Solving differential equations using parametric methods

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1.1 Introduction

This method is meant to be used for solving first order ode's that are non-linear. Let the ode be

$$f(x, y, y') = 0$$

In the parametric method we let $y' = p$ and let x, y be functions of parameter p. Hence $x \equiv x(p), y \equiv y(p)$. The above becomes

$$f(x, y, p) = 0 \tag{1}$$

To find the solution of the original ode, the idea is to generate two equations in $x(p), y(p)$ and use these two equations to eliminate p.

The first equation is easy to find. It is found by either isolating $x(p)$ or $y(p)$ the original ode itself. The second equation is differential equation, either in $\frac{dy}{dp}$ or $\frac{dx}{dp}$. Which one to find depends if we have isolated x or y at the start. If we have isolated x then we need to generate $\frac{dy}{dp}$ equation in order to solve it for $y(p)$. If we have isolated y instead, then we need to find $\frac{dx}{dp}$ equation to solve it for $x(p)$. But how to obtain $\frac{dy}{dp}$or $\frac{dx}{dp}$? This is described below.

Let us assume we wanted to generate $\frac{dx}{dp}$ ode. Then taking derivative of the original ode w.r.t.

p gives

$$
\begin{aligned}
0 &= \frac{\partial f(x,y,p)}{\partial p} \\
&= \frac{df}{dp} + \frac{\partial f}{\partial x}\frac{dx}{dp} + \frac{\partial f}{\partial y}\frac{dy}{dp} \\
&= \frac{df}{dp} + \frac{\partial f}{\partial x}\frac{dx}{dp} + \frac{\partial f}{\partial y}\frac{dy}{dx}\frac{dx}{dp} \\
&= \frac{df}{dp} + \frac{\partial f}{\partial x}\frac{dx}{dp} + p\frac{\partial f}{\partial y}\frac{dx}{dp} \\
-\frac{\partial f}{\partial x}\frac{dx}{dp} - p\frac{\partial f}{\partial y}\frac{dx}{dp} &= \frac{df}{dp} \\
\frac{dx}{dp}\left(-\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial y}\right) &= \frac{df}{dp} \\
\frac{dx}{dp} &= \frac{\frac{df}{dp}}{-\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial y}} \\
&= \frac{-\frac{df}{dp}}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}} \qquad \text{(2A)}
\end{aligned}
$$

If instead we wanted $\frac{dy}{dp}$, then instead we do the following

$$
\begin{aligned}
0 &= \frac{\partial f(x,y,p)}{\partial p} \\
&= \frac{df}{dp} + \frac{\partial f}{\partial x}\frac{dx}{dp} + \frac{\partial f}{\partial y}\frac{dy}{dp} \\
&= \frac{df}{dp} + \frac{\partial f}{\partial x}\frac{dx}{dy}\frac{dy}{dp} + \frac{\partial f}{\partial y}\frac{dy}{dp} \\
&= \frac{df}{dp} + \frac{\partial f}{\partial x}\frac{1}{p}\frac{dy}{dp} + \frac{\partial f}{\partial y}\frac{dy}{dp} \\
-\frac{\partial f}{\partial x}\frac{1}{p}\frac{dy}{dp} - \frac{\partial f}{\partial y}\frac{dy}{dp} &= \frac{df}{dp} \\
\frac{dy}{dp}\left(-\frac{\partial f}{\partial x}\frac{1}{p} - \frac{\partial f}{\partial y}\right) &= \frac{df}{dp} \\
\frac{dy}{dp} &= \frac{\frac{df}{dp}}{-\frac{\partial f}{\partial x}\frac{1}{p} - \frac{\partial f}{\partial y}} \\
&= \frac{p\frac{df}{dp}}{-\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial y}} \\
&= \frac{-p\frac{df}{dp}}{p\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}} \qquad \text{(3A)}
\end{aligned}
$$

Eqs. (2A) or Eq. (3A) can be used as the second equation we talked above above.

It is important to note the following. If given the original ode $f(x,y,p)=0$ and we decided to isolate x to obtain $x=f(y,p)$ then for the second equation we must use (3A) and now it becomes

$$\frac{dy}{dp}=\frac{p\frac{df}{dp}}{1-p\frac{\partial f}{\partial y}} \tag{3A}$$

Then we use the solution $y(p)$ of the above, with $x=f(y,p)$ to eliminate p and find y. The ode generated using (3A) should be simple to solve (quadrature or separable).

