Kovacic Algorithm Outline

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1 Kovacic Algorithm for solving second order ode

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1.1 Introduction

Detailed description of Kovacic algorithm for solving second order linear ode with rational coefficients is given with many solved examples showing how the algorithm works step by step.

The algorithm is first described based on Kovacic original 1985 paper (1) and later described in separate section based on modified Saunders/Smith algorithm in papers (2,3). The same ode examples are solved using both algorithms to show the difference.

Given the ode

$$y''(x) + ay'(x) + by(x) = 0 \qquad a, b \in \mathbb{C}(x) \tag{1}$$

It is transformed to the following ode by eliminating the first derivative

$$z'' = rz \qquad r \in \mathbb{C}(x) \tag{2}$$

This is done using what is known as the Liouville transformation given by

$$z = ye^{\frac{1}{2}\int a dx} \tag{3}$$

Where r in (2) is given by

$$r = \frac{1}{4}a^2 + \frac{1}{2}a' - b \tag{4}$$

It is equation (2) (called the DE from now on) which is solved using the Kovacic algorithm and not Eq. (1). The solution to (1) can be obtained using (3) once y is found. Kovacic algorithm finds a Liouvillian solution to (2) if one exists. There are 4 cases

1. DE has solution $z = e^{\int \omega dx}$ where $\omega \in \mathbb{C}(x)$.
2. DE has solution $z = e^{\int \omega dx}$ where ω is polynomial over $\mathbb{C}(x)$ of degree 2 and case (1) does not hold.
3. Solutions of DE are algebraic over $\mathbb{C}(x)$ and case 1,2 do not hold.
4. DE has no Liouvillian solution.

Before describing how the algorithm works, there are necessary (but not sufficient) conditions that should be checked to determine which case of the above the ode satisfies.

The following are the necessary conditions for each case. To check each case, let $r = \frac{s}{t}$ where $\gcd(s, t) = 1$. This means there is no common factor between s, t. The order of r at ∞ is defined as $\deg(t) - \deg(s)$.

For an example, if $r = \frac{1}{x^2}$ then $O(\infty) = 2 - 0 = 2$. And if $r = \frac{1+x}{3x^2}$ then $O(\infty) = 2 - 1 = 1$. The poles of r and the order of each pole needs to be determined.

The poles of r are the zeros of t. For example if $t = (1-x)^2(x)$ then there is one pole is at $x = 1$ of order 2 and one pole at $x = 0$ of order 1.

Knowing these two pieces of information is all what is needed to determine the necessary conditions for each case. The necessary conditions for each case are the following

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

1. Case 1. Every pole of r must have even order or its order is 1. And $O(\infty)$ is even or greater than 2. For an example, given $r = (x^2 + 3)$, this has a pole of order zero (since no poles), therefore $O(\infty) = 0 - 2 = -2$ which is even. Hence it satisfies case 1. (pole order zero, is even, since zero is even number).
2. Case 2. r has at least one pole of order 2 or the order is odd and greater than 2. There are no conditions related to $O(\infty)$ for this case.
3. r has only poles of order 1 or 2. And $O(\infty)$ must be at least 2.

If the conditions are not satisfied then there is no need to try that specific case as there will be no solution. However if the conditions are satisfied, this does not necessarily mean a solution exists for that case. This is what necessary but not sufficient conditions means.

The following table summarizes the above conditions and the possible L list (to be described later) for each case.

Some observations: In case one no odd order pole is allowed except for order 1. And for case 3, only poles of order 1,2 are allowed. If $O(\infty) = 0$, which means s and t have same degree, then only possibility is case one or case two. Case 3 is not possible. For case one, if $O(\infty)$ is negative, then it has to be even. For example if $r = \frac{x^6}{(x-1)^2}$ then now $O(\infty) = 2 - 6 = -4$. But if $r = \frac{x^5}{(x-1)^2}$ then $O(\infty) = 2 - 5 = -3$ and hence this can not be case 1.

The following are examples to help understand these conditions. Notice that if a pole is of order 2 and $O(\infty)$ is say 2, then all three cases are met.

1.2 Examples how to determine which case the ode belongs into

1.2.1 Example 1

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}$$

There is one pole at $x = 0$ of order 4. And $O(\infty) = 4 - 6 = -2$. Conditions for case 1 are met. Since it has a pole of even order. Also $O(\infty)$ is even. Case 2 are not satisfied, since there is no pole of order 2 and no odd pole of order greater than 2 exist. Case 3 is also not met, since the pole is order 4 and case 3 will only work if pole is order 1 or 2. Hence $L = [1]$

1.2.2 Example 2

$$r = x$$

There is one pole of order zero (an even pole). So case 1 or 3 qualify. But $O(\infty) = 0 - 1 = -1$ which is odd. But case 1 and 3 require $O(\infty)$ be even. Hence case 1,2,3 all fail. This is case 4 where there is no Liouvillian solution. This is known already, because this is the known Airy ode $y'' = xy$. Its solution are the Airy special functions. These are not Liouvillian solutions. Hence $L = []$

1.2.3 Example 3

$$\begin{aligned} r &= \frac{1}{x} - \frac{3}{16x^2} \\ &= \frac{16x - 3}{16x^2} \end{aligned}$$

There is pole at $x = 0$ of order 2. And $O(\infty) = 2 - 1 = 1$. Case 1 is not satisfied, since $O(\infty)$ is not greater than 2. Also case 3 can not hold, since case 3 requires $O(\infty)$ be at least order 2 and here it is 1. Only possibility left is case 2. There is one pole of order 2. Since case 2 have no conditions on $O(\infty)$ to satisfy, then case 2 has been met. So this is case 2 only. $L = [2]$

1.2.4 Example 4

$$\begin{aligned} r &= -\frac{3}{16x^2} - \frac{2}{9(x-1)^2} + \frac{3}{16x(x-1)} \\ &= -\frac{-32x^2 + 27x - 27}{144x^2 (x-1)^2} \end{aligned}$$

There is pole at $x = 0$ of order 2, and pole at $x = 1$ of order 2. And $O(\infty) = 4 - 2 = 2$.we see that $O(\infty)$ is satisfied for case 1 and case 3. Recall that case 2 has no $O(\infty)$ conditions. The pole order is satisfied for case 1 (must have even order or order 1), also the pole order is satisfied for case 2 (have at least one pole of order 2), and pole order is satisfied for case 3 (can only have poles of order 1 or 2). So all three cases are satisfied. Remember that just because the necessary conditions are met, this does not mean a Liouvillian solution exists. Hence $L = [1, 2, 4, 6, 12]$.

1.2.5 Example 5

$$r = \frac{-5x + 27}{36(x-1)^2}$$

$O(\infty) = 2 - 1 = 1$. And r has pole at $x = 1$ of order 2. We see that $O(\infty)$ is not satisfied for case 1 and case 3 (case 1 requires even or greater than 2 for $O(\infty)$ and case 3 requires $O(\infty) = 2$.). So our only hope is case 2. Case 2 has no $O(\infty)$ conditions. But it needs to have at least one pole of order 2 or a pole which is odd order and greater than 2. This is satisfied here, since pole is order 2. Hence only case 2 is possible. Hence $L = [2]$. I do not understand why paper says all three cases are possible for this. This seems to be an error in the paper (1).

1.2.6 Example 6

$$r = (x^2 + 3)$$

There is zero order pole. (even order). $O(\infty) = 0 - 2 = -2$. Hence only case 1 is possible. $L = [1]$

1.2.7 Example 7

$$r = \frac{1}{x^2}$$

One pole at $x = 0$ of order 2. And $O(\infty) = 2 - 0 = 2$. Case 1 is satisfied. Also case 2 since pole is even order. Also case 3 is satisfied. Hence all three cases are satisfied. $L = [1, 2, 4, 6, 12]$

1.2.8 Example 8

$$r = \frac{4x^2 - 15}{4x^2}$$

We see that $O(\infty) = 0$. From the table this means only case 1 and 2 are possible. (since case 2 has no conditions on $O(\infty)$ and only case 1 allows zero order for $O(\infty)$). We see there is a pole at $x = 0$ of order 2. This is allowed by both case 1 and case 2. Hence case 1,2 are possible and $L = [1, 2]$

2 Examples of ODE's for each case

The following table gives an example ode for each case of the above. Recall there are 4 cases. Case 1 $(n = 1)$, and case 2 $(n = 2)$ and case 3 $(n = 4, 6, 12)$ and case 4 which means no Liouvillian solution exist. Recall also that if ode belong to case 1,2 or 3, this does not imply that Liouvillian solution exists. For example, below $x^2y'' - 2xy' + (x^2 + 2x + 2)\,y = 0$ satisfies conditions for case 1, however, we can find out that no Liouvillian solution exists.

case number	ODE
One $L = [1]$	$x^2y'' + 4xy' + (x^2 + 2)\,y = 0$
Two $L = [2]$	$2xy'' - y' + 2y = 0$
One and Two $L = [1, 2]$	$4x^2y'' + 4x(1 - x)\,y' + (2x - 9)\,y = 0$
One $L = [1]$	$x^2y'' - 2xy' + (x^2 + 2x + 2)\,y = 0$
One and Two and Three $L = [1, 2, 4, 6, 12]$	$y'' - \frac{1}{(4x^2)}y = 0$
Case 4, (i.e. No Liouvillian solution exist)	$y'' - xy = 0$

3 Algorithm implementation based on original Kovacic 1985 paper

The following describes the algorithm for each case separately. The easiest one is for case 1, and the hardest is for case 3. Many examples will also be given at the end of the algorithm describing to show how it works.

3.1 Case one algorithm

3.1.1 Step 1

This description is based on KOVACIC 1985 paper and not based on the Saunders paper.

We are given $y'' = ry$. It is assumed that the necessary conditions for case 1 have been met as given in the table above and $r = \frac{s}{t}$ where $\gcd(s,t) = 1$ (in Maple this is done using the normal() command). The first step is to find the poles of r and the order of each pole. If there are no poles, then let the set of poles Γ will be empty.

If a pole $x = c$ is of order 1 which means there is a factor $\frac{1}{(x-c)}$ in the partial fractions decomposition of r, then let

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^+ &= 1 \\ \alpha_c^- &= 1 \end{aligned}$$

If the pole c is of order 2, which means there is a factor $\frac{1}{(x-c)^2}$ in the partial fractions decomposition of r, then let

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_c^- &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction decomposition of r. For example, if $r = \frac{3}{(x-2)^2}$, then $x = 2$ is a pole of order 2 and $b = 3$. The coefficients are found using undetermined coefficients method. (Examples below show how).

If the pole is of order 4 or 6 or 8 and so on, then it is a little bit more complicated. We write $2v = \mathit{order}$. For example, if the pole was order 4, then $v = 2$ and if the pole was order 6, then $v = 3$ and so on. Notice that for case 1, which we are discussing here, if pole is of order larger than 2, then only poles of order $4, 6, 8, \cdots$ are allowed. This is from the necessary condition. In this case, we add all terms involving $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ (not r) as follows

$$\begin{aligned} \left[\sqrt{r}\right]_c &= \sum_{i=2}^{v} \frac{a_i}{(x-c)^i} \\ &= \frac{a_2}{(x-c)^2} + \frac{a_3}{(x-c)^3} + \cdots + \frac{a_v}{(x-c)^v} \end{aligned} \tag{1}$$

For an example if the pole was of order 6, then $v = 3$. Therefore we need to add all terms in the Laurent series expansion of $\sqrt{r}$ from $v = 3$ down to 2. As follows

$$\begin{aligned} \left[\sqrt{r}\right]_c &= \sum_{i=2}^{3} \frac{a_i}{(x-c)^i} \\ &= \frac{a_2}{(x-c)^2} + \frac{a_3}{(x-c)^3} \end{aligned}$$

Lets look at an example of the above before going to the next step. Assume

$$r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}$$

There is only one pole at $x = 0$ of order 4. Hence $v = 2$. We need to find the Laurent series of $\sqrt{r}$ expanding around the specific pole c of order $2v$. In Maple this is done using series($\sqrt{r}, x = c$).

$$\sqrt{r} \approx \frac{1}{x^2} - \frac{5}{2}\frac{1}{x} - \frac{9}{4} - \frac{41}{8}x - \frac{443}{32}x^2 + \cdots \tag{2}$$

Hence

$$\left[\sqrt{r}\right]_c = \sum_{i=2}^{2} \frac{a_i}{(x-c)^i} \\ = \frac{a_2}{(x-c)^2} \\ = \frac{a_2}{(x-0)^2} \tag{3}$$

Comparing the above to Eq. 2 shows that the coefficient is a_2 is (written now as just a to make it match the paper and use it in the following equation later on)

$$a = 1$$

So the term a is the coefficient of $\frac{a_v}{(x-c)^v}$ in the Laurent series expansion of $\sqrt{r}$ around $x = c$. In implementation of the algorithm the method of undetermined coefficients is used instead of actually finding Laurent series for $\sqrt{r}$ at $x = c$.

Now that we found $\left[\sqrt{r}\right]_c$ for poles > 2, we need to find its α_c^+, α_c^- also. In this case $\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right)$ and $\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right)$. Where a is the one we just found above. But what is b here? b is the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $\left[\sqrt{r}\right]_c$ which we found above in (2). For an example, using the above r its Laurent series expansion around $x = 0$ is

$$r \approx x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{x^2} - \frac{5}{x^3} + \frac{1}{x^4}$$

Then since $a = 1$ from earlier and since $v = 2$ here (since pole of order 4) then we look above for the coefficient of the term $\frac{1}{(x-0)^{v+1}} = \frac{1}{(x-0)^3}$ in r itself. We see this is -5. Now we need to subtract from this value the coefficient of $\frac{1}{(x-0)^{v+1}} = \frac{1}{(x-0)^3}$ from $\left[\sqrt{r}\right]_c$ series from Eq (3). But since $\left[\sqrt{r}\right]_c = \frac{1}{(x-0)^2}$ then there is no term $\frac{1}{(x-0)^3}$. Which means

$$b = -5 - 0 \\ = -5$$

Therefore for this example

$$\alpha_c^+ = \frac{1}{2}\left(\frac{b}{a} + v\right) = \frac{1}{2}\left(-\frac{5}{1} + 2\right) = -\frac{3}{2}$$
$$\alpha_c^- = \frac{1}{2}\left(-\frac{b}{a} + v\right) = \frac{1}{2}\left(\frac{5}{1} + 2\right) = \frac{7}{2}$$

We are now done with finding everything we need related to poles. The above needs to be done for each pole c in r.

We see that for each pole, we need to calculate 3 items. They are $\left[\sqrt{r}\right]_c, \alpha_c^+, \alpha_c^-$.

Now we switch attention to the $O(\infty)$ order. This is much easier. This is the order of $r = \frac{s}{t}$ at infinity which is found from $\deg(t) - \deg(s)$. There are also three cases to consider.

If $O(\infty) > 2$ then we write

$$\left[\sqrt{r}\right]_\infty = 0 \\ \alpha_\infty^+ = 0 \\ \alpha_\infty^- = 1$$

If $O(\infty) = 2$ then $\left[\sqrt{r}\right]_\infty = 0$. Now we calculate b for this case. This is given by the leading coefficient of s divided by the leading coefficient of t when $\gcd(s,t) = 1$. In this case

$$\left[\sqrt{r}\right]_\infty = 0 \\ \alpha_\infty^+ = \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_\infty^- = \frac{1}{2} - \frac{1}{2}\sqrt{1+4b}$$

Where here b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. But we do not need to find Laurent series expansion of r at ∞ to find b here. It can be found using $b = \frac{lcoeff(s)}{lcoeff(t)}$ where $r = \frac{s}{t}$ and $\gcd(s,t) = 1$. And $lcoeff$ is the leading coefficient. For example, if $r = \frac{1+5x}{2x^2}$ then $b = \frac{1}{2}$. If we took the Laurent series of r at ∞ which in Maple can be done using the command $series(r, x = \infty)$ then we will get $\frac{5}{2x} + \frac{1}{2}\frac{1}{x^2}$ which also give $b = \frac{1}{2}$.

And finally, if $\underline{O(\infty) \leq 0}$, then $O(\infty)$ has to be negative and even number (conditions for case 1). Let the order of r at ∞ be $-2v \leq 0$. Then now $\left[\sqrt{r}\right]_\infty$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞.

$$\left[\sqrt{r}\right]_\infty = ax^v + z_1 x^{v-1} + \cdots + z_n$$

And b is the coefficient of x^{v-1} in r minus the coefficient of x^{v-1} in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. Then

$$\begin{aligned}\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right)\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right)\end{aligned}$$

This completes step 1 of the algorithm. We have found $\left[\sqrt{r}\right]_c$ for each pole and associated α_c^+, α_c^- and also found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^+, \alpha_\infty^-$. So, what will we do with these? In step 2 these are used to find all possible values of what is called d. For each non negative d, we will find a candidate ω. And use this candidate ω to find $P(x)$ by solving $P'' + 2\omega P' + (\omega' + \omega^2 - r)\, P = 0$ (linear algebra problem). If we are able to find a $P(x)$ for any one candidate ω then we stop and we have found the solution $y = p(x)\, e^{\int \omega dx}$ to the $y'' = ry$. Examples below will show how all this works.

3.1.2 Step 2

Recall that from step 1 we have found $\left[\sqrt{r}\right]_c$ and its associated α_c^+, α_c^- (this is done for each pole of r) and we have found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^+, \alpha_\infty^-$. From these we now found a possible d values and trying each $d \geq 0$. The value of d is found using the following for each combination of $s(c)$ where $s(c)$ is $+$ or $-$

$$d = \alpha_\infty^\pm - \sum_c \alpha_c^\pm$$

We keep only the non negative values of d. It is important to note that we have to find an integer positive value for d to continue. If no such value is found from the above, then we stop here as this means no Liouvillian solution exist using case 1. Then we go to case two or case three if it is available.

If we do find $d \geq 0$, then we now find corresponding candidate ω_d using

$$\omega_d = \sum_c \left((\pm) \left[\sqrt{r}\right]_c + \frac{\alpha_c^\pm}{x - c} \right) + (\pm) \left[\sqrt{r}\right]_\infty$$

3.1.3 Step 3

In this step we first attempt to find a polynomial $p(x)$ of degree d, for ω found in step 2. This is done by solving

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0$$

For example, if $d = 2$, then we let $p(x) = x^2 + ax + b$ and if ω happened to be say $\frac{1}{x^2} - \frac{3}{2x} + x - 1$, then by substituting these in the above, we can solve for a, b (if a solution exist). Then the solution to $y'' = ry$ is $y = p(x)\, e^{\int \omega dx}$. If the degree $d = 1$ then we guess $p(x) = x + a$ and try to solve for a. If the degree $d = 0$, then we let $p(x) = 1$, a constant. In the special case of $p(x) = 1$, there is no coefficients a_i to solve for. So we would just need to verify that

$$\omega' + \omega^2 - r = 0$$

In this case.

This completes the full algorithm for case 1. We will now go over many examples for case 1, showing how to implement this algorithm for each example.

The hardest part of the kovacic algorithm is just finding all the $\left[\sqrt{r}\right]_c, \alpha_c^{\pm}, \left[\sqrt{r}\right]_\infty, \alpha_\infty^{\pm}$. Once these are found, the rest of the algorithm is much more direct.

3.1.4 Case one algorithm diagram

The following diagram summarized the above for case one.

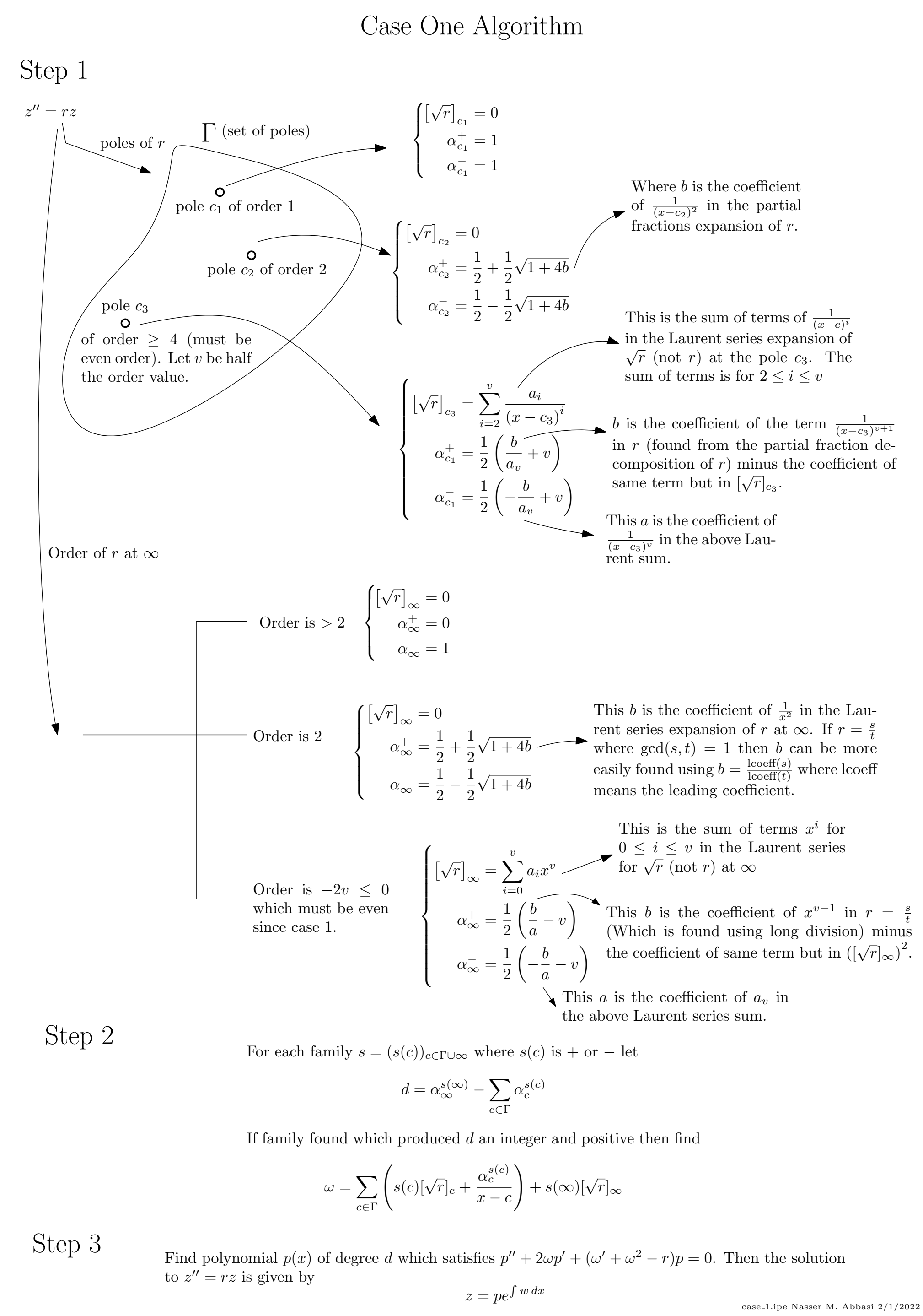

Figure 1: Case 1 Kovacic algorithm

3.2 worked examples for case one

3.2.1 Example 1

Let

$$y'' = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} y$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \\ &= x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{x^2} - \frac{5}{x^3} + \frac{1}{x^4} \end{aligned} \tag{1}$$

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is only one pole at $x=0$ of order 4. Hence $2v=4$. And $v=2$. This is step (C3) now in the paper (1).

We need now to find Laurent series of $\sqrt{r}$ expanded around $x=c=0$. This is given by (using series command on the computer)

$$\frac{1}{x^2}-\frac{5}{2}\frac{1}{x}-\frac{9}{4}-\frac{41}{8}x-\frac{443}{32}x^2+\cdots \tag{2}$$

We need to add all terms in the Laurent series expansion of $\sqrt{r}$ from $v=2$ down to 2. Hence

$$\left[\sqrt{r}\right]_c=\frac{1}{(x-0)^2} \tag{3}$$

Is only term from 2. Comparing the above to $\frac{a}{(x-0)^2}$ shows that

$$a=1 \tag{4}$$

Hence

$$\left[\sqrt{r}\right]_c=\frac{1}{x^2} \tag{5}$$

$$\alpha_c^+=\frac{1}{2}\left(\frac{b}{a}+v\right)$$

$$\alpha_c^-=\frac{1}{2}\left(-\frac{b}{a}+v\right) \tag{6}$$

Where $v=2$ and $a=1$. We still need to find b. But b is the coefficient of the term $\frac{1}{(x-0)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-0)^{v+1}}$ in $\left[\sqrt{r}\right]_c$ which we just found above. Looking at r from Eq (1) we see that the term $\frac{1}{(x-0)^{v+1}}=\frac{1}{(x-0)^3}$ has coefficient -5. And looking at Eq (3) we see that there is no term $\frac{1}{(x-0)^3}$ in it. Hence

$$\begin{aligned}b&=-5-0\\&=-5\end{aligned} \tag{7}$$

Now we found a,b, then (5,6) becomes (since $v=2$)

$$\alpha_c^+=\frac{1}{2}(-5+2)=\frac{-3}{2} \tag{8}$$

$$\alpha_c^-=\frac{1}{2}(5+2)=\frac{7}{2} \tag{9}$$

We are done with all the poles.

Now we consider $O(\infty)$ which is $\deg(t)-\deg(s)=4-6=-2$. Since this is even order and negative then $-2v=-2$ or

$$v=1$$

We need the Laurent series of $\sqrt{r}$ around ∞. Using the computer this is

$$\left[\sqrt{r}\right]_\infty=x-1+\frac{1}{x}+\frac{3}{2}\frac{1}{x^2}+\frac{15}{8x^3}+\cdots$$

Now we only want the terms x^i where $0\leq i\leq v$. This implies the above is reduced to

$$\left[\sqrt{r}\right]_\infty=x-1$$

The a is the coefficient of $x^v=x$ which is

$$a=1$$

Now we need to find $\alpha_\infty^{\pm}$ associated with $\left[\sqrt{r}\right]_\infty$. For this we need to first find b. Recall from above that b is the coefficient of x^{v-1} or x^0 in r minus the coefficient of $x^{v-1}=x^0$

in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. Since $v = 1$ then we want the coefficient of x^0 in r and subtract from it the coefficient of x^0 in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. But

$$\begin{aligned}\left(\left[\sqrt{r}\right]_\infty\right)^2 &= (x-1)^2\\ &= x^2 + 1 - 2x\end{aligned}$$

Hence the coefficient of x^0 in $\left(\left[\sqrt{r}\right]_\infty\right)^2$ is 1. To find the coefficient of x^0 in r long division is done

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}\\ &= Q + \frac{R}{4x^2}\end{aligned}$$

Where Q is the quotient and R is the remainder. This gives

$$r = \left(x^2 - 2x + 3\right) + \frac{4x^3 + 7x^2 - 20x + 4}{4x^2}$$

For the case of $v \neq 0$ then the coefficient is read from Q above. Which is 3. Hence

$$\begin{aligned}b &= 3 - 1\\ &= 2\end{aligned}$$

For the other case of $v = 0$ then the coefficient of x^{-1} in r is found using $\frac{lcoeff(R)}{lcoeff(t)}$ which will give 1 in this case. (More examples below).

