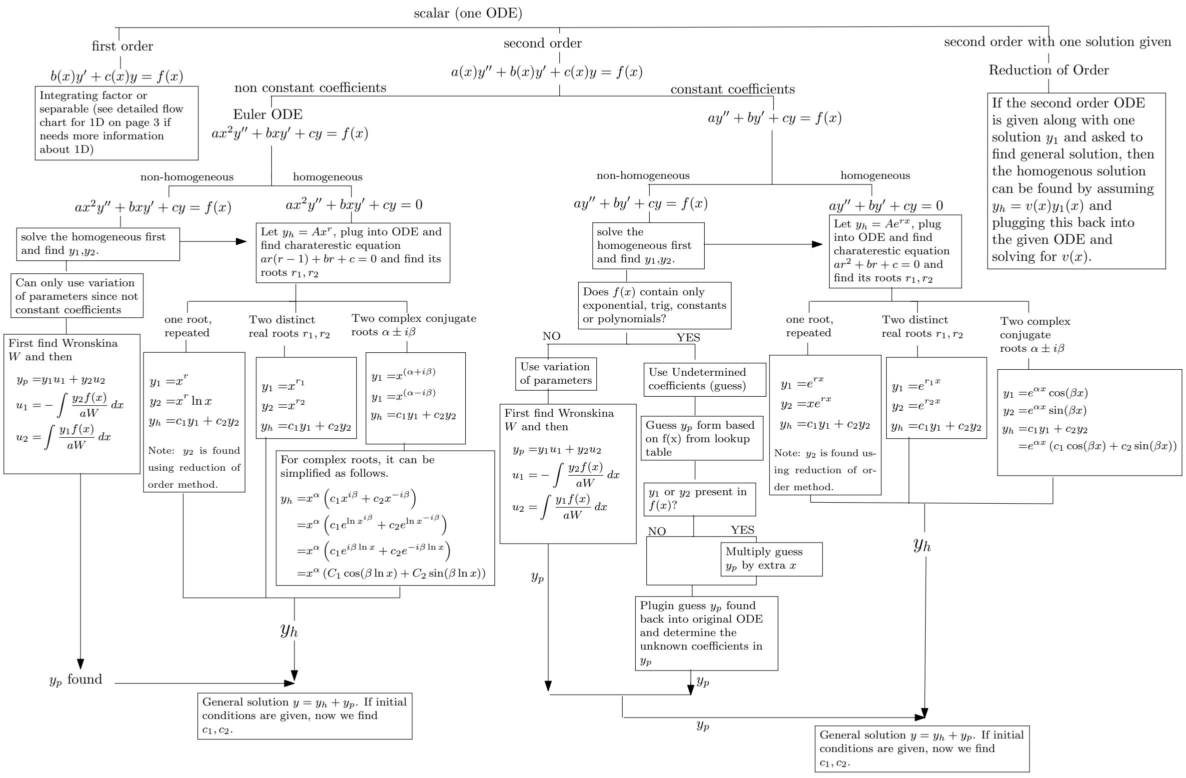
ODE classification

Nasser M. Abbasi

December 29, 2020 Compiled on December 29, 2020 at 12:05am



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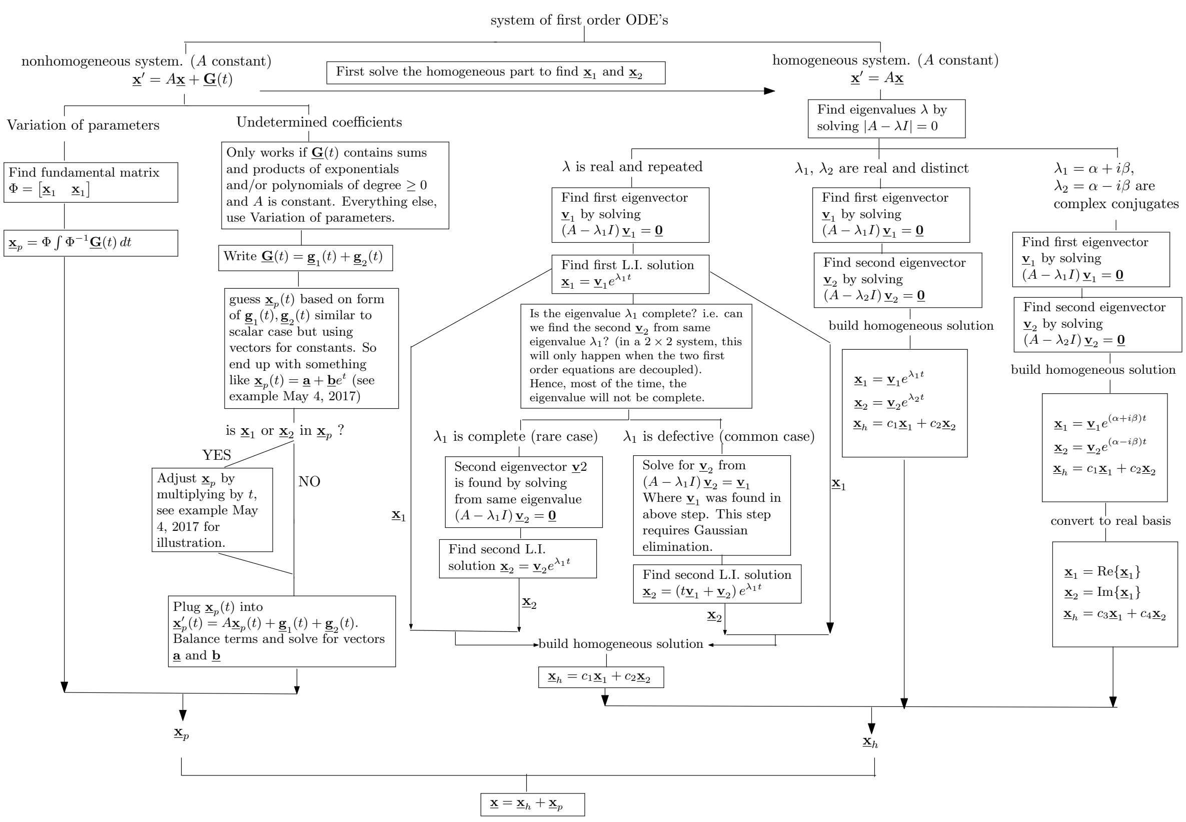