If instead, we have an ode $f(x,y,p)=0$ and we isolated y instead to obtain $y=f(x,p)$ then we must now use (2A) to obtain the second equation. (2A) now becomes

$$\frac{dx}{dp}=\frac{\frac{df}{dp}}{p-\frac{\partial f}{\partial x}} \tag{2A}$$

Then we use the solution $x(p)$ in the above. This gives us the second equation, and with the equation $y=f(x,p)$ to eliminate p and find y.

In summary: we use one of these two equations

$$\begin{aligned} x &= f(y,p) \\ \frac{dy}{dp} &= \frac{p\frac{df}{dp}}{1-p\frac{\partial f}{\partial y}} \end{aligned} \tag{3A}$$

Or use the following two equations

$$\begin{aligned} y &= f(x,p) \\ \frac{dx}{dp} &= \frac{\frac{df}{dp}}{p-\frac{\partial f}{\partial x}} \end{aligned} \tag{2A}$$

Once we solve the ode in one of the above two cases, next we have to elminate p from these. Once p is elminated, then the solution y is found. Examples below help illustrate this method.

Sometimes it is easier to isolate x and use (3A) and sometimes it is easier to isolate y and use (2A). In theory, both should give same answer, but eliminating p can be easier using one method compared to the other. Only way is to try and find out. If it is not possible to isolate x nor y from the original ode, then this method will not work.

The main difficulties in this method are not in solving the ode (3A) nor the ode (2A) (typically these come out to be basic types) but in eliminating p from the two equations we obtained. If we ar unable to isolate p then we just leave the solution in terms of the two equations as is in a parametric form. Normally p is labeled as t at the end in this case.

This method only works if in (2A) $p-\frac{\partial f}{\partial x}\neq 0$ and in (3A) $1-p\frac{\partial f}{\partial y}\neq 0$.

As always in math, the best way to learn a method is to solve some examples.

1.2 Examples

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1.2.1 Example 1 $y(y')^2 - 4xy' + y = 0$

Given

$$y(y')^2 - 4xy' + y = 0 \tag{1}$$

Let $y' = p$. The above becomes

$$\begin{aligned} yp^2 - 4xp + y &= 0 \\ f(x, y, p) &= 0 \end{aligned} \tag{2}$$

We have to either isolate x or y as both can be inside f. Let us start by isolating x. The above becomes

$$x = \frac{yp^2 + y}{4p}$$

Therefore $f(y, p) = \frac{yp^2+y}{4p}$ now, We now apply (3A) (not 2A) and obtain

$$\frac{dy}{dp} = \frac{p\frac{df}{dp}}{1 - p\frac{\partial f}{\partial y}}$$

Hence

$$\begin{aligned} \frac{dy}{dp} &= \frac{\frac{y(p^2-1)}{4p}}{1 - \frac{p^2}{4} - \frac{1}{4}} \\ &= -\frac{y}{p}\frac{p^2 - 1}{p^2 - 3} \\ &= \frac{y}{p}\frac{(p^2 - 1)}{(3 - p^2)} \end{aligned}$$

This is separable. Solving gives

$$y = \frac{c_1}{(p\,(p^2 - 3))^{\frac{1}{3}}}$$

Therefore the parametric solution is

$$
\begin{aligned}
x &= \frac{y p^2 + y}{4p} \\
y &= \frac{c_1}{\left(p\left(p^2-3\right)\right)^{\frac{1}{3}}}
\end{aligned}
$$

The above is the solution to (1) in parametric form where the dependency between y and x is via p. We can stop here. But let see if we can get the solution as $y(x)$ as the normal case is. Eliminating p between 4(1) and 4(2) results in the solution

$$
c^6 - 64c^3x^3 + 24c^3xy^2 - 48x^2y^4 + 16y^6 = 0
$$

And the above is the final nonparametric solution. It is an implicit solution.