Now that we found a, b, then from the above section describing the algorithm, we see in this case that

$$\begin{aligned}\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{2}{1} - 1\right) = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{2}{1} - 1\right) = -\frac{3}{2}\end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the d's.

step 2 Since we have a pole at zero, and we have one $O(\infty)$, each with $\pm$ signs, then we set up this table to make it easier to work with. This implements

$$d = \alpha_\infty^\pm - \sum_{i=1}^{1} \alpha_{c_i}^\pm$$

Therefore we obtain 4 possible d values.

pole c	α_c value	$O(\infty)$ value	d	d value
$x = 0$	$\alpha_c^+ = \frac{-3}{2}$	$\alpha_\infty^+ = \frac{1}{2}$	$\alpha_\infty^+ - (\alpha_c^+) = \frac{1}{2} - \left(\frac{-3}{2}\right)$	2
$x = 0$	$\alpha_c^+ = \frac{-3}{2}$	$\alpha_\infty^- = -\frac{3}{2}$	$\alpha_\infty^- - (\alpha_c^+) = -\frac{3}{2} - \left(\frac{-3}{2}\right)$	0
$x = 0$	$\alpha_c^- = \frac{7}{2}$	$\alpha_\infty^+ = \frac{1}{2}$	$\alpha_\infty^+ - (\alpha_c^-) = \frac{1}{2} - \left(\frac{7}{2}\right)$	-3
$x = 0$	$\alpha_c^- = \frac{7}{2}$	$\alpha_\infty^- = -\frac{3}{2}$	$\alpha_\infty^- - (\alpha_c^-) = -\frac{3}{2} - \left(\frac{7}{2}\right)$	-5

We see from the above that we took each pole in this problem (there is only one pole here at $x = 0$) and its associated $\alpha_c^\pm$ with each $\alpha_\infty^\pm$ and generated all possible d values from all the combinations. Hence we obtain 4 possible d values in this case. If we had 2 poles, then we would have 8 possible d values. Hence the maximum possible d values we can get is 2^{p+1} where p is number of poles. Now we remove all negative d values. Hence the trial d values remaining is

$$d = \{0, 2\}$$

Now for each d value, we generate ω using

$$\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$$

To apply the above, we update the table above, now using only $d = 0, d = 2$ values, but also add columns for $\left[\sqrt{r}\right]_c, \left[\sqrt{r}\right]_\infty$ to make the computation easier. Here is the new table

pole c	α_c value	$s(c)$	$\left[\sqrt{r}\right]_c$	$O(\infty)$ value	$s(\infty)$	$\left[\sqrt{r}\right]_\infty$	d value	ω value $\left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$
$x=0$	$\alpha_c^+ = \frac{-3}{2}$	$+$	$\frac{1}{x^2}$	$\alpha_\infty^- = -\frac{3}{2}$	$-$	$x-1$	0	$\left(+\left(\frac{1}{x^2}\right) + \frac{\frac{-3}{2}}{x-0}\right) + (-)(x-1) = \frac{1}{x^2} - \frac{3}{2x} - x + 1$
$x=0$	$\alpha_c^+ = \frac{-3}{2}$	$+$	$\frac{1}{x^2}$	$\alpha_\infty^+ = \frac{1}{2}$	$+$	$x-1$	2	$\left(+\left(\frac{1}{x^2}\right) + \frac{\frac{-3}{2}}{x-0}\right) + (+)(x-1) = x - \frac{3}{2x} + \frac{1}{x^2} - 1$

The above are the two candidate ω values. For each ω we need to find polynomial P by solving

$$P'' + 2\omega P' + \left(\omega' + \omega^2 - r\right)P = 0 \tag{8}$$

If we are able to find P, then we stop and the ode $y'' = ry$ is solved. If we try all candidate ω and can not find P then this case is not successful and we go to the next case.

step 3 Now for each candidate ω we solve the above Eq (8). Starting with $\omega = \frac{1}{x^2} - \frac{3}{2x} - x + 1$ associated with $d = 0$ in the table, then (8) becomes

$$P'' + 2\left(\frac{1}{x^2} - \frac{3}{2x} - x + 1\right)P' + \left(\left(\frac{3}{2x^2} - \frac{2}{x^3} - 1\right) + \left(\frac{1}{x^2} - \frac{3}{2x} - x + 1\right)^2 - \left(\frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}\right)\right)P = 0$$
$$P'' + \left(\frac{2}{x^2} - \frac{3}{2x} - 2x + 2\right)P' + \left(\frac{4}{x^2} - \frac{6}{x}\right)P = 0$$

Since this the case for $d = 0$, then P has zero degree, Hence P is constant. Therefore the above simplifies to

$$\left(\frac{4}{x^2} - \frac{6}{x}\right)P = 0$$

Which means

$$\frac{4}{x^2} - \frac{6}{x} = 0$$

Which is not possible for all x. Hence $d = 0$ do not generate valid P polynomial. We now try the case of $d = 2$. Since $d = 2$, it means the polynomial d is of degree two. Let

$$P = x^2 + ax + b$$

Substituting this in (8) using $\omega = x - \frac{3}{2x} + \frac{1}{x^2} - 1$. This gives

$$P'' + 2\left(x - \frac{3}{2x} + \frac{1}{x^2} - 1\right)P' + \left(\left(\frac{3}{2x^2} - \frac{2}{x^3} + 1\right) + \left(x - \frac{3}{2x} + \frac{1}{x^2} - 1\right)^2 - \left(\frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}\right)\right)P = 0$$
$$P'' + 2\left(x - \frac{3}{2x} + \frac{1}{x^2} - 1\right)P' + \left(\frac{4}{x} - 4\right)P = 0$$

Using $P = x^2 + ax + b$ the above becomes

$$2 + 2\left(x - \frac{3}{2x} + \frac{1}{x^2} - 1\right)(2x + a) + \left(\frac{4}{x} - 4\right)(x^2 + ax + b) = 0$$
$$2a - 4b - 3\frac{a}{x} + 2\frac{a}{x^2} + 4\frac{b}{x} - 2ax + \frac{4}{x} - 4 = 0$$
$$(2a - 4b - 4) + \frac{1}{x}\left(-\frac{3}{4}a + 4b + 4\right) + \frac{1}{x^2}(2a) - 2ax = 0$$

Therefore

$$2a - 4b - 4 = 0$$
$$-\frac{3}{4}a + 4b + 4 = 0$$
$$2a = 0$$
$$2a = 0$$

hence $a = 0$ from last equation. Using first or second equation gives $b = -1$. Therefore a solution is found. Hence

$$p(x) = x^2 - 1$$

Therefore the solution to $y'' = ry$ is

$$\begin{aligned} y &= p(x)\, e^{\int \omega dx} \\ &= \left(x^2 - 1\right) e^{\int x - \frac{3}{2x} + \frac{1}{x^2} - 1 \, dx} \\ &= \left(x^2 - 1\right) e^{-\frac{1}{x} - \frac{3}{2}\ln x + \frac{x^2}{2} - x} \\ &= \left(x^2 - 1\right) x^{\frac{-3}{2}} e^{-\frac{1}{x} + \frac{x^2}{2} - x} \end{aligned}$$

The second solution can be found by reduction of order.

3.2.2 Example 2

Let

$$y'' = \left(\frac{2}{x^2} - 1\right) y$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2 - x^2}{x^2} \end{aligned} \tag{1}$$

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x = 0$ of order 2. In this case, from the description of the algorithm earlier, we write

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4b} \\ \alpha_c^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-0)^2}$ in the partial fraction decomposition of r which is $\frac{2}{x^2} - 1$. Hence $b = 2$. Therefore

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1 + 8} = \frac{1}{2} + \frac{3}{2} = 2 \\ \alpha_c^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1 + 8} = \frac{1}{2} - \frac{3}{2} = -1 \end{aligned}$$

We are done with all the poles. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 2 - 2 = 0$. This falls in the case $-2v \leq 0$. Hence

$$v = 0$$

We need the Laurent series of $\sqrt{r}$ around ∞. Using the computer this is $i - \frac{i}{x^2} - \frac{1}{2x^4} i + \cdots$. Hence we need the coefficient of x^0 in this series (0 because that is value of v).

$$\left[\sqrt{r}\right]_\infty = i x^0$$

Recall that $\left[\sqrt{r}\right]_\infty$ is the sum of terms of x^j for $0 \leq j \leq v$ or for $j = 0$ since $v = 0$. Looking at the series above, we see that

$$a = i$$

Which is the coefficient of the term x^0. Now we need to find $\alpha_\infty^{\pm}$ associated with $\left[\sqrt{r}\right]_\infty$. For this we need to first find b which is the coefficient of $x^{v-1} = \frac{1}{x}$ in r minus the coefficient of $x^{v-1} = \frac{1}{x}$ in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. But

$$\left(\left[\sqrt{r}\right]_\infty\right)^2 = i^2 = -1$$

Hence the coefficient of x^{-1} in $\left(\left[\sqrt{r}\right]_\infty\right)^2$ is 0. To find the coefficient of x^{-1} in r long division is done

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2-x^2}{x^2} \\
&= Q + \frac{R}{x^2}
\end{aligned}
$$

Where Q is the quotient and R is the remainder. This gives

$$
r = -1 + \frac{2}{x^2}
$$

For the other case of $v=0$ then the coefficient of x^{-1} is found by looking up the coefficient in R of x to the degree of of t then subtracting one and dividing result by $lcoeff(t)$. But degree of t is 2. Therefore we want the coefficient of x^{2-1} or x in R which is zero. Hence $b=0-0=0$.

Now that we found a, b, then from the above section describing the algorithm, we see in this case that

$$
\begin{aligned}
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}(0-0) = 0 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}(-0-0) = 0
\end{aligned}
$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the d's.

step 2 Since we have a pole at zero, and we have one $O(\infty)$, each with $\pm$ signs, then we set up this table to make it easier to work with. This implements

$$
d = \alpha_\infty^\pm - \sum_{i=1}^{1} \alpha_{c_i}^\pm
$$

Therefore we obtain 4 possible d values.

pole c	α_c value	$O(\infty)$ value	d	d value
$x=0$	$\alpha_c^+ = 2$	$\alpha_\infty^+ = 0$	$\alpha_\infty^+ - (\alpha_c^+) = 0-(2)$	-2
$x=0$	$\alpha_c^+ = 2$	$\alpha_\infty^- = 0$	$\alpha_\infty^- - (\alpha_c^+) = 0-(2)$	-2
$x=0$	$\alpha_c^- = -1$	$\alpha_\infty^+ = 0$	$\alpha_\infty^+ - (\alpha_c^-) = 0-(-1)$	1
$x=0$	$\alpha_c^- = -1$	$\alpha_\infty^- = 0$	$\alpha_\infty^- - (\alpha_c^-) = 0-(-1)$	1

We see from the above that we took each pole in this problem (there is only one pole here at $x=0$) and its associated $\alpha_c^\pm$ with each $\alpha_\infty^\pm$ and generated all possible d values from all the combinations. Hence we obtain 4 possible d values in this case. If we had 2 poles, then we would have 8 possible d values. Hence the maximum possible d values we can get is 2^{p+1} where p is number of poles. Now we remove all negative d values. Hence the trial d values remaining is

$$
d = \{1\}
$$

There is one d value to try. We can pick any one of the two values of d generated since there are both $d=1$. Both will give same solution. We generate ω using

$$
\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty
$$

To apply the above, we update the table above, now using only the first $d=1$ value in the above table. (selecting the second $d=1$ row, will not change the final solution). but we also add columns for $\left[\sqrt{r}\right]_c, \left[\sqrt{r}\right]_\infty$ to make the computation easier. Here is the new table

pole c	α_c value	$s(c)$	$\left[\sqrt{r}\right]_c$	$O(\infty)$ value	$s(\infty)$	$\left[\sqrt{r}\right]_\infty$	d value	ω value $\left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$
$x=0$	$\alpha_c^- = -1$	$-$	0	$\alpha_\infty^+ = 0$	$+$	i	1	$\left(-(0) + \frac{-1}{x-0}\right) + (+)(i) = \frac{-1}{x} + i$

The above gives one candidate ω value to try. For this ω we need to find polynomial P by solving

$$P'' + 2\omega P' + \left(\omega' + \omega^2 - r\right) P = 0 \tag{8}$$

If we are able to find P, then we stop and the ode $y'' = ry$ is solved. If we try all candidate ω and can not find P then this case is not successful and we go to the next case.

step 3 Now for each candidate ω we solve the above Eq (8). Starting with $\omega = \frac{-1}{x} + i$ associated with first $d = 1$ in the table, then (8) becomes

$$P'' + 2\left(\frac{-1}{x} + i\right) P' + \left(\left(\frac{-1}{x} + i\right)' + \left(\frac{-1}{x} + i\right)^2 - \left(\frac{2}{x^2} - 1\right)\right) P = 0$$
$$P'' + 2\left(\frac{-1}{x} + i\right) P' + \left(\frac{-2i}{x}\right) P = 0$$

This needs to be solved for P. Since degree of $p(x)$ is $d = 1$. Let $p = x + a$. The above becomes

$$\begin{aligned} 2\left(\frac{-1}{x} + i\right) + \left(\frac{-2i}{x}\right)(x + a) &= 0 \\ \frac{-2}{x} + 2i - 2i - \frac{2ia}{x} &= 0 \\ \frac{-2}{x} - \frac{2ia}{x} &= 0 \end{aligned}$$

Which means

$$a = i$$

Hence we found the polynomial

$$p(x) = x + i$$

Therefore the solution to $y'' = ry$ is

$$\begin{aligned} y &= pe^{\int \omega dx} \\ &= (x + i)\, e^{\int \frac{-1}{x} + i \, dx} \\ &= (x + i)\, \frac{1}{x} e^{ix} \\ &= \frac{x + i}{x} (\cos x + i \sin x) \end{aligned}$$

The second solution can be found by reduction of order. The full general solution to $y'' = \left(\frac{2}{x^2} - 1\right) y$ is

$$y(x) = \frac{c_1}{x}(x \cos x - \sin x) + \frac{c_2}{x}(\cos x + x \sin x)$$

3.2.3 Example 3

Let

$$y'' = \left(x^2 + 3\right) y$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 3}{1} \end{aligned} \tag{1}$$

Step 1 In this step we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^\pm$ for each pole. There are no poles. In this case $\alpha_c^\pm = 0$ (paper was not explicit in saying this, but from example 3 in paper this can be inferred). Hence the value of d is decided by $\alpha_\infty^\pm$ only in this case.

Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 0 - 2 = -2$. This falls in the case $-2v \leq 0$. Hence $2v = -2$ or

$$v = 1$$

We need the Laurent series of $\sqrt{r}$ around ∞. Using the computer this is

$$x + \frac{3}{2x} - \frac{9}{8x^3} + \cdots$$

Hence we need the coefficient of x^1 in this series (1 because that is value of v). Recall that $[\sqrt{r}]_\infty$ is the sum of terms of x^j for $0 \leq j \leq v$ or for $j = 0, 1$ since $v = 1$. Looking at the series above, we see that

$$a = 1$$

Which is the coefficient of the term x. There is no term x^0. Hence

$$[\sqrt{r}]_\infty = x$$

Now we need to find $\alpha_\infty^\pm$ associated with $[\sqrt{r}]_\infty$. For this we need to first find b. Recall from above that b is the coefficient of x^{v-1} or x^0 in r minus the coefficient of $x^{v-1} = x^0$ in $\left([\sqrt{r}]_\infty\right)^2$. But $\left([\sqrt{r}]_\infty\right)^2 = x^2$. Hence the coefficient of x^0 is zero. To find the coefficient of x^0 in r long division is done

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 3}{1} \\ &= Q + \frac{R}{1} \end{aligned}$$

Where Q is the quotient and R is the remainder. This gives

$$r = x^2 + 3 + \frac{0}{1}$$

For the case of $v \neq 0$ then the coefficient is read from Q above. Which is 3. Hence

$$\begin{aligned} b &= 3 - 0 \\ &= 3 \end{aligned}$$

Now that we found a, b, then from the above section describing the algorithm, we see in this case that

$$\begin{aligned} \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}(3 - 1) = 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}(-3 - 1) = -2 \end{aligned}$$

This completes step 1 of the solution. We have found $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ (these are zero, in this example, since there are no poles) and found $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the d's.

step 2 We set up this table to make it easier to work with. This implements

$$d = \alpha_\infty^\pm - \sum_{i=1}^{0} \alpha_{c_i}^\pm$$

Therefore we obtain 2 possible d values.

pole c	α_c values (all zero)	$O(\infty)$ value	d	d value
$x = \text{N/A}$	$\alpha_c = 0$	$\alpha_\infty^+ = 1$	$\alpha_\infty^+ = 1$	1
$x = \text{N/A}$	$\alpha_c = 0$	$\alpha_\infty^- = -2$	$\alpha_\infty^- = -2$	-2

Picking the positive d integers, this gives

$$d = \{1\}$$

There is one d value to try. We can pick any one of the two values of d generated since there are both $d = 1$. Both will give same solution. We generate ω using

$$\omega = \left(\sum_c s(c) \left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty) \left[\sqrt{r}\right]_\infty$$

To apply the above, we update the table above, now using only the first $d = 1$ value in the above table. (selecting the first $d = 1$ row). but we also add columns for $\left[\sqrt{r}\right]_c, \left[\sqrt{r}\right]_\infty$ to make the computation easier. Here is the new table

pole c	α_c value	$s(c)$	$\left[\sqrt{r}\right]_c$	$O(\infty)$ value	$s(\infty)$	$\left[\sqrt{r}\right]_\infty$	d value	ω value $\left(\sum_c s(c) \left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty) \left[\sqrt{r}\right]_\infty$
$x = 0$	$\alpha_c = 0$	$+$	0	$\alpha_\infty^+ = 1$	$+$	x	1	$(+(0) + 0) + (+)(x) = x$

The above gives one candidate ω value to try. For this ω we need to find polynomial P by solving

$$P'' + 2\omega P' + \left(\omega' + \omega^2 - r\right) P = 0 \tag{8}$$

If we are able to find P, then we stop and the ode $y'' = ry$ is solved. If we try all candidate ω and can not find P then this case is not successful and we go to the next case.

step 3 Now for each candidate ω (there is only one in this example) we solve the above Eq (8). Starting with $\omega = x$ associated with first $d = 1$ in the table, then (8) becomes

$$\begin{aligned} P'' + 2(x)\,P' + \left((x)' + (x)^2 - \left(x^2 + 3\right)\right) P &= 0 \\ P'' + 2xP' + \left(1 + x^2 - x^2 - 3\right) P &= 0 \\ P'' + 2xP' - 2P &= 0 \end{aligned}$$

This needs to be solved for P. Since degree of $p(x)$ is $d = 1$. Let $p = x + a$. The above becomes

$$\begin{aligned} 2x - 2(x + a) &= 0 \\ 2x - 2x - 2a &= 0 \\ 2a &= 0 \end{aligned}$$

Which means

$$a = 0$$

Hence we found the polynomial

$$p(x) = x$$

Therefore the solution to $y'' = ry$ is

$$\begin{aligned} y &= pe^{\int \omega dx} \\ &= xe^{\int x\,dx} \\ &= xe^{\frac{x^2}{2}} \end{aligned}$$

The second solution can be found by reduction of order. The full general solution to $y'' = (x^2 + 3)\,y$ is

$$y(x) = c_1 xe^{\frac{x^2}{2}} + c_2\left(xe^{\frac{x^2}{2}}\sqrt{\pi}\,\mathrm{erf}\,(x) + e^{\frac{-x^2}{2}}\right)$$

3.2.4 Example 4

Let

$$y'' = \frac{2}{x^2}y$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2}{x^2} \end{aligned} \tag{1}$$

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x = 0$ of order 2. In this case, from the description of the algorithm earlier, we write

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_c^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-0)^2}$ in the partial fraction decomposition of r which is $\frac{2}{x^2}$. Hence $b = 2$. Therefore

$$\begin{aligned} \left[\sqrt{r}\right]_c &= 0 \\ \alpha_c^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+8} = \frac{1}{2} + \frac{3}{2} = 2 \\ \alpha_c^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+8} = \frac{1}{2} - \frac{3}{2} = -1 \end{aligned}$$

We are done with all the poles. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 2 - 0 = 2$. Since $O(\infty) = 2$ then from the algorithm above

$$\left[\sqrt{r}\right]_\infty = 0$$

Now we calculate b for this case. This is given by the leading coefficient of s divided by the leading coefficient of t when $\gcd(s,t) = 1$. In this case $r = \frac{2}{x^2}$, hence $b = \frac{2}{1} = 2$. Therefore

$$\begin{aligned} \left[\sqrt{r}\right]_\infty &= 0 \\ \alpha_\infty^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} = \frac{1}{2} + \frac{1}{2}\sqrt{1+8} = 2 \\ \alpha_\infty^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} = \frac{1}{2} - \frac{1}{2}\sqrt{1+8} = -1 \end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^{\pm}$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^{\pm}$. Now we go to step 2 which is to find the d's.

step 2 Since we have a pole at zero, and we have one $O(\infty)$, each with $\pm$ signs, then we set up this table to make it easier to work with. This implements

$$d = \alpha_\infty^{\pm} - \sum_{i=1}^{1} \alpha_{c_i}^{\pm}$$

Therefore we obtain 4 possible d values.

pole c	α_c value	$O(\infty)$ value	d	d value
$x = 0$	$\alpha_c^{+} = 2$	$\alpha_\infty^{+} = 2$	$\alpha_\infty^{+} - (\alpha_c^{+}) = 2 - (2)$	0
$x = 0$	$\alpha_c^{+} = 2$	$\alpha_\infty^{-} = -1$	$\alpha_\infty^{-} - (\alpha_c^{+}) = -1 - (2)$	-3
$x = 0$	$\alpha_c^{-} = -1$	$\alpha_\infty^{+} = 2$	$\alpha_\infty^{+} - (\alpha_c^{-}) = 2 - (-1)$	3
$x = 0$	$\alpha_c^{-} = -1$	$\alpha_\infty^{-} = -1$	$\alpha_\infty^{-} - (\alpha_c^{-}) = -1 - (-1)$	0

Hence the trial d values which are not negative are

$$d = \{0, 3\}$$

For $d = 0$, since it shows in two rows, we take the first row. Now we generate ω for each d using

$$\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$$

To apply the above, we update the table above, but we also add columns for $\left[\sqrt{r}\right]_c, \left[\sqrt{r}\right]_\infty$ to make the computation easier. Here is the new table

pole c	α_c value	$s(c)$	$\left[\sqrt{r}\right]_c$	$O(\infty)$ value	$s(\infty)$	$\left[\sqrt{r}\right]_\infty$	d value	ω value $\left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$
$x = 0$	$\alpha_c^+ = 2$	$+$	0	$\alpha_\infty^+ = 2$	$+$	0	0	$\left(+(0) + \frac{2}{x-0}\right) + (+)(0) = \frac{2}{x}$
$x = 0$	$\alpha_c^- = -1$	$-$	0	$\alpha_\infty^+ = 2$	$+$	0	3	$\left(-(0) + \frac{-1}{x-0}\right) + (+)(0) = \frac{-1}{x}$

The above gives two candidate $\omega = \left\{\frac{2}{x}, \frac{-1}{x}\right\}$ value to try. For this ω we need to find polynomial P by solving

$$P'' + 2\omega P' + \left(\omega' + \omega^2 - r\right)P = 0 \tag{8}$$

If we are able to find P, then we stop and the ode $y'' = ry$ is solved. If we try all candidate ω and can not find P then this case is not successful and we go to the next case.

step 3 Now for each candidate ω we solve the above Eq (8). Starting with $\omega = \frac{2}{x}$ associated with first $d = 0$ in the table, then (8) becomes

$$\begin{aligned} P'' + 2\left(\frac{2}{x}\right)P' + \left(\left(\frac{2}{x}\right)' + \left(\frac{2}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right)P &= 0 \\ P'' + \frac{4}{x}P' + \left(-\frac{2}{x^2} + \frac{4}{x^2} - \frac{2}{x^2}\right)P &= 0 \\ P'' + \frac{4}{x}P' &= 0 \end{aligned}$$

This needs to be solved for P. Since degree of $p(x)$ is $d = 0$. Let $p = a$. The above becomes

$$0 = 0$$

No unique solution. Hence $d = 0$ did not work. Now we try the second $\omega = \frac{-1}{x}$ associated with $d = 3$. Substituting in 8 gives

$$\begin{aligned} P'' + 2\left(\frac{-1}{x}\right)P' + \left(\left(\frac{-1}{x}\right)' + \left(\frac{-1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right)P &= 0 \\ P'' + \frac{-2}{x}P' + \left(\frac{1}{x^2} + \frac{1}{x^2} - \frac{2}{x^2}\right)P &= 0 \\ P'' - \frac{2}{x}P' &= 0 \end{aligned}$$

Since $d = 3$, let

$$p(x) = x^3 + ax^2 + bx + c$$

Then $P'' - \frac{2}{x}P' = 0$ becomes

$$\begin{aligned} (6x + 2a) - \frac{2}{x}\left(3x^2 + 2ax + b\right) &= 0 \\ -2a - 2\frac{b}{x} &= 0 \end{aligned}$$

Hence $a = 0, b = 0$ is solution. c is arbitrary. Taking $c = 0$ then

$$p(x) = x^3$$

Therefore the solution to $y'' = ry$ is

$$\begin{aligned} y &= p(x)\, e^{\int \omega dx} \\ &= x^3 e^{\int \frac{-1}{x}\, dx} \\ &= x^3 e^{-\ln x} \\ &= x^2 \end{aligned}$$

The second solution can be found by reduction of order. The full general solution to $y'' = \frac{2}{x^2}y$ is

$$y(x) = c_1\frac{1}{x} + c_2x^2$$

3.2.5 Example 5

Let

$$y'' = \frac{32x^2 - 27x + 27}{144x^4 - 288x^3 + 144x^2}y$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{32x^2 - 27x + 27}{144x^4 - 288x^3 + 144x^2} \qquad (1) \\ &= \frac{3}{16x} - \frac{3}{16}\frac{1}{(x-1)} + \frac{3}{16}\frac{1}{x^2} + \frac{2}{9}\frac{1}{(x-1)^2} \qquad (2) \end{aligned}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x = 0$ of order 2 and one pole at $x = 1$ of order 2. For the pole at $x = 0$ since order is 2 then

$$\begin{aligned} \left[\sqrt{r}\right]_{c=0} &= 0 \\ \alpha_{c=0}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c=0}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-0)^2}$ in the partial fraction decomposition of r given in Eq (2) which is $\frac{3}{16}$. Hence $b = \frac{3}{16}$. Therefore

$$\begin{aligned} \left[\sqrt{r}\right]_{c=0} &= 0 \\ \alpha_{c=0}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{16}\right)} = \frac{1}{4}\sqrt{7} + \frac{1}{2} \\ \alpha_{c=0}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{3}{16}\right)} = \frac{1}{2} - \frac{1}{4}\sqrt{7} \end{aligned}$$

And for the pole at $x = 1$ which is order 2,

$$\begin{aligned} \left[\sqrt{r}\right]_{c=1} &= 0 \\ \alpha_{c=1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c=1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-1)^2}$ in the partial fraction decomposition of r given in Eq (2) which is $\frac{2}{9}$. Hence $b = \frac{2}{9}$. Therefore the above becomes

$$\begin{aligned} \left[\sqrt{r}\right]_{c=1} &= 0 \\ \alpha_{c=1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{2}{9}\right)} = \frac{1}{6}\sqrt{17} + \frac{1}{2} \\ \alpha_{c=1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{2}{9}\right)} = \frac{1}{2} - \frac{1}{6}\sqrt{17} \end{aligned}$$

We are done with all the poles. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 4 - 2 = 2$. Since $O(\infty) = 2$ then from the algorithm above

$$\left[\sqrt{r}\right]_{\infty} = 0$$

Now we calculate b for this case. This is given by the leading coefficient of s divided by the leading coefficient of t when $\gcd(s,t) = 1$. In this case $r = \frac{32x^2-27x+27}{144x^4-288x^3+144x^2}$ from Eq (1) , hence $b = \frac{32}{144} = \frac{2}{9}$. Therefore

$$
\begin{aligned}
\left[\sqrt{r}\right]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{2}{9}\right)} = \frac{1}{6}\sqrt{17} + \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} = \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{2}{9}\right)} = \frac{1}{2} - \frac{1}{6}\sqrt{17}
\end{aligned}
$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the $d's$.

step 2 Since we have a pole at $x = c_1 = 0$ and pole at $x = c_2 = 1$, and we have one $O(\infty)$, each with $\pm$ signs. The following now implements

$$
d = \alpha_\infty^\pm - \sum_{i=1}^{2} \alpha_{c_i}^\pm
$$

By trying all possible combinations. There are 8 possible d values. This gives

$$
\begin{aligned}
d_1 &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) = \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) - \left(\frac{1}{4}\sqrt{7} + \frac{1}{2}\right) - \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) = -\frac{1}{4}\sqrt{7} - \frac{1}{2} \\
d_2 &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) = \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) - \left(\frac{1}{4}\sqrt{7} + \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) = \frac{1}{3}\sqrt{17} - \frac{1}{4}\sqrt{7} - \frac{1}{2} \\
d_3 &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) = \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{4}\sqrt{7}\right) - \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) = \frac{1}{4}\sqrt{7} - \frac{1}{2} \\
d_4 &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) = \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{4}\sqrt{7}\right) - \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) = \frac{1}{4}\sqrt{7} + \frac{1}{3}\sqrt{17} - \frac{1}{2} \\
d_5 &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) = \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) - \left(\frac{1}{4}\sqrt{7} + \frac{1}{2}\right) - \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) = -\frac{1}{4}\sqrt{7} - \frac{1}{3}\sqrt{17} - \frac{1}{2} \\
d_6 &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) = \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) - \left(\frac{1}{4}\sqrt{7} + \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) = -\frac{1}{4}\sqrt{7} - \frac{1}{2} \\
d_7 &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) = \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) - \left(\frac{1}{2} - \frac{1}{4}\sqrt{7}\right) - \left(\frac{1}{6}\sqrt{17} + \frac{1}{2}\right) = \frac{1}{4}\sqrt{7} - \frac{1}{3}\sqrt{17} - \frac{1}{2} \\
d_8 &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) = \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) - \left(\frac{1}{2} - \frac{1}{4}\sqrt{7}\right) - \left(\frac{1}{2} - \frac{1}{6}\sqrt{17}\right) = \frac{1}{4}\sqrt{7} - \frac{1}{2}
\end{aligned}
$$

None of the d found are integer. Hence case 1 did not work we need to try case 2 and if that also fail, try case 3. We will find all three cases fail on this ode..

