Automated function to solve Ax=b showing all matrix transformations at each step

by Nasser M. Abbasi

Introduction

This function runs inside Mathematica. It accepts matrix A and vector b, and solves Ax=b for x, showing each step all the Gaussian eliminations steps made, all the way to the Reduced Echelon form. This works only on matrices with non-zero determinant. Examples below are shown from textbook Differential equations and linear Algebra, 3rd edition by Edwards and Penney.

```
displayRREF[mat_?(MatrixQ[#] &), b_?(VectorQ[#] &)] :=
  Module[{i, j, multiplier, pivot, augmented, m = Length@mat, lu, p, c, tmp, inverse},
   (*version 3/10/2017*)

   (*check if matrix is singular.Per Daniel Lichtblau post seen Wolfram site*)
   (*this is better method than using Det*)
   If[MatrixQ[mat] && MatrixRank[mat] == Length[mat] == Length[mat[[1]]] === False,
    Return["Sorry, but matrix is singular!"]
   ];

   If[Length@b ≠ Length@mat,
    Return["Size of b vector not the same as number of rows in A matrix"]
   ];

   {lu, p, c} = LUDecomposition[mat];
   tmp = lu SparseArray[{i_, j_} /; j ≥ i → 1, {Length@mat, Length@mat}];
   (*Print["Mathematica says Echelon form is ", MatrixForm@tmp];*)
   tmp = mat[[p, All]];
   (*Print["Mathematica says inverse Matrix is", MatrixForm@Inverse@mat];*)
   augmented = Join[mat, Transpose[{b}], 2];
   augmented = ArrayFlatten[{{augmented, IdentityMatrix[Length@tmp]}}];

   Print[">>>>>>Starting forward Gaussian elimination phase using ",
    augmented[[All, 1 ;; m + 1]] // matWithDiv[m + 1, Background → LightOrange] ,
    MatrixForm[augmented[[All, m + 2 ;;]]]];
   Do[
    Print["pivot now is (", pivot, ",", pivot, ")" ];
    Do[
     multiplier = augmented[[j, pivot]] / augmented[[pivot, pivot]];
     Print["will now zero out element (", j, ",", pivot,
      ") by subtracting ", multiplier, " times row ", pivot, " from row ", j];
     augmented[[j, pivot ;;]] = augmented[[j, pivot ;;]] -
       multiplier * augmented[[pivot, pivot ;;]];
     Print[augmented[[All, 1 ;; m + 1]] // matWithDiv[m + 1, Background → LightOrange] ,
```

```
    MatrixForm[augmented[[All, m + 2 ;;]]]]
   , {j, pivot + 1, m}
  ]
  , {pivot, 1, m}
 ];

 Print[">>>>>>Starting backward elimination phase"];

 Do[
  Do[
   multiplier = augmented[[j, pivot]] / augmented[[pivot, pivot]];
   Print["will now zero out element (", j, ",", pivot,
    ") by subtracting ", multiplier, " times row ", pivot, " from row ", j];
   augmented[[j, pivot ;;]] = augmented[[j, pivot ;;]] -
     multiplier * augmented[[pivot, pivot ;;]];
   Print[augmented[[All, 1 ;; m + 1]] // matWithDiv[m + 1, Background → LightOrange] ,
    MatrixForm[augmented[[All, m + 2 ;;]]]]
   , {j, 1, pivot - 1}
  ]
  , {pivot, 2, m}
 ];

 Print[">>>>>>Starting Final phase, convert reduced echelon to identity matrix"];
 Do[
  augmented[[j, ;;]] = augmented[[j, ;;]] / augmented[[j, j]]
  , {j, 1, m}
 ];

 Print[augmented[[All, 1 ;; m + 1]] // matWithDiv[m + 1, Background → LightOrange] ,
  MatrixForm[augmented[[All, m + 2 ;;]]]];
 (*flip at inverse Matrix now using p from LUDecomposition, but using column wise*)
 Print["Inverse Matrix is ", MatrixForm[ augmented[[All, m + 2 ;;]] ]];
 Print["Solution  vector is ", MatrixForm[augmented[[All, m + 1]]]]
];

(*thanks to http://mathematica.stackexchange.com/questions/60613/how-
  to-add-a-vertical-line-to-a-matrix*)
(*makes a dash line inside Matrix*)
Format[matWithDiv[n_, opts : OptionsPattern[Grid]][m_?MatrixQ]] :=
  MatrixForm[{{Grid[m, opts, Dividers → {n → {Red, Dashed}}]}}];
```

