For Maple 2018.1, there are improvements in pdsolve's ability to solve PDE with boundary and initial conditions. This is work done together with E.S. Cheb-Terrab. The improvements include an extended ability to solve problems involving non-homogeneous PDE and/or non-homogeneous boundary and initial conditions, as well as improved simplification of solutions and better handling of functions such as piecewise in the arguments and in the processing of solutions. This is also an ongoing project, with updates being distributed regularly within the Physics Updates.

Solving more problems involving non-homogeneous PDE and/or non-homogeneous boundary and initial conditions

Example 1: Pinchover and Rubinstein's exercise 6.17: we have a non-homogenous PDE and boundary and initial conditions that are also non-homogeneous:

> $pde_1 := \frac{\partial}{\partial t} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = 1 + x\cos(t)$:

$iv_1 := D_1(u)(0,t) = \sin(t), D_1(u)(1,t) = \sin(t), u(x,0) = 1 + \cos(2\pi x)$:

> $pdsolve([pde_1, iv_1]);$

$$u(x,t) = 1 + \cos(2\pi x)\, e^{-4\pi^2 t} + t + x\sin(t)$$

How we solve the problem, step by step:

Example 2: the PDE is homogeneous but the boundary conditions are not. We solve the problem through the same process, which means we end up solving a nonhomogeneous pde with homogeneous BC as an intermediate step:

> $pde_2 := \frac{\partial}{\partial t} u(x,t) = 13\left(\frac{\partial^2}{\partial x^2} u(x,t)\right)$:

$iv_2 := D_1(u)(0,t) = 0, D_1(u)(1,t) = 1, u(x,0) = \frac{1}{2}x^2 + x$:

> $pdsolve([pde_2, iv_2]);$

$$u(x,t) = \frac{1}{2} + \left(\sum_{n=1}^{\infty} \frac{2\left(-1+(-1)^n\right)\cos(n\pi x)\, e^{-13\pi^2 n^2 t}}{\pi^2 n^2}\right) + 13t + \frac{x^2}{2}$$

How we solve the problem, step by step:

Example 3: a wave PDE with a source that does not depend on time:

> $pde_3 := \left(\frac{\partial^2}{\partial x^2} u(x,t)\right) a^2 + 1 = \frac{\partial^2}{\partial t^2} u(x,t)$:

$iv_3 := u(0,t) = 0, u(L,t) = 0, u(x,0) = f(x), D_2(u)(x,0) = g(x)$:

> $pdsolve([pde_3, iv_3])$ assuming $L > 0$;

$$u(x,t) = \frac{1}{2a^2}\left(2\left(\sum_{n=1}^{\infty} \frac{1}{\pi n a^2 L}\left(\sin\left(\frac{n\pi x}{L}\right)\left(2L\sin\left(\frac{a\pi t n}{L}\right)\left(\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)\,dx\right) a\right.\right.\right.\right.$$

$$-\pi\cos\left(\frac{a\,\pi\,t\,n}{L}\right)\left(\left(\int_0^L \sin\left(\frac{n\,\pi\,x}{L}\right)\left(-2\,f(x)\,a^2+L\,x-x^2\right)\mathrm{d}x\right)n\right)\Bigg)\Bigg)\,a^2+L\,x-x^2\Bigg)$$

▶ How we solve the problem, step by step:

Example 4: Pinchover and Rubinstein's exercise 6.23 - we have a non-homogenous PDE and initial condition:

$> pde_4 := \frac{\partial}{\partial t}\,u(x,t) - \frac{\partial^2}{\partial x^2}\,u(x,t) = g(x,t)$:

$iv_4 := \mathrm{D}_1(u)(0,t)=0,\ \mathrm{D}_1(u)(1,t)=0,\ u(x,0)=f(x)$:

$>$ $pdsolve([pde_4, iv_4], u(x,t))$;

$$u(x,t) = \int_0^1 f(\tau 1)\,\mathrm{d}\tau 1 + \left(\sum_{n=1}^{\infty} 2\left(\int_0^1 f(\tau 1)\cos(n\,\pi\,\tau 1)\,\mathrm{d}\tau 1\right)\cos(n\,\pi\,x)\,\mathrm{e}^{-\pi^2 n^2 t}\right) + \int_0^t \left(\left(\int_0^1 g(x,\tau 1)\,\mathrm{d}x + \left(\sum_{n1=1}^{\infty} 2\left(\int_0^1 g(x,\tau 1)\cos(n1\,\pi\,x)\,\mathrm{d}x\right)\cos(n1\,\pi\,x)\,\mathrm{e}^{-\pi^2 n1^2 (t-\tau 1)}\right)\right)\right)\mathrm{d}\tau 1$$

If we now make the functions f and g into specific mappings, we can compare pdsolve's solutions to the general and specific problems:

$> f := x \rightarrow 3\cos(42\,x\,\mathrm{Pi})$:

