

Using Simulink to analyze 2 degrees of freedom system

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Abstract

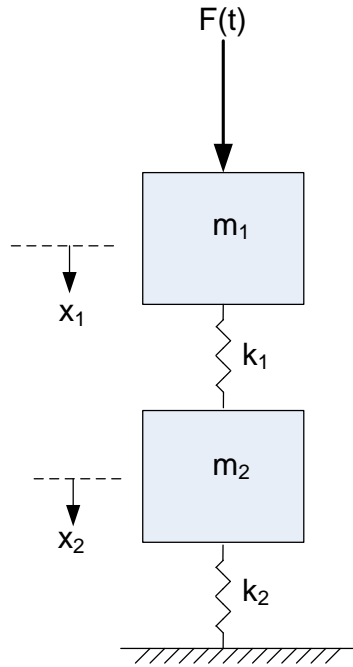
A two degrees of freedom system consisting of two masses connected by springs and subject to 3 different type of input forces is analyzed and simulated using Simulink

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1 Introduction and Theory

The system that is being analyzed is shown in the following diagram



In the above, $F(t)$ is to be taken as each of the following

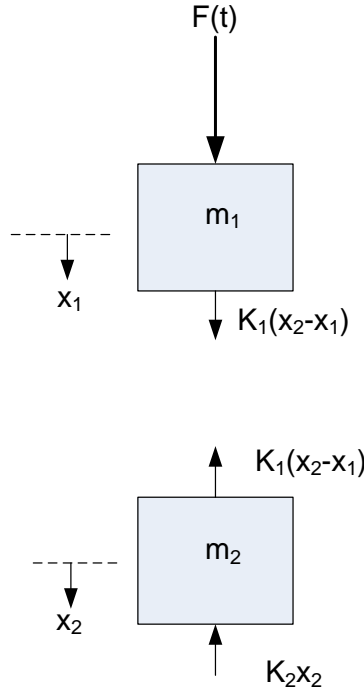
1. Unit impulse force.
2. Unit step force.
3. $\sin \omega t$

It is required to find $x_1(t)$ and $x_2(t)$ analytically and then to use Matlab's Simulink software for the analysis.

The mathematical model of the system is first developed and the equation of motions obtained using Lagrangian formulation then the analytical solution is found by solving the resulting coupled second order differential equations for m_1 and m_2 . Next, a simulink model is developed to implement the differential equations and the output $x_1(t)$ and $x_2(t)$ from Simulink is shown and compared to the output from the analytical solution.

2 Analytical solution

The following is the free body diagram of the above system



Assuming positive is downwards and that $x_2 > x_1$, force-balance equations for m_1 results in

$$m_1 \ddot{x}_1 = F(t) + k_1(x_2 - x_1)$$

And force-balance equations for m_2 results in

$$m_2 \ddot{x}_2 = -k_1(x_2 - x_1) - k_2x_2$$

Hence the EQM for the system become

$$\begin{aligned} m_1 \ddot{x}_1 - k_1x_2 + k_1x_1 &= F(t) \\ m_2 \ddot{x}_2 + (k_1 + k_2)x_2 - k_1x_1 &= 0 \end{aligned}$$

Or in matrix form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F(t) \\ 0 \end{pmatrix}$$

The above can be written in matrix form as

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}$$

Where $\ddot{\mathbf{x}}, \mathbf{x}, \mathbf{F}$ are 2 by 1 vectors and M and K are the mass and stiffness matrices. The solution to the above is

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \quad (1)$$

2.1 Finding the homogenous solution

We start by finding \mathbf{x}_h from the following

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now assume $x_1(t) = A_1 \cos(\omega t + \phi)$ and $x_2(t) = A_2 \cos(\omega t + \phi)$, hence $\dot{x}_1(t) = -\omega A_1 \sin(\omega t + \phi)$ and $\dot{x}_2(t) = -\omega A_2 \sin(\omega t + \phi)$ and $\ddot{x}_1(t) = -\omega^2 A_1 \cos(\omega t + \phi)$ and $\ddot{x}_2(t) = -\omega^2 A_2 \cos(\omega t + \phi)$. Substituting the above values in the above system results in

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} -\omega^2 A_1 \cos(\omega t + \phi) \\ -\omega^2 A_2 \cos(\omega t + \phi) \end{pmatrix} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{pmatrix} \begin{pmatrix} A_1 \cos(\omega t + \phi) \\ A_2 \cos(\omega t + \phi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Divide by $\cos(\omega t + \phi)$ since not zero (else no solution exist) we obtain

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} -\omega^2 A_1 \\ -\omega^2 A_2 \end{pmatrix} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rewrite the above as

$$\begin{aligned} \begin{pmatrix} -\omega^2 m_1 A_1 \\ -\omega^2 m_2 A_2 \end{pmatrix} + \begin{pmatrix} k_1 A_1 - k_1 A_2 \\ -k_1 A_1 + (k_1 + k_2) A_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\omega^2 m_1 A_1 + k_1 A_1 - k_1 A_2 \\ -\omega^2 m_2 A_2 - k_1 A_1 + (k_1 + k_2) A_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} (-\omega^2 m_1 + k_1) A_1 - k_1 A_2 \\ -\omega^2 m_2 A_2 + (k_1 + k_2) A_2 - k_1 A_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -\omega^2 m_2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \quad (2)$$

