Solving first and second order linear differential equations using power series method

Ordinary and Regular singular point using Frobenius method

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Contents

1 First order differential equations

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1.1 Algorithm charts

1.1.1 First chart

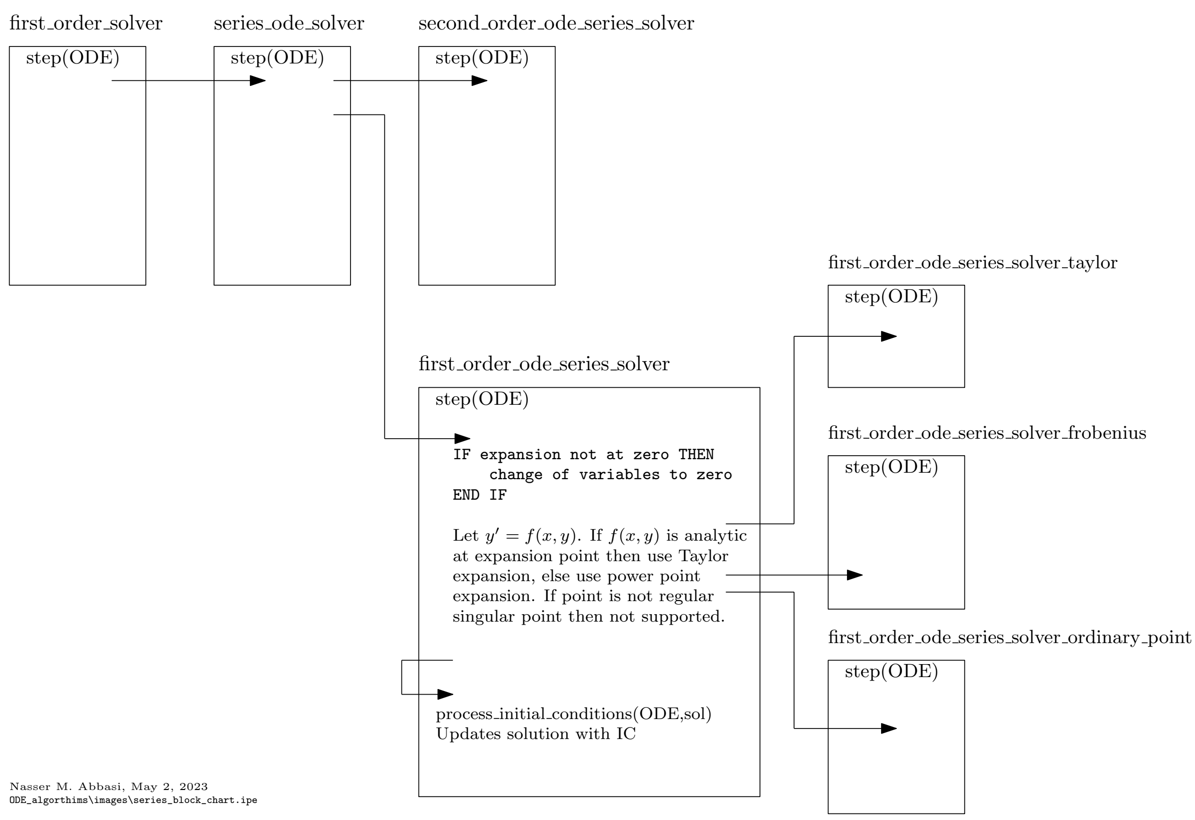

Figure 1: Flow chart for series solution for first order

1.1.2 Second chart

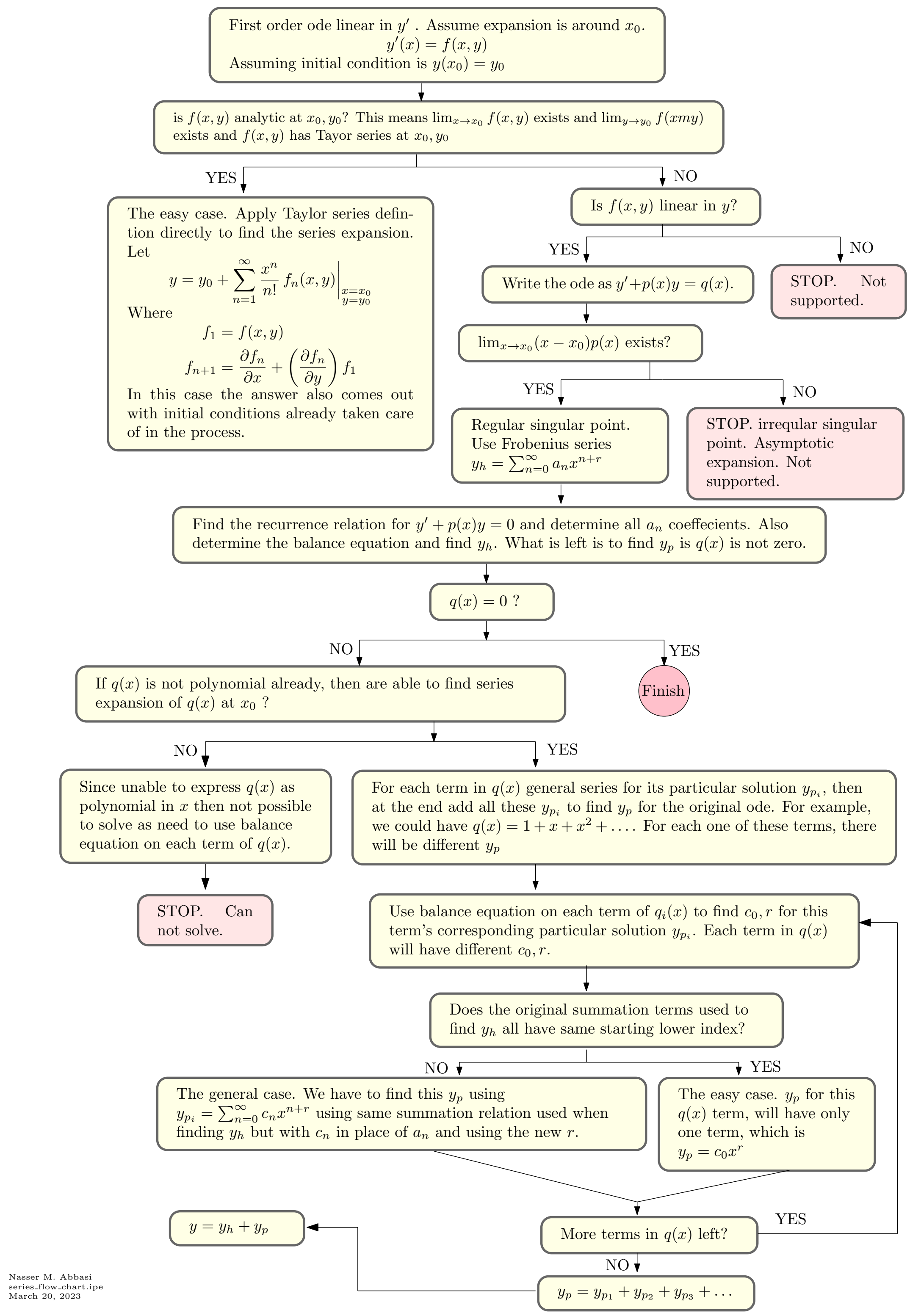

Figure 2: Algorithm for series solution for first orde

1.2 Ordinary point using Taylor series method and power series method

Local contents

1.2.1 Taylor series algorithm

Let

$$y' = f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\ \frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\ \frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\ &\vdots \end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned} F_0 &= f(x,y) && (4) \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5) \end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\ &= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

1.2.2 Examples for ordinary point

Local contents

1.2.2.1 Example 1 $y' + 2xy = x$

$$y' + 2xy = x$$

Solved using power series

Expansion is around $x = 0$. The (homogeneous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x) = 2x$ is defined as is at $x = 0$. Hence this is an ordinary point. also the RHS has series expansion at $x = 0$. It is very important to check that the RHS has series expansion at $x = 0$ otherwise this method will fail and we must use Frobenius even if $x = 0$ is ordinary point. For example for the ode $y' + 2xy = \sqrt{x}$ or the ode $y' + 2xy = \frac{1}{x}$ the standard power series will fail. See examples below. Using standard power series, let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned}$$

The ode now becomes

$$\begin{aligned} \sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n &= x \\ \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} &= x \end{aligned}$$

Reindex so that all powers on x are n gives

$$\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=1}^{\infty} 2 a_{n-1} x^n = x$$

For $n = 0$, for balance we see the RHS has no x^0, which results in

$$a_1 = 0$$

For $n = 1$, for balance we see the RHS has x^1, which results in

$$\begin{aligned} (n+1)\, a_{n+1} + 2a_{n-1} &= 1 \\ 2a_2 + 2a_0 &= 1 \\ a_2 &= \frac{1 - 2a_0}{2} \end{aligned}$$

For $n \geq 2$, for balance, since there are no more terms on right side, then we have the recursive relation

$$\begin{aligned} (n+1)\, a_{n+1} + 2a_{n-1} &= 0 \\ a_{n+1} &= \frac{-a_{n-1}}{(n+1)} \end{aligned}$$

For $n = 2$ we find

$$\begin{aligned} a_3 &= \frac{-2a_1}{3} \\ &= 0 \end{aligned}$$

For $n = 3$

$$\begin{aligned} a_4 &= \frac{-2a_2}{4} \\ &= -\frac{1}{2}\left(\frac{1 - 2a_0}{2}\right) \\ &= \frac{2a_0 - 1}{4} \end{aligned}$$

And so on. Hence we obtain

$$
\begin{aligned}
y_h &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= a_0 + \left(\frac{1-2a_0}{2}\right) x^2 + \left(\frac{2a_0 - 1}{4}\right) x^4 + \cdots \\
&= a_0 \left(1 - x^2 + \frac{1}{2} x^4 + \cdots\right) + \left(\frac{1}{2} x^2 - \frac{1}{4} x^4 + \cdots\right) \\
&= y(0) \left(1 - x^2 + \frac{1}{2} x^4 + \cdots\right) + \left(\frac{1}{2} x^2 - \frac{1}{4} x^4 + \cdots\right)
\end{aligned}
$$

Solved using Taylor series

$$
\begin{aligned}
y' + 2xy &= x \\
y' &= x - 2xy \\
&= f(x, y)
\end{aligned}
$$

For this method to work, $f(x,y)$ must be analytic at $x = x_0$, the expansion point. Let expansion point be $x = 0$. Let $y(0) = y_0$. Then

$$
y = y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x, y)\big|_{x=0, y_0}
$$

Where $F_0 = f(x,y)$ and $F_n = \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0$. Hence

$$
\begin{aligned}
F_0 &= (x - 2xy) \\
F_1 &= \frac{d}{dx} F_0 \\
&= \left(\frac{\partial F_0}{\partial x}\right) + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \left(\frac{\partial (x - 2xy)}{\partial x}\right) + \left(\frac{\partial (x - 2xy)}{\partial y}\right) (x - 2xy) \\
&= (1 - 2y) - 2x(x - 2xy) \\
&= 4x^2 y - 2y - 2x^2 + 1 \\
F_2 &= \frac{d^2}{dx^2} F_1 \\
&= \left(\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \left(\frac{\partial}{\partial x} (4x^2 y - 2y - 2x^2 + 1)\right) + \left(\frac{\partial}{\partial y} 4x^2 y - 2y - 2x^2 + 1\right) (x - 2xy) \\
&= (8xy - 4x) + \left(4x^2 - 2\right)(x - 2xy) \\
&= 12xy - 8x^3 y - 6x + 4x^3 \\
F_3 &= \frac{d^3}{dx^3} F_2 \\
&= \left(\frac{\partial F_2}{\partial x}\right) + \left(\frac{\partial F_2}{\partial y}\right) F_0 \\
&= \left(\frac{\partial}{\partial x} \left(12xy - 8x^3 y - 6x + 4x^3\right)\right) + \left(\frac{\partial}{\partial y} \left(12xy - 8x^3 y - 6x + 4x^3\right)\right) (x - 2xy) \\
&= 12y - 24x^2 y - 6 + 12x^2 + \left(12x - 8x^3\right)(x - 2xy) \\
&= 12y - 48x^2 y + 16x^4 y + 24x^2 - 8x^4 - 6
\end{aligned}
$$

And so on. Evaluating the above at $x = 0, y = y_0$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y_0 + 1 \\
F_2 &= 0 \\
F_3 &= 12y_0 - 6
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0} \\
&= y_0 + xF_0 + \frac{x^2}{2}F_1 + \frac{x^3}{6}F_2 + \frac{x^4}{24}F_3 + \cdots \\
&= y_0 + 0 + \frac{x^2}{2}(-2y_0+1) + 0 + \frac{x^4}{24}(12y_0 - 6) + \cdots \\
&= y_0 - 2y_0\frac{x^2}{2} + \frac{x^2}{2} + \frac{1}{2}y_0x^4 - \frac{x^4}{4} + \\
&= y_0\left(1 - x^2 + \frac{1}{2}x^4\right) + \frac{x^2}{2} - \frac{x^4}{4} + \cdots
\end{aligned}
$$

1.2.2.2 Example 2 $y' + 2xy = 1 + x + x^2$

Solved using Taylor series

Another example using Taylor series method.

$$
\begin{aligned}
y' + 2xy &= 1 + x + x^2 \\
y' &= 1 + x + x^2 - 2xy \\
&= f(x,y)
\end{aligned}
$$

Let expansion point be $x = 0$. Let $y(0) = y_0$. Then

$$
y = y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0}
$$

Where $F_0 = f(x,y)$ and $F_n = \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0$. Hence

$$
\begin{aligned}
F_0 &= 1 + x + x^2 - 2xy \\
F_1 &= \left(\frac{\partial F_0}{\partial x}\right) + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= 1 + 2x - 2y + (-2x)\left(1 + x + x^2 - 2xy\right) \\
&= 4x^2y - 2y - 2x^2 - 2x^3 + 1 \\
F_2 &= \left(\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \left(8xy - 4x - 6x^2\right) + \left(4x^2 - 2\right)(x - 2xy) \\
&= 12xy - 8x^3y - 6x - 6x^2 + 4x^3 \\
F_3 &= \left(\frac{\partial F_2}{\partial x}\right) + \left(\frac{\partial F_2}{\partial y}\right) F_0 \\
&= 12y - 24x^2y - 6 - 12x + 12x^2 + \left(12x - 8x^3\right)\left(1 + x + x^2 - 2xy\right) \\
&= 12y - 48x^2y + 16x^4y + 24x^2 + 4x^3 - 8x^4 - 8x^5 - 6
\end{aligned}
$$

And so on. Evaluating the above at $x = 0, y = y_0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -2y_0 + 1 \\
F_2 &= 0 \\
F_3 &= 12y_0 - 6
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x,y)\big|_{x=0,y_0} \\
&= y_0 + F_0x + F_1\frac{x^2}{2} + F_2\frac{x^3}{6} + F_3\frac{x^4}{24} + \cdots \\
&= y_0 + x + (-2y_0 + 1)\frac{x^2}{2} + (12y_0 - 6)\frac{x^4}{24} + \cdots \\
&= y_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + \left(x + \frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots\right)
\end{aligned}
$$