Example of usage and how to call the function

Section 3.1, problem 11

```
mat = {{2, 7, 3}, {1, 3, 2}, {3, 7, 9}};
b = {11, 2, -12};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 2 & 7 & 3 & 11 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 9 & -12 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

pivot now is (1,1)

will now zero out element (2,1) by subtracting $\frac{1}{2}$ times row 1 from row 2

$\left(\begin{array}{ccc|c} 2 & 7 & 3 & 11 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{7}{2} \\ 3 & 7 & 9 & -12 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

will now zero out element (3,1) by subtracting $\frac{3}{2}$ times row 1 from row 3

$\left(\begin{array}{ccc|c} 2 & 7 & 3 & 11 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{7}{2} \\ 0 & -\frac{7}{2} & \frac{9}{2} & -\frac{57}{2} \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{array}\right)$

pivot now is (2,2)

will now zero out element (3,2) by subtracting 7 times row 2 from row 3

$\left(\begin{array}{ccc|c} 2 & 7 & 3 & 11 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & -4 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 2 & -7 & 1 \end{array}\right)$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting -14 times row 2 from row 1

$\left(\begin{array}{ccc|c} 2 & 0 & 10 & -38 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & -4 \end{array}\right)\left(\begin{array}{ccc} -6 & 14 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 2 & -7 & 1 \end{array}\right)$

will now zero out element (1,3) by subtracting 10 times row 3 from row 1

$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & -4 \end{array}\right)\left(\begin{array}{ccc} -26 & 84 & -10 \\ -\frac{1}{2} & 1 & 0 \\ 2 & -7 & 1 \end{array}\right)$

will now zero out element (2,3) by subtracting $\frac{1}{2}$ times row 3 from row 2

$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & -4 \end{array}\right)\left(\begin{array}{ccc} -26 & 84 & -10 \\ -\frac{3}{2} & \frac{9}{2} & -\frac{1}{2} \\ 2 & -7 & 1 \end{array}\right)$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right)\left(\begin{array}{ccc} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{array}\right)$

Inverse Matrix is $\left(\begin{array}{ccc} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{array}\right)$

Solution vector is $\left(\begin{array}{c} 1 \\ 3 \\ -4 \end{array}\right)$

Section 3.1, problem 21

```
mat = {{1, 1, -1}, {3, 1, 3}, {4, 1, 5}};
b = {0, 0, 0};
displayRREF[mat, b]
```

```
Sorry, but matrix is singular!
```

Section 3.2, problem 11

```
mat = {{2, 8, 3}, {1, 3, 2}, {2, 7, 4}};
b = {2, 5, 8};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

pivot now is (1,1)

will now zero out element (2,1) by subtracting $\frac{1}{2}$ times row 1 from row 2

$$\left(\begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 0 & -1 & \frac{1}{2} & 4 \\ 2 & 7 & 4 & 8 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will now zero out element (3,1) by subtracting 1 times row 1 from row 3

$$\left(\begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 0 & -1 & \frac{1}{2} & 4 \\ 0 & -1 & 1 & 6 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

pivot now is (2,2)

will now zero out element (3,2) by subtracting 1 times row 2 from row 3

$$\left(\begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 0 & -1 & \frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & 2 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting −8 times row 2 from row 1

$$\left(\begin{array}{ccc|c} 2 & 0 & 7 & 34 \\ 0 & -1 & \frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & 2 \end{array}\right)\begin{pmatrix} -3 & 8 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

will now zero out element (1,3) by subtracting 14 times row 3 from row 1

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & -1 & \frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & 2 \end{array}\right)\begin{pmatrix} 4 & 22 & -14 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

will now zero out element (2,3) by subtracting 1 times row 3 from row 2

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & \frac{1}{2} & 2 \end{array}\right)\begin{pmatrix} 4 & 22 & -14 \\ 0 & 2 & -1 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix}$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right)\begin{pmatrix} 2 & 11 & -7 \\ 0 & -2 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

Inverse Matrix is $\begin{pmatrix} 2 & 11 & -7 \\ 0 & -2 & 1 \\ -1 & -2 & 2 \end{pmatrix}$