3.2.6 Example 6

Let

$$
y'' = y
$$

Therefore

$$
r = \frac{s}{t} = 1
$$

The necessary conditions for case 1 are met since zero order pole and $O(\infty) = 0$.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^\pm$ for each pole. There are no poles. In this case $\left[\sqrt{r}\right]_c = 0$ and $\alpha_c^\pm = 0$. Since $O(\infty) = 0$, we are in case $2v \le 0$. Hence $v = 0$. Then now $\left[\sqrt{r}\right]_\infty$ is the sum of all terms x^i for $0 \le i \le v$ in the Laurent series expansion of $\sqrt{r}$ at ∞.

$$
\left[\sqrt{r}\right]_\infty = 1
$$

Hence $a = 1$. And b is the coefficient of $x^{v-1} = x^{-1}$ in r minus the coefficient of $x^{v-1} = x^{-1}$ in $\left(\left[\sqrt{r}\right]_{\infty}\right)^2$. Hence $b = 0$. Then

$$\alpha_{\infty}^{+} = \frac{1}{2}\left(\frac{b}{a} - v\right) = 0$$
$$\alpha_{\infty}^{-} = \frac{1}{2}\left(-\frac{b}{a} - v\right) = 0$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^{\pm}$ and found $\left[\sqrt{r}\right]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$. Now we go to step 2 which is to find the $d's$.

step 2 Since we have a pole at zero and pole at $x = 1$, and we have one $O(\infty)$, each with $\pm$ signs, then we set up this table to make it easier to work with. This implements

$$d = \alpha_{\infty}^{\pm} - \sum_{i=1}^{0} \alpha_{c_i}^{\pm}$$

Therefore we obtain 2 possible d values.

pole c	α_c value	$O(\infty)$ value	d	d value
$x = NA$	$\alpha_c = 0$	$\alpha_{\infty}^{+} = 0$	$\alpha_{\infty}^{+} = 0$	0
$x = NA$	$\alpha_c = 0$	$\alpha_{\infty}^{-} = 0$	$\alpha_{\infty}^{+} = 0$	0

Hence the trial d values which are not negative integers are

$$d = \{0\}$$

For $d = 0$, since it shows in two rows, we take the first row. Now we generate ω for each d using

$$\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x - c}\right) + s(\infty)\left[\sqrt{r}\right]_{\infty}$$

To apply the above, we update the table above, but we also add columns for $\left[\sqrt{r}\right]_c, \left[\sqrt{r}\right]_{\infty}$ to make the computation easier. Here is the new table

pole c	α_c value	$s(c)$	$\left[\sqrt{r}\right]_c$	$O(\infty)$ value	$s(\infty)$	$\left[\sqrt{r}\right]_{\infty}$	d value	ω value $\left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_{\infty}$
$x = NA$	$\alpha_c = 0$	$+$	0	$\alpha_{\infty}^{+} = 0$	$+$	0	0	$(+(0) + 0) + (+)(1) = 1$

The above gives candidate $\omega = 1$ value to try. For this ω we need to find polynomial P by solving

$$P'' + 2\omega P' + \left(\omega' + \omega^2 - r\right) P = 0 \tag{8}$$

If we are able to find P, then we stop and the ode $y'' = ry$ is solved.

step 3 Now for each candidate ω we solve the above Eq (8). Starting with $\omega = 1$ associated with $d = 0$ in the table. Let $p(x) = 1$ since degree is zero, then (8) becomes

$$\begin{aligned} P'' + 2(1)\,P' + \left((1)' + (1)^2 - (1)\right) P &= 0 \\ (0 + 1 - 1) &= 0 \\ 0 &= 0 \end{aligned}$$

Hence $p(x) = 1$ is valid solution. Therefore the solution to $y'' = y$ is

$$\begin{aligned} y &= p(x)\, e^{\int \omega dx} \\ &= e^{\int 1\, dx} \\ &= e^{x} \end{aligned}$$

The second solution can be found by reduction of order. The full general solution to $y'' = y$ is

$$y(x) = c_1 e^{x} + c_2 e^{-x}$$

3.2.7 Example 7

Let

$$\left(x^2-2x\right)y''+\left(2-x^2\right)y'+(2x-2)\,y=0$$

Normalizing so that coefficient of y'' is one gives

$$y''+\frac{(2-x^2)}{(x^2-2x)}y'+\frac{(2x-2)}{(x^2-2x)}y=0$$

$$y''+ay'(x)+by=0 \tag{1}$$

Hence

$$a=\frac{(2-x^2)}{(x^2-2x)}$$
$$b=\frac{(2x-2)}{(x^2-2x)}$$

It is first transformed to the following ode by eliminating the first derivative

$$z''=rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y=ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r&=\frac{1}{4}a^2+\frac{1}{2}a'-b\\ &=\frac{1}{4}\left(\frac{(2-x^2)}{(x^2-2x)}\right)^2+\frac{1}{2}\frac{d}{dx}\left(\frac{(2-x^2)}{(x^2-2x)}\right)-\frac{(2x-2)}{(x^2-2x)}\\ &=\frac{(x^4-8x^3+24x^2-24x+12)}{4x^2\,(x-2)^2} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z''=\frac{(x^4-8x^3+24x^2-24x+12)}{4x^2\,(x-2)^2}z \tag{5}$$

Therefore

$$\begin{aligned} r&=\frac{s}{t}\\ &=\frac{(x^4-8x^3+24x^2-24x+12)}{4x^2\,(x-2)^2} \end{aligned}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x=0$ of order 2 and one pole at $x=2$ of order 2. For the pole at $x=0$ since order is 2 then

$$\begin{aligned} \left[\sqrt{r}\right]_{c=0}&=0\\ \alpha_{c=0}^{+}&=\frac{1}{2}+\frac{1}{2}\sqrt{1+4b}\\ \alpha_{c=0}^{-}&=\frac{1}{2}-\frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-0)^2}$ in the partial fraction decomposition of r which is

$$\frac{x^4-8x^3+24x^2-24x+12}{4x^2\,(x-2)^2}=\frac{1}{4}-\frac{3}{4}\frac{1}{x}-\frac{1}{4}\frac{1}{(x-2)}+\frac{3}{4}\frac{1}{(x-2)^2}+\frac{3}{4}\frac{1}{x^2} \tag{6}$$

Hence $b = \frac{3}{4}$. Therefore

$$\begin{aligned}
[\sqrt{r}]_{c=0} &= 0 \\
\alpha_{c=0}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left(\frac{3}{4}\right)} = \frac{3}{2} \\
\alpha_{c=0}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\left(\frac{3}{4}\right)} = -\frac{1}{2}
\end{aligned}$$

And for the pole at $x = 2$ which is order 2,

$$\begin{aligned}
[\sqrt{r}]_{c=2} &= 0 \\
\alpha_{c=2}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4b} \\
\alpha_{c=2}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4b}
\end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-2)^2}$ in the partial fraction decomposition of r given in Eq (6). Hence $b = \frac{3}{4}$. Therefore the above becomes

$$\begin{aligned}
[\sqrt{r}]_{c=2} &= 0 \\
\alpha_{c=2}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\left(\frac{3}{4}\right)} = \frac{3}{2} \\
\alpha_{c=2}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\left(\frac{3}{4}\right)} = -\frac{1}{2}
\end{aligned}$$

We are done with all the poles. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 4 - 4 = 0$. Since $O(\infty) = 0$, we are in case $2v \leq 0$. Hence $v = 0$. Then now $[\sqrt{r}]_{\infty}$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$[\sqrt{r}]_{\infty} = \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \dots \tag{7}$$

We want only terms for $0 \leq i \leq v$ but

$$v = 0$$

Therefore only the constant term. Hence

$$[\sqrt{r}]_{\infty} = \frac{1}{2} \tag{8}$$

Which means

$$a = \frac{1}{2}$$

As it is the the term that matches $[\sqrt{r}]_{\infty} = ax^v + \cdots$. Hence $\left([\sqrt{r}]_{\infty}\right)^2 = \frac{1}{4}$ and the coefficient of $\frac{1}{x}$ is zero. To find the coefficient of $\frac{1}{x}$ in r long division is done

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2}
\end{aligned}$$

Where Q is the quotient and R is the remainder. This gives

$$r = \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}$$

Since $v = 0$ then the coefficient of x^{-1} in r is found using $\frac{lcoeff(R)}{lcoeff(t)}$. This gives -1 as seen from above. Hence $b = -1 - 0 = -1$. Therefore

$$\begin{aligned}
\alpha_{\infty}^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) = -1 \\
\alpha_{\infty}^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) = 1
\end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^{\pm}$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^{\pm}$. Now we go to step 2 which is to find the $d's$.

step 2 Since we have a pole at $x = c_1 = 0$ and pole at $x = c_2 = 1$, and we have one $O(\infty)$, each with $\pm$ signs. The following now implements

$$d = \alpha_\infty^{\pm} - \sum_{i=1}^{2} \alpha_{c_i}^{\pm}$$

By trying all possible combinations. There are 8 possible d values. This gives

$$\begin{aligned}
d_1 &= 1 - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) = 1 - \left(\frac{3}{2} + \frac{3}{2}\right) = -2 \\
d_2 &= 1 - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) = 1 - \left(\frac{3}{2} - \frac{1}{2}\right) = 0 \\
d_3 &= 1 - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) = 1 - \left(-\frac{1}{2} + \frac{3}{2}\right) = 0 \\
d_4 &= 1 - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) = 1 - \left(-\frac{1}{2} - \frac{1}{2}\right) = 2 \\
d_5 &= -1 - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{+}\right) = -1 - \left(\frac{3}{2} + \frac{3}{2}\right) = -4 \\
d_6 &= -1 - \left(\alpha_{c_1}^{+} + \alpha_{c_2}^{-}\right) = -1 - \left(\frac{3}{2} - \frac{1}{2}\right) = -2 \\
d_7 &= -1 - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) = -1 - \left(-\frac{1}{2} + \frac{3}{2}\right) = -2 \\
d_8 &= -1 - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) = -1 - \left(-\frac{1}{2} - \frac{1}{2}\right) = 0
\end{aligned}$$

Need to complete the solution next.

3.2.8 Example 8

Let

$$\left(x^2 + 1\right) y'' + 2xy' - 2y = 0$$

Normalizing so that coefficient of y'' is one gives

$$\begin{aligned}
y'' + \frac{2x}{(x^2+1)} y' - \frac{2}{(x^2+1)} y &= 0 \\
y'' + ay'(x) + by &= 0 \qquad (1)
\end{aligned}$$

Hence

$$\begin{aligned}
a &= \frac{2x}{(x^2+1)} \\
b &= -\frac{2}{(x^2+1)}
\end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \qquad (2)$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \qquad (3)$$

Where it can be found that r in (2) is given by

$$\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{1}{4}\left(\frac{2x}{(x^2+1)}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{2x}{(x^2+1)}\right) - \left(-\frac{2}{(x^2+1)}\right) \\
&= \frac{2x^2+3}{(x^2+1)^2} \qquad (4)
\end{aligned}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{2x^2+3}{(x^2+1)^2} z \tag{5}$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2x^2+3}{(x^2+1)^2} \\ &= \frac{2x^2+3}{x^4+2x^2+1} \end{aligned} \tag{5A}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $[\sqrt{r}]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x = -i$ of order 2 and one pole at $x = i$ of order 2. For the pole at $x = -i$ since order is 2 then

$$\begin{aligned} [\sqrt{r}]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x+i)^2}$ in the partial fraction decomposition of r which is (in Maple this can be found using fullparfrac.

$$\frac{2x^2+3}{(x^2+1)^2} = -\frac{1}{4}\frac{1}{(x-i)^2} - \frac{1}{4}\frac{1}{(x+i)^2} - \frac{5i}{4}\frac{1}{(x-i)} + \frac{5i}{4}\frac{1}{x+i} \tag{6}$$

Hence $b = -\frac{1}{4}$. Therefore

$$\begin{aligned} [\sqrt{r}]_{c=0} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(-\frac{1}{4}\right)} = \frac{1}{2} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(-\frac{1}{4}\right)} = \frac{1}{2} \end{aligned}$$

And for the pole at $x = +i$ which is order 2,

$$\begin{aligned} [\sqrt{r}]_{c_2} &= 0 \\ \alpha_{c_2}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c_2}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-i)^2}$ in the partial fraction decomposition of r given in Eq (6). Hence $b = -\frac{1}{4}$. Therefore the above becomes

$$\begin{aligned} [\sqrt{r}]_{c=2} &= 0 \\ \alpha_{c=2}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(-\frac{1}{4}\right)} = \frac{1}{2} \\ \alpha_{c=2}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(-\frac{1}{4}\right)} = \frac{1}{2} \end{aligned}$$

We are done with all the poles. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 4 - 2 = 2$. Since $O(\infty) = 2$, then $[\sqrt{r}]_\infty = 0$. Now b is the coefficient of $\frac{1}{x^2}$ in r minus coefficient of $\frac{1}{x^2}$

in $\left[\sqrt{r}\right]_\infty^2$ which is zero. the coefficient of $\frac{1}{x^2}$ in r is found from $\frac{lcoeff(s)}{lcoeff(t)}$ which from Eq (5A) above is $\frac{2}{1}=2$. Hence $b=2-0=2$.

$$
\begin{aligned}
\alpha_\infty^+ &= \frac{1}{2}+\frac{1}{2}\sqrt{1+4b}=\frac{1}{2}+\frac{1}{2}\sqrt{1+8}=2\\
\alpha_\infty^- &= \frac{1}{2}-\frac{1}{2}\sqrt{1+4b}=\frac{1}{2}-\frac{1}{2}\sqrt{1+8}=-1
\end{aligned}
$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the $d's$.

step 2 Since we have a pole at $x=-i$ and pole at $x=+i$ each of order 2, and we have one $O(\infty)$, each with $\pm$ signs. The following now implements

$$
d=\alpha_\infty^\pm-\sum_{i=1}^{2}\alpha_{c_i}^\pm
$$

By trying all possible combinations. There are 8 possible d values. This gives

$$
\begin{aligned}
d_1 &= 2-\left(\alpha_{c_1}^+ +\alpha_{c_2}^+\right)=2-\left(\frac{1}{2}+\frac{1}{2}\right)=1\\
d_2 &= 2-\left(\alpha_{c_1}^+ +\alpha_{c_2}^-\right)=2-\left(\frac{1}{2}+\frac{1}{2}\right)=1\\
d_3 &= 2-\left(\alpha_{c_1}^- +\alpha_{c_2}^+\right)=2-\left(\frac{1}{2}+\frac{1}{2}\right)=1\\
d_4 &= 2-\left(\alpha_{c_1}^- +\alpha_{c_2}^-\right)=2-\left(\frac{1}{2}+\frac{1}{2}\right)=1\\
d_5 &= -1-\left(\alpha_{c_1}^+ +\alpha_{c_2}^+\right)=-1-\left(\frac{1}{2}+\frac{1}{2}\right)=-2\\
d_6 &= -1-\left(\alpha_{c_1}^+ +\alpha_{c_2}^-\right)=-1-\left(\frac{1}{2}+\frac{1}{2}\right)=-2\\
d_7 &= -1-\left(\alpha_{c_1}^- +\alpha_{c_2}^+\right)=-1-\left(\frac{1}{2}+\frac{1}{2}\right)=-2\\
d_8 &= -1-\left(\alpha_{c_1}^- +\alpha_{c_2}^-\right)=-1-\left(\frac{1}{2}+\frac{1}{2}\right)=-2
\end{aligned}
$$

Need to complete the solution next.

3.2.9 Example 9

Let

$$
(1-x)\,y''+xy'-y=0
$$

Normalizing so that coefficient of y'' is one gives

$$
\begin{aligned}
y''+\frac{x}{(1-x)}y'-\frac{1}{(1-x)}y&=0\\
y''+ay'(x)+by&=0 \qquad (1)
\end{aligned}
$$

Hence

$$
\begin{aligned}
a&=\frac{x}{(1-x)}\\
b&=-\frac{1}{(1-x)}
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$
z''=rz \qquad (2)
$$

Using what is known as the Liouville transformation given by

$$
y=ze^{\frac{-1}{2}\int a\,dx} \qquad (3)
$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a + \frac{1}{2}a' - b \\ &= \frac{1}{4}\left(\frac{x}{(1-x)}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{x}{(1-x)}\right) - \left(-\frac{1}{(1-x)}\right) \\ &= \frac{x^2 - 4x + 6}{4(x-1)^2} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{x^2 - 4x + 6}{4(x-1)^2} z \tag{5}$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4(x-1)^2} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \end{aligned} \tag{5A}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $[\sqrt{r}]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x = 1$ of order 2. Hence

$$\begin{aligned} [\sqrt{r}]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-1)^2}$ in the partial fraction decomposition of r which is (in Maple this can be found using fullparfrac).

$$\frac{x^2 - 4x + 6}{4(x-1)^2} = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2}\frac{1}{x-1} \tag{6}$$

Hence $b = \frac{3}{4}$. Therefore

$$\begin{aligned} [\sqrt{r}]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = \frac{3}{2} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = -\frac{1}{2} \end{aligned}$$

We are done with all the poles. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 2 - 2 = 0$. Since $O(\infty) = 0$, we are in case $2v \leq 0$. Hence

$$v = 0$$

Then now $[\sqrt{r}]_\infty$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$[\sqrt{r}]_\infty = \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \cdots \tag{7}$$

We want only terms for $0 \leq i \leq v$ but $v = 0$. Therefore only the constant term. Hence

$$[\sqrt{r}]_\infty = \frac{1}{2} \tag{8}$$

Which means

$$a = \frac{1}{2}$$

As it is the term that matches $\left[\sqrt{r}\right]_\infty = ax^v + \cdots$. Now we need to find b. This will be the coefficient of $x^{v-1} = \frac{1}{x}$ in $r - \left(\left[\sqrt{r}\right]_\infty\right)^2$. But $\left(\left[\sqrt{r}\right]_\infty\right)^2 = \frac{1}{4}$. So coefficient of $\frac{1}{x}$ is zero in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. To find the coefficient of $\frac{1}{x}$ in r long division is done

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \end{aligned}$$

Where Q is the quotient and R is the remainder. This gives

$$r = \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4}$$

Hence the coefficient of $\frac{1}{x}$ is $\frac{lcoeff(R)}{lcoeff(t)} = \frac{-2}{4} = -\frac{1}{2}$. Therefore $b = -\frac{1}{2} - 0 = -\frac{1}{2}$. Hence

$$\begin{aligned} \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) = -\frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) = \frac{1}{2} \end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the $d's$.

<u>step 2</u> Now d is found using

$$d = \alpha_\infty^\pm - \sum_{i=1}^{1} \alpha_{c_i}^\pm$$

By trying all possible combinations. There are 4 possible d values. This gives

$$\begin{aligned} d_1 &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) = -\frac{1}{2} - \frac{3}{2} = -2 \\ d_2 &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0 \\ d_3 &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) = \frac{1}{2} - \frac{3}{2} = -1 \\ d_4 &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1 \end{aligned}$$

Using entry $d = 1$ entry above now we find ω using

$$\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$$

Hence

$$\begin{aligned} \omega &= \left(0 + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)\frac{1}{2} \\ &= \frac{-\frac{1}{2}}{x - 1} - \frac{1}{2} \\ &= -\frac{1}{2}\frac{1}{x - 1} - \frac{1}{2} \end{aligned}$$

If this ω fails to find $p(x)$, then we will try the entry $d = 0$. Which will give

$$\omega = -\frac{1}{2}\frac{1}{x - 1} + \frac{1}{2}$$

Will finish the solution later.

3.2.10 Example 10

Let

$$y'' - x^2 y' - 3xy = 0 \tag{1}$$
$$y'' + ay'(x) + by = 0$$

Hence

$$a = -x^2$$
$$b = -3x$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\ &= \frac{1}{4}(-x^2)^2 + \frac{1}{2}\frac{d}{dx}(-x^2) - (-3x) \\ &= \frac{1}{4}x^4 + 2x \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \left(\frac{1}{4}x^4 + 2x\right) z \tag{5}$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{1}{4}x^4 + 2x \end{aligned}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $[\sqrt{r}]_c$ and associated $\alpha_c^{\pm}$ for each pole. There are no poles. Hence set Γ of poles is empty. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 0 - 4 = -4$. We are in case $2v \leq 0$. Hence $-2v = -4$ or

$$v = 2$$

Then now $[\sqrt{r}]_\infty$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$[\sqrt{r}]_\infty = \frac{x^2}{2} + \frac{2}{x} - \frac{4}{x^4} + \frac{16}{x^7} - \cdots \tag{7}$$

We want only terms for $0 \leq i \leq v$ but $v = 2$. Therefore need to sum terms x^0, x^1, x^2. From the above we see that

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x^2}{2} + 0x^1 + 0x^0 \\ &= \frac{x^2}{2} \end{aligned}$$

Which means

$$a = \frac{1}{2}$$

As that is the term which matches $\left[\sqrt{r}\right]_\infty = ax^2 + \cdots$. Now we need to find b. This will be the coefficient of $x^{v-1} = x$ in r minus coefficient of x in$\left(\left[\sqrt{r}\right]_\infty\right)^2$. But

$$\left(\left[\sqrt{r}\right]_\infty\right)^2 = \frac{x^4}{4}$$

Hence the coefficient is zero here. Now we find coefficient of x in r. But $r = \frac{1}{4}x^4 + 2x$ hence the coefficient of x is 2. Therefore

$$\begin{aligned} b &= 2 - 0 \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{2}{\frac{1}{2}} - 2\right) = 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{2}{\frac{1}{2}} - 2\right) = -3 \end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the $d's$.

step 2 Now d is found using

$$d = \alpha_\infty^\pm - \sum_{i=1}^{0} \alpha_{ci}^\pm$$

By trying all possible combinations. There are 4 possible d values. This gives

$$\begin{aligned} d_1 &= \alpha_\infty^+ = 1 \\ d_2 &= \alpha_\infty^- = -3 \end{aligned}$$

Using $d = 1$ entry above now we find ω using

$$\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$$

Hence, since there are no poles, only last term above survives giving

$$\omega = s(\infty)\left[\sqrt{r}\right]_\infty = (+)\frac{x^2}{2} = \frac{x^2}{2}$$

Will finish the solution next.

3.2.11 Example 11

Let

$$\begin{aligned} y'' - \frac{2}{5}xy' + 2y &= 0 \qquad (1) \\ y'' + ay'(x) + by &= 0 \end{aligned}$$

Hence

$$\begin{aligned} a &= -\frac{2}{5}x \\ b &= 2 \end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \qquad (2)$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a\,dx} \qquad (3)$$

Where it can be found that r in (2) is given by

$$
\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{1}{4}\left(-\frac{2}{5}x\right)^2 + \frac{1}{2}\frac{d}{dx}\left(-\frac{2}{5}x\right) - (2) \\
&= \frac{x^2 - 55}{25}
\end{aligned} \tag{4}
$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$
z'' = \left(\frac{x^2 - 55}{25}\right) z \tag{5}
$$

Therefore

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 - 55}{25}
\end{aligned}
$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $[\sqrt{r}]_c$ and associated $\alpha_c^{\pm}$ for each pole. There are no poles. Hence set Γ of poles is empty. Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 0 - 2 = -2$. We are in case $2v \leq 0$. Hence $-2v = -2$ or

$$
v = 1
$$

Then now $[\sqrt{r}]_\infty$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$
[\sqrt{r}]_\infty = \frac{x}{5} - \frac{11}{2}\frac{1}{x} - \cdots \tag{7}
$$

We want only terms for $0 \leq i \leq v$ but $v = 1$. Therefore need to sum terms x^0, x^1. From the above we see that

$$
[\sqrt{r}]_\infty = \frac{x}{5}
$$

Which means

$$
a = \frac{1}{5}
$$

As it is the the term that matches $[\sqrt{r}]_\infty = ax$. Now we need to find b. This will be the coefficient of $x^{v-1} = x^0$ in r minus the coefficient of x^0 in $\left([\sqrt{r}]_\infty\right)^2$. But

$$
\begin{aligned}
\left([\sqrt{r}]_\infty\right)^2 &= \left(\frac{x}{5}\right)^2 \\
&= \frac{x^2}{25}
\end{aligned}
$$

Hence the coefficient of x^0 is zero. Now we find coefficient of x^0 in r. Since $r = \frac{x^2}{25} - \frac{55}{25}$ then coefficient of x^0 is $\frac{-55}{25} = -\frac{11}{5}$. Hence $b = -\frac{11}{5} - 0 = -\frac{11}{5}$. Therefore

$$
\begin{aligned}
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) = -6 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{-\frac{11}{5}}{\frac{1}{5}} - 1\right) = 5
\end{aligned}
$$

This completes step 1 of the solution. We have found $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ and found $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$. Now we go to step 2 which is to find the d's.

step 2 Now d is found using

$$
d = \alpha_\infty^{\pm} - \sum_{i=1}^{0} \alpha_{c_i}^{\pm}
$$

By trying all possible combinations. There are 2 possible d values since no poles.

$$
\begin{aligned}
d_1 &= \alpha_\infty^+ = -6 \\
d_2 &= \alpha_\infty^- = 5
\end{aligned}
$$

Using $d = 5$ entry above now we find ω using

$$
\omega = \left(\sum_c s(c) \left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) \left[\sqrt{r}\right]_\infty
$$

Hence, since there are no poles, only last term above survives giving

$$
\omega = s(\infty) \left[\sqrt{r}\right]_\infty = (-) \frac{x}{5} = -\frac{x}{5}
$$

Will finish the solution next.

3.2.12 Example 12

Let

$$
\begin{aligned}
y'' + \frac{x^2 - 1}{x} y' + x^2 y &= 0 \qquad (1) \\
y'' + a y'(x) + b y &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
a &= \frac{x^2 - 1}{x} \\
b &= x^2
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$
z'' = rz \qquad (2)
$$

Using what is known as the Liouville transformation given by

$$
y = z e^{\frac{-1}{2} \int a dx} \qquad (3)
$$

Where it can be found that r in (2) is given by

$$
\begin{aligned}
r &= \frac{1}{4} a^2 + \frac{1}{2} a' - b \\
&= \frac{1}{4} \left(\frac{x^2 - 1}{x} \right)^2 + \frac{1}{2} \frac{d}{dx} \left(\frac{x^2 - 1}{x} \right) - x^2 \\
&= -\frac{3(x^4 - 1)}{4x^2} \qquad (4)
\end{aligned}
$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$
z'' = -\frac{3(x^4 - 1)}{4x^2} z \qquad (5)
$$

Therefore

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-3(x^4 - 1)}{4x^2}
\end{aligned}
$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^\pm$ for each pole. There is one pole at $x = 0$ of order 2. Hence

$$
\begin{aligned}
\left[\sqrt{r}\right]_{c_1} &= 0\\
\alpha_{c_1}^{+} &= \frac{1}{2}+\frac{1}{2}\sqrt{1+4b}\\
\alpha_{c_1}^{-} &= \frac{1}{2}-\frac{1}{2}\sqrt{1+4b}
\end{aligned}
$$

Where b is the coefficient of $\frac{1}{(x-0)^2}$ in the partial fraction decomposition of r which is (in Maple this can be found using fullparfrac).

$$
\frac{-3(x^4-1)}{4x^2} = -\frac{3}{4}x^2+\frac{3}{4}\frac{1}{x^2} \tag{6}
$$

Hence $b=\frac{3}{4}$. Therefore

$$
\begin{aligned}
\left[\sqrt{r}\right]_{c_1} &= 0\\
\alpha_{c_1}^{+} &= \frac{1}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = \frac{3}{2}\\
\alpha_{c_1}^{-} &= \frac{1}{2}-\frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = -\frac{1}{2}
\end{aligned}
$$

Now we consider $O(\infty)$ which is $\deg(t)-\deg(s)=2-4=-2$. We are in case $2v\leq 0$. Hence $-2v=-2$ or

$$
v=1
$$

Then now $\left[\sqrt{r}\right]_{\infty}$ is the sum of all terms x^i for $0\leq i\leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$
\left[\sqrt{r}\right]_{\infty} = \frac{i\sqrt{3}}{2}x-\frac{i\sqrt{3}}{4}\frac{1}{x^3}+\cdots \tag{7}
$$

We want only terms for $0\leq i\leq v$ but $v=1$. Therefore need to sum terms x^0, x^1. From the above we see that

$$
\left[\sqrt{r}\right]_{\infty} = \frac{i\sqrt{3}}{2}x
$$

Which means

$$
a=\frac{i\sqrt{3}}{2}
$$

As it is the the term that matches $\left[\sqrt{r}\right]_{\infty}=ax$. Now we need to find b. This will be the coefficient of $x^{v-1}=x^0$ in r minus the coefficient of x^0 in $\left(\left[\sqrt{r}\right]_{\infty}\right)^2$. But

$$
\begin{aligned}
\left(\left[\sqrt{r}\right]_{\infty}\right)^2 &= \left(\frac{i\sqrt{3}}{2}x\right)^2\\
&= -\frac{3}{4}x^2
\end{aligned}
$$

Hence the coefficient is zero here. Now we find coefficient of x^0 in r. Since $r=-\frac{3}{4}x^2+\frac{3}{4}\frac{1}{x^2}$ then coefficient of x^0 is zero also. Hence $b=0-0=0$. Therefore

$$
\begin{aligned}
\alpha_{\infty}^{+} &= \frac{1}{2}\left(\frac{b}{a}-v\right) = \frac{1}{2}(0-1) = -\frac{1}{2}\\
\alpha_{\infty}^{-} &= \frac{1}{2}\left(-\frac{b}{a}-v\right) = \frac{1}{2}(0-1) = -\frac{1}{2}
\end{aligned}
$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^{\pm}$ and found $\left[\sqrt{r}\right]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$. Now we go to step 2 which is to find the $d's$.

step 2 Now d is found using

$$
d=\alpha_{\infty}^{\pm}-\sum_{i=1}^{0}\alpha_{c_i}^{\pm}
$$

By trying all possible combinations. There are 4 possible d values. This gives

$$
\begin{aligned}
d_1 &= \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) = -\frac{1}{2} - \frac{3}{2} = -1 \\
d_2 &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0 \\
d_3 &= \alpha_\infty^- - \left(\alpha_{c_1}^+\right) = -\frac{1}{2} - \frac{3}{2} = -2 \\
d_4 &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0
\end{aligned}
$$

Using first $d = 0$ entry above now we find ω using

$$
\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty
$$

Hence

$$
\omega = (-1)(0) + \frac{-\frac{1}{2}}{x-0} + (+)\left(\frac{i\sqrt{3}}{2}x\right) = \frac{-1}{2x} + \frac{i\sqrt{3}}{2}x
$$

Notice that if have taken the last $d = 0$ entry, we will get

$$
\omega = (-1)(0) + \frac{-\frac{1}{2}}{x-0} + (-)\left(\frac{i\sqrt{3}}{2}x\right) = \frac{-1}{2x} - \frac{i\sqrt{3}}{2}x
$$

In practice, we will try the second one if the first fails. Will finish the solution next.

