}}{3} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{4c_1\mathrm{e}^{10t}}{3} - c_2\mathrm{e}^{11t} - \frac{\mathrm{e}^{2t}}{3} - \frac{13\,\mathrm{e}^{t}}{9} \\ c_1\mathrm{e}^{10t} + c_2\mathrm{e}^{11t} - \frac{5\,\mathrm{e}^{2t}}{12} + \frac{\mathrm{e}^{t}}{3} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = -1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

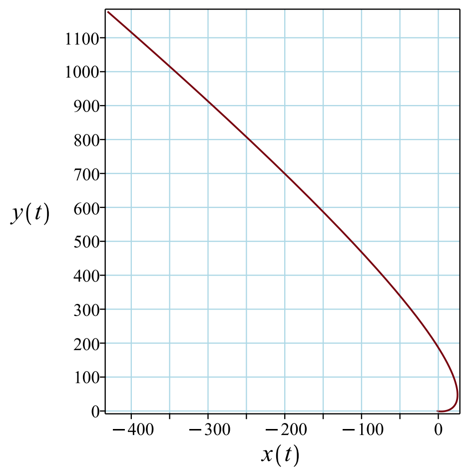
$$
\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{4c_1}{3} - c_2 - \frac{16}{9} \\ c_1 + c_2 - \frac{1}{12} \end{bmatrix}
$$

Solving for the constants of integrations gives

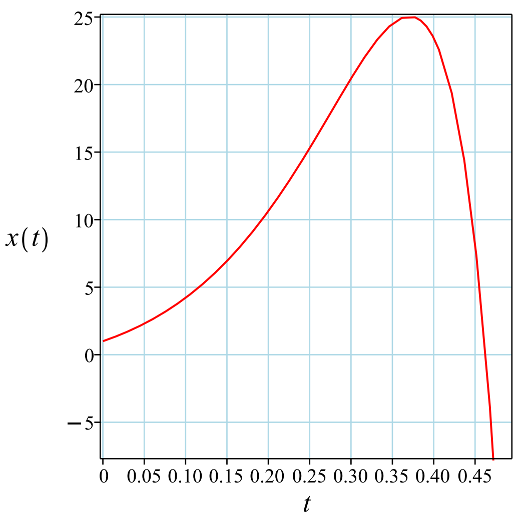
$$
\begin{bmatrix} c_1 = -\frac{67}{12} \\ c_2 = \frac{14}{3} \end{bmatrix}
$$

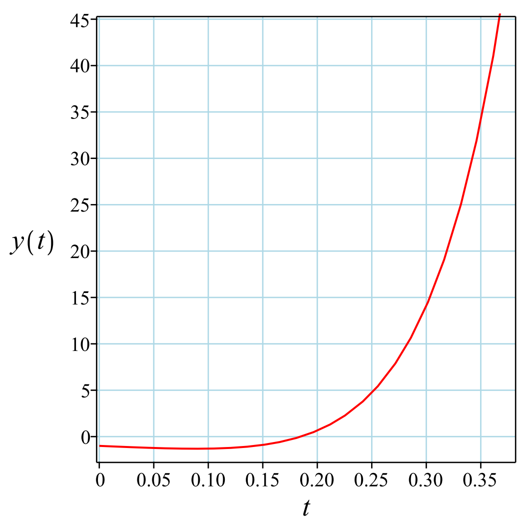
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{67\,e^{10t}}{9} - \frac{14\,e^{11t}}{3} - \frac{e^{2t}}{3} - \frac{13\,e^{t}}{9} \\ -\frac{67\,e^{10t}}{12} + \frac{14\,e^{11t}}{3} - \frac{5\,e^{2t}}{12} + \frac{e^{t}}{3} \end{bmatrix}$$




The following are plots of each solution.



✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 54

```
dsolve([diff(x(t),t) = 7*x(t)-4*y(t)+10*exp(t), diff(y(t),t) = 3*x(t)+14*y(t)+6*exp(2*t), x(
```

$$x(t) = -\frac{14\,\mathrm{e}^{11t}}{3} + \frac{67\,\mathrm{e}^{10t}}{9} - \frac{\mathrm{e}^{2t}}{3} - \frac{13\,\mathrm{e}^{t}}{9}$$
$$y(t) = \frac{14\,\mathrm{e}^{11t}}{3} - \frac{67\,\mathrm{e}^{10t}}{12} - \frac{5\,\mathrm{e}^{2t}}{12} + \frac{\mathrm{e}^{t}}{3}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 54

```
DSolve[{x'[t]==7*x[t]-4*y[t]+10*Exp[t],y'[t]==3*x[t]+14*y[t]+6*Exp[2*t]},{x[0]==1,y[0]==-1},
```

$$x(t) \to -\frac{1}{9}e^t\left(-40e^{9t} + 18e^{10t} + 13\right)$$
$$y(t) \to \frac{1}{3}e^t\left(-10e^{9t} + 6e^{10t} + 1\right)$$

## 7.12 problem Problem 5(d)



Internal problem ID [12387]
Internal file name [`OUTPUT/11039_Wednesday_October_04_2023_01_27_16_AM_39771329/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 5(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -7x(t) + 4y + 6\,\mathrm{e}^{3t} \\ y' &= -5x(t) + 2y + 6\,\mathrm{e}^{2t} \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = -1]$$

### 7.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 6\,\mathrm{e}^{3t} \\ 6\,\mathrm{e}^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} 5\,e^{-3t} - 4\,e^{-2t} & 4\,e^{-2t} - 4\,e^{-3t} \\ -5\,e^{-2t} + 5\,e^{-3t} & -4\,e^{-3t} + 5\,e^{-2t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \begin{bmatrix} 5\,e^{-3t} - 4\,e^{-2t} & 4\,e^{-2t} - 4\,e^{-3t} \\ -5\,e^{-2t} + 5\,e^{-3t} & -4\,e^{-3t} + 5\,e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 9\,e^{-3t} - 8\,e^{-2t} \\ -10\,e^{-2t} + 9\,e^{-3t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} (-4 + 5\,e^{t})\,e^{2t} & -4(e^{t} - 1)\,e^{2t} \\ 5(e^{t} - 1)\,e^{2t} & (-4\,e^{t} + 5)\,e^{2t} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} 5\,e^{-3t} - 4\,e^{-2t} & 4\,e^{-2t} - 4\,e^{-3t} \\ -5\,e^{-2t} + 5\,e^{-3t} & -4\,e^{-3t} + 5\,e^{-2t} \end{bmatrix} \int \begin{bmatrix} (-4 + 5\,e^{t})\,e^{2t} & -4(e^{t} - 1)\,e^{2t} \\ 5(e^{t} - 1)\,e^{2t} & (-4\,e^{t} + 5)\,e^{2t} \end{bmatrix} \begin{bmatrix} 6\,e^{3t} \\ 6\,e^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} 5\,e^{-3t} - 4\,e^{-2t} & 4\,e^{-2t} - 4\,e^{-3t} \\ -5\,e^{-2t} + 5\,e^{-3t} & -4\,e^{-3t} + 5\,e^{-2t} \end{bmatrix} \begin{bmatrix} -\frac{48\,e^{5t}}{5} + 5\,e^{6t} + 6\,e^{4t} \\ \frac{15\,e^{4t}}{2} - \frac{54\,e^{5t}}{5} + 5\,e^{6t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{6\,e^{2t}}{5} + \frac{e^{3t}}{5} \\ \frac{27\,e^{2t}}{10} - e^{3t} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(e^{6t}+6\,e^{5t}-40\,e^{t}+45)\,e^{-3t}}{5} \\ -\frac{(10\,e^{6t}-27\,e^{5t}+100\,e^{t}-90)\,e^{-3t}}{10} \end{bmatrix}$$

### 7.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 6\,e^{3t} \\ 6\,e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -7 & 4 \\ -5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -7-\lambda & 4 \\ -5 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = -3$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-2$ | 1 | real eigenvalue |
| $-3$ | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -7 & 4 \\ -5 & 2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -4 & 4 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ -5 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -7 & 4 \\ -5 & 2 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -5 & 4 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -5 & 4 & 0 \\ -5 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \implies \left[\begin{array}{cc|c} -5 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{4t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{4t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{4t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{4t}{5} \\ t \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{4t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-2$ | 1 | 1 | No | $\begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}$ |
| $-3$ | 1 | 1 | No | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-2$ is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-2t}$$
$$= \begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix} e^{-2t}$$

Since eigenvalue $-3$ is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-3t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{4\,e^{-2t}}{5} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{4\,e^{-2t}}{5} & e^{-3t} \\ e^{-2t} & e^{-3t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -5\,e^{2t} & 5\,e^{2t} \\ 5\,e^{3t} & -4\,e^{3t} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{4\,\mathrm{e}^{-2t}}{5} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-3t} \end{bmatrix} \int \begin{bmatrix} -5\,\mathrm{e}^{2t} & 5\,\mathrm{e}^{2t} \\ 5\,\mathrm{e}^{3t} & -4\,\mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} 6\,\mathrm{e}^{3t} \\ 6\,\mathrm{e}^{2t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{4\,\mathrm{e}^{-2t}}{5} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-3t} \end{bmatrix} \int \begin{bmatrix} -30\,\mathrm{e}^{5t} + 30\,\mathrm{e}^{4t} \\ 30\,\mathrm{e}^{6t} - 24\,\mathrm{e}^{5t} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{4\,\mathrm{e}^{-2t}}{5} & \mathrm{e}^{-3t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{-3t} \end{bmatrix} \begin{bmatrix} \frac{15\,\mathrm{e}^{4t}}{2} - 6\,\mathrm{e}^{5t} \\ -\frac{24\,\mathrm{e}^{5t}}{5} + 5\,\mathrm{e}^{6t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{6\,\mathrm{e}^{2t}}{5} + \frac{\mathrm{e}^{3t}}{5} \\ \frac{27\,\mathrm{e}^{2t}}{10} - \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y \end{bmatrix} &= \begin{bmatrix} \frac{4c_1\mathrm{e}^{-2t}}{5} \\ c_1\mathrm{e}^{-2t} \end{bmatrix} + \begin{bmatrix} c_2\mathrm{e}^{-3t} \\ c_2\mathrm{e}^{-3t} \end{bmatrix} + \begin{bmatrix} \frac{6\,\mathrm{e}^{2t}}{5} + \frac{\mathrm{e}^{3t}}{5} \\ \frac{27\,\mathrm{e}^{2t}}{10} - \mathrm{e}^{3t} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(\mathrm{e}^{6t}+6\,\mathrm{e}^{5t}+4c_1\mathrm{e}^{t}+5c_2)\mathrm{e}^{-3t}}{5} \\ \frac{(-10\,\mathrm{e}^{6t}+27\,\mathrm{e}^{5t}+10c_1\mathrm{e}^{t}+10c_2)\mathrm{e}^{-3t}}{10} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 1 \\ y(0) = -1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

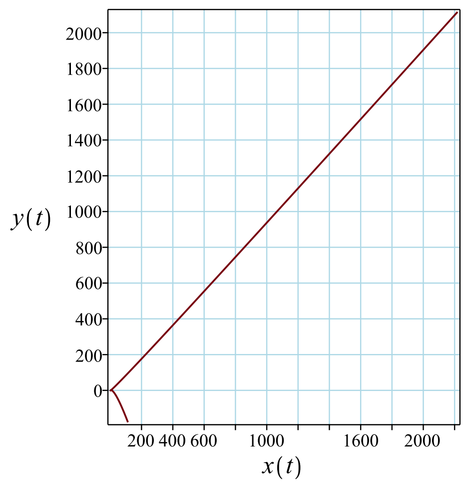
$$
\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} + \frac{4c_1}{5} + c_2 \\ \frac{17}{10} + c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

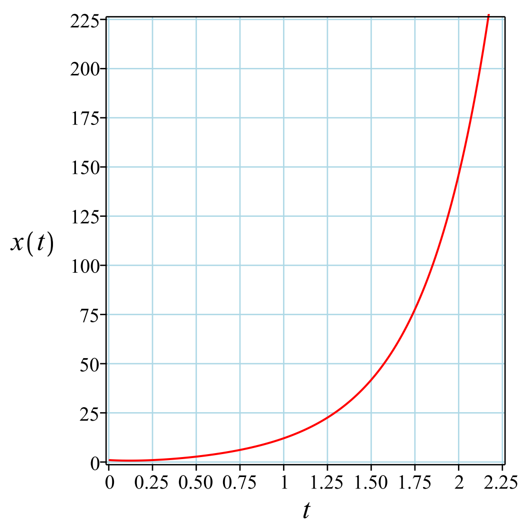
$$
\begin{bmatrix} c_1 = -\frac{23}{2} \\ c_2 = \frac{44}{5} \end{bmatrix}
$$

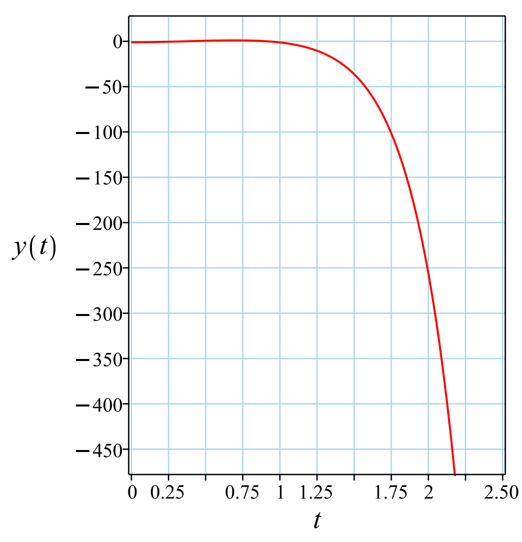
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(e^{6t}+6\,e^{5t}-46\,e^{t}+44)e^{-3t}}{5} \\ \frac{(-10\,e^{6t}+27\,e^{5t}-115\,e^{t}+88)e^{-3t}}{10} \end{bmatrix}$$




The following are plots of each solution.



✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 58

```
dsolve([diff(x(t),t) = -7*x(t)+4*y(t)+6*exp(3*t), diff(y(t),t) = -5*x(t)+2*y(t)+6*exp(2*t),
```

$$x(t) = -\frac{46\,\mathrm{e}^{-2t}}{5} + \frac{44\,\mathrm{e}^{-3t}}{5} + \frac{\mathrm{e}^{3t}}{5} + \frac{6\,\mathrm{e}^{2t}}{5}$$
$$y(t) = -\mathrm{e}^{3t} - \frac{23\,\mathrm{e}^{-2t}}{2} + \frac{44\,\mathrm{e}^{-3t}}{5} + \frac{27\,\mathrm{e}^{2t}}{10}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 48

```
DSolve[{x'[t]==-7*x[t]+4*y[t]+6*Exp[3*t],y'[t]==-5*x[t]+2*y[t]+6*Exp[2*t]},{x[0]==1,y[0]==-1
```

$$x(t) \to \frac{1}{5}e^{-3t}\left(-16e^{t} + e^{6t} + 20\right)$$
$$y(t) \to -e^{-3t}\left(4e^{t} + e^{6t} - 4\right)$$

## 7.13 problem Problem 6(a)



Internal problem ID [12388]
Internal file name [`OUTPUT/11040_Wednesday_October_04_2023_01_27_17_AM_85648862/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 6(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -3x(t) - 3y + z(t) \\
y' &= 2y + 2z(t) + 29\,\mathrm{e}^{-t} \\
z'(t) &= 5x(t) + y + z(t) + 39\,\mathrm{e}^{t}
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 2, z(0) = 3]
$$

### 7.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 29\,\mathrm{e}^{-t} \\ 39\,\mathrm{e}^{t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \text{Expression too large to display} \\ &= \text{Expression too large to display} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{x}_0 \\ &= \text{Expression too large to display} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \text{Expression too large to display} \\ &= \text{Expression too large to display} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \text{Expression too large to display} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \text{Expression too large to display} \int \text{Expression too large to display} \begin{bmatrix} 0 \\ 29\,\mathrm{e}^{-t} \\ 39\,\mathrm{e}^{t} \end{bmatrix} dt$$

$$= \text{Expression too large to display}\text{Expression too large to display}$$

$$= \begin{bmatrix} \frac{7\left(261\,\mathrm{e}^{\frac{t\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}-689\,\mathrm{e}^{\frac{t\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}\right)\mathrm{e}^{-\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}}{477} \\ -\frac{\left(2349\,\mathrm{e}^{\frac{t\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}-5512\,\mathrm{e}^{\frac{t\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}\right)\mathrm{e}^{-\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}}{477} \\ -\frac{13\left(261\,\mathrm{e}^{\frac{t\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}+212\,\mathrm{e}^{\frac{t\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}\right)\mathrm{e}^{-\frac{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+42\right)t}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}}{477} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \text{Expression too large to display} \end{aligned}$$

### 7.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 29\,\mathrm{e}^{-t} \\ 39\,\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & -3 & 1 \\ 0 & 2-\lambda & 2 \\ 5 & 1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 14\lambda + 40 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{\left(540 + 6\sqrt{6042}\right)^{\frac{1}{3}}}{3} - \frac{14}{\left(540 + 6\sqrt{6042}\right)^{\frac{1}{3}}}$$

$$\lambda_2 = \frac{\left(540 + 6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540 + 6\sqrt{6042}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = \frac{\left(540 + 6\sqrt{6042}\right)^{\frac{1}{3}}}{6} + \frac{7}{\left(540 + 6\sqrt{6042}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3} + \frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigen |
|---|---|---|
| $-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$ | 1 | real eigenval |
| $\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$ | 1 | complex eige |
| $\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$ | 1 | complex eige |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc}-3 & -3 & 1\\ 0 & 2 & 2\\ 5 & 1 & 1\end{array}\right]-\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\left[\right.$$

$$\left[\begin{array}{ccc}\frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -3 & 1\\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & 2\\ 5 & 1 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\end{array}\right]\left[\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\begin{bmatrix} -3+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -3 & 1 \\ 0 & 2+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & 2 \\ 5 & 1 & 1+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{604\cdots}\right.} \end{bmatrix}$$

$$R_3 = R_3 - \frac{5R_1}{-3+\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}} \Longrightarrow \begin{bmatrix} \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -\cdots \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(\cdots\right.}{3\left(540+6\cdots\right.} \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+36\left(\cdots\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(\cdots\right.} \end{bmatrix}$$

$$R_3 = R_3 - \frac{3\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+36\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)\left(540+6\sqrt{6042}\right)^{\frac{1}{3}} R_2}{\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42\right)\left(\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+\cdots\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-9\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & -3 & 1 \\ 0 & \frac{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42}{3\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}} & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\sqrt{6042}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+24}, v_2 = - \cdots \right.$

Hence the solution is

$$\left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\sqrt{6042}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\cdots\right.}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)\cdots} \\ -\frac{6t\left(540+6\sqrt{\cdots}\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(5\cdots\right.} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\sqrt{6042}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right] = t\left[\begin{array}{c} -\frac{3\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{604\cdots}\right.}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{604\cdots}\right.} \\ -\frac{6\left(540+6\sqrt{\cdots}\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(\cdots\right.} \\ 1 \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\sqrt{6042}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{3\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}\cdots\right.}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)\cdots} \\ -\frac{6\left(540+6\sqrt{\cdots}\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(5\cdots\right.} \\ 1 \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} -\frac{3t\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+7\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+90\right)}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-3\sqrt{6042}+69\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+24} \\ -\frac{6t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{3\left(4\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+\sqrt{6042}+\cdots\right.}{\sqrt{6042}\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+5\left(540+6\sqrt{6042}\right)\cdots} \\ -\frac{6\left(540+6\sqrt{6\cdots}\right.}{\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}+6\left(5\cdots\right.} \\ 1 \end{array}\right.$$

Considering the eigenvalue $\lambda_2 = \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{array}\right] - \left(\frac{(540\cdots}{\cdots}\right.\right.$$

$$\left[\begin{array}{cc} -\frac{42+18\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}-i\left(\left(540+6\sqrt{3}\,\sqrt{20\cdots}\right.\right.}{6\left(540+6\sqrt{3}\,\sqrt{20\cdots}\right.} \\ 5 & 1 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\left[\begin{array}{ll} -3-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2} & \\ 0 & 2-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042\right.}} \\ 5 & \end{array}\right.$$

$$R_3 = R_3 - \frac{5R_1}{-3-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[-\frac{42+18\left(540+6\sqrt{3}\right.}{}\right.$$

$$R_3 = R_3 - \frac{6\left(i\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42i\right.}{\left(i\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42i\sqrt{3}+18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{array}{ll} -\frac{42+18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}-i\left(\left(540+6\sqrt{3}\sqrt{20\right.\right.}{6\left(540+6\sqrt{3}\sqrt{20\right.}} \\ 0 & 0 \end{array}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{3t\left(3i\sqrt{2014}-7i\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+\sqrt{3}\sqrt{2014}+90i\sqrt{3}-8\right.}{3i\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69i\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+9} \right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}-8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\sqrt{201\right.\right.}}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}+3\sqrt{3}\sqrt{2014}+10\left(540+6\sqrt{3}\sqrt{2014}\right.} \\ \frac{12t\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}-12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}-8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\sqrt{201\right.\right.}}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}+3\sqrt{3}\sqrt{2014}+10\left(540+6\sqrt{3}\sqrt{2014}\right.} \\ \frac{12t\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}-12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}-8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\sqrt{201\right.\right.}}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}+3\sqrt{3}\sqrt{2014}+10\left(540+6\sqrt{3}\sqrt{2014}\right.} \\ \frac{12t\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}-12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Which is normalized to

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}-8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}+7\left(540+6\sqrt{3}\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}+3\sqrt{3}\sqrt{2014}+10\left(540+6\sqrt{3}\sqrt{2014}\right.} \\ \frac{12t\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}\sqrt{3}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}-12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+42} \\ t \end{array}\right.$$

Considering the eigenvalue $\lambda_3 = \frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\left[\begin{array}{ccc} -3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{array}\right] - \left(\frac{(540}{}\right.\right.$$

$$\left[\begin{array}{cc} \frac{-42-18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{201\right.\right.}{6\left(540+6\sqrt{3}\sqrt{201\right.} \\ 5 & 1 \end{array}\right.$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented

matrix is

$$\left[\begin{matrix} -3-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2} & \\ 0 & 2-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042\right.}} \\ 5 & \end{matrix}\right.$$

$$R_3 = R_3 - \frac{5R_1}{-3-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}-\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}} \Longrightarrow \left[\begin{matrix} \frac{-42-18\left(540+6\sqrt{3}\right.}{} \\ \\ \\ \end{matrix}\right.$$

$$R_3 = R_3 - \frac{6\left(i\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42i\sqrt{3}+7\right.}{\left(i\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42i\sqrt{3}-18\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42\right)\left(-42+1\right.}$$

Therefore the system in Echelon form is

$$\left[\begin{matrix} \frac{-42-18\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-42\right)\sqrt{3}-\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}}{6\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}} & -3 \\ 0 & \frac{-42+12\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+i\left(\left(540+6\sqrt{3}\sqrt{201\right.\right.}{6\left(540+6\sqrt{3}\sqrt{201\right.}} \\ 0 & 0 \end{matrix}\right.$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{ v_1 = \frac{3t\left(3i\sqrt{2014}-7i\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}-\sqrt{3}\sqrt{2014}+90i\sqrt{3}+8\right.}{3i\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69i\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}+9} \right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}-\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}+8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}-3\sqrt{3}\sqrt{2014}-10\left(540+6\sqrt{3}\sqrt{2014}\right)} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right]$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}-\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}+8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}-3\sqrt{3}\sqrt{2014}-10\left(540+6\sqrt{3}\sqrt{2014}\right)} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right]$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{3t\left(3\,\mathrm{I}\sqrt{2014}-7\,\mathrm{I}\sqrt{3}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}-\sqrt{3}\sqrt{2014}+90\,\mathrm{I}\sqrt{3}+8\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\sqrt{2014}\right.\right.}{3\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\sqrt{2014}+69\,\mathrm{I}\left(540+6\sqrt{3}\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9\,\mathrm{I}\sqrt{2014}-3\sqrt{3}\sqrt{2014}-10\left(540+6\sqrt{3}\sqrt{2014}\right)} \\ -\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42\,\mathrm{I}\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\ t \end{array}\right]$$

Which is normalized to

$$
\left[\begin{array}{c}
\frac{3t\left(3 \,\mathrm{I}\,\sqrt{2014}-7 \,\mathrm{I}\,\sqrt{3}\,\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}-\sqrt{3}\,\sqrt{2014}+90 \,\mathrm{I}\,\sqrt{3}+8\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{2}{3}}-7\left(540+6\sqrt{3}\,\sqrt{2014}\right.\right.}{3 \,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{2014}+\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}\,\sqrt{2014}+69 \,\mathrm{I}\left(540+6\sqrt{3}\,\sqrt{2014}\right)^{\frac{1}{3}}\sqrt{3}+9 \,\mathrm{I}\,\sqrt{2014}-3\sqrt{3}\,\sqrt{2014}-10\left(540+6\sqrt{3}\,\sqrt{2014}\right)} \\
-\frac{12t\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{\mathrm{I}\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\sqrt{3}-\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}-42 \,\mathrm{I}\,\sqrt{3}+12\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}-42} \\
t
\end{array}\right]
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| eigenvalue | multiplicity | | defectiv |
|---|---|---|---|
| | algebraic $m$ | geometric $k$ | |
| $-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$ | 1 | 1 | No |
| $\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$ | 1 | 1 | No |
| $\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\frac{7}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-\frac{i\sqrt{3}\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}+\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{2}$ | 1 | 1 | No |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \\
&= \left[\begin{array}{c} \frac{-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+3\right)} \\ \frac{2}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ 1 \end{array}\right] e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right.}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \left(-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+3\right)} \\ \frac{2\,e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \end{bmatrix} + c_2 \left[ \frac{e^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\cdots\right.}}{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\cdots\right.} \cdots \right.
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \text{Expression too large to display}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$
\Phi^{-1} = \text{Expression too large to display}
$$

Hence

$$\vec{x}_p(t) = \text{Expression too large to display} \int \text{Expression too large to display} \begin{bmatrix} 0 \\ 29\,\mathrm{e}^{-t} \\ 39\,\mathrm{e}^{t} \end{bmatrix} dt$$

$$= \text{Expression too large to display} \int \text{Expression too large to display}\, dt$$

$$= \text{Expression too large to display}\text{Expression too large to display}$$

$$= \left[\begin{array}{c} \dfrac{7\left(540+6\sqrt{6042}\right)^{\frac{2}{3}} \left(689\,\mathrm{e}^{t}-261\,\mathrm{e}^{-t}\right) \left(21\left(20199392243-20199392243 i\sqrt{3}-779599731 i\sqrt{2014}+259866577\sqrt{6042}\right)\left(540+6\cdots\right.\right.}{1431\left(-142298484724368+444\left(-51544877093-663124410\sqrt{6042}\right)\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+\left(-1830670135266+59502370013\left(540+6\sqrt{\cdots}\right.\right.\right.} \\ \dfrac{43165007182128\left(-\frac{2349\,\mathrm{e}^{-t}}{5512}+\mathrm{e}^{t}\right)\left(540+6\sqrt{6042}\right)^{\frac{2}{3}}\left(\left(i\sqrt{3}+\frac{50374407 i\sqrt{2014}}{1305182849}+\frac{16791469\sqrt{6042}}{1305182849}\cdots\right.\right.}{\left(663490237622796 i\sqrt{3}+25607422183572 i\sqrt{2014}-8535807394524\sqrt{6042}-663490237622796\right)\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}+\cdots} \\ \end{array}\right.$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}\left(-8-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)}{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2\right)\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}+3\right)} \\ \frac{2c_1 e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t}}{-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}-2} \\ c_1 e^{\left(-\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{3}-\frac{14}{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}\right)t} \end{bmatrix} + \begin{bmatrix} \frac{c_2 e^{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\cdots\right.}}{\left(\frac{\left(540+6\sqrt{6042}\right)^{\frac{1}{3}}}{6}+\cdots\right.} \\ \cdots \end{bmatrix}$$

Which becomes

Expression too large to display

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \\ z(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

Expression too large to display

Solving for the constants of integrations gives

Expression too large to display

Substituting these constants back in original solution in Eq. (1) gives

Expression too large to display

The following are plots of each solution against another.

The following are plots of each solution.

