

# Animation of swinging pendulum on spring

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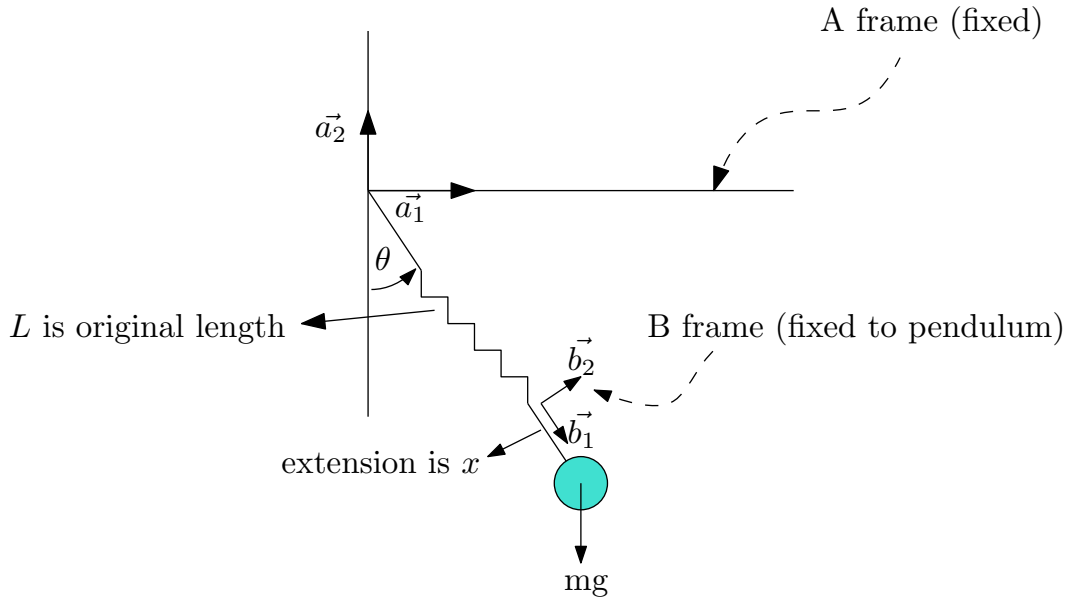


Figure 1: the Problem statement

Let original length of pendulum be  $L$  and extension at instance shown be  $x$ . The position vector of bob is

$$\vec{r} = (L + x) \vec{b}_1$$

Hence the velocity vector is

$$\begin{aligned} \left( \frac{d}{dt} \vec{r} \right)_A &= \left( \frac{d}{dt} \vec{r} \right)_B + \vec{\omega} \times \vec{r} \\ \vec{v}_A &= \left( \frac{d}{dt} (L + x) \vec{b}_1 \right)_B + \vec{\omega} \times ((L + x) \vec{b}_1) \\ &= \dot{x} \vec{b}_1 + \dot{\theta} \vec{b}_3 \times ((L + x) \vec{b}_1) \\ &= \dot{x} \vec{b}_1 + \dot{\theta} (L + x) \vec{b}_2 \end{aligned}$$

And the acceleration is

$$\left( \frac{d}{dt} \vec{v} \right)_A = \left( \frac{d}{dt} \vec{v} \right)_B + \vec{\omega} \times \vec{v}$$

But

$$\begin{aligned}\left(\frac{d}{dt}\bar{v}\right)_B &= \frac{d}{dt}(\dot{x}\bar{b}_1 + \dot{\theta}(L+x)\bar{b}_2)_B \\ &= \ddot{x}\bar{b}_1 + (\ddot{\theta}(L+x) + \dot{\theta}\dot{x})\bar{b}_2\end{aligned}\quad (2)$$

And

$$\begin{aligned}\bar{\omega} \times \bar{v} &= \bar{\omega} \times (\dot{x}\bar{b}_1 + \dot{\theta}(L+x)\bar{b}_2) \\ &= \dot{\theta}\bar{b}_3 \times (\dot{x}\bar{b}_1 + \dot{\theta}(L+x)\bar{b}_2) \\ &= \dot{\theta}\dot{x}\bar{b}_2 + \dot{\theta}^2(L+x)\bar{b}_1\end{aligned}\quad (3)$$

Substituting (2,3) into (1) gives the acceleration of the bob in frame  $A$  as

$$\begin{aligned}\bar{a}_A &= \ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\bar{b}_2 + \dot{\theta}\dot{x}\bar{b}_2 + \dot{\theta}^2(L+x)\bar{b}_1 \\ &= \ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\bar{b}_2 + 2\dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\bar{b}_1\end{aligned}$$

To obtain  $\bar{F} = m\bar{a}$  we now just need to find  $\bar{F}$ . The force on bob is given by just the weight and the spring force acting on it. The spring force is proportional to spring coefficient  $k$  times the extension.  $F_s = -kx$ . Hence the force vector is

$$\bar{F} = mg \cos \theta \bar{b}_1 - mg \sin \theta \bar{b}_2 - kx \bar{b}_1$$

Therefore the equation of motion is

$$\begin{aligned}\bar{F} &= m\bar{a} \\ mg \cos \theta \bar{b}_1 - mg \sin \theta \bar{b}_2 - kx \bar{b}_1 &= m(\ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\bar{b}_2 + 2\dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\bar{b}_1) \\ g \cos \theta \bar{b}_1 - g \sin \theta \bar{b}_2 - \frac{k}{m}x \bar{b}_1 &= \ddot{x}\bar{b}_1 + \ddot{\theta}(L+x)\bar{b}_2 + 2\dot{\theta}\dot{x}\bar{b}_2 - \dot{\theta}^2(L+x)\bar{b}_1\end{aligned}$$