From the last equation above, we see that to obtain a solution we must have

$$\begin{vmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -\omega^2 m_2 + (k_1 + k_2) \end{vmatrix} = 0$$

since if we had $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$ then no solution will exist. Therefore, taking the determinant and setting it to zero results in

$$\begin{aligned} (-\omega^2 m_1 + k_1)(-\omega^2 m_2 + (k_1 + k_2)) - k_1^2 &= 0 \\ k_1^2 + k_1 k_2 - \omega^2 k_1 m_1 - \omega^2 k_1 m_2 - \omega^2 k_2 m_1 + \omega^4 m_1 m_2 - k_1^2 &= 0 \\ \omega^4 m_1 m_2 + \omega^2(-k_1 m_1 - k_2 m_1 - k_1 m_2) + (k_1^2 + k_1 k_2 - k_1^2) &= 0 \end{aligned}$$

Let $\omega^4 = \lambda^2$, hence the above becomes

$$\lambda^2 m_1 m_2 + \lambda(-k_1 m_1 - k_2 m_1 - k_1 m_2) + (k_1^2 + k_1 k_2 - k_1^2) = 0$$

Solving for λ gives

$$\begin{aligned} \lambda_1 &= \frac{1}{2m_1 m_2} \left(k_1 m_1 + k_1 m_2 + k_2 m_1 + \sqrt{k_1^2 m_1^2 + k_1^2 m_2^2 + k_2^2 m_1^2 + 2k_1 k_2 m_1^2 + 2k_1^2 m_1 m_2 - 2k_1 k_2 m_1 m_2} \right) \\ \lambda_2 &= \frac{1}{2m_1 m_2} \left(k_1 m_1 + k_1 m_2 + k_2 m_1 - \sqrt{k_1^2 m_1^2 + k_1^2 m_2^2 + k_2^2 m_1^2 + 2k_1 k_2 m_1^2 + 2k_1^2 m_1 m_2 - 2k_1 k_2 m_1 m_2} \right) \end{aligned} \quad (3)$$

For each of the above solutions, we obtain a different $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ from equation (2) as follows

For λ_1 , (2) becomes

$$\begin{pmatrix} -\lambda_1 m_1 + k_1 & -k_1 \\ -k_1 & -\lambda_1 m_2 + (k_1 + k_2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (-\lambda_1 m_1 + k_1) A_1 - k_1 A_2 \\ -k_1 A_1 - \lambda_1 m_2 A_2 + (k_1 + k_2) A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first equation above, we have

$$(-\lambda_1 m_1 + k_1) A_1 - k_1 A_2 = 0$$

$$\frac{-\lambda_1 m_1 + k_1}{k_1} = \left(\frac{A_2}{A_1} \right)^{(1)}$$

Similarly for λ_2 ,

$$\frac{-\lambda_2 m_1 + k_1}{k_1} = \left(\frac{A_2}{A_1} \right)^{(2)}$$

Let

$$r_1 = \frac{-\lambda_1 m_1 + k_1}{k_1} = \left(\frac{A_2}{A_1} \right)^{(1)} \quad (4)$$

$$r_2 = \frac{-\lambda_2 m_1 + k_1}{k_1} = \left(\frac{A_2}{A_1} \right)^{(2)}$$

Hence now \mathbf{x}_h can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_h = \begin{pmatrix} A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2) \\ A_2^{(1)} \cos(\omega_1 t + \phi_1) + A_2^{(2)} \cos(\omega_2 t + \phi_2) \end{pmatrix}$$

But $A_2^{(1)} = r_1 A_1^{(1)}$ and $A_2^{(2)} = r_2 A_1^{(2)}$, hence the above becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_h = \begin{pmatrix} A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_1 A_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2) \end{pmatrix} \quad (5)$$

Now, given numerical values for k_1, k_2, m_1, m_2 we can find ω_1, ω_2 from (3) above, and next find r_1, r_2 from (4). Hence (5) contains 4 unknowns, $A_1^{(1)}, A_1^{(2)}, \phi_1, \phi_2$ which now can be found from initial conditions (after we find the particular solution) which we will now proceed to do.

2.2 Finding particular solutions

There are 3 different $F(t)$ which we are asked to consider

1. Unit impulse force.
2. Unit step force.
3. $\sin \omega t$

For each of the above, we find \mathbf{x}_p and then add it to \mathbf{x}_h found above in (5) to obtain (1).