✓ Solution by Maple

Time used: 5.609 (sec). Leaf size: 949416

```
dsolve([diff(x(t),t) = -3*x(t)-3*y(t)+z(t), diff(y(t),t) = 2*y(t)+2*z(t)+29*exp(-t), diff(z(
```

Expression too large to display
Expression too large to display
Expression too large to display

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 3462

```
DSolve[{x'[t]==-3*x[t]-3*y[t]+z[t],y'[t]==2*y[t]+2*z[t]+29*Exp[-t],z'[t]==5*x[t]+y[t]+z[t]+3
```

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## 7.14 problem Problem 6(b)



Internal problem ID [12389]
Internal file name [`OUTPUT/11041_Wednesday_October_04_2023_01_27_28_AM_42015280/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
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**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) + y - z(t) + 5\sin(t) \\
y' &= y + z(t) - 10\cos(t) \\
z'(t) &= x(t) + z(t) + 2
\end{aligned}
$$

With initial conditions

$$[x(0) = 1, y(0) = 2, z(0) = 3]$$

### 7.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 5\sin(t) \\ -10\cos(t) \\ 2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^t \cos(t)}{2} + \frac{e^t \sin(t)}{2} & -\frac{e^t \cos(t)}{2} + \frac{e^t \sin(t)}{2} + \frac{e^{2t}}{2} & -e^t \sin(t) \\ -\frac{e^t \cos(t)}{2} - \frac{e^t \sin(t)}{2} + \frac{e^{2t}}{2} & \frac{e^{2t}}{2} + \frac{e^t \cos(t)}{2} - \frac{e^t \sin(t)}{2} & e^t \sin(t) \\ -\frac{e^t \cos(t)}{2} + \frac{e^t \sin(t)}{2} + \frac{e^{2t}}{2} & -\frac{e^t \cos(t)}{2} - \frac{e^t \sin(t)}{2} + \frac{e^{2t}}{2} & e^t \cos(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^t(\cos(t)+\sin(t))}{2} & \frac{e^{2t}}{2} + \frac{(-\cos(t)+\sin(t))e^t}{2} & -e^t \sin(t) \\ \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(\cos(t)-\sin(t))e^t}{2} & e^t \sin(t) \\ \frac{e^{2t}}{2} + \frac{(-\cos(t)+\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & e^t \cos(t) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{e^{2t}}{2} + \frac{e^t(\cos(t)+\sin(t))}{2} & \frac{e^{2t}}{2} + \frac{(-\cos(t)+\sin(t))e^t}{2} & -e^t \sin(t) \\ \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(\cos(t)-\sin(t))e^t}{2} & e^t \sin(t) \\ \frac{e^{2t}}{2} + \frac{(-\cos(t)+\sin(t))e^t}{2} & \frac{e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} & e^t \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3\,e^{2t}}{2} + \frac{e^t(\cos(t)+\sin(t))}{2} + (-\cos(t) + \sin(t))\,e^t - 3\,e^t \sin(t) \\ \frac{3\,e^{2t}}{2} + \frac{(-\cos(t)-\sin(t))e^t}{2} + (\cos(t) - \sin(t))\,e^t + 3\,e^t \sin(t) \\ \frac{3\,e^{2t}}{2} + \frac{(-\cos(t)+\sin(t))e^t}{2} + (-\cos(t) - \sin(t))\,e^t + 3\,e^t \cos(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{3\,e^{2t}}{2} + \frac{(-\cos(t)-3\sin(t))e^t}{2} \\ \frac{3\,e^{2t}}{2} + \frac{(\cos(t)+3\sin(t))e^t}{2} \\ \frac{3\,e^{2t}}{2} + \frac{(3\cos(t)-\sin(t))e^t}{2} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{e^{-2t}(1+(\cos(t)-\sin(t))e^{t})}{2} & -\frac{(-1+e^{t}(\cos(t)+\sin(t)))e^{-2t}}{2} & e^{-t}\sin(t) \\ -\frac{(-1+(\cos(t)-\sin(t))e^{t})e^{-2t}}{2} & \frac{(1+e^{t}(\cos(t)+\sin(t)))e^{-2t}}{2} & -e^{-t}\sin(t) \\ -\frac{(-1+e^{t}(\cos(t)+\sin(t)))e^{-2t}}{2} & -\frac{(-1+(\cos(t)-\sin(t))e^{t})e^{-2t}}{2} & e^{-t}\cos(t) \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{2t}}{2}+\frac{e^{t}(\cos(t)+\sin(t))}{2} & \frac{e^{2t}}{2}+\frac{(-\cos(t)+\sin(t))e^{t}}{2} & -e^{t}\sin(t) \\ \frac{e^{2t}}{2}+\frac{(-\cos(t)-\sin(t))e^{t}}{2} & \frac{e^{2t}}{2}+\frac{(\cos(t)-\sin(t))e^{t}}{2} & e^{t}\sin(t) \\ \frac{e^{2t}}{2}+\frac{(-\cos(t)+\sin(t))e^{t}}{2} & \frac{e^{2t}}{2}+\frac{(-\cos(t)-\sin(t))e^{t}}{2} & e^{t}\cos(t) \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2t}(1+(\cos(t)-\sin(t))e^{t})}{2} & -\frac{(-1+e^{t} \cdots}{\cdots} \\ -\frac{(-1+(\cos(t)-\sin(t))e^{t})e^{-2t}}{2} & \frac{(1+e^{t}(\cos \cdots}{\cdots} \\ -\frac{(-1+e^{t}(\cos(t)+\sin(t)))e^{-2t}}{2} & -\frac{(-1+(\cos \cdots}{\cdots} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}}{2}+\frac{e^{t}(\cos(t)+\sin(t))}{2} & \frac{e^{2t}}{2}+\frac{(-\cos(t)+\sin(t))e^{t}}{2} & -e^{t}\sin(t) \\ \frac{e^{2t}}{2}+\frac{(-\cos(t)-\sin(t))e^{t}}{2} & \frac{e^{2t}}{2}+\frac{(\cos(t)-\sin(t))e^{t}}{2} & e^{t}\sin(t) \\ \frac{e^{2t}}{2}+\frac{(-\cos(t)+\sin(t))e^{t}}{2} & \frac{e^{2t}}{2}+\frac{(-\cos(t)-\sin(t))e^{t}}{2} & e^{t}\cos(t) \end{bmatrix} \begin{bmatrix} -\frac{9\left(\left(\cos(t)^{2}+\left(-\frac{\sin(t)}{3}+\frac{2}{9}\right)\cos(t)+\frac{2\sin(t)}{9}-\frac{2}{9}\right) \cdots}{2} \\ \frac{\left(\left(9\cos(t)^{2}+(-3\sin(t)+2)\cos(t)+2\sin(t)-2\right)e^{t} \cdots}{2} \\ \frac{3\left(\left(\cos(t)^{2}+(3\sin(t)-\frac{2}{3})\cos(t)+\frac{2\sin(t)}{3}-\frac{4}{3}\right)e^{t} \cdots}{2} \end{bmatrix} \\
&= \begin{bmatrix} -1-2\cos(t) \\ 1+5\cos(t)-4\sin(t) \\ -1+\cos(t)-\sin(t) \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
&= \begin{bmatrix} -\frac{e^{t}\cos(t)}{2}-\frac{3e^{t}\sin(t)}{2}+\frac{3e^{2t}}{2}-2\cos(t)-1 \\ \frac{3e^{2t}}{2}+\frac{(e^{t}+10)\cos(t)}{2}+\frac{3e^{t}\sin(t)}{2}-4\sin(t)+1 \\ \frac{3e^{t}\cos(t)}{2}-\frac{e^{t}\sin(t)}{2}+\frac{3e^{2t}}{2}+\cos(t)-\sin(t)-1 \end{bmatrix}
\end{aligned}
$$

### 7.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)
$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 5\sin(t) \\ -10\cos(t) \\ 2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 + 6\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = 1 - i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $2$ | 1 | real eigenvalue |
| $1+i$ | 1 | complex eigenvalue |
| $1-i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - (1-i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 1 & -1 \\ 0 & i & 1 \\ 1 & 0 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1+i & 1 & -1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 1 & -1 & 0 \\ 0 & i & 1 & 0 \\ 0 & -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 1 & -1 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 & -1 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_2 = it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} - (1+i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 & -1 \\ 0 & -i & 1 \\ 1 & 0 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-i & 1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & -i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & -\frac{1}{2} - \frac{i}{2} & \frac{1}{2} - \frac{i}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 1 & -1 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 & -1 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_2 = -it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ \text{-}\mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| eigenvalue | multiplicity | | defective? | eigenvectors |
|---|---|---|---|---|
| | algebraic $m$ | geometric $k$ | | |
| 2 | 1 | 1 | No | $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ |
| $1+i$ | 1 | 1 | No | $\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$ |
| $1-i$ | 1 | 1 | No | $\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{(1+i)t} \\ -i\mathrm{e}^{(1+i)t} \\ \mathrm{e}^{(1+i)t} \end{bmatrix} + c_3 \begin{bmatrix} -i\mathrm{e}^{(1-i)t} \\ i\mathrm{e}^{(1-i)t} \\ \mathrm{e}^{(1-i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{2t} & i e^{(1+i)t} & -i e^{(1-i)t} \\ e^{2t} & -i e^{(1+i)t} & i e^{(1-i)t} \\ e^{2t} & e^{(1+i)t} & e^{(1-i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} & 0 \\ \left(-\frac{1}{4} - \frac{i}{4}\right) e^{(-1-i)t} & \left(-\frac{1}{4} + \frac{i}{4}\right) e^{(-1-i)t} & \frac{e^{(-1-i)t}}{2} \\ \left(-\frac{1}{4} + \frac{i}{4}\right) e^{(-1+i)t} & \left(-\frac{1}{4} - \frac{i}{4}\right) e^{(-1+i)t} & \frac{e^{(-1+i)t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t} & i e^{(1+i)t} & -i e^{(1-i)t} \\ e^{2t} & -i e^{(1+i)t} & i e^{(1-i)t} \\ e^{2t} & e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} & 0 \\ \left(-\frac{1}{4} - \frac{i}{4}\right) e^{(-1-i)t} & \left(-\frac{1}{4} + \frac{i}{4}\right) e^{(-1-i)t} & \frac{e^{(-1-i)t}}{2} \\ \left(-\frac{1}{4} + \frac{i}{4}\right) e^{(-1+i)t} & \left(-\frac{1}{4} - \frac{i}{4}\right) e^{(-1+i)t} & \frac{e^{(-1+i)t}}{2} \end{bmatrix} \begin{bmatrix} 5\sin(t) \\ -10\cos(t) \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & i e^{(1+i)t} & -i e^{(1-i)t} \\ e^{2t} & -i e^{(1+i)t} & i e^{(1-i)t} \\ e^{2t} & e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \int \begin{bmatrix} \frac{5\, e^{-2t}(\sin(t) - 2\cos(t))}{2} \\ -\frac{5\left((-1+i)\cos(t) - \frac{2}{5} + \left(\frac{1}{2} + \frac{i}{2}\right)\sin(t)\right) e^{(-1-i)t}}{2} \\ \frac{((10+10i)\cos(t) + 4 + (-5+5i)\sin(t)) e^{(-1+i)t}}{4} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & i e^{(1+i)t} & -i e^{(1-i)t} \\ e^{2t} & -i e^{(1+i)t} & i e^{(1-i)t} \\ e^{2t} & e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \begin{bmatrix} \frac{e^{-2t}(3\cos(t) - 4\sin(t))}{2} \\ \frac{((-1+7i)\cos(t) - 2 + 2i + (2-4i)\sin(t)) e^{(-1-i)t}}{4} \\ \frac{((-1-7i)\cos(t) - 2 - 2i + (2+4i)\sin(t)) e^{(-1+i)t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} -1 - 2\cos(t) \\ 1 + 5\cos(t) - 4\sin(t) \\ -1 + \cos(t) - \sin(t) \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_1 e^{2t} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} i c_2 e^{(1+i)t} \\ -i c_2 e^{(1+i)t} \\ c_2 e^{(1+i)t} \end{bmatrix} + \begin{bmatrix} -i c_3 e^{(1-i)t} \\ i c_3 e^{(1-i)t} \\ c_3 e^{(1-i)t} \end{bmatrix} + \begin{bmatrix} -1 - 2\cos(t) \\ 1 + 5\cos(t) - 4\sin(t) \\ -1 + \cos(t) - \sin(t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + i c_2 e^{(1+i)t} - i c_3 e^{(1-i)t} - 1 - 2\cos(t) \\ c_1 e^{2t} - i c_2 e^{(1+i)t} + i c_3 e^{(1-i)t} + 1 + 5\cos(t) - 4\sin(t) \\ c_1 e^{2t} + c_2 e^{(1+i)t} + c_3 e^{(1-i)t} - 1 + \cos(t) - \sin(t) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \\ z(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} i c_2 - i c_3 + c_1 - 3 \\ -i c_2 + i c_3 + c_1 + 6 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

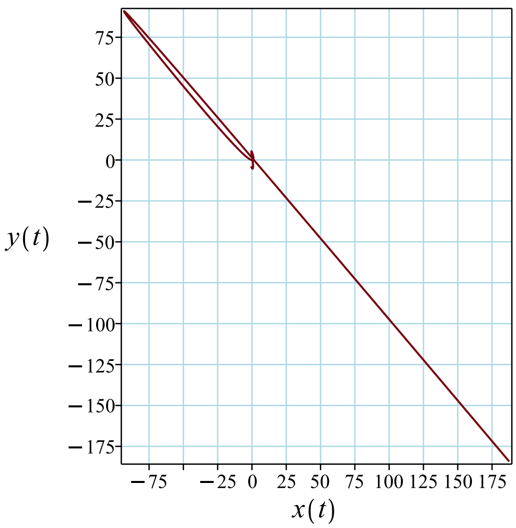
Solving for the constants of integrations gives

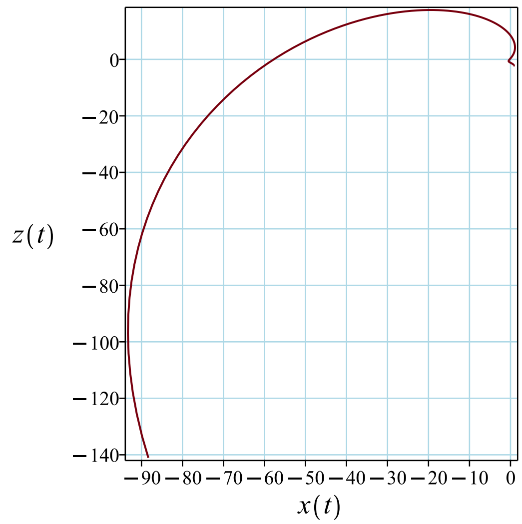
$$\begin{bmatrix} c_1 = 0 \\ c_2 = \frac{3}{2} - 2i \\ c_3 = \frac{3}{2} + 2i \end{bmatrix}$$

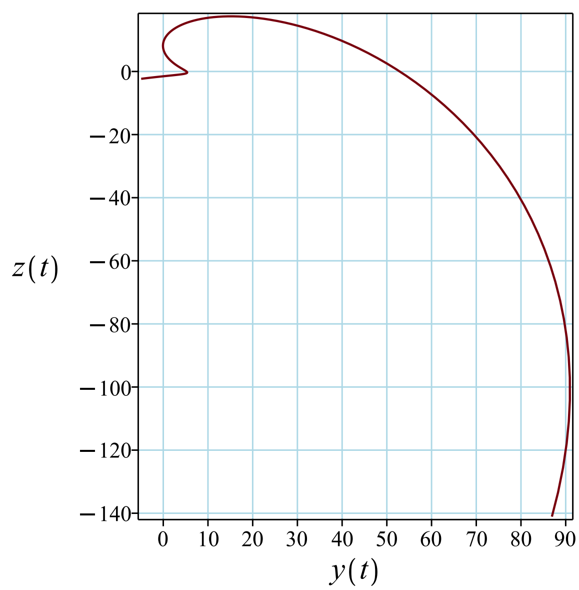
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -1 + \left(2 + \frac{3i}{2}\right) e^{(1+i)t} + \left(2 - \frac{3i}{2}\right) e^{(1-i)t} - 2\cos(t) \\ 1 + \left(-2 - \frac{3i}{2}\right) e^{(1+i)t} + \left(-2 + \frac{3i}{2}\right) e^{(1-i)t} + 5\cos(t) - 4\sin(t) \\ -1 + \left(\frac{3}{2} - 2i\right) e^{(1+i)t} + \left(\frac{3}{2} + 2i\right) e^{(1-i)t} + \cos(t) - \sin(t) \end{bmatrix}$$

The following are plots of each solution against another.