Solution vector is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$

Section 3.2, problem 24

```
mat = {{3, 2}, {6, k}};
b = {0, 0};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{cc|c} 3 & 2 & 0 \\ 6 & k & 0 \end{array}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

pivot now is (1,1)

will now zero out element (2,1) by subtracting 2 times row 1 from row 2

$\left(\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & -4+k & 0 \end{array}\right)\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

pivot now is (2,2)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting $\frac{2}{-4+k}$ times row 2 from row 1

$\left(\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & -4+k & 0 \end{array}\right)\begin{pmatrix} 1+\frac{4}{-4+k} & -\frac{2}{-4+k} \\ -2 & 1 \end{pmatrix}$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)\begin{pmatrix} \frac{1}{3}\left(1+\frac{4}{-4+k}\right) & -\frac{2}{3\,(-4+k)} \\ -\frac{2}{-4+k} & \frac{1}{-4+k} \end{pmatrix}$

Inverse Matrix is $\begin{pmatrix} \frac{1}{3}\left(1+\frac{4}{-4+k}\right) & -\frac{2}{3\,(-4+k)} \\ -\frac{2}{-4+k} & \frac{1}{-4+k} \end{pmatrix}$

Solution vector is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Section 3.2, problem 27

```
mat = {{1, 2, 1}, {2, -1, -3}, {4, 3, -1}};
b = {3, 5, k};
displayRREF[mat, b]
```

Sorry, but matrix is singular!

Section 3.3, problem 8

```
mat = {{1, -4, -5}, {3, -9, 3}, {1, -2, 3}};
b = {0, 0, 0};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 1 & -4 & -5 & 0 \\ 3 & -9 & 3 & 0 \\ 1 & -2 & 3 & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

pivot now is (1,1)

will now zero out element (2,1) by subtracting 3 times row 1 from row 2

$$\left(\begin{array}{ccc|c} 1 & -4 & -5 & 0 \\ 0 & 3 & 18 & 0 \\ 1 & -2 & 3 & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will now zero out element (3,1) by subtracting 1 times row 1 from row 3

$$\left(\begin{array}{ccc|c} 1 & -4 & -5 & 0 \\ 0 & 3 & 18 & 0 \\ 0 & 2 & 8 & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

pivot now is (2,2)

will now zero out element (3,2) by subtracting $\frac{2}{3}$ times row 2 from row 3

$$\left(\begin{array}{ccc|c} 1 & -4 & -5 & 0 \\ 0 & 3 & 18 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting $-\frac{4}{3}$ times row 2 from row 1

$$\left(\begin{array}{ccc|c} 1 & 0 & 19 & 0 \\ 0 & 3 & 18 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)\begin{pmatrix} -3 & \frac{4}{3} & 0 \\ -3 & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

will now zero out element (1,3) by subtracting $-\frac{19}{4}$ times row 3 from row 1

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 3 & 18 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)\begin{pmatrix} \frac{7}{4} & -\frac{11}{6} & \frac{19}{4} \\ -3 & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

will now zero out element (2,3) by subtracting $-\frac{9}{2}$ times row 3 from row 2

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)\begin{pmatrix} \frac{7}{4} & -\frac{11}{6} & \frac{19}{4} \\ \frac{3}{2} & -2 & \frac{9}{2} \\ 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)\begin{pmatrix} \frac{7}{4} & -\frac{11}{6} & \frac{19}{4} \\ \frac{1}{2} & -\frac{2}{3} & \frac{3}{2} \\ -\frac{1}{4} & \frac{1}{6} & -\frac{1}{4} \end{pmatrix}$$

Inverse Matrix is $\begin{pmatrix} \frac{7}{4} & -\frac{11}{6} & \frac{19}{4} \\ \frac{1}{2} & -\frac{2}{3} & \frac{3}{2} \\ -\frac{1}{4} & \frac{1}{6} & -\frac{1}{4} \end{pmatrix}$

