

Notes on Using Morse function in dynamics

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Contents

1	Introduction	1
2	Examples	2
2.1	Example 1 (non linear pendulum)	2
2.2	Example 2	5
2.3	Example 3	9

1 Introduction

Morse function is used as follows. We have set of nonlinear first order ODE's. We'd like to determine if the nonlinear system is stable or not at its critical (or equilibrium) points. This is done by first finding the critical points and then linearizing the system around each one of these points, since it is much easier to check if the system is stable or not when it is linear by finding the eigenvalues of the Jacobian matrix.

But how do we know that behavior of the linear system around the critical points is the same as the nonlinear system? In other words, if the linearized system is stable at a critical point, does this mean also the nonlinear is stable? By finding Morse function, called $F(x_1, x_2, \dots)$ below, and then by finding the determinant of its Hessian evaluated at each critical point, i.e. $\det(\nabla^2 F)$, then if the determinant is not zero, then the critical point is called non-degenerate, which means the nonlinear system behaves the same as the linear system around the critical point. So if the linear system says the critical point is stable, we know the nonlinear system is also stable. But if $\det(\nabla^2 F) = 0$ then the point is called degenerate. In this case, the nonlinear system is not guaranteed to behave the same as the linearized system.

Morse function is basically the first integral at the critical point, and it always come out having a quadratic form (in the state variables of the system). If all terms in the quadratic form have positive sign, then the system is stable at the critical point. If one term in the quadratic form have negative sign, then the system is unstable there. But this all makes sense only if the point is non-degenerate, so we should always check the point is non-degenerate before trying to find Morse function.

To find Morse function, we need to first find what is called the first integral, called $\tilde{F}(x_1, x_2, \dots)$ below, of the set of given ODE's. The first integral $\tilde{F}(x_1, x_2, \dots)$ normally represents the energy of the system. For example, the Hamiltonian is a first integral. But in general, it is any function $\tilde{F}(x_1, x_2, \dots)$ which is constant along each solution path. To check that a given $\tilde{F}(x_1, x_2, \dots)$ is a first integral of the system, we can do this test

$$\frac{\partial \tilde{F}}{\partial x_1} \dot{x}_1 + \frac{\partial \tilde{F}}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial \tilde{F}}{\partial x_n} \dot{x}_n = 0$$

If the above test fail to give zero then $\tilde{F}(x_1, x_2, \dots)$ is not a first integral of the set of ODE's, and most likely we made mistake in finding it in first place. Also, not all set of ODE's have a first integral and it can be hard sometimes to find the first integral analytically since it involves solving nonlinear set of ODE's if the system is nonlinear.

2 Examples

2.1 Example 1 (non linear pendulum)

$$\ddot{x} + \sin x = 0$$

step 1 convert to state space. Let $x_1 = x, x_2 = \dot{x}$. Taking derivative w.r.t. gives $\dot{x}_1 = x_2, \dot{x}_2 = \ddot{x} = -\sin x_1$. Hence

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\sin x_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

step 2 Find first integral

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1} = \frac{-\sin x_1}{x_2}$$

Integrating

$$\int x_2 dx_2 = - \int \sin x_1 dx_1$$

$$\frac{1}{2} x_2^2 - \cos x_1 = E$$

Where E is the constant of integration. Let $\bar{F}(x_1, x_2) \equiv \frac{1}{2} x_2^2 - \cos x_1$.

step 2.1 Verify that the above is indeed first integral by checking that $\frac{\partial \bar{F}}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial \bar{F}}{\partial x_n} \dot{x}_n = 0$. Hence

$$\frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} \dot{x}_2 = \sin x_1 \dot{x}_1 + x_2 \dot{x}_2$$

But $\dot{x}_2 = \ddot{x}, x_1 = x, \dot{x}_1 = x_2$. Hence the above becomes

$$\begin{aligned} \frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} \dot{x}_2 &= \sin x x_2 + x_2 \ddot{x} \\ &= x_2 (\sin x + \ddot{x}) \end{aligned}$$

But $\sin x + \ddot{x} = 0$ since this is the ODE given. Therefore $\frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} \dot{x}_2 = 0$ as expected.