Let try the other way. Let us start by isolating y. This gives

$$
\begin{aligned}
y p^2 - 4xp + y &= 0 \\
f(x, y, p) &= 0
\end{aligned}
$$

Isolating y. The above becomes

$$
y = \frac{4xp}{1+p^2}
$$

Therefore $f(y,p) = \frac{4xp}{1+p^2}$ now, We now apply (2A) (not 3A) and obtain

$$
\frac{dx}{dp} = \frac{\frac{df}{dp}}{p - \frac{\partial f}{\partial x}}
$$

Hence

$$
\begin{aligned}
\frac{dx}{dp} &= \frac{\frac{-4xp^2+4x}{\left(p^2+1\right)^2}}{p - \frac{4p}{p^2+1}} \\
&= 4x \frac{p^2-1}{-p^5+2p^3+3p}
\end{aligned}
$$

Solving gives

$$
x = \frac{\left(p^2+1\right) c_1}{\left(p^2-3\right)^{\frac{1}{3}} p^{\frac{4}{3}}}
$$

Thus the parametric solution is

$$
\begin{aligned}
y &= \frac{4xp}{1+p^2} \\
x &= \frac{\left(p^2+1\right) c_1}{\left(p^2-3\right)^{\frac{1}{3}} p^{\frac{4}{3}}}
\end{aligned}
$$

Eliminating p gives the solution. But it was harder to eliminate p using this approach.

We might think this method is complicated, but it is actually much simpler than direct method. How would we solve original ode directly? We start by solving for y' in (1) which gives two ode's that we need to solve each on its own.

$$\begin{aligned} y' &= \frac{2x + \sqrt{4x^2 - y^2}}{y} \\ y' &= \frac{2x - \sqrt{4x^2 - y^2}}{y} \end{aligned} \tag{5}$$

Starting with the first one above, we notice it is homogeneous ode. Let $u = \frac{y}{x}$ and it becomes

$$u' = \frac{-u^2 + \sqrt{-u^2 + 4} + 2}{ux}$$

This is separable which results in

$$\int \frac{u}{-u^2 + \sqrt{-u^2 + 4} + 2} du = \int \frac{1}{x} dx$$

The above integrals gives a very complicated antiderivative. After that we have to replace u back by $\frac{y}{x}$ and simplify. We now do the same for the second ode in (5). It is clear here that the parametric method is simpler. But for the parametric method to work, we would have to be able isolate p.

1.2.2 Example 2 $y - xy' - y' + (y')^2 = 0$

Given

$$y - xy' - y' + (y')^2 = 0 \tag{1}$$

This problem from chapter 7, problem 7. From Boole book, page 137. This is actually a clairaut ode. Let $y' = p$. The above becomes

$$y - xp - p + p^2 = 0 \tag{1}$$

We start by isolating x which gives

$$\begin{aligned} -xp &= \frac{p - p^2}{y} \\ x &= \frac{p - 1}{y} \\ &= f(y, p) \end{aligned}$$

Using (3A) and not (2A)

$$
\begin{aligned}
\frac{dy}{dp} &= \frac{p\frac{df}{dp}}{1-p\frac{\partial f}{\partial y}} \\
&= \frac{p\frac{1}{y}}{1-p\left(\frac{1-p}{y^2}\right)} \\
&= \frac{py}{p^2+y^2-p}
\end{aligned}
$$

This is non-linear ode in y. So this is no better than what we started.

Let try to isolate y instead. Solving (1) for y gives

$$
\begin{aligned}
y &= xp+p-p^2 \\
&= f(x,p)
\end{aligned}
$$

Therefore, using (2A) gives

$$
\frac{dx}{dp} = \frac{\frac{df}{dp}}{p-\frac{\partial f}{\partial x}}
$$

But $p - \frac{\partial f}{\partial x} = 0$. Hence this method does not work for this ode. The method of Clairaut works on this, since there we apply a different algorithm. It is best to keep the parmateric algorithm separate. So we try this and if this fails, then try other approaches.

1.2.3 Example 3 $y - ay' - \sqrt{1+(y')^2} = 0$

Here, we can only isolate y and use (2A). (note, this can also be solved as dAlembert). Solve

$$
y - ay' - \sqrt{1+(y')^2} = 0 \tag{1}
$$

This is problem chapter 7, problem 7. From Boole book, page 137. Let $y' = p$. The above becomes

$$
\begin{aligned}
y - ap - \sqrt{1+p^2} &= 0 \qquad (2) \\
y &= f(p) \\
&= ap + \sqrt{1+p^2}
\end{aligned}
$$