Or, by equating each vector component

$$\begin{aligned}\ddot{x} - \dot{\theta}^2(L+x) + \frac{k}{m}x &= g \cos \theta \\ \ddot{\theta}(L+x) + 2\dot{\theta}\dot{x} &= -g \sin \theta\end{aligned}$$

Or

$$\begin{aligned}\ddot{x} &= \dot{\theta}^2(L+x) - \frac{k}{m}x + g \cos \theta \\ \ddot{\theta} &= -\frac{2\dot{\theta}\dot{x}}{L+x} - \frac{g}{L+x} \sin \theta\end{aligned}$$

Let initial conditions be  $L = 1, x(0) = 0.1, \dot{x}(0) = 0, \theta(0) = 20^\circ, \dot{\theta}(0) = .1$  (rad/sec).

We can now solve for  $x, \theta$  as function of time and make the animations. We can assume values for  $m, k$  and  $g = 9.81$ .

The solution will give  $x(t), \theta(t)$ . To plot the path, we need to express  $x(t)$  in frame  $A$  coordinates of course which is the fixed frame. But this is easy, since the pendulum frame  $B$  is fixed at the base on the frame  $A$  origin.

The following is plot of the solution using  $k = 0.99, m = 0.09$

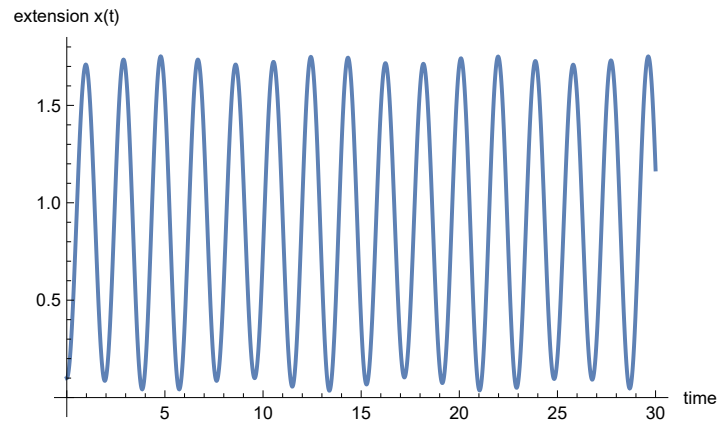


Figure 2: position of bob

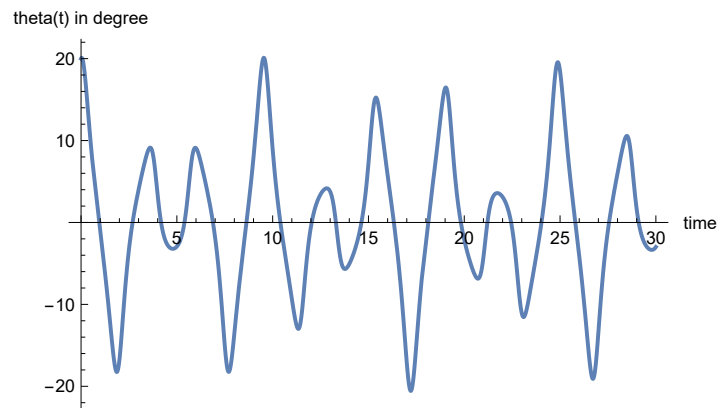


Figure 3: Angle  $\theta$  of pendulum

The following is a quick animation of the above

The following is the code used

```
(*Nasser M. Abbasi, Jan 25, 2025*)

SetDirectory[NotebookDirectory[]]

L = 1;
k = .99;
m = .09;
g = 9.81;
ode1 = x'[t] == z'[t]^2*(L + x[t]) + g*Cos[z[t]] - k/m*x[t];
ode2 = z'[t] == -2 z'[t]*x'[t]/(L + x[t]) - g*Sin[z[t]]/(L + x[t]);
IC = {x[0] == .1, x'[0] == 0, z[0] == 20 Degree, z'[0] == .1};
sol = NDSolve[{ode1, ode2, IC}, {x, z}, {t, 0, 100}]

Manipulate[
Module[{currentX, currentY},
currentX = (L + Evaluate[x[t0] /. sol])*Sin[Evaluate[z[t0] /. sol]];

```

```

currentY = (L + Evaluate[x[t0] /. sol])*Cos[Evaluate[z[t0] /. sol]];
Grid[{
  {Graphics[{
    Line[{0, 0}, {First@currentX, -First@currentY}],
    {Blue, Disk[{First@currentX, -First@currentY}, .1]}
  ]},
  PlotRange -> {{-2, 2}, {-4, 1}}, GridLines -> Automatic,
  GridLinesStyle -> LightGray
  ]}],
{{t0, 0, "time"}, 0, 10, .01},
TrackedSymbols -> {t0}
]

```