The following are plots of each solution.

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 71

```
dsolve([diff(x(t),t) = 2*x(t)+y(t)-z(t)+5*sin(t), diff(y(t),t) = y(t)+z(t)-10*cos(t), diff(z
```

$$x(t) = -2\cos(t) - 1 - 3\,e^{t}\sin(t) + 4\,e^{t}\cos(t)$$
$$y(t) = -4\sin(t) + 5\cos(t) + 1 + 3\,e^{t}\sin(t) - 4\,e^{t}\cos(t)$$
$$z(t) = -1 - \sin(t) + \cos(t) + 3\,e^{t}\cos(t) + 4\,e^{t}\sin(t)$$

✓ Solution by Mathematica

Time used: 4.398 (sec). Leaf size: 74

```
DSolve[{x'[t]==2*x[t]+y[t]-z[t]+5*Sin[t],y'[t]==y[t]+z[t]-10*Cos[t],z'[t]==x[t]+z[t]+2},{x[0
```

$$x(t) \to -3e^{t}\sin(t) + \left(4e^{t} - 2\right)\cos(t) - 1$$
$$y(t) \to \left(3e^{t} - 4\right)\sin(t) + \left(5 - 4e^{t}\right)\cos(t) + 1$$
$$z(t) \to \left(4e^{t} - 1\right)\sin(t) + \left(3e^{t} + 1\right)\cos(t) - 1$$

## 7.15 problem Problem 6(c)



Internal problem ID [12390]
Internal file name [`OUTPUT/11042_Wednesday_October_04_2023_01_27_29_AM_4657738/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 6(c).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -3x(t) + 3y + z(t) + 10\cos(t)\sin(t) \\
y' &= x(t) - 5y - 3z(t) + 10\cos(t)^2 - 5 \\
z'(t) &= -3x(t) + 7y + 3z(t) + 23\,e^t
\end{aligned}
$$

With initial conditions

$$
[x(0) = 1, y(0) = 2, z(0) = 3]
$$

### 7.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 10\cos(t)\sin(t) \\ 10\cos(t)^2 - 5 \\ 23\,e^t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t}\sin(2t) & \mathrm{e}^{-2t}\cos(2t) + 2\,\mathrm{e}^{-2t}\sin(2t) - \mathrm{e}^{-t} & \mathrm{e}^{-2t}\cos(2t) + \mathrm{e}^{-2t}\sin(2t \\ -\mathrm{e}^{-2t}\cos(2t) + \mathrm{e}^{-t} & -\mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t}\cos(2t) - \mathrm{e}^{-2t}\sin(2t) & \mathrm{e}^{-2t}\cos(2t) - \mathrm{e}^{-2t}\sin(2t \\ \mathrm{e}^{-2t}\cos(2t) - \mathrm{e}^{-2t}\sin(2t) - \mathrm{e}^{-t} & -\mathrm{e}^{-2t}\cos(2t) + 3\,\mathrm{e}^{-2t}\sin(2t) + \mathrm{e}^{-t} & \mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t}\sin(2t \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t}\sin(2t) & -\mathrm{e}^{-t} + (2\sin(2t) + \cos(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\sin(2t) + \cos(2t \\ -\mathrm{e}^{-2t}\cos(2t) + \mathrm{e}^{-t} & -\mathrm{e}^{-t} + (-\sin(2t) + 2\cos(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t \\ -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + (3\sin(2t) - \cos(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t}\sin(2t) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \mathrm{e}^{-t} - \mathrm{e}^{-2t}\sin(2t) & -\mathrm{e}^{-t} + (2\sin(2t) + \cos(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\sin(2t) + \cos( \\ -\mathrm{e}^{-2t}\cos(2t) + \mathrm{e}^{-t} & -\mathrm{e}^{-t} + (-\sin(2t) + 2\cos(2t))\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} + (\cos(2t) - \sin( \\ -\mathrm{e}^{-t} + (\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + (3\sin(2t) - \cos(2t))\,\mathrm{e}^{-2t} & \mathrm{e}^{-t} + 2\,\mathrm{e}^{-2t}\sin(2t \end{bmatrix} \\
&= \begin{bmatrix} -4\,\mathrm{e}^{-t} - \mathrm{e}^{-2t}\sin(2t) + 2(2\sin(2t) + \cos(2t))\,\mathrm{e}^{-2t} + 3(\sin(2t) + \cos(2t))\,\mathrm{e}^{-2t} \\ -\mathrm{e}^{-2t}\cos(2t) - 4\,\mathrm{e}^{-t} + 2(-\sin(2t) + 2\cos(2t))\,\mathrm{e}^{-2t} + 3(\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} \\ 4\,\mathrm{e}^{-t} + (\cos(2t) - \sin(2t))\,\mathrm{e}^{-2t} + 2(3\sin(2t) - \cos(2t))\,\mathrm{e}^{-2t} + 6\,\mathrm{e}^{-2t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} -4\,\mathrm{e}^{-t} + (6\sin(2t) + 5\cos(2t))\,\mathrm{e}^{-2t} \\ -4\,\mathrm{e}^{-t} + (-5\sin(2t) + 6\cos(2t))\,\mathrm{e}^{-2t} \\ 4\,\mathrm{e}^{-t} + (11\sin(2t) - \cos(2t))\,\mathrm{e}^{-2t} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \begin{bmatrix} \mathrm{e}^{t}(\mathrm{e}^{t}\sin(2t) + 1) & \mathrm{e}^{t}(-2\,\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}\cos(2t) - 1) & \mathrm{e}^{t}(-\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}\cos(2t) - \\ -\mathrm{e}^{t}(\mathrm{e}^{t}\cos(2t) - 1) & \mathrm{e}^{t}(\mathrm{e}^{t}\sin(2t) + 2\,\mathrm{e}^{t}\cos(2t) - 1) & \mathrm{e}^{t}(\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}\cos(2t) - 1 \\ \mathrm{e}^{t}(\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}\cos(2t) - 1) & -\mathrm{e}^{t}(3\,\mathrm{e}^{t}\sin(2t) + \mathrm{e}^{t}\cos(2t) - 1) & -2\,\mathrm{e}^{2t}\sin(2t) + \mathrm{e}^{t} \end{bmatrix}
\end{aligned}
$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-t} - e^{-2t}\sin(2t) & -e^{-t} + (2\sin(2t) + \cos(2t))\,e^{-2t} & -e^{-t} + (\sin(2t) + \cos(2 \\ -e^{-2t}\cos(2t) + e^{-t} & -e^{-t} + (-\sin(2t) + 2\cos(2t))\,e^{-2t} & -e^{-t} + (\cos(2t) - \sin(2 \\ -e^{-t} + (\cos(2t) - \sin(2t))\,e^{-2t} & e^{-t} + (3\sin(2t) - \cos(2t))\,e^{-2t} & e^{-t} + 2\,e^{-2t}\sin(2t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} - e^{-2t}\sin(2t) & -e^{-t} + (2\sin(2t) + \cos(2t))\,e^{-2t} & -e^{-t} + (\sin(2t) + \cos(2 \\ -e^{-2t}\cos(2t) + e^{-t} & -e^{-t} + (-\sin(2t) + 2\cos(2t))\,e^{-2t} & -e^{-t} + (\cos(2t) - \sin(2 \\ -e^{-t} + (\cos(2t) - \sin(2t))\,e^{-2t} & e^{-t} + (3\sin(2t) - \cos(2t))\,e^{-2t} & e^{-t} + 2\,e^{-2t}\sin(2t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos(2t)}{2} + \sin(2t) - \frac{69\,e^{t}}{26} \\ -\frac{5\sin(2t)}{2} - \frac{253\,e^{t}}{26} \\ \frac{7\cos(2t)}{2} + \frac{9\sin(2t)}{2} + \frac{483\,e^{t}}{26} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{e^{-2t}\left((e^{2t}+10)\cos(2t)+2\,e^{2t}\sin(2t)-8\,e^{t}-\frac{69\,e^{3t}}{13}+12\sin(2t)\right)}{2} \\ \frac{e^{-2t}\left(-5(e^{2t}+2)\sin(2t)-8\,e^{t}+12\cos(2t)-\frac{253\,e^{3t}}{13}\right)}{2} \\ \frac{(117\,e^{2t}\sin(2t)+91\,e^{2t}\cos(2t)+483\,e^{3t}+286\sin(2t)-26\cos(2t)+104\,e^{t})e^{-2t}}{26} \end{bmatrix} \end{aligned}$$