Using (2A) gives

$$
\begin{aligned}
\frac{dx}{dp} &= \frac{\frac{df}{dp}}{p - \frac{\partial f}{\partial x}} \\
&= \frac{a + \frac{p}{\sqrt{1+p^2}}}{p} \\
&= \frac{a\sqrt{1+p^2} + p}{p\sqrt{1+p^2}} \\
&= \frac{a}{p} + \frac{1}{\sqrt{1+p^2}}
\end{aligned}
$$

This is quadrature ode. Solving gives

$$
x = a \ln p + \operatorname{arcsinh} p + c_1
$$

Hence the parameteric solution is

$$
\begin{aligned}
y &= ap + \sqrt{1+p^2} \\
x &= a \ln p + \operatorname{arcsinh} p + c_1
\end{aligned}
$$

Eliminating p gives

$$
p = \frac{ya + \sqrt{a^2 + y^2 - 1}}{a^2 - 1}
$$

Which results in the following implicit solution for y

$$
x = a \ln \left(\frac{ay + \sqrt{a^2 + y^2 - 1}}{a^2 - 1} \right) + \operatorname{arcsinh} \left(\frac{ay + \sqrt{a^2 + y^2 - 1}}{a^2 - 1} \right) + c_1
$$

Another solution is

$$
p = -\frac{-ya + \sqrt{a^2 + y^2 - 1}}{a^2 - 1}
$$

Which results in the following implicit solution for y

$$
x = a \ln \left(-\frac{-ya + \sqrt{a^2 + y^2 - 1}}{a^2 - 1} \right) - \operatorname{arcsinh} \left(-\frac{-ya + \sqrt{a^2 + y^2 - 1}}{a^2 - 1} \right) + c_1
$$

1.2.4 Example 4 $(y')^2 + 2(y')^3 + y = 0$

Solve (This can also be solved as dAlembert)

$$
(y')^2 + 2(y')^3 + y = 0 \tag{1}
$$

Let $p = \frac{dy}{dx}$

$$
\begin{aligned}
y &= p^2 + 2p^3 \qquad (2) \\
y &= f(p)
\end{aligned}
$$

Where $p = \frac{dy}{dx}$. Hence $dx = \frac{1}{p}dy$. But from the above $dy = f'(p)\,dp = (2p + 6p^2)\,dp$. Hence

$$\begin{aligned} dx &= \frac{1}{p}(2p + 6p^2)\,dp \\ &= (2 + 6p)\,dp \\ x &= \int (2 + 6p)\,dp \\ &= 2p + 3p^2 + c \end{aligned}$$

Solving for p gives

$$p = \frac{-1 \pm \sqrt{3x + c}}{3}$$

Hence the solution from (2) becomes (for the first root)

$$\begin{aligned} y &= p^2 + 2p^3 \\ &= \left(\frac{-1 + \sqrt{3x + c}}{3}\right)^2 + 2\left(\frac{-1 + \sqrt{3x + c}}{3}\right)^3 \end{aligned}$$

And for the second root

$$\begin{aligned} y &= p^2 + 2p^3 \\ &= \left(\frac{-1 - \sqrt{3x + c}}{3}\right)^2 + 2\left(\frac{-1 - \sqrt{3x + c}}{3}\right)^3 \end{aligned}$$

These methods produce simpler solution if we can solve for p easily in the above.

1.2.5 Example 5 $x = 1 + y' + (y')^3$

Note: This can also be solved as quadrature, but solving for y' which will generate 3 ode's, each can be directly integrated. We can only isolate x here and use (3A) since there is no y.

$$\begin{aligned} x &= 1 + y' + (y')^3 \\ &= 1 + p + p^3 \end{aligned} \tag{1}$$

Using (3A)

$$\begin{aligned} \frac{dy}{dp} &= \frac{p\frac{df}{dp}}{1 - p\frac{\partial f}{\partial y}} \\ &= p(1 + 3p^2) \end{aligned}$$

This ode is quadrature. Solving gives

$$\begin{aligned} y &= \int p(1 + 3p^2)\,dp \\ &= \frac{p^2}{2} + \frac{3}{4}p^4 + c \end{aligned} \tag{2}$$

p is eliminated between (1,2) to obtain the final solution. This gives implicit solution for y as

$27x^4 + 64c^3 - 192c^3y + 72cx^2 + 192cy^2 - 108x^3 - 72x^2y - 64y^3 + 32c^2 - 144cx - 64cy + 164x^2 + 144xy + 32y^2 + 76c - 11$