step 3 Find critical points by solving

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ -\sin x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence $x_2 = 0$ and $x_1 = \{0, \pi, -\pi\}$. Therefore the critical points are $(0, 0), (\pi, 0), (-\pi, 0)$

step 4 Find Hessian of $\bar{F}(x_1, x_2) = \frac{1}{2}x_2^2 - \cos x_1$ at each critical point and determine if the determinant of the Hessian at each critical point is non-degenerate.

$$\begin{aligned} \nabla^2 \bar{F}(x_1, x_2) &= \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x_1}(\sin x_1) & \frac{\partial F}{\partial x_1} x_2 \\ \frac{\partial F}{\partial x_2}(\sin x_1) & \frac{\partial F}{\partial x_2} x_2 \end{pmatrix} \\ &= \begin{pmatrix} \cos x_1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Hence $\nabla^2 \bar{F}(x_1, x_2)$ evaluated at $x_1 = 0, x_2 = 0$ is $\nabla^2 \bar{F}(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The determinant of this is $1 \neq 0$. Hence non-degenerate.

$\nabla^2 \bar{F}(x_1, x_2)$ evaluated at $x_1 = \pi, x_2 = 0$ is $\nabla^2 \bar{F}(x_1, x_2) = \begin{pmatrix} \cos \pi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The determinant of this is $-1 \neq 0$. Hence non-degenerate.

$\nabla^2 \bar{F}(x_1, x_2)$ evaluated at $x_1 = -\pi, x_2 = 0$ is $\nabla^2 \bar{F}(x_1, x_2) = \begin{pmatrix} \cos(-\pi) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

The determinant of this is $-1 \neq 0$. Hence non-degenerate.

Since $\det(\nabla^2 \bar{F}(x_1, x_2)) \neq 0$ at each critical point, then Morse function exist.

step 5 Find Morse function $F(x_1, x_2)$ associated with $\bar{F}(x_1, x_2)$ at each critical point.

For (0,0): Taylor expansion of $F(x_1, x_2)$ around $a = (0, 0)$ gives

$$F(x_1, x_2) = \bar{F}(a) + (x - a) \nabla F(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T$$

But $\nabla F(a) = 0$ since a is critical point. The above becomes

$$\begin{aligned}
F(x_1, x_2) &= \bar{F}(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T \\
&= \bar{F}(x_1, x_2)_{\substack{x_1=0 \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \nabla^2 \bar{F}(x_1, x_2)_{\substack{x_1=0 \\ x_2=0}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \left(\frac{1}{2} x_2^2 - \cos x_1 \right)_{\substack{x_1=0 \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \cos x_1 & 0 \\ 0 & 1 \end{pmatrix}_{\substack{x_1=0 \\ x_2=0}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= -1 + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 1 + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= 1 + \frac{1}{2} (x_1^2 + x_2^2)
\end{aligned}$$

Since $F(x_1, x_2)$ is constant, say E , then the above can be written as

$$\begin{aligned}
F(x_1, x_2) &= x_1^2 + x_2^2 \\
&= E
\end{aligned}$$

Where 1 and $\frac{1}{2}$ are combined into the constant E . Since there is no negative term in the quadratic form above, then the index of the critical point is 0 which means this is stable critical point.

For $(\pi, 0)$: Taylor expansion of $F(x_1, x_2)$ around $a = (\pi, 0)$ gives

$$F(x_1, x_2) = \bar{F}(a) + (x - a) \nabla F(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T$$

But $\nabla F(a) = 0$ since a is critical point. The above becomes

$$\begin{aligned}
F(x_1, x_2) &= \bar{F}(x_1, x_2)_{\substack{x_1=\pi \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 - \pi & x_2 \end{pmatrix} \nabla^2 \bar{F}(x_1, x_2)_{\substack{x_1=\pi \\ x_2=0}} \begin{pmatrix} x_1 - \pi \\ x_2 \end{pmatrix} \\
&= \left(\frac{1}{2} x_2^2 - \cos x_1 \right)_{\substack{x_1=\pi \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 - \pi & x_2 \end{pmatrix} \begin{pmatrix} \cos x_1 & 0 \\ 0 & 1 \end{pmatrix}_{\substack{x_1=\pi \\ x_2=0}} \begin{pmatrix} x_1 - \pi \\ x_2 \end{pmatrix} \\
&= 1 + \frac{1}{2} \begin{pmatrix} x_1 - \pi & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 - \pi \\ x_2 \end{pmatrix} \\
&= 1 + \frac{1}{2} \begin{pmatrix} -x_1 + \pi & x_2 \end{pmatrix} \begin{pmatrix} x_1 - \pi \\ x_2 \end{pmatrix} \\
&= 1 + \frac{1}{2} ((-x_1 + \pi)(x_1 - \pi) + x_2^2) \\
&= 1 + \frac{1}{2} (-(x_1 - \pi)^2 + x_2^2)
\end{aligned}$$