$g := (x,t) \rightarrow \mathrm{e}^{3t}\cos(17\,x\,\mathrm{Pi})$:

Here is what pdsolve's solution to the general problem looks like when taking into account the new values of f(x) and g(x,t):

$>$ *value*(*simplify*(*evalindets*(**Function Call**, *specfunc*(*Int*), *u* → `*PDEtools/int*`(*op*(*u*), *AllSolutions*))));

$$u(x,t) = 3\cos(42\,\pi\,x)\,\mathrm{e}^{-1764\,\pi^2 t} + \frac{1}{289\,\pi^2+3}\Big(\cos(\pi\,x)\,\big(65536\cos(\pi\,x)^{16} - 278528\cos(\pi\,x)^{14} + 487424\cos(\pi\,x)^{12} - 452608\cos(\pi\,x)^{10} + 239360\cos(\pi\,x)^{8} - 71808\cos(\pi\,x)^{6} + 11424\cos(\pi\,x)^{4} - 816\cos(\pi\,x)^{2} + 17\big)\left(\mathrm{e}^{289\,\pi^2 t+3t} - 1\right)\mathrm{e}^{-289\,\pi^2 t}\Big)$$

Here is pdsolve's solution to the specific problem:

$>$ $pdsolve([pde_4, iv_4], u(x,t))$;

$$u(x,t) = \frac{1}{289\,\pi^2+3}\Big(\big(-65536\cos(\pi\,x)^{17} + 278528\cos(\pi\,x)^{15} - 487424\cos(\pi\,x)^{13} + 452608\cos(\pi\,x)^{11} - 239360\cos(\pi\,x)^{9} + 71808\cos(\pi\,x)^{7} - 11424\cos(\pi\,x)^{5} + 816\cos(\pi\,x)^{3} - 17\cos(\pi\,x)\big)\,\mathrm{e}^{-289\,\pi^2 t} + \left(867\,\pi^2+9\right)\cos(42\,\pi\,x)\,\mathrm{e}^{-1764\,\pi^2 t}$$

$$+65536\, e^{3t} \left(\cos(\pi x)^{16} - \frac{17 \cos(\pi x)^{14}}{4} + \frac{119 \cos(\pi x)^{12}}{16} - \frac{221 \cos(\pi x)^{10}}{32} + \frac{935 \cos(\pi x)^{8}}{256} - \frac{561 \cos(\pi x)^{6}}{512} + \frac{357 \cos(\pi x)^{4}}{2048} - \frac{51 \cos(\pi x)^{2}}{4096} + \frac{17}{65536} \right) \cos(\pi x) \Bigg)$$

And the two solutions are equal:

\> *simplify*(**Equality**−**Equality**);

$$0 = 0$$

\> $f := 'f' : g := 'g' :$

Improved simplification in integrals, piecewise functions, and sums in the solutions returned by pdsolve

Example 1: exercise 6.21 from Pinchover and Rubinstein is a non-homogeneous heat problem. Its solution used to include unevaluated integrals and sums, but is now returned in a significantly simpler format.

\> $pde_5 := \frac{\partial}{\partial t} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = t \cos(2001\, x) :$

$iv_5 := D_1(u)(0, t) = 0, D_1(u)(\pi, t) = 0, u(x, 0) = \pi \cos(2\, x) :$

\> $pdsolve([pde_5, iv_5])$

$$u(x, t) = \frac{\left(4004001\, t + e^{-4004001\, t} - 1\right) \cos(2001\, x)}{16032024008001} + \pi \cos(2\, x)\, e^{-4\, t}$$

\> $pdetest(\%, [pde_5, iv_5])$

$$[0, 0, 0, 0]$$

Example 2: example 6.46 from Pinchover and Rubinstein is a non-homogeneous heat equation with non-homogeneous boundary and initial conditions. Its solution used to involve two separate sums with unevaluated integrals, but is now returned with only one sum and unevaluated integral.

\> $pde_6 := \frac{\partial}{\partial t} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = e^{-t} \sin(3\, x) :$

$iv_6 := u(0, t) = 0, u(\pi, t) = 1, u(x, 0) = \phi(x) :$

\> $pdsolve([pde_6, iv_6], u(x, t))$

$u(x, t)$

$$= \frac{1}{8\, \pi} \left(8 \left(\sum_{n=1}^{\infty} \frac{2 \left(\int_0^{\pi} -\left(-\phi(x)\, \pi + x\right) \sin(n\, x)\, dx \right) \sin(n\, x)\, e^{-n^2 t}}{\pi^2} \right) \pi - \pi \left(e^{-9\, t} \right.\right.$$

$$-\,\mathrm{e}^{-t}\big)\,\sin(3\,x)\,+8\,x\Bigg)$$

> $pdetest(\%, [pde_6, iv_6])$

$$\left[0,0,0,\frac{-\phi(x)\,\pi^2+\pi\,x+2\left(\sum_{n=1}^{\infty}\left(\int_0^{\pi}-(-\phi(x)\,\pi+x)\,\sin(n\,x)\,\mathrm{d}x\right)\sin(n\,x)\right)}{\pi^2}\right]$$