### 7.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 10\cos(t)\sin(t) \\ 10\cos(t)^2 - 5 \\ 23\,e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 3 & 1 \\ 1 & -5-\lambda & -3 \\ -3 & 7 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 5\lambda^2 + 12\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -2 + 2i$$
$$\lambda_3 = -2 - 2i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| $-1$ | 1 | real eigenvalue |
| $-2+2i$ | 1 | complex eigenvalue |
| $-2-2i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & 1 \\ 1 & -4 & -3 \\ -3 & 7 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 1 & -4 & -3 & 0 \\ -3 & 7 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ -3 & 7 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ 0 & \frac{5}{2} & \frac{5}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 3 & 1 \\ 0 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - (-2 - 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 + 2i & 3 & 1 \\ 1 & -3 + 2i & -3 \\ -3 & 7 & 5 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 + 2i & 3 & 1 & 0 \\ 1 & -3 + 2i & -3 & 0 \\ -3 & 7 & 5 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} + \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1+2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} & 0 \\ -3 & 7 & 5+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{3}{5} - \frac{6i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1+2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} & 0 \\ 0 & \frac{26}{5} - \frac{18i}{5} & \frac{22}{5} + \frac{4i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{3}{2} + \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1+2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+2i & 3 & 1 \\ 0 & -\frac{12}{5} + \frac{16i}{5} & -\frac{14}{5} + \frac{2i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right) t, v_2 = -\frac{1}{2}t - \frac{1}{2}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{\mathrm{I}}{2}\right) t \\ -\frac{t}{2} - \frac{\mathrm{I}t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ -\frac{1}{2}t - \frac{1}{2}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{\mathrm{I}}{2}\right) t \\ -\frac{t}{2} - \frac{\mathrm{I}t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ \frac{-\frac{1}{2}t - \frac{1}{2}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -\frac{t}{2} - \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ -\frac{t}{2} - \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ -1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -2 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -3 & 3 & 1 \\ 1 & -5 & -3 \\ -3 & 7 & 3 \end{bmatrix} - (-2 + 2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - 2i & 3 & 1 \\ 1 & -3 - 2i & -3 \\ -3 & 7 & 5 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 - 2i & 3 & 1 & 0 \\ 1 & -3 - 2i & -3 & 0 \\ -3 & 7 & 5 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{5} - \frac{2i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 - 2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} & 0 \\ -3 & 7 & 5 - 2i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{3}{5} + \frac{6i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 - 2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} & 0 \\ 0 & \frac{26}{5} + \frac{18i}{5} & \frac{22}{5} - \frac{4i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{3}{2} - \frac{i}{2}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1-2i & 3 & 1 & 0 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-2i & 3 & 1 \\ 0 & -\frac{12}{5} - \frac{16i}{5} & -\frac{14}{5} - \frac{2i}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t, v_2 = -\frac{1}{2}t + \frac{1}{2}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ -\frac{1}{2}t + \frac{1}{2}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ \frac{-\frac{1}{2}t + \frac{1}{2}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ -\frac{t}{2} + \frac{It}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1+i \\ -1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-1$ | 1 | 1 | No | $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ |
| $-2+2i$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{2}+\frac{i}{2} \\ -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$ |
| $-2-2i$ | 1 | 1 | No | $\begin{bmatrix} \frac{1}{2}-\frac{i}{2} \\ -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-1$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-2+2i)t} \\ \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-2+2i)t} \\ \mathrm{e}^{(-2+2i)t} \end{bmatrix} + c_3 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-2-2i)t} \\ \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-2-2i)t} \\ \mathrm{e}^{(-2-2i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-t} & \left(\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-2+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-2-2i)t} \\ -\mathrm{e}^{-t} & \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-2+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-2-2i)t} \\ \mathrm{e}^{-t} & \mathrm{e}^{(-2+2i)t} & \mathrm{e}^{(-2-2i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\mathrm{e}^{t} & \mathrm{e}^{t} & \mathrm{e}^{t} \\ \left(\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(2-2i)t} & \left(-\frac{1}{2}-\frac{3i}{2}\right)\mathrm{e}^{(2-2i)t} & -i\mathrm{e}^{(2-2i)t} \\ \left(\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(2+2i)t} & \left(-\frac{1}{2}+\frac{3i}{2}\right)\mathrm{e}^{(2+2i)t} & i\mathrm{e}^{(2+2i)t} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{-t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ -e^{-t} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ e^{-t} & e^{(-2+2i)t} & e^{(-2-2i)t} \end{bmatrix} \int \begin{bmatrix} -e^{t} & e^{t} & e^{t} \\ \left(\frac{1}{2}+\frac{i}{2}\right) e^{(2-2i)t} & \left(-\frac{1}{2}-\frac{3i}{2}\right) e^{(2-2i)t} & -ie^{(2} \\ \left(\frac{1}{2}-\frac{i}{2}\right) e^{(2+2i)t} & \left(-\frac{1}{2}+\frac{3i}{2}\right) e^{(2+2i)t} & ie^{(2+} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ -e^{-t} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ e^{-t} & e^{(-2+2i)t} & e^{(-2-2i)t} \end{bmatrix} \int \begin{bmatrix} e^{t}(23\,e^{t}+5\cos(2t)-5\sin(2t) \\ \frac{5\left(1+3i+(-2-6i)\cos(t)^2+(2+2i)\sin(t)\cos(t)\right)e^{(2-2i)t}}{2} \\ \left(-\frac{5}{2}+\frac{5i}{2}\right) e^{(2+4i)t} + 5ie^{2t} + 23ie^{(3} \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ -e^{-t} & \left(-\frac{1}{2}+\frac{i}{2}\right) e^{(-2+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{(-2-2i)t} \\ e^{-t} & e^{(-2+2i)t} & e^{(-2-2i)t} \end{bmatrix} \begin{bmatrix} \frac{e^{t}\left(23\,e^{t}+6\cos(2t)+2\sin(2t)\right)}{2} \\ \left(\frac{1}{4}-\frac{3i}{4}\right) e^{(2-4i)t} + \left(\frac{46}{13}-\frac{69i}{13}\right) e^{(3-2i)t} - \frac{5ie^{2t}}{2} \\ \left(\frac{1}{4}+\frac{3i}{4}\right) e^{(2+4i)t} + \left(\frac{46}{13}+\frac{69i}{13}\right) e^{(3+2i)t} + \frac{5ie^{2t}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos(2t)}{2} + \sin(2t) - \frac{69\,e^{t}}{26} \\ -\frac{5\sin(2t)}{2} - \frac{253\,e^{t}}{26} \\ \frac{7\cos(2t)}{2} + \frac{9\sin(2t)}{2} + \frac{483\,e^{t}}{26} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{-t} \\ -c_1 e^{-t} \\ c_1 e^{-t} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-2+2i)t} \\ \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-2+2i)t} \\ c_2 e^{(-2+2i)t} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-2-2i)t} \\ \left(-\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-2-2i)t} \\ c_3 e^{(-2-2i)t} \end{bmatrix} + \begin{bmatrix} \frac{\cos(2t)}{2} + \sin(2t) - \\ -\frac{5\sin(2t)}{2} - \frac{253\,e^{t}}{26} \\ \frac{7\cos(2t)}{2} + \frac{9\sin(2t)}{2} + \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{-t} + \left(\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-2+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-2-2i)t} + \frac{\cos(2t)}{2} + \sin(2t) - \frac{69\,e^{t}}{26} \\ -c_1 e^{-t} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-2+2i)t} + \left(-\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-2-2i)t} - \frac{5\sin(2t)}{2} - \frac{253\,e^{t}}{26} \\ c_1 e^{-t} + c_2 e^{(-2+2i)t} + c_3 e^{(-2-2i)t} + \frac{7\cos(2t)}{2} + \frac{9\sin(2t)}{2} + \frac{483\,e^{t}}{26} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \\ z(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right] = \left[\begin{array}{c} \left(\frac{1}{2}+\frac{i}{2}\right) c_{2}+\left(\frac{1}{2}-\frac{i}{2}\right) c_{3}-c_{1}-\frac{28}{13} \\ \left(-\frac{1}{2}+\frac{i}{2}\right) c_{2}+\left(-\frac{1}{2}-\frac{i}{2}\right) c_{3}-c_{1}-\frac{253}{26} \\ c_{1}+c_{2}+c_{3}+\frac{287}{13} \end{array}\right]$$

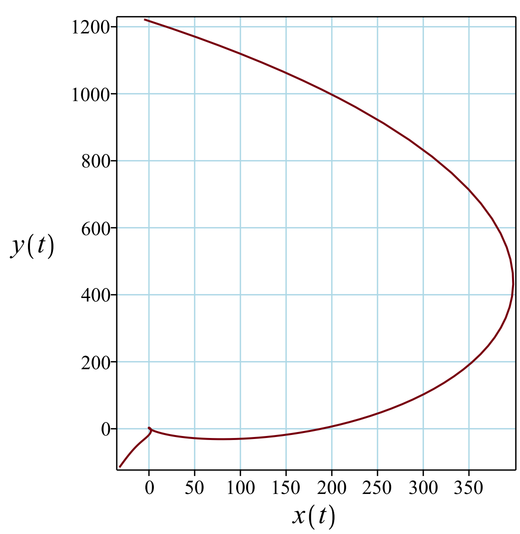
Solving for the constants of integrations gives

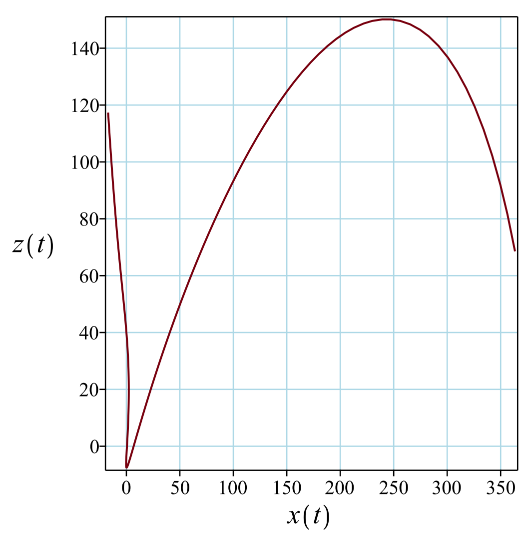
$$\left[\begin{array}{c} c_{1}=-\frac{21}{2} \\ c_{2}=-\frac{223}{52}+\frac{159 i}{52} \\ c_{3}=-\frac{223}{52}-\frac{159 i}{52} \end{array}\right]$$

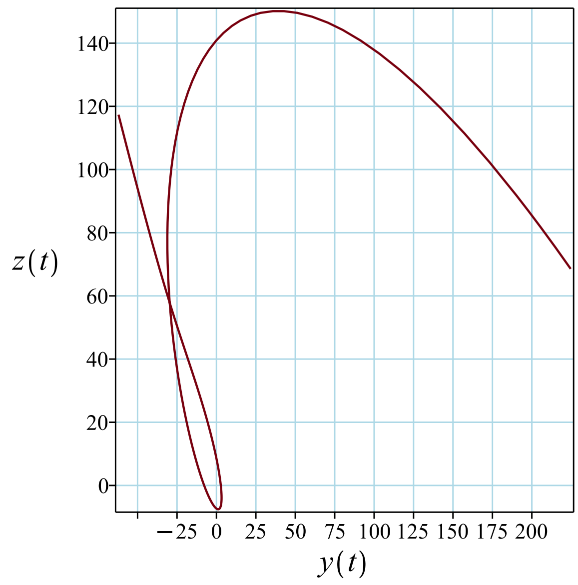
Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} x(t) \\ y \\ z(t) \end{array}\right] = \left[\begin{array}{c} \frac{21\,\mathrm{e}^{-t}}{2}+\left(-\frac{191}{52}-\frac{8 i}{13}\right) \mathrm{e}^{(-2+2 i) t}+\left(-\frac{191}{52}+\frac{8 i}{13}\right) \mathrm{e}^{(-2-2 i) t}+\frac{\cos(2 t)}{2}+\sin(2 t)-\frac{69\,\mathrm{e}^{t}}{26} \\ \frac{21\,\mathrm{e}^{-t}}{2}+\left(\frac{8}{13}-\frac{191 i}{52}\right) \mathrm{e}^{(-2+2 i) t}+\left(\frac{8}{13}+\frac{191 i}{52}\right) \mathrm{e}^{(-2-2 i) t}-\frac{5 \sin(2 t)}{2}-\frac{253\,\mathrm{e}^{t}}{26} \\ -\frac{21\,\mathrm{e}^{-t}}{2}+\left(-\frac{223}{52}+\frac{159 i}{52}\right) \mathrm{e}^{(-2+2 i) t}+\left(-\frac{223}{52}-\frac{159 i}{52}\right) \mathrm{e}^{(-2-2 i) t}+\frac{7 \cos(2 t)}{2}+\frac{9 \sin(2 t)}{2}+\frac{483\,\mathrm{e}^{t}}{26} \end{array}\right]$$

The following are plots of each solution against another.






The following are plots of each solution.

✓ Solution by Maple

Time used: 0.829 (sec). Leaf size: 132

```
dsolve([diff(x(t),t) = -3*x(t)+3*y(t)+z(t)+5*sin(2*t), diff(y(t),t) = x(t)-5*y(t)-3*z(t)+5*c
```

$$x(t) = -\frac{69\,e^{t}}{26} + \sin(2t) + \frac{\cos(2t)}{2} + \frac{21\,e^{-t}}{2} - \frac{191\,e^{-2t}\cos(2t)}{26} + \frac{16\,e^{-2t}\sin(2t)}{13}$$

$$y(t) = -\frac{253\,e^{t}}{26} - \frac{5\sin(2t)}{2} + \frac{21\,e^{-t}}{2} + \frac{191\,e^{-2t}\sin(2t)}{26} + \frac{16\,e^{-2t}\cos(2t)}{13}$$

$$z(t) = \frac{483\,e^{t}}{26} + \frac{7\cos(2t)}{2} + \frac{9\sin(2t)}{2} - \frac{21\,e^{-t}}{2} - \frac{223\,e^{-2t}\cos(2t)}{26} - \frac{159\,e^{-2t}\sin(2t)}{26}$$

✓ Solution by Mathematica

Time used: 14.393 (sec). Leaf size: 197

```
DSolve[{x'[t]==-3*x[t]+3*y[t]+z[t]+5*Sin[3*t],y'[t]==x[t]-5*y[t]-3*z[t]+5*Cos[2*t],z'[t]==-3
```

$$x(t) \to \frac{1}{754}\left(7540e^{-t} - 2001e^{t} + 603e^{-2t}\sin(2t) + 377\sin(2t) + 429\sin(3t) + \left(1131 - 5409e^{-2t}\right)\cos(2t) - 507\cos(3t)\right)$$

$$y(t) \to \frac{1}{754}\left(7540e^{-t} - 7337e^{t} + 5409e^{-2t}\sin(2t) - 1508\sin(2t) - 507\sin(3t) + \left(603e^{-2t} + 1131\right)\cos(2t) - 429\cos(3t)\right)$$

$$z(t) \to -10e^{-t} + \frac{483e^{t}}{26} - \frac{2403}{377}e^{-2t}\sin(2t) + \frac{43}{58}\sin(3t) + \left(1 - \frac{3006e^{-2t}}{377}\right)\cos(2t) + \frac{81}{58}\cos(3t) + 9\sin(t)\cos(t)$$

## 7.16 problem Problem 6(d)



Internal problem ID [12391]
Internal file name [OUTPUT/11043_Wednesday_October_04_2023_01_27_32_AM_62598434/index.tex]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.3 Systems of Linear Differential Equations (Variation of Parameters). Problems page 514
**Problem number**: Problem 6(d).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) &= -3x(t) + y - 3z(t) + 2\,\mathrm{e}^t \\ y' &= 4x(t) - y + 2z(t) + 4\,\mathrm{e}^t \\ z'(t) &= 4x(t) - 2y + 3z(t) + 4\,\mathrm{e}^t \end{aligned}$$

With initial conditions

$$[x(0) = 1, y(0) = 2, z(0) = 3]$$

### 7.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^t \\ 4\,\mathrm{e}^t \\ 4\,\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{-t}\sin(2t) + e^{-t}\cos(2t) & -\frac{e^{-t}\cos(2t)}{2} + \frac{e^{t}}{2} & \frac{e^{-t}\cos(2t)}{2} - e^{-t}\sin(2t) - \frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{t}}{2} + \frac{e^{-t}\cos(2t)}{2} - \frac{e^{-t}\sin(2t)}{2} & \frac{e^{-t}\cos(2t)}{2} + \frac{3\,e^{-t}\sin(2t)}{2} - \frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}\cos(2t)}{2} - \frac{e^{-t}\sin(2t)}{2} - \frac{e^{t}}{2} & \frac{e^{t}}{2} + \frac{e^{-t}\cos(2t)}{2} + \frac{3\,e^{-t}\sin(2t)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\cos(2t) - \sin(2t)) & -\frac{e^{-t}\cos(2t)}{2} + \frac{e^{t}}{2} & \frac{e^{-t}(\cos(2t)-2\sin(2t))}{2} - \frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2} + \frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2} - \frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2} - \frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2} + \frac{e^{t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{x}_0$$

