The Mathematics of HYPR

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imagination at work

Introduction

A promising new method for accelerated imaging, HighlY constrained backPRojection (HYPR) [1] uses the spatial information of a time averaged image (composite) to improve images at individual time frames. The original HYPR algorithm was derived heuristically.

The HW-HYPR method [2] modified HYPR so that the method converges to the composite when the time frame uses all projections. The HYPR approach has also been extended to an iterative method (I-HYPR) [3] in which that version of HYPR was identified by O'Halloran as the expectation maximization (EM) algorithm.

The objective of this work is to derive the original HYPR algorithm and HW-HYPR as the first step of iterative methods based on maximum likelihood estimation and fixed-point iteration.

Mathematical Background

Data Model:

$$\mathbf{g} = \mathrm{H}\mathbf{f} + \mathbf{n}$$

g: Data vector (in radial acquisitions, it would be the projections)
H: Imaging operator which maps the discrete object to the discrete projections in radial acquisition but can be more general (**see poster # 2834**)
f: Discrete object
n: Noise vector

Maximum Likelihood Estimation (MLE):

• Choose **f** such that P(**g**|**f**) is maximized, i.e. choose the object that maximizes the probability (likelihood) of the observed data.
• Finding the MLE requires choosing a noise model (for example, Gaussian or Poisson) and solving a large system of equations.

Fixed Point Iteration (FPI):

• Large systems of equation of the form **b=Ax** can be solved by iterating until a fixed point ($\mathbf{x}^{n+1} = \mathbf{x}^n$) is reached, for example:

$$\mathbf{x}_k^{n+1} = \mathbf{x}_k^n \left(\frac{\mathbf{b}}{\mathrm{A}\mathbf{x}^n} \right)_k$$

where k is the index for each vector element (voxel) and n is the iteration number.

Results

• MLE (with Poisson noise) and FPI leads to the Maximum Likelihood Expectation Maximization (MLEM) algorithm [4,5] commonly used for limited angle tomography in nuclear medicine:

$$f_k^{n+1} = f_k^n \frac{1}{s_k} \left(\mathrm{H}^t \left[\frac{\mathbf{g}}{\mathrm{H}\mathbf{f}^n} \right] \right)_k$$

where s_k is the a scaling constant that depends on the pixel k and H^t is the transpose of the imaging operator (in a radial acquisition, the backprojection operation).

• **The original HYPR algorithm is approximately the first step of the MLEM algorithm with the composite image as the initial guess.**

• The approximation is that the multiplicative updates (weights of the initial guess) are the same:

$$\frac{1}{N_p} \left[\sum_{i=1}^{N_p} \frac{\mathrm{H}_i^t(\mathbf{g}_i)}{\mathrm{H}_i^t([\mathrm{H}\mathbf{f}]_i)} \right]_k = \frac{1}{s_k} \left[\mathrm{H}^t \left(\frac{\mathbf{g}}{\mathrm{H}\mathbf{f}} \right) \right]_k$$

where N_p is the number of projections per time frame and the index i refers to the individual projections. Up to a normalization constant this equation implies that the ratio of backprojections is the backprojection of the ratio. O'Halloran et al. [3] identify I-HYPR as the EM algorithm with a formulation of HYPR as the backprojection of the ratio.

• Figure 1 shows that the approximation is very good for a disk. Figure 2 shows subtle differences in the multiplicative updates of both methods for the same simulation as Figure 1 .

•MLE (with Gaussian noise) and FPI applied to the normal equation (**$\mathbf{H^t g = H^t H f}$**) leads to the Multiplicative Arithmetic Reconstruction Technique (MART) [6]:

$$f_k^{n+1} = f_k^n \left(\frac{\mathrm{H}^t \mathbf{g}}{\mathrm{H}^t \mathrm{H} \mathbf{f}^n} \right)_k$$

• **HW-HYPR is the first step of MART applied to the normal equation with the composite image as the initial guess.**

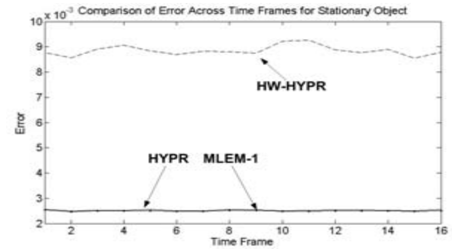

Fig. 1 The noise-free simulation of a stationary disk with 4 projections shows HYPR and the first step of MLEM (using composite as the initial guess) are the same but HW-HYPR is different. The lines for HYPR and MLEM-1 overlap and all methods have small errors.

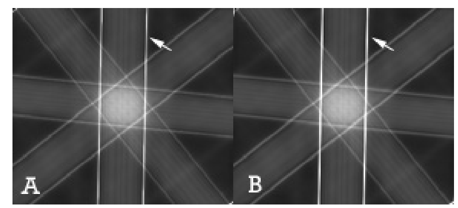

Fig. 2 Comparison of multiplicative update (weighting image) for HYPR (A) and first step of MLEM (B). There are subtle differences outside of the object support as shown by the arrow.

Discussion

• The connection of HYPR with MLEM (for Poisson noise) and HW-HYPR with MART (for Gaussian noise) helps to understand these algorithms within the context of algorithms previously studied in other imaging modalities.

• The success of these new algorithms can be understood in terms of the prior information they utilize, i.e. the compact support given by the composite image and the positivity constraint (estimates with a non-negative initial guess remain non-negative) for the radial acquisitions.

• The iterative reconstruction beyond the first step further enforces agreement between the object and the data leading to a trade-off between data agreement and noise amplification.

References: 1. Mistretta et al,MRM,2006;55: 30-40. **2**. Huang et al,MRM,2007;58 :316-325. **3**. O'Halloran et al,MRM,2008;59 :132-139. **4.** Barrett and Myers,Foundations of Image Science,2004. **5.** Shepp et al,IEEE TMI,1982;MI-1:113-122. **6.** Gordon et al,J Theor Biol, 1970; 29:471-481.