3.2.13 Example 13

Let

$$
\begin{aligned}
(1-x)\,y'' + xy' - y &= 0 \qquad (1) \\
y'' + ay'(x) + by &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
a &= \frac{x}{1-x} \\
b &= -\frac{1}{1-x}
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$
z'' = rz \qquad (2)
$$

Using what is known as the Liouville transformation given by

$$
y = ze^{\frac{-1}{2}\int a dx} \qquad (3)
$$

Where it can be found that r in (2) is given by

$$
\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{1}{4}\left(\frac{x}{1-x}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{x}{1-x}\right) - \left(-\frac{1}{1-x}\right) \\
&= \frac{x^2 - 4x + 6}{4\left(x-1\right)^2} \qquad (4)
\end{aligned}
$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$
z'' = \frac{x^2 - 4x + 6}{4\left(x-1\right)^2}z \qquad (5)
$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4\left(x-1\right)^2} \end{aligned}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There is one pole at $x = 1$ of order 2. Hence

$$\begin{aligned} \left[\sqrt{r}\right]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b} \end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-1)^2}$ in the partial fraction decomposition of r which is

$$\frac{x^2 - 4x + 6}{4\left(x-1\right)^2} = \frac{1}{4} + \frac{3}{4\left(x-1\right)^2} - \frac{1}{2}\frac{1}{x-1} \tag{6}$$

Hence $b = \frac{3}{4}$. Therefore

$$\begin{aligned} \left[\sqrt{r}\right]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = \frac{3}{2} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)} = -\frac{1}{2} \end{aligned}$$

Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 2 - 2 = 0$. We are in case $2v \leq 0$. Hence $-2v = 0$ or

$$v = 0$$

Then now $\left[\sqrt{r}\right]_\infty$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$\left[\sqrt{r}\right]_\infty = \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \cdots \tag{7}$$

But we want only terms for $0 \leq i \leq v$ but $v = 0$. Therefore need to sum terms x^0. Which is the constant term

$$\left[\sqrt{r}\right]_\infty = \frac{1}{2} \tag{8}$$

Which means

$$a = \frac{1}{2}$$

Now we need to find b. Which is given by the coefficient of $\frac{1}{x}$ in r minus coefficient of $\frac{1}{x}$ in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. But $\left(\left[\sqrt{r}\right]_\infty\right)^2 = \frac{1}{4}$ Hence the coefficient is zero here. To find the coefficient of $\frac{1}{x}$ in r long division is done (here paper is not clear at all what it means by coefficient of x^{v-1} in r as that depends on the form of r and how it is represented). This method of using long division to find the coefficient works to obtain the correct result. But it is still not clear what the paper actually means by this.

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \end{aligned}$$

Where Q is the quotient and R is the remainder. This gives

$$r = \frac{1}{4} + \frac{-2x+5}{4x^2-8x+4}$$

For the case of $v=0$ then the coefficient of x^{-1} is $\frac{lcoeff(R)}{lcoeff(t)} = \frac{-2}{4} = -\frac{1}{2}$. Notice that if we just expanded r it will give $\frac{x^2}{4(x-1)^2} - \frac{x}{(x-1)^2} + \frac{3}{2(x-1)^2}$ and we see there is no coefficient of $\frac{1}{x}$ in this representation. So we would have obtain wrong value of b if we just used what the paper said. Now $b = -\frac{1}{2} - 0 = -\frac{1}{2}$. Therefore

$$\alpha_\infty^+ = \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) = \frac{1}{2}$$
$$\alpha_\infty^- = \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} - 0\right) = -\frac{1}{2}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^\pm$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^\pm$. Now we go to step 2 which is to find the $d's$.

step 2 Now d is found using

$$d = \alpha_\infty^\pm - \sum_{i=1}^{1} \alpha_{c_i}^\pm$$

By trying all possible combinations. There are 4 possible d values. This gives

$$d_1 = \alpha_\infty^+ - \left(\alpha_{c_1}^+\right) = \frac{1}{2} - \left(\frac{3}{2}\right) = -1$$
$$d_2 = \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$
$$d_3 = \alpha_\infty^- - \left(\alpha_{c_1}^+\right) = -\frac{1}{2} - \left(\frac{3}{2}\right) = -2$$
$$d_4 = \alpha_\infty^- - \left(\alpha_{c_1}^-\right) = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0$$

Using first $d=1$ entry from above we find ω using

$$\omega = \left(\sum_c s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$$

Hence

$$\omega = \left((-1)(0) + \frac{-\frac{1}{2}}{x-1}\right) + (+)\left(\frac{1}{2}\right) = \frac{-1}{2(x-1)} + \frac{1}{2}$$

And if use the last entry $d=0$ then

$$\omega\left(= (-1)(0) + \frac{-\frac{1}{2}}{x-1}\right) + (-)\left(\frac{1}{2}x\right) = \frac{-1}{2(x-1)} - \frac{1}{2}$$

In practice, we will try the second one if the first fails. Will finish the solution next.

3.2.14 Example 14

Let

$$3y'' + xy' - 4y = 0 \tag{1}$$
$$y'' + ay'(x) + by = 0$$

Hence

$$a = \frac{x}{3}$$
$$b = -\frac{4}{3}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a\,dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{1}{4}\left(\frac{x}{3}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{x}{3}\right) - \left(-\frac{4}{3}\right) \\
&= \frac{x^2}{36} + \frac{3}{2}
\end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{x^2 + 54}{36}z \tag{5}$$

Therefore

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 54}{36}
\end{aligned}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There are no poles. Hence Γ is empty.

Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 0 - 2 = -2$. We are in case $2v \leq 0$. Hence $-2v = -2$ or

$$v = 1$$

Then now $\left[\sqrt{r}\right]_\infty$ is the sum of all terms x^i for $0 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ at ∞ which is

$$\left[\sqrt{r}\right]_\infty = \frac{x}{6} + \frac{9}{2}\frac{1}{x} + \cdots \tag{7}$$

But we want only terms for $0 \leq i \leq v$ but $v = 1$. Therefore need to sum terms x^0, x^1. Therefore

$$\left[\sqrt{r}\right]_\infty = \frac{x}{6} \tag{8}$$

Which means

$$a = \frac{1}{6}$$

Now we need to find b. Which is given by the coefficient of $x^{v-1} = x^0$ or the constant term in r minus coefficient of x^0 in $\left(\left[\sqrt{r}\right]_\infty\right)^2$. But $\left(\left[\sqrt{r}\right]_\infty\right)^2 = \frac{x^2}{36}$. Hence the coefficient of x^0 is zero here. Now we find coefficient of x^0 in r. Since $r = \frac{x^2}{36} + \frac{54}{36}$ then the coefficient of x^0 is $\frac{54}{36} = \frac{3}{2}$. Hence $b = \frac{3}{2} - 0 = \frac{3}{2}$. Therefore

$$\begin{aligned}
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) = \frac{1}{2}\left(\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) = 4 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) = \frac{1}{2}\left(-\frac{\frac{3}{2}}{\frac{1}{6}} - 1\right) = -5
\end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^{\pm}$ and found $\left[\sqrt{r}\right]_\infty$ and its associated $\alpha_\infty^{\pm}$. Now we go to step 2 which is to find the d's.

step 2 Now d is found using

$$d = \alpha_\infty^{\pm} - \sum_{i=1}^{0} \alpha_{c_i}^{\pm}$$

By trying all possible combinations. There are 2 possible d values (since no poles). This gives

$$\begin{aligned} d_1 &= \alpha_\infty^+ = 4 \\ d_2 &= \alpha_\infty^- = -5 \end{aligned}$$

Using $d = 4$ entry from above we find ω using

$$\omega = \left(\sum_c s(c) \left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) \left[\sqrt{r}\right]_\infty$$

Hence

$$\omega = (0) + (+) \left(\frac{x}{6}\right) = \frac{x}{6}$$

3.2.15 Example 15

Let

$$\begin{aligned} \left(4 - x^2\right) y'' + x y' + 2y &= 0 \\ y'' + a y'(x) + b y &= 0 \end{aligned} \tag{1}$$

Hence

$$\begin{aligned} a &= \frac{x}{(4 - x^2)} \\ b &= \frac{2}{(4 - x^2)} \end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = z e^{\frac{-1}{2} \int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4} a^2 + \frac{1}{2} a' - b \\ &= \frac{1}{4} \left(\frac{x}{(4 - x^2)} \right)^2 + \frac{1}{2} \frac{d}{dx} \left(\frac{x}{(4 - x^2)} \right) - \left(\frac{2}{(4 - x^2)} \right) \\ &= \frac{11x^2 - 24}{4 \left(x^2 - 4\right)^2} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{11x^2 - 24}{4 \left(x^2 - 4\right)^2} z \tag{5}$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{11x^2 - 24}{4 \left(x^2 - 4\right)^2} = \frac{11x^2 - 24}{4x^4 - 32x^2 + 64} \end{aligned} \tag{5A}$$

The necessary conditions for case 1 are met.

Step 1 In this we find all $\left[\sqrt{r}\right]_c$ and associated $\alpha_c^{\pm}$ for each pole. There are two poles at ± 2 each of order 2. For pole at $x = 2 = c_1$

$$\begin{aligned}\left[\sqrt{r}\right]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4b} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4b}\end{aligned}$$

Where b is the coefficient of $\frac{1}{(x-2)^2}$ in the partial fraction decomposition of r which is

$$\frac{11x^2-24}{4\left(x^2-4\right)^2} = \frac{5}{16}\frac{1}{\left(x+2\right)^2} + \frac{5}{16}\frac{1}{\left(x-2\right)^2} - \frac{17}{32}\frac{1}{\left(x+2\right)} + \frac{17}{32}\frac{1}{\left(x-2\right)} \tag{6}$$

Hence $b = \frac{5}{16}$. Therefore

$$\begin{aligned}\left[\sqrt{r}\right]_{c_1} &= 0 \\ \alpha_{c_1}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{5}{16}\right)} = \frac{5}{4} \\ \alpha_{c_1}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{5}{16}\right)} = -\frac{1}{4}\end{aligned}$$

And for pole at $x = -2 = c_2$

$$\begin{aligned}\left[\sqrt{r}\right]_{c_2} &= 0 \\ \alpha_{c_2}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{5}{16}\right)} = \frac{5}{4} \\ \alpha_{c_2}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{5}{16}\right)} = -\frac{1}{4}\end{aligned}$$

Now we consider $O(\infty)$ which is $\deg(t) - \deg(s) = 4 - 2 = 2$. Therefore $v = 1$. In this case

$$\left[\sqrt{r}\right]_{\infty} = 0 \tag{7}$$

And

$$\alpha_{\infty}^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$$

The coefficient of $x^{v-1} = x^0$ is zero in $\left[\sqrt{r}\right]_{\infty}^2$. To find coefficient of x^0 in $r = \frac{11x^2-24}{4(x^2-4)^2}$ and since $v = 0$ then using $b = \frac{lcoeff(s)}{lcoeff(t)}$ gives $\frac{11}{4}$. Hence $b = \frac{11}{4} - 0 = \frac{11}{4}$. Therefore

$$\begin{aligned}\alpha_{\infty}^{+} &= \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(\frac{11}{4}\right)} = \frac{1}{2} + \sqrt{3} \\ \alpha_{\infty}^{-} &= \frac{1}{2} - \frac{1}{2}\sqrt{1+4\left(\frac{11}{4}\right)} = \frac{1}{2} - \sqrt{3}\end{aligned}$$

This completes step 1 of the solution. We have found $\left[\sqrt{r}\right]_c$ and its associated $\alpha_c^{\pm}$ and found $\left[\sqrt{r}\right]_{\infty}$ and its associated $\alpha_{\infty}^{\pm}$. Now we go to step 2 which is to find the $d's$.

step 2 Now d is found using

$$d = \alpha_{\infty}^{\pm} - \sum_{i=1}^{0} \alpha_{c_i}^{\pm}$$

By trying all possible combinations. There are 8 possible d values. These are

$$
\begin{aligned}
d_1 &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) = \frac{1}{2} + \sqrt{3} - \left(\frac{5}{4} + \frac{5}{4}\right) = \sqrt{3} - 2 \\
d_2 &= \alpha_\infty^+ - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) = \frac{1}{2} + \sqrt{3} - \left(\frac{5}{4} - \frac{1}{4}\right) = \sqrt{3} - \frac{1}{2} \\
d_3 &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) = \frac{1}{2} + \sqrt{3} - \left(-\frac{1}{4} + \frac{5}{4}\right) = \sqrt{3} - \frac{1}{2} \\
d_4 &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) = \frac{1}{2} + \sqrt{3} - \left(-\frac{1}{4} - \frac{1}{4}\right) = \sqrt{3} + 1 \\
d_5 &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) = \frac{1}{2} - \sqrt{3} - \left(\frac{5}{4} + \frac{5}{4}\right) = -\sqrt{3} - 2 \\
d_6 &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^-\right) = \frac{1}{2} - \sqrt{3} - \left(\frac{5}{4} - \frac{1}{4}\right) = -\sqrt{3} - \frac{1}{2} \\
d_7 &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) = \frac{1}{2} - \sqrt{3} - \left(-\frac{1}{4} + \frac{5}{4}\right) = -\sqrt{3} - \frac{1}{2} \\
d_8 &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^-\right) = \frac{1}{2} - \sqrt{3} - \left(-\frac{1}{4} - \frac{1}{4}\right) = 1 - \sqrt{3}
\end{aligned}
$$

There are no $d \geq 0$ integers. This means case 1 does not apply. We need to try case 2 now.

3.3 Case two Kovacic algorithm

The following diagram shows the algorithm for case two.

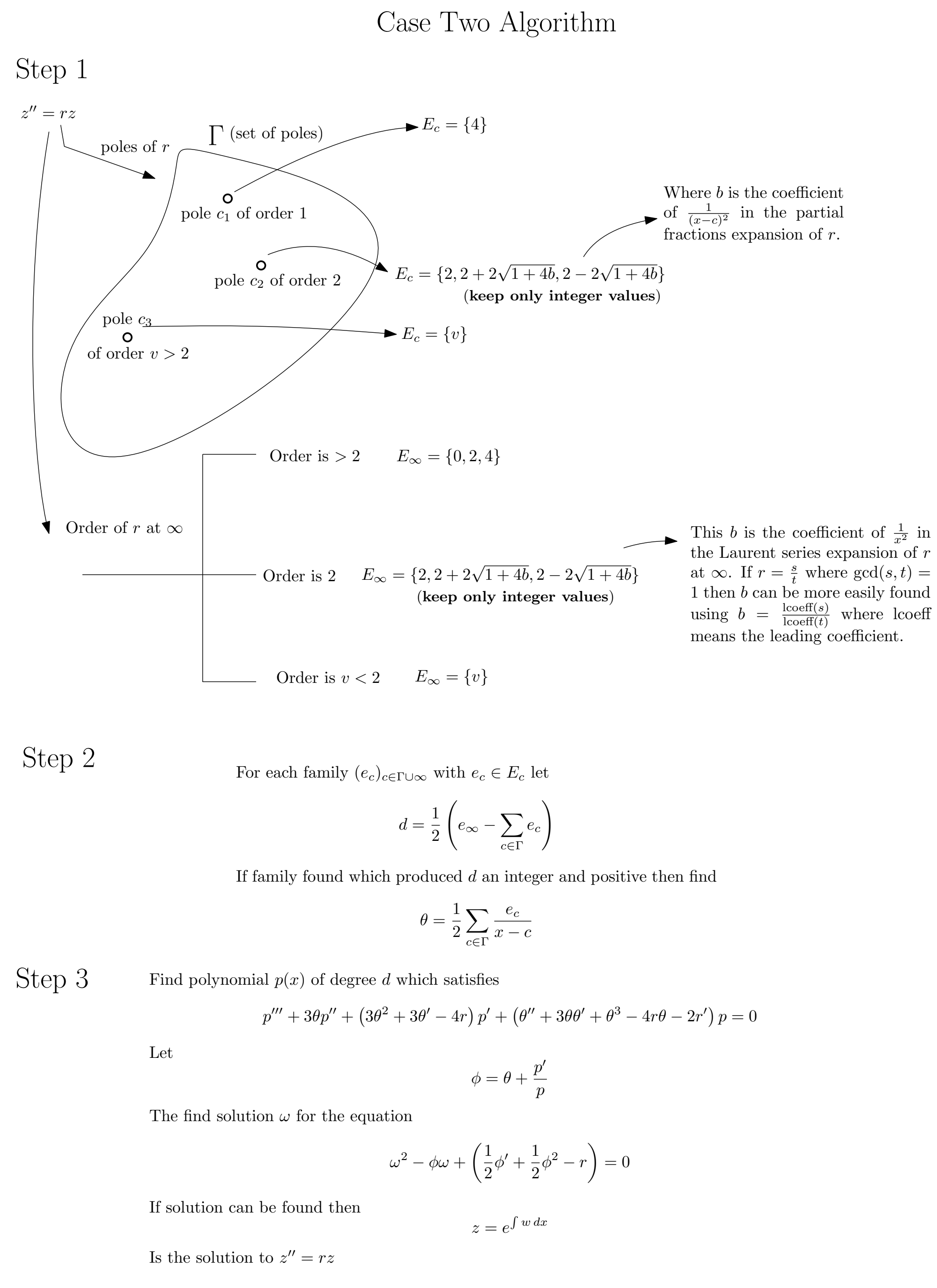

Figure 2: Case 2 Kovacic algorithm

3.4 Case 3 Kovacic algorithm

The following diagram shows the algorithm for case 3.

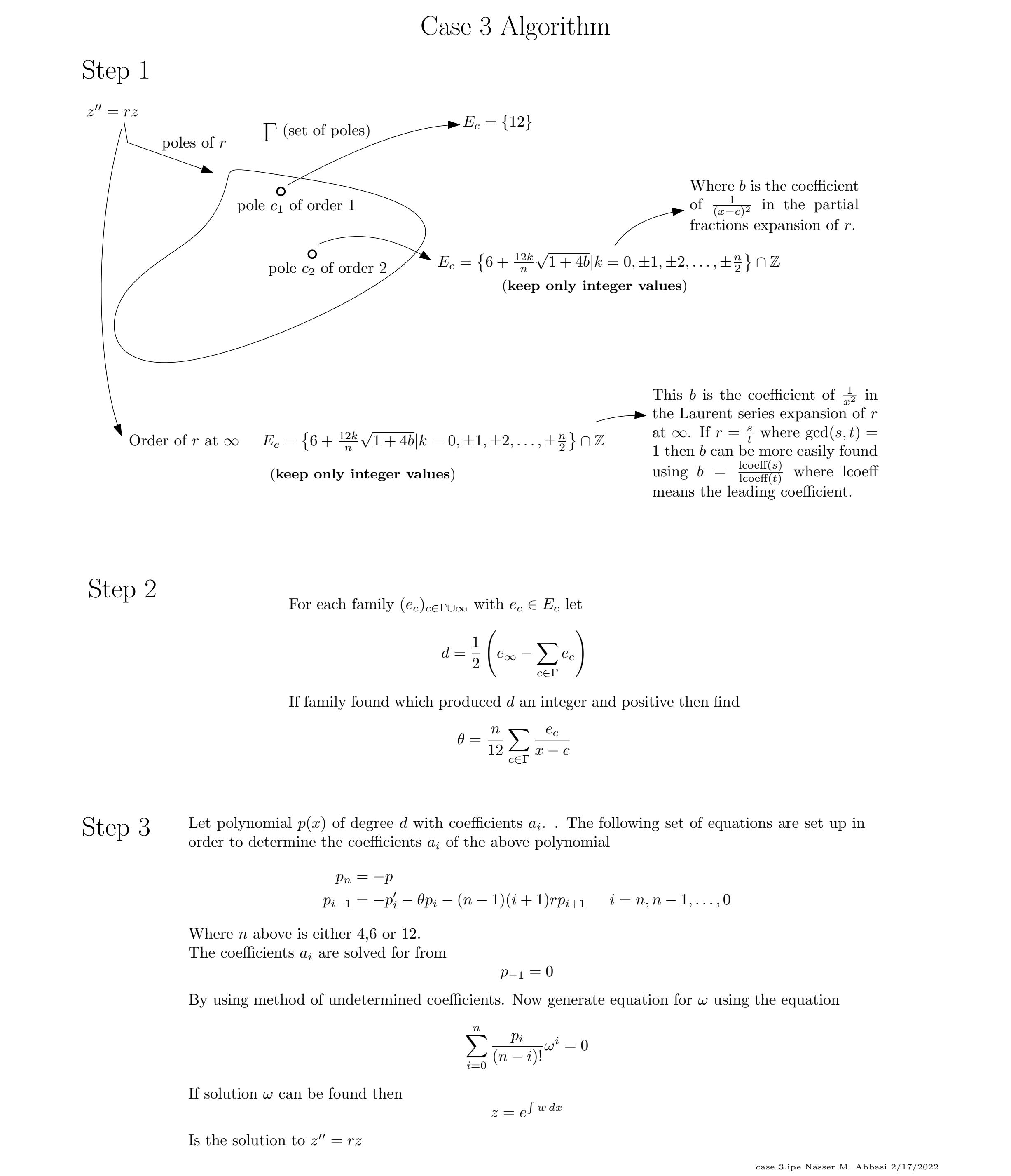

Figure 3: Case 3 Kovacic algorithm

4 Algorithm implementation based on modified Saunders and Smith algorithm

As with original Kovacic algorithm, an input ode

$$y''(x) + ay'(x) + by(x) = 0 \qquad (1)$$
$$a, b \in \mathbb{C}(x)$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \qquad r \in \mathbb{C}(x) \qquad (2)$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \qquad (3)$$

Where it can be found that r in (2) is given by

$$r = \frac{1}{4}a^2 + \frac{1}{2}a' - b \tag{4}$$

It is Eq. (2) (called DE from now on) which is solved and not Eq. (1). There are three steps for Saunders version. Case 1 and case 2 are handled in same way. In showing the steps, both Saunders paper (2) and Carolyn J. Smith paper (3) were used. Smith paper is more detailed and has some corrections to Saunders algorithm also.

There are 4 steps to the algorithm. Step 0 determines which case r belongs to (case 1 or 2 or 3 or non of these). Step 1 determines the fixed $e_{fixed}, \theta_{fixed}$ and also e_i, θ_i. Where i can be 0 and higher depending. (see below). Step 2 uses the e_{fixed}, θ_{fixed} and all the e_i, θ_i found in step 1 to determine the trial d, Θ. Here Θ is used instead of θ as in Smith paper so not confuse it with the θ_i found in step 1. If trial number d can be found which is integer and positive then step 3 is now called. It is in step 3 where the minimal polynomial $p(x)$ is found using the d and Θ. If such $p(x)$ can be found then ω is solved for and the solution for the ode $z'' = rz$ is now $z = e^{\int \omega dx}$. If no solution $p(x)$ can be found, then the next trials d, Θ tried in order to find $p(x)$. This continues until all trials are tried or if solution is found. Below shows more details on each step. The trials d, Θ are found by iterating over all possible set of values called s. These sets of values are generated depending on case number and m value, where m is the number of terms in the square free factorization of $t = t_1 t_2^2 t_3^3 \cdots t_m^m$. How this is all done is given below in examples.

4.0.1 Step 0

This step is similar to Kovacic algorithm. In it we determine necessary conditions for each case but it is done is more direct way in this version. Given $y'' = ry$, we write $d = \frac{s}{t}$ then now we do square free factorization on t which gives

$$t = t_1 t_2^2 t_3^3 \cdots t_m^m$$

For example, if $t = x^2$, then $t_1 = 1, t_2 = x$. And if $t = 3 - x^3$ then $t_1 = -1, t_2 = x^3 - 3$. And $O(\infty) = \deg(t) - \deg(s)$. Then now we determine which case we are in by finding necessary conditions, This is done slightly different from the original Kovacic. So at the end of this step we know if $L = [1]$ (case 1) or $L = [1, 2]$ (case 1 and case 2), or $L = [2]$ (case 2 only) or $L = [4, 6, 12]$ (case 3 only) and so on.

The necessary conditions are based on the square free factorization on $t = t_1 t_2^2 t_3^3 \cdots t_m^m$ and is summarized in Carolyn J. Smith paper (3) as (these are all the same necessary conditions as from original Kovacic paper) but expressed in terms of the square free factorization of $t = t_1 t_2^2 \cdots t_m^m$ where $r = \frac{s}{t}$.

1. $L = [1]$ (meaning case 1) if $t_i = 1$ for all odd $i \geq 3$ (i.e. no odd order poles allowed other than 1) and $O(\infty)$ is even only. (i.e. allowed $O(\infty)$ are $\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots$.
2. $L = [2]$ (meaning case 2) if $t_2 \neq 1$ or $t_i \neq 1$ for any odd $i \geq 3$. (i.e. only pole of order 2 is allowed, and then poles of order $3, 5, 7, \cdots$ are allowed.
3. $L = [4, 6, 12]$ (meaning case 3), if $t_i = 1$ for all $i > 2$ and $O(\infty) \geq 2$. (i.e. poles of order 1,2 is only allowed).

4.0.2 Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$e_{fixed} = \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \tag{5}$$

$$\theta_{fixed} = \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right) \tag{6}$$

For an example, lets say that $r = \frac{16x-3}{16x^2}$. Hence $t = 16x^2 = t_1 t_2^2$ where $t_1 = 16, t_2 = x$. And $O(\infty) = 2 - 1 = 1$. Therefore $\deg(t) = 2, \deg(t_1) = 0$ and $t' = \frac{d}{dx}16x^2 = 32x$ and $t_1' = 0$. The above becomes

$$\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(1,2) - 2 - 3(0)) \\
&= \frac{1}{4}(1-2) \\
&= -\frac{1}{4} \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{32x}{16x^2} + 3(0)\right) \\
&= \frac{1}{2x}
\end{aligned}$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2. These will be the zeros of t_2 in the above square free factorization of t. Label these poles $c_1, c_2, \cdots, c_{k_2}$. For each c_i then $e_i = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{(x-c_i)^2}$ in the partial fraction expansion of r and $\theta_i = \frac{e_i}{x-c_i}$. For an example, if $t = x^2$. Then $t = t_1 t_2^2$ and $t_1 = 1, t_2 = x$. Hence the zeros of t_2 are $c_1 = 0$. There is only one zero. Hence $k_2 = 1$ and we only have one iteration to do. Hence $b = 1, e_1 = \sqrt{1+4(1)} = \sqrt{5}$ and $\theta_e = \frac{\sqrt{5}}{x-0} = \frac{\sqrt{5}}{x}$.

Part (c)

This part applied only to case 1. It generates additional values of e_i, θ_i in addition to what was generated in part (b). This is done for poles of r of order $4, 6, 8, \cdots, M$ if any exist. These are the roots of $t_4, t_6, \cdots, t_M$. These poles are labeled $c_{k_2+1}, c_{k_2+2}, \cdots, c_k$. The labeling starts from k_2 since for the pole of order 2 in part b we used $c_1, c_2, \cdots, c_{k_2}$ for its zeros. Now we iterate for i from $k_2 + 1$ to k. For each pole we find its e_i and θ_i. These are found similar to original Kovacic paper. Examples below will illustrate better how this is done.

Note that if there are no poles of order $4, 6, \cdots$ then $k = k_2 = M$. The value M is used below to generate s sequences in step 2. What this means is that for case 1, if there are no poles of order $4, 6, \cdots$, then M is just the number of poles of r of order 2. For cases other than case 1, M is always number of poles of r of order 2. The checking on poles of order $4, 6, \cdots$ is only done for case 1.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. This is the same as was done in original Kovacic algorithm. If $O(\infty)$ is none of the above two cases, then case 1 is handled on its own and examples below show how this is done. Otherwise for all other cases $e_0 = 0, e_0 = 0$.

The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

4.0.3 Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}. Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \tag{7}$$

Where n is the case number. For case 1, it will be $n = 1$. For case 2 it will be $n = 2$. For case 3 it will be 4 and 6 and 12. If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \tag{8}$$

What is s in the above? These are sets of all combinations generated based on case number n and m values which will be described below. So the above trial d and Θ are generated for each such set s. Each set s has values $\{s_0, s_1, \cdots s_m\}$ in it. So s_i above means the i^{th} element is the current set s. There will be $(n+1)^{m+1}$ such different s sets. For example, case 1, means $n = 1$ and if $m = 2$, which means $t = t_1 t_2^2$ then there will be $2^3 = 8$ sets of s to try. For each such set, we generate d, Θ. If one set s gives a d which is an integer and positive, then Θ is generated and then step 3 is called to calculate ω. If step 3 is successful then we stop since a solution is found. Hence step 3 takes as input the trial d and Θ and is called repeatedly from step 2 until either solution $y = e^{\int \omega dx}$ is found or until all sets s are used. This is done for each case number n which can be $1, 2, 4, 6$ or 12. Starting from case 1 to case 3 (recall that case 3 has $n = 4, 6, 12$ in it). Of course if any one case manages to find a solution, then the algorithm stops.

