Finding image forward projection and its transpose matrix

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Problem

Write the matrix which implements the forward projection and its transpose.

A simple case would be to consider a 2-D object made up of only 4 pixels and one projection. After that think about an object with many pixels and many projections.

Answer

I will use the convention used by the radon transform in Matlab in setting up the coordinates system which is as shown below (diagram from Matlab documentation page).

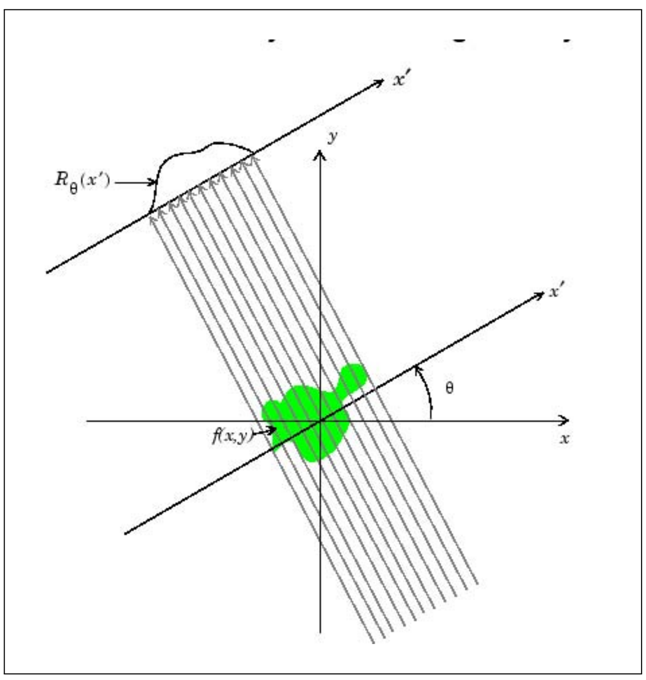

Figure 1: radon transform convention

In our case, we need to perform the following projection, which is at angle $\theta = -90^0$ as follows

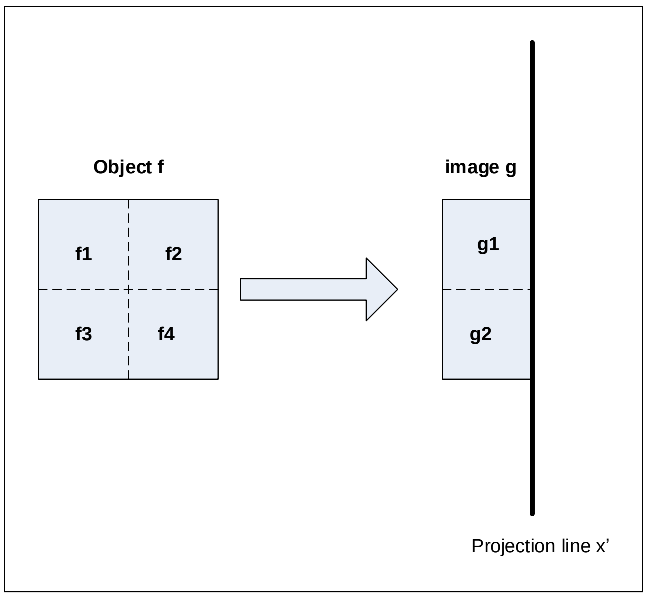

Figure 2: Projection at 90 degrees angle

The equation for the above mapping is $g = Hf$, hence we write

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

Hence

$$g_1 = h_{11}f_1 + h_{12}f_2 + h_{13}f_3 + h_{14}f_4$$
$$g_2 = h_{21}f_1 + h_{22}f_2 + h_{23}f_3 + h_{24}f_4$$

But $g_1 = f_1 + f_2$ from the line integral at the above projection and $g_2 = f_3 + f_4$, hence the above 2 equations becomes

$$f_1 + f_2 = h_{11}f_1 + h_{12}f_2 + h_{13}f_3 + h_{14}f_4$$
$$f_3 + f_4 = h_{21}f_1 + h_{22}f_2 + h_{23}f_3 + h_{24}f_4$$

By comparing coefficients on the LHS and RHS for each of the above equations, we see that for the first equation we obtain

$$h_{11} = 1, h_{12} = 1, h_{13} = 0, h_{14} = 0$$

For the second equation we obtain

$$h_{21} = 0, h_{22} = 0, h_{23} = 1, h_{24} = 1$$

Hence the H matrix is

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Taking the transpose

$$H^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Hence if we apply H^T operator onto the image g, we obtain back a 2×2 image, which is written as

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}$$

Hence $k_1 = g_1, k_2 = g_1, k_3 = g_2, k_4 = g_2$. In other words, the image is a 4 pixels $\begin{bmatrix} g_1 & g_1 \\ g_2 & g_2 \end{bmatrix}$

H^T can now be viewed as back projecting the image g into a plane by smearing each pixel g_i value over the plane along the line of sight as illustrated below