1.2.2.3 Example 3 $y' + 2xy^2 = 1 + x + x^2$

Solved using Taylor series

$$
\begin{aligned}
y' + 2xy^2 &= 1 + x + x^2 \\
y' &= 1 + x + x^2 - 2xy^2 \\
&= f(x, y)
\end{aligned}
$$

Let expansion point be $x = 0$. Let $y(0) = y_0$. Then

$$
y = y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x, y)\big|_{x=0, y_0}
$$

Where $F_0 = f(x, y)$ and $F_n = \frac{\partial F_{n-1}}{\partial x} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0$. Hence

$$
\begin{aligned}
F_0 &= 1 + x + x^2 - 2xy^2 \\
F_1 &= \left(1 + 2x - 2y^2\right) + (-4xy)\left(1 + x + x^2 - 2xy^2\right) \\
&= -4x^3y + 8x^2y^3 - 4x^2y - 4xy + 2x - 2y^2 + 1 \\
F_2 &= \left(\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \left(-12x^2y + 16xy^3 - 8xy - 4y + 2\right) + \left(-4x^3 + 24x^2y^2 - 4x^2 - 4x - 4y\right)\left(1 + x + x^2 - 2xy^2\right) \\
&= -4x^5 + 32x^4y^2 - 8x^4 - 48x^3y^4 + 32x^3y^2 - 12x^3 + 32x^2y^2 - 16x^2y - 8x^2 + 24xy^3 - 12xy - 4x - 8y + 2 \\
F_3 &= \left(\frac{\partial F_2}{\partial x}\right) + \left(\frac{\partial F_2}{\partial y}\right) F_0
\end{aligned}
$$

And so on. Evaluating the above at $x = 0, y = y_0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= -2y_0^2 + 1 \\
F_2 &= -8y_0 + 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= y(0) + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n(x, y)\big|_{x=0, y_0} \\
&= y_0 + F_0 x + F_1 \frac{x^2}{2} + F_2 \frac{x^3}{6} + F_3 \frac{x^4}{24} + \cdots \\
&= y_0 + x + \left(-2y_0^2 + 1\right) \frac{x^2}{2} + (-8y_0 + 2) \frac{x^3}{6} + \cdots \\
&= y_0 \left(1 - \frac{4}{3}x^3 + \cdots\right) + y_0^2\left(-x^2 + \cdots\right) + \cdots + \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)
\end{aligned}
$$

1.2.2.4 Example 4 $y' + y = \sin x$

Solved using power series

$$
y' + y = \sin x
$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x)$ is defined as is at $x = 0$. Hence this is ordinary point, also the RHS has series expansion at $x = 0$.

Let $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$. The ode becomes

$$
\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sin x
$$

Indexing so all powers of x start at n gives

$$
\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sin x
$$

Expanding $\sin x$ in series gives

$$
\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots
$$

For $n = 0$, there is no term on RHS with x^0, hence we obtain

$$
\begin{aligned}
a_1 + a_0 &= 0 \\
a_1 &= -a_0
\end{aligned}
$$

For $n=1$ there is one term x^1 on RHS, hence

$$
\begin{aligned}
2a_2 + a_1 &= 1\\
a_2 &= \frac{1-a_1}{2} = \frac{1+a_0}{2}
\end{aligned}
$$

For $n=2$ there is no term on RHS with x^2 hence

$$
\begin{aligned}
3a_3 + a_2 &= 0\\
a_3 &= -\frac{a_2}{3} = -\frac{\frac{1+a_0}{2}}{3} = -\frac{1}{6}a_0 - \frac{1}{6}
\end{aligned}
$$

For $n=3$ there is term $-\frac{1}{6}x^3$ on RHS, hence

$$
\begin{aligned}
4a_4 + a_3 &= -\frac{1}{6}\\
a_4 &= \frac{-\frac{1}{6} - a_3}{4} = \frac{-\frac{1}{6} - \left(-\frac{1}{6}a_0 - \frac{1}{6}\right)}{4} = \frac{1}{24}a_0
\end{aligned}
$$

And so on. The solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n\\
&= a_0 + a_1 x + a_2 x^2 + \cdots\\
&= a_0 - a_0 x + \left(\frac{1+a_0}{2}\right)x^2 + \left(-\frac{1}{6}a_0 - \frac{1}{6}\right)x^3 + \left(\frac{1}{24}a_0\right)x^4 + \cdots\\
&= a_0\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \cdots\right) + \left(\frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots\right)
\end{aligned}
$$

1.2.2.5 Example 5. $y' + xy = x^2$

$$
y' + xy = x^2 \tag{1}
$$

Looking first at $y' + xy = 0$. Expansion around $x = 0$. This is an ordinary point. Hence standard power series will be used. let $y = \sum_{n=0}^{\infty} a_n x^n$. Substituting into the ode gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n &= 0\\
\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} &= 0
\end{aligned}
$$

We always want to power on x be the same in each sum and be x^n. Adjusting gives

$$
\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \tag{3A}
$$

For $n=0$ we have

$$
a_1 x^0 = x^0
$$

Hence $a_1 = 1$. The recurrence relation is for $n > 0$. From (3A) we have

$$
\begin{aligned}
(n+1)\, a_{n+1} x^n + a_{n-1} x^n &= 0\\
((n+1)\, a_{n+1} + a_{n-1})\, x^n &= 0\\
(n+1)\, a_{n+1} + a_{n-1} &= 0
\end{aligned}
$$

For $n=1$

$$
\begin{aligned}
2a_2 + a_0 &= 0\\
a_2 &= -\frac{a_0}{2}
\end{aligned}
$$

For $n=2$

$$
\begin{aligned}
3a_3 + a_1 &= 0\\
a_3 &= 0
\end{aligned}
$$

For $n=3$

$$
\begin{aligned}
4a_4 + a_2 &= 0\\
a_4 &= -\frac{a_2}{4}\\
&= \frac{a_0}{8}
\end{aligned}
$$

And so on. Hence

$$\begin{aligned}
y_h &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
&= a_0 - \frac{a_0}{2} x^2 + \frac{a_0}{8} x^4 - \frac{a_0}{48} x^6 - \cdots \\
&= a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{48} x^6 - \cdots\right)
\end{aligned}$$

Now we find y_p. Let

$$y = y_h + y_p$$

Let us start by trying the undetermined coefficients method. We will see this will fail. Let $y_p = c_2 x^2 + c_1 x + c_0$. Substituting this into the ode gives

$$\begin{aligned}
\left(c_2 x^2 + c_1 x + c_0\right)' + x\left(c_2 x^2 + c_1 x + c_0\right) &= x^2 \\
2c_2 x + c_1 + c_2 x^3 + c_1 x^2 + c_0 x &= x^2 \\
c_1 + x(2c_2 + c_0) + x^2(c_1) + x^3(c_2) &= x^2
\end{aligned}$$

Hence $c_1 = 1, c_2 = 0, c_1 = 0$. We see this did not work. We get both $c_1 = 0$ and $c_1 = 1$. What went wrong? The problem is that we used undetermined coefficients on an ode with non constant coefficients. And this is a no-no. Undetermined coefficients can sometimes work on ode with non constant coefficients but by chance as we see in earlier examples.

In solving an ode not the series method and if the ode have non constant coefficients, we should also not use undetermined coefficient but use the variation of parameters method which works on constant and non constant coefficients. We can not use variation of parameters here, since we are solving using series and do not have basis functions for integration.

So what to do now? How to find y_p? We use the same balance equation method as was done in earlier examples for singular point. This example was added here to show that undetermined coefficients can fail when finding y_p even on ordinary point. Assuming y_p is

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^n \\
y_p' &= \sum_{n=0}^{\infty} n c_n x^{n-1}
\end{aligned}$$

Substituting this into the ode gives

$$\begin{aligned}
\sum_{n=0}^{\infty} n c_n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n &= x^2 \\
\sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} &= x^2 \\
\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} &= x^2 \\
\sum_{n=0}^{\infty} (n+1)\, c_{n+1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n &= x^2
\end{aligned}$$

And the above is what will be used to obtain the c_n and y_p. For $n = 0$

$$c_1 x^0 = x^2$$

No balance. Hence $c_1 = 0$. For $n = 1$

$$\begin{aligned}
2c_2 x + c_0 x &= x^2 \\
(2c_2 + c_0)\, x &= x^2
\end{aligned}$$

No balance. Hence $2c_2 + c_0 = 0$. Here, we are free to choose $c_0 = 0$. Therefore $c_2 = 0$. For $n = 2$

$$\begin{aligned}
3c_3 x^2 + c_1 x^2 &= x^2 \\
(3c_3 + c_1)\, x^2 &= x^2
\end{aligned}$$

Balance exist. Hence $3c_3 + c_1 = 1$. But $c_1 = 0$, which gives $c_3 = \frac{1}{3}$. For $n = 3$

$$4c_4 x^3 + c_2 x^3 = x^2$$

No balance. Hence $4c_4 + c_2 = 0$ But $c_2 = 0$ then $c_4 = 0$. For $n = 4$

$$5c_5 x^4 + c_3 x^4 = x^2$$

No balance. Hence $5c_5 + c_3 = 0$ or $c_5 = -\frac{1}{15}$. For $n = 5$

$$6c_6x^6 + c_4x^6 = x^2$$

No balance. Hence $6c_6 + c_4 = 0$. But $c_4 = 0$, hence $c_6 = 0$. For $n = 6$

$$7c_7x^6 + c_5x^6 = x^2$$

No balance. Hence $7c_7 + c_5 = 0$. But $c_5 = -\frac{1}{15}$, therefore $c_7 = \frac{1}{105}$ and so on. Therefore

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^n \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \\
&= 0 + 0 + 0 + \frac{1}{3}x^3 + 0 - \frac{1}{15}x^5 + 0 + \frac{1}{105}x^7 - \cdots \\
&= \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{105}x^7 - \frac{1}{945}x^9 + \cdots
\end{aligned}$$

The final solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 - \cdots\right) + \left(\frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{105}x^7 - \frac{1}{945}x^9 + \cdots\right)
\end{aligned}$$

In this example, we choose $c_0 = 0$. This was arbitrary choice.

1.3 Examples for Frobenius series for first order ode

Local contents

1.3.1 Example 1. $xy' + y = 0, y(0) = 1$

$$xy' + y = 0 \tag{1}$$

With expansion around $x = 0$. Since x is regular singular point, then let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}
\end{aligned}$$

Then (1) becomes

$$x\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{2}$$

$$\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{3A}$$

All x in sums start from same index, so there is no adjustments needed. For $n = 0$, EQ (3) gives

$$\begin{aligned}
r a_0 x^r + a_0 x^r &= 0 \qquad (*)\\
(r + 1)\, a_0 x^r &= 0
\end{aligned}$$

Eq (*) above is important. It is called balance equation and will be used to find c_0, r for particular solution for ode when the RHS is not zero. In this case it will not be needed but will be used in next problems. Eq (3A) is also very important. It will be used also to find particular solution when right side is not zero. From now will call this the summation equation. It is the final equation when all sums have been normalized such that power of x inside each sum all have same power. Both the balance equation and the summation equation are the core of the algorithm for solving first order series equation with regular singular point and will be used over and over in all the problems.

Now we start by finding y_h. From balance equation and since $a_0 \neq 0$, hence $r+1=0$ or

$$r = -1$$

Therefore summation equation becomes

$$\sum_{n=0}^{\infty} (n-1)\, a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n-1} = 0 \tag{3B}$$

For $n > 0$ (because $n = 0$ was used to find r) EQ (3B) gives the recursive relation

$$\begin{aligned} (n-1)\, a_n + a_n &= 0 \\ n a_n &= 0 \end{aligned} \tag{3C}$$

We see that for all $n > 0$ then $a_n = 0$. Hence solution is for $n = 0$ only, which is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= a_0 x^{-1} \\ &= \frac{a_0}{x} \end{aligned}$$

Initial condition now applied. We see that at $x = 0$ we get undefined value. Hence not possible to solve for a_0. Therefore no solution exist with this initial condition.

1.3.2 Example 2. $xy' + y = x$

$$xy' + y = x \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\, c_0 x^r = x$$

For balance we see that

$$r = 1$$

Hence

$$\begin{aligned} (r+1)\, c_0 &= 1 \\ 2c_0 &= 1 \\ c_0 &= \frac{1}{2} \end{aligned}$$

Now that we found r, c_0 we will use the summation equation in first example to find all c_n for $n > 0$. We see that all summations terms start from the same index. This implies that only term exist for y_p which is

$$\begin{aligned} y_p &= c_0 x^r \\ &= \frac{1}{2} x \end{aligned}$$

If the summation equation did not have all the sums in it start from same lower index, then we would had to apply the summation equation to find all c_n for $n > 0$ just like we did for finding a_n. But here we got lucky. Therefore

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + \frac{1}{2} x \end{aligned}$$

Last problem below gives case when not all sums in the summation equation have the same starting index.

1.3.3 Example 3. $xy' + y = 1$

$$xy' + y = 1 \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\,c_0 x^r = 1$$

Balance gives $r = 0$. Hence

$$\begin{aligned}(r+1)\,c_0 &= 1\\ c_0 &= 1\end{aligned}$$

Since all sum terms in the summation equation of the first example have same starting index, then we know that all $c_n = 0$ for $n > 0$. Therefore

$$\begin{aligned}y_p &= c_0 x^r\\ &= 1\end{aligned}$$

Therefore

$$\begin{aligned}y &= y_h + y_p\\ &= \frac{a_0}{x} + 1\end{aligned}$$

1.3.4 Example 4. $xy' + y = k$

$$xy' + y = k \tag{1}$$

This is similar to above example. Carrying out same steps shows that $c_0 = k$ and all other $c_n = 0$. Hence $y_p = k$. The solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \frac{a_0}{x} + k\end{aligned}$$

1.3.5 Example 5. $xy' + y = \sin x$

$$xy' + y = \sin x \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$\begin{aligned}(r+1)\,c_0 x^r &= \sin x\\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\end{aligned}$$

For each term on the right side, there is different balance equation. Hence we have

$$(r+1)\,c_0 x^r = x \tag{2}$$
$$(r+1)\,c_0 x^r = -\frac{1}{6}x^3 \tag{3}$$
$$(r+1)\,c_0 x^r = \frac{1}{120}x^5 \tag{4}$$
$$(r+1)\,c_0 x^r = -\frac{1}{5040}x^7 \tag{5}$$
$$\vdots$$

Each one equation above, gives different y_p then at the end we will add them all. Starting with (2)

$$(r+1)\,c_0 x^r = x$$

Hence $r = 1$ the therefore $(r+1)\,c_0 = 1$ or $c_0 = \frac{1}{2}$. Now, since the summation equation have all the sums in it start at the same lower index, then we know there is only one term. Hence the first particular solution is $y_{p_1} = c_0 x^r = \frac{1}{2}x$. Now we will look at the next term on the right side, which is (3) and do the same.

$$(r+1)\,c_0 x^r = -\frac{1}{6}x^3$$

Then $r=3$ and $(r+1)\,c_0 = -\frac{1}{6}$ or $4c_0 = -\frac{1}{6}$ or $c_0 = \frac{-1}{24}$. Then the second particular solution is $y_{p_2} = \frac{-1}{24}x^3$. Now we will look at the next term in (4) and do the same

$$(r+1)\,c_0 x^r = \frac{1}{120}x^5$$

Hence $r=5$ the therefore $(r+1)\,c_0 = \frac{1}{120}$ or $6c_0 = \frac{1}{120}$ or $c_0 = \frac{1}{720}$. Hence the first particular solution is $y_{p_3} = \frac{1}{720}x^5$. Now we will look at (5)

$$(r+1)\,c_0 x^r = -\frac{1}{5040}x^7$$

Hence $r=7$ the therefore $(r+1)\,c_0 = -\frac{1}{5040}$ or $8c_0 = -\frac{1}{5040}$ or $c_0 = -\frac{1}{40320}$. Hence the first particular solution is $y_{p_3} = -\frac{1}{40320}x^7$ and so on. Hence

$$y_p = \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{720}x^5 - \frac{1}{5040}x^7 + \cdots$$

We found y_p. The solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + \left(\frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{720}x^5 - \frac{1}{5040}x^7 + \cdots\right) \end{aligned}$$

Notice here we also used that fact that since all sum term in the summation equation in the first example have same starting index, then we know that all $c_n = 0$ for $n > 0$ for each y_p we found above, so we only needed to find c_0 only for each y_p.