1.2.6 Example 6 $x(y')^2 - 2yy' + 4x = 0$

This can also be solved as dAlembert ode.

$$
\begin{aligned}
x(y')^2 - 2yy' + 4x &= 0 \\
xp^2 - 2yp + 4x &= 0
\end{aligned} \tag{1}
$$

We start by trying to isolate x

$$
\begin{aligned}
x\left(4 + p^2\right) &= 2yp \\
x &= \frac{2yp}{4 + p^2} \\
&= f(y, p)
\end{aligned} \tag{2}
$$

Eq (3A) gives

$$
\begin{aligned}
\frac{dy}{dp} &= \frac{p\frac{df}{dp}}{1 - p\frac{\partial f}{\partial y}} \\
&= \frac{2yp}{p^2 + 4}
\end{aligned}
$$

This is separable (and linear). solving gives

$$
y = cp^2 + 4c \tag{3}
$$

Eliminating p from (2,3) gives

$$
p = \frac{x}{2c}
$$

And implicit solution for y

$$
y\left(16c^2 - 4cy + x^2\right) = 0
$$

Which implies

$$
\begin{aligned}
y &= 0 \\
16c^2 - 4cy + x^2 &= 0
\end{aligned}
$$

Or

$$
\begin{aligned}
y &= 0 \\
y &= \frac{x^2 + 16c^2}{4c} \\
&= \frac{x^2 + c_2^2}{c_2}
\end{aligned}
$$

But the solution $y = 0$ does not satisfy the ode. So it is removed. So the solution is

$$
y = \frac{x^2 + c_2^2}{c_2}
$$

This is the same general solution obtained using dAlembert, but with dAlembert method, we also obtain singular solutions $y = \pm 2x$. This method does not find these, only the general solution. See my dAlembert showing how this problem is solved using that method.

Let us now solving this problem by isolating y instead of x and see if we get same solution. From original ode

$$\begin{aligned} xp^2 - 2yp + 4x = 0 \\ y &= \frac{4x + xp^2}{2p} \\ &= f(x,p) \end{aligned} \tag{4}$$

Using (2A) gives

$$\begin{aligned} \frac{dx}{dp} &= \frac{\frac{df}{dp}}{p - \frac{\partial f}{\partial x}} \\ &= \frac{x}{p} \end{aligned}$$

This is easilt solved giving

$$x = c_1 p \tag{5}$$

Eliminating p between (4,5) gives

$$\begin{aligned} y &= \frac{4c_1^2 + x^2}{2c_1} \\ &= \frac{c^2 + x^2}{c} \end{aligned}$$

Which is the same solution found earlier.

1.2.7 Example 8 $e^{2y}(y')^3 + (e^{2x} + e^{3x})\, y' - e^{3x} = 0$

$$e^{2y}p^3 + \left(e^{2x} + e^{3x}\right) p - e^{3x} = 0$$

Isolating y

$$\begin{aligned} e^{2y}p^3 &= e^{3x} - \left(e^{2x} + e^{3x}\right) p \\ e^{2y} &= \frac{e^{3x} - \left(e^{2x} + e^{3x}\right) p}{p^3} \\ 2y &= \ln\left(\frac{e^{3x} - \left(e^{2x} + e^{3x}\right) p}{p^3}\right) \\ y &= \frac{1}{2}\ln\left(\frac{e^{3x} - \left(e^{2x} + e^{3x}\right) p}{p^3}\right) = f(x,p) \end{aligned} \tag{1}$$

Using (2A)

$$
\begin{aligned}
\frac{dx}{dp} &= \frac{\frac{df}{dp}}{p-\frac{\partial f}{\partial x}} \\
&= \frac{-1}{p\left(p-1\right)}
\end{aligned}
$$

This ode is just quadrature. Integrating gives

$$
x = -\ln\left(p-1\right)+\ln\left(p\right)+c \tag{2}
$$

Eliminating p between (1,2) gives

$$
\begin{aligned}
y &= \frac{1}{2}\ln\left(-\left(e^{-x+c}-1\right)^{2}\left(e^{3x}e^{-x+c}+e^{2x}\right)\right) \\
&= \frac{1}{2}\ln\left(-\left(c_1e^{-x}-1\right)^{2}\left(c_1e^{3x}e^{-x}+e^{2x}\right)\right) \\
&= \frac{1}{2}\ln\left(-\left(c_1e^{-x}-1\right)^{2}\left(c_1e^{2x}+e^{2x}\right)\right) \\
&= \frac{1}{2}\ln\left(-\left(c_1e^{-x}-1\right)^{2}e^{2x}(1+c_1)\right) \\
&= \frac{1}{2}\ln\left(-(1+c_1)\left(c_1e^{-x}-1\right)^{2}\right)+\frac{1}{2}\ln\left(e^{2x}\right) \\
&= \frac{1}{2}\ln\left(-(1+c_1)\left(c_1e^{-x}-1\right)^{2}\right)+x
\end{aligned}
$$