Before going to step 3 description, We will show how the sets s are generated. This depends on value of n and M. Recall that M is number of poles of r of order 2 for case 1 if there are no higher order poles. For example, for $n = 1$ and say $M = 2$ then 8 different sets s are generated. Based on all different permutations of $\left\{\pm\frac{n}{2}, \pm\frac{n}{2}, \cdots, \pm\frac{n}{2}\right\}$. There are $M + 1$ entries, because entries are indexed from 0 to M. Hence for $M = 0$ (which will happen if where are no poles), then there are $(n+1)^1$ entries. For example for $n = 1$ this means two entries given by $\left\{\pm\frac{1}{2}\right\}$ which $s = \left\{\frac{1}{2}\right\}$ and $s = \left\{-\frac{1}{2}\right\}$ to try

$$\Lambda = \begin{matrix} -\frac{1}{2} \\ +\frac{1}{2} \end{matrix}$$

For $n = 1, M = 2$ then there are $(1+1)^3 = 8$ entries. We have all combinations of $\left\{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right\}$. This results in the following matrix

$$\Lambda = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

But if $n = 1$ and $M = 1$ then there are $(1+1)^2 = 4$ entries. All combinations of$\left\{\pm\frac{1}{2}, \pm\frac{1}{2}\right\}$ and the matrix is

$$\Lambda = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

Each row in the above matrix S is one set s to try. To be more clear, in the equation $\Theta = (n)\left(e_{fix}\right) + \sum_{i=0}^{M} s_i e_i$ the s_i in the equation means the i^{th} entry in that specific s set we are using at the moment, which happens to be one row of the matrix Λ. For example, if we are trying the second row in Λ, then $s_0 = -\frac{1}{2}, s_1 = -\frac{1}{2}, s_2 = +\frac{1}{2}$. For the case $n = 1$ and $M = 3$, then $\left\{\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right\}$. There is $(2)^4 = 16$ different sets s (or 16 rows in the matrix Λ). The matrix Λ is

$$\Lambda = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

If it was case 2 which means $n = 2$, then if $M = 2$, then we have all different permutations of $1, -1, 0$ in each entry. This gives $3^3 = 27$ different sets to try

$$\Lambda = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 0 \\ -1 & -1 & +1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & +1 \\ \vdots & \vdots & \vdots \\ +1 & +1 & +1 \end{bmatrix}$$

And if $n = 2, M = 3$ then this gives $3^4 = 81$ different sets s

$$\Lambda = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & +1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & +1 \\ \vdots & \vdots & \vdots & \vdots \\ +1 & +1 & +1 & +1 \end{bmatrix}$$

And if $n = 4, M = 2$ then this gives $5^3 = 125$ different sets s

$$\Lambda = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \\ \vdots & \vdots & \vdots \\ +2 & +2 & +2 \end{bmatrix}$$

And if $n = 4, M = 3$ then this gives $5^4 = 625$ different sets s

$$\Lambda = \begin{bmatrix} -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -1 \\ -2 & -2 & -2 & 0 \\ -2 & -2 & -2 & +1 \\ -2 & -2 & -2 & +2 \\ \vdots & \vdots & \vdots & \vdots \\ +2 & +2 & +2 & +2 \end{bmatrix}$$

And if $n = 6, M = 2$ then this gives $7^3 = 343$ different sets s

$$\Lambda = \begin{bmatrix} -3 & -3 & -3 \\ -2 & -2 & -2 \\ -2 & -2 & -1 \\ -2 & -2 & -0 \\ -2 & -2 & +1 \\ -2 & -2 & +2 \\ -2 & -2 & +3 \\ \vdots & \vdots & \vdots \\ +3 & +3 & +3 \end{bmatrix}$$

And if $n = 6, M = 3$ then this gives $7^4 = 2401$ different sets s

$$\Lambda = \begin{bmatrix} -3 & -3 & -3 & -3 \\ -3 & -3 & -3 & -2 \\ -3 & -3 & -3 & -1 \\ -3 & -3 & -3 & 0 \\ -3 & -3 & -3 & +1 \\ -3 & -3 & -3 & +2 \\ -3 & -3 & -3 & +3 \\ \vdots & \vdots & \vdots & \vdots \\ +3 & +3 & +3 & +3 \end{bmatrix}$$

And if $n = 12, M = 2$ then this gives $13^3 = 2197$ different sets s

$$\Lambda = \begin{bmatrix} -12 & -12 & -12 \\ -12 & -12 & -11 \\ -12 & -12 & -10 \\ \vdots & \vdots & \vdots \\ -12 & -12 & +12 \\ \vdots & \vdots & \vdots \\ +12 & +12 & +12 \end{bmatrix}$$

And if $n = 12, M = 3$ then this gives $13^4 = 28561$ different sets s

$$\Lambda = \begin{bmatrix} -12 & -12 & -12 & -12 \\ -12 & -12 & -12 & -11 \\ -12 & -12 & -12 & -10 \\ \vdots & \vdots & \vdots & \vdots \\ -12 & -12 & -12 & +12 \\ \vdots & \vdots & \vdots & \vdots \\ +12 & +12 & +12 & +12 \end{bmatrix}$$

And so on.

4.0.4 step 3

The input to this step is the integer d and Θ found from Eq (7,8) as described in step 2 and also r which comes from $z'' = rz$.

This step is broken into these parts. First we find the $p_{-1}(x)$ polynomial. If we are to solve for its coefficients, then next we build the minimal polynomial from the $p_i(x)$ polynomials constructed during finding $p_{-1}(x)$. The minimal polynomial $p_{\min}(x)$ will be a function of ω. Next we solve for ω from $p_{\min}(x) = 0$. If this is successful, then we have found ω and the first solution to the ode $z'' = rz$ is $e^{\int \omega dx}$. Below shows how this is done.

We start by forming a polynomial

$$p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$$

Notice that x^d has coefficient 1. The goal is to solve for the a_i coefficient. Now depending on case number n, we do the following. If case $n = 1$ then

$$\begin{aligned} p_1 &= -p \\ p_0 &= -p_1' - \Theta p_1 \\ p_{-1} &= -p_0' - \Theta p_0 - (1)(1)\, r p_1 \end{aligned} \tag{9}$$

And it is $p_{-1} = 0$ which is solved for the coefficients a_i. For the case $n = 2$ we find p as follows

$$\begin{aligned} p_2 &= -p \\ p_1 &= -p_2' - \Theta p_2 \\ p_0 &= -p_1' - \Theta p_1 - (1)(2)\, r p_2 \\ p_{-1} &= -p_0' - \Theta p_0 - (2)(1)\, r p_1 \end{aligned} \tag{10}$$

And it is $p_{-1} = 0$ which is solved for the coefficients a_i. For the case $n = 4$ we find p as follows

$$\begin{aligned} p_4 &= -p \\ p_3 &= -p_4' - \Theta p_4 \\ p_2 &= -p_3' - \Theta p_3 - (1)(4)\, r p_4 \\ p_1 &= -p_2' - \Theta p_2 - (2)(3)\, r p_3 \\ p_0 &= -p_1' - \Theta p_1 - (3)(2)\, r p_2 \\ p_{-1} &= -p_0' - \Theta p_0 - (4)(1)\, r p_1 \end{aligned} \tag{11}$$

And it is $p_{-1} = 0$ which is solved for the coefficients a_i. For the case $n = 6$ we find p as follows

$$\begin{aligned} p_6 &= -p \\ p_5 &= -p_6' - \Theta p_6 \\ p_4 &= -p_5' - \Theta p_5 - (1)(6)\, r p_6 \\ p_3 &= -p_4' - \Theta p_4 - (2)(5)\, r p_5 \\ p_2 &= -p_3' - \Theta p_3 - (3)(4)\, r p_4 \\ p_1 &= -p_2' - \Theta p_2 - (4)(3)\, r p_3 \\ p_0 &= -p_1' - \Theta p_1 - (5)(2)\, r p_2 \\ p_{-1} &= -p_0' - \Theta p_0 - (6)(1)\, r p_1 \end{aligned} \tag{12}$$

And it is $p_{-1} = 0$ which is solved for the coefficients a_i. For the case $n = 12$ we find p as follows

$$
\begin{aligned}
p_{12} &= -p \\
p_{11} &= -p_{12}' - \Theta p_{12} \\
p_{10} &= -p_{11}' - \Theta p_{11} - (1)(12)\, r p_{12} \\
p_{9} &= -p_{10}' - \Theta p_{10} - (2)(11)\, r p_{11} \\
p_{8} &= -p_{9}' - \Theta p_{9} - (3)(10)\, r p_{10} \\
p_{7} &= -p_{8}' - \Theta p_{8} - (4)(9)\, r p_{9} \\
p_{6} &= -p_{7}' - \Theta p_{7} - (5)(8)\, r p_{8} \\
p_{5} &= -p_{6}' - \Theta p_{6} - (6)(7)\, r p_{7} \\
p_{4} &= -p_{5}' - \Theta p_{5} - (7)(6)\, r p_{6} \\
p_{3} &= -p_{4}' - \Theta p_{4} - (8)(5)\, r p_{5} \\
p_{2} &= -p_{3}' - \Theta p_{3} - (9)(4)\, r p_{4} \\
p_{1} &= -p_{2}' - \Theta p_{2} - (10)(3)\, r p_{3} \\
p_{0} &= -p_{1}' - \Theta p_{1} - (11)(2)\, r p_{2} \\
p_{-1} &= -p_{0}' - \Theta p_{0} - (12)(1)\, r p_{1}
\end{aligned} \tag{13}
$$

If we are able to solve for all the a_i by solving

$$p_{-1}(x) = 0$$

Then we now have determined $p(x)$. This is used to find ω as follows. For the <u>case $n = 1$</u>

$$\omega = \frac{p'}{p} + \Theta$$

For the <u>case $n = 2$</u> ω is found by solving

$$\omega^2 - \phi\omega + \frac{\phi'}{2} + \frac{\phi^2}{2} - r = 0$$

Where $\phi = \frac{p'}{p} + \Theta$.

Where p_0, p_1, p_2 are from Eq (10) above. For the <u>case $n = 4$</u> then

$$p_{\min} = \frac{1}{4!}p_0 + \frac{1}{3!}p_1\omega + \frac{1}{2!}p_2\omega^2 + p_3\omega^3 + p_4\omega^4$$

Where p_0, p_1, p_2, p_3, p_4 are from Eq (11) above. For the <u>case $n = 6$</u> then

$$p_{\min} = \frac{1}{6!}p_0 + \frac{1}{5!}p_1\omega + \frac{1}{4!}p_2\omega^2 + \frac{1}{3!}p_3\omega^3 + \frac{1}{2!}p_4\omega^4 + p_5\omega^5 + p_6\omega^6$$

Where $p_0, p_1, p_2, p_3, p_4, p_5, p_6$ are from Eq (12) above. For the <u>case $n = 12$</u> then

$$p_{\min} = \frac{1}{12!}p_0 + \frac{1}{11!}p_1\omega + \frac{1}{10!}p_2\omega^2 + \frac{1}{9!}p_3\omega^3 + \frac{1}{8!}p_4\omega^4 + \frac{1}{7!}p_5\omega^5 + \frac{1}{6!}p_6\omega^6 + \frac{1}{5!}p_7\omega^7 + \frac{1}{4!}p_8\omega^8 + \frac{1}{3!}p_9\omega^9 + \frac{1}{2!}p_{10}\omega^{10} + p_{11}\omega^{11} +$$

Where p_i for $i = 0 \cdots 12$ are from Eq (13) above. In each case, we now solve for

$$p_{\min}(x) = 0$$

For ω. If this is successful, then now we have to verify the solution satisfies the Riccati ODE. ω must satisfy in all case the following equation

$$\omega' + \omega^2 = r$$

If it does, then we have solved the $z'' = rz$ ode $z = e^{\int \omega dx}$ and also the original $y'' + ay' + by = 0$ ode. This completes the Kovacic algorithm. Examples are given below showing how to implement the above to solve number of ode's.

4.1 Worked examples

4.1.1 Example 1 case one

Solve

$$\left(1-x^2\right)y''-2xy'+6y=0$$

Normalizing so that coefficient of u'' is one gives (assuming $x\neq 1$)

$$y''-2\frac{x}{(1-x^2)}y'+\frac{6}{(1-x^2)}y=0$$
$$y''+ay'(x)+by=0 \tag{1}$$

Hence

$$a=\frac{-2x}{(1-x^2)}$$
$$b=\frac{6}{(1-x^2)}$$

It is first transformed to the following ode by eliminating the first derivative

$$z''=rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y=ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2+\frac{1}{2}a'-b\\ &= \frac{1}{4}\left(\frac{-2x}{(1-x^2)}\right)^2+\frac{1}{2}\frac{d}{dx}\left(\frac{-2x}{(1-x^2)}\right)-\frac{6}{(1-x^2)}\\ &= \frac{6x^2-7}{\left(x^2-1\right)^2} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z''=\frac{6x^2-7}{\left(x^2-1\right)^2}z \tag{5}$$

Step 0 We need to find which case it is. $r=\frac{s}{t}$. The free square factorization of t is $t=t_1t_2^2$. Hence

$$m=2 \tag{6}$$

And $t_1=1, t_2=(x^2-1)$. Now $O(\infty)=\deg(t)-\deg(s)=4-2=2$. The poles of r are $x=1,-1$ each of order 2. Looking at the cases table giving up, reproduced here

case	allowed pole order for $r=\frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-8,-6,-4,-2,0,2,3,4,5,6,7,\cdots\}$	[1]
2	$\{2,3,5,7,9,\cdots\}$	no condition	[2]
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$	[4, 6, 12]

Shows that all cases are possible. Hence $L=\{1,2,4,6,12\}$. So $n=1, n=2, n=4, n=6, n=12$ will be tried until one is successful.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts e_{fixed},θ_{fixed} are calculated using

$$e_{fixed}=\frac{1}{4}\left(\min\left(O(\infty),2\right)-\deg(t)-3\deg(t_1)\right)$$
$$\theta_{fixed}=\frac{1}{4}\left(\frac{t'}{t}+3\frac{t_1'}{t_1}\right)$$

Using $O(\infty)=2, t=\left(x^2-1\right)^2, t_1=1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(2,2)-4-3(0)) \\
&= \frac{1}{4}(2-4) \\
&= -\frac{1}{2} \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}\left(\left(x^2-1\right)^2\right)}{\left(x^2-1\right)^2}+3(0)\right) \\
&= \frac{x}{x^2-1}
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i=1\cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2. These will be the zeros of t_2 in the above square free factorization of t. From above we found that $t_2=x^2-1$. Label these poles $c_1, c_2, \cdots, c_{k_2}$. The zeros of t_2 are $\{1,-1\}$ therefore $c_1=1, c_2=-1$ and $k_2=2$. For each c_i then $e_i=\sqrt{1+4b}$ where b is the coefficient of $\frac{1}{(x-c_i)^2}$ in the partial fraction expansion of r which is

$$
\begin{aligned}
\frac{6x^2-7}{\left(x^2-1\right)^2} &= \left(\sum_i \frac{\alpha_i}{(x-c_i)^2}\right)+\left(\sum_j \frac{\beta_j}{x-d_j}\right) \\
&= \left(-\frac{1}{4}\frac{1}{(x-1)^2}-\frac{1}{4}\frac{1}{(x+1)^2}\right)+\left(\frac{13}{4}\frac{1}{(x-1)}-\frac{13}{4}\frac{1}{x+1}\right)
\end{aligned}
$$

Therefore for $c_1=1$, looking at the above, we see that the coefficient of $\frac{1}{(x-1)^2}$ is $\frac{-1}{4}$. Hence $b=\frac{-1}{4}$ and $e_1=\sqrt{1+4b}=\sqrt{1-4\frac{1}{4}}=0$. For $c_2=-1$ looking at the above, we see that the coefficient of $\frac{1}{(x+1)^2}$ is $\frac{-1}{4}$. Hence $b=\frac{-1}{4}$ and $e_2=\sqrt{1+4b}=\sqrt{1-4\frac{1}{4}}=0$. Therefore

$$
\begin{aligned}
e_1 &= 0 \\
e_2 &= 0
\end{aligned}
$$

Now, $\theta_i=\frac{e_i}{x-c_i}$. For $c_1=1$ this gives $\theta_1=\frac{e_1}{x-1}=0$ since $e_1=0$. And $\theta_2=\frac{e_2}{x-c_2}$. For $c_2=-1$ this gives $\theta_2=\frac{e_2}{x+1}=0$ since $e_2=0$. Hence

$$
\begin{aligned}
\theta_1 &= 0 \\
\theta_2 &= 0
\end{aligned}
$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4,6,8,\cdots,M$ if any exist. Since non exist here. This is skipped. This means

$$
M=2
$$

since 2 is the number of poles of order 2.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty)>2$ then $e_0=1, \theta_0=0$. But if $O(\infty)=2$ then $\theta_0=0$ and $e_0=\sqrt{1+4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since in this example $O(\infty)=2$ then

$$
\begin{aligned}
e_0 &= \sqrt{1+4b} \\
\theta_0 &= 0
\end{aligned}
$$

Now we need to find b. The Laurent series expansion of r at ∞ is $\frac{6}{x^2}+\frac{5}{x^4}+\frac{4}{x^6}+\cdots$. Hence $b=6$. Therefore

$$\begin{aligned} e_0 &= \sqrt{1+4(6)} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Now we have found all e_i, θ_i. They are

$$\begin{aligned} e &= \{5,0,0\} \\ \theta &= \{0,0,0\} \end{aligned}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}. Now we generate trial d using

$$d=(n)(e_{fix})+s_0e_0-\sum_{i=1}^{M}s_ie_i \tag{7}$$

Where n is the case number. For case 1, it will be $n=1$. For case 2 it will be $n=2$. For case 3 it will be 4 and 6 and 12. If $d\geq 0$ then we go and find a trial Θ. We need to have both d,Θ to go to the next step. Θ is found using

$$\Theta=(n)(\theta_{fix})+\sum_{i=0}^{M}s_i\theta_i \tag{8}$$

We need to first generate s sets. For $n=1$ and since $M=2$ in this example (number of poles of order 2), then these these are given by

$$\Lambda=\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

We go over each row one at a time. Trying the first row $s=\left\{\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right\}$ which means $s_0=\frac{-1}{2}, s_1=\frac{-1}{2}, s_2=-\frac{1}{2}$. Hence the first trial d is (using Eq (7)) and recalling that $e_{fix}=-\frac{1}{2}, \theta_{fixed}=\frac{x}{x^2-1}$ then

$$\begin{aligned} d &= (n)(e_{fix})+s_0e_0-\sum_{i=1}^{M}s_ie_i \\ &= (1)\left(-\frac{1}{2}\right)+\left(\frac{-1}{2}\right)(5)-(s_1e_1+s_2e_2) \\ &= (1)\left(-\frac{1}{2}\right)+\left(\frac{-1}{2}\right)(5)-\left(\frac{-1}{2}(0)-\frac{1}{2}(0)\right) \\ &= -3 \end{aligned}$$

Since this is negative, then we skip this set s. Now we try the second row of Λ which is $s = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{+1}{2}\right\}$. Then above now gives

$$
\begin{aligned}
d &= (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(5) - (s_1 e_1 + s_2 e_2) \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(5) - \left(\frac{-1}{2}(0) + \frac{1}{2}(0)\right) \\
&= -3
\end{aligned}
$$

Since this is negative, then we skip this set s. Now we try the third row of Λ which is $s = \left\{\frac{-1}{2}, \frac{+1}{2}, \frac{-1}{2}\right\}$. Then above now gives

$$
\begin{aligned}
d &= (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(5) - (s_1 e_1 + s_2 e_2) \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(5) - \left(\frac{+1}{2}(0) - \frac{1}{2}(0)\right) \\
&= -3
\end{aligned}
$$

Since this is negative, then we skip this set s. Now we try the row 4 of Λ which is $s = \left\{\frac{-1}{2}, \frac{+1}{2}, \frac{+1}{2}\right\}$. Then above now gives

$$
\begin{aligned}
d &= (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(5) - (s_1 e_1 + s_2 e_2) \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(5) - \left(\frac{+1}{2}(0) + \frac{1}{2}(0)\right) \\
&= -3
\end{aligned}
$$

Since this is negative, then we skip this set s. Now we try the row 5 of Λ which is $s = \left\{\frac{+1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}$. Then above now gives

$$
\begin{aligned}
d &= (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{+1}{2}\right)(5) - (s_1 e_1 + s_2 e_2) \\
&= (1)\left(-\frac{1}{2}\right) + \left(\frac{+1}{2}\right)(5) - \left(\frac{+1}{2}(0) + \frac{1}{2}(0)\right) \\
&= +2
\end{aligned}
$$

Since $d \geq 0$, then we can use it. Now using Eq (8) gives

$$
\begin{aligned}
\Theta &= (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \\
&= (1)\left(\frac{x}{x^2-1}\right) + (s_0\theta_0 + s_1\theta_1 + s_2\theta_2) \\
&= \frac{x}{x^2-1} + \left(\frac{+1}{2}\theta_0 - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right)
\end{aligned}
$$

But all $\theta_i = 0$. Therefore

$$
\Theta = \frac{x}{x^2-1}
$$

Now that we have good trial d and Θ, then step 3 is called to generate ω if possible.

Step 3

The input to this step is the integer $d = 2$ and $\Theta = \frac{x}{x^2-1}$ found from step 2 and also $r = \frac{6x^2-7}{(x^2-1)^2}$ which comes from $z'' = rz$. This step is broken into these parts. First we find the $p_{-1}(x)$ polynomial. If we are to solve for its coefficients, then next we build the minimal polynomial from the $p_i(x)$ polynomials constructed during finding $p_{-1}(x)$. The minimal polynomial $p_{\min}(x)$ will be a function of ω. Next we solve for ω from $p_{\min}(x) = 0$. If this is successful, then we have found ω and the first solution to the ode $z'' = rz$ is $z = e^{\int \omega dx}$. Below shows how this is done.

We start by forming a polynomial

$$\begin{aligned} p(x) &= x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\ &= x^2 + a_1 x + a_0 \end{aligned}$$

The goal is to solve for the a_i coefficient. Now depending on case number n, we do the following. Since we are in case $n = 1$ then

$$\begin{aligned} p_1 &= -p \\ &= -x^2 - a_1 x - a_0 \\ p_0 &= -p_1' - \Theta p_1 \\ &= -(-x^2 - a_1 x - a_0)' - \left(\frac{x}{x^2-1}\right)(-x^2 - a_1 x - a_0) \\ &= \frac{(x a_0 - a_1 - 2x + 2x^2 a_1 + 3x^3)}{x^2-1} \\ p_{-1} &= -p_0' - \Theta p_0 - (1)(1) r p_1 \\ &= -\left(\frac{(x a_0 - a_1 - 2x + 2x^2 a_1 + 3x^3)}{x^2-1}\right)' - \left(\frac{x}{x^2-1}\right)\left(\frac{(x a_0 - a_1 - 2x + 2x^2 a_1 + 3x^3)}{x^2-1}\right) - \frac{6x^2-7}{(x^2-1)^2}(- \\ &= 2\frac{(2x a_1 + 3a_0 + 1)}{x^2-1} \end{aligned}$$

Now we try to solve for a_i using $p_{-1}(x) = 0$. This gives $2x a_1 + 3a_0 + 1 = 0$ which gives $a_1 = 0, a_0 = -\frac{1}{3}$. Hence this implies

$$\begin{aligned} p(x) &= x^2 + a_1 x + a_0 \\ &= x^2 - \frac{1}{3} \end{aligned}$$

Since this is case $n = 1$ then

$$\begin{aligned} \omega &= \frac{p'}{p} + \Theta \\ &= \frac{2x}{x^2 - \frac{1}{3}} + \frac{x}{x^2-1} \\ &= x\frac{9x^2-7}{3x^4 - 4x^2 + 1} \end{aligned}$$

Before using this, we will verify it is correct. For case 1 the above should satisfy

$$\omega' + \omega^2 = r$$

Let us see if this is the case or not.

$$\begin{aligned} \frac{d}{dx}\left(x\frac{9x^2-7}{3x^4-4x^2+1}\right) + \left(x\frac{9x^2-7}{3x^4-4x^2+1}\right)^2 &= \frac{6x^2-7}{(x^2-1)^2} \\ \frac{6x^2-7}{(x^2-1)^2} &= \frac{6x^2-7}{(x^2-1)^2} \end{aligned}$$

Verified. Since solution ω is found and verified, then first solution to the ode is

$$
\begin{aligned}
z &= e^{\int \omega dx} \\
&= e^{\int \frac{x(9x^2-7)}{3x^4-4x^2+1}dx} \\
&= e^{\frac{1}{2}\ln(x^2-1)+\ln(x^2-\frac{1}{3})} \\
&= \sqrt{x^2-1}\left(x^2-\frac{1}{3}\right)
\end{aligned}
$$

Hence first solution to the original ode is

$$
\begin{aligned}
y &= z e^{\frac{-1}{2}\int a dx} \\
&= \sqrt{x^2-1}\left(x^2-\frac{1}{3}\right) e^{\frac{-1}{2}\int \frac{-2x}{(1-x^2)}dx} \\
&= \sqrt{x^2-1}\left(x^2-\frac{1}{3}\right) e^{\int \frac{x}{(1-x^2)}dx} \\
&= \sqrt{x^2-1}\left(x^2-\frac{1}{3}\right) e^{\frac{-1}{2}\ln(x^2-1)} \\
&= \frac{\sqrt{x^2-1}(x^2-\frac{1}{3})}{\sqrt{x^2-1}} \\
&= \left(x^2-\frac{1}{3}\right)
\end{aligned}
$$

4.1.2 Example 2 case one

Solve

$$
x(x-1)^2 y'' - 2y = 0
$$

Normalizing so that coefficient of u'' is one gives (assuming $x \neq 1$ and $x \neq 0$)

$$
\begin{aligned}
y'' - \frac{2}{x(x-1)^2}y &= 0 \\
y'' + ay'(x) + by &= 0 \qquad (1)
\end{aligned}
$$

Hence

$$
\begin{aligned}
a &= 0 \\
b &= -\frac{2}{x(x-1)^2}
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$
z'' = rz \qquad (2)
$$

Using what is known as the Liouville transformation given by

$$
y = z e^{\frac{-1}{2}\int a dx} \qquad (3)
$$

Where it can be found that r in (2) is given by

$$
\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{2}{x(x-1)^2} \qquad (4)
\end{aligned}
$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$
z'' = \frac{2}{x(x-1)^2}z \qquad (5)
$$

Step 0 We need to find which case it is. $r=\frac{s}{t}$. The free square factorization of t is $t=t_1t_2^2$. Hence

$$m=2 \tag{6}$$

And

$$\begin{aligned} t_1 &= x \\ t_2 &= x-1 \end{aligned}$$

Now $O(\infty)=\deg(t)-\deg(s)=3-0=3$. The poles of r are $x=0$ of order 1 and $x=1$ of order 2. Looking at the cases table giving up, reproduced here

case	allowed pole order for $r=\frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-8,-6,-4,-2,0,2,3,4,5,6,7,\cdots\}$	[1]
2	$\{2,3,5,7,9,\cdots\}$	no condition	[2]
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$	[4, 6, 12]

Shows that all cases are possible. Hence $L=\{1,2,4,6,12\}$. So $n=1,n=2,n=4,n=6,n=12$ will be tried until one is successful. Starting with $n=1$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts e_{fixed},θ_{fixed} are calculated using

$$\begin{aligned} e_{fixed} &= \frac{1}{4}\left(\min\left(O(\infty),2\right)-\deg(t)-3\deg(t_1)\right) \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t}+3\frac{t_1'}{t_1}\right) \end{aligned}$$

Using $O(\infty)=3,t=x(x-1)^2,t_1=x$ the above gives

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(3,2)-3-3(1)) \\ &= \frac{1}{4}(2-3-3) \\ &= -1 \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}\left(x(x-1)^2\right)}{x\left(x-1\right)^2}+3\left(\frac{x'}{x}\right)\right) \\ &= \frac{1}{4}\left(\frac{3x^2-4x+1}{x\left(x-1\right)^2}+\frac{3}{x}\right) \\ &= \frac{1}{2x}\frac{3x-2}{x-1} \end{aligned}$$

part (b)

Here the values e_i,θ_i are found for $i=1\cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2. These will be the zeros of t_2 in the above square free factorization of t. From above we found that $t_2=x-1$. Label these poles $c_1,c_2,\cdots,c_{k_2}$. The zeros of t_2 are $\{1\}$ therefore $c_1=1$ and $k_2=1$ since one zero. Hence

$$M=1$$

For each c_i then $e_i=\sqrt{1+4b}$ where b is the coefficient of $\frac{1}{(x-c_i)^2}$ in the partial fraction expansion of r which is

$$\begin{aligned} r &= \frac{2}{x\left(x-1\right)^2} \\ &= \left(\sum_i\frac{\alpha_i}{\left(x-c_i\right)^2}\right)+\left(\sum_j\frac{\beta_j}{x-d_j}\right) \\ &= \left(\frac{2}{\left(x-1\right)^2}\right)+\left(-\frac{2}{x-1}+\frac{2}{x}\right) \end{aligned}$$

Therefore for $c_1 = 1$, looking at the above, we see that the coefficient of $\frac{1}{(x-1)^2}$ is 2. Hence $b = 2$ and $e_1 = \sqrt{1+4b} = \sqrt{1+8} = 3$. Hence

$$e_1 = 3$$

Now, $\theta_i = \frac{e_i}{x-c_i}$. For $c_1 = 1$ this gives $\theta_1 = \frac{e_1}{x-1} = \frac{3}{x-1}$. Hence

$$\theta_1 = \frac{3}{x-1}$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, M$ if any exist. Since non exist here. This is skipped. Hence M stays 1.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since in this example $O(\infty) = 3$ then

$$\begin{aligned} e_0 &= 1 \\ \theta_0 &= 0 \end{aligned}$$

Now we have found all e_i, θ_i. They are

$$\begin{aligned} e &= \{1, 3\} \\ \theta &= \left\{0, \frac{3}{x-1}\right\} \end{aligned}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}. Now we generate trial d using

$$d = (n)(e_{fix}) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \tag{7}$$

Where n is the case number. For case 1, it will be $n = 1$. For case 2 it will be $n = 2$. For case 3 it will be 4 and 6 and 12. If $d \geq 0$, then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)(\theta_{fix}) + \sum_{i=0}^{M} s_i \theta_i \tag{8}$$

We need to first generate s sets. For $n = 1$ and since $M = 1$ in this example, then these these are given by

$$\Lambda = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} \\ +\frac{1}{2} & +\frac{1}{2} \end{bmatrix}$$

We go over each row one at a time. Trying the first row $s = \left\{\frac{-1}{2}, \frac{-1}{2}\right\}$ which means $s_0 = \frac{-1}{2}, s_1 = \frac{-1}{2}$. Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -1, \theta_{fixed} =$

$\frac{1}{2x}\frac{3x-2}{x-1}$ then

$$\begin{aligned}
d &= (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i \\
&= (1)(-1) + \left(\frac{-1}{2}\right)(1) - (s_1 e_1) \\
&= -1 - \frac{1}{2} - \left(\frac{-1}{2}(3)\right) \\
&= 0
\end{aligned}$$

Since $d \geq 0$, then we can use it. Now using Eq (8) gives

$$\begin{aligned}
\Theta &= (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \\
&= (1)\left(\frac{1}{2x}\frac{3x-2}{x-1}\right) + (s_0\theta_0 + s_1\theta_1) \\
&= \frac{1}{2x}\frac{3x-2}{x-1} + \left(\frac{-1}{2}\theta_0 - \frac{1}{2}\theta_1\right) \\
&= \frac{1}{2x}\frac{3x-2}{x-1} - \frac{1}{2}(0) - \frac{1}{2}\frac{3}{x-1}
\end{aligned}$$

Therefore

$$\Theta = -\frac{1}{x(x-1)}$$

Now that we have good trial d and Θ, then step 3 is called to generate ω if possible.