1.3.6 Example 6. $xy' + y = x + x^3 + 2x^4$

$$xy' + y = x + x^3 + 2x^4 \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\,c_0 x^r = x + x^3 + 2x^4$$

For each term on the right side, there is different balance equation. Hence we have

$$(r+1)\,c_0 x^r = x \tag{2}$$
$$(r+1)\,c_0 x^r = x^3 \tag{3}$$
$$(r+1)\,c_0 x^r = 2x^4 \tag{4}$$

Each one equation above, gives different y_p then at the end we will add them all. Starting with (2)

$$(r+1)\,c_0 x^r = x$$

Hence $r=1$, therefore $(r+1)\,c_0 = 1$ or $c_0 = \frac{1}{2}$. Hence the first particular solution is $y_{p_1} = \frac{1}{2}x$. Now we will look at (3)

$$(r+1)\,c_0 x^r = x^3$$

Hence $r=3$, therefore $(r+1)\,c_0 = 1$ or $4c_0 = 1$ or $c_0 = \frac{1}{4}$. Hence the second particular solution is $y_{p_2} = \frac{1}{4}x^3$. Now we will look at (3)

$$(r+1)\,c_0 x^r = 2x^4$$

Hence $r=4$, therefore $(r+1)\,c_0 = 1$ or $5c_0 = 2$ or $c_0 = \frac{2}{5}$. Hence the third particular solution is $y_{p_3} = \frac{2}{5}x^4$. Hence adding all the above gives

$$y_p = \frac{1}{2}x + \frac{1}{4}x^3 + \frac{2}{5}x^4$$

We found y_p. The solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + \frac{1}{2}x + \frac{1}{4}x^3 + \frac{2}{5}x^4 \end{aligned}$$

Notice here we also used that fact that since all sum term in the summation equation in the first example have same starting index, then we know that all $c_n = 0$ for $n > 0$ for each y_p we found above, so we only needed to find c_0 only for each y_p.

1.3.7 Example 7. $xy' + y = \frac{1}{x}$

$$xy' + y = \frac{1}{x} \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\, c_0 x^r = \frac{1}{x}$$

Hence $r = -1$. Therefore $(r+1)\, c_0 = 1$ or $0c_0 = 1$. No solution exist. Can not find y_p. This is an example where there is *no series solution.* This ode of course can be easily solved directly which gives solution $y = \frac{c_1}{x} + \frac{1}{x}\ln x$, but not using series method. The next problems shows that when changing $\frac{1}{x}$ to $\frac{1}{x^2}$ then the balance equation is able to find c_0.

1.3.8 Example 8. $xy' + y = \frac{1}{x^2}$

$$xy' + y = x^{-2} \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\, c_0 x^r = \frac{1}{x^2}$$

Hence $r = -2$, therefore $(r+1)\, c_0 = 1$ or $-c_0 = 1$ or $c_0 = -1$. Hence the first particular solution is $y_p = -x^{-2}$. Hence the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} - \frac{1}{x^2} \end{aligned}$$

1.3.9 Example 9. $xy' + y = 3 + x$

$$xy' + y = 3 + x \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\, c_0 x^r = 3 + x$$

For each term on the right side, there is different balance equation. Hence we have

$$(r+1)\, c_0 x^r = 3 \tag{2}$$
$$(r+1)\, c_0 x^r = x \tag{3}$$

Each one equation above, gives different y_p then at the end we will add them all. Starting with (2)

$$(r+1)\, c_0 x^r = 3$$

Hence $r = 0$, therefore $(r+1)\, c_0 = 3$ or $c_0 = 3$. Hence the first particular solution is $y_{p_1} = 3$. Now we will look at (3)

$$(r+1)\, c_0 x^r = x$$

Hence $r = 1$, therefore $(r+1)\, c_0 = 1$ or $c_0 = \frac{1}{2}$. Hence the second particular solution is $y_{p_2} = \frac{1}{2}x$. Adding all the particular solutions gives

$$y_p = 2 + \frac{1}{2}x$$

The complete solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + 3 + \frac{1}{2}x \end{aligned}$$

1.3.10 Example 10. $xy' + y = \frac{1}{x^3}$

$$xy' + y = x^{-3} \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we will use the balance equation, EQ (*) found in the first example when finding y_h. We just need to rename a_0 to c_0 and add the x on the right side of the balance equation.

$$(r+1)\, c_0 x^r = x^{-3}$$

Hence $r = -3$, therefore $(r+1)\, c_0 = 1$ or $c_0 = -\frac{1}{2}$. Hence the particular solution is $y_p = -\frac{1}{2}x^{-3}$. The solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} - \frac{1}{2x^3} \end{aligned}$$

1.3.11 Example 11 $y' + 2xy = \sqrt{x}$

$$y' + 2xy = \sqrt{x}$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\, y = 0$. We see that $p(x)$ is analytic at $x = 0$. However the RHS has no series expansion at $x = 0$ (not analytic there). Therefore we must use Frobenius series in this case. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \end{aligned}$$

The (homogenous) ode becomes

$$\begin{aligned} \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + 2x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r+1} &= 0 \end{aligned}$$

Reindex so all powers on x are the lowest gives

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r-1} = 0 \tag{1}$$

For $n = 0$, Eq(1) gives

$$r a_0 x^{r-1} = 0 \tag{*}$$

Hence $r = 0$ since $a_0 \neq 0$. For $n = 1$, Eq(1) gives

$$\begin{aligned} (1+r)\, a_1 x^r &= 0 \\ a_1 &= 0 \end{aligned}$$

For $n \geq 2$ the recurrence relation is from (1)

$$\begin{aligned} (n+r)\, a_n + 2a_{n-2} &= 0 \\ a_n &= -\frac{2a_{n-2}}{(n+r)} \end{aligned} \tag{2}$$

Or for $r = 0$ the above simplifies to

$$a_n = -\frac{2}{n} a_{n-2} \tag{2A}$$

Eq (2A) is what is used to find all a_n for For $n \geq 2$. Hence for $n = 2$ and remembering that $a_0 = 1$ gives

$$a_2 = -1$$

For $n = 3$

$$a_3 = -\frac{2}{3} a_1 = 0$$

For $n = 4$

$$a_4 = -\frac{1}{2} a_2 = \frac{1}{2}$$

For $n = 5, 7, \cdots$ and all odd n then $a_n = 0$. For $n = 6$

$$a_6 = -\frac{1}{3}a_4 = -\frac{1}{6}$$

And so on. Hence (using $a_0 = 1$)

$$\begin{aligned}
y_h &= c_1 \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= c_1 \sum_{n=0}^{\infty} a_n x^n \\
&= c_1 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \\
&= c_1 \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots\right)
\end{aligned}$$

Now we need to find y_p using the balance equation. From above we found that

$$r a_0 x^{r-1} = x^{\frac{1}{2}}$$

Renaming a to c the above becomes

$$r c_0 x^{r-1} = x^{\frac{1}{2}}$$

Hence $r - 1 = \frac{1}{2}$ or $r = \frac{3}{2}$. Therefore $r c_0 = 1$ or $c_0 = \frac{2}{3}$. To find the rest of c_n, we see that summation terms in (1) above, do not all have same starting index, then here, we can not just find c_0 and assume that all $c_n = 0$ for $n > 0$ as we did in the earlier example. In this case we have to use the same relation (1) above to find all the c_n like we did when finding a_n. We just have to use c_n for a_n and use r found from the balance equation (*). EQ (1) becomes

$$\begin{aligned}
\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} + \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r-1} &= 0 \\
\sum_{n=0}^{\infty} \left(n + \frac{3}{2}\right) c_n x^{n+\frac{3}{2}-1} + \sum_{n=2}^{\infty} 2c_{n-2} x^{n+\frac{3}{2}-1} &= 0 \\
\sum_{n=0}^{\infty} \left(n + \frac{3}{2}\right) c_n x^{n+\frac{1}{2}} + \sum_{n=2}^{\infty} 2c_{n-2} x^{n+\frac{1}{2}} &= 0 \qquad (1B)
\end{aligned}$$

And it is (1B) that will be used to find all c_n for $n > 0$. For $n = 1$

$$\begin{aligned}
\left(n + \frac{3}{2}\right) c_n &= 0 \\
\left(1 + \frac{3}{2}\right) c_1 &= 0 \\
c_1 &= 0
\end{aligned}$$

For $n = 2$ we have recursion relation

$$\begin{aligned}
\left(n + \frac{3}{2}\right) c_n + 2c_{n-2} &= 0 \\
c_n &= \frac{-2c_{n-2}}{n + \frac{3}{2}}
\end{aligned}$$

Hence for $n = 2$

$$\begin{aligned}
c_2 &= \frac{-2c_0}{\left(2 + \frac{3}{2}\right)} \\
&= \frac{-2\left(\frac{2}{3}\right)}{\left(2 + \frac{3}{2}\right)} \\
&= -\frac{8}{21}
\end{aligned}$$

For $n = 3$

$$\begin{aligned}
c_3 &= \frac{-2c_1}{\left(3 + \frac{3}{2}\right)} \\
&= 0
\end{aligned}$$

For $n = 4$

$$\begin{aligned}
c_4 &= \frac{-2c_2}{\left(4 + \frac{3}{2}\right)} \\
&= \frac{-2\left(-\frac{8}{21}\right)}{\left(4 + \frac{3}{2}\right)} \\
&= \frac{32}{231}
\end{aligned}$$

And so on. Hence

$$\begin{aligned} y_p &= x^{\frac{3}{2}} \sum_{n=0}^{\infty} c_n x^n \\ &= x^{\frac{3}{2}} \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \left(1 - x^2 + \frac{1}{2} x^4 - \frac{1}{6} x^6 + \cdots\right) + x^{\frac{3}{2}} \left(\frac{2}{3} + -\frac{8}{21} x^2 + \frac{32}{231} x^4 - \cdots\right) \end{aligned}$$

1.3.12 Example 12 $y' + 2xy = \frac{1}{x}$

$$y' + 2xy = \frac{1}{x}$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\, y = 0$. We see that $p(x)$ is defined as is at $x = 0$. However the RHS has no series expansion at $x = 0$. Therefore we must use Frobenius series. This is the same ode as example 1. So we go straight to find y_p as y_h is the same. Now we need to find y_p using the balance equation. From above we found that balance equation (*) is

$$r c_0 x^{r-1} = \frac{1}{x}$$

Hence $r - 1 = -1$ or $r = 0$. Therefore $r c_0 = 1$. But since $r = 0$ then there is no solution for c_0. Hence it is not possible to find series solution. This is an example where the balance equation fails and so we have to use asymptotic expansion to find solution, which is not supported now.

1.3.13 Example 13 $y' = \frac{1}{x}$

$$y' = \frac{1}{x}$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\, y = 0$. We see that $p(x) = 0$ is analytic at $x = 0$. However the RHS has no series expansion at $x = 0$ (not analytic there). Therefore we must use Frobenius series in this case. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} \end{aligned}$$

The (homogenous) ode becomes

$$\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} = 0 \tag{1}$$

For $n = 0$

$$r a_0 x^{r-1} = 0 \tag{*}$$

Hence $r = 0$ since $a_0 \neq 0$. Therefore the ode satisfies

$$y' = r a_0 x^{r-1}$$

Eq (1) becomes

$$\begin{aligned} \sum_{n=0}^{\infty} n a_n x^{n-1} &= 0 \\ n a_n x^{n-1} &= 0 \end{aligned} \tag{2}$$

Therefore for all $n \geq 1$ we have $a_n = 0$. Hence

$$y_h = a_0$$

Now we need to find y_p using the balance equation (*). From above we see that (where we rename a_0 to c_0)

$$r c_0 x^{r-1} = x^{-1}$$

Hence $r - 1 = -1$ or $r = 0$. Hence

$$\begin{aligned} r c_0 &= 1 \\ 0 c_0 &= 1 \end{aligned}$$

Therefore there is no solution for c_0. Unable to find y_p therefore no series solution exists. Asymptotic methods are needed to solve this. Mathematica AsymptoticDSolveValue gives the solution as $y(x) = c + \ln x$.