Since $F(x_1, x_2)$ is constant, say E , then the above can be written as

$$\begin{aligned} F(x_1, x_2) &= -(x_1 - \pi)^2 + x_2^2 \\ &= E \end{aligned}$$

Since there is a negative term in the quadratic form above, then the index of this critical point is -1 which means this is unstable critical point. (index must be zero for stable critical point)

For $(-\pi, 0)$: Taylor expansion of $F(x_1, x_2)$ around $a = (-\pi, 0)$ gives

$$F(x_1, x_2) = \bar{F}(a) + (x - a) \nabla F(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T$$

But $\nabla F(a) = 0$ since a is critical point. The above becomes

$$\begin{aligned} F(x_1, x_2) &= \bar{F}(x_1, x_2)_{\substack{x_1=-\pi \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 + \pi & x_2 \end{pmatrix} \nabla^2 \bar{F}(x_1, x_2)_{\substack{x_1=-\pi \\ x_2=0}} \begin{pmatrix} x_1 + \pi \\ x_2 \end{pmatrix} \\ &= \left(\frac{1}{2} x_2^2 - \cos x_1 \right)_{\substack{x_1=-\pi \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 + \pi & x_2 \end{pmatrix} \begin{pmatrix} \cos x_1 & 0 \\ 0 & 1 \end{pmatrix}_{\substack{x_1=-\pi \\ x_2=0}} \begin{pmatrix} x_1 + \pi \\ x_2 \end{pmatrix} \\ &= 1 + \frac{1}{2} \begin{pmatrix} x_1 + \pi & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 + \pi \\ x_2 \end{pmatrix} \\ &= 1 + \frac{1}{2} \begin{pmatrix} -(x_1 + \pi) & x_2 \end{pmatrix} \begin{pmatrix} x_1 + \pi \\ x_2 \end{pmatrix} \\ &= 1 + \frac{1}{2} (-(x_1 + \pi)^2 + x_2^2) \end{aligned}$$

Since $F(x_1, x_2)$ is constant, say E , then the above can be written as

$$\begin{aligned} F(x_1, x_2) &= -(x_1 + \pi)^2 + x_2^2 \\ &= E \end{aligned}$$

Since there is a negative term in the quadratic form above, then the index of this critical point is -1 which means this is unstable critical point. (index must be zero for stable critical point)

2.2 Example 2

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 \end{aligned}$$

step 1

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 - 2x_1^3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

step 2 Find first integral

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1} = \frac{x_1 - 2x_1^3}{x_2}$$

Integrating

$$\begin{aligned} \int x_2 dx_2 &= \int x_1 - 2x_1^3 dx_1 \\ \frac{1}{2}x_2^2 &= \frac{1}{2}x_1^2 - \frac{2}{4}x_1^4 + E \\ \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4 &= E \end{aligned}$$

Where E is the constant of integration. Let $\bar{F}(x_1, x_2) \equiv \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4$. The above is also the equation of orbit.

step 2.1 Verify that the above is indeed first integral by checking that $\frac{\partial \bar{F}}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial \bar{F}}{\partial x_n} \dot{x}_n = 0$. Hence

$$\frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} \dot{x}_2 = (-x_1 + 2x_1^3) \dot{x}_1 + x_2 \dot{x}_2$$

But $x_1 = x, \dot{x}_1 = y, x_2 = y, \dot{x}_2 = \dot{y}$ Hence the above becomes

$$\begin{aligned} \frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} \dot{x}_2 &= (-x_1 + 2x_1^3) y + y \dot{y} \\ &= y((-x_1 + 2x_1^3) + \dot{y}) \end{aligned}$$