Solution vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Section 3.3, problem 32

```
ClearAll[a, b, c, d, e];
mat = {{a, b}, {c, d}};
e = {0, 0};
displayRREF[mat, e]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \end{array}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

pivot now is (1,1)

will now zero out element (2,1) by subtracting $\frac{c}{a}$ times row 1 from row 2

$$\left(\begin{array}{cc|c} a & b & 0 \\ 0 & -\frac{b\,c}{a}+d & 0 \end{array}\right)\begin{pmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{pmatrix}$$

pivot now is (2,2)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting $\frac{b}{-\frac{b\,c}{a}+d}$ times row 2 from row 1

$$\left(\begin{array}{cc|c} a & 0 & 0 \\ 0 & -\frac{b\,c}{a}+d & 0 \end{array}\right)\begin{pmatrix} 1+\frac{b\,c}{a\left(-\frac{b\,c}{a}+d\right)} & -\frac{b}{-\frac{b\,c}{a}+d} \\ -\frac{c}{a} & 1 \end{pmatrix}$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)\begin{pmatrix} \frac{1+\frac{b\,c}{a\left(-\frac{b\,c}{a}+d\right)}}{a} & -\frac{b}{a\left(-\frac{b\,c}{a}+d\right)} \\ -\frac{c}{a\left(-\frac{b\,c}{a}+d\right)} & \frac{1}{-\frac{b\,c}{a}+d} \end{pmatrix}$$

Inverse Matrix is $\begin{pmatrix} \frac{1+\frac{b\,c}{a\left(-\frac{b\,c}{a}+d\right)}}{a} & -\frac{b}{a\left(-\frac{b\,c}{a}+d\right)} \\ -\frac{c}{a\left(-\frac{b\,c}{a}+d\right)} & \frac{1}{-\frac{b\,c}{a}+d} \end{pmatrix}$

Solution vector is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example 7, page 105

```
mat = {{4, 3, 2}, {5, 6, 3}, {3, 5, 2}};
b = {0, 0, 0};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 4 & 3 & 2 & 0 \\ 5 & 6 & 3 & 0 \\ 3 & 5 & 2 & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

pivot now is (1,1)

will now zero out element (2,1) by subtracting $\frac{5}{4}$ times row 1 from row 2

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & 0 \\ 0 & \frac{9}{4} & \frac{1}{2} & 0 \\ 3 & 5 & 2 & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{5}{4} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will now zero out element (3,1) by subtracting $\frac{3}{4}$ times row 1 from row 3

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & 0 \\ 0 & \frac{9}{4} & \frac{1}{2} & 0 \\ 0 & \frac{11}{4} & \frac{1}{2} & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{5}{4} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{pmatrix}$$

pivot now is (2,2)

will now zero out element (3,2) by subtracting $\frac{11}{9}$ times row 2 from row 3

$$\left(\begin{array}{ccc|c} 4 & 3 & 2 & 0 \\ 0 & \frac{9}{4} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{9} & 0 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -\frac{5}{4} & 1 & 0 \\ \frac{7}{9} & -\frac{11}{9} & 1 \end{pmatrix}$$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting $\frac{4}{3}$ times row 2 from row 1

$$\left(\begin{array}{ccc|c} 4 & 0 & \frac{4}{3} & 0 \\ 0 & \frac{9}{4} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{9} & 0 \end{array}\right)\begin{pmatrix} \frac{8}{3} & -\frac{4}{3} & 0 \\ -\frac{5}{4} & 1 & 0 \\ \frac{7}{9} & -\frac{11}{9} & 1 \end{pmatrix}$$

will now zero out element (1,3) by subtracting −12 times row 3 from row 1

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & \frac{9}{4} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{9} & 0 \end{array}\right)\begin{pmatrix} 12 & -16 & 12 \\ -\frac{5}{4} & 1 & 0 \\ \frac{7}{9} & -\frac{11}{9} & 1 \end{pmatrix}$$

will now zero out element (2,3) by subtracting $-\frac{9}{2}$ times row 3 from row 2

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & \frac{9}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{9} & 0 \end{array}\right)\begin{pmatrix} 12 & -16 & 12 \\ \frac{9}{4} & -\frac{9}{2} & \frac{9}{2} \\ \frac{7}{9} & -\frac{11}{9} & 1 \end{pmatrix}$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)\begin{pmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{pmatrix}$$

Inverse Matrix is $\begin{pmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{pmatrix}$