2.2.1 Finding the particular solution for unit impulse input

Using the standard response for a unit impulse which for a single degree of freedom system is $x(t) = \frac{1}{m\omega_n} \sin \omega_n t$, then we write \mathbf{x}_p as

$$\mathbf{x}_p = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_p = \begin{pmatrix} \frac{1}{m\omega_1} \sin \omega_1 t + \frac{1}{m\omega_2} \sin \omega_2 t \\ 0 \end{pmatrix}$$

Hence, the general solution becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_1 A_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2) \end{pmatrix} + \begin{pmatrix} \frac{1}{m\omega_1} \sin \omega_1 t + \frac{1}{m\omega_2} \sin \omega_2 t \\ 0 \end{pmatrix} \quad (6)$$

2.2.2 Finding the particular solution for unit step input

Since unit step is 1 for $t > 0$, then, using convolution we write

$$\begin{aligned} x_p(t) &= \int_0^t f(\tau) h(t-\tau) d\tau \\ &= \int_0^t h(t-\tau) d\tau \\ &= \int_0^t \frac{1}{m\omega_n} \sin \omega_n(t-\tau) d\tau \\ &= \frac{1}{m\omega_n} \left[\frac{-\cos(\omega_n(t-\tau))}{-\omega_n} \right]_0^t \\ &= \frac{1}{m\omega_n^2} [\cos(\omega_n(t-\tau))]_0^t \\ &= \frac{1}{m\omega_n^2} [1 - \cos(\omega_n t)] \end{aligned}$$

Then, since now we have 2 natural frequencies, we can write \mathbf{x}_p as

$$\mathbf{x}_p = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_p = \begin{pmatrix} \frac{1}{m\omega_1^2} [1 - \cos(\omega_1 t)] + \frac{1}{m\omega_2^2} [1 - \cos(\omega_2 t)] \\ 0 \end{pmatrix}$$

Hence, the general solution becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_1 A_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2) \end{pmatrix} + \begin{pmatrix} \frac{1}{m\omega_1^2} [1 - \cos(\omega_1 t)] + \frac{1}{m\omega_2^2} [1 - \cos(\omega_2 t)] \\ 0 \end{pmatrix}$$

2.2.3 Finding the particular solution for $\sin \omega t$

In this case, we guess that $x_{1p} = c_1 \cos \omega t + c_2 \sin \omega t$, and since there is no forcing function being applied directly on m_2 then $x_{2p} = 0$ hence

$$\mathbf{x}_p = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}_p = \begin{pmatrix} c_1 \cos \omega t + c_2 \sin \omega t \\ 0 \end{pmatrix}$$

Then $\dot{x}_{1p}(t) = -\omega c_1 \sin \omega t + \omega c_2 \cos \omega t$ and $\ddot{x}_{1p}(t) = -\omega^2 c_1 \cos \omega t + \omega^2 c_2 \sin \omega t$ and now we substitute these into the original ODE for x_1 which is

$$\begin{aligned} m_1 \ddot{x}_1 - k_1 x_2 + k_1 x_1 &= F(t) \\ m_1 \ddot{x}_{1p} - k_1 x_{2p} + k_1 x_{1p} &= \sin \omega t \end{aligned}$$

We obtain the following

$$\begin{aligned} m_1 (-\omega^2 c_1 \cos \omega t + \omega^2 c_2 \sin \omega t) - k_1 (0) + k_1 (c_1 \cos \omega t + c_2 \sin \omega t) &= \sin \omega t \\ \cos \omega t (-\omega^2 c_1 m_1 + k_1 c_1) + \sin \omega t (m_1 \omega^2 c_2 + k_1 c_2) &= \sin \omega t \end{aligned}$$

Hence by comparing coefficients, we obtain

$$\begin{aligned} -\omega^2 c_1 m_1 + k_1 c_1 &= 0 \\ m_1 \omega^2 c_2 + k_1 c_2 &= 1 \end{aligned}$$

or

$$\begin{aligned} c_1 (-\omega^2 m_1 + k_1) &= 0 \\ c_2 (m_1 \omega^2 + k_1) &= 1 \end{aligned}$$

c_1 must be zero since $k_1 - \omega^2 m_1 = 0$ only when $\omega = \omega_n$ and we assume that this is not the case here. Hence

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \frac{1}{(m_1 \omega^2 + k_1)} \end{aligned}$$

Therefore \mathbf{x}_p becomes

$$\mathbf{x}_p = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}_p = \begin{pmatrix} \frac{1}{(m_1 \omega^2 + k_1)} \sin \omega t \\ 0 \end{pmatrix}$$

And the general solution becomes

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_1 A_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2) \end{pmatrix} + \begin{pmatrix} \frac{1}{(m_1 \omega^2 + k_1)} \sin \omega t \\ 0 \end{pmatrix} \quad (8)$$