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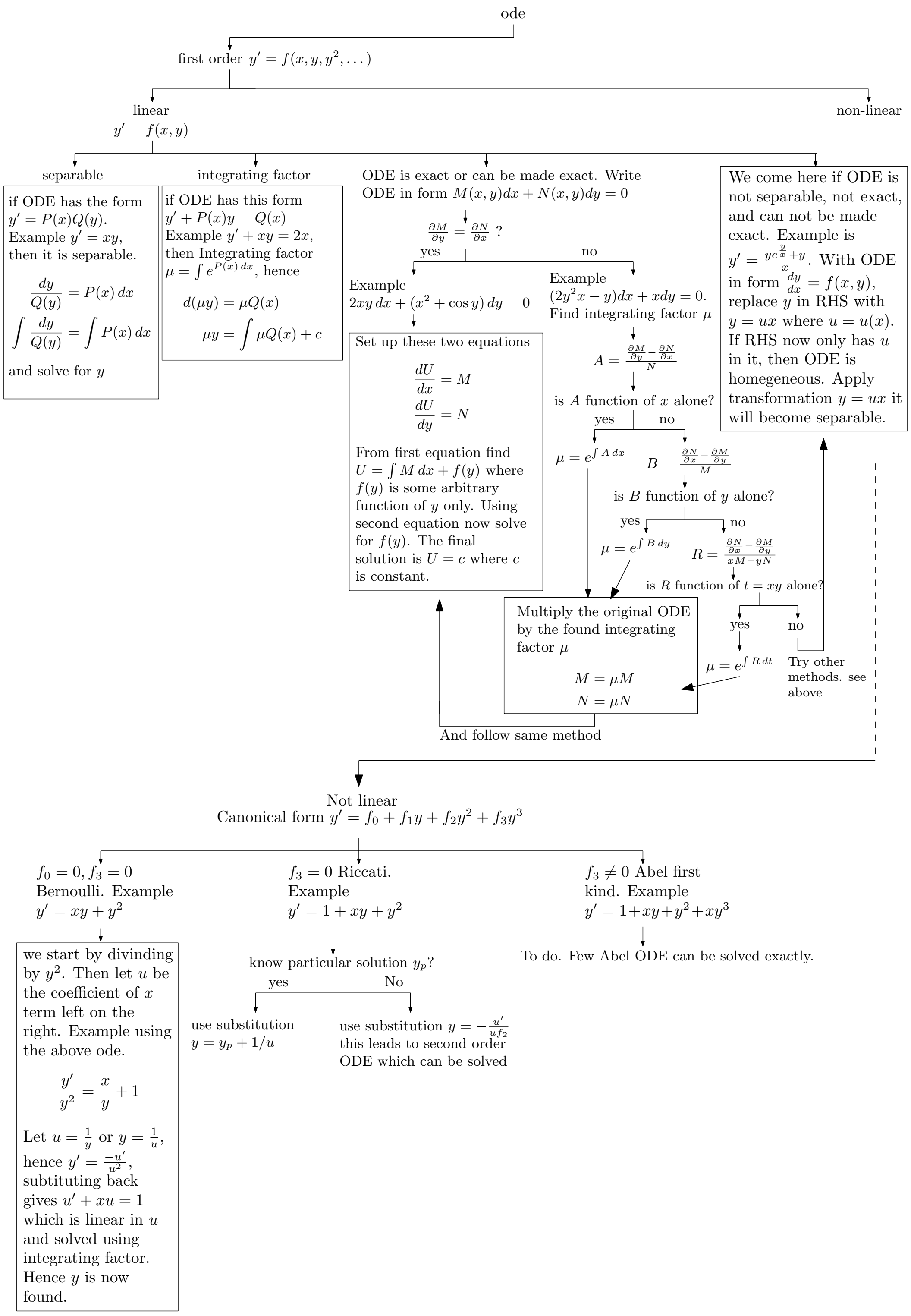
ode
first order $y' = f(x,y,y^2,\dots)$
linear
$y' = f(x,y)$
non-linear
separable
if ODE has the form $y' = P(x)Q(y)$. Example $y' = xy$, then it is separable.
$\frac{dy}{Q(y)} = P(x)\,dx$
$\int \frac{dy}{Q(y)} = \int P(x)\,dx$
and solve for y
integrating factor
if ODE has this form $y' + P(x)y = Q(x)$ Example $y' + xy = 2x$, then Integrating factor $\mu = \int e^{P(x)\,dx}$, hence
$d(\mu y) = \mu Q(x)$
$\mu y = \int \mu Q(x) + c$
ODE is exact or can be made exact. Write ODE in form $M(x,y)dx + N(x,y)dy = 0$
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?
yes
no
Example $2xy\,dx + (x^2 + \cos y)\,dy = 0$
Set up these two equations
$\frac{dU}{dx} = M$
$\frac{dU}{dy} = N$
From first equation find $U = \int M\,dx + f(y)$ where $f(y)$ is some arbitrary function of y only. Using second equation now solve for $f(y)$. The final solution is $U = c$ where c is constant.
Example $(2y^2x - y)dx + xdy = 0$. Find integrating factor μ
$A = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$
is A function of x alone?
yes
no
$\mu = e^{\int A\,dx}$
$B = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$
is B function of y alone?
yes
no
$\mu = e^{\int B\,dy}$
$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$