1.3.14 Example 14 $y' = \frac{1}{x^2}$

$$y' = \frac{1}{x^2}$$

This is the same as above problem where we found

$$y_h = a_0$$

To find y_p we will use the balance equation (*) from the above problem which is

$$rc_0 x^{r-1} = x^{-2}$$

Hence $r - 1 = -2$ or $r = -1$. Therefore $rc_0 = 1$ or $c_0 = -1$. The particular solution is therefore

$$y_p = -x^{-1}$$

Hence the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\left(1 + O\left(x^2\right)\right) - \frac{1}{x} \end{aligned}$$

1.3.15 Example 15 $y' + \frac{y}{x} = 0$

$$y' + \frac{y}{x} = 0$$

Expansion is around $x = 0$. The (homogenous) ode has the form $y' + p(x)\,y = 0$. We see that $p(x) = \frac{1}{x}$ is not analytic at $x = 0$ but $\lim_{x\to 0} xp(x) = 0$ is analytic. Therefore we must use Frobenius series in this case. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{A}$$

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

The ode becomes

$$\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \frac{1}{x}\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty} \left((n+r)\,a_n + a_n\right) x^{n+r-1} = 0$$

$$\sum_{n=0}^{\infty} (n+r+1)\,a_n x^{n+r-1} = 0 \tag{1}$$

For $n = 0$

$$(r+1)\,a_0 = 0$$

Hence $r = -1$ since $a_0 \neq 0$. Eq (1) becomes, where $r = -1$ now

$$\sum_{n=0}^{\infty} n a_n x^n = 0$$

$$n a_n x^{n-1} = 0 \tag{2}$$

$n = 0$ is not used since that was used to find r. Therefore we start from $n = 1$. For $n = 1$ the above gives $a_1 = 0$ and same for all $n \geq 1$. Hence from Eq (A), since $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ then (note: When there is only one $\sum$ term left in (1) as in this case, then this means there is no recurrence relation and all $a_n = 0$ for $n > 0$).

$$\begin{aligned} y &= c_1\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) \\ &= c_1\left(\sum_{n=0}^{\infty} a_n x^{n-1}\right) \\ &= c_1\left(a_0 x^{-1} + 0 + 0 + \cdots + O(x)\right) \end{aligned}$$

Letting $a_0 = 1$ the above becomes

$$y = c_1\left(x^{-1} + O(x)\right)$$

1.3.16 Example 16 $\cos(x)\, y' + \frac{1}{x} y = x$

$$\cos(x)\, y' + \frac{1}{x} y = x$$

Expansion is around $x = 0$. Since regular singular point, then we must use Frobenius series in this case. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \text{(A)} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \end{aligned}$$

The homogeneous ode becomes

$$\begin{aligned} \cos(x) \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + x^{-1} \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r-1} &= 0 \\ \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - \frac{1}{2}x^2 \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \frac{1}{24}x^4 \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \cdots\right) + \sum_{n=0}^{\infty} a_n x^{n+r-1} &= 0 \\ \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - \sum_{n=0}^{\infty} \frac{1}{2}(n+r)\, a_n x^{n+r+1} + \sum_{n=0}^{\infty} \frac{1}{24}(n+r)\, a_n x^{n+r+3} + \cdots\right) + \sum_{n=0}^{\infty} a_n x^{n+r-1} &= 0 \end{aligned}$$

Making all powers on x the lowest, which is $n + r - 1$ gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - \sum_{n=2}^{\infty} \frac{1}{2}(n+r-2)\, a_{n-2} x^{n+r-1} + \sum_{n=4}^{\infty} \frac{1}{24}(n+r-4)\, a_{n-4} x^{n+r-1} + \cdots\right) + \sum_{n=0}^{\infty} a_n x^{n+r-1} = 0 \qquad (1)$$

The indicial equation is when $n = 0$. Hence

$$\begin{aligned} (n+r)\, a_n x^{n+r-1} + a_n x^{n+r-1} &= 0 \\ r a_0 x^{r-1} + a_0 x^{r-1} &= 0 \\ a_0 (r+1)\, x^{r-1} &= 0 \qquad (*) \end{aligned}$$

Hence $r = -1$ is the root. EQ (*) is the balance equation. Now we find all a_n. For $n = 1$, EQ (1) gives (and using $r = -1$)

$$\begin{aligned} (n+r)\, a_n x^{n+r-1} + a_n x^{n+r-1} &= 0 \\ (1-1)\, a_1 x^{1-1-1} + a_1 x^{1-1-1} &= 0 \\ a_1 x^{-1} &= 0 \end{aligned}$$

Hence $a_1 = 0$. For $n = 2$

$$\begin{aligned} (n+r)\, a_n x^{n+r-1} - \frac{1}{2}(n+r-2)\, a_{n-2} x^{n+r-1} + a_n x^{n+r-1} &= 0 \\ (2-1)\, a_2 - \frac{1}{2}(2-1-2)\, a_{2-2} + a_2 &= 0 \\ a_2 + \frac{1}{2} a_0 + a_2 &= 0 \\ 2a_2 &= -\frac{1}{2} a_0 \\ a_2 &= -\frac{1}{4} a_0 \end{aligned}$$

For $n = 3$

$$\begin{aligned} (n+r)\, a_n x^{n+r-1} - \frac{1}{2}(n+r-2)\, a_{n-2} x^{n+r-1} + a_n x^{n+r-1} &= 0 \\ (3-1)\, a_3 - \frac{1}{2}(3-1-2)\, a_{3-2} + a_3 &= 0 \\ 2a_3 + a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

For $n=4$

$$
\begin{aligned}
(n+r)\,a_n x^{n+r-1} - \frac{1}{2}(n+r-2)\,a_{n-2}x^{n+r-1} + \frac{1}{24}(n+r-4)\,a_{n-4}x^{n+r-1} + a_n x^{n+r-1} &= 0\\
(4-1)\,a_4 - \frac{1}{2}(4-1-2)\,a_2 + \frac{1}{24}(4-1-4)\,a_0 + a_4 &= 0\\
3a_4 - \frac{1}{2}a_2 - \frac{1}{24}a_0 + a_4 &= 0\\
4a_4 &= \frac{1}{2}a_2 + \frac{1}{24}a_0\\
4a_4 &= \frac{1}{2}\left(-\frac{1}{4}a_0\right) + \frac{1}{24}a_0\\
&= -\frac{1}{8}a_0 + \frac{1}{24}a_0\\
&= -\frac{1}{12}a_0\\
a_4 &= -\frac{1}{48}a_0
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_h &= \sum_{n=0}^{\infty} a_n x^{n+r}\\
&= \sum_{n=0}^{\infty} a_n x^{n-1}\\
&= \frac{1}{x}\sum_{n=0}^{\infty} a_n x^{n}\\
&= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)\\
&= \frac{1}{x}\left(a_0 + 0x - \frac{1}{4}a_0 x^2 + 0x^3 - \frac{1}{48}a_0 x^4 + \cdots\right)\\
&= \frac{1}{x}\left(a_0 - \frac{1}{4}a_0 x^2 - \frac{1}{48}a_0 x^4 + \cdots\right)\\
&= a_0\left(\frac{1}{x} - \frac{1}{4}x - \frac{1}{48}x^3 + \cdots\right)
\end{aligned}
$$

Now that we found y_h, we need to find y_p. The balance equation is (*). Using c_0 instead of a_0 it becomes (now with x on the right side)

$$
c_0(r+1)\,x^{r-1} = x
$$

For balance we need $r-1=1$ or $r=2$. Hence $c_0(r+1)=1$ or $c_0=\frac{1}{3}$. Since the summation terms in (1) do not all have the same starting index, then we have to use (1) to find all c_n and we can not just use c_0 like we did in earlier problem. Only when starting index is the same on all summation terms, we can do that.

To find all c_n we repeat the same process used to find a_n. We use EQ (1) again but replace all a_n by c_n and now use $r=2$ instead of $r=-1$. EQ (1) now becomes

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r-1} - \sum_{n=2}^{\infty}\frac{1}{2}(n+r-2)\,c_{n-2}x^{n+r-1} + \sum_{n=4}^{\infty}\frac{1}{24}(n+r-4)\,c_{n-4}x^{n+r-1} + \cdots\right) + \sum_{n=0}^{\infty} c_n x^{n+r-1} &= 0\\
\left(\sum_{n=0}^{\infty}(n+2)\,c_n x^{n+1} - \sum_{n=2}^{\infty}\frac{1}{2}n c_{n-2}x^{n+1} + \sum_{n=4}^{\infty}\frac{1}{24}(n-2)\,c_{n-4}x^{n+1} + \cdots\right) + \sum_{n=0}^{\infty} c_n x^{n+1} &= 0
\end{aligned}
\tag{1B}
$$

Notice that we kept the right side zero, since we used $n=0$ for balance and hence there is no more terms to balance any more for $n>0$. Now we use (1B) to find all c_n for $n>0$. For $n=1$

$$
\begin{aligned}
(n+2)\,c_n + c_n &= 0\\
3c_1 + c_1 &= 0\\
c_1 &= 0
\end{aligned}
$$

For $n=2$

$$
\begin{aligned}
(n+2)\,c_n - \frac{1}{2}n c_{n-2} + c_n &= 0\\
4c_2 - c_0 + c_2 &= 0\\
5c_2 &= c_0\\
c_2 &= \frac{c_0}{5}\\
&= \frac{1}{15}
\end{aligned}
$$

Since $c_0 = \frac{1}{3}$. For $n = 3$

$$
\begin{aligned}
(n+2)\,c_n - \frac{1}{2} n c_{n-2} + c_n &= 0\\
5c_3 - \frac{3}{2}c_1 + c_3 &= 0\\
6c_3 &= 0\\
c_3 &= 0
\end{aligned}
$$

For $n = 4$

$$
\begin{aligned}
(n+2)\,c_n - \frac{1}{2} n c_{n-2} + \frac{1}{24}(n-2)\,c_{n-4} + c_n &= 0\\
6c_4 - 2c_2 + \frac{1}{12}c_0 + c_4 &= 0\\
7c_4 &= 2c_2 - \frac{1}{12}c_0\\
&= 2\left(\frac{1}{15}\right) - \frac{1}{12}\left(\frac{1}{3}\right)\\
&= \frac{19}{180}\\
c_4 &= \frac{19}{180\,(7)}\\
&= \frac{19}{1260}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+r}\\
&= \sum_{n=0}^{\infty} c_n x^{n+2}\\
&= x^2 \sum_{n=0}^{\infty} c_n x^{n}\\
&= x^2\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\right)\\
&= x^2\left(\frac{1}{3} + 0x + \frac{1}{15}x^2 + 0x^3 + \frac{19}{1260}x^4 + \cdots\right)\\
&= x^2\left(\frac{1}{3} + \frac{1}{15}x^2 + \frac{19}{1260}x^4 + \cdots\right)\\
&= \frac{x^2}{3} + \frac{x^4}{15} + \frac{19}{1260}x^6 + \cdots
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= a_0\left(\frac{1}{x} - \frac{1}{4}x - \frac{1}{48}x^3 + \cdots\right) + \left(\frac{x^2}{3} + \frac{x^4}{15} + \frac{19}{1260}x^6 + \cdots\right)\\
&= c_1\left(\frac{1}{x} - \frac{1}{4}x - \frac{1}{48}x^3 + \cdots\right) + \left(\frac{x^2}{3} + \frac{x^4}{15} + \frac{19}{1260}x^6 + \cdots\right)
\end{aligned}
$$

1.3.17 Example 17 $xy' + 2xy = \sqrt{x}$

$$
xy' + 2xy = \sqrt{x}
$$

Expansion is around $x = 0$. Since regular singular point, then we must use Frobenius series in this case. Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \text{(A)}\\
y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}
\end{aligned}
$$

The homogeneous ode becomes

$$
\begin{aligned}
x\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + 2x\sum_{n=0}^{\infty} a_n x^{n+r} &= 0\\
\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+1} &= 0
\end{aligned}
$$

Making all powers on x the lowest, which is $n+r$ gives

$$\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r}+\sum_{n=1}^{\infty}2a_{n-1}x^{n+r}=0 \tag{1}$$

The indicial equation is when $n=0$. Hence

$$\begin{aligned}(n+r)\,a_n x^{n+r}&=0\\ ra_0x^r&=0\\ ra_0&=0 \qquad (*)\end{aligned}$$

Hence $r=0$ since $a_0\neq 0$. EQ (*) is the balance equation. Now we find all a_n. For $n=1$, EQ (1) gives (and using $r=0$)

$$\begin{aligned}a_1+2a_0&=0\\ a_1&=-2a_0\end{aligned}$$

For $n=2$

$$\begin{aligned}2a_2+2a_1&=0\\ 2a_2&=-2a_1\\ a_2&=-a_1\\ &=2a_0\end{aligned}$$

For $n=3$

$$\begin{aligned}3a_3+2a_2&=0\\ a_3&=-\frac{2a_2}{3}\\ a_3&=-\frac{4a_0}{3}\end{aligned}$$

And so on. Hence

$$\begin{aligned}y_h&=\sum_{n=0}^{\infty}a_n x^{n+r}\\ &=\sum_{n=0}^{\infty}a_n x^{n}\\ &=\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots\right)\\ &=a_0\left(1-2x+2x^2-\frac{4}{3}x^3+\cdots\right)\end{aligned}$$

Now that we found y_h, we need to find y_p. The balance equation is (*). Using c_0 instead of a_0 it becomes (now with x on the right side)

$$rc_0x^r=\sqrt{x}$$

For balance we need $r=\frac{1}{2}$. Hence $c_0r=1$ or $c_0=2$. Since the summation terms in (1) do not all have the same starting index, then we have to use (1) to find all c_n and we can not just use c_0 like we did in earlier problem. Only when starting index is the same on all summation terms, we can do that.

To find all c_n we repeat the same process used to find a_n. We use EQ (1) again but replace all a_n by c_n and now use $r=\frac{1}{2}$ instead of $r=0$. EQ (1) now becomes

$$\begin{aligned}\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r}+\sum_{n=1}^{\infty}2c_{n-1}x^{n+r}&=0\\ \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)c_n x^{n+\frac{1}{2}}+\sum_{n=1}^{\infty}2c_{n-1}x^{n+\frac{1}{2}}&=0 \qquad \text{(1B)}\end{aligned}$$

Notice that we kept the right side zero, since we used $n=0$ for balance and hence there is no more terms to balance any more for $n>0$. Now we use (1B) to find all c_n for $n>0$. For $n=1$

$$\begin{aligned}\left(n+\frac{1}{2}\right)c_n+2c_{n-1}&=0\\ \frac{3}{2}c_1+2c_0&=0\\ c_1&=-\frac{4}{3}c_0\\ &=-\frac{4}{3}2\\ &=-\frac{8}{3}\end{aligned}$$

For $n = 2$

$$
\begin{aligned}
\left(2+\frac{1}{2}\right) c_2 + 2c_1 &= 0 \\
\frac{5}{2}c_2 &= -2c_1 \\
c_2 &= -\frac{4c_1}{5} \\
c_2 &= \left(-\frac{4}{5}\right)\left(-\frac{8}{3}\right) \\
&= \frac{32}{15}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} \\
&= \sqrt{x}\sum_{n=0}^{\infty} c_n x^{n} \\
&= \sqrt{x}\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) \\
&= \sqrt{x}\left(2 - \frac{8}{3}x + \frac{32}{15}x^2 - \frac{128}{105}x^3 + \frac{512}{945}x^4 - \cdots\right)
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= a_0\left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots\right) + \sqrt{x}\left(2 - \frac{8}{3}x + \frac{32}{15}x^2 - \frac{128}{105}x^3 + \frac{512}{945}x^4 - \cdots\right)
\end{aligned}
$$

1.4 Irregular singular point for first order

ode internal name "first order ode series method. Irregular singular point"

expansion point is an irregular singular point. Not supported.

2 Second order differential equation

Local contents

2.1 Algorithm flow chart

This chart gives the algorithm for second order ode where expansion point is regular singular point. This uses Frobenius series. There are 4 different cases to consider.