Solution vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Example 2, page 169

```
mat = {{1, 2, 1}, {3, 8, 7}, {2, 7, 9}};
b = {4, 20, 23};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

pivot now is (1,1)

will now zero out element (2,1) by subtracting 3 times row 1 from row 2

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 2 & 7 & 9 & 23 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

will now zero out element (3,1) by subtracting 2 times row 1 from row 3

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

pivot now is (2,2)

will now zero out element (3,2) by subtracting $\frac{3}{2}$ times row 2 from row 3

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{5}{2} & -\frac{3}{2} & 1 \end{pmatrix}$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting 1 times row 2 from row 1

$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)\begin{pmatrix} 4 & -1 & 0 \\ -3 & 1 & 0 \\ \frac{5}{2} & -\frac{3}{2} & 1 \end{pmatrix}$

will now zero out element (1,3) by subtracting −3 times row 3 from row 1

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)\begin{pmatrix} \frac{23}{2} & -\frac{11}{2} & 3 \\ -3 & 1 & 0 \\ \frac{5}{2} & -\frac{3}{2} & 1 \end{pmatrix}$

will now zero out element (2,3) by subtracting 4 times row 3 from row 2

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right)\begin{pmatrix} \frac{23}{2} & -\frac{11}{2} & 3 \\ -13 & 7 & -4 \\ \frac{5}{2} & -\frac{3}{2} & 1 \end{pmatrix}$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)\begin{pmatrix} \frac{23}{2} & -\frac{11}{2} & 3 \\ -\frac{13}{2} & \frac{7}{2} & -2 \\ \frac{5}{2} & -\frac{3}{2} & 1 \end{pmatrix}$

Inverse Matrix is $\begin{pmatrix} \frac{23}{2} & -\frac{11}{2} & 3 \\ -\frac{13}{2} & \frac{7}{2} & -2 \\ \frac{5}{2} & -\frac{3}{2} & 1 \end{pmatrix}$

Solution vector is $\begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$

problem 1, page 175

```
mat = {{17, 42, -36}, {13, 45, -34}, {12, 47, -35}};
b = {213, 226, 197};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 17 & 42 & -36 & 213 \\ 13 & 45 & -34 & 226 \\ 12 & 47 & -35 & 197 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

pivot now is (1,1)

will now zero out element (2,1) by subtracting $\frac{13}{17}$ times row 1 from row 2

$$\left(\begin{array}{ccc|c} 17 & 42 & -36 & 213 \\ 0 & \frac{219}{17} & -\frac{110}{17} & \frac{1073}{17} \\ 12 & 47 & -35 & 197 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{13}{17} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

will now zero out element (3,1) by subtracting $\frac{12}{17}$ times row 1 from row 3

$$\left(\begin{array}{ccc|c} 17 & 42 & -36 & 213 \\ 0 & \frac{219}{17} & -\frac{110}{17} & \frac{1073}{17} \\ 0 & \frac{295}{17} & -\frac{163}{17} & \frac{793}{17} \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{13}{17} & 1 & 0 \\ -\frac{12}{17} & 0 & 1 \end{array}\right)$$

pivot now is (2,2)

will now zero out element (3,2) by subtracting $\frac{295}{219}$ times row 2 from row 3

$$\left(\begin{array}{ccc|c} 17 & 42 & -36 & 213 \\ 0 & \frac{219}{17} & -\frac{110}{17} & \frac{1073}{17} \\ 0 & 0 & -\frac{191}{219} & -\frac{8404}{219} \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{13}{17} & 1 & 0 \\ \frac{71}{219} & -\frac{295}{219} & 1 \end{array}\right)$$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting $\frac{238}{73}$ times row 2 from row 1

$$\left(\begin{array}{ccc|c} 17 & 0 & -\frac{1088}{73} & \frac{527}{73} \\ 0 & \frac{219}{17} & -\frac{110}{17} & \frac{1073}{17} \\ 0 & 0 & -\frac{191}{219} & -\frac{8404}{219} \end{array}\right)\left(\begin{array}{ccc} \frac{255}{73} & -\frac{238}{73} & 0 \\ -\frac{13}{17} & 1 & 0 \\ \frac{71}{219} & -\frac{295}{219} & 1 \end{array}\right)$$

will now zero out element (1,3) by subtracting $\frac{3264}{191}$ times row 3 from row 1

$$\left(\begin{array}{ccc|c} 17 & 0 & 0 & 663 \\ 0 & \frac{219}{17} & -\frac{110}{17} & \frac{1073}{17} \\ 0 & 0 & -\frac{191}{219} & -\frac{8404}{219} \end{array}\right)\left(\begin{array}{ccc} -\frac{391}{191} & \frac{3774}{191} & -\frac{3264}{191} \\ -\frac{13}{17} & 1 & 0 \\ \frac{71}{219} & -\frac{295}{219} & 1 \end{array}\right)$$

will now zero out element (2,3) by subtracting $\frac{24\,090}{3247}$ times row 3 from row 2

$$\left(\begin{array}{ccc|c} 17 & 0 & 0 & 663 \\ 0 & \frac{219}{17} & 0 & \frac{5913}{17} \\ 0 & 0 & -\frac{191}{219} & -\frac{8404}{219} \end{array}\right)\left(\begin{array}{ccc} -\frac{391}{191} & \frac{3774}{191} & -\frac{3264}{191} \\ -\frac{10\,293}{3247} & \frac{35\,697}{3247} & -\frac{24\,090}{3247} \\ \frac{71}{219} & -\frac{295}{219} & 1 \end{array}\right)$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & 27 \\ 0 & 0 & 1 & 44 \end{array}\right)\left(\begin{array}{ccc} -\frac{23}{191} & \frac{222}{191} & -\frac{192}{191} \\ -\frac{47}{191} & \frac{163}{191} & -\frac{110}{191} \\ -\frac{71}{191} & \frac{295}{191} & -\frac{219}{191} \end{array}\right)$$