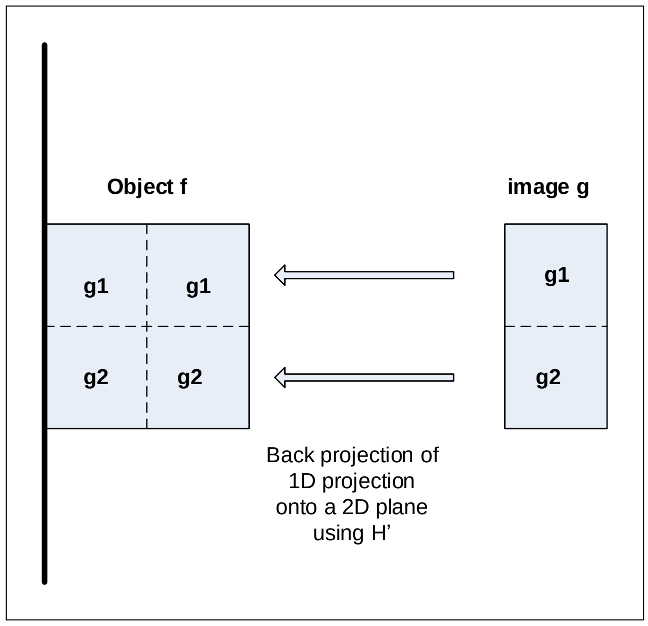

Figure 3: back projection

1 Case of 45 degree

We repeat the above for $\theta = 45^0$

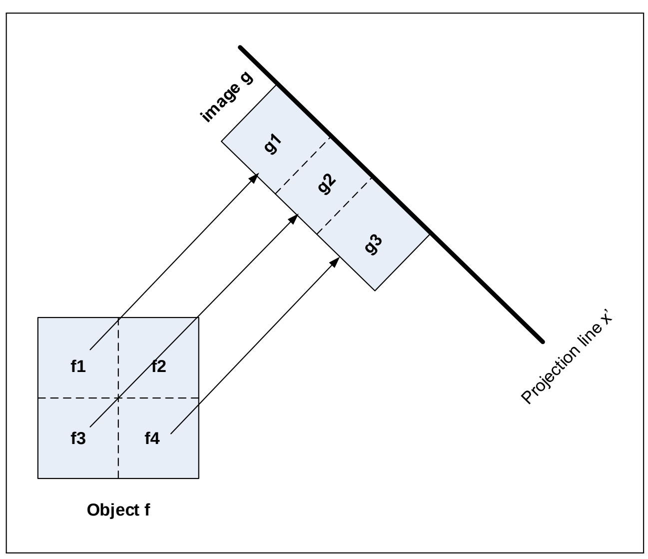

Figure 4: back projection at 45 degrees

The equation for the above mapping is $g = Hf$, hence we write

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

Therefore

$$g_1 = h_{11}f_1 + h_{12}f_2 + h_{13}f_3 + h_{14}f_4$$
$$g_2 = h_{21}f_1 + h_{22}f_2 + h_{23}f_3 + h_{24}f_4$$
$$g_3 = h_{31}f_1 + h_{32}f_2 + h_{33}f_3 + h_{34}f_4$$

We see from projection diagram that $f_1 = g_1, f_3 + f_2 = g_2$ and $f_4 = g_3$, hence the above 3 equations become

$$\begin{aligned} f_1 &= h_{11}f_1 + h_{12}f_2 + h_{13}f_3 + h_{14}f_4 \\ f_3 + f_2 &= h_{21}f_1 + h_{22}f_2 + h_{23}f_3 + h_{24}f_4 \\ f_4 &= h_{31}f_1 + h_{32}f_2 + h_{33}f_3 + h_{34}f_4 \end{aligned}$$

By comparing coefficients, we obtain from the first equation $h_{11} = 1, h_{12} = 0, h_{13} = 0, h_{14} = 0$ and from the second equation $h_{21} = 0, h_{22} = 1, h_{23} = 1, h_{24} = 0$ and from the last equation $h_{31} = 0, h_{32} = 0, h_{33} = 0, h_{34} = 1$. Hence the H matrix is

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using H^T to project the image g we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}$$

Hence $k_1 = g_1, k_2 = g_2, k_3 = g_2, k_4 = g_3$, hence the back projection plane is

$$K = \begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix}$$

This also can be interpreted as back projecting the image g on a 45^0 onto a plane by smearing each pixel value g_i on each pixel along its line of sight as illustrated below

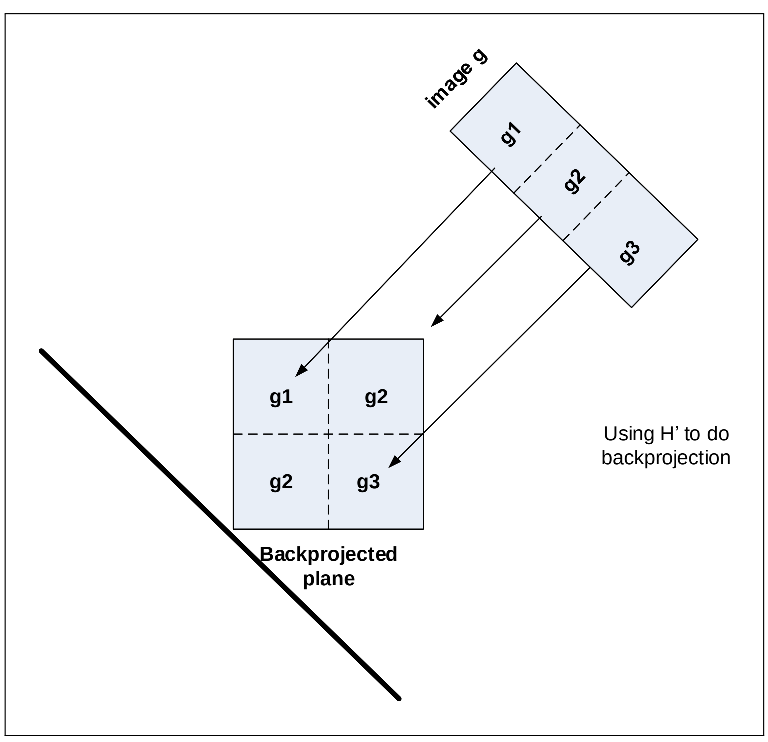

Figure 5: back projection at 45 degrees