is R function of $t = xy$ alone?
yes
no
$\mu = e^{\int R\,dt}$
Try other methods. see above
Multiply the original ODE by the found integrating factor μ
$M = \mu M$
$N = \mu N$
And follow same method
We come here if ODE is not separable, not exact, and can not be made exact. Example is $y' = \frac{ye^{\frac{y}{x}} + y}{x}$. With ODE in form $\frac{dy}{dx} = f(x,y)$, replace y in RHS with $y = ux$ where $u = u(x)$. If RHS now only has u in it, then ODE is homegeneous. Apply transformation $y = ux$ it will become separable.
Not linear
Canonical form $y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$
$f_0 = 0, f_3 = 0$ Bernoulli. Example $y' = xy + y^2$
we start by divinding by y^2. Then let u be the coefficient of x term left on the right. Example using the above ode.
$\frac{y'}{y^2} = \frac{x}{y} + 1$
Let $u = \frac{1}{y}$ or $y = \frac{1}{u}$, hence $y' = \frac{-u'}{u^2}$, subtituting back gives $u' + xu = 1$ which is linear in u and solved using integrating factor. Hence y is now found.
$f_3 = 0$ Riccati. Example $y' = 1 + xy + y^2$
know particular solution y_p?
yes
No
use substitution $y = y_p + 1/u$
use substitution $y = -\frac{u'}{uf_2}$ this leads to second order ODE which can be solved
$f_3 \neq 0$ Abel first kind. Example $y' = 1 + xy + y^2 + xy^3$
To do. Few Abel ODE can be solved exactly.