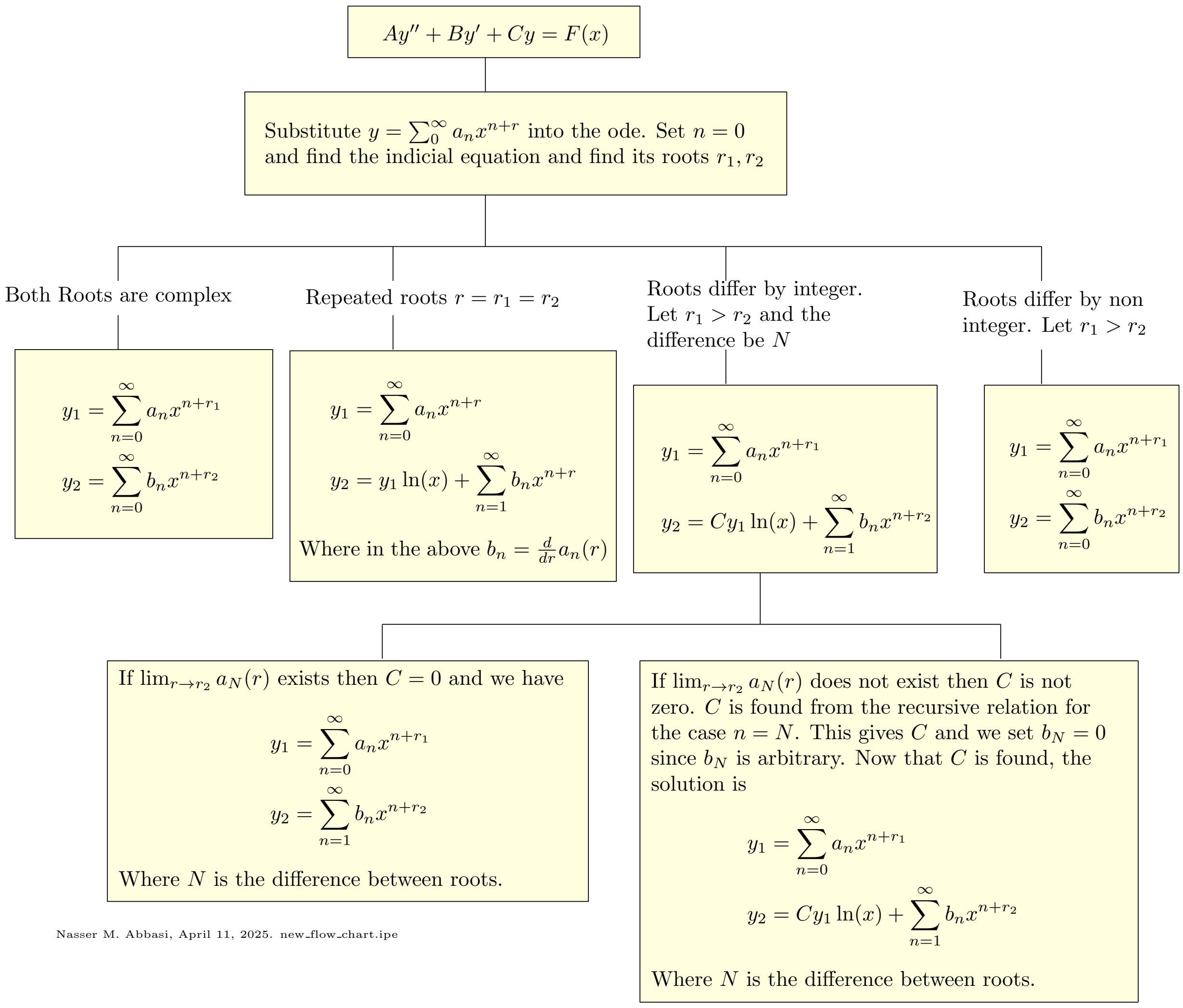

Figure 3: Series algorithm for second order ode for regular singular point

This gives the algorithm for second order ode where expansion point is ordinary.

if $f(x, y, y')$ is analytic at expansion point x_0 then this means x_0 is an ordinary point. We Apply Taylor series defintion directly to find the series expansion. Let $y_0 = y(x_0), y'(x_0) = y'_0$ and

$$y = y_0 + y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n(x, y)\bigg|_{\substack{x_0 \\ y_0 \\ y'_0}}$$

Where

$$\begin{aligned}
F_0 &= f(x, y, y') \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0
\end{aligned}$$

Ordinary point and regular singular point are supported. irregular singular point support will be added in the future. Expansion around point other than zero is also supported, including initial conditions. All 4 cases of regular point are supported, these are when the roots on indicial equation are repeated, or differ by an integer, or differ by non integer. case of Complex roots of indicial equation is also supported. Only second order and first order series solution is supported. Higher order ode support will be added in the future.

2.2 Second order Frobenius series. Indicial equation with repeated root

Local contents

2.2.1 Algorithm

ode internal name "second_order_series_method_regular_singular_point_repeated_root".

In this case the solution is

$$y = c_1 y_1 + c_2 y_2$$

Where

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\ y_2 &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2} \end{aligned}$$

r_1, r_2 are roots of the indicial equation. a_0, b_0 are set to 1 as arbitrary. The coefficients b_n are not found from the recurrence relation but found using using $b_n = \frac{d}{dr} a_n(r)$ after finding a_n first, and the result evaluated at root r_2. (notice that $r = r_1 = r_2$ in this case). Notice there is no C term in from of the ln in this case as when root differ by an integer and the sum on b_n starts at 1 since b_0 is always zero due to $\frac{d}{dr} a_0(r) = 0$ always as $a_0 = 1$ by default.

2.2.2 Examples

Local contents

2.2.2.1 Example 1. $x^2y'' + xy' + x^2y = 0$

Solve

$$x^2y'' + xy' + x^2y = 0 \tag{1}$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$y'' + \frac{1}{x}y' + y = 0$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x\frac{1}{x} = 1$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation.

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

Substituting the above in (1) gives

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0 \tag{2}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Eq (2) becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{3}$$

Eq (3) is the base equation which is used to find roots of indicial equation. For $n = 0$ the above gives

$$\begin{aligned}
(n+r)(n+r-1) a_0 x^r + (n+r) a_0 x^r &= 0 \\
(r)(r-1) a_0 x^r + (r) a_0 x^r &= 0 \\
(r(r-1) a_0 + r a_0) x^r &= 0 \\
(r(r-1) + r) a_0 x^r &= 0 \\
r^2 a_0 x^r &= 0
\end{aligned} \tag{4}$$

Eq (4) above is important. Called the balance equation and will be used to find r, c_0 for particular solutions. Let call it equation (*). Since $a_0 \neq 0$ then (*) gives

$$\begin{aligned}
r(r-1) + r &= 0 \\
r^2 - r + r &= 0 \\
r^2 &= 0
\end{aligned}$$

Hence the roots of the indicial equation are $r = 0$ which is a double root. Hence $r_1 = r_2 = 0$. This is the case when roots of indicial equation are repeated. In this case the solution y_h is given by

$$y_h = c_1 y_1 + c_2 y_2$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{5}$$

This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} \tag{6}$$

Something important to notice. In the sum above, it starts from 1 and not from 0. The main issue is how to find b_n. Since that is the only thing we need to be able to complete the solution as $y_1(x)$ is easily found. It turns out that there is a relation between the b_n and the a_n. The b_n can be found by taking just derivative of a_n as function of r for each n and then evaluating the result at $r = r_1$. How this is done will be shown below.

First we need to find $y_1(x)$. Substituting (5) in the ODE gives (3) again (but now with r having specific value r_1).

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{7}$$

Now we are ready to find a_n for $n > 0$. We skip $n = 0$ since that was used to obtain the indicial equation.

For $n = 1$. Eq (7) gives

$$\begin{aligned}
(n+r)(n+r-1) a_1 + (n+r) a_1 &= 0 \\
(1+r)(1+r-1) a_1 + (1+r) a_1 &= 0 \\
((1+r)(1+r-1) + (1+r)) a_1 &= 0 \\
(r+1)^2 a_1 &= 0
\end{aligned}$$

But $r = r_1 = 0$. The above gives $a_1 = 0$. It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	a_n	$a_n(r = r_1)$
0	a_0	a_0
1	0	0

For $n \geq 2$ now we obtain the recursive equation. Notice that the recursive equation starts from n which is the largest lower summation index. In this case it is $n = 2$. For all lower index, we have to find a_n without the use of recursive equation as we did above for a_1. Using (7), the recursive equation is

$$(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} = 0$$
$$a_n = -\frac{a_{n-2}}{(n+r)(n+r-1)+(n+r)} \tag{8}$$

Now we find few more a_n terms. From above and for $n = 2$

$$a_2 = -\frac{a_0}{(2+r)(2+r-1)+(2+r)}$$
$$= -\frac{a_0}{(r+2)^2}$$

and $r = r_1 = 0$ then the above becomes

$$a_2 = -\frac{a_0}{(2)^2} = -\frac{a_0}{4}$$

The table now becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	1
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$

And for $n = 3$

$$a_3 = -\frac{a_1}{(3+r)(3+r-1)+(3+r)}$$

But $a_1 = 0$. Then $a_3 = 0$. The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0

For $n = 4$ Eq (8) gives

$$a_4 = -\frac{a_2}{(4+r)(4+r-1)+(4+r)}$$

But a_2 from the table is $-\frac{1}{(r+2)^2}$. Hence

$$a_4(r) = -\frac{-\frac{a_0}{(r+2)^2}}{(4+r)(4+r-1)+(4+r)} = \frac{a_0}{(r^2+6r+8)^2}$$

The above becomes at $r = r_1 = 0$

$$a_4 = \frac{a_0}{(8)^2} = \frac{a_0}{64}$$

The Table now becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$

For $n = 5$ Eq (8) gives

$$a_5 = -\frac{a_3}{(n+r)(n+r-1)+(n+r)}$$

But $a_3 = 0$, hence $a_5 = 0$. The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$
5	0	0

For $n = 6$ Eq (8) gives

$$a_6(r) = -\frac{a_4(r)}{(6+r)(6+r-1)+(6+r)}$$

But from the table $a_4 = \frac{a_0}{(r^2+6r+8)^2}$, so the above becomes

$$a_6(r) = -\frac{\frac{a_0}{(r^2+6r+8)^2}}{(6+r)(6+r-1)+(6+r)} = -\frac{a_0}{(r+6)^2(r^2+6r+8)^2}$$

At $r = r_1 = 0$ the above becomes

$$a_6 = -\frac{a_0}{(6)^2(8)^2} = -\frac{a_0}{2304}$$

The table becomes

n	a_n	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$
5	0	0
6	$-\frac{a_0}{(r+6)^2(r^2+6r+8)^2}$	$-\frac{a_0}{2304}$

And so on. Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= a_0 \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \cdots\right) \end{aligned} \tag{9}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (6) it is given by

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

To find b_n, we will use the following

$$b_n = \frac{d}{dr}(a_n) \tag{10}$$

Notice that n starts from 1. Hence

$$b_1(r) = \frac{d}{dr}(a_1(r))\bigg|_{r=r_1}$$

What the above says, is that we first take derivative of a_n w.r.t. r and evaluate the result at the root of the indicial equation. Using the table above, we obtain (recalling that $r_1 = r = 0$ in this example)

n	a_n	$a_n(r = r_1 = 0)$	$b_n = \frac{d}{dr}(a_n)$	$b_n(r = r_1 = 0)$
0	a_0	a_0	N/A since b starts from $n = 1$	N/A
1	0	0	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$	$\frac{d}{dr}\left(-\frac{a_0}{(r+2)^2}\right) = \frac{2a_0}{(r+2)^3}$	$\frac{2a_0}{(2)^3} = \frac{a_0}{4}$
3	0	0	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$	$\frac{d}{dr}\left(\frac{a_0}{(r^2+6r+8)^2}\right) = -2a_0\frac{2r+6}{(r^2+6r+8)^3}$	$-2a_0\frac{6}{(8)^3} = -\frac{3a_0}{128}$
5	0	0	0	0
6	$-\frac{a_0}{(r+6)^2(r^2+6r+8)^2}$	$-\frac{a_0}{2304}$	$\frac{d}{dr}\left(-\frac{a_0}{(r+6)^2(r^2+6r+8)^2}\right) = 2a_0\frac{3r^2+24r+44}{(r^3+12r^2+44r+48)^3}$	$2a_0\frac{44}{(48)^3} = \frac{11a_0}{13824}$

We have found all b_n terms. Hence using (6) and since $r = r_1 = 0$ and using $a_0 = 1$ then

$$y_2 = y_1(x)\ln(x) + \left(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots\right)$$

But from the above table, we see that $b_1 = 0, b_2 = \frac{a_0}{4}, b_3 = 0, b_4 = -\frac{3a_0}{128}, b_5 = 0, b_6 = \frac{11a_0}{13824}$. The above becomes

$$y_2 = y_1 \ln(x) + a_0\left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right) \tag{11}$$

And we know what $y_1(x)$ is from Eq (9). Hence the above becomes

$$y_2(x) = a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right)\ln(x) + a_0\left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right)$$

Therefore the general solution is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= a_0 c_1\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right) \\ &+ a_0 c_2\left(\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right)\right) \end{aligned}$$

We can now absorb a_0 into the constants c_1, c_2 and the above becomes

$$
\begin{aligned}
y(x) &= c_1\left(1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+O(x^8)\right) \\
&+ c_2\left(\left(1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+O(x^8)\right)\ln(x)+\left(\frac{1}{4}x^2-\frac{3}{128}x^4+\frac{11}{13\,824}x^6+O(x^8)\right)\right)
\end{aligned}
$$

It it easier in implementation to just make $a_0 = 1$ at the start of this process so we do not have to carry it around, and that is what we will do from now on.

This completes the solution. The only difficulty in this method, is to make sure when finding the b_n is to have access to the a_n with r being unevaluated form in order to take derivatives correctly. This was done above by keeping a table of these quantities updated.