$$= \begin{bmatrix} e^{-t}(\cos(2t) - \sin(2t)) & -\frac{e^{-t}\cos(2t)}{2} + \frac{e^{t}}{2} & \frac{e^{-t}(\cos(2t)-2\sin(2t))}{2} - \frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2} + \frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2} - \frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2} - \frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2} + \frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\cos(2t) - \sin(2t)) - e^{-t}\cos(2t) - \frac{e^{t}}{2} + \frac{3\,e^{-t}(\cos(2t)-2\sin(2t))}{2} \\ 2\,e^{-t}\sin(2t) + e^{-t}(\cos(2t) - \sin(2t)) - \frac{e^{t}}{2} + \frac{3\,e^{-t}(3\sin(2t)+\cos(2t))}{2} \\ 2\,e^{-t}\sin(2t) + e^{-t}(\cos(2t) - \sin(2t)) + \frac{e^{t}}{2} + \frac{3\,e^{-t}(3\sin(2t)+\cos(2t))}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(3\cos(2t)-8\sin(2t))e^{-t}}{2} - \frac{e^{t}}{2} \\ \frac{(11\sin(2t)+5\cos(2t))e^{-t}}{2} - \frac{e^{t}}{2} \\ \frac{(11\sin(2t)+5\cos(2t))e^{-t}}{2} + \frac{e^{t}}{2} \end{bmatrix}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{t}(\sin(2t)+\cos(2t)) & -\frac{e^{t}\cos(2t)}{2}+\frac{e^{-t}}{2} & \frac{e^{t}\cos(2t)}{2}+e^{t}\sin(2t)-\frac{e^{-t}}{2} \\ -2\,e^{t}\sin(2t) & \frac{e^{t}\cos(2t)}{2}+\frac{e^{t}\sin(2t)}{2}+\frac{e^{-t}}{2} & \frac{e^{t}\cos(2t)}{2}-\frac{3\,e^{t}\sin(2t)}{2}-\frac{e^{-t}}{2} \\ -2\,e^{t}\sin(2t) & \frac{e^{t}\cos(2t)}{2}+\frac{e^{t}\sin(2t)}{2}-\frac{e^{-t}}{2} & \frac{e^{t}\cos(2t)}{2}-\frac{3\,e^{t}\sin(2t)}{2}+\frac{e^{-t}}{2} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{-t}(\cos(2t)-\sin(2t)) & -\frac{e^{-t}\cos(2t)}{2}+\frac{e^{t}}{2} & \frac{e^{-t}(\cos(2t)-2\sin(2t))}{2}-\frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}+\frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2}-\frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}-\frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2}+\frac{e^{t}}{2} \end{bmatrix} \int \begin{bmatrix} e^{t}(\sin(2t)+\cos(2 \\ -2\,e^{t}\sin(2t) \\ -2\,e^{t}\sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(2t)-\sin(2t)) & -\frac{e^{-t}\cos(2t)}{2}+\frac{e^{t}}{2} & \frac{e^{-t}(\cos(2t)-2\sin(2t))}{2}-\frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}+\frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2}-\frac{e^{t}}{2} \\ 2\,e^{-t}\sin(2t) & \frac{e^{-t}(\cos(2t)-\sin(2t))}{2}-\frac{e^{t}}{2} & \frac{e^{-t}(3\sin(2t)+\cos(2t))}{2}+\frac{e^{t}}{2} \end{bmatrix} \begin{bmatrix} e^{2t}(-\cos(2t)+2\sin \\ e^{2t}(3\cos(2t)-\sin( \\ e^{2t}(3\cos(2t)-\sin( \end{bmatrix} \\
&= \begin{bmatrix} -e^{t} \\ 3\,e^{t} \\ 3\,e^{t} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{(3\cos(2t)-8\sin(2t))e^{-t}}{2}-\frac{3\,e^{t}}{2} \\ \frac{(11\sin(2t)+5\cos(2t))e^{-t}}{2}+\frac{5\,e^{t}}{2} \\ \frac{(11\sin(2t)+5\cos(2t))e^{-t}}{2}+\frac{7\,e^{t}}{2} \end{bmatrix}
\end{aligned}
$$

### 7.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \\ z'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} + \begin{bmatrix} 2\,e^{t} \\ 4\,e^{t} \\ 4\,e^{t} \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -3-\lambda & 1 & -3 \\ 4 & -1-\lambda & 2 \\ 4 & -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + \lambda^2 + 3\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1 + 2i$$
$$\lambda_3 = -1 - 2i$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 1 | 1 | real eigenvalue |
| $-1 - 2i$ | 1 | complex eigenvalue |
| $-1 + 2i$ | 1 | complex eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & -3 \\ 4 & -2 & 2 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 4 & -2 & 2 & 0 \\ 4 & -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 4 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 1 & -3 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - (-1-2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2+2i & 1 & -3 \\ 4 & 2i & 2 \\ 4 & -2 & 4+2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 4 & 2i & 2 & 0 \\ 4 & -2 & 4+2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 4 & -2 & 4+2i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 0 & -1+i & 1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{5} - \frac{2i}{5}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2+2i & 1 & -3 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2+2i & 1 & -3 \\ 0 & 1+3i & -1-3i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 2 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1+2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} -3 & 1 & -3 \\ 4 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix} - (-1+2i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2-2i & 1 & -3 \\ 4 & -2i & 2 \\ 4 & -2 & 4-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2-2i & 1 & -3 & 0 \\ 4 & -2i & 2 & 0 \\ 4 & -2 & 4-2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2-2i & 1 & -3 & 0 \\ 0 & 1-3i & -1+3i & 0 \\ 4 & -2 & 4-2i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2-2i & 1 & -3 & 0 \\ 0 & 1-3i & -1+3i & 0 \\ 0 & -1-i & 1+i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{5}+\frac{2i}{5}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2-2i & 1 & -3 & 0 \\ 0 & 1-3i & -1+3i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2-2i & 1 & -3 \\ 0 & 1-3i & -1+3i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right) t, v_2 = t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| eigenvalue | multiplicity: algebraic $m$ | multiplicity: geometric $k$ | defective? | eigenvectors |
|---|---|---|---|---|
| 1 | 1 | 1 | No | $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ |
| $-1+2i$ | 1 | 1 | No | $\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$ |
| $-1-2i$ | 1 | 1 | No | $\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^t \\ -\mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \end{bmatrix} + c_3 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{t} & \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-1+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-1-2i)t} \\ -\mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} 0 & -\frac{\mathrm{e}^{-t}}{2} & \frac{\mathrm{e}^{-t}}{2} \\ -i\mathrm{e}^{(1-2i)t} & \left(\frac{1}{4}+\frac{i}{4}\right)\mathrm{e}^{(1-2i)t} & \left(\frac{1}{4}-\frac{3i}{4}\right)\mathrm{e}^{(1-2i)t} \\ i\mathrm{e}^{(1+2i)t} & \left(\frac{1}{4}-\frac{i}{4}\right)\mathrm{e}^{(1+2i)t} & \left(\frac{1}{4}+\frac{3i}{4}\right)\mathrm{e}^{(1+2i)t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\mathrm{e}^{t} & \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-1+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-1-2i)t} \\ -\mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \end{bmatrix} \int \begin{bmatrix} 0 & -\frac{\mathrm{e}^{-t}}{2} & \frac{\mathrm{e}^{-t}}{2} \\ -i\mathrm{e}^{(1-2i)t} & \left(\frac{1}{4}+\frac{i}{4}\right)\mathrm{e}^{(1-2i)t} & \left(\frac{1}{4}-\frac{3i}{4}\right)\mathrm{e}^{(1-2i)t} \\ i\mathrm{e}^{(1+2i)t} & \left(\frac{1}{4}-\frac{i}{4}\right)\mathrm{e}^{(1+2i)t} & \left(\frac{1}{4}+\frac{3i}{4}\right)\mathrm{e}^{(1+2i)t} \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{t} & \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-1+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-1-2i)t} \\ -\mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \end{bmatrix} \int \begin{bmatrix} 0 \\ (2-4i)\,\mathrm{e}^{(2-2i)t} \\ (2+4i)\,\mathrm{e}^{(2+2i)t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\mathrm{e}^{t} & \left(-\frac{1}{2}+\frac{i}{2}\right)\mathrm{e}^{(-1+2i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right)\mathrm{e}^{(-1-2i)t} \\ -\mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{t} & \mathrm{e}^{(-1+2i)t} & \mathrm{e}^{(-1-2i)t} \end{bmatrix} \begin{bmatrix} 0 \\ \left(\frac{3}{2}-\frac{i}{2}\right)\mathrm{e}^{(2-2i)t} \\ \left(\frac{3}{2}+\frac{i}{2}\right)\mathrm{e}^{(2+2i)t} \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{t} \\ 3\,\mathrm{e}^{t} \\ 3\,\mathrm{e}^{t} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^t \\ -c_1 e^t \\ c_1 e^t \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-1+2i)t} \\ c_2 e^{(-1+2i)t} \\ c_2 e^{(-1+2i)t} \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-1-2i)t} \\ c_3 e^{(-1-2i)t} \\ c_3 e^{(-1-2i)t} \end{bmatrix} + \begin{bmatrix} -e^t \\ 3\,e^t \\ 3\,e^t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_3 e^{(-1-2i)t} + \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 e^{(-1+2i)t} - e^t(1+c_1) \\ c_3 e^{(-1-2i)t} + c_2 e^{(-1+2i)t} - e^t(c_1-3) \\ c_3 e^{(-1-2i)t} + c_2 e^{(-1+2i)t} + e^t(c_1+3) \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \\ z(0) = 3 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) c_2 + \left(-\frac{1}{2}-\frac{i}{2}\right) c_3 - c_1 - 1 \\ c_3 + c_2 - c_1 + 3 \\ c_3 + c_2 + c_1 + 3 \end{bmatrix}$$

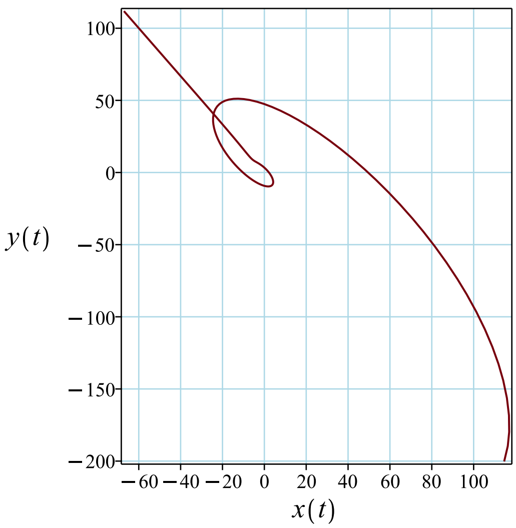
Solving for the constants of integrations gives

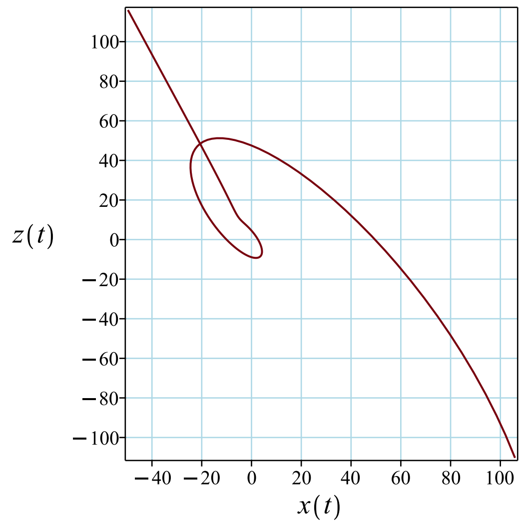
$$\begin{bmatrix} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{4} - \frac{9i}{4} \\ c_3 = -\frac{1}{4} + \frac{9i}{4} \end{bmatrix}$$

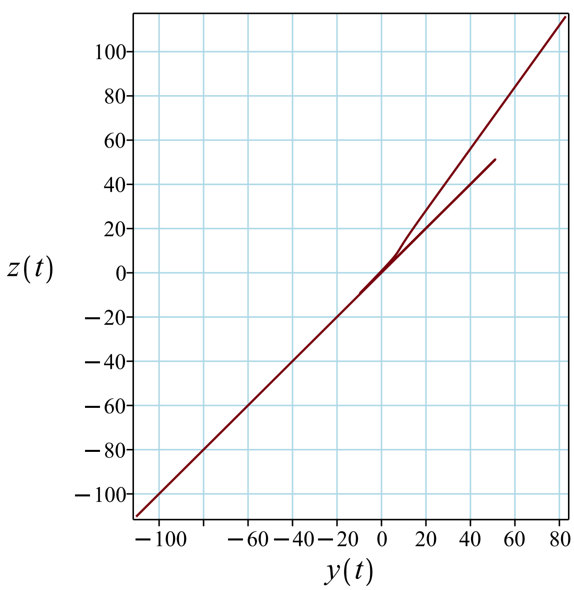
Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y \\ z(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{5}{4}-i\right) e^{(-1-2i)t} + \left(\frac{5}{4}+i\right) e^{(-1+2i)t} - \frac{3\,e^t}{2} \\ \left(-\frac{1}{4}+\frac{9i}{4}\right) e^{(-1-2i)t} + \left(-\frac{1}{4}-\frac{9i}{4}\right) e^{(-1+2i)t} + \frac{5\,e^t}{2} \\ \left(-\frac{1}{4}+\frac{9i}{4}\right) e^{(-1-2i)t} + \left(-\frac{1}{4}-\frac{9i}{4}\right) e^{(-1+2i)t} + \frac{7\,e^t}{2} \end{bmatrix}$$

The following are plots of each solution against another.




The following are plots of each solution.