3 Simulink simulation and block diagrams

In simulink, we will directly solve the system from the original formulation

$$\begin{aligned} m_1 \ddot{x}_1 - k_1 x_2 + k_1 x_1 &= F(t) \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 &= 0 \end{aligned}$$

or

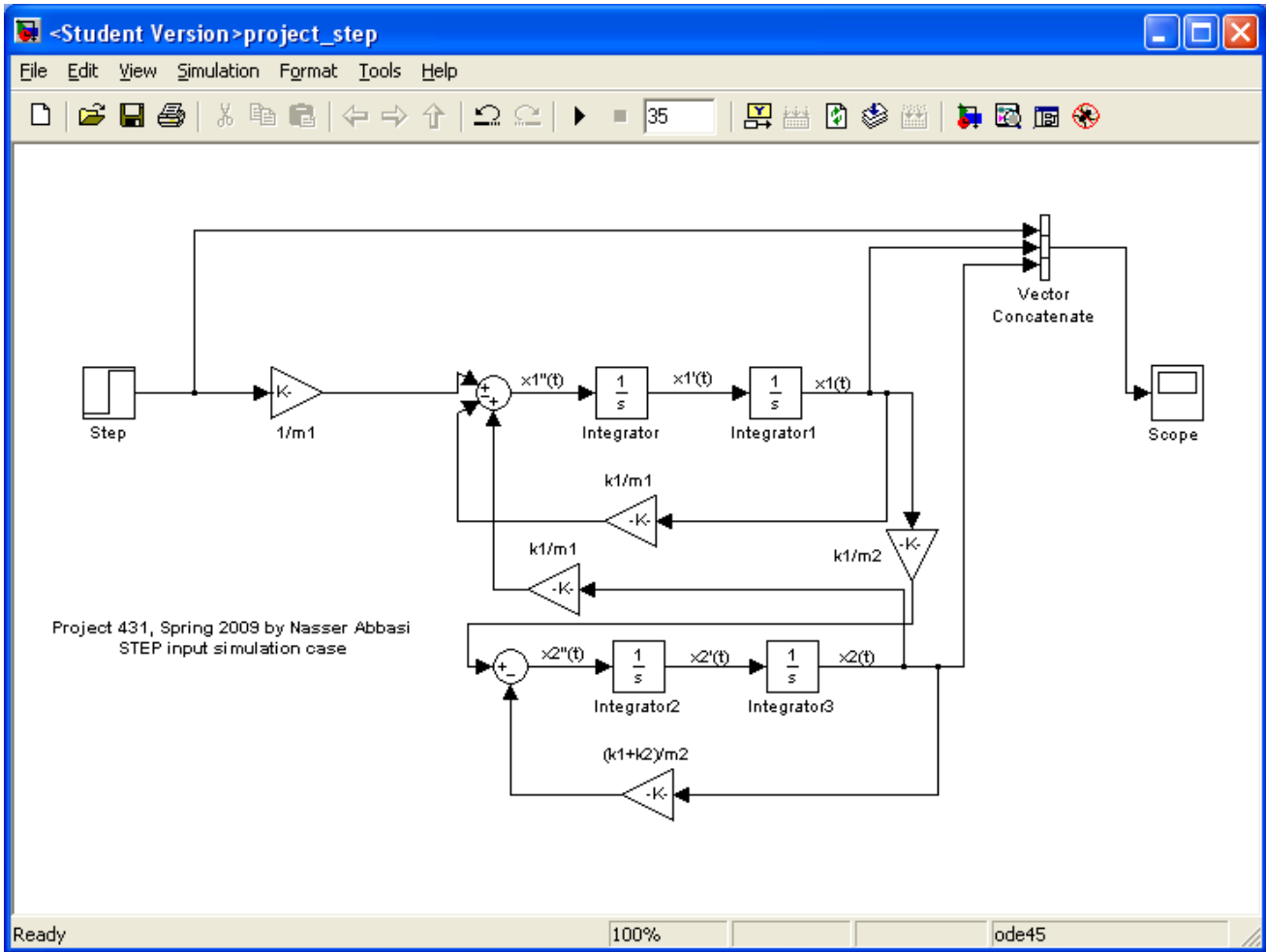
$$\begin{aligned} \ddot{x}_1 - \frac{k_1}{m_1} x_2 + \frac{k_1}{m_1} x_1 &= \frac{F(t)}{m_1} \\ \ddot{x}_2 + \frac{(k_1 + k_2)}{m_2} x_2 - \frac{k_1}{m_2} x_1 &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \ddot{x}_1 &= \frac{F(t)}{m_1} + \frac{k_1}{m_1} x_2 - \frac{k_1}{m_1} x_1 \\ \ddot{x}_2 &= -\frac{(k_1 + k_2)}{m_2} x_2 + \frac{k_1}{m_2} x_1 \end{aligned}$$