1 Detailed algorithm

Algorithm 1 solve_first_order_ode

```
procedure SOLVE_FIRST_ORDER_ODE(ode,y,x,ic)
    if ode missing y'(x) then
        if ode missing y(x) then
            return (error)
        else
            this is algebraic equation and not ode. solve for y(x)
        end if
    else
        if ode missing y(x) then
            solve for y'(x) and integrate each solution directly to find y(x)
        else
            general case, where both y'(x) and y(x) are present.
            if ode is linear in y'(x) then
                solve for y'(x) and write ode as y'(x) = f(x, y)
                if f(x, y) is linear in y(x) then
                    write f(x, y) in the form f(x, y) = A + By(x)
                    solve y' = A + By using integrating factor
                else
                    ode is linear in y'(x) but not linear in y(x). Most common case.
                    if is_class_A_ode(ode) then
                        return solution
                    else
                        unable to solve
                    end if
                end if
            else
                the hard case. ode not linear in y'(x)
                if is_class_B_ode(ode) then
                    return solution
                else
                    unable to solve
                end if
            end if
        end if
    end if
end procedure
```

Algorithm 2 is_class_A_ode

procedure IS_CLASS_A_ODE(*ode*,y,x)
  ode is 1st order, linear in $y'(x)$ but not linear in $y(x)$
  let ode be $y'(x) = f(x,y)$
  if is_Bernoulli_ode(ode) **then**
    solve as Bernoulli. Example is $y' = xy + y^3$
    return **true**
  else if is_Riccati_ode(ode) **then**
    solve as Riccati.
    return **true**
  else if is_Abel_first_kind(ode) **then**
    solve as Abel first kind
    return **true**
  else if is_Abel_second_kind(ode) **then**
    solve as Abel second kind
    return **true**
  else if is_Chini_(ode) **then**
    solve as Chini ode
    return **true**
  else if is_separable_(ode) **then**
    solve as seperable ode
    return **true**
  else if is_homogeneous_first_order(ode) **then**
    solve as homogeneous first order ode
    return **true**
  else if is_polynomial_form_ode(ode) **then**
    solve as special polynomial form ode
    return **true**
  else if is_exact_ode(ode) **then**
    solve as exact ode
    return **true**
  else if is_dAlembert_ode(ode) **then**
    Special check. Check for dAlembert ODE even if the input ODE was linear in $y'(x)$ this is becuase of cases like $y'(x) = \sqrt{1 + x + y(x)}$, which can be written as $y'(x)^2 = 1 + x + y(x)$ or $y(x) = -1 - x + y'(x)^2$ which is dAlembert ODE.
    Call dAlembert ODE solver to solve it.
    return **true**
  else if is_Clairaut_ode(ode) **then**
    Special check. Check for Clairaut ODE even if the input ODE was linear in $y'(x)$ this is becuase of cases like $y'(x) + 1 + \frac{x}{2} = \sqrt{x^2 + 4x + 4y}$, which can be written as $y(x) = xy'(x) + \left((y')^2 + 1 + 2y'\right)$ which is Clairaut ODE ODE.
    Call Clairaut ODE solver to solve it.
    return **true**
  else
    return **false**
  end if
end procedure