Step 3

The input to this step is the integer $d = 0$ and $\Theta = -\frac{1}{x(x-1)}$ found from step 2 and also $r = \frac{2}{x(x-1)^2}$ which comes from $z'' = rz$. This step is broken into these parts. First we find the $p_{-1}(x)$ polynomial. If we are to solve for its coefficients, then next we build the minimal polynomial from the $p_i(x)$ polynomials constructed during finding $p_{-1}(x)$. The minimal polynomial $p_{\min}(x)$ will be a function of ω. Next we solve for ω from $p_{\min}(x) = 0$. If this is successful, then we have found ω and the first solution to the ode $z'' = rz$ is $z = e^{\int \omega dx}$. Below shows how this is done.

We start by forming a polynomial

$$\begin{aligned}
p(x) &= x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\
&= 1
\end{aligned}$$

Since this is case $n = 1$ then

$$\begin{aligned}
\omega &= \frac{p'}{p} + \Theta \\
&= 0 - \frac{1}{x(x-1)} \\
&= \frac{-1}{x(x-1)}
\end{aligned}$$

Before using this, we will verify it is correct. For case 1 the above should satisfy

$$\omega' + \omega^2 = r$$

Let us see if this is the case or not.

$$\begin{aligned}
\frac{d}{dx}\left(-\frac{1}{x(x-1)}\right) + \left(-\frac{1}{x(x-1)}\right)^2 &= \frac{2}{x(x-1)^2} \\
\frac{1}{x^2}\frac{2x-1}{(x-1)^2} + \left(-\frac{1}{x(x-1)}\right)^2 &= \frac{2}{x(x-1)^2} \\
\frac{2}{x(x-1)^2} &= \frac{2}{x(x-1)^2}
\end{aligned}$$

Verified. Since solution ω is found and verified, then first solution to the ode is

$$
\begin{aligned}
z &= e^{\int -\frac{1}{x(x-1)} dx} \\
&= e^{\ln x - \ln(x-1)} \\
&= \frac{x}{x-1}
\end{aligned}
$$

Hence first solution to given ODE is

$$
\begin{aligned}
y &= z e^{\frac{-1}{2} \int a dx} \\
&= \frac{x}{x-1} e^{-\frac{1}{2} \int 0 dx} \\
&= \frac{x}{x-1}
\end{aligned}
$$

4.1.3 Example 3 case one

Solve

$$
\begin{aligned}
y'' - x^2 y' - x^2 y &= 0 \\
y'' + a y'(x) + b y &= 0
\end{aligned}
\tag{1}
$$

Hence

$$
\begin{aligned}
a &= -x^2 \\
b &= -x^2
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$
z'' = r z \tag{2}
$$

Using what is known as the Liouville transformation given by

$$
y = z e^{\frac{-1}{2} \int a dx} \tag{3}
$$

Where it can be found that r in (2) is given by

$$
\begin{aligned}
r &= \frac{1}{4} a^2 + \frac{1}{2} a' - b \\
&= \frac{1}{4} x^4 - x + x^2 \\
&= \frac{x^4 + 4x^2 - 4x}{4}
\end{aligned}
\tag{4}
$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$
z'' = \left(\frac{x^4 + 4x^2 - 4x}{4} \right) z \tag{5}
$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$
\begin{aligned}
s &= x^4 + 4x^2 - 4x \\
t &= 4
\end{aligned}
$$

The free square factorization of t is $t = [[]]$. Hence

$$
m = 0 \tag{6}
$$

Since m is number of elements in the free square factorization. in this special case we set

$$
\begin{aligned}
t_1 &= 1 \\
t_2 &= 1
\end{aligned}
$$

Now

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 4 \\
&= -4
\end{aligned}
$$

There are no poles. Looking at the cases table giving up, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that only case 1 meets the necessary conditions. Hence $L = [1]$. So $n = 1$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min\left(O(\infty), 2\right) - \deg\left(t\right) - 3\deg\left(t_1\right)) \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty) = -4, t = 4, t_1 = 4$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min\left(-4, 2\right) - 0 - 3(0)) \\
&= \frac{1}{4}(-4) \\
&= -1 \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(4)}{(4)} + 3\left(\frac{(4)'}{4}\right)\right) \\
&= 0
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2. These will be the zeros of t_2 in the above square free factorization of t. From above we found that $t_2 = 1$. Label these poles $c_1, c_2, \cdots, c_{k_2}$. The zeros of t_2 are $\{\}$. There are no zeros since constant. Therefore $k_2 = 0$ since one zero. Hence

$$
M = 0
$$

No e_i, θ_i are generated. i.e. $e = \{\}, \theta = \{\}$ so far.

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, M$ if any exist. Since non exist here. This is skipped. Hence M stays 0.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1 + 4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since in this example $O(\infty) = -4$ then none of these cases apply. We fall into the case that handles $n = 1$ only which is the current case which results in $e_0 = -2, \theta_0 = 2 + x^2$. Hence

$$
\begin{aligned}
e &= \{-2\} \\
\theta &= \left\{2 + x^2\right\}
\end{aligned}
$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}. Since $n = 1$ and $M = 0$ then we have $(n+1)^{M+1} = 2^1 = 2$ sets s to try. These are given by

$$\Lambda = \begin{bmatrix} -\frac{1}{2} \\ +\frac{1}{2} \end{bmatrix}$$

Now we generate trial d using

$$d = (n)(e_{fix}) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i$$

Since $M = 0$ then the above becomes

$$d = (n)(e_{fix}) + s_0 e_0 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)(\theta_{fix}) + \sum_{i=0}^{M} s_i \theta_i \tag{8}$$

Since $M = 0$. Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -1, \theta_{fixed} = 0$ then

$$\begin{aligned} d &= (n)(e_{fix}) + \left(\frac{-1}{2}\right)(-2) \\ &= (1)(-1) + 1 \\ &= 0 \end{aligned}$$

Since $d \geq 0$, then we can use it. Using Eq (8) gives

$$\begin{aligned} \Theta &= (n)(\theta_{fix}) + s_0 \theta_0 \\ &= 0 - \frac{1}{2}(2 + x^2) \\ &= -1 - \frac{1}{2}x^2 \end{aligned}$$

Now that we have good trial d and Θ, then step 3 is called to generate ω if possible.

Step 3

The input to this step is the integer $d = 0$ and $\Theta = -1 - \frac{1}{2}x^2$ found from step 2 and also $r = \frac{x^4+4x^2-4x}{4}$ which comes from $z'' = rz$. This step is broken into these parts. First we find the $p_{-1}(x)$ polynomial. If we are to solve for its coefficients, then next we build the minimal polynomial from the $p_i(x)$ polynomials constructed during finding $p_{-1}(x)$. The minimal polynomial $p_{\min}(x)$ will be a function of ω. Next we solve for ω from $p_{\min}(x) = 0$. If this is successful, then we have found ω and the first solution to the ode $y'' = ry$ is $e^{\int \omega dx}$. Below shows how this is done.

We start by forming a polynomial

$$\begin{aligned} p(x) &= x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\ &= 1 \end{aligned}$$

Since this is case $n = 1$ then

$$\begin{aligned} \omega &= \frac{p'}{p} + \Theta \\ &= -1 - \frac{1}{2}x^2 \end{aligned}$$

Before using this, we will verify it is correct. For case 1 the above should satisfy

$$\omega' + \omega^2 = r$$

Let us see if this is the case or not.

$$\begin{aligned}
\frac{d}{dx}\left(-1-\frac{1}{2}x^2\right)+\left(-1-\frac{1}{2}x^2\right)^2 &= \left(\frac{x^4+4x^2-4x}{4}\right)\\
-x+\frac{1}{4}x^4+x^2+1 &= \frac{x^4+4x^2-4x}{4}\\
\frac{x^4+4x^2-4x}{4}+\frac{1}{4} &= \frac{x^4+4x^2-4x}{4}
\end{aligned}$$

It did not verify. This means this solution can not be used. If we try the next row in Λ we will find it gives negative d. This means there is no Liouvillian solution. This is an example where even if we find $d \geq 0$ we still can end up not finding a solution.

4.1.4 Example 4 case one

Solve

$$\begin{aligned}
\left(x^3+1\right)y''+7x^2y'+9xy &= 0 \qquad (1)\\
y''+ay'(x)+by &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
a &= \frac{7x^2}{(x^3+1)}\\
b &= \frac{9x}{(x^3+1)}
\end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z''=rz \qquad (2)$$

Using what is known as the Liouville transformation given by

$$y=ze^{\frac{-1}{2}\int a dx} \qquad (3)$$

Where it can be found that r in (2) is given by

$$\begin{aligned}
r &= \frac{1}{4}a^2+\frac{1}{2}a'-b\\
&= \frac{1}{4}\left(\frac{7x^2}{(x^3+1)}\right)^2+\frac{1}{2}\left(\frac{d}{dx}\left(\frac{7x^2}{(x^3+1)}\right)\right)-\frac{9x}{(x^3+1)}\\
&= \frac{1}{4}\left(\frac{7x^2}{(x^3+1)}\right)^2+\frac{1}{2}\left(\frac{-7x(x^3-2)}{(x^3+1)^2}\right)-\frac{9x}{(x^3+1)}\\
&= \frac{-x(x^3+8)}{4\left(x^3+1\right)^2} \qquad (4)
\end{aligned}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z''=-\frac{x(x^3+8)}{4\left(x^3+1\right)^2}z \qquad (5)$$

Step 0 We need to find which case it is. $r=\frac{s}{t}$ where

$$\begin{aligned}
s &= -x\left(x^3+8\right)\\
t &= 4\left(x^3+1\right)^2
\end{aligned}$$

The free square factorization of t is $t=[1,x^3+1]$. Hence

$$m=2 \qquad (6)$$

Since m is number of elements in the free square factorization. in this special case we set

$$\begin{aligned}
t_1 &= 1\\
t_2 &= x^3+1
\end{aligned}$$

Now

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

There are three poles each of order 2. Looking at the cases table giving up, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that all three cases are possible. Hence $L = [1, 2, 4, 6, 12]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty) = 2, t = 4(x^3+1)^2, t_1 = 1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(2, 2) - 6 - 3(0)) \\
&= \frac{1}{4}(2 - 6) \\
&= -1 \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}\left(4(x^3+1)^2\right)}{4(x^3+1)^2} + 3(0)\right) \\
&= \frac{3}{2}\frac{x^2}{x^3+1}
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2.

$$
r = -\frac{x(x^3+8)}{4(x^3+1)^2}
$$

These will be the zeros of t_2 in the above square free factorization of t. From above we found that

$$
t_2 = x^3 + 1
$$

Label these zeros of t_2 as $c_1, c_2, \cdots, c_{k_2}$. The zeros of t_2 are $\left\{-1, (-1)^{\frac{1}{3}}, -(-1)^{\frac{1}{3}}\right\} = \left\{-1, \frac{1}{2} - \frac{1}{2}i\sqrt{3}, \frac{1}{2} + \frac{1}{2}i\sqrt{3}\right\}$. Therefore $k_2 = 3$. Hence

$$
M = 3
$$

Now we iterate over each zero c_i times finding e_i and θ_i from each. These are found to be (following formula in paper) to be

$$
\begin{aligned}
b_1 &= \frac{7}{36} \\
e_1 &= \frac{4}{3} \\
b_2 &= \frac{7}{36} \\
e_2 &= \frac{4}{3} \\
b_3 &= \frac{7}{36} \\
e_3 &= \frac{4}{3}
\end{aligned}
$$

And

$$
\begin{aligned}
\theta_1 &= \frac{4}{3(x+1)} \\
\theta_2 &= \frac{8}{3\left(i\sqrt{3}+2x-1\right)} \\
\theta_3 &= \frac{-8}{3\left(i\sqrt{3}-2x+1\right)}
\end{aligned}
$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, M$ if any exist. Since non exist here. This is skipped. Hence M stays 3.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since in this example $O(\infty) = 2$ then this case applies. $b = \frac{lcoeff(s)}{lcoeff(t)}$ where $lcoeff$ gives the leading coefficient. Since $s = -x(x^3+8) = -x^4 - 8x$ then $lcoeff(s) = -1$. And since $t = 4(x^3+1)^2 = 4x^6 + 8x^3 + 4$ then $lcoeff(t) = 4$. Therefore $b = \frac{-1}{4}$ and therefore

$$
\begin{aligned}
e_0 &= \sqrt{1+4b} \\
&= 0
\end{aligned}
$$

Hence now we have

$$
\begin{aligned}
e &= \left\{0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right\} \\
\theta &= \left\{0, \frac{4}{3(x+1)}, \frac{8}{3\left(i\sqrt{3}+2x-1\right)}, \frac{-8}{3\left(i\sqrt{3}-2x+1\right)}\right\}
\end{aligned}
$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $\underline{n=1}$. And since we have $M = 3$ then there are $(n+1)^{M+1} = 2^4 = 16$ sets s to try. The first set s is

$$
s = \left\{\frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2}\right\} = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}
$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i$$

Since $M = 3$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 - s_2 e_2 - s_3 e_3 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -1, \theta_{fixed} = \frac{3}{2}\frac{x^2}{x^3+1}$ gives

$$\begin{aligned} d &= (1)(-1) + \left(\frac{-1}{2}\right)(0) - \left(\frac{-1}{2}\right)\left(\frac{4}{3}\right) - \left(-\frac{1}{2}\right)\left(\frac{4}{3}\right) - \left(-\frac{1}{2}\right)\left(\frac{4}{3}\right) \\ &= 1 \end{aligned}$$

Since $d \geq 0$, then we can use it. Using Eq (8) gives (using $M = 3$)

$$\begin{aligned} \Theta &= (n)\left(\theta_{fix}\right) + s_0\theta_0 + s_1\theta_1 + s_2\theta_2 + s_3\theta_3 \\ &= (1)\left(\frac{3}{2}\frac{x^2}{x^3+1}\right) + \left(\frac{-1}{2}\right)(0) + \left(\frac{-1}{2}\right)\left(\frac{4}{3(x+1)}\right) + \left(\frac{-1}{2}\right)\left(\frac{8}{3\left(i\sqrt{3}+2x-1\right)}\right) + \left(\frac{-1}{2}\right)\left(\frac{}{3\left(i\sqrt{3}\right.}\right. \\ &= \left(\frac{3}{2}\frac{x^2}{x^3+1}\right) - \frac{2}{3(x+1)} - \frac{4}{3\left(i\sqrt{3}+2x-1\right)} + \frac{4}{3\left(i\sqrt{3}-2x+1\right)} \\ &= \frac{2x^2}{(x+1)\left(i\sqrt{3}+2x-1\right)\left(i\sqrt{3}-2x+1\right)} \\ &= \frac{-x^2}{2x^3+2} \end{aligned}$$

Now that we have good trial d and Θ, then step 3 is called to generate ω if possible.

<u>Step 3</u>

The input to this step is the integer $d = 1$ and $\Theta = \frac{-x^2}{2x^3+2}$ found from step 2 and also $r = -\frac{x(x^3+8)}{4(x^3+1)^2}$ which comes from $z'' = rz$. This step is broken into these parts. First we find the $p_{-1}(x)$ polynomial. If we are to solve for its coefficients, then next we build the minimal polynomial from the $p_i(x)$ polynomials constructed during finding $p_{-1}(x)$. The minimal polynomial $p_{\min}(x)$ will be a function of ω. Next we solve for ω from $p_{\min}(x) = 0$. If this is successful, then we have found ω and the first solution to the ode $y'' = ry$ is $e^{\int \omega dx}$. Below shows how this is done.

We start by forming a polynomial

$$\begin{aligned} p(x) &= x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\ &= x \end{aligned}$$

Since this is case $n = 1$ then

$$\begin{aligned} \omega &= \frac{p'}{p} + \Theta \\ &= \frac{1}{x} - \frac{x^2}{2x^3+2} \\ &= \frac{x^3+2}{2x\left(x^3+1\right)} \end{aligned}$$

Before using this, we will verify it is correct. For case 1 the above should satisfy

$$\omega' + \omega^2 = r$$

Let us see if this is the case or not.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^3+2}{2x\,(x^3+1)}\right) + \left(\frac{x^3+2}{2x\,(x^3+1)}\right)^2 &= -\frac{x(x^3+8)}{4\,(x^3+1)^2} \\
-\frac{(x^6+6x^3+2)}{2x^2\,(x^3+1)^2} + \frac{(x^3+2)^2}{4x^2\,(x^3+1)^2} &= -\frac{x(x^3+8)}{4\,(x^3+1)^2} \\
-\frac{x(x^3+8)}{4\,(x^3+1)^2} &= -\frac{x(x^3+8)}{4\,(x^3+1)^2}
\end{aligned}$$

Verified. Since solution ω is found and verified, then first solution to the ode is

$$\begin{aligned}
z &= e^{\int \omega dx} \\
&= e^{\int \frac{x^3+2}{2x\left(x^3+1\right)} dx} \\
&= \frac{x}{\sqrt[6]{x^3+1}}
\end{aligned}$$

Hence first solution to given ODE is

$$\begin{aligned}
y &= z e^{\frac{-1}{2}\int a dx} \\
&= \frac{x}{\sqrt[6]{x^3+1}} e^{-\frac{1}{2}\int \frac{7x^2}{\left(x^3+1\right)} dx} \\
&= \frac{x}{\sqrt[6]{x^3+1}} \frac{1}{(x^3+1)^{\frac{7}{6}}} \\
&= \frac{x}{(x^3+1)^{\frac{4}{3}}}
\end{aligned}$$

4.1.5 Example 5 case one

Solve Bessel ode (from Kovacic original paper)

$$\begin{aligned}
y'' - \frac{4(m^2-x^2)-1}{4x^2} y &= 0 \qquad (1)\\
y'' + ay' + by &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
a &= 0 \\
b &= \frac{4(m^2-x^2)-1}{4x^2}
\end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \qquad (2)$$

Using what is known as the Liouville transformation given by

$$y = z e^{\frac{-1}{2}\int a dx} \qquad (3)$$

Where it can be found that r in (2) is given by

$$\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{4(m^2-x^2)-1}{4x^2} \qquad (4)
\end{aligned}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{4(m^2-x^2)-1}{4x^2} z \qquad (5)$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$
\begin{aligned}
s &= 4(m^2 - x^2) - 1 \\
t &= 4x^2
\end{aligned}
$$

The free square factorization of t is $t = [1, x]$. Hence

$$
m = 2 \tag{6}
$$

Since m is number of elements in the free square factorization. in this case we set

$$
\begin{aligned}
t_1 &= 1 \\
t_2 &= x
\end{aligned}
$$

Now

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

There is one pole at $x = 0$ of order 2. Looking at the cases table giving up, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	$[1]$
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	$[2]$
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	$[4, 6, 12]$

Shows that only case 1,2 are possible. $L = [1, 2]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty) = 0, t = 4x^2, t_1 = 1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(0, 2) - 2 - 3(0)) \\
&= \frac{1}{4}(0 - 2) \\
&= -\frac{1}{2} \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(4x^2)}{4x^2} + 3(0)\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2.

$$
r = \frac{4(m^2 - x^2) - 1}{4x^2}
$$

These will be the zeros of t_2 in the above square free factorization of t. From above we found that

$$
t_2 = x
$$

Label these zeros of t_2 as $c_1, c_2, \cdots, c_{k_2}$. The zeros of t_2 are $\{0\}$. Therefore $k_2 = 1$. Hence

$$M = 1$$

Now we iterate over each zero c_i times finding e_i and θ_i from each. These are found to be (following formula in paper) to be

$$b_1 = m^2 - \frac{1}{4}$$
$$e_1 = \sqrt{1 + 4b} = \sqrt{1 + 4\left(m^2 - \frac{1}{4}\right)} = 2m \qquad m > 0$$

Where b_1 is the coefficient of $\frac{1}{(x-c_1)^2}$ in the partial fractions decomposition of r which is $r = -1 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}$. And

$$\theta_1 = \frac{e_1}{x - c_1} = \frac{2m}{x}$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, M$ if any exist. Since non exist here. This is skipped. Hence M stays 1.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1 + 4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since $O(\infty) = 0$ here then none of these cases applies. For case 1 ($n = 1$) following the method in the paper we find

$$e_0 = 0$$
$$\theta_0 = 2i$$

Hence now we have

$$e = \{0, 2m\}$$
$$\theta = \left\{2i, \frac{2m}{x}\right\}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 1$. And since we have $M = 1$ then there are $(n+1)^{M+1} = 2^2 = 4$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}, \frac{-n}{2}\right\} = \left\{\frac{-1}{2}, \frac{-1}{2}\right\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i$$

Since $M = 1$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{2}, \theta_{fixed} = \frac{1}{2x}$ gives

$$\begin{aligned} d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(0) - \left(\frac{-1}{2}\right)(2m) \\ &= m - \frac{1}{2} \end{aligned}$$

We now have to assume something about m to be to continue otherwise we will not be able to decide if d is integer and $d \geq 0$. If we assume that m is half of all positive odd integers $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots)$ then $d \geq 0$. We can also assume that m is half of all negative odd integers and the other s set will match. So under the assumption that m is half of all positive odd integers the above d can be used for the next step. To continue, we assume m takes some specific value to simplify the steps. Let $m = \frac{3}{2}$ from now on. Hence $d = 1$. Therefore e, θ become

$$\begin{aligned} e &= \{0, 3\} \\ \theta &= \left\{2i, \frac{3}{x}\right\} \end{aligned}$$

Using Eq (8) gives (using $M = 1$)

$$\begin{aligned} \Theta &= (n)\left(\theta_{fix}\right) + s_0\theta_0 + s_1\theta_1 \\ &= (1)\left(\frac{1}{2x}\right) + \left(\frac{-1}{2}\right)(2i) + \left(\frac{-1}{2}\right)\left(\frac{3}{x}\right) \\ &= -\frac{1}{x}(ix + 1) \end{aligned}$$

Now that we have good trial d and Θ, then step 3 is called to generate ω if possible.

Step 3

The input to this step is the integer $d = 1$ and $\Theta = -\frac{1}{x}(ix + 1)$ found from step 2 and also $r = \frac{4(n^2 - x^2) - 1}{4x^2}$ which comes from $z'' = rz$. This step is broken into these parts. First we find the $p_{-1}(x)$ polynomial. If we are to solve for its coefficients, then next we build the minimal polynomial from the $p_i(x)$ polynomials constructed during finding $p_{-1}(x)$. The minimal polynomial $p_{\min}(x)$ will be a function of ω. Next we solve for ω from $p_{\min}(x) = 0$. If this is successful, then we have found ω and the first solution to the ode $y'' = ry$ is $e^{\int \omega dx}$. Below shows how this is done.

We start by forming a polynomial

$$\begin{aligned} p(x) &= x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\ &= x + a_0 \end{aligned}$$

The goal is to solve for the a_0 coefficient. Now depending on case number n, we do the following. Since we are in case $n = 1$ then

$$\begin{aligned} p_1 &= -p \\ &= -x - a_0 \\ p_0 &= -p_1' - \Theta p_1 \\ &= -(-x - a_0)' - \left(-\frac{1}{x}(ix + 1)\right)(-x - a_0) \\ &= 1 - \left(-\frac{1}{x}(ix + 1)\right)(-x - a_0) \\ &= -\frac{1}{x}\left(ix^2 + ia_0x + a_0\right) \\ p_{-1} &= -p_0' - \Theta p_0 - (1)(1)rp_1 \\ &= -\left(-\frac{1}{x}\left(ix^2 + ia_0x + a_0\right)\right)' - \left(-\frac{1}{x}(ix + 1)\right)\left(-\frac{1}{x}\left(ix^2 + ia_0x + a_0\right)\right) - \frac{4(n^2 - x^2) - 1}{4x^2}(-x - a_0) \\ &= -\frac{2}{x}(ia_0 - 1) \end{aligned}$$

Now we try to solve for a_i using $p_{-1}(x) = 0$. This gives $a_0 = -i$. Hence this implies

$$p(x) = x - i$$

Since this is case $n = 1$ then

$$\begin{aligned}
\omega &= \frac{p'}{p} + \Theta \\
&= \frac{(x-i)'}{x-i} - \frac{1}{x}(ix+1) \\
&= \frac{-i(ix - x^2 + 1)}{(-x+i)\,x} \\
&= -\frac{(ix^3+1)}{x^3+x}
\end{aligned}$$

Before using this, we will verify it is correct. For case 1 the above should satisfy

$$\omega' + \omega^2 = r$$

Let us see if this is the case or not.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{-i(ix - x^2 + 1)}{(-x+i)\,x}\right) + \left(\frac{-i(ix - x^2 + 1)}{(-x+i)\,x}\right)^2 &= \frac{4\left(\left(\frac{3}{2}\right)^2 - x^2\right) - 1}{4x^2} \\
\frac{(-2ix^3 + 3x^2 + 1)}{x^2\,(x^2+1)^2} + \frac{(ix^2 + x - i)^2}{x^2\,(x-i)^2} &= -\frac{1}{x^2}(x^2 - 2) \\
-\frac{1}{x^2}(x^2 - 2) &= -\frac{1}{x^2}(x^2 - 2)
\end{aligned}$$

Verified. Hence $\omega = \frac{-i(ix-x^2+1)}{(-x+i)x} = -\frac{(ix^3+1)}{x^3+x}$ will give the solution to the ode $y'' - \frac{4(m^2-x^2)-1}{4x^2}y = 0$ when $m = \frac{3}{2}$. Since solution ω is found and verified, then first solution to the ode is

$$\begin{aligned}
z &= e^{\int \omega dx} \\
&= e^{\int -\frac{(ix^3+1)}{x^3+x} dx} \\
&= \frac{1}{x} e^{-ix} (x - i)
\end{aligned}$$

Hence first solution to given ODE is

$$\begin{aligned}
y &= z e^{\frac{-1}{2}\int a dx} \\
&= \frac{1}{x} e^{-ix} (x - i)\, e^{-\frac{1}{2}\int 0 dx} \\
&= \frac{1}{x} e^{-ix} (x - i)
\end{aligned}$$

One difficulty in implementation of Kovacic algorithm using an ode with a parameter m like in this Bessel ode example, is that it makes it hard to decide if $d \geq 0$ or not. So in practice, it is better to use this algorithm for specific values of any parameters that can be involved.

4.1.6 Example 6 case one

Solve

$$y'' = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} y \tag{1}$$

Hence

$$\begin{aligned}
a &= 0 \\
b &= -\frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}
\end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a\,dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\ &= \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4} z \tag{5}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$\begin{aligned} s &= 4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4 \\ t &= 4x^4 \end{aligned}$$

The free square factorization of t is $t = [1, 1, 1, x]$. Hence

$$m = 4 \tag{6}$$

Since m is number of elements in the free square factorization. in this case we set

$$\begin{aligned} t_1 &= 1 \\ t_2 &= 1 \end{aligned}$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$

There is one pole at $x = 0$ of order 4. Looking at the cases table

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that only case 1 is possible. $L = [1]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right) \end{aligned}$$

Using $O(\infty) = 0, t = 4x^4, t_1 = 1$ the above gives

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(-2, 2) - 4 - 3(0)) \\ &= \frac{1}{4}(-2 - 4) \\ &= -\frac{3}{2} \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(4x^4)}{4x^4} + 3(0)\right) \\ &= \frac{1}{x} \end{aligned}$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2. Since $t_2 = 1$ then there are poles. Hence $k_2 = 0$ and

$$M = 0$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, M$ if any exist. Here we have pole $x = 0$ of order 4. Following the paper (need to document), we find $e_1 = -5, \theta_1 = \frac{2}{x^2} - \frac{5}{x}$. We start from index 1 since $k_2 = 0$ from part (b). Now $k_1 = 1$. Note that for case 1, we use k_1. Hence

$$M = 1$$

And now not $M = 0$ (for case 1 only). For other cases, we use k_2 for M.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1 + 4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since $O(\infty) = 0$ here then none of these cases applies. For case 1 ($n = 1$) following the method in the paper we find (need to document)

$$\begin{aligned} e_0 &= 2 \\ \theta_0 &= -2 + 2x \end{aligned}$$

Hence now we have

$$\begin{aligned} e &= \{2, -5\} \\ \theta &= \left\{-2 + 2x, \frac{2}{x^2} - \frac{5}{x}\right\} \end{aligned}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 1$. And since we have $M = k_1 = 1$ then there are $(n+1)^{M+1} = 2^2 = 4$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}, \frac{-n}{2}\right\} = \left\{\frac{-1}{2}, \frac{-1}{2}\right\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i$$

Since $M = 1$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{3}{2}, \theta_{fixed} = \frac{1}{x}$ gives

$$
\begin{aligned}
d &= (1)\left(-\frac{3}{2}\right) + \left(\frac{-1}{2}\right)(2) - \left(\frac{-1}{2}\right)(-5) \\
&= -5
\end{aligned}
$$

Since this is not ≥ 0, we go to next set s $\left\{\frac{+1}{2}, \frac{+1}{2}\right\}$ and try again

$$
\begin{aligned}
d &= (1)\left(-\frac{3}{2}\right) + \left(\frac{+1}{2}\right)(2) - \left(\frac{+1}{2}\right)(-5) \\
&= 2
\end{aligned}
$$

This works. Using Eq (8) gives (using $M = 1$)

$$
\begin{aligned}
\Theta &= (n)\left(\theta_{fix}\right) + s_0\theta_0 + s_1\theta_1 \\
&= (1)\left(\frac{1}{x}\right) + \left(\frac{+1}{2}\right)(-2+2x) + \left(\frac{+1}{2}\right)\left(\frac{2}{x^2} - \frac{5}{x}\right) \\
&= -\frac{\left(-2x^3 + 2x^2 + 3x - 2\right)}{2x^2} \\
&= x - 1 - \frac{3}{2x} + \frac{1}{x^2}
\end{aligned}
$$

Now that we have good trial d and Θ, then step 3 is called to generate $P(x)$ if possible.