2.2.2.2 Example 2. $x^2y'' + xy' + x^2y = \sin(x)$

Solve

$$
x^2y'' + xy' + x^2y = \sin(x) \tag{1}
$$

This is the same example as example 1, but with non-zero on RHS. Expanding $\sin(x)$ gives

$$
x^2y'' + xy' + x^2y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots \tag{1A}
$$

The solution is

$$
y = y_h + y_p
$$

Where we found y_h above. We just need to find y_p now. To find y_p we have to find y_p for each term on the right side of (1A). Starting with x term, we have

$$
x^2y'' + xy' + x^2y = x
$$

Using the balance equation (*) found above we then have (renaming a_0 to c_0)

$$
r^2c_0x^r = x
$$

Balance gives $r = 1$. And $r^2c_0 = 1$ or $c_0 = 1$. Then EQ (3) in first example becomes

$$
\sum_{n=0}^{\infty} n(n+1)\,c_nx^{n+1} + \sum_{n=0}^{\infty}(n+1)\,c_nx^{n+1} + \sum_{n=2}^{\infty} c_{n-2}x^{n+1} = x \tag{3}
$$

Now that we used $n = 0$ to find r by matching against the RHS which is x, from now on, for all $n > 0$ we will use (3) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x on the right. For $n = 1$ then (3) gives (remember to make RHS zero)

$$
\begin{aligned}
2c_1x^2 + 2c_1x^2 &= 0 \\
4c_1x^2 &= 0
\end{aligned}
$$

Hence $c_1 = 0$. For $n = 2$ EQ (3) gives

$$
\begin{aligned}
2(3)\,c_3x^3 + 3c_3x^3 + c_0x^3 &= 0 \\
x^3(9c_3 + c_0) &= 0
\end{aligned}
$$

Hence $9c_3 + c_0 = 0$ or $c_3 = -\frac{1}{9}$ since we found $c_1 = 1$. We continue this way and find that $c_3 = 0, c_4 = \frac{1}{225}, c_5 = 0, c_6 = -\frac{1}{11\,025}$ and so on. Hence

$$
\begin{aligned}
y_{p_1} &= \sum_{n=0}^{\infty} c_nx^{n+r} \\
&= \sum_{n=0}^{\infty} c_nx^{n+1} \\
&= c_0x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5 + c_5x^6 + c_6x^7 + \cdots \\
&= x - \frac{1}{9}x^3 + \frac{1}{225}x^5 - \frac{1}{11\,025}x^7 + \cdots
\end{aligned}
$$

Now we repeat the above for the second term on the right side, which is $-\frac{1}{6}x^3$

$$
x^2y'' + xy' + x^2y = -\frac{1}{6}x^3
$$

Balance equation (*) gives

$$
r^2c_0x^r = -\frac{1}{6}x^3
$$

Hence $r=3$ and $r^2 c_0 = -\frac{1}{6}$ or $9c_0 = -\frac{1}{6}$ or $c_0 = -\frac{1}{54}$. Using EQ (3) from above problem gives

$$\sum_{n=0}^{\infty}(n+3)(n+2)\,c_n x^{n+3} + \sum_{n=0}^{\infty}(n+3)\,c_n x^{n+3} + \sum_{n=2}^{\infty} c_{n-2} x^{n+3} = -\frac{1}{6}x^3 \tag{4}$$

From now on, for all $n>0$ we will use (4) to solve for all other c_n. From now on, we just need to solve (4) with RHS zero, since there can be no more matches for any x^3 on the right. For $n=1$ the above gives

$$\begin{aligned}(4)(3)\,c_1 x^4 + 4c_1 x^4 &= 0\\ 16c_1 x^4 &= 0\end{aligned}$$

Hence $c_1 = 0$. For $n=2$ EQ (4) gives

$$\begin{aligned}(5)(4)\,c_2 x^5 + 5c_2 x^5 + c_0 x^5 &= 0\\ x^5(25c_2 + c_0) &= 0\end{aligned}$$

Hence $25c_2 + c_0 = 0$ or $c_2 = -\frac{c_0}{25} = -\frac{-\frac{1}{54}}{25} = \frac{1}{1350}$. For $n=3$ then (4) gives

$$(6)(5)\,c_3 x^6 + 6c_3 x^6 + c_1 x^6 = 0$$

Which gives $c_3 = 0$ since $c_1 = 0$. For $n=4$ then (4) gives

$$\begin{aligned}(7)(6)\,c_4 x^7 + 7c_4 x^7 + c_2 x^7 &= 0\\ x^7(49c_4 + c_2) &= 0\end{aligned}$$

Hence $49c_4 + c_2 = 0$ or $c_4 = -\frac{c_2}{49} = -\frac{\frac{1}{1350}}{49} = -\frac{1}{66\,150}$. We continue this way. Hence

$$\begin{aligned}y_{p_2} &= \sum_{n=0}^{\infty} c_n x^{n+r}\\ &= \sum_{n=0}^{\infty} c_n x^{n+3}\\ &= c_0 x^3 + c_1 x^4 + c_2 x^5 + c_3 x^6 + c_4 x^7 + \cdots\\ &= -\frac{1}{54}x^3 + \frac{1}{1350}x^5 - \frac{1}{66\,150}x^7 + \cdots\end{aligned}$$

Now we repeat the above for the next term on the right, which is $\frac{1}{120}x^5$

$$x^2y'' + xy' + x^2y = \frac{1}{120}x^5$$

And if we carry the same steps as above we will find that

$$\begin{aligned}y_{p_3} &= \sum_{n=0}^{\infty} c_n x^{n+r}\\ &= \sum_{n=0}^{\infty} c_n x^{n+5}\\ &= c_0 x^5 + c_1 x^6 + c_2 x^7 + c_3 x^8 + c_4 x^9 + \cdots\\ &= \frac{1}{3000}x^5 - \frac{1}{147\,000}x^7 + \frac{1}{11\,907\,000}x^9 + \cdots\end{aligned}$$

We keep doing this for as many terms as we have on the right side. At the end, all y_{p_i} are added. This gives the final y_p

$$\begin{aligned}y_p &= y_{p_1} + y_{p2} + y_{p3} + \cdots\\ &= x - \frac{1}{9}x^3 + \frac{1}{225}x^5 - \frac{1}{11\,025}x^7 + \cdots\\ &\quad - \frac{1}{54}x^3 + \frac{1}{1350}x^5 - \frac{1}{66\,150}x^7 + \cdots\\ &\quad + \frac{1}{3000}x^5 - \frac{1}{147\,000}x^7 + \frac{1}{11\,907\,000}x^9 + \cdots\\ &\quad - \frac{1}{246960}x^7 + \frac{1}{20003760}x^9 + \cdots\end{aligned}$$

Which results in

$$\begin{aligned}y_p &= x + x^3\left(-\frac{1}{9} - \frac{1}{54}\right) + x^5\left(\frac{1}{225} + \frac{1}{1350} + \frac{1}{3000}\right) + x^7\left(-\frac{1}{11\,025} - \frac{1}{66\,150} - \frac{1}{147\,000} - \frac{1}{246960}\right) + \cdots\\ &= x + x^3\left(-\frac{7}{54}\right) + x^5\left(\frac{149}{27\,000}\right) + x^7\left(-\frac{2161}{18\,522\,000}\right) + \cdots\end{aligned}$$

Hence the solution is

$$y = y_h + y_p$$

Using y_h from the above problem gives the total solution as

$$\begin{aligned} y &= c_1\left(1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+O(x^8)\right) \\ &+ c_2\left(\left(1-\frac{1}{4}x^2+\frac{1}{64}x^4-\frac{1}{2304}x^6+O(x^8)\right)\ln(x)+\left(\frac{1}{4}x^2-\frac{3}{128}x^4+\frac{11}{13824}x^6+O(x^8)\right)\right) \\ &+ \left(x-\frac{7}{54}x^3+\frac{149}{27000}x^5-\frac{2161}{18522000}x^7+\cdots\right) \end{aligned}$$

2.2.2.3 Example 3. $(e^x-1)\,y''+e^x y'+y=0$

Solve

$$(e^x-1)\,y''+e^x y'+y=0 \tag{1}$$

Using power series method by expanding around $x=0$. Writing the ode as

$$y''+\frac{e^x}{(e^x-1)}y'+\frac{1}{(e^x-1)}y=0$$

Shows that $x=0$ is a singular point. But $\lim_{x\to 0} x\frac{e^x}{(e^x-1)}=1$ and $\lim_{x\to 0} x^2\frac{1}{(e^x-1)}=0$ Hence the singularity is removable. This means $x=0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned} y' &= \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2} \end{aligned}$$

Substituting the above in (1) gives

$$(e^x-1)\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+e^x\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Expanding e^x in Taylor series around x gives $e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots$. The above becomes

$$\begin{aligned} &\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots\right)\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2} \\ &\quad+\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots\right)\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0 \end{aligned} \tag{1}$$

Expanding gives (and keeping only terns up to x^4 gives

$$\begin{aligned} &x\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+\frac{1}{2}x^2\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+\frac{1}{6}x^3\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2} \\ &+\frac{1}{24}x^4\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+x\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\frac{1}{2}x^2\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} \\ &+\frac{1}{6}x^3\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\frac{1}{24}x^4\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0 \end{aligned}$$

Moving the x inside the sum, the above becomes

$$\begin{aligned} &\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}\frac{1}{2}(n+r)\,(n+r-1)\,a_n x^{n+r}+\sum_{n=0}^{\infty}\frac{1}{6}(n+r)\,(n+r-1)\,a_n x^{n+r+1} \\ &+\sum_{n=0}^{\infty}\frac{1}{24}(n+r)\,(n+r-1)\,a_n x^{n+r+2}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r}+\sum_{n=0}^{\infty}\frac{1}{2}(n+r)\,a_n x^{n+r+1} \\ &+\sum_{n=0}^{\infty}\frac{1}{6}(n+r)\,a_n x^{n+r+2}+\sum_{n=0}^{\infty}\frac{1}{24}(n+r)\,a_n x^{n+r+3}+\sum_{n=0}^{\infty}a_n x^{n+r}=0 \end{aligned}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=1}^{\infty} \frac{1}{2}(n+r-1)(n+r-2) a_{n-1} x^{n+r-1} + \sum_{n=2}^{\infty} \frac{1}{6}(n+r-2)(n+r-3) a_{n-2} x^{n+r-1}$$
$$+ \sum_{n=3}^{\infty} \frac{1}{24}(n+r-3)(n+r-4) a_{n-3} x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r-1} + \sum_{n=2}^{\infty} \frac{1}{2}(n+r-2) a_{n-2} x^{n+r-}$$
$$+ \sum_{n=3}^{\infty} \frac{1}{6}(n+r-3) a_{n-3} x^{n+r-1} + \sum_{n=4}^{\infty} \frac{1}{24}(n+r-4) a_{n-4} x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0 \quad (2)$$

The case $n=0$ gives the indicial equation

$$\begin{aligned}(n+r)(n+r-1) + (n+r) &= 0\\ (r)(r-1) + (r) &= 0\\ r^2 &= 0\end{aligned}$$

Hence the roots of the indicial equation are $r=0$ which is a double root. Hence $r_1 = r_2 = 0$. When this happens, the solution is given by

$$y = c_1 y_1 + c_2 y_2$$

Where y_1 is the first solution, which is assumed to be

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (3)$$

Where we take $a_0 = 1$ as it is arbitrary and where $r = r_1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} \quad (3)$$

Something important to notice. In the sum above, it starts from 1 and not from 0. The main issue is how to find b_n. Since that is the only thing we need to be able to complete the solution as $y_1(x)$ is easily found. It turns out that there is a relation between the b_n and the a_n. The b_n can be found by taking just derivative of a_n as function of r for each n and then evaluate the result at $r = r_1$. How this is done will be shown below. First we need to find $y_1(x)$. We take Eq(3) and substitute it in the original ODE. This will result in Eq (2) which we found above so no need to repeat that. We just need to remember that now we now what r is. It has a numerical value unlike the above phase where we still did not know its value.

Now we are ready to find a_n. We skip $n=0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose. We start from $n=1$.

For $n=1$ only, using Eq (2) gives

$$\begin{aligned}(n+r)(n+r-1) a_1 + \frac{1}{2}(n+r-1)(n+r-2) a_0 + (n+r) a_1 + (n+r-1) a_0 + a_0 &= 0\\ (1+r)(1+r-1) a_1 + \frac{1}{2}(1+r-1)(1+r-2) a_0 + (1+r) a_1 + (1+r-1) a_0 + a_0 &= 0\\ ((1+r)(1+r-1) + (1+r)) a_1 + \left(\frac{1}{2}(1+r-1)(1+r-2) + (1+r-1) + 1\right) a_0 &= 0\end{aligned}$$

But $a_0 = 1$. The above becomes

$$\begin{aligned}((1+r)(1+r-1) + (1+r)) a_1 &= -\left(\frac{1}{2}(1+r-1)(1+r-2) + (1+r-1) + 1\right)\\ a_1 &= \frac{-\left(\frac{1}{2}(1+r-1)(1+r-2) + (1+r-1) + 1\right)}{((1+r)(1+r-1) + (1+r))}\\ &= -\frac{(r^2+r+2)}{2r^2+4r+2}\end{aligned}$$

Which at $r=0$ gives

$$a_1 = -1$$

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1

For $n=2$ only, using Eq (2) gives

$$(n+r)(n+r-1)a_2+\frac{1}{2}(n+r-1)(n+r-2)a_1+\frac{1}{6}(n+r-2)(n+r-3)a_0+(n+r)a_2+(n+r-1)a_1+\frac{1}{2}(n+r-2)$$

$$(2+r)(2+r-1)a_2+\frac{1}{2}(2+r-1)(2+r-2)a_1+\frac{1}{6}(2+r-2)(2+r-3)a_0+(2+r)a_2+(2+r-1)a_1+\frac{1}{2}(2+r-2)$$

$$((2+r)(2+r-1)+(2+r))a_2+\left(\frac{1}{2}(2+r-1)(2+r-2)+(2+r-1)+1\right)a_1+\left(\frac{1}{6}(2+r-2)(2+r-3)+\frac{1}{2}(2+r-$$

$$(r+2)^2a_2+\left(\frac{1}{2}r^2+\frac{3}{2}r+2\right)a_1+\frac{1}{6}r($$

But $a_0=1$ and $a_1=-\frac{(r^2+r+2)}{2r^2+4r+2}$. The above becomes

$$(r+2)^2a_2+\left(\frac{1}{2}r^2+\frac{3}{2}r+2\right)\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right)+\frac{1}{6}r(r+2)=0$$

$$(r+2)^2a_2=\frac{r^4+4r^3+17r^2+26r+24}{2(r+1)^2}$$

$$a_2=\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$$

At $r=0$ the above becomes

$$a_2=\frac{24}{12(2)^2}=\frac{1}{2}$$

The table becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$

For $n=3$ only, using Eq (2) gives

$$(n+r)(n+r-1)a_3+\frac{1}{2}(n+r-1)(n+r-2)a_2+\frac{1}{6}(n+r-2)(n+r-3)a_1$$
$$+\frac{1}{24}(n+r-3)(n+r-4)a_0+(n+r)a_3+(n+r-1)a_2+\frac{1}{2}(n+r-2)a_1+\frac{1}{6}(n+r-3)a_0+a_2=0$$

Or

$$(3+r)(2+r)a_3+\frac{1}{2}(2+r)(1+r)a_2+(3+r)a_3+(2+r)a_2+\frac{1}{2}(1+r)a_1+a_2=0$$

Or

$$((3+r)(2+r)+(3+r))a_3+\left(\frac{1}{2}(2+r)(1+r)+(2+r)+1\right)a_2+\frac{1}{2}(1+r)a_1=0$$

$$(r+3)^2a_3+\left(\frac{1}{2}r^2+\frac{5}{2}r+4\right)a_2+\frac{1}{2}(1+r)a_1=0$$

But $a_1=-\frac{(r^2+r+2)}{2r^2+4r+2}, a_2=\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$. The above becomes

$$\begin{aligned}(r+3)^2a_3&=-\left(\frac{1}{2}r^2+\frac{5}{2}r+4\right)a_2-\frac{1}{2}(1+r)a_1\\&=-\left(\frac{1}{2}r^2+\frac{5}{2}r+4\right)\left(\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}\right)-\frac{1}{2}(1+r)\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right)\\&=-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2}\\a_3&=-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}\end{aligned}$$

For $r=0$ the above reduces to

$$a_3=-\frac{144}{24(2)^2(3)^2}=-\frac{1}{6}$$

The table becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$
3	$-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}$	$-\frac{1}{6}$

And so on. Recursion starts at $n \geq 5$ but we have enough terms, so we stop here. $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= 1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + \cdots \end{aligned} \tag{6A}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = y_1(x) \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

To find b_n, we will use the following

$$b_n(r) = \frac{d}{dr}(a_n(r)) \tag{7}$$

Notice that n starts from 1. Hence

$$b_1(r) = \frac{d}{dr}(a_1(r))\bigg|_{r=r_1}$$

What the above says, is that we first take derivative of $a_n(r)$ w.r.t. r and evaluate the result at the root of the indicial equation. Using the table above, we obtain (recalling that $r_1 = 0$ in this example)

n	$a_n(r)$	$a_n(r = r_1 = 0)$	$b_n(r) = \frac{d}{dr}(a_n(r))$
0	1	1	N/A since b starts from $n = 1$
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1	$\frac{d}{dr}\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) = -\frac{(r-3)}{2(r+1)^3}$
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$	$\frac{d}{dr}\left(\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}\right) = -\frac{(-r^4+7r^3+27r^2+53r+46)}{6(r^2+3r+2)^3}$
3	$-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}$	$-\frac{1}{6}$	$\frac{d}{dr}\left(-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}\right) = \frac{(-9r^7-44r^6+24r^5+662r^4}{24(r^3+}$

We have found all b_n terms. Hence

$$y_2 = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

And since $r = r_1 = 0$ then

$$y_2 = y_1 \ln(x) + \left(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots\right)$$

But from the above table, we see that $b_1 = \frac{3}{2}, b_2 = -\frac{23}{24}, b_3 = \frac{3}{8}$, The above becomes

$$y_2 = y_1 \ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O(x^4)\right)$$

And we know what $y_1(x)$ is from Eq (6A). Hence

$$y_2 = \left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O(x^4)\right)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O(x^4)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O(x^4)\right) + c_2\left(\left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O(x^4)\right)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O(x^4)\right)\right) \end{aligned}$$

This completes the solution.