Algorithm 3 is_class_B_ode

procedure IS_CLASS_B_ODE(*ode*,y,x,ic)
    ode is first order ODE, not linear in $y'(x)$
    if is_Clairaut_ode(ode) **then**
        solve as clairaut ode
        return **true**
    else if is_dAlembert_ode(ode) **then**
        solve as dAlembert_ode
        return **true**
    else if ode has form $(y')^n = f(x,y)$ **then**
        if $|n| > 1$ **then**
            sols = solve for y' by finding the roots. This will generate n new ode's
            for all sol_i in sols **do**
                call solve_first_order_ode(sol_i, y, x, ic)
            end for
            return **true**
        else
            since $|n| < 1$, then ode has form $(y')^{1/m} = f(x,y)$ for some integer $m > 1$. raise power of both side by m to obtain $y' = f(x,y)^m$ and call solve_first_order_ode() on this ode.
            return **true**
        end if
    else
        if ode has form $\left(A(y')^n + B(y')^m + ...\right) = f(x,y)$ **then**
            solve for y'. For each root solution call solve_first_order_ode()
            return **true**
        else
            return **false**
        end if
    end if
end procedure

Algorithm 4 check if ODE is separable type

procedure IS_SEPARABLE_ODE(*ode*,y,x,ic)
    ode is first order ODE linear in $y'(x)$
    if ode has form $y' = A(x)B(y)$ where $A(x)$ is function of x only and $B(y)$ is function of y only, both $A(x), B(y)$ can be linear or nonlinear functions **then**
        return **true**
    else
        return **false**
    end if
end procedure

Algorithm 5 check if ODE is homogeneous first order type

procedure IS_HOMOGENEOUS_FIRST_ORDER(*ode*,y,x,ic)
    ode is first order ODE linear in $y'(x)$
    if ode has form $y' = F(x,y)$ where $F(x,y)$ is homogeneous function **then**
        A homogeneous is one which $F((tx),(ty)) = F(x,y)$ for all t
        return **true**
    else
        return **false**
    end if
end procedure

Algorithm 6 check if ODE is ratio of two polynomials type

procedure IS_POLYNOMIAL_FORM_ODE(*ode*,*y*,*x*,*ic*)
    ode is first order ODE linear in $y'(x)$
    if ode has form $y' = \frac{Ax+By+C}{Dx+Ey+F}$ for non-zero A, B, C, D, E, F **then**
        return **true**
    else
        return **false**
    end if
end procedure

Algorithm 7 check if ODE is exact or can be made exact using special integrating factor

procedure IS_EXACT_ODE(*ode*,*y*,*x*,*ic*)
    ode is first order ODE linear in $y'(x)$
    write the ode as $M(x,y) + N(x,y)y' = 0$ or $M(x,y)\,dx + N(x,y)\,dy = 0$
    if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ **then**
        example is $2xy + (x^2 + \cos y)y' = 0$
        return **true**
    else
        example is $(2y^2x - y)\,dx + x\,dy = 0$
        Let $A = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$
        if A function of x only **then**
            return **true**
        else
            Let $B = \frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$
            if B function of y only **then**
                return **true**
            else
                Let $R = \frac{-1}{xM-yN}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$
                if R function of t only where $t = xy$ **then**
                    return **true**
                else
                    return **false**
                end if
            end if
        end if
    end if
end procedure

Algorithm 8 check if ODE is Bernoulli type

procedure IS_BERNOULLI_ODE(*ode*,*y*,*x*,*ic*)
    ode is first order ODE linear in $y'(x)$
    if ode has form $y' = Ay + By^n$ with $n \neq 0, n \neq 1$ **then**
        if at least one of A, B is function of x **then**
            return **true**
        else
            return **false**
        end if
    else
        return **false**
    end if
end procedure

