

Small note on recursive formula for integral of trigonometric functions

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After struggling in deriving this, I found similar one on wikipedia. References below. May be I will add Mathematica implementation for this later....

The goal is to find recursive formula for $\int \cos(x)^n dx$. Starting by rewriting it as

$$\int \cos(x)^n dx = \int \cos(x)^{n-1} \cos(x) dx \quad (1)$$

Integrating by parts $\int u dv = (uv) - \int v du$ and letting $u = \cos(x)^{n-1}$, $dv = \cos(x)$, hence $du = -(n-1)\cos(x)^{n-2}\sin(x)$ and $v = \sin(x)$ the above becomes

$$\begin{aligned} \int \cos(x)^n dx &= \cos(x)^{n-1} \sin(x) + \int \sin(x)(n-1)\cos(x)^{n-2}\sin(x) dx \\ &= \cos(x)^{n-1} \sin(x) + \int (n-1)\cos(x)^{n-2}\sin^2(x) dx \\ &= \cos(x)^{n-1} \sin(x) + \int (n-1)\cos(x)^{n-2}(1-\cos(x)^2) dx \\ &= \cos(x)^{n-1} \sin(x) + (n-1) \int \cos(x)^{n-2} - \cos(x)^n dx \\ &= \cos(x)^{n-1} \sin(x) + (n-1) \int \cos(x)^{n-2} dx - (n-1) \int \cos(x)^n dx \end{aligned}$$

The $\int \cos(x)^n dx$ in the RHS above is what is being solved for. Moving it to the LHS gives

$$\begin{aligned} \int \cos(x)^n dx + (n-1) \int \cos(x)^n dx &= \cos(x)^{n-1} \sin(x) + (n-1) \int \cos(x)^{n-2} dx \\ n \int \cos(x)^n dx &= \cos(x)^{n-1} \sin(x) + (n-1) \int \cos(x)^{n-2} dx \end{aligned}$$

Therefore the recursive formula is

$$\int \cos(x)^n dx = \frac{\cos(x)^{n-1} \sin(x)}{n} + \frac{(n-1)}{n} \int \cos(x)^{n-2} dx$$

References:

1. <http://www.integraltec.com/math/math.php?f=cosPower.html#cos>
2. http://en.wikipedia.org/wiki/Integration_by_reduction_formulae#Examples