Inverse Matrix is $\left(\begin{array}{ccc} -\frac{23}{191} & \frac{222}{191} & -\frac{192}{191} \\ -\frac{47}{191} & \frac{163}{191} & -\frac{110}{191} \\ -\frac{71}{191} & \frac{295}{191} & -\frac{219}{191} \end{array}\right)$

Solution vector is $\begin{pmatrix} 39 \\ 27 \\ 44 \end{pmatrix}$

problem 2, page 175

```
mat = {{32, 57, -41}, {23, 43, -37}, {42, -61, 39}};
b = {713, 130, 221};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 32 & 57 & -41 & 713 \\ 23 & 43 & -37 & 130 \\ 42 & -61 & 39 & 221 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

pivot now is (1,1)

will now zero out element (2,1) by subtracting $\frac{23}{32}$ times row 1 from row 2

$$\left(\begin{array}{ccc|c} 32 & 57 & -41 & 713 \\ 0 & \frac{65}{32} & -\frac{241}{32} & -\frac{12239}{32} \\ 42 & -61 & 39 & 221 \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{23}{32} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

will now zero out element (3,1) by subtracting $\frac{21}{16}$ times row 1 from row 3

$$\left(\begin{array}{ccc|c} 32 & 57 & -41 & 713 \\ 0 & \frac{65}{32} & -\frac{241}{32} & -\frac{12239}{32} \\ 0 & -\frac{2173}{16} & \frac{1485}{16} & -\frac{11437}{16} \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{23}{32} & 1 & 0 \\ -\frac{21}{16} & 0 & 1 \end{array}\right)$$

pivot now is (2,2)

will now zero out element (3,2) by subtracting $-\frac{4346}{65}$ times row 2 from row 3

$$\left(\begin{array}{ccc|c} 32 & 57 & -41 & 713 \\ 0 & \frac{65}{32} & -\frac{241}{32} & -\frac{12239}{32} \\ 0 & 0 & -\frac{26698}{65} & -\frac{1708672}{65} \end{array}\right)\left(\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{23}{32} & 1 & 0 \\ -\frac{3209}{65} & \frac{4346}{65} & 1 \end{array}\right)$$

pivot now is (3,3)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting $\frac{1824}{65}$ times row 2 from row 1

$$\left(\begin{array}{ccc|c} 32 & 0 & \frac{11072}{65} & \frac{743968}{65} \\ 0 & \frac{65}{32} & -\frac{241}{32} & -\frac{12239}{32} \\ 0 & 0 & -\frac{26698}{65} & -\frac{1708672}{65} \end{array}\right)\left(\begin{array}{ccc} \frac{1376}{65} & -\frac{1824}{65} & 0 \\ -\frac{23}{32} & 1 & 0 \\ -\frac{3209}{65} & \frac{4346}{65} & 1 \end{array}\right)$$

will now zero out element (1,3) by subtracting $-\frac{5536}{13349}$ times row 3 from row 1

$$\left(\begin{array}{ccc|c} 32 & 0 & 0 & 544 \\ 0 & \frac{65}{32} & -\frac{241}{32} & -\frac{12239}{32} \\ 0 & 0 & -\frac{26698}{65} & -\frac{1708672}{65} \end{array}\right)\left(\begin{array}{ccc} \frac{9280}{13349} & -\frac{4448}{13349} & \frac{5536}{13349} \\ -\frac{23}{32} & 1 & 0 \\ -\frac{3209}{65} & \frac{4346}{65} & 1 \end{array}\right)$$

will now zero out element (2,3) by subtracting $\frac{15665}{854336}$ times row 3 from row 2

$$\left(\begin{array}{ccc|c} 32 & 0 & 0 & 544 \\ 0 & \frac{65}{32} & 0 & \frac{3185}{32} \\ 0 & 0 & -\frac{26698}{65} & -\frac{1708672}{65} \end{array}\right)\left(\begin{array}{ccc} \frac{9280}{13349} & -\frac{4448}{13349} & \frac{5536}{13349} \\ \frac{159315}{854336} & -\frac{96525}{427168} & -\frac{15665}{854336} \\ -\frac{3209}{65} & \frac{4346}{65} & 1 \end{array}\right)$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & 49 \\ 0 & 0 & 1 & 64 \end{array}\right)\left(\begin{array}{ccc} \frac{290}{13349} & -\frac{139}{13349} & \frac{173}{13349} \\ \frac{2451}{26698} & -\frac{1485}{13349} & -\frac{241}{26698} \\ \frac{3209}{26698} & -\frac{2173}{13349} & -\frac{65}{26698} \end{array}\right)$$

Inverse Matrix is $\left(\begin{array}{ccc} \frac{290}{13349} & -\frac{139}{13349} & \frac{173}{13349} \\ \frac{2451}{26698} & -\frac{1485}{13349} & -\frac{241}{26698} \\ \frac{3209}{26698} & -\frac{2173}{13349} & -\frac{65}{26698} \end{array}\right)$

Solution vector is $\begin{pmatrix} 17 \\ 49 \\ 64 \end{pmatrix}$