Algorithm 9 check if ODE is Riccati type

```
procedure IS_RICCATI_ODE(ode,y,x,ic)
    ode is first order ODE linear in y'(x)
    if ode has the form y' = A + By + Cy^2 or y' = A + Cy^2 then
        if at least one of A, B, C is function of x then
            return true
        else
            return false
        end if
    else
        return false
    end if
end procedure
```

Algorithm 10 check if ODE is Abel first kind type

```
procedure IS_ABEL_FIRST_KIND_ODE(ode,y,x,ic)
    ode is first order ODE linear in y'(x)
    if ode has one of the following forms
      y' = A + By + Cy^2 + Dy^3
      y' = By + Cy^2 + Dy^3
      y' = A + Cy^2 + Dy^3
      y' = A + By + Dy^3
      y' = A + Dy^3
      y' = Cy^2 + Dy^3                      then
        if at least one of A, B, C, D is function of x then
            return true
        else
            return false
        end if
    else
        return false
    end if
end procedure
```

Algorithm 11 check if ODE is Abel first kind type

```
procedure IS_ABEL_FIRST_KIND_ODE(ode,y,x,ic)
    ode is first order ODE linear in y'(x)
    if ode has the form (E + Fy)y' = A + By + Cy^2 + Dy^3 then
        if is_Abel_first_kind_ode(y' = A + By + Cy^2 + Dy^3) and F ≠ 0 then
            return true
        else
            return false
        end if
    else
        return false
    end if
end procedure
```

Algorithm 12 check if ODE is Chini type

```
procedure IS_CHINI_ODE(ode,y,x,ic)
    ode is first order ODE linear in y'(x)
    if ode has one of these forms
      y' = A + By^n
      y' = A + By + Cy^n
      where n ≠ 0, n ≠ 1, n ≠ 2, n ≠ 3 then
        if at least one of A, B, C is function of x then
            return true
        else
            return false
        end if
    else
        return false
    end if
end procedure
```

2 examples

2.1 second order, constant coeff.

2.1.1 second order, constant coeff. homogeneous

second order, constant coeff. homogeneous, one root repeated

$$y'' - 2y' + 1 = 0$$

Let $y = Ae^{rx}$ and plug into the above and simplify, we obtain the charaterstic equation

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \\ r &= 1 \end{aligned}$$

Repated root. Hence the two L.I. basis solutions are

$$\begin{aligned} y_1 &= e^x \\ y_2 &= xe^x \end{aligned}$$

And the homogeneous solution is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 e^x \end{aligned}$$

second order, constant coeff. homogeneous, two real distinct roots

$$y'' + y' - 2y = 0$$

Let $y = Ae^{rx}$ and plug into the above and simplify, we obtain the charaterstic equation

$$\begin{aligned} r^2 + r - 2 &= 0 \\ (r-1)(r+2) &= 0 \\ r_1 &= 1 \\ r_2 &= -2 \end{aligned}$$

Hence the L.I. basis solutions are

$$\begin{aligned} y_1 &= e^x \\ y_2 &= e^{-2x} \end{aligned}$$

And the homogeneous solution is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 e^{-2x} \end{aligned}$$

second order, constant coeff. homogeneous, two complex conjugate roots

$$y'' - 6y' + 13y = 0$$

Let $y = Ae^{rx}$ and plug into the above and simplify, we obtain the charaterstic equation

$$r^2 - 6r + 13 = 0$$

Whose roots are

$$\begin{aligned} r_1 &= 3 + 2i \\ r_2 &= 3 - 2i \end{aligned}$$

Hence the L.I. basis solutions are

$$\begin{aligned} y_1 &= e^{(3+2i)x} \\ y_2 &= e^{(3-2i)x} \end{aligned}$$

the homogeneous solution is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{(3+2i)x} + c_2 e^{(3-2i)x} \end{aligned}$$

This can be converted to real basis using Euler relation which results in

$$\begin{aligned} y_h &= C_1 e^{3x} \cos 2x + C_2 e^3 \sin 2x \\ &= e^{3x}(C_1 \cos 2x + C_2 \sin 2x) \end{aligned}$$