More accuracy when returning series solutions that have exceptions for certain values of the summation index or a parameter

Example 1: the answer to this problem was previously given with $n=0\,..\,\infty$ instead of $n=1\,..\,\infty$ as it should be:

> $pde_7 := \frac{\partial^2}{\partial t^2}\,v(x,t)-\frac{\partial^2}{\partial x^2}\,v(x,t):$

$$iv_7 := v(0,t)=0,\ v(x,0)=-\frac{\mathrm{e}^2\,x-\mathrm{e}^{x+1}-x+\mathrm{e}^{1-x}}{\mathrm{e}^2-1},\ \mathrm{D}_2(v)(x,0)=1+\frac{\mathrm{e}^2\,x-\mathrm{e}^{x+1}-x+\mathrm{e}^{1-x}}{\mathrm{e}^2-1},\ v(1,t)=0:$$

> $pdsolve([pde_7, iv_7])$

$$v(x,t)=\sum_{n=1}^{\infty}-\frac{2\,\sin(n\,\pi\,x)\,\left(\left(\pi^2\,(-1)^n\,n^2-\pi^2\,n^2+2\,(-1)^n-1\right)\sin(\pi\,t\,n)-(-1)^n\cos(\pi\,t\,n)\,\pi\,n\right)}{\pi^2\,n^2\,\left(\pi^2\,n^2+1\right)}$$

Example 2: the answer to exercise 6.25 from Pinchover and Rubinstein is now given in a much simpler format, with the special limit case for w = 0 calculated separately:

> $pde_8 := \frac{\partial}{\partial t}\,u(x,t)=k\left(\frac{\partial^2}{\partial x^2}\,u(x,t)\right)+\cos(w\,t):$
> $iv_8 := \mathrm{D}_1(u)(L,t)=0,\ \mathrm{D}_1(u)(0,t)=0,\ u(x,0)=x:$

> $pdsolve([pde_8, iv_8], u(x,t))$ assuming $L>0$

$u(x,t)$

$$= \begin{cases} \frac{L}{2} + \left(\sum_{n=1}^{\infty} \frac{2 L \left(-1 + (-1)^n\right) \cos\left(\frac{n \pi x}{L}\right) e^{-\frac{k \pi^2 n^2 t}{L^2}}}{n^2 \pi^2} \right) + t & w = 0 \\ \frac{L w + 2 \left(\sum_{n=1}^{\infty} \frac{2 L \left(-1 + (-1)^n\right) \cos\left(\frac{n \pi x}{L}\right) e^{-\frac{k \pi^2 n^2 t}{L^2}}}{n^2 \pi^2} \right) w + 2 \sin(w t)}{2 w} & \mathit{otherwise} \end{cases}$$

Improved handling of piecewise, eval/diff in the given problem

Example 1: this problem, which contains a piecewise function in the initial condition, can now be solved:

$$> pde_9 := \frac{\partial}{\partial t} f(t, x) = \frac{\partial^2}{\partial x^2} f(t, x) :$$

$$iv_9 := f(t, 0) = 0, f(t, 1) = 1, f(0, x) = \begin{cases} 1 & x = 0 \\ 0 & \mathit{otherwise} \end{cases} :$$

$$> pdsolve([pde_9, iv_9])$$

$$f(t, x) = \left(\sum_{n=1}^{\infty} \frac{2 (-1)^n \sin(n \pi x) e^{-\pi^2 n^2 t}}{n \pi} \right) + x$$

Example 2: this problem, which contains a derivative written using eval/diff, can now be solved:

$$> pde_{10} := -\frac{\partial^2}{\partial t^2} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) + u(x, t) = 2 e^{-t} \left(x - \frac{1}{2} x^2 + \frac{1}{2} t - 1 \right) :$$

$$iv_{10} := u(x, 0) = x^2 - 2 x, u(x, 1) = u\left(x, \frac{1}{2}\right) + \left(\frac{1}{2} x^2 - x\right) e^{-1} - \left(\frac{3}{4} x^2 - \frac{3}{2} x\right) e^{-\frac{1}{2}}, u(0, t) = 0, \left. \left(\frac{\partial}{\partial x} u(x, t) \right) \right|_{\{x = 1\}} = 0 :$$

$$> pdsolve([pde_{10}, iv_{10}], u(x, t))$$

$$u(x, t) = -\frac{e^{-t} x (x - 2) (t - 2)}{2}$$

References:

Pinchover, Y. and Rubinstein, J.. *An Introduction to Partial Differential Equations.* Cambridge UP, 2005.