3.1 Unit step simulink diagram and output

The simulink block diagram will be as follows for the unit step input

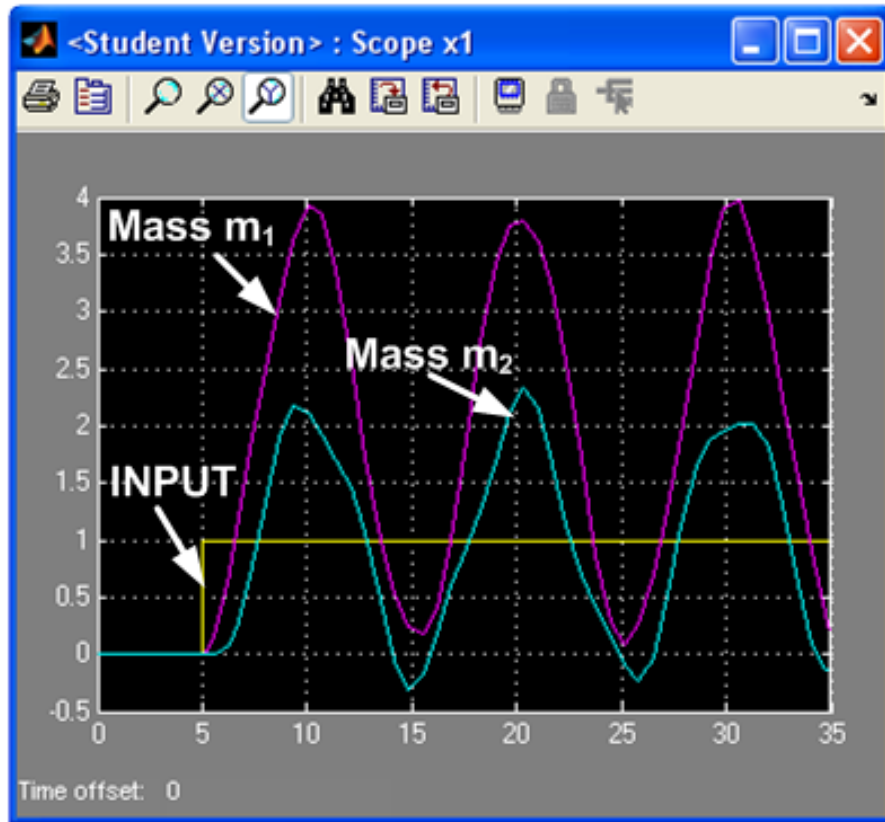


For an initial run with parameters $m_1 = m_2 = k_1 = k_2 = 1$ I get this warning below

```
EDU>> simulink
Warning: Using a default value of 0.2 for maximum step size. The simulation
step size will be equal to or less than this value. You can disable this
diagnostic by setting 'Automatic solver parameter selection' diagnostic to
'none' in the Diagnostics page of the configuration parameters dialog.
```

And this is the output for $x_1(t)$ and $x_2(t)$ for the unit step response

run with parameters $m_1 = m_2 = k_1 = k_2 = 1$



Solution to the following coupled second order differential equations when $F(t)$ is unit step

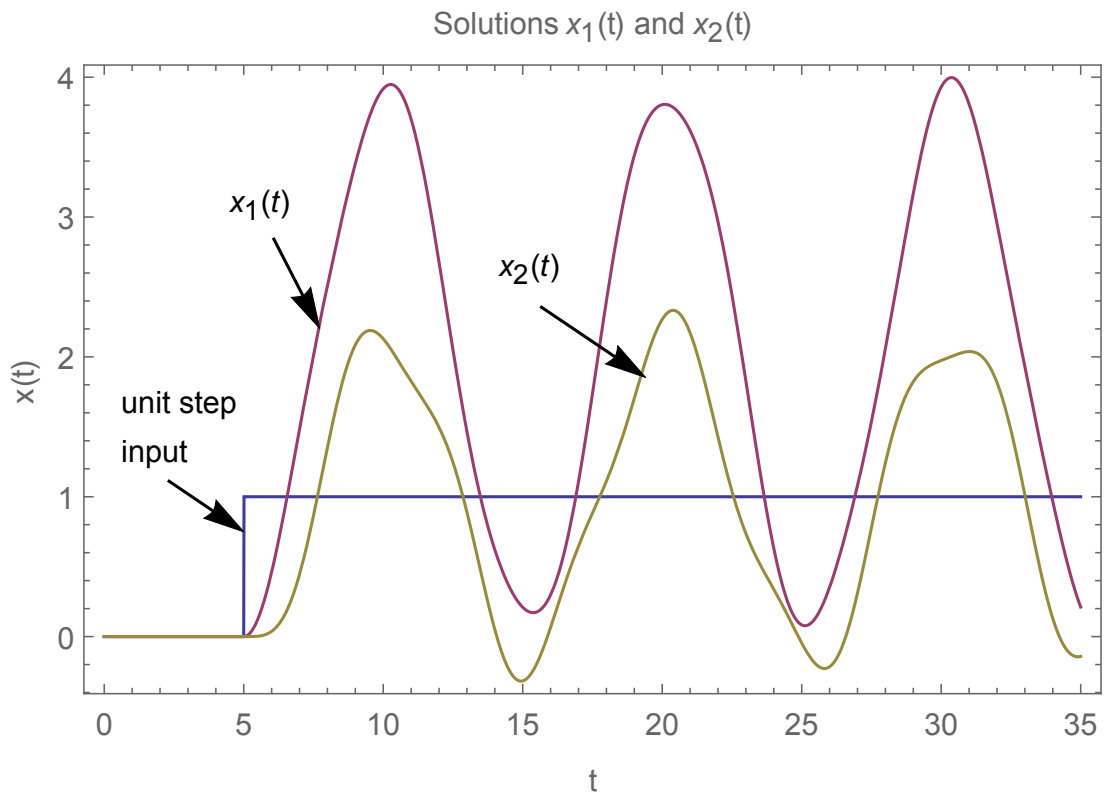
$$m_1 \ddot{x}_1 - k_1 x_2 + k_1 x_1 = F(t)$$