But $\dot{y} = x - 2x^3$ as given. The above becomes

$$\begin{aligned} \frac{\partial \bar{F}(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial \bar{F}(x_1, x_2)}{\partial x_2} \dot{x}_2 &= y(-x_1 + 2x_1^3 + x - 2x^3) \\ &= 0 \end{aligned}$$

as expected for first integral.

step 3 Find critical points by solving

$$\begin{aligned} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} x_2 \\ x_1 - 2x_1^3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Therefore the critical points are $(x_1, x_2) = \left\{ (0, 0), \left(\frac{1}{\sqrt{2}}, 0 \right), \left(-\frac{1}{\sqrt{2}}, 0 \right) \right\}$

step 4 Find Hessian of $\bar{F}(x_1, x_2) = \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4$ at each critical point and determine if the determinant of the Hessian at each critical point is non-degenerate.

$$\begin{aligned}\nabla^2 \bar{F}(x_1, x_2) &= \begin{pmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x_1}(-x_1 + 2x_1^3) & \frac{\partial F}{\partial x_1} x_2 \\ \frac{\partial F}{\partial x_2}(-x_1 + 2x_1^3) & \frac{\partial F}{\partial x_2} x_2 \end{pmatrix} \\ &= \begin{pmatrix} -1 + 6x_1^2 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

Hence $\nabla^2 \bar{F}(x_1, x_2)$ evaluated at $x_1 = 0, x_2 = 0$ is $\nabla^2 \bar{F}(x_1, x_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The determinant of this is $-1 \neq 0$. Hence non-degenerate.

$\nabla^2 \bar{F}(x_1, x_2)$ evaluated at $x_1 = \frac{1}{\sqrt{2}}, x_2 = 0$ is $\nabla^2 \bar{F}(x_1, x_2) = \begin{pmatrix} -1 + 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The determinant of this is $2 \neq 0$. Hence non-degenerate.

$\nabla^2 \bar{F}(x_1, x_2)$ evaluated at $x_1 = -\frac{1}{\sqrt{2}}, x_2 = 0$ is $\nabla^2 \bar{F}(x_1, x_2) = \begin{pmatrix} -1 + 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The determinant of this is $2 \neq 0$. Hence non-degenerate.

Since $\det(\nabla^2 \bar{F}(x_1, x_2)) \neq 0$ at each critical point, then Morse function exist.

step 5 Find Morse function $F(x_1, x_2)$ associated with $\bar{F}(x_1, x_2)$ at each critical point.

For $(0, 0)$: Taylor expansion of $F(x_1, x_2)$ around $a = (0, 0)$ gives

$$F(x_1, x_2) = \bar{F}(a) + (x - a) \nabla F(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T$$

But $\nabla F(a) = 0$ since a is critical point. The above becomes

$$\begin{aligned}F(x_1, x_2) &= \bar{F}(x_1, x_2)_{\substack{x_1=0 \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \nabla^2 \bar{F}(x_1, x_2)_{\substack{x_1=0 \\ x_2=0}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \left(\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4 \right)_{\substack{x_1=0 \\ x_2=0}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -1 + 6x_1^2 & 0 \\ 0 & 1 \end{pmatrix}_{\substack{x_1=0 \\ x_2=0}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= 0 + \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \frac{1}{2} (-x_1^2 + x_2^2)\end{aligned}$$

Since $F(x_1, x_2)$ is constant, say E , then the above can be written as

$$\begin{aligned} F(x_1, x_2) &= -x_1^2 + x_2^2 \\ &= E \end{aligned}$$

Since there is a negative term in the quadratic form above, then the index of this critical point is -1 which means this is unstable critical point. It is a saddle. (index must be zero for stable critical point)

For $\left(0, \frac{1}{\sqrt{2}}\right)$: Taylor expansion of $F(x_1, x_2)$ around $a = \left(0, \frac{1}{\sqrt{2}}\right)$ gives

$$F(x_1, x_2) = \bar{F}(a) + (x - a) \nabla F(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T$$