This is good example where solving using parametric method is much easier than otherwise. Actually, sympy 1.13.3 and Mathematica V 14.2 were not able to solve this ode. They probably do not have this method implemented.

1.2.8 Example 9 $y' = -\frac{x}{2}-1+\frac{1}{2}\sqrt{x^2+4x+4y}$

This is clairaut ode.

$$
\begin{aligned}
p &= -\frac{x}{2}-1+\frac{1}{2}\sqrt{x^2+4x+4y} \\
\frac{1}{2}\sqrt{x^2+4x+4y} &= p+1+\frac{x}{2} \\
\sqrt{x^2+4x+4y} &= 2p+2+x \\
x^2+4x+4y &= (2p+2+x)^2 \\
x^2+4x+4y &= 4p^2+4px+8p+x^2+4x+4
\end{aligned}
$$

$$
4y-8p-4px-4p^2-4=0 \tag{1}
$$

Isolating x gives

$$
\begin{aligned}
x &= -\frac{p^2+2p-y+1}{p} \\
&= f(y,p) \qquad (2)
\end{aligned}
$$

Using (3A)

$$\frac{dy}{dp} = \frac{p\frac{df}{dp}}{1 - p\frac{\partial f}{\partial y}}$$

But $1 - p\frac{\partial f}{\partial y} = 0$, so this did not work. Let us try to isolate y to see if we get better luck. Isolating y from (1) gives

$$\begin{aligned} y &= p^2 + xp + 2p + 1 \\ &= f(x,p) \end{aligned}$$

Using (2A)

$$\frac{dx}{dp} = \frac{\frac{df}{dp}}{p - \frac{\partial f}{\partial x}}$$

But here we also get $p - \frac{\partial f}{\partial x} = 0$. So it is not possible to use parametric method on this ode. Does this happen always on clairaut ode? No, we solved clairaut using this method in example 2 above. It is just by chance we get denominator zero for this specific ode. I think it is because p is linear in the ode. This method should be used for ode which has non linear $y'(x)$ in it. But I need to double check more on this. For now, I check that p is nonlinear before using this method.

1.2.9 Example 10 $y' = -y - \sin(x)$

$$p = -y - \sin(x)$$

This example was added to show that parametric method works also for standard first order ode's such as linear, separable and so on. But it should not be used for these, as the generated ode can be more complicated than the original ode and there is no point of using this method here. This method should be used only for ode's which are non-linear first order as in all the examples above.

Isolating y gives

$$\begin{aligned} y &= -\sin x - p \qquad (1) \\ &= f(x,p) \end{aligned}$$

Using (2A)

$$\begin{aligned} \frac{dx}{dp} &= \frac{\frac{df}{dp}}{p - \frac{\partial f}{\partial x}} \\ &= \frac{-1}{p - \cos x} \end{aligned}$$

We see that this ode is much more complicated to solve that the original ode (which is linear ode and can be easily solved using integrating factor). Solving this ode gives

$$p - \frac{\cos x}{2} - \frac{\sin x}{2} - e^{-x}c = 0 \qquad (2)$$

Eliminating p from (1,2) gives

$$
\begin{aligned}
y &= -\frac{e^x(3\sin x + \cos x) + 2c}{2e^x} \\
&= -\frac{1}{2}(3\sin x + \cos x) + ce^{-x}
\end{aligned}
$$

Which is the same solution we could found by solving the original ode using standard integrating factor method much more easily.

1.3 References

There is almost no mention of this method in any books I've looked at. Only a glimps of this method is given on 2 pages in the book by Ames and this is where I expanded it from. The book by Boole does not explicitly mention this but page 133 could be related.

1. Nonlinear ordinary differential equations in transport processes. William F. Ames. Academic press 1968. page 40-41.

2. Differential equations by George Boole. 1865. page 133.