```
mat = {{5, 2, 18, 4}, {0, 1, 2, 5}, {4, 1, 12, 6}, {2, 3, 8, 9}};
b = {1, 2, 3, 4};
displayRREF[mat, b]
```

>>>>>>Starting forward Gaussian elimination phase using $\left(\begin{array}{ccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 4 & 1 & 12 & 6 & 3 \\ 2 & 3 & 8 & 9 & 4 \end{array}\right)\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

pivot now is (1,1)

will now zero out element (2,1) by subtracting 0 times row 1 from row 2

$$\left(\begin{array}{ccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 4 & 1 & 12 & 6 & 3 \\ 2 & 3 & 8 & 9 & 4 \end{array}\right)\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

will now zero out element (3,1) by subtracting $\frac{4}{5}$ times row 1 from row 3

$$\left(\begin{array}{ccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 0 & -\frac{3}{5} & -\frac{12}{5} & \frac{14}{5} & \frac{11}{5} \\ 2 & 3 & 8 & 9 & 4 \end{array}\right)\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

will now zero out element (4,1) by subtracting $\frac{2}{5}$ times row 1 from row 4

$$\left(\begin{array}{ccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 0 & -\frac{3}{5} & -\frac{12}{5} & \frac{14}{5} & \frac{11}{5} \\ 0 & \frac{11}{5} & \frac{4}{5} & \frac{37}{5} & \frac{18}{5} \end{array}\right)\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & 0 & 1 & 0 \\ -\frac{2}{5} & 0 & 0 & 1 \end{array}\right)$$

pivot now is (2,2)

will now zero out element (3,2) by subtracting $-\frac{3}{5}$ times row 2 from row 3

$$\left(\begin{array}{cccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & \frac{11}{5} & \frac{4}{5} & \frac{37}{5} & \frac{18}{5} \end{array}\right) \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{2}{5} & 0 & 0 & 1 \end{array}\right)$$

will now zero out element (4,2) by subtracting $\frac{11}{5}$ times row 2 from row 4

$$\left(\begin{array}{cccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & -\frac{18}{5} & -\frac{18}{5} & -\frac{4}{5} \end{array}\right) \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ -\frac{2}{5} & -\frac{11}{5} & 0 & 1 \end{array}\right)$$

pivot now is (3,3)

will now zero out element (4,3) by subtracting 3 times row 3 from row 4

$$\left(\begin{array}{cccc|c} 5 & 2 & 18 & 4 & 1 \\ 0 & 1 & 2 & 5 & 2 \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right) \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ 2 & -4 & -3 & 1 \end{array}\right)$$

pivot now is (4,4)

>>>>>>Starting backward elimination phase

will now zero out element (1,2) by subtracting 2 times row 2 from row 1

$$\left(\begin{array}{cccc|c} 5 & 0 & 14 & -6 & -3 \\ 0 & 1 & 2 & 5 & 2 \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right) \left(\begin{array}{cccc} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ 2 & -4 & -3 & 1 \end{array}\right)$$

will now zero out element (1,3) by subtracting $-\frac{35}{3}$ times row 3 from row 1

$$\left(\begin{array}{cccc|c} 5 & 0 & 0 & \frac{185}{3} & \frac{110}{3} \\ 0 & 1 & 2 & 5 & 2 \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right) \left(\begin{array}{cccc} -\frac{25}{3} & 5 & \frac{35}{3} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ 2 & -4 & -3 & 1 \end{array}\right)$$