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 86

```
dsolve([diff(x(t),t) = -3*x(t)+y(t)-3*z(t)+2*exp(t), diff(y(t),t) = 4*x(t)-y(t)+2*z(t)+4*exp
```

$$x(t) = -\frac{3\,\mathrm{e}^{t}}{2} - 2\,\mathrm{e}^{-t}\sin(2t) + \frac{5\,\mathrm{e}^{-t}\cos(2t)}{2}$$
$$y(t) = \frac{5\,\mathrm{e}^{t}}{2} + \frac{9\,\mathrm{e}^{-t}\sin(2t)}{2} - \frac{\mathrm{e}^{-t}\cos(2t)}{2}$$
$$z(t) = \frac{7\,\mathrm{e}^{t}}{2} + \frac{9\,\mathrm{e}^{-t}\sin(2t)}{2} - \frac{\mathrm{e}^{-t}\cos(2t)}{2}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 98

```
DSolve[{x'[t]==-3*x[t]+y[t]-3*z[t]+2*Exp[t],y'[t]==4*x[t]-y[t]+2*z[t]+4*Exp[t],z'[t]==4*x[t]
```

$$x(t) \to -\frac{1}{2}e^{-t}\left(3e^{2t} + 4\sin(2t) - 5\cos(2t)\right)$$
$$y(t) \to \frac{1}{2}e^{-t}\left(5e^{2t} + 9\sin(2t) - \cos(2t)\right)$$
$$z(t) \to \frac{1}{2}e^{-t}\left(7e^{2t} + 9\sin(2t) - \cos(2t)\right)$$

# 8 Chapter 8.4 Systems of Linear Differential Equations (Method of Undetermined Coefficients). Problems page 520

## 8.1 problem Problem 1(a)



Internal problem ID [12392]
Internal file name [`OUTPUT/11044_Wednesday_October_04_2023_01_27_33_AM_15180600/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.4 Systems of Linear Differential Equations (Method of Undetermined Coefficients). Problems page 520
**Problem number**: Problem 1(a).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + 5y + 10\sinh(t) \\
y' &= 19x(t) - 13y + 24\sinh(t)
\end{aligned}
$$

### 8.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 19 & -13 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 10\sinh(t) \\ 24\sinh(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(19\,\mathrm{e}^{24t}+5)\mathrm{e}^{-18t}}{24} & \frac{5(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} \\ \frac{19(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} & \frac{(5\,\mathrm{e}^{24t}+19)\mathrm{e}^{-18t}}{24} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{(19\,\mathrm{e}^{24t}+5)\mathrm{e}^{-18t}}{24} & \frac{5(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} \\ \frac{19(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} & \frac{(5\,\mathrm{e}^{24t}+19)\mathrm{e}^{-18t}}{24} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(19\,\mathrm{e}^{24t}+5)\mathrm{e}^{-18t}c_1}{24} + \frac{5(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}c_2}{24} \\ \frac{19(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}c_1}{24} + \frac{(5\,\mathrm{e}^{24t}+19)\mathrm{e}^{-18t}c_2}{24} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{-18t}((19c_1+5c_2)\mathrm{e}^{24t}+5c_1-5c_2)}{24} \\ \frac{19\left(\left(c_1+\frac{5c_2}{19}\right)\mathrm{e}^{24t}+c_2-c_1\right)\mathrm{e}^{-18t}}{24} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{5\,\mathrm{e}^{18t}}{24} + \frac{19\,\mathrm{e}^{-6t}}{24} & -\frac{5\,\mathrm{e}^{18t}}{24} + \frac{5\,\mathrm{e}^{-6t}}{24} \\ -\frac{19\,\mathrm{e}^{18t}}{24} + \frac{19\,\mathrm{e}^{-6t}}{24} & \frac{19\,\mathrm{e}^{18t}}{24} + \frac{5\,\mathrm{e}^{-6t}}{24} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{(19\,\mathrm{e}^{24t}+5)\mathrm{e}^{-18t}}{24} & \frac{5(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} \\ \frac{19(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} & \frac{(5\,\mathrm{e}^{24t}+19)\mathrm{e}^{-18t}}{24} \end{bmatrix} \int \begin{bmatrix} \frac{5\,\mathrm{e}^{18t}}{24} + \frac{19\,\mathrm{e}^{-6t}}{24} & -\frac{5\,\mathrm{e}^{18t}}{24} + \frac{5\,\mathrm{e}^{-6t}}{24} \\ -\frac{19\,\mathrm{e}^{18t}}{24} + \frac{19\,\mathrm{e}^{-6t}}{24} & \frac{19\,\mathrm{e}^{18t}}{24} + \frac{5\,\mathrm{e}^{-6t}}{24} \end{bmatrix} \begin{bmatrix} 10\sinh(t) \\ 24\sinh(t) \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{(19\,\mathrm{e}^{24t}+5)\mathrm{e}^{-18t}}{24} & \frac{5(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} \\ \frac{19(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} & \frac{(5\,\mathrm{e}^{24t}+19)\mathrm{e}^{-18t}}{24} \end{bmatrix} \begin{bmatrix} \frac{31\sinh(5t)}{24} - \frac{155\sinh(7t)}{168} + \frac{35\sinh(17t)}{408} - \frac{35\sinh(19t)}{456} - \frac{31\cosh(5t)}{24} + 155 \\ \frac{31\sinh(5t)}{24} - \frac{155\sinh(7t)}{168} - \frac{133\sinh(17t)}{408} + \frac{7\sinh(19t)}{24} - \frac{31\cosh(5t)}{24} + 155 \end{bmatrix} \\ &= \begin{bmatrix} \frac{35\left(\left(-\frac{527\cosh(5t)}{35}+\frac{527\cosh(7t)}{49}+\frac{527\sinh(5t)}{35}-\frac{527\sinh(7t)}{49}\right)\mathrm{e}^{24t}+\sinh(17t)-\frac{17\sinh(19t)}{19}+\cosh(17t)-\frac{17\cosh(19t)}{19}\right)\mathrm{e}^{-18t}}{408} \\ -\frac{133\left(\frac{527\left(\cosh(5t)-\frac{5\cosh(7t)}{7}-\sinh(5t)+\frac{5\sinh(7t)}{7}\right)\mathrm{e}^{24t}}{133}+\sinh(17t)-\frac{17\sinh(19t)}{19}+\cosh(17t)-\frac{17\cosh(19t)}{19}\right)\mathrm{e}^{-18t}}{408} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
= \left[\begin{array}{c} \frac{35\,\mathrm{e}^{-18t}\left(\frac{17\left(\frac{19c_1}{5}+c_2-\frac{31\cosh(5t)}{5}+\frac{31\cosh(7t)}{7}+\frac{31\sinh(5t)}{5}-\frac{31\sinh(7t)}{7}\right)\mathrm{e}^{24t}}{7}+\cosh(17t)-\frac{17\cosh(19t)}{19}+\sinh(17t)-\frac{17\sinh(19t)}{19}+ \cdots}{408} \\ -\frac{133\,\mathrm{e}^{-18t}\left(\left(-\frac{17c_1}{7}-\frac{85c_2}{133}+\frac{527\cosh(5t)}{133}-\frac{2635\cosh(7t)}{931}-\frac{527\sinh(5t)}{133}+\frac{2635\sinh(7t)}{931}\right)\mathrm{e}^{24t}+\cosh(17t)-\frac{17\cosh(19t)}{19}+\sinh(17t)-\frac{17\sinh( \cdots}{19}}{408} \end{array}\right]
$$

### 8.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 19 & -13 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} 10\sinh(t) \\ 24\sinh(t) \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 5 \\ 19 & -13 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 5 \\ 19 & -13-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 12\lambda - 108 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -18$$
$$\lambda_2 = 6$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 6 | 1 | real eigenvalue |
| −18 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -18$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 5 \\ 19 & -13 \end{bmatrix} - (-18)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 19 & 5 \\ 19 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 19 & 5 & 0 \\ 19 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 19 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 19 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{19}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{19} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{19} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{19} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{19} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{5t}{19} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{19} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{5t}{19} \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 19 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 1 & 5 \\ 19 & -13 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 5 \\ 19 & -19 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -5 & 5 & 0 \\ 19 & -19 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{19R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| $-18$ | 1 | 1 | No | $\begin{bmatrix} -\frac{5}{19} \\ 1 \end{bmatrix}$ |
| 6 | 1 | 1 | No | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $-18$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-18t} \\
&= \begin{bmatrix} -\frac{5}{19} \\ 1 \end{bmatrix} e^{-18t}
\end{aligned}
$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{6t} \\
&= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{5\,\mathrm{e}^{-18t}}{19} \\ \mathrm{e}^{-18t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{6t} \\ \mathrm{e}^{6t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} -\frac{5\,\mathrm{e}^{-18t}}{19} & \mathrm{e}^{6t} \\ \mathrm{e}^{-18t} & \mathrm{e}^{6t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{19\,e^{18t}}{24} & \frac{19\,e^{18t}}{24} \\ \frac{19\,e^{-6t}}{24} & \frac{5\,e^{-6t}}{24} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{5\,e^{-18t}}{19} & e^{6t} \\ e^{-18t} & e^{6t} \end{bmatrix} \int \begin{bmatrix} -\frac{19\,e^{18t}}{24} & \frac{19\,e^{18t}}{24} \\ \frac{19\,e^{-6t}}{24} & \frac{5\,e^{-6t}}{24} \end{bmatrix} \begin{bmatrix} 10\sinh(t) \\ 24\sinh(t) \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{5\,e^{-18t}}{19} & e^{6t} \\ e^{-18t} & e^{6t} \end{bmatrix} \int \begin{bmatrix} \frac{133\sinh(t)e^{18t}}{12} \\ \frac{155\sinh(t)e^{-6t}}{12} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{5\,e^{-18t}}{19} & e^{6t} \\ e^{-18t} & e^{6t} \end{bmatrix} \begin{bmatrix} -\frac{133\sinh(17t)}{408} + \frac{7\sinh(19t)}{24} - \frac{133\cosh(17t)}{408} + \frac{7\cosh(19t)}{24} \\ \frac{31\sinh(5t)}{24} - \frac{155\sinh(7t)}{168} - \frac{31\cosh(5t)}{24} + \frac{155\cosh(7t)}{168} \end{bmatrix} \\
&= \begin{bmatrix} \frac{35\left(\left(-\frac{527\cosh(5t)}{35} + \frac{527\cosh(7t)}{49} + \frac{527\sinh(5t)}{35} - \frac{527\sinh(7t)}{49}\right)e^{24t} + \sinh(17t) - \frac{17\sinh(19t)}{19} + \cosh(17t) - \frac{17\cosh(19t)}{19}\right)e^{-18t}}{408} \\ -\frac{133\left(\frac{527\left(\cosh(5t) - \frac{5\cosh(7t)}{7} - \sinh(5t) + \frac{5\sinh(7t)}{7}\right)e^{24t}}{133} + \sinh(17t) - \frac{17\sinh(19t)}{19} + \cosh(17t) - \frac{17\cosh(19t)}{19}\right)e^{-18t}}{408} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -\frac{5c_1 e^{-18t}}{19} \\ c_1 e^{-18t} \end{bmatrix} + \begin{bmatrix} c_2 e^{6t} \\ c_2 e^{6t} \end{bmatrix} + \begin{bmatrix} \frac{35\left(\left(-\frac{527\cosh(5t)}{35} + \frac{527\cosh(7t)}{49} + \frac{527\sinh(5t)}{35} - \frac{527\sinh(7t)}{49}\right)e^{24t} + \sinh(17t) - \frac{17\sinh(1}{19}}{408} \\ -\frac{133\left(\frac{527\left(\cosh(5t) - \frac{5\cosh(7t)}{7} - \sinh(5t) + \frac{5\sinh(7t)}{7}\right)e^{24t}}{133} + \sinh(17t) - \frac{17\sinh(19t)}{19} +}{408} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{35\,e^{-18t}\left(\left(\frac{408c_2}{35} - \frac{527\cosh(5t)}{35} + \frac{527\cosh(7t)}{49} + \frac{527\sinh(5t)}{35} - \frac{527\sinh(7t)}{49}\right)e^{24t} + \cosh(17t) - \frac{17\cosh(19t)}{19} + \sinh(17t) - \frac{17\sinh(19t)}{19} -}{408} \\ -\frac{133\left(\frac{17\left(-24c_2 + 31\cosh(5t) - \frac{155\cosh(7t)}{7} - 31\sinh(5t) + \frac{155\sinh(7t)}{7}\right)e^{24t}}{133} + \cosh(17t) - \frac{17\cosh(19t)}{19} + \sinh(17t) - \frac{17\sinh(19t)}{19} - \frac{408}{13}}{408} \end{bmatrix}$$

### 8.1.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + 5y + 10\sinh(t), y' = 19x(t) - 13y + 24\sinh(t)]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y \end{array}\right]$$

- Convert system into a vector equation

$$\vec{x}'(t) = \left[\begin{array}{cc} 1 & 5 \\ 19 & -13 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 10\sinh(t) \\ 24\sinh(t) \end{array}\right]$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} 1 & 5 \\ 19 & -13 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 10\sinh(t) \\ 24\sinh(t) \end{array}\right]$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} 10\sinh(t) \\ 24\sinh(t) \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 1 & 5 \\ 19 & -13 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[-18, \left[\begin{array}{c} -\frac{5}{19} \\ 1 \end{array}\right]\right], \left[6, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-18, \left[\begin{array}{c} -\frac{5}{19} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^{-18t} \cdot \left[\begin{array}{c} -\frac{5}{19} \\ 1 \end{array}\right]$$

- Consider eigenpair

  $$\left[6, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{6t} \cdot \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $$\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \overrightarrow{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \left[\begin{array}{cc} -\frac{5\,\mathrm{e}^{-18t}}{19} & \mathrm{e}^{6t} \\ \mathrm{e}^{-18t} & \mathrm{e}^{6t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where $I$ is th

  $$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \left[\begin{array}{cc} -\frac{5\,\mathrm{e}^{-18t}}{19} & \mathrm{e}^{6t} \\ \mathrm{e}^{-18t} & \mathrm{e}^{6t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -\frac{5}{19} & 1 \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \left[\begin{array}{cc} \frac{(19\,\mathrm{e}^{24t}+5)\mathrm{e}^{-18t}}{24} & \frac{5(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} \\ \frac{19(\mathrm{e}^{24t}-1)\mathrm{e}^{-18t}}{24} & \frac{(5\,\mathrm{e}^{24t}+19)\mathrm{e}^{-18t}}{24} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$$

- Take the derivative of the particular solution

  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \left[\begin{array}{c} \frac{35\,\mathrm{e}^{-18t}\left(\left(-\frac{527\cosh(5t)}{35}+\frac{527\cosh(7t)}{49}+\frac{527\sinh(5t)}{35}-\frac{527\sinh(7t)}{49}+\frac{1054}{245}\right)\mathrm{e}^{24t}+\sinh(17t)-\frac{17\sinh(19t)}{19}+\cosh(17t)-\frac{17\cos}{1} \right.}{408} \\ -\frac{133\,\mathrm{e}^{-18t}\left(-\frac{2}{19}+\frac{527\left(-\frac{2}{7}+\cosh(5t)-\frac{5\cosh(7t)}{7}-\sinh(5t)+\frac{5\sinh(7t)}{7}\right)\mathrm{e}^{24t}}{133}+\sinh(17t)-\frac{17\sinh(19t)}{19}+\cosh(17t)-\frac{17\,\mathrm{co}}{} \right.}{408} \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{35\,\mathrm{e}^{-18t}\left(\left(-\frac{527\cosh(5t)}{35}+\frac{527\cosh(7t)}{49}+\frac{527\sinh(5t)}{35}-\frac{527\sinh(7t)}{49}+\frac{1054}{245}\right)\mathrm{e}^{24t}+\sinh(17t)-\frac{17\sinh(19t}{19} \right.}{408} \\ -\frac{133\,\mathrm{e}^{-18t}\left(-\frac{2}{19}+\frac{527\left(-\frac{2}{7}+\cosh(5t)-\frac{5\cosh(7t)}{7}-\sinh(5t)+\frac{5\sinh(7t)}{7}\right)\mathrm{e}^{24t}}{133}+\sinh(17t)-\frac{17\sinh(19}{19} \right.}{408} \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y \end{array}\right] = \left[\begin{array}{c} \frac{35\left(\left(\frac{408c_2}{35}-\frac{527\cosh(5t)}{35}+\frac{527\cosh(7t)}{49}+\frac{527\sinh(5t)}{35}-\frac{527\sinh(7t)}{49}+\frac{1054}{245}\right)\mathrm{e}^{24t}+\cosh(17t)-\frac{17\cosh(19t)}{19}+\sinh(17t) \right.}{408} \\ -\frac{133\left(-\frac{2}{19}+\frac{17\left(-\frac{62}{7}-24c_2+31\cosh(5t)-\frac{155\cosh(7t)}{7}-31\sinh(5t)+\frac{155\sinh(7t)}{7}\right)\mathrm{e}^{24t}}{133}+\cosh(17t)-\frac{17\cosh(19t)}{19}+\sinh \right.}{408} \end{array}\right.$$