2.2.2.4 Example 4 $x^2y'' + xy' + xy = 0$

$$x^2y'' + xy' + xy = 0$$

Comparing the ode to

$$y'' + py' + qy = 0$$

Hence $p = \frac{1}{x}, q = \frac{1}{x}$. Therefore $p_0 = \lim_{x\to 0} xp = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q = \lim_{x\to 0} x = 0$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r &= 0 \\ r^2 &= 0 \\ r &= 0, 0 \end{aligned}$$

Therefore $r_1 = 0, r_2 = 0$.

Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0
\end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \tag{1}$$

The indicial equation is obtained from $n = 0$. The above reduces to

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} &= 0 \\
(n+r)\,(n+r-1)\, a_n + (n+r)\, a_n &= 0 \\
(r)\,(r-1)\, a_0 + r a_0 &= 0 \\
a_0 \left((r^2 - r) + r \right) &= 0 \\
a_0 r^2 &= 0
\end{aligned}$$

Since $a_0 \neq 0$ then

$$r^2 = 0$$

Hence $r_1 = 0, r_2 = 0$. Since the roots are repeated then two linearly independent solutions can be constructed using

$$\begin{aligned}
y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \\
y_2 &= y_1 \ln(x) + x^{r_2} \sum_{n=1}^{\infty} b_n x^n = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n
\end{aligned}$$

For $n \geq 1$ the recurrence relation is

$$\begin{aligned}
(n+r)\,(n+r-1)\, a_n + (n+r)\, a_n + a_{n-1} &= 0 \\
a_n &= -\frac{a_{n-1}}{(n+r)\,(n+r-1) + (n+r)} \\
&= -\frac{a_{n-1}}{(n+r)^2} \qquad (1)
\end{aligned}$$

Starting with y_1. From (1) with $r = 0$ gives

$$a_n = -\frac{a_{n-1}}{n^2}$$

For $n = 1$ and using $a_0 = 1$

$$a_1 = -1$$

For $n = 2$

$$a_2 = -\frac{a_1}{4} = \frac{1}{4}$$

And so on. Hence

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= 1 - x + \frac{1}{4} x^2 - \frac{1}{36} x^3 + \cdots
\end{aligned}$$

In the case of duplicate roots, b_n is found using $b_n = \frac{d}{dr} a_n(r)$. And this is evaluated at $r = r_0 = 0$ in this case since $r_0 = 0$ here. So we need to find $a_n(r)$. This is done from (1). For $n = 1$

$$
\begin{aligned}
b_1 &= \frac{d}{dr}(a_1(r)) \\
b_1 &= \frac{d}{dr}\left(-\frac{a_0}{(1+r)^2}\right) = \frac{d}{dr}\left(-\frac{1}{(1+r)^2}\right) = \frac{2}{(r+1)^3}
\end{aligned}
$$

Evaluated at $r = 0$ gives

$$
b_1 = 2
$$

For $n = 2$ then (2) becomes

$$
\begin{aligned}
b_2 &= \frac{d}{dr}(a_2(r)) \\
b_2 &= \frac{d}{dr}\left(-\frac{a_1}{(2+r)^2}\right) \\
&= \frac{d}{dr}\left(-\frac{-\frac{1}{(1+r)^2}}{(2+r)^2}\right) \\
&= \frac{d}{dr}\left(\frac{1}{(r+1)^2 (r+2)^2}\right) \\
&= -2\frac{2r+3}{(r^2+3r+2)^3}
\end{aligned}
$$

At $r = 0$ the above becomes

$$
\begin{aligned}
b_2 &= -2\frac{3}{(2)^3} \\
&= -\frac{3}{4}
\end{aligned}
$$

And so on. Just remember when replacing the a_n in the above, is to use the original $a_n(r)$ as function of r and not the actual a_n values from above. It has to be function of r first before taking derivatives, Hence

$$
\begin{aligned}
y_2 &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n \\
&= y_1 \ln(x) + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\
&= y_1 \ln(x) + 2x - \frac{3}{4}x^2 + \cdots \\
&= y_1 \ln(x) + \left(2x - \frac{3}{4}x^2 + \cdots\right)
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots\right) + c_2\left(y_1 \ln(x) + \left(2x - \frac{3}{4}x^2 + \cdots\right)\right)
\end{aligned}
$$

2.2.2.5 Example 5 $x^2y'' + xy' + xy = 1$

$$
x^2y'' + xy' + xy = 1
$$

The homogenous ode was solved up, so we just need to find y_p. To find y_p, and using c in place of a so not to confuse terms with the y_h terms, then from the above problem, we found the indicial equation. Hence the balance equation is

$$
c_0 r^2 x^r = 1
$$

To balance this we need $r = 0$. Hence $0c_0 = 1$ which is not possible. Hence no particular solution exists. No solution in series exists.

2.2.2.6 Example 6 $x^2y'' + xy' + xy = \frac{1}{x}$

$$
x^2y'' + xy' + xy = \frac{1}{x}
$$

This is the same ode as above but with different RHS. So we will go directly to finding y_p. From above we found that the balance equation is

$$
c_0 r^2 x^r = x^{-1}
$$

Which implies $r = -1$ and therefore $r^2 c_0 = 1$ or $c_0 = 1$. Using the recurrence equation (1) in the above problem using using c_n in place of a_n gives

$$c_n = -\frac{c_{n-1}}{(n+r)^2}$$

For $m = -1$

$$c_n = -\frac{c_{n-1}}{(n-1)^2}$$

Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \frac{1}{x}\sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Now to find few c_n terms. For $n = 1$

$$\begin{aligned} c_1 &= -\frac{c_0}{(1-1)^2} \\ &= \infty \end{aligned}$$

Hence series does not converge. No y_p exist. There is no solution in terms of series solution.

2.2.2.7 Example 7 $x^2y'' + xy' + xy = x$

$$x^2y'' + xy' + xy = x$$

This is the same ode as above, where we found y_h but with different RHS. So we will go directly to finding y_p. From above we found that the balance equation is

$$r^2 c_0 x^r = x$$

Which implies $r = 1$ and therefore $r^2 c_0 = 1$ or $c_0 = 1$. Using the recurrence equation (1) in the above problem and using c_n in place of a_n gives

$$c_n = -\frac{c_{n-1}}{(n+r)^2}$$

For $r = 1$

$$c_n = -\frac{c_{n-1}}{(n+1)^2}$$

Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= x\sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Now to find few c_n terms. For $n = 1$

$$c_1 = -\frac{c_0}{(2)^2} = -\frac{1}{4}$$

For $n = 2$

$$c_2 = -\frac{c_1}{(2+1)^2} = \frac{\frac{1}{4}}{9} = \frac{1}{36}$$

For $n = 3$

$$c_3 = -\frac{c_2}{(3+1)^2} = -\frac{\frac{1}{36}}{16} = -\frac{1}{576}$$

And so on. Hence

$$\begin{aligned} y_p &= x\sum_{n=0}^{\infty} c_n x^n \\ &= x(c_0 + c_1 x + c_2 x^2 + \cdots) \\ &= x\left(1 - \frac{1}{4}x + \frac{1}{36}x^2 - \frac{1}{576}x^3 + \cdots\right) \\ &= \left(x - \frac{1}{4}x^2 + \frac{1}{36}x^3 - \frac{1}{576}x^4 + \cdots\right) \end{aligned}$$

Using y_h found in the above problem since that does not change, then the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{1}{576}x^4 + \cdots\right) \\
&+ c_2\left(\ln(x)\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \frac{1}{576}x^4 + \cdots\right) + \left(2x - \frac{3}{4}x^2 + \frac{14}{108}x^3 + \cdots\right)\right) \\
&+ \left(x - \frac{1}{4}x^2 + \frac{1}{36}x^3 - \frac{1}{576}x^4 + \cdots\right)
\end{aligned}$$

2.2.2.8 Example 8 $xy'' + y' - xy = 0$

$$xy'' + y' - xy = 0$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = -1$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 = 0$. Hence the indicial equation is

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + r &= 0 \\
r^2 &= 0 \\
r &= 0, 0
\end{aligned}$$

Therefore $r_1 = 0, r_2 = 0$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
x\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} - x\sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0
\end{aligned}$$

Re indexing to lowest powers on x gives

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} &= 0 \\
\sum_{n=0}^{\infty} \left((n+r)(n+r-1) + (n+r)\right) a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} &= 0 \\
\sum_{n=0}^{\infty} (n+r)^2\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} &= 0 \qquad (1)
\end{aligned}$$

The indicial equation is obtained from $n = 0$. The above reduces to

$$r^2 a_0 x^{n+r-1} = 0$$

Since $a_0 \neq 0$ then

$$r^2 = 0$$

Hence $r_1 = 0, r_2 = 0$ as found earlier. Since the roots are repeated then two linearly independent solutions can be constructed using

$$\begin{aligned}
y_1 &= x^{r_1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \\
y_2 &= y_1 \ln(x) + x^{r_2}\sum_{n=1}^{\infty} b_n x^n = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n
\end{aligned}$$

$n = 1$ gives

$$\begin{aligned}
(1+r)(r)\,a_1 + (1+r)\,a_1 &= 0 \\
(r+1)^2\,a_1 &= 0
\end{aligned}$$

Hence $a_1 = 0$. The recurrence relation is obtained for $n \geq 2$. From (1)

$$n + r(n+r-1)\, a_n + (n+r)\, a_n - a_{n-2} = 0$$
$$a_n = \frac{a_{n-2}}{(n+r)^2} \tag{1}$$

Since we need to differentiate y_1 to obtain y_2 and the differentiation is w.r.t r, we will carry the calculations with r in place and at the end replace r by its value (which happened to be *zero* in this example). We do this only in the case of repeated roots.

For $n = 2$

$$a_2 = \frac{a_0}{(2+r)^2} = \frac{1}{(2+r)^2}$$

For $n = 3$

$$a_3 = \frac{a_1}{(3+r)^2} = 0$$

For $n = 4$

$$a_4 = \frac{a_2}{(4+r)^2} = \frac{\frac{1}{(2+r)^2}}{(4+r)^2} = \frac{1}{(2+r)^2 (4+r)^2}$$

For $n = 5$, we will find $a_5 = 0$ (for all odd n this is the case). For $n = 6$

$$a_6 = \frac{a_4}{(6+r)^2} = \frac{1}{(2+r)^2 (4+r)^2 (6+r)^2}$$

And so on. We see that n^{th} term is $a_n = \Pi_{j=1}^{k} \frac{1}{(2j+r)^2}$. Now we can substitute the $r = 0$ value into the above to obtain

$$\begin{aligned} a_2 &= \frac{1}{4} \\ a_4 &= \frac{1}{64} \\ a_6 &= \frac{1}{2304} \end{aligned}$$

Hence

$$\begin{aligned} y_1 &= \sum a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= 1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \cdots \end{aligned}$$

To find y_2 we use $b_n = \frac{d}{dr} a_n$ and evaluate this at $r = r_2$ which in this case is zero. Hence

$$b_2 = \frac{d}{dr} a_2 = \frac{d}{dr}\left(\frac{1}{(2+r)^2}\right) = \left(-\frac{2}{(r+2)^3}\right)_{r=0} = -\frac{2}{8} = -\frac{1}{4}$$

$$b_4 = \frac{d}{dr} a_4 = \frac{d}{dr}\left(\frac{1}{(2+r)^2 (4+r)^2}\right) = \left(-4\frac{r+3}{(r^2+6r+8)^3}\right)_{r=0} = \left(-4\frac{3}{(8)^3}\right) = -\frac{3}{128}$$

$$\begin{aligned} b_6 &= \frac{d}{dr} a_6 \\ &= \frac{d}{dr}\left(\frac{1}{(2+r)^2 (4+r)^2 (6+r)^2}\right) \\ &= \left(-2\frac{3r^2 + 24r + 44}{(r^3 + 12r^2 + 44r + 48)^3}\right)_{r=0} \\ &= -2\frac{44}{(48)^3} \\ &= -\frac{11}{13\,824} \end{aligned}$$

And so on. Hence

$$\begin{aligned} y_1 &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2} \\ &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n \\ &= y_1 \ln(x) + \left(b_2 x^2 + b_4 x^4 + b_6 x^6 + \cdots\right) \\ &= y_1 \ln(x) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 + -\frac{11}{13\,824}x^6 + \cdots\right) \end{aligned}$$

Therefore the complete solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \cdots\right) \\
&+ c_2 \left(\ln(x)\left(1 + \frac{1}{4}x^2 + \frac{1}{64}x^4 + \frac{1}{2304}x^6 + \cdots\right) + \left(-\frac{1}{4}x^2 - \frac{3}{128}x^4 + -\frac{11}{13824}x^6 + \cdots\right)\right)
\end{aligned}$$

2.2.2.9 Example 9 $\sin(x)\,y'' + y' + y = 0$

$$\sin(x)\,y'' + y' + y = 0$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{1}{\sin(x)}, q(x) = \frac{1}{\sin x}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x}{x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots} = \frac{1}{1 - \frac{x}{3!} + \frac{x^4}{5!} -} = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{x^2}{x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots} = \frac{x}{1 - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots} = 0$. Hence the indicial equation is

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + r &= 0 \\
r^2 &= 0 \\
r &= 0, 0
\end{aligned}$$