will now zero out element (2,3) by subtracting $-\frac{5}{3}$ times row 3 from row 2

$$\left(\begin{array}{cccc|c} 5 & 0 & 0 & \frac{185}{3} & \frac{110}{3} \\ 0 & 1 & 0 & \frac{44}{3} & \frac{23}{3} \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right) \left(\begin{array}{cccc} -\frac{25}{3} & 5 & \frac{35}{3} & 0 \\ -\frac{4}{3} & 2 & \frac{5}{3} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ 2 & -4 & -3 & 1 \end{array}\right)$$

will now zero out element (1,4) by subtracting $-\frac{185}{63}$ times row 4 from row 1

$$\left(\begin{array}{cccc|c} 5 & 0 & 0 & 0 & \frac{275}{63} \\ 0 & 1 & 0 & \frac{44}{3} & \frac{23}{3} \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right) \left(\begin{array}{cccc} -\frac{155}{63} & -\frac{425}{63} & \frac{20}{7} & \frac{185}{63} \\ -\frac{4}{3} & 2 & \frac{5}{3} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ 2 & -4 & -3 & 1 \end{array}\right)$$

will now zero out element (2,4) by subtracting $-\frac{44}{63}$ times row 4 from row 2

$$\left(\begin{array}{cccc|c} 5 & 0 & 0 & 0 & \frac{275}{63} \\ 0 & 1 & 0 & 0 & -\frac{1}{63} \\ 0 & 0 & -\frac{6}{5} & \frac{29}{5} & \frac{17}{5} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right) \left(\begin{array}{cccc} -\frac{155}{63} & -\frac{425}{63} & \frac{20}{7} & \frac{185}{63} \\ \frac{4}{63} & -\frac{50}{63} & -\frac{3}{7} & \frac{44}{63} \\ -\frac{4}{5} & \frac{3}{5} & 1 & 0 \\ 2 & -4 & -3 & 1 \end{array}\right)$$

will now zero out element (3,4) by subtracting $-\frac{29}{105}$ times row 4 from row 3

$$\left(\begin{array}{cccc|c} 5 & 0 & 0 & 0 & \frac{275}{63} \\ 0 & 1 & 0 & 0 & -\frac{1}{63} \\ 0 & 0 & -\frac{6}{5} & 0 & \frac{38}{105} \\ 0 & 0 & 0 & -21 & -11 \end{array}\right)\left(\begin{array}{cccc} -\frac{155}{63} & -\frac{425}{63} & \frac{20}{7} & \frac{185}{63} \\ \frac{4}{63} & -\frac{50}{63} & -\frac{3}{7} & \frac{44}{63} \\ -\frac{26}{105} & -\frac{53}{105} & \frac{6}{35} & \frac{29}{105} \\ 2 & -4 & -3 & 1 \end{array}\right)$$

>>>>>>Starting Final phase, convert reduced echelon to identity matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{55}{63} \\ 0 & 1 & 0 & 0 & -\frac{1}{63} \\ 0 & 0 & 1 & 0 & -\frac{19}{63} \\ 0 & 0 & 0 & 1 & \frac{11}{21} \end{array}\right)\left(\begin{array}{cccc} -\frac{31}{63} & -\frac{85}{63} & \frac{4}{7} & \frac{37}{63} \\ \frac{4}{63} & -\frac{50}{63} & -\frac{3}{7} & \frac{44}{63} \\ \frac{13}{63} & \frac{53}{126} & -\frac{1}{7} & -\frac{29}{126} \\ -\frac{2}{21} & \frac{4}{21} & \frac{1}{7} & -\frac{1}{21} \end{array}\right)$$

Inverse Matrix is $\left(\begin{array}{cccc} -\frac{31}{63} & -\frac{85}{63} & \frac{4}{7} & \frac{37}{63} \\ \frac{4}{63} & -\frac{50}{63} & -\frac{3}{7} & \frac{44}{63} \\ \frac{13}{63} & \frac{53}{126} & -\frac{1}{7} & -\frac{29}{126} \\ -\frac{2}{21} & \frac{4}{21} & \frac{1}{7} & -\frac{1}{21} \end{array}\right)$

Solution vector is $\left(\begin{array}{c} \frac{55}{63} \\ -\frac{1}{63} \\ -\frac{19}{63} \\ \frac{11}{21} \end{array}\right)$