But $\nabla F(a) = 0$ since a is critical point. The above becomes

$$\begin{aligned} F(x_1, x_2) &= \bar{F}(x_1, x_2) \Big|_{\substack{x_1=0 \\ x_2=\frac{1}{\sqrt{2}}}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \nabla^2 \bar{F}(x_1, x_2) \Big|_{\substack{x_1=0 \\ x_2=\frac{1}{\sqrt{2}}}} \begin{pmatrix} x_1 \\ x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \left(\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4 \right) \Big|_{\substack{x_1=0 \\ x_2=\frac{1}{\sqrt{2}}}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 + 6x_1^2 & 0 \\ 0 & 1 \end{pmatrix} \Big|_{\substack{x_1=0 \\ x_2=\frac{1}{\sqrt{2}}}} \begin{pmatrix} x_1 \\ x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{4} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{4} + \frac{1}{2} \begin{pmatrix} 2x_1 & x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{4} + \frac{1}{2} \left(2x_1^2 + \left(x_2 - \frac{1}{\sqrt{2}} \right)^2 \right) \end{aligned}$$

Since $F(x_1, x_2)$ is constant, say E , then the above can be written as

$$\begin{aligned} F(x_1, x_2) &= 2x_1^2 + \left(x_2 - \frac{1}{\sqrt{2}} \right)^2 \\ &= E \end{aligned}$$

Since there is no negative term in the quadratic form above, then the index of the critical point is 0 which means this is stable critical point. (Center node)

For $\left(0, \frac{-1}{\sqrt{2}}\right)$: Taylor expansion of $F(x_1, x_2)$ around $a = \left(0, \frac{-1}{\sqrt{2}}\right)$ gives

$$F(x_1, x_2) = \bar{F}(a) + (x - a) \nabla F(a) + \frac{1}{2}(x - a) \nabla^2 F(a) (x - a)^T$$

But $\nabla F(a) = 0$ since a is critical point. The above becomes

$$\begin{aligned}
 F(x_1, x_2) &= \bar{F}(x_1, x_2) \Big|_{\substack{x_1=0 \\ x_2=\frac{-1}{\sqrt{2}}}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \nabla^2 \bar{F}(x_1, x_2) \Big|_{\substack{x_1=0 \\ x_2=\frac{-1}{\sqrt{2}}}} \begin{pmatrix} x_1 \\ x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \\
 &= \left(\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4 \right) \Big|_{\substack{x_1=0 \\ x_2=\frac{-1}{\sqrt{2}}}} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 + 6x_1^2 & 0 \\ 0 & 1 \end{pmatrix} \Big|_{\substack{x_1=0 \\ x_2=\frac{-1}{\sqrt{2}}}} \begin{pmatrix} x_1 \\ x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \\
 &= \frac{1}{4} + \frac{1}{2} \begin{pmatrix} x_1 & x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \\
 &= \frac{1}{4} + \frac{1}{2} \begin{pmatrix} 2x_1 & x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 + \frac{1}{\sqrt{2}} \end{pmatrix} \\
 &= \frac{1}{4} + \frac{1}{2} \left(2x_1^2 + \left(x_2 + \frac{1}{\sqrt{2}} \right)^2 \right)
 \end{aligned}$$

Since $F(x_1, x_2)$ is constant, say E , then the above can be written as

$$\begin{aligned}
 F(x_1, x_2) &= 2x_1^2 + \left(x_2 + \frac{1}{\sqrt{2}} \right)^2 \\
 &= E
 \end{aligned}$$

Since there is no negative term in the quadratic form above, then the index of the critical point is 0 which means this is stable critical point. (center node)

2.3 Example 3

$$\begin{aligned}
 \dot{x}_1 &= x_1 - x_1x_2 - x_2^3 + x_3(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3) \\
 \dot{x}_2 &= x_1 - x_3(x_1 - x_2 + 2x_1x_2) \\
 \dot{x}_3 &= (x_3 - 1)(x_3 + 2x_3x_2^2 + x_3^3)
 \end{aligned} \tag{A}$$

step 1

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_1x_2 - x_2^3 + x_3(x_1^2 + x_2^2 - 1 - x_1 + x_1x_2 + x_2^3) \\ x_1 - x_3(x_1 - x_2 + 2x_1x_2) \\ (x_3 - 1)(x_3 + 2x_3x_2^2 + x_3^3) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

step 2 Find first integral

TO DO.

References

1. Lecture notes. Math 5525, Spring 2020. Professor M. Carme Calderer.
2. Nonlinear Differential Equations and Dynamical Systems by F.Verhulst.