Step 3

The input to this step is the integer $d = 2$ and $\Theta = x - 1 - \frac{3}{2x} + \frac{1}{x^2}$ found from step 2 and also $r = \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + 4}{4x^4}$ which comes from $z'' = rz$. Since degree $d = 2$, then let $p(x) = x^2 + ax + b$. Therefore we need to now find $P(x)$ that solves

$$
\begin{aligned}
P'' + 2\Theta P' \cdots \\
2 + 2\Theta(2x+a) + \left(\Theta' + \Theta^2 \cdots \right. \\
2 + 2\left(x - 1 - \frac{3}{2x} + \frac{1}{x^2}\right)(2x+a) + \left(\left(x - 1 - \frac{3}{2x} + \frac{1}{x^2}\right)' + \left(x - 1 - \frac{3}{2x} + \frac{1}{x^2}\right)^2 - \frac{4x^6 - 8x^5 + 12x^4 + 4x^3 + 7x^2 - 20x + \cdots}{4x^4}\right.
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
-\frac{1}{x^2}\left(2ax^3 - 4x - 2ax^2 - 2a + 4bx^2 + 3ax - 4bx + 4x^2\right) &= 0 \\
-2ax + \frac{4}{x} + 2a + 2\frac{a}{x^2} - 4b - 3a\frac{1}{x} + 4\frac{b}{x} - 4 &= 0
\end{aligned}
$$

Hence by comparing coefficients

$$
x(-2a) + \frac{1}{x}(4 - 3a + 4b) + \frac{1}{x^2}(2a) + (2a - 4b - 4) = 0
$$

Therefore $a = 0$. And $4 - 3a + 4b = 0$ gives $b = -1$. Same if we used $2a - 4b - 4 = 0$, So consistent equations. Therefore

$$
P(x) = x^2 - 1
$$

And the solution is

$$
\begin{aligned}
z &= P(x)\, e^{\int \Theta dx} \\
&= \left(x^2 - 1\right) e^{\int x - 1 - \frac{3}{2x} + \frac{1}{x^2} dx} \\
&= \left(x^2 - 1\right) x^{-\frac{3}{2}} e^{-\frac{1}{x} + \frac{x^2}{2} - x}
\end{aligned}
$$

Hence first solution to given ODE is

$$
\begin{aligned}
y &= \left(x^2 - 1\right) x^{-\frac{3}{2}} e^{-\frac{1}{x} + \frac{x^2}{2} - x} e^{\frac{-1}{2}\int a dx} \\
&= \left(x^2 - 1\right) x^{-\frac{3}{2}} e^{-\frac{1}{x} + \frac{x^2}{2} - x} e^{-\frac{1}{2}\int 0 dx} \\
&= \left(x^2 - 1\right) x^{-\frac{3}{2}} e^{-\frac{1}{x} + \frac{x^2}{2} - x}
\end{aligned}
$$

4.1.7 Example 7 case two

Solve

$$y'' = \frac{16x-3}{16x^2}y \tag{1}$$

$$y'' + ay'(x) + by = 0$$

Hence

$$a = 0$$
$$b = -\frac{16x-3}{16x^2}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\ &= \frac{16x-3}{16x^2} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{16x-3}{16x^2}z \tag{5}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$s = 16x - 3$$
$$t = 16x^2$$

The free square factorization of t is $t = [1, x]$. Hence

$$m = 2 \tag{6}$$

Since m is number of elements in the free square factorization. in this special case we set

$$t_1 = 1$$
$$t_2 = x$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

There is pole $x = 0$ of order 2. Looking at the cases table, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that only case 2 is possible (due to $O(\infty) = 1$ which is not allowed other than for case 2). Hence $L = [2]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty) = 1, t = 16x^2, t_1 = 1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(1, 2) - 2 - 3(0)) \\
&= \frac{1}{4}(1 - 2) \\
&= -\frac{1}{4} \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(16x^2)}{16x^2} + 3(0)\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2.

$$
r = \frac{16x - 3}{16x^2}
$$

These will be the zeros of t_2 in the above square free factorization of t. From above we found that

$$
t_2 = x
$$

Label these zeros of t_2 as $c_1, c_2, \cdots, c_{k_2}$. The zeros of t_2 are $\{0\}$. Therefore $k_2 = 1$. Hence

$$
M = 1
$$

Now we iterate over each zero c_i times finding e_i and θ_i from each. These are found to be (following formula in paper) to be

$$
e_1 = \frac{1}{2}
$$

And

$$
\theta_1 = \frac{1}{2x}
$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, M$ if any exist. Since only case 2 exist in this example. This is skipped. Hence M stays 1.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1 + 4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. None of these apply, and this is not case 1. Hence

$$
\begin{aligned}
e_0 &= 0 \\
\theta_0 &= 0
\end{aligned}
$$

Hence now we have

$$
\begin{aligned}
e &= \left\{0, \frac{1}{2}\right\} \\
\theta &= \left\{0, \frac{1}{2x}\right\}
\end{aligned}
$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 2$. Since case 2 only applies here. And since we have $M = 1$ then there are $(n+1)^{M+1} = 3^2 = 9$ sets s to try. The first set s is

$$
s = \left\{\frac{-n}{2}, \frac{-n}{2}\right\} = \{-1, -1\}
$$

Now we generate trial d using

$$
d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{M} s_i e_i
$$

Since $M = 1$ then the above becomes

$$
d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 \tag{7}
$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$
\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{M} s_i \theta_i \tag{8}
$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{4}, \theta_{fixed} = \frac{1}{2x}$ gives

$$
\begin{aligned}
d &= (2)\left(-\frac{1}{4}\right) + (-1)(0) - (-1)\left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since $d \geq 0$, then we can use it. Using Eq (8) gives (using $M = 1$)

$$
\begin{aligned}
\Theta &= (n)\left(\theta_{fix}\right) + s_0\theta_0 + s_1\theta_1 \\
&= (2)\left(\frac{1}{2x}\right) + (-1)(0) + (-1)\left(\frac{1}{2x}\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that we have good trial d and Θ, then step 3 is called to generate $P(x)$ if possible.

Step 3

The input to this step is the integer $d = 0$ and $\Theta = \frac{1}{2x}$ found from step 2 and also $r = \frac{16x-3}{16x^2}$ which comes from $z'' = rz$. We need now to find $P(x)$ of degree $d = 0$ which is a constant such that

$$
P''' + 3\Theta P'' + \left(3\Theta^2 + 3\Theta' - 4r\right)P' + \left(\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r'\right)P = 0
$$

Since $P = 1$ then above simplifies to

$$
\left(\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r'\right) = 0
$$

We know Θ and r. If this verifies, then we can use $P = 1$. Substituting the above becomes

$$
\begin{aligned}
\left(\left(\frac{1}{2x}\right)'' + 3\frac{1}{2x}\left(\frac{1}{2x}\right)' + \left(\frac{1}{2x}\right)^3 - 4\left(\frac{16x-3}{16x^2}\right)\left(\frac{1}{2x}\right) - 2\left(\frac{16x-3}{16x^2}\right)'\right) &= 0 \\
\left(\frac{d^2}{dx^2}\left(\frac{1}{2x}\right) + 3\frac{1}{2x}\frac{d}{dx}\left(\frac{1}{2x}\right) + \left(\frac{1}{2x}\right)^3 - 4\left(\frac{16x-3}{16x^2}\right)\left(\frac{1}{2x}\right) - 2\frac{d}{dx}\left(\frac{16x-3}{16x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

Verified. Hence

$$
P(x) = 1
$$

Let

$$
\begin{aligned}
\phi &= \Theta + \frac{P'}{P} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now we solve for ω from

$$
\begin{aligned}
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi - r\right) &= 0 \\
\omega^2 - \frac{1}{2x}\omega + \left(\frac{1}{2}\left(\frac{1}{2x}\right)' + \frac{1}{2}\left(\frac{1}{2x}\right) - \frac{16x-3}{16x^2}\right) &= 0 \\
\omega^2 - \frac{1}{2x}\omega + \frac{1}{16x^2} - \frac{1}{x} &= 0
\end{aligned}
$$

The solution $\omega = \frac{1}{4x} \pm \frac{1}{\sqrt{x}}$. We pick either solution. Hence the solution is

$$
\begin{aligned}
z &= e^{\int \omega dx} \\
&= e^{\int \frac{1}{4x} + \frac{1}{\sqrt{x}} dx} \\
&= x^{\frac{1}{4}} e^{2\sqrt{x}}
\end{aligned}
$$

Hence first solution to given ODE is

$$
\begin{aligned}
y &= z e^{\frac{-1}{2} \int a dx} \\
&= x^{\frac{1}{4}} e^{2\sqrt{x}} e^{-\frac{1}{2} \int 0 dx} \\
&= x^{\frac{1}{4}} e^{2\sqrt{x}}
\end{aligned}
$$

4.1.8 Example 8 case two

Solve

$$
\begin{aligned}
y'' &= \frac{1}{x^3} y \qquad (1) \\
y'' + ay'(x) + by &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
a &= 0 \\
b &= -\frac{1}{x^3}
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$
z'' = rz \qquad (2)
$$

Using what is known as the Liouville transformation given by

$$
y = z e^{\frac{-1}{2} \int a dx} \qquad (3)
$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\ &= \frac{1}{x^3} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{1}{x^3}z \tag{5}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$\begin{aligned} s &= 1 \\ t &= x^3 \end{aligned}$$

The free square factorization of t is $t = [1, 1, x]$. Hence

$$m = 3 \tag{6}$$

Since m is number of elements in the free square factorization. in this special case we set

$$\begin{aligned} t_1 &= 1 \\ t_2 &= 1 \end{aligned}$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

There is pole $x = 0$ of order 3. Looking at the cases table, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that only case 2 is possible (since odd pole is only allowed in case 2). Hence $L = [2]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right) \end{aligned}$$

Using $O(\infty) = 1, t = x^3, t_1 = 1$ the above gives

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(3, 2) - 3 - 3(0)) \\ &= \frac{1}{4}(2 - 3) \\ &= -\frac{1}{4} \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(x^3)}{x^3} + 3(0)\right) \\ &= \frac{3}{4x} \end{aligned}$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of t_2. In other words, the number of poles of r that are of order 2. There are no poles of order 2. Hence $k_2 = 0$.

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots$ if any exist. Since only case 2 exist in this example. This is skipped. Hence k_2 stays 0.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. Hence

$$\begin{aligned} e_0 &= 1 \\ \theta_0 &= 0 \end{aligned}$$

Hence now we have

$$\begin{aligned} e &= \{1\} \\ \theta &= \{0\} \end{aligned}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 2$. Since case 2 only applies here. And since we have $k_2 = 0$ then there are $(n+1)^{k_2+1} = 3$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}\right\} = \{-1\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 0$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{4}, \theta_{fixed} = \frac{3}{4x}$ gives

$$\begin{aligned} d &= (2)\left(-\frac{1}{4}\right) + (-1)(1) \\ &= -\frac{3}{2} \end{aligned}$$

Since negative then we can not use it. Now we try the next set $s = \{0\}$. Then Eq(7) gives

$$\begin{aligned} d &= (2)\left(-\frac{1}{4}\right) + (0)(1) \\ &= -\frac{1}{2} \end{aligned}$$

Since negative then we can not use it. Now we try the last set $s = \{+1\}$. Then Eq(7) gives

$$\begin{aligned} d &= (2)\left(-\frac{1}{4}\right) + (+1)(1) \\ &= \frac{1}{2} \end{aligned}$$

Since not an integer, then we can not use it. We are run out of sets s to try. Therefore there is no Liouvillian solution.

4.1.9 Example 9 case two

Solve

$$\begin{aligned} 2x^2y'' - xy' + (1-2x)\,y &= 0 & (1) \\ y'' - \frac{1}{2x}y' + \frac{(1-2x)}{2x^2}y &= 0 \qquad x \neq 0 \\ y'' + ay'(x) + by &= 0 \end{aligned}$$

Hence

$$\begin{aligned} a &= -\frac{1}{2x} \\ b &= \frac{(1-2x)}{2x^2} \end{aligned}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\ &= \frac{1}{4}\left(-\frac{1}{2x}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(-\frac{1}{2x}\right) - \frac{(1-2x)}{2x^2} \\ &= \frac{16x-3}{16x^2} & (4) \end{aligned}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{16x-3}{16x^2}z \tag{5}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$\begin{aligned} s &= 16x - 3 \\ t &= 16x^2 \end{aligned}$$

The free square factorization of t is $t = [1, x]$. Hence

$$m = 2 \tag{6}$$

Since m is number of elements in the free square factorization. in this special case we set

$$\begin{aligned} t_1 &= 1 \\ t_2 &= x \end{aligned}$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

There is pole $x = 0$ of order 2. Looking at the cases table, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that only case 2 is possible ($O(\infty) = 1$ is only possible for case 2). Hence $L = [2]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min\left(O(\infty), 2\right) - \deg(t) - 3\deg(t_1)) \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty) = 1, t = 16x^2, t_1 = 1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(1, 2) - 2 - 3(0)) \\
&= \frac{1}{4}(1 - 2) \\
&= -\frac{1}{4} \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(16x^2)}{16x^2} + 3(0)\right) \\
&= \frac{1}{2x}
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of $t_2 = x$. In other words, the number of poles of r that are of order 2. There is one pole of order 2. Hence $k_2 = 1$. the coefficient of $\frac{1}{(x-0)^2}$ in the partial fractions of $r = \frac{16x-3}{16x^2} = \frac{1}{x} - \frac{3}{16}\frac{1}{(x-0)^2}$. Therefore $b = -\frac{3}{16}$. Hence $e_1 = \sqrt{1+4b} = \sqrt{1+4\left(-\frac{3}{16}\right)} = \frac{1}{2}$ and $\theta_1 = \frac{e_1}{x-0} = \frac{1}{2x}$. Hence

$$
\begin{aligned}
e &= \left\{\frac{1}{2}\right\} \\
\theta &= \left\{\frac{1}{2x}\right\}
\end{aligned}
$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots$ if any exist. Since only case 2 exist in this example. This is skipped. Hence k_2 stays 0.

Part(d)

Now we need to find e_0, θ_0. Since this is not case 1 and since it is not $O(\infty) > 2$ and not $O(\infty) = 2$, then

$$
\begin{aligned}
e_0 &= 0 \\
\theta_0 &= 0
\end{aligned}
$$

Hence now we have

$$
\begin{aligned}
e &= \left\{0, \frac{1}{2}\right\} \\
\theta &= \left\{0, \frac{1}{2x}\right\}
\end{aligned}
$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 2$. Since case 2 only applies here. And since we have $k_2 = 1$ then there are $(n+1)^{k_2+1} = 3^2 = 9$ sets s to try. The first set s is

$$s = \left\{ \frac{-n}{2}, \frac{-n}{2} \right\} = \{-1, -1\}$$

Now we generate trial d using

$$d = (n)(e_{fix}) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 1$ then the above becomes

$$d = (n)(e_{fix}) + s_0 e_0 - s_1 e_1 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)(\theta_{fix}) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{4}, \theta_{fixed} = \frac{1}{2x}$ gives

$$\begin{aligned} d &= (2)\left(-\frac{1}{4}\right) + (-1)(0) - (-1)\left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

We can use this d.From Eq (8)

$$\begin{aligned} \Theta &= (2)\left(\frac{1}{2x}\right) + s_0\theta_0 + s_1\theta_1 \\ &= (2)\left(\frac{1}{2x}\right) + (-1)(0) + (-1)\frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Since this is case 2 ($n = 2$) then we need to first find $P(x)$. The degree is $d = 0$. Hence constant. Say $P = 1$. But we need to verify this is valid. Setting up the equation

$$P''' + 3\Theta P'' + \left(3\Theta^2 + 3\Theta' - 4r\right) P' + \left(\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r'\right) P = 0$$

Which simplifies to (since $P = 1$)

$$\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r' = 0$$

Using $\Theta = \frac{1}{2x}, r = \frac{16x-3}{16x^2}$ the above reduces to

$$0 = 0$$

Hence $P(x) = 1$ can be used. Now let

$$\begin{aligned} \phi &= \Theta + \frac{P'}{P} \\ &= \frac{1}{2x} \end{aligned}$$

We now need to solve for ω from (notice that original Kovacic paper has $+$ and not $-$ after first term in the following equation. The $+$ is from Smith paper. It seems to have been a typo in original paper as this version gives the correct solution).

$$\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) = 0$$
$$\omega^2 - \frac{1}{2x}\omega - \frac{1}{8x^2} - \frac{1}{16x^2}(16x-3) = 0$$

Solving (and picking first root) gives

$$\omega = \frac{1}{4x}\left(1+4\sqrt{x}\right)$$

Before using this, we verify it satisfies $\omega' + \omega^2 = r$

$$\frac{d}{dx}\left(\frac{1}{4x}\left(1+4\sqrt{x}\right)\right) + \left(\frac{1}{4x}\left(1+4\sqrt{x}\right)\right)^2 = \frac{16x-3}{16x^2}$$
$$\frac{1}{16x^2}(16x-3) = \frac{16x-3}{16x^2}$$

Verified OK. Hence solution is

$$z = e^{\int \omega dx}$$
$$= x^{\frac{1}{4}}e^{2\sqrt{x}}$$

Hence first solution to given ODE is

$$y_1 = ze^{\frac{-1}{2}\int a dx}$$
$$= x^{\frac{1}{4}}e^{2\sqrt{x}}e^{-\frac{1}{2}\int -\frac{1}{2x}dx}$$
$$= \sqrt{x}e^{2\sqrt{x}}$$

Second solution y_2 can now be find by reduction of order.

4.1.10 Example 10 case two

Solve

$$\left(x^2+2\right)y'' + 3xy' - y = 0 \tag{1}$$
$$y'' + \frac{3x}{\left(x^2+2\right)}y' - \frac{1}{\left(x^2+2\right)}y = 0$$
$$y'' + ay'(x) + by = 0$$

Hence

$$a = \frac{3x}{\left(x^2+2\right)}$$
$$b = -\frac{1}{\left(x^2+2\right)}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$r = \frac{1}{4}a^2 + \frac{1}{2}a' - b$$
$$= \frac{1}{4}\left(\frac{3x}{\left(x^2+2\right)}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{3x}{\left(x^2+2\right)}\right) + \frac{1}{\left(x^2+2\right)}$$
$$= \frac{7x^2+20}{4\left(x^2+2\right)^2} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{7x^2 + 20}{4\left(x^2 + 2\right)^2} z \tag{5}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$\begin{aligned} s &= 7x^2 + 20 \\ t &= 4\left(x^2 + 2\right)^2 = 16 + 16x^2 + 4x^4 \end{aligned}$$

The free square factorization of t is $t = [1, (x^2 + 2)]$. Hence

$$m = 2 \tag{6}$$

Since m is number of elements in the free square factorization. in this special case we set

$$\begin{aligned} t_1 &= 1 \\ t_2 &= \left(x^2 + 2\right) \end{aligned}$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

There is pole $x = \pm i\sqrt{2}$ of order 2. Looking at the cases table, reproduced here

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that all cases are possible. Hence $L = [1, 2, 4, 6, 12]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right) \end{aligned}$$

Using $O(\infty) = 2, t = 4(x^2 + 2)^2, t_1 = 1$ the above gives

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(2, 2) - 4 - 3(0)) \\ &= \frac{1}{4}(2 - 4) \\ &= -\frac{1}{2} \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}\left(4(x^2 + 2)^2\right)}{4\left(x^2 + 2\right)^2} + 3(0)\right) \\ &= \frac{x}{x^2 + 2} \end{aligned}$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of $t_2 = (x^2 + 2)$. In other words, the number of poles of r that are of order 2. There are two poles of order 2.

Hence $k_2 = 2$. These poles at $x = \pm i\sqrt{2}$. The coefficient of $\frac{1}{(x-c_1)^2}$ where c_1 is first pole is $b_1 = -\frac{3}{16}$. Hence $e_1 = \sqrt{1+4b} = \sqrt{1+4\left(-\frac{3}{16}\right)} = \frac{1}{2}$ and $\theta_1 = \frac{e_1}{x-c_1} = \frac{1}{2\left(x-i\sqrt{2}\right)}$. The coefficient of $\frac{1}{(x-c_2)^2}$ where c_2 is second pole is $b_2 = -\frac{3}{16}$. Hence $e_2 = \sqrt{1+4b} = \sqrt{1+4\left(-\frac{3}{16}\right)} = \frac{1}{2}$ and $\theta_2 = \frac{e_1}{x-c_2} = \frac{1}{2\left(x+i\sqrt{2}\right)}$ Hence

$$e = \left\{\frac{1}{2}, \frac{1}{2}\right\}$$
$$\theta = \left\{\frac{1}{2\left(x-i\sqrt{2}\right)}, \frac{1}{2\left(x+i\sqrt{2}\right)}\right\}$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8$ if any exist. Since only order 2 pole exist, then this is skipped. Hence k_2 stays 2.

Part(d)

Now we need to find e_0, θ_0. Since this is case $O(\infty) = 2$, then $e_0 = \sqrt{1+4b}$ where $b = \frac{lcoeff(s)}{lcoeff(t)}$ where $lcoeff(s)$ is leading coefficient of $s = 7x^2 + 20$ which is 7 and $lcoeff(t)$ is leading coefficient of $t = 16 + 16x^2 + 4x^4$ which is 4. Hence $b = \frac{7}{4}$. Therefore

$$e_0 = \sqrt{1+4b} = \sqrt{1+4\left(\frac{7}{4}\right)} = 2\sqrt{2}$$
$$\theta_0 = 0$$

Hence now we have

$$e = \left\{2\sqrt{2}, \frac{1}{2}, \frac{1}{2}\right\}$$
$$\theta = \left\{0, \frac{1}{2\left(x-i\sqrt{2}\right)}, \frac{1}{2\left(x+i\sqrt{2}\right)}\right\}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 1$. And since we have $k_2 = 2$ then there are $(n+1)^{k_2+1} = 2^3 = 8$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2}\right\} = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 2$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 - s_2 e_2 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{2}, \theta_{fixed} = \frac{x}{x^2+2}$ gives

$$\begin{aligned} d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)\left(2\sqrt{2}\right) - \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) \\ &= -\sqrt{2} \end{aligned}$$

Since not an integer, we try next set $s = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{+1}{2}\right\}$ and now Eq (7) gives

$$\begin{aligned} d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)\left(2\sqrt{2}\right) - \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{+1}{2}\right)\left(\frac{1}{2}\right) \\ &= -\sqrt{2} - \frac{1}{2} \end{aligned}$$

Since not an integer, we try next set $s = \left\{\frac{-1}{2}, \frac{+1}{2}, \frac{-1}{2}\right\}$. If we continue this way we will find that all sets s will fail to generate a $d \geq 0$. Hence case one did not work. Now we go to case 2 ($n = 2$).

Starting with $\underline{n = 2}$. And since we have $k_2 = 2$ then there are $(n+1)^{k_2+1} = 3^3$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2}\right\} = \{-1, -1, -1\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 2$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 - s_2 e_2 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{2}, \theta_{fixed} = \frac{x}{x^2+2}$ gives

$$\begin{aligned} d &= (2)\left(-\frac{1}{2}\right) + (-1)\left(2\sqrt{2}\right) - (-1)\left(\frac{1}{2}\right) - (-1)\left(\frac{1}{2}\right) \\ &= -2\sqrt{2} \end{aligned}$$

Since not an integer, we try next set $s = \{-1, -1, +1\}$. If we continue this way we will find that set $s = \{0, -1, -1\}$ works.

$$\begin{aligned} d &= (2)\left(-\frac{1}{2}\right) + (0)\left(2\sqrt{2}\right) - (-1)\left(\frac{1}{2}\right) - (-1)\left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

We can use this d. From Eq (8)

$$\begin{aligned} \Theta &= (2)\left(\frac{1}{2x}\right) + s_0\theta_0 + s_1\theta_1 + s_2\theta_2 \\ &= (2)\left(\frac{x}{x^2+2}\right)(0)(0) - (-1)\left(\frac{1}{2\left(x - i\sqrt{2}\right)}\right) - (-1)\left(\frac{1}{2\left(x + i\sqrt{2}\right)}\right) \\ &= \frac{x}{x^2+2} \end{aligned}$$

Since this is case 2 ($n = 2$) then we need to first find $P(x)$. The degree is $d = 0$. Hence constant. Say $P = 1$. But we need to verify this is valid. Setting up the equation

$$P''' + 3\Theta P'' + \left(3\Theta^2 + 3\Theta' - 4r\right)P' + \left(\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r'\right)P = 0$$

Which simplifies to (since $P = 1$)

$$\Theta'' + 3\Theta\Theta' + \Theta^3 - 4r\Theta - 2r' = 0$$

Using $\Theta = \frac{x}{x^2+2}, r = \frac{7x^2+20}{4(x^2+2)^2}$ the above reduces to

$$0 = 0$$

Hence $P(x) = 1$ can be used. Now let

$$\begin{aligned}
\phi &= \Theta + \frac{P'}{P} \\
&= \frac{x}{x^2+2}
\end{aligned}$$

We now need to solve for ω from (notice that original Kovacic paper has $+$ and not $-$ after first term in the following equation. The $+$ is from Smith paper. It seems to have been a typo in original paper as this version gives the correct solution).

$$\begin{aligned}
\omega^2 - \phi\omega + \left(\frac{1}{2}\phi' + \frac{1}{2}\phi^2 - r\right) &= 0 \\
\omega^2 - \left(\frac{x}{x^2+2}\right) + \omega\left(\frac{1}{2}\left(\frac{x}{x^2+2}\right)' + \frac{1}{2}\left(\frac{x}{x^2+2}\right)^2 - \frac{7x^2+20}{4\left(x^2+2\right)^2}\right) &= 0 \\
\frac{4\omega^2x^4 - 4\omega x^3 + \left(16\omega^2 - 7\right)x^2 - 8x\omega + 16\omega^2 - 16}{4\left(x^2+2\right)^2} &= 0 \\
4\omega^2x^4 - 4\omega x^3 + \left(16\omega^2 - 7\right)x^2 - 8x\omega + 16\omega^2 - 16 &= 0
\end{aligned}$$

Solving for ω (and picking first root) gives

$$\omega = \frac{x + 2\sqrt{2x^2+4}}{2\left(x^2+2\right)}$$

Before using this, we verify it satisfies $\omega' + \omega^2 = r$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x + 2\sqrt{2x^2+4}}{2\left(x^2+2\right)}\right) + \left(\frac{x + 2\sqrt{2x^2+4}}{2\left(x^2+2\right)}\right)^2 &= \frac{7x^2+20}{4\left(x^2+2\right)^2} \\
\frac{7x^2+20}{4x^4+16x^2+16} &= \frac{7x^2+20}{4\left(x^2+2\right)^2}
\end{aligned}$$

Verified OK. Hence solution is

$$\begin{aligned}
z &= e^{\int \omega dx} \\
&= e^{\int \frac{x+2\sqrt{2x^2+4}}{2\left(x^2+2\right)} dx} \\
&= \left(x^2+2\right)^{\frac{1}{4}} e^{\sqrt{2}\operatorname{arcsinh}\left(\frac{\sqrt{2}}{2}x\right)}
\end{aligned}$$

Hence first solution to given ODE is

$$\begin{aligned}
y_1 &= ze^{\frac{-1}{2}\int a dx} \\
&= \left(x^2+2\right)^{\frac{1}{4}} e^{\sqrt{2}\operatorname{arcsinh}\left(\frac{\sqrt{2}}{2}x\right)} e^{-\frac{1}{2}\int \frac{3x}{\left(x^2+2\right)} dx} \\
&= \frac{e^{\sqrt{2}\operatorname{arcsinh}\left(\frac{\sqrt{2}}{2}x\right)}}{\sqrt{\left(x^2+2\right)}}
\end{aligned}$$

Second solution y_2 can now be find by reduction of order.