$$m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 = 0$$

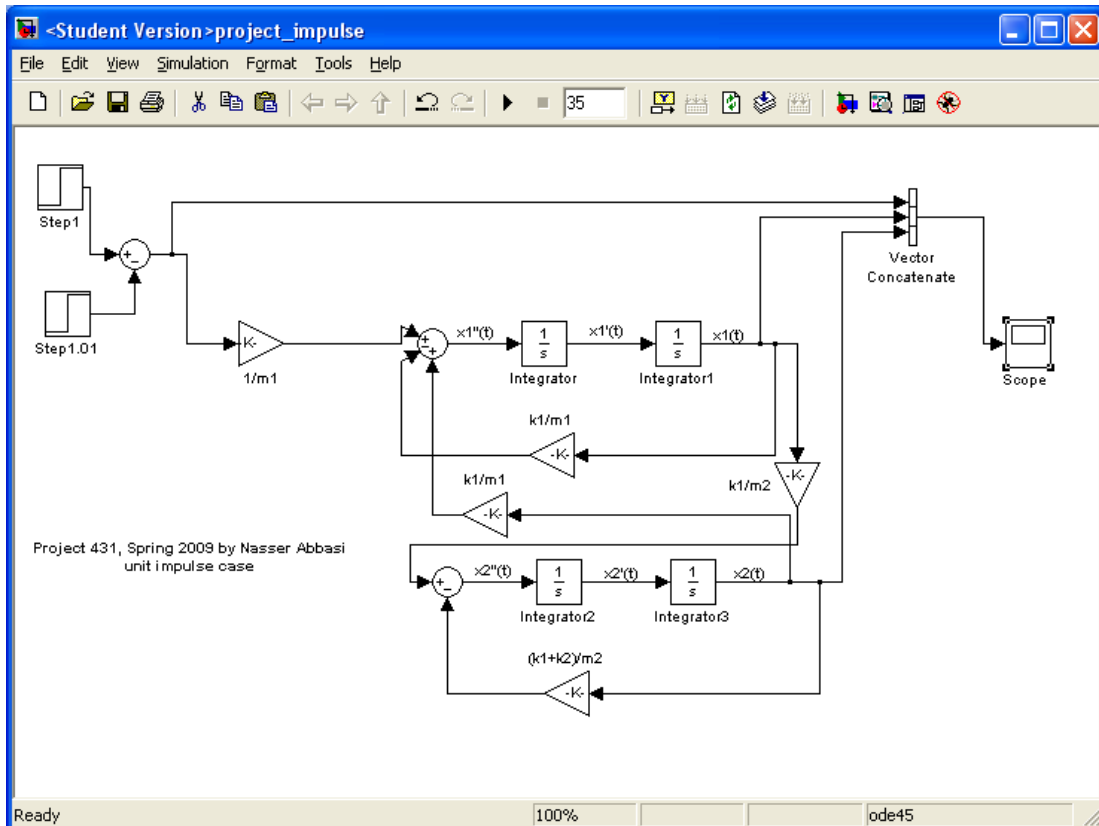
3.1.1 Verification of result from Simulink by Numerically solving the differential equations

To verify the above output from Simulink, I solved the same coupled differential equations for zero initial conditions numerically (using a numerical differential equation solver) and plotted the solution for $x_1(t)$ and $x_2(t)$ and the result matches that shown above by simulink. Here is the code the plot as a result of this verification

```
k1 = 1; k2 = 1; m1 = 1; m2 = 1;
eq1 = m1 x1''[t] - k1 x2[t] + k1 x1[t] == f[t]
eq2 = m2 x2''[t] + (k1 + k2) x2[t] - k1 x1[t] == 0
sol = NDSolve[{eq1, eq2, x1[0] == 0, x1'[0] == 0, x2[0] == 0, x2'[0] == 0}, {x1[t], x2[t]},
  {t, 0, 35}];
```