- Solution to the system of ODEs

$$\begin{cases} x(t) = \frac{35\left(\left(\frac{408c_2}{35}-\frac{527\cosh(5t)}{35}+\frac{527\cosh(7t)}{49}+\frac{527\sinh(5t)}{35}-\frac{527\sinh(7t)}{49}+\frac{1054}{245}\right)e^{24t}+\cosh(17t)-\frac{17\cosh(19t)}{19}+\sinh(17t)-\frac{17\sinh(1}{19}}{408} \end{cases}$$

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 136

```
dsolve([diff(x(t),t)=x(t)+5*y(t)+10*sinh(t),diff(y(t),t)=19*x(t)-13*y(t)+24*sinh(t)],singsol
```

$$x(t) = e^{-18t}c_2 + e^{6t}c_1 + \frac{5\,e^{-18t}\left(\left(-\frac{221\cosh(5t)}{60}+\frac{17\cosh(7t)}{7}+\frac{221\sinh(5t)}{60}-\frac{17\sinh(7t)}{7}\right)e^{24t}+\sinh(17t)-\frac{221\sinh(19t)}{228}+\cosh(17t)-}{17}$$

$$y(t) = -\frac{2\cosh(7t)\,e^{6t}}{7}+\frac{2\sinh(7t)\,e^{6t}}{7}-\frac{2\,e^{-18t}\sinh(17t)}{17} - \frac{2\,e^{-18t}\cosh(17t)}{17}-\frac{19\,e^{-18t}c_2}{5}+e^{6t}c_1-2\sinh(t)$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 108

```
DSolve[{x'[t]==x[t]+5*y[t]+10*Sinh[t],y'[t]==19*x[t]-13*y[t]+24*Sinh[t]},{x[t],y[t]},t,Inclu
```

$$x(t) \to \frac{120e^{-t}}{119}-\frac{26e^t}{19}+\frac{5}{24}(c_1-c_2)e^{-18t}+\frac{1}{24}(19c_1+5c_2)e^{6t}$$
$$y(t) \to \frac{71e^{-t}}{119}-e^t-\frac{19}{24}(c_1-c_2)e^{-18t}+\frac{1}{24}(19c_1+5c_2)e^{6t}$$

## 8.2 problem Problem 1(b)



Internal problem ID [12393]
Internal file name [`OUTPUT/11045_Wednesday_October_04_2023_01_27_34_AM_8872665/index.tex`]

**Book**: APPLIED DIFFERENTIAL EQUATIONS The Primary Course by Vladimir A. Dobrushkin. CRC Press 2015
**Section**: Chapter 8.4 Systems of Linear Differential Equations (Method of Undetermined Coefficients). Problems page 520
**Problem number**: Problem 1(b).
**ODE order**: 1.
**ODE degree**: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 9x(t) - 3y - 6t \\
y' &= -x(t) + 11y + 10t
\end{aligned}
$$

### 8.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential $e^{At}$ allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} -6t \\ 10t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix $A$, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{8t}}{4} + \frac{e^{12t}}{4} & -\frac{3\,e^{12t}}{4} + \frac{3\,e^{8t}}{4} \\ -\frac{e^{12t}}{4} + \frac{e^{8t}}{4} & \frac{e^{8t}}{4} + \frac{3\,e^{12t}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{3\,e^{8t}}{4} + \frac{e^{12t}}{4} & -\frac{3\,e^{12t}}{4} + \frac{3\,e^{8t}}{4} \\ -\frac{e^{12t}}{4} + \frac{e^{8t}}{4} & \frac{e^{8t}}{4} + \frac{3\,e^{12t}}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{3\,e^{8t}}{4} + \frac{e^{12t}}{4}\right) c_1 + \left(-\frac{3\,e^{12t}}{4} + \frac{3\,e^{8t}}{4}\right) c_2 \\ \left(-\frac{e^{12t}}{4} + \frac{e^{8t}}{4}\right) c_1 + \left(\frac{e^{8t}}{4} + \frac{3\,e^{12t}}{4}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(c_1 - 3c_2)e^{12t}}{4} + \frac{3\,e^{8t}(c_1 + c_2)}{4} \\ \frac{(-c_1 + 3c_2)e^{12t}}{4} + \frac{e^{8t}(c_1 + c_2)}{4} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{e^{-12t}(1+3\,e^{4t})}{4} & \frac{3\,e^{-12t}(e^{4t}-1)}{4} \\ \frac{e^{-12t}(e^{4t}-1)}{4} & \frac{e^{-12t}(e^{4t}+3)}{4} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{3\,e^{8t}}{4} + \frac{e^{12t}}{4} & -\frac{3\,e^{12t}}{4} + \frac{3\,e^{8t}}{4} \\ -\frac{e^{12t}}{4} + \frac{e^{8t}}{4} & \frac{e^{8t}}{4} + \frac{3\,e^{12t}}{4} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-12t}(1+3\,e^{4t})}{4} & \frac{3\,e^{-12t}(e^{4t}-1)}{4} \\ \frac{e^{-12t}(e^{4t}-1)}{4} & \frac{e^{-12t}(e^{4t}+3)}{4} \end{bmatrix} \begin{bmatrix} -6t \\ 10t \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{3\,e^{8t}}{4} + \frac{e^{12t}}{4} & -\frac{3\,e^{12t}}{4} + \frac{3\,e^{8t}}{4} \\ -\frac{e^{12t}}{4} + \frac{e^{8t}}{4} & \frac{e^{8t}}{4} + \frac{3\,e^{12t}}{4} \end{bmatrix} \begin{bmatrix} \frac{(1+12t)e^{-12t}}{16} + \frac{3(-1-8t)e^{-8t}}{64} \\ \frac{(-48t-4)e^{-12t}}{64} + \frac{(-1-8t)e^{-8t}}{64} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3t}{8} + \frac{1}{64} \\ -\frac{7t}{8} - \frac{5}{64} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(c_1-3c_2)e^{12t}}{4} + \frac{3\,e^{8t}(c_1+c_2)}{4} + \frac{3t}{8} + \frac{1}{64} \\ \frac{(-c_1+3c_2)e^{12t}}{4} + \frac{e^{8t}(c_1+c_2)}{4} - \frac{7t}{8} - \frac{5}{64} \end{bmatrix}$$

### 8.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y' \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} x(t) \\ y \end{bmatrix} + \begin{bmatrix} -6t \\ 10t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of $A$. This is done by solving the following equation for the eigenvalues $\lambda$

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & -3 \\ -1 & 11-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 20\lambda + 96 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 12$$
$$\lambda_2 = 8$$

This table summarises the above result

| eigenvalue | algebraic multiplicity | type of eigenvalue |
|---|---|---|
| 8 | 1 | real eigenvalue |
| 12 | 1 | real eigenvalue |

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 8$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} - (8)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ -1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 12$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left( \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} - (12) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity $m$, and its geometric multiplicity $k$ and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity $k$) does not equal the algebraic multiplicity $m$, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

| | multiplicity | | | |
|---|---|---|---|---|
| eigenvalue | algebraic $m$ | geometric $k$ | defective? | eigenvectors |
| 12 | 1 | 1 | No | $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ |
| 8 | 1 | 1 | No | $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ |

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 12 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{12t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{12t} \end{aligned}$$

Since eigenvalue 8 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{8t}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{8t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = c_1 \begin{bmatrix} -e^{12t} \\ e^{12t} \end{bmatrix} + c_2 \begin{bmatrix} 3\,e^{8t} \\ e^{8t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -e^{12t} & 3\,e^{8t} \\ e^{12t} & e^{8t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{e^{-12t}}{4} & \frac{3\,e^{-12t}}{4} \\ \frac{e^{-8t}}{4} & \frac{e^{-8t}}{4} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{12t} & 3\,e^{8t} \\ e^{12t} & e^{8t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-12t}}{4} & \frac{3\,e^{-12t}}{4} \\ \frac{e^{-8t}}{4} & \frac{e^{-8t}}{4} \end{bmatrix} \begin{bmatrix} -6t \\ 10t \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{12t} & 3\,e^{8t} \\ e^{12t} & e^{8t} \end{bmatrix} \int \begin{bmatrix} 9\,e^{-12t}t \\ e^{-8t}t \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{12t} & 3\,e^{8t} \\ e^{12t} & e^{8t} \end{bmatrix} \begin{bmatrix} -\frac{(1+12t)e^{-12t}}{16} \\ -\frac{(8t+1)e^{-8t}}{64} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3t}{8} + \frac{1}{64} \\ -\frac{7t}{8} - \frac{5}{64} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 e^{12t} \\ c_1 e^{12t} \end{bmatrix} + \begin{bmatrix} 3c_2 e^{8t} \\ c_2 e^{8t} \end{bmatrix} + \begin{bmatrix} \frac{3t}{8} + \frac{1}{64} \\ -\frac{7t}{8} - \frac{5}{64} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} -c_1 e^{12t} + 3c_2 e^{8t} + \frac{3t}{8} + \frac{1}{64} \\ c_1 e^{12t} + c_2 e^{8t} - \frac{7t}{8} - \frac{5}{64} \end{bmatrix}$$

### 8.2.3 Maple step by step solution

Let's solve

$[x'(t) = 9x(t) - 3y - 6t, y' = -x(t) + 11y + 10t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -6t \\ 10t \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} 9 & -3 \\ -1 & 11 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} -6t \\ 10t \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(t) = \left[\begin{array}{c} -6t \\ 10t \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 9 & -3 \\ -1 & 11 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of $A$
- Eigenpairs of $A$

$$\left[\left[8, \left[\begin{array}{c} 3 \\ 1 \end{array}\right]\right], \left[12, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[8, \left[\begin{array}{c} 3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{8t} \cdot \left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[12, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{12t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t)=c_1\overrightarrow{x}_1+c_2\overrightarrow{x}_2+\overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t)=\left[\begin{array}{cc} 3\,\mathrm{e}^{8t} & -\mathrm{e}^{12t} \\ \mathrm{e}^{8t} & \mathrm{e}^{12t} \end{array}\right]$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where $I$ is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t)=\left[\begin{array}{cc} 3\,\mathrm{e}^{8t} & -\mathrm{e}^{12t} \\ \mathrm{e}^{8t} & \mathrm{e}^{12t} \end{array}\right]\cdot\frac{1}{\left[\begin{array}{cc} 3 & -1 \\ 1 & 1 \end{array}\right]}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t)=\left[\begin{array}{cc} \frac{3\,\mathrm{e}^{8t}}{4}+\frac{\mathrm{e}^{12t}}{4} & -\frac{3\,\mathrm{e}^{12t}}{4}+\frac{3\,\mathrm{e}^{8t}}{4} \\ -\frac{\mathrm{e}^{12t}}{4}+\frac{\mathrm{e}^{8t}}{4} & \frac{\mathrm{e}^{8t}}{4}+\frac{3\,\mathrm{e}^{12t}}{4} \end{array}\right]$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t)=\Phi(t)\cdot\overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p^{\prime}(t)=\Phi'(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)=A\cdot\Phi(t)\cdot\overrightarrow{v}(t)+\overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\overrightarrow{v}^{\prime}(t)=\overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{3t}{8} + \frac{1}{64} + \frac{3\,\mathrm{e}^{8t}}{64} - \frac{\mathrm{e}^{12t}}{16} \\ \frac{\mathrm{e}^{12t}}{16} - \frac{7t}{8} - \frac{5}{64} + \frac{\mathrm{e}^{8t}}{64} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} \frac{3t}{8} + \frac{1}{64} + \frac{3\,\mathrm{e}^{8t}}{64} - \frac{\mathrm{e}^{12t}}{16} \\ \frac{\mathrm{e}^{12t}}{16} - \frac{7t}{8} - \frac{5}{64} + \frac{\mathrm{e}^{8t}}{64} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \frac{(-64c_2-4)\mathrm{e}^{12t}}{64} + \frac{(192c_1+3)\mathrm{e}^{8t}}{64} + \frac{3t}{8} + \frac{1}{64} \\ \frac{(64c_2+4)\mathrm{e}^{12t}}{64} + \frac{(1+64c_1)\mathrm{e}^{8t}}{64} - \frac{7t}{8} - \frac{5}{64} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-64c_2-4)\mathrm{e}^{12t}}{64} + \frac{(192c_1+3)\mathrm{e}^{8t}}{64} + \frac{3t}{8} + \frac{1}{64}, y = \frac{(64c_2+4)\mathrm{e}^{12t}}{64} + \frac{(1+64c_1)\mathrm{e}^{8t}}{64} - \frac{7t}{8} - \frac{5}{64}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
dsolve([diff(x(t),t)=9*x(t)-3*y(t)-6*t,diff(y(t),t)=-x(t)+11*y(t)+10*t],singsol=all)
```

$$x(t) = c_2\mathrm{e}^{8t} + \mathrm{e}^{12t}c_1 + \frac{3t}{8} + \frac{1}{64}$$

$$y(t) = \frac{c_2\mathrm{e}^{8t}}{3} - \mathrm{e}^{12t}c_1 - \frac{5}{64} - \frac{7t}{8}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 78

```
DSolve[{x'[t]==9*x[t]-3*y[t]-6*t,y'[t]==-x[t]+11*y[t]+10*t},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to \frac{1}{64}\left(24t + 16(c_1 - 3c_2)e^{12t} + 48(c_1 + c_2)e^{8t} + 1\right)$$
$$y(t) \to \frac{1}{64}\left(-56t - 16(c_1 - 3c_2)e^{12t} + 16(c_1 + c_2)e^{8t} - 5\right)$$