4.1.11 Example 11 case one

Solve

$$
\begin{aligned}
x^2(x^2+x+1)\,y''-x(-2x^2-4x+1)\,y'+y&=0 \qquad (1)\\
y''-\frac{x(-2x^2-4x+1)}{x^2(x^2+x+1)}y'+\frac{1}{x^2(x^2+x+1)}y&=0\\
y''+ay'(x)+by&=0
\end{aligned}
$$

Hence

$$
\begin{aligned}
a&=-\frac{x(-2x^2-4x+1)}{x^2(x^2+x+1)}\\
b&=\frac{1}{x^2(x^2+x+1)}
\end{aligned}
$$

It is first transformed to the following ode by eliminating the first derivative

$$z''=rz \qquad (2)$$

Using what is known as the Liouville transformation given by

$$y=ze^{\frac{-1}{2}\int a dx} \qquad (3)$$

Where it can be found that r in (2) is given by

$$
\begin{aligned}
r&=\frac{1}{4}a^2+\frac{1}{2}a'-b\\
&=\frac{1}{4}\left(-\frac{x(-2x^2-4x+1)}{x^2(x^2+x+1)}\right)^2+\frac{1}{2}\frac{d}{dx}\left(-\frac{x(-2x^2-4x+1)}{x^2(x^2+x+1)}\right)-\frac{1}{x^2(x^2+x+1)}\\
&=\frac{10x^2-8x-1}{4x^2(x^2+x+1)^2} \qquad (4)
\end{aligned}
$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z''=\frac{10x^2-8x-1}{4x^2(x^2+x+1)^2}z \qquad (5)$$

Step 0 We need to find which case it is. $r=\frac{s}{t}$ where

$$
\begin{aligned}
s&=10x^2-8x-1\\
t&=4x^2(x^2+x+1)^2\\
&=(x^3+x^2+x)^2
\end{aligned}
$$

The free square factorization of t is $t=[1,x^3+x^2+x]$. Hence

$$m=2 \qquad (6)$$

Since m is number of elements in the free square factorization. in this special case we set

$$
\begin{aligned}
t_1&=1\\
t_2&=x^3+x^2+x
\end{aligned}
$$

Now

$$
\begin{aligned}
O(\infty)&=\deg(t)-\deg(s)\\
&=6-2\\
&=4
\end{aligned}
$$

There are poles of order 2. Looking at the cases table, reproduced here

case	allowed pole order for $r=\frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-8,-6,-4,-2,0,2,3,4,5,6,7,\cdots\}$	$[1]$
2	$\{2,3,5,7,9,\cdots\}$	no condition	$[2]$
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$	$[4,6,12]$

Shows that all cases are possible. Hence $L=[1,2,4,6,12]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts e_{fixed},θ_{fixed} are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(O(\infty),2)-\deg(t)-3\deg(t_1))\\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t}+3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty)=2, t=(x^3+x^2+x)^2, t_1=1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(6,2)-6-3(0))\\
&= \frac{1}{4}(2-6)\\
&= -1\\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}\left((x^3+x^2+x)^2\right)}{(x^3+x^2+x)^2}+3(0)\right)\\
&= \frac{1}{2}\frac{3x^2+2x+1}{x^3+x^2+x}
\end{aligned}
$$

part (b)

Here the values e_i,θ_i are found for $i=1\cdots k_2$ where k_2 is the number of roots of $t_2=x^3+x^2+x$. In other words, the number of poles of r that are of order 2. There are three poles of order 2. Hence $k_2=3$. These poles at $x=\left\{0,-\frac{1}{2}\pm\frac{1}{2}i\sqrt{3}\right\}$. The coefficient of $\frac{1}{(x-c_1)^2}$ where c_1 is first pole is $b_1=-\frac{1}{4}$. Hence $e_1=\sqrt{1+4b}=\sqrt{1+4\left(-\frac{1}{4}\right)}=0$ and $\theta_1=\frac{e_1}{x-c_1}=0$. The coefficient of $\frac{1}{(x-c_2)^2}$ where c_2 is second pole is $b_2=\frac{9i\sqrt{3}+2}{3\left(-1+i\sqrt{3}\right)^2}$. Hence $e_2=\sqrt{1+4b}=\frac{\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)}$ and $\theta_2=\frac{e_2}{x-c_2}=\frac{-2\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x-1\right)}$. The coefficient of $\frac{1}{(x-c_3)^2}$ where c_3 is the third pole is $b_3=\frac{-9i\sqrt{3}+2}{3\left(1+i\sqrt{3}\right)^2}$. Hence $e_3=\sqrt{1+4b}=\frac{\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)}$ and $\theta_3=\frac{e_3}{x-c_3}=\frac{2\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)\left(i\sqrt{3}+2x+1\right)}$. Hence

$$
\begin{aligned}
e &= \left\{0,\frac{\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)},\frac{\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)}\right\}\\
\theta &= \left\{0,\frac{-2\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x-1\right)},\frac{2\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)\left(i\sqrt{3}+2x+1\right)}\right\}
\end{aligned}
$$

Part (c)

This part applied only to case 1. It is used to generate e_i,θ_i for poles of r order $4,6,8$ if any exist. Since only order 2 pole exist, then this is skipped. Hence k_2 stays 3.

Part(d)

Now we need to find e_0,θ_0. Since this is case $O(\infty)=4>2$ and since there are no poles or order $4,6,8,\cdots$ then we do not need to handle case $n=1$. Instead we use

$$
\begin{aligned}
e_0 &= 1\\
\theta_0 &= 0
\end{aligned}
$$

Hence now we have

$$e = \left\{1, 0, \frac{\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)}, \frac{\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)}\right\}$$

$$\theta = \left\{0, 0, \frac{-2\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x-1\right)}, \frac{2\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)\left(i\sqrt{3}+2x+1\right)}\right\}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n=1$. And since we have $k_2 = 3$ then there are $(n+1)^{k_2+1} = 2^4 = 16$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2}\right\} = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 3$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 - s_2 e_2 - s_3 e_3 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -1, \theta_{fixed} = \frac{1}{2}\frac{3x^2+2x+1}{x^3+x^2+x}$ gives

$$\begin{aligned} d &= (1)(-1) + \left(\frac{-1}{2}\right)(+1) - \left(\frac{-1}{2}\right)(0) - \left(\frac{-1}{2}\right)\left(\frac{\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)}\right) - \left(\frac{-1}{2}\right)\left(\frac{\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)}\right) \\ &= -\frac{7}{6}i\sqrt{3} - \frac{3}{2} \end{aligned}$$

Since not an integer, we try next set $s = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{+1}{2}\right\}$. If we continue this process we will find that set $s = \left\{\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{+1}{2}\right\}$ works and generates

$$\begin{aligned} d &= (1)(-1) + \left(\frac{1}{2}\right)(+1) - \left(\frac{-1}{2}\right)(0) - \left(\frac{-1}{2}\right)\left(\frac{\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}\sqrt{2-30i\sqrt{3}}}{3\left(1+i\sqrt{3}\right)}\right) \\ &= 0 \end{aligned}$$

We can use this d. From Eq (8)

$$\begin{aligned} \Theta &= (n)\left(\theta_{fix}\right) + s_0\theta_0 + s_1\theta_1 + s_2\theta_2 + s_3\theta_3 \\ &= (1)\left(\frac{1}{2}\frac{3x^2+2x+1}{x^3+x^2+x}\right) + \left(\frac{-1}{2}\right)(0) + \left(\frac{-1}{2}\right)(0) + \left(\frac{-1}{2}\right)\left(\frac{-2\sqrt{3}\sqrt{2+30i\sqrt{3}}}{3\left(-1+i\sqrt{3}\right)\left(i\sqrt{3}-2x-1\right)}\right) + \left(\frac{1}{2}\right)\Bigg(\\ &= \frac{1}{2x}\frac{2x^2-2x+1}{x^2+x+1} \end{aligned}$$

Now that we have good trial d and Θ, then step 3 is called to generate $P(x)$ if possible.

Step 3

The input to this step is the integer $d = 0$ and $\Theta = \frac{1}{2x}\frac{2x^2-2x+1}{x^2+x+1}$ found from step 2 and also $r = \frac{10x^2-8x-1}{4x^2(x^2+x+1)^2}$. Since degree $d = 0$, then let $p(x) = 1$. A constant. We need to verify

$$P'' + 2\Theta P' + \left(\Theta' + \Theta^2 - r\right) P = 0$$
$$\Theta' + \Theta^2 - r = 0$$

Substituting gives

$$\frac{d}{dx}\left(\frac{1}{2x}\frac{2x^2-2x+1}{x^2+x+1}\right) + \left(\frac{1}{2x}\frac{2x^2-2x+1}{x^2+x+1}\right)^2 - \frac{10x^2-8x-1}{4x^2\left(x^2+x+1\right)^2} = 0$$
$$0 = 0$$

Verified. The solution is

$$\begin{aligned}
z &= P(x)\, e^{\int \Theta dx} \\
&= e^{\int \frac{1}{2x}\frac{2x^2-2x+1}{x^2+x+1} dx} \\
&= \left(x^2+x+1\right)^{\frac{1}{4}} \sqrt{x}e^{-\frac{7}{6}\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}
\end{aligned}$$

Hence first solution to given ODE is

$$\begin{aligned}
y_1 &= ze^{\frac{-1}{2}\int a dx} \\
&= \left(x^2+x+1\right)^{\frac{1}{4}} \sqrt{x}e^{-\frac{7}{6}\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)} e^{\frac{-1}{2}\int\left(\frac{1}{x^2\left(x^2+x+1\right)}\right)dx} \\
&= \frac{xe^{-\frac{7}{3}\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)}}{\sqrt{x^2+x+4}}
\end{aligned}$$

Second solution y_2 can now be find by reduction of order.

4.1.12 Example 12 case one

Let

$$\left(x^2-2x\right) y'' + \left(2-x^2\right) y' + (2x-2)\, y = 0$$

Normalizing so that coefficient of y'' is one gives

$$y'' + \frac{(2-x^2)}{(x^2-2x)} y' + \frac{(2x-2)}{(x^2-2x)} y = 0$$
$$y'' + ay'(x) + by = 0 \tag{1}$$

Hence

$$a = \frac{(2-x^2)}{(x^2-2x)}$$
$$b = \frac{(2x-2)}{(x^2-2x)}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned}
r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\
&= \frac{1}{4}\left(\frac{(2-x^2)}{(x^2-2x)}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{(2-x^2)}{(x^2-2x)}\right) - \frac{(2x-2)}{(x^2-2x)} \\
&= \frac{x^4-8x^3+24x^2-24x+12}{4x^2\left(x-2\right)^2}
\end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^2 (x-2)^2} z \tag{5}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$\begin{aligned} s &= x^4 - 8x^3 + 24x^2 - 24x + 12 \\ t &= 4x^2(x-2)^2 \end{aligned}$$

The square free factorization of t is $t = [1, x(x-2)]$. Hence

$$m = 2 \tag{6}$$

Since m is number of elements in the free square factorization. in this case we set

$$\begin{aligned} t_1 &= 1 \\ t_2 &= x(x-2) \end{aligned}$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

There is one pole at $x = 0$ of order 2 and one pole at $x = 2$ also of order 2. Looking at the cases table

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	[1]
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	[2]
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	[4, 6, 12]

Shows that only case 1,2 are possible. $L = [1, 2]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right) \end{aligned}$$

Using $O(\infty) = 0, t = 4x^4, t_1 = 1$ the above gives

$$\begin{aligned} e_{fixed} &= \frac{1}{4}(\min(0, 2) - 4 - 3(0)) \\ &= \frac{1}{4}(0 - 4) \\ &= -1 \\ \theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(4x^2(x-2)^2)}{4x^2(x-2)^2} + 3(0)\right) \\ &= \frac{x^2 - 3x + 2}{x(x-2)^2} \\ &= \frac{x-1}{x(x-2)} \end{aligned}$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of $t_2 = x(x-2)$. In other words, the number of poles of r that are of order 2. There are two poles. Hence $k_2 = 2$. These poles c_i where $i = 1, 2$ at $x = \{0, 2\}$. For each c_i then $e_i = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{(x-c_i)^2}$ in the partial fraction expansion of r and $\theta_i = \frac{e_i}{x-c_i}$. The partial fraction expansion of r is

$$\begin{aligned} r &= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^2 (x-2)^2} \\ &= \frac{1}{4} - \frac{3}{4}\frac{1}{x} - \frac{1}{4}\frac{1}{(x-2)} + \frac{3}{4}\frac{1}{(x-2)^2} + \frac{3}{4}\frac{1}{x^2} \end{aligned}$$

The coefficient of $\frac{1}{(x-0)^2}$ where $c_1 = 0$ is first pole is $b_1 = \frac{3}{4}$ from looking at the above. Hence $e_1 = \sqrt{1+4b} = \sqrt{1+4\left(\frac{3}{4}\right)} = 2$ and $\theta_1 = \frac{e_1}{x-c_1} = \frac{2}{x}$. The coefficient of $\frac{1}{(x-c_2)^2}$ where $c_2 = 2$ is second pole is $b_2 = \frac{3}{4}$. Hence $e_2 = \sqrt{1+4b} = \sqrt{1+4\left(\frac{3}{4}\right)} = 2$ and $\theta_2 = \frac{2}{x-c_2} = \frac{2}{x-2}$. Therefore the lists e, θ are

$$\begin{aligned} e &= \{2, 2\} \\ \theta &= \left\{\frac{2}{x}, \frac{2}{x-2}\right\} \end{aligned}$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, k$ if any exist. There are none. This step is skipped.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since $O(\infty) = 0$ here then none of these cases applies. For case 1 ($n = 1$) we first find $[r]_\infty$ the sum of terms x^i for $i = -\frac{v}{2}, \cdots 0$ where v is the $O(\infty)$ which is zero here. Hence

$$v = 0$$

The following is sum of terms from the Laurent series expansion of $\sqrt{r}$ at $x = \infty$ which is

$$[\sqrt{r}]_\infty = \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \cdots$$

We want only terms for $0 \leq i \leq v$ but $v = 0$. Therefore only the constant term. Hence

$$[\sqrt{r}]_\infty = \frac{1}{2}$$

Then a is the coefficient of $x^{-\frac{v}{2}} = x^0$ or constant term. Hence

$$a = \frac{1}{2}$$

And b is the coefficient of $x^{\frac{-v}{2}+1} = x$ in $r - \left([\sqrt{r}]_\infty\right)^2$. This comes out to be

$$b = -1$$

Hence

$$\begin{aligned} e_0 &= \frac{b}{a} = -2 \\ \theta_0 &= 2[\sqrt{r}]_\infty = 1 \end{aligned}$$

Hence now we have

$$\begin{aligned} e &= \{-2, 2, 2\} \\ \theta &= \left\{1, \frac{2}{x}, \frac{2}{x-2}\right\} \end{aligned}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $\underline{n=1}$. And since we have $k_2 = 2$ then there are $(n+1)^{k_2+1} = 2^3 = 8$ sets s to try. The first set s is

$$s = \left\{ \frac{-n}{2}, \frac{-n}{2}, \frac{-n}{2} \right\} = \left\{ \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2} \right\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 2$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 - s_2 e_2 \tag{7}$$

If $d \geq 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -1, \theta_{fixed} = \frac{x^2-3x+2}{x(x-2)^2}$ gives

$$\begin{aligned} d &= (1)(-1) + \left(\frac{-1}{2}\right)(-2) - \left(\frac{-1}{2}\right)(2) - \left(\frac{-1}{2}\right)(2) \\ &= 2 \end{aligned}$$

This will work. Let us find all of the d so to compare with the solution to same ode using original kovacic algorithm given earlier to see if we get same $d's$. We try next set $s = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{-1}{2}\right)(-2) - \left(\frac{-1}{2}\right)(2) - \left(\frac{-1}{2}\right)(2) \\ &= 2 \end{aligned}$$

Trying next set $s = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{+1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{-1}{2}\right)(-2) - \left(\frac{-1}{2}\right)(2) - \left(\frac{+1}{2}\right)(2) \\ &= 0 \end{aligned}$$

Trying next set $s = \left\{\frac{-1}{2}, \frac{+1}{2}, \frac{+1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{-1}{2}\right)(-2) - \left(\frac{+1}{2}\right)(2) - \left(\frac{+1}{2}\right)(2) \\ &= -2 \end{aligned}$$

Trying next set $s = \left\{\frac{+1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{+1}{2}\right)(-2) - \left(\frac{-1}{2}\right)(2) - \left(\frac{-1}{2}\right)(2) \\ &= 0 \end{aligned}$$

Trying next set $s = \left\{\frac{+1}{2}, \frac{-1}{2}, \frac{+1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{+1}{2}\right)(-2) - \left(\frac{-1}{2}\right)(2) - \left(\frac{+1}{2}\right)(2) \\ &= -2 \end{aligned}$$

Trying the next set $s = \left\{\frac{+1}{2}, \frac{+1}{2}, \frac{-1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{+1}{2}\right)(-2) - \left(\frac{+1}{2}\right)(2) - \left(\frac{-1}{2}\right)(2) \\ &= -2 \end{aligned}$$

Trying the next set $s = \left\{\frac{+1}{2}, \frac{+1}{2}, \frac{+1}{2}\right\}$

$$\begin{aligned} d &= (1)(-1) + \left(\frac{+1}{2}\right)(-2) - \left(\frac{+1}{2}\right)(2) - \left(\frac{+1}{2}\right)(2) \\ &= -4 \end{aligned}$$

OK, we have all d values. We now try the ones which are $d \geq 0$ and these are $d - 0, d = 2$. Trying $d = 2$ first which used the set $s = \left\{\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\}$ gives $\left\{1, \frac{2}{x}, \frac{2}{x-2}\right\}$

$$\begin{aligned} \Theta &= (n)(\theta_{fix}) + s_0\theta_0 + s_1\theta_1 + s_{21}\theta_1 \\ &= (1)\left(\frac{x^2 - 3x + 2}{x(x-2)^2}\right) + \left(\frac{-1}{2}\right)(1) + \left(\frac{-1}{2}\right)\left(\frac{2}{x}\right) + \left(\frac{-1}{2}\right)\left(\frac{2}{x-2}\right) \\ &= -\frac{1}{2x}\frac{x^2 - 2}{x - 2} \end{aligned}$$

Now that we have good trial d and Θ, then step 3 is called to generate $P(x)$ if possible.

Step 3

The input to this step is the integer $d = 0$ and $\Theta = -\frac{1}{2x}\frac{x^2-2}{x-2}$ found from step 2 and also $r = \frac{x^4-8x^3+24x^2-24x+12}{4x^2(x-2)^2}$ which comes from $z'' = rz$. Since degree $d = 2$, then let $p(x) = x^2 + a_1x + a_2$. Solving for $p(x)$ from

$$P'' + 2\Theta P' + \left(\Theta' + \Theta^2 - r\right)P = 0$$

gives $p(x) = x^2$ as solution. Hence the solution is

$$\begin{aligned} z &= P(x)\, e^{\int \Theta dx} \\ &= x^2 e^{\int -\frac{1}{2x}\frac{x^2-2}{x-2} dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x-2}} e^{-\frac{x}{2}} \end{aligned}$$

Hence first solution to given ODE is

$$\begin{aligned} y_1 &= z e^{\frac{-1}{2}\int a dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x-2}} e^{-\frac{x}{2}} e^{\frac{-1}{2}\int \frac{(2-x^2)}{(x^2-2x)} dx} \\ &= x^2 \end{aligned}$$

The second solution can be found by reduction of order.

4.1.13 Example 13 case one

Let

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = 0 \tag{1}$$
$$y'' + ay'(x) + by = 0$$

Hence

$$a = \frac{x}{1-x}$$
$$b = -\frac{1}{1-x}$$

It is first transformed to the following ode by eliminating the first derivative

$$z'' = rz \tag{2}$$

Using what is known as the Liouville transformation given by

$$y = ze^{\frac{-1}{2}\int a dx} \tag{3}$$

Where it can be found that r in (2) is given by

$$\begin{aligned} r &= \frac{1}{4}a^2 + \frac{1}{2}a' - b \\ &= \frac{1}{4}\left(\frac{x}{1-x}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{x}{1-x}\right) - \left(-\frac{1}{1-x}\right) \\ &= \frac{x^2-4x+6}{4(x-1)^2} \end{aligned} \tag{4}$$

Hence the DE we will solve using Kovacic algorithm is Eq (2) which is

$$z'' = \frac{x^2-4x+6}{4(x-1)^2}z \tag{5}$$

Therefore

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2-4x+6}{4(x-1)^2} \end{aligned}$$

Step 0 We need to find which case it is. $r = \frac{s}{t}$ where

$$s = x^2 - 4x + 6$$
$$t = 4(x-1)^2$$

The square free factorization of t is $t = [1, (x-1)]$. Hence

$$m = 2 \tag{6}$$

Since m is number of elements in the free square factorization. in this case we set

$$t_1 = 1$$
$$t_2 = (x-1)$$

Now

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

There is one pole at $x = 1$ of order 2. Looking at the cases table

case	allowed pole order for $r = \frac{s}{t}$	allowed $O(\infty)$ order	L
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -8, -6, -4, -2, 0, 2, 3, 4, 5, 6, 7, \cdots\}$	$[1]$
2	$\{2, 3, 5, 7, 9, \cdots\}$	no condition	$[2]$
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$	$[4, 6, 12]$

Shows that only case 1,2 are possible. Hence $L = [1, 2]$.

Step 1

This step has 4 parts (a,b,c,d).

part (a) Here the fixed parts $e_{fixed}, \theta_{fixed}$ are calculated using

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(O(\infty), 2) - \deg(t) - 3\deg(t_1)) \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{t'}{t} + 3\frac{t_1'}{t_1}\right)
\end{aligned}
$$

Using $O(\infty) = 0, t = 4(x-1)^2, t_1 = 1$ the above gives

$$
\begin{aligned}
e_{fixed} &= \frac{1}{4}(\min(0, 2) - 2 - 3(0)) \\
&= \frac{1}{4}(0 - 2) \\
&= -\frac{1}{2} \\
\theta_{fixed} &= \frac{1}{4}\left(\frac{\frac{d}{dx}(4(x-1)^2)}{4(x-1)^2} + 3(0)\right) \\
&= \frac{1}{2x-2}
\end{aligned}
$$

part (b)

Here the values e_i, θ_i are found for $i = 1 \cdots k_2$ where k_2 is the number of roots of $t_2 = (x-1)$. In other words, the number of poles of r that are of order 2. There is one pole of order 2. Hence $k_2 = 1$. For each pole c_i then $e_i = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{(x-c_i)^2}$ in the partial fraction expansion of r and $\theta_i = \frac{e_i}{x-c_i}$. The partial fraction expansion of r is

$$
\frac{x^2 - 4x + 6}{4(x-1)^2} = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2}\frac{1}{x-1}
$$

The coefficient of $\frac{1}{(x-1)^2}$ is $b_1 = \frac{3}{4}$ from looking at the above. Hence $e_1 = \sqrt{1+4b} = \sqrt{1+4\left(\frac{3}{4}\right)} = 2$ and $\theta_1 = \frac{e_1}{x-c_1} = \frac{2}{x-1}$.Therefore the lists e, θ are

$$
\begin{aligned}
e &= \{2\} \\
\theta &= \left\{\frac{2}{x-1}\right\}
\end{aligned}
$$

Part (c)

This part applied only to case 1. It is used to generate e_i, θ_i for poles of r order $4, 6, 8, \cdots, k$ if any exist. There are none. This step is skipped.

Part(d)

Now we need to find e_0, θ_0. If $O(\infty) > 2$ then $e_0 = 1, \theta_0 = 0$. But if $O(\infty) = 2$ then $\theta_0 = 0$ and $e_0 = \sqrt{1+4b}$ where b is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. Since $O(\infty) = 0$ here then none of these cases applies. For case 1 ($n = 1$) we first find $[r]_\infty$ the sum of terms x^i for $i = -\frac{v}{2}, \cdots 0$ where v is $O(\infty)$ which is zero here. Hence $v = 0$. This sum of terms is from the Laurent series expansion of $\sqrt{r}$ at $x = \infty$ which is

$$
[\sqrt{r}]_\infty = \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \cdots
$$

We want only terms for $0 \le i \le v$ but $v = 0$. Therefore only the constant term. Hence

$$\left[\sqrt{r}\right]_\infty = \frac{1}{2}$$

Then a is the coefficient of $x^{-\frac{v}{2}} = x^0$ or constant term. Hence

$$a = \frac{1}{2}$$

And b is the coefficient of $x^{\frac{-v}{2}+1} = x$ in $r - \left(\left[\sqrt{r}\right]_\infty\right)^2$. This comes out to be

$$b = -\frac{1}{2}$$

Therefore

$$e_0 = \frac{b}{a} = \frac{-\frac{1}{2}}{\frac{1}{2}} = -1$$
$$\theta_0 = 2\left[\sqrt{r}\right]_\infty = 1$$

Hence now we have

$$e = \{-1, 2\}$$
$$\theta = \left\{1, \frac{2}{x-1}\right\}$$

The above are arranged such that e_0 is the first entry. Same for θ. This to keep the same notation as in the paper. The above complete step 1, which is to generates the candidate $e's$ and $\theta's$. In step 2, these are used to generate trials d and θ and find from them $P(x)$ polynomial if possible.

Step 2

In this step, we now have all the e_i, θ_i values found above in addition to e_{fix}, θ_{fix}.

Starting with $n = 1$. And since we have $k_2 = 1$ then there are $(n+1)^{k_2+1} = 2^2 = 4$ sets s to try. The first set s is

$$s = \left\{\frac{-n}{2}, \frac{-n}{2}\right\} = \left\{\frac{-1}{2}, \frac{-1}{2}\right\}$$

Now we generate trial d using

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - \sum_{i=1}^{k_2} s_i e_i$$

Since $k_2 = 1$ then the above becomes

$$d = (n)\left(e_{fix}\right) + s_0 e_0 - s_1 e_1 \tag{7}$$

If $d \ge 0$ then we go and find a trial Θ. We need to have both d, Θ to go to the next step. Θ is found using

$$\Theta = (n)\left(\theta_{fix}\right) + \sum_{i=0}^{k_2} s_i \theta_i \tag{8}$$

Hence the first trial d is (using Eq (7)) and recalling that $e_{fix} = -\frac{1}{2}, \theta_{fixed} = \frac{1}{2x-2}$ gives using set $\left\{\frac{-1}{2}, \frac{-1}{2}\right\}$

$$\begin{aligned} d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(-1) - \left(\frac{-1}{2}\right)(2) \\ &= 1 \end{aligned}$$

This will work. The corresponding Θ is from (8)

$$
\begin{aligned}
\Theta &= (1)\left(\frac{1}{2x-2}\right) + s_0\theta_0 + s_1\theta_1 \\
&= \frac{1}{2x-2} - \frac{1}{2}(1) - \frac{1}{2}\frac{2}{x-1} \\
&= -\frac{1}{2}\frac{x}{x-1}
\end{aligned}
$$

Let us find all of the d and Θ so to compare with the solution to same ode using original kovacic algorithm given earlier to see if we get same $d's$. We try next set $s = \left\{\frac{-1}{2}, \frac{+1}{2}\right\}$

$$
\begin{aligned}
d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{-1}{2}\right)(-1) - \left(\frac{+1}{2}\right)(2) \\
&= -1
\end{aligned}
$$

We skip this d since negative. Next is $s = \left\{\frac{+1}{2}, \frac{-1}{2}\right\}$

$$
\begin{aligned}
d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)(-1) - \left(\frac{-1}{2}\right)(2) \\
&= 0
\end{aligned}
$$

The corresponding Θ is from (8)

$$
\begin{aligned}
\Theta &= (1)\left(\frac{1}{2x-2}\right) + s_0\theta_0 + s_1\theta_1 \\
&= \frac{1}{2x-2} + \frac{1}{2}(1) - \frac{1}{2}\frac{2}{x-1} \\
&= \frac{x-2}{2\,(x-1)}
\end{aligned}
$$

The next set is $\left\{\frac{+1}{2}, \frac{+1}{2}\right\}$

$$
\begin{aligned}
d &= (1)\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)(-1) - \left(\frac{1}{2}\right)(2) \\
&= -2
\end{aligned}
$$

OK, we have all d values. We now try the ones which are $d \geq 0$ and these are $d = 0, d = 1$. Let us use $d = 1$ case. Now that we have good trial d and Θ, then step 3 is called to generate $P(x)$ if possible.

Step 3

The input to this step is the integer $d = 1$ and $\Theta = -\frac{1}{2}\frac{x}{x-1}$ found from step 2 and also $r = \frac{x^2-4x+6}{4(x-1)^2}$ which comes from $z'' = rz$. Since degree $d = 1$, then let $p(x) = x + a$. Solving for $p(x)$ from

$$
P'' + 2\Theta P' + \left(\Theta' + \Theta^2 - r\right) P = 0
$$

gives $p(x) = x$ as solution. Hence the solution is

$$
\begin{aligned}
z &= P(x)\, e^{\int \Theta dx} \\
&= x e^{\int -\frac{1}{2}\frac{x}{x-1} dx} \\
&= x\frac{e^{-\frac{1}{2}x}}{\sqrt{x-1}}
\end{aligned}
$$

Hence first solution to given ODE is

$$
\begin{aligned}
y_1 &= x\frac{e^{-\frac{1}{2}x}}{\sqrt{x-1}} e^{\frac{-1}{2}\int a dx} \\
&= x\frac{e^{-\frac{1}{2}x}}{\sqrt{x-1}} e^{\frac{-1}{2}\int \frac{x}{1-x} dx} \\
&= x
\end{aligned}
$$

The second solution can be found by reduction of order.

5 Notation mapping between Saunders/Smith algorithm and original Kovacis algorithm

I have implemented the original Kovacis algorithm using Maple 2021 based on the original paper (1). The following are notation difference between the two algorithms and the implementation by Smith [3] that I found.

1. Kovacis algorithm uses $\alpha_{\infty}^{\pm}$ defined as $\alpha_{\infty}^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$ for the case when $O(\infty) = 2$. Smith algorithm uses e_0 for the $\sqrt{1+4b}$ part only. In both algorithms the b value is calculated in the same way. It is the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. But we do not need to find Laurent series expansion of r at ∞ to find b here. It can be found using $b = \frac{lcoeff(s)}{lcoeff(t)}$ where $r = \frac{s}{t}$ and $\gcd(s,t) = 1$.

2. Smith algorithm finds $e_1, e_2, \cdots$ values for each pole. This is part b of step 1 for poles of order 2, these correspond to only the $\sqrt{1+4b}$ part in Kovacis algorithm (this is part c2 of step1), where there it finds $\left[\sqrt{r}\right]_c$ for each pole and $\alpha_c^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4b}$ where b is the coefficient of $\frac{1}{(x-c)^2}$ in the partial fraction decomposition of r. This b value is also the same for Smith algorithm in its $e's$.

More mappings to be added next.

6 References

1. An Algorithm for Solving Second Order Linear Homogeneous Differential Equations (1985 version). By JERALD J. KOVACIC.

2. An Implementation of Kovacic's Algorithm for Solving Second Order Linear Homogeneous Differential Equations. By David Saunders.

3. A DISCUSSION AND IMPLEMENTATION OF KOVACICS ALGORITHM FOR ORDINARY DIFFERENTIAL EQUATIONS. By Carolyn J. Smith.