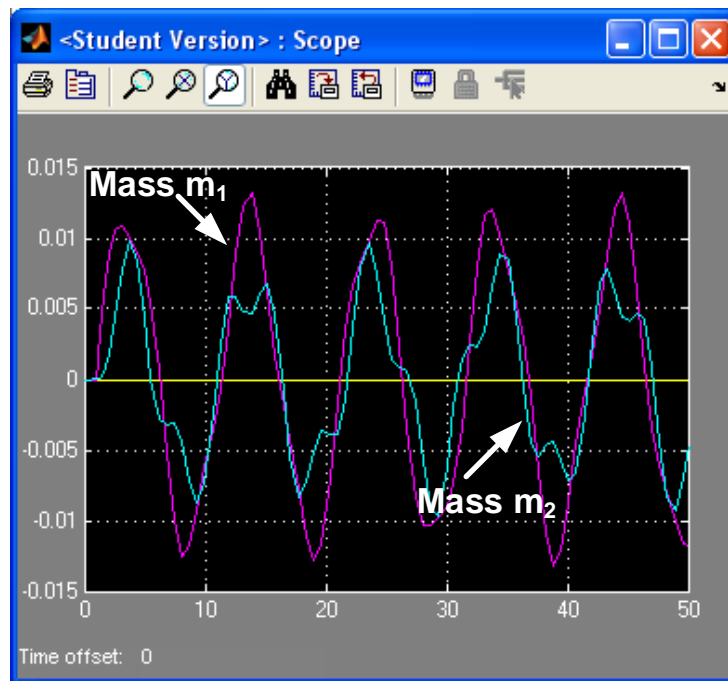


4 Unit impulse simulink diagram and output



And the output for $x_1(t)$ and $x_2(t)$ is as follows

run with parameters $m_1 = m_2 = k_1 = k_2 = 1$



Solution to the following coupled second order differential equations when $F(t)$ is a unit impulse

$$\begin{aligned} m_1 \ddot{x}_1 - k_1 x_2 + k_1 x_1 &= F(t) \\ m_2 \ddot{x}_2 + (k_1 + k_2) x_2 - k_1 x_1 &= 0 \end{aligned}$$

4.0.1 Verification of result from Simulink by Numerically solving the differential equations

To verify the above output from Simulink, The same coupled differential equations were solved numerically for zero initial conditions numerically and the solution plotted for $x_1(t)$ and $x_2(t)$ and the result was found to match that shown above by simulink. Here is the code used to do the verification

verification

```
f[x_] := Piecewise[{{0, x < 0}, {0, x > 0.01}, {1, True}}]
```

```
k1 = 1; k2 = 1; m1 = 1; m2 = 1;
```

```
eq1 = m1 x1''[t] - k1 x2[t] + k1 x1[t] == f[t]
```

```
eq2 = m2 x2''[t] + (k1 + k2) x2[t] - k1 x1[t] == 0
```

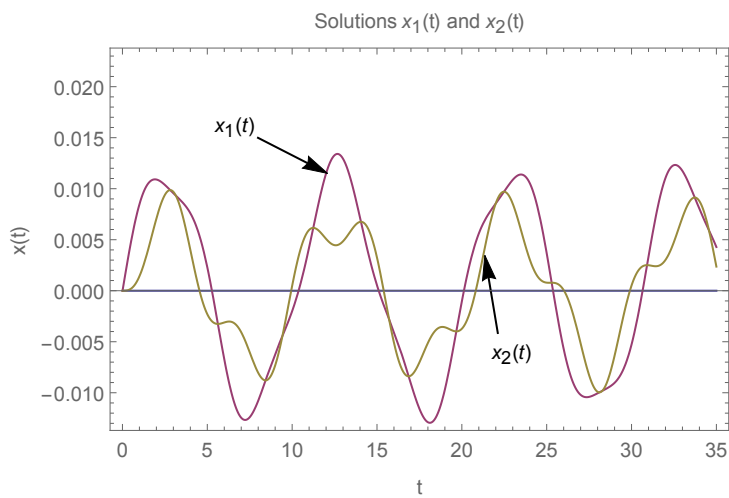
```
sol = NDSolve[{eq1, eq2, x1[0] == 0, x1'[0] == 0, x2[0] == 0, x2'[0] == 0}, {x1[t], x2[t]}, {t, 0, 35}];
```

```
sol = Chop[sol]
```

```
sol1 = x1[t] /. sol
```

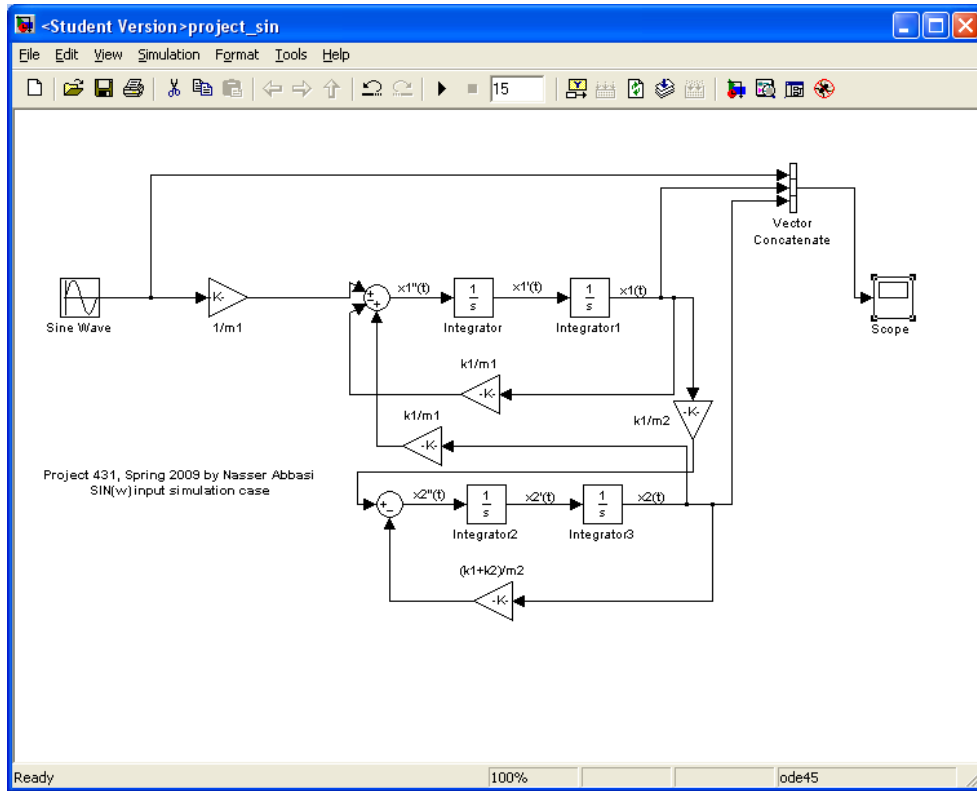
```
sol2 = x2[t] /. sol
```

```
Plot[Evaluate[{f[t], sol1, sol2}], {t, 0, 35}, Frame → True, FrameLabel → {"x(t)", None}, {"t", "Solutions x1(t) and x2(t)"}]
```



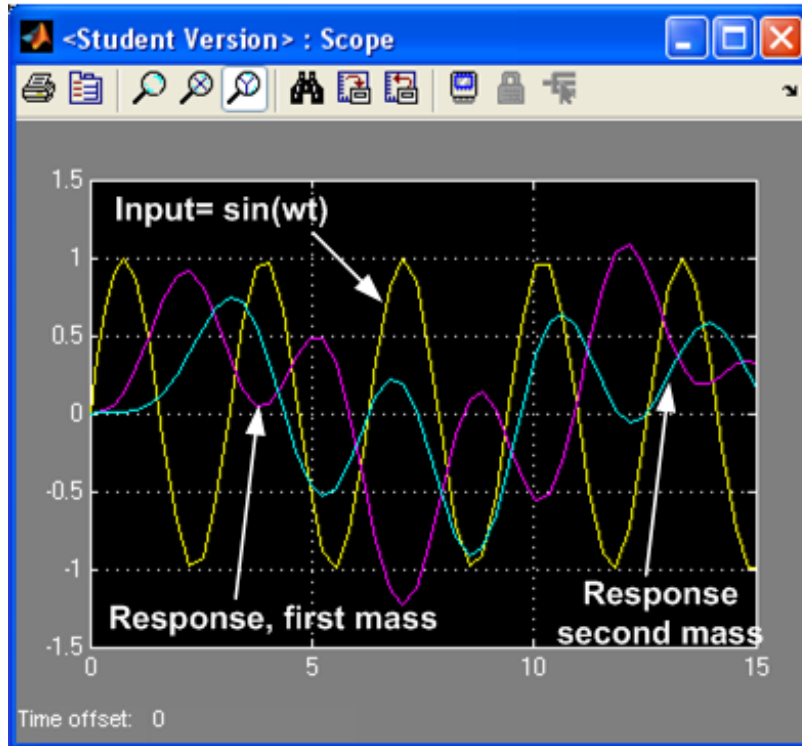
4.1 $\sin(\omega t)$ input simulink diagram and output

The simulink block diagram will be as follows for the $\sin \omega t$ input



For an initial run with parameters $m_1 = m_2 = k_1 = k_2 = 1$ this is the output for $x_1(t)$ and $x_2(t)$ and showing the input signal at the same time

Using $\sin(\omega t)$ as forcing function. Case for $\omega=2$ rad/sec



Simulink output for parameters: $m_1=1, m_2=1, k_1=1, k_2=1$

5 Discussion

A coupled system of two masses and springs was analyzed using Simulink. The simulation was done for one set of parameters (masses and stiffness). Simulink made the simulation of this system under different loading conditions easy to do. The 2 masses response were recorded using simulink scope and the signals captured on the same plot to make it easy to compare the response of the first mass to the second mass.

The analytical analysis was more time consuming than actually making the simulation in simulink. The ability to easily change different sources to the system was useful as well as the ability to change the frequency of the input and immediately see the effect on the response.

This was my first project using Simulink, and I can see that this tool will be useful to learn more as it allows one to easily analyze engineering problems.