Therefore $r_1 = 0, r_2 = 0$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\sin(x) \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0
\end{aligned}$$

Using $O(x^7)$ terms as the Order of the series (if more terms are needed we will use more terms from the $\sin x$ series). This means we have to now only expand up to $n = 7$ as that is the order used for the series of $\sin x$. The above becomes

$$\begin{aligned}
&x \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} - \frac{x^3}{3!} \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} \\
&\quad + \frac{x^5}{5!} \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\end{aligned}$$

Which becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-1} - \sum_{n=0}^{\infty} \frac{1}{6}(n+r)\,(n+r-1)\,a_n x^{n+r+1} \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{120}(n+r)\,(n+r-1)\,a_n x^{n+r+3} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
\end{aligned}$$

Re indexing to lowest powers on x gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} \frac{1}{6}(n+r-2)\,(n+r-3)\,a_{n-2} x^{n+r-1} \\
&\quad + \sum_{n=4}^{\infty} \frac{1}{120}(n+r-4)\,(n+r-5)\,a_{n-4} x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0
\end{aligned}$$

Simplifying gives

$$\sum_{n=0}^{\infty} (n+r)^2\,a_n x^{n+r-1} - \sum_{n=2}^{\infty} \frac{(n+r-2)\,(n+r-3)}{6} a_{n-2} x^{n+r-1} + \sum_{n=4}^{\infty} \frac{(n+r-4)\,(n+r-5)}{120} a_{n-4} x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0 \tag{1}$$

The indicial equation is obtained from $n=0$. The above reduces to

$$r^2 a_0 x^{r-1} = 0$$

Since $a_0 \neq 0$ then

$$r^2 = 0$$

Hence $r_1 = 0, r_2 = 0$ as found earlier. Since the roots are repeated then two linearly independent solutions can be constructed using

$$\begin{aligned}
y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n \\
y_2 &= y_1 \ln(x) + x^{r_2} \sum_{n=1}^{\infty} b_n x^n = y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n
\end{aligned}$$

$n=1$ gives from (1) and by taking $a_0 = 1$

$$\begin{aligned}
(1+r)^2 a_1 + a_0 &= 0 \\
a_1 &= -\frac{a_0}{(1+r)^2} \\
&= -\frac{1}{(1+r)^2}
\end{aligned}$$

For $n=2$ gives from (1)

$$\begin{aligned}
(2+r)^2 a_2 - \frac{(r)(r-1)}{6} a_0 + a_1 &= 0 \\
(2+r)^2 a_2 &= -a_1 + \frac{(r)(r-1)}{6} a_0 \\
a_2 &= \frac{1}{(1+r)^2 (2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2}
\end{aligned}$$

For $n=3$

$$\begin{aligned}
(3+r)^2 a_3 - \frac{(1+r)(r)}{6} a_1 + a_2 &= 0 \\
a_3 &= -\frac{a_2}{(3+r)^2} + \frac{(1+r)(r)}{6(3+r)^2} a_1 \\
&= -\frac{\frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2}}{(3+r)^2} - \frac{(1+r)(r)}{6(3+r)^2} \frac{1}{(1+r)^2} \\
&= -\frac{(r^4 + r^3 - r^2 - r + 6)}{6(r+3)^2 (r^2+3r+2)^2} - \frac{(1+r)(r)}{6(3+r)^2 (1+r)^2}
\end{aligned}$$

For $n \geq 4$ the recurrence relation is

$$(n+r)^2 a_n - \frac{(n+r-2)(n+r-3)}{6} a_{n-2} + \frac{(n+r-4)(n+r-5)}{120} a_{n-4} + a_{n-1} = 0$$

Or

$$a_n = -\frac{a_{n-1}}{(n+r)^2} + \frac{(n+r-2)(n+r-3)}{6(n+r)^2} a_{n-2} - \frac{(n+r-4)(n+r-5)}{120(n+r)^2} a_{n-4} \tag{2}$$

Since we need to differentiate y_1 to obtain y_2 and the differentiation is w.r.t r, we will carry the calculations with r in place and at the end replace r by its value (which happened to be *zero* in this example). We do this only in the case of repeated roots.

For $n=4$ then (2) gives

$$\begin{aligned}
a_4 &= -\frac{a_3}{(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2} a_2 - \frac{(r)(-1+r)}{120(4+r)^2} a_0 \\
&= -\frac{-\frac{1}{(r+1)^2(r+2)^2(r+3)^2}}{(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2} a_2 - \frac{(r)(-1+r)}{120(4+r)^2} a_0 \\
&= \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2} \frac{1}{(r+1)^2 (r+2)^2} - \frac{(r)(-1+r)}{120(4+r)^2}
\end{aligned}$$

And so on. Now we replace $r=0$ to find y_1. Just remember not to use anything over $n=5$ since we cut off the series for $\sin(x)$ at x^5.

Using $r=0$, then the above values for a_i found become

$$
\begin{aligned}
a_1 &= -\frac{1}{(1+r)^2} = -1\\
a_2 &= \frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2} = \frac{1}{4}\\
a_3 &= -\frac{(r^4+r^3-r^2-r+6)}{6(r+3)^2(r^2+3r+2)^2} - \frac{(1+r)(r)}{6(3+r)^2(1+r)^2} = -\frac{1}{(2)^2(3)^2} = -\frac{1}{36}\\
a_4 &= \frac{1}{(r+1)^2(r+2)^2(r+3)^2(4+r)^2} + \frac{(2+r)(1+r)}{6(4+r)^2}\frac{1}{(r+1)^2(r+2)^2} - \frac{(r)(-1+r)}{120(4+r)^2}\\
&= \frac{1}{(2)^2(3)^2(4)^2} + \frac{(2)}{6(4)^2}\frac{1}{(2)^2}\\
&= \frac{1}{144}
\end{aligned}
$$

Let find one more term. For $n=5$ then (2) gives

$$
\begin{aligned}
a_5 &= -\frac{a_4}{(5+r)^2} + \frac{(3+r)(2+r)}{6(5+r)^2}a_3 - \frac{(1+r)(r)}{120(5+r)^2}a_1\\
&= -\frac{\frac{1}{144}}{5^2} + \frac{(3)(2)}{6(5)^2}\left(-\frac{1}{36}\right)\\
&= -\frac{1}{720}
\end{aligned}
$$

For $n=6$ the above recurrence relation gives

$$
\begin{aligned}
a_6 &= -\frac{a_5}{(6+r)^2} + \frac{(4+r)(3+r)}{6(6+r)^2}a_4 - \frac{(2+r)(1+r)}{120(6+r)^2}a_2\\
&= -\frac{-\frac{1}{720}}{6^2} + \frac{(4)(3)}{6(6)^2}\frac{1}{144} - \frac{(2)}{120(6)^2}\frac{1}{4}\\
&= \frac{1}{3240}
\end{aligned}
$$

For $n=7$

$$
\begin{aligned}
a_7 &= -\frac{a_6}{(7+r)^2} + \frac{(5+r)(4+r)}{6(7+r)^2}a_5 - \frac{(3+r)(2+r)}{120(7+r)^2}a_3\\
&= -\frac{\frac{1}{3240}}{(7)^2} + \frac{(5)(4)}{6(7)^2}\left(-\frac{1}{720}\right) - \frac{(3)(2)}{120(7)^2}\left(-\frac{1}{36}\right)\\
&= -\frac{23}{317520}
\end{aligned}
$$

For $n=8$

$$
\begin{aligned}
a_8 &= -\frac{a_7}{(8+r)^2} + \frac{(6+r)(5+r)}{6(8+r)^2}a_6 - \frac{(4+r)(3+r)}{120(8+r)^2}a_4\\
&= -\frac{\left(-\frac{23}{317520}\right)}{(8)^2} + \frac{(6)(5)}{6(8)^2}\left(\frac{1}{3240}\right) - \frac{(4)(3)}{120(8)^2}\left(\frac{1}{144}\right)\\
&= \frac{13}{903168}
\end{aligned}
$$

Which is now the wrong value. It should be $\frac{1}{62720}$. So using 3 terms from $\sin x$ we obtain up to a_7 correct terms. Hence

$$
\begin{aligned}
y_1 &= \sum a_n x^n\\
&= a_0 + a_1 x + a_2 x^2 + \cdots\\
&= 1 - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{144}x^4 - \frac{1}{720}x^5 + \frac{1}{3240}x^6 - \frac{23}{317520}x^7 + \cdots
\end{aligned}
$$

What would have happened if we expanded $\sin(x)$ only for two terms? Lets find out. The ode becomes

$$
\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0
$$

$$
\left(x - \frac{x^3}{3!} + \cdots\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0
$$

The above becomes

$$
x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - \frac{x^3}{3!}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0
$$

$$
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}\frac{1}{6}(n+r)(n+r-1)a_n x^{n+r+1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0
$$

Reindex

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}-\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

$$\sum_{n=0}^{\infty}(n+r)^2 a_n x^{n+r-1}-\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0$$

For $n=0$ we obtain the indicial equation as we did above. For $n=1$

$$\begin{aligned}\left(1+r^2\right)a_1+a_0&=0\\ a_1&=-\frac{a_0}{(1+r^2)}\\ &=-\frac{1}{(1+r^2)}\end{aligned}$$

For $r=0$ this gives

$$a_1=-1$$

$n\geq 2$ gives

$$\begin{aligned}(n+r)^2 a_n-\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}+a_{n-1}&=0\\ a_n&=-\frac{a_{n-1}}{(n+r)^2}+\frac{1}{6}\frac{(n+r-2)(n+r-3)}{(n+r)^2}a_{n-2}\end{aligned}\tag{2A}$$

Hence for $n=2$

$$\begin{aligned}a_2&=-\frac{a_1}{(2+r)^2}+\frac{1}{6}\frac{r(-1+r)}{(2+r)^2}a_0\\ &=-\frac{-\frac{1}{(1+r^2)}}{(2+r)^2}+\frac{1}{6}\frac{r(-1+r)}{(2+r)^2}\end{aligned}$$

For $r=0$ the above gives

$$a_2=-\frac{-\frac{1}{(1)}}{(2)^2}=\frac{1}{4}$$

$n=3$ gives

$$\begin{aligned}a_3&=-\frac{a_2}{(3+r)^2}+\frac{1}{6}\frac{(1+r)(r)}{(3+r)^2}a_1\\ &=-\frac{\frac{1}{4}}{(3+r)^2}-\frac{1}{6}\frac{(1+r)(r)}{(3+r)^2}\end{aligned}$$

For $r=0$

$$a_3=-\frac{\frac{1}{4}}{(3)^2}=-\frac{1}{36}$$

For $n=4$

$$\begin{aligned}a_4&=-\frac{a_3}{(4+r)^2}+\frac{1}{6}\frac{(2+r)(1+r)}{(4+r)^2}a_2\\ &=-\frac{a_3}{(4)^2}+\frac{1}{6}\frac{(2)(1)}{(4)^2}a_2\\ &=-\frac{\left(-\frac{1}{36}\right)}{(4)^2}+\frac{1}{6}\frac{(2)(1)}{(4)^2}\left(\frac{1}{4}\right)\\ &=\frac{1}{144}\end{aligned}$$

For $n=5$

$$\begin{aligned}a_5&=-\frac{a_4}{(5+r)^2}+\frac{1}{6}\frac{(3+r)(2+r)}{(5+r)^2}a_3\\ &=-\frac{\frac{1}{144}}{(5)^2}+\frac{1}{6}\frac{(3)(2)}{(5)^2}\left(-\frac{1}{36}\right)=-\frac{1}{720}\end{aligned}$$

For $n=6$

$$\begin{aligned}a_6&=-\frac{a_5}{(6+r)^2}+\frac{1}{6}\frac{(6+r-2)(6+r-3)}{(6+r)^2}a_4\\ &=-\frac{\left(-\frac{1}{720}\right)}{(6)^2}+\frac{1}{6}\frac{(4)(3)}{6^2}\frac{1}{144}\\ &=\frac{11}{25\,920}\end{aligned}$$

Which is the wrong value. We see that using two terms only from the $\sin(x)$ gave up correct a_n values up to a_5. What if we used only one term? Lets find out.

$$
\begin{aligned}
\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
(x+\cdots)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}&=0\\
\sum_{n=0}^{\infty}(n+r)^2a_nx^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}&=0
\end{aligned}
$$

$n=0$ gives the indicial equation. For $n\geq 1$ the recurrence relation is

$$
\begin{aligned}
(n+r)^2a_n+a_{n-1}&=0\\
a_n&=-\frac{a_{n-1}}{(n+r)^2}
\end{aligned}
$$

For $n=1$

$$
\begin{aligned}
a_1&=-\frac{a_0}{(1+r)^2}\\
&=-\frac{1}{(1+r)^2}
\end{aligned}
$$

For $r=0$

$$a_1=-1$$

For $n=2$

$$a_2=-\frac{a_1}{(2+r)^2}=\frac{1}{(2+r)^2}$$

For $r=0$

$$a_2=\frac{1}{4}$$

For $n=3$

$$a_3=-\frac{a_2}{(3+r)^2}=-\frac{\frac{1}{4}}{(3+r)^2}$$

For $r=0$

$$a_3=-\frac{\frac{1}{4}}{(3)^2}=-\frac{1}{36}$$

For $n=4$

$$a_4=-\frac{a_3}{(4+r)^2}=-\frac{-\frac{1}{36}}{(4+r)^2}$$

For $r=0$

$$a_4=-\frac{-\frac{1}{36}}{(4)^2}=\frac{1}{576}$$

We see that this is the wrong value. So when using one term only we obtain correct a_n up to a_3. What do we learn from all the above? It is that if we expand $f(x)$ up to $O(x^n)$ order, then we can only determine correct terms up to a_n and no more. In the above when we used $\sin(x)=x-\frac{x^3}{6}+\frac{x^5}{120}+O(x^7)$ then we obtained correct terms up to a_7. And when we used $\sin(x)=x-\frac{x^3}{6}+O(x^5)$ then we obtained correct terms up to a_5 and when we used $\sin(x)=x+O(x^3)$ then we obtained correct terms up to a_3. So we should keep this in mind from now on,.

To find y_2 we use $b_n = \frac{d}{dr}a_n$ and evaluate this at $r = r_2$ which in this case is zero. Hence

$$
\begin{aligned}
b_1 &= \frac{d}{dr}a_1 = \frac{d}{dr}\left(-\frac{1}{(1+r)^2}\right)_{r=0} = \frac{2}{(r+1)^3} = 2 \\
b_2 &= \frac{d}{dr}a_2 = \frac{d}{dr}\left(\frac{1}{(1+r)^2(2+r)^2} + \frac{(r)(r-1)}{6(2+r)^2}\right) = \left(\frac{5r^4+13r^3+9r^2-25r-38}{6(r^2+3r+2)^3}\right)_{r=0} = \frac{-38}{6(2)^3} = -\frac{19}{24} \\
b_3 &= \frac{d}{dr}a_3 \\
&= \frac{d}{dr}\left(-\frac{(r^4+r^3-r^2-r+6)}{6(r+3)^2(r^2+3r+2)^2} - \frac{(1+r)(r)}{6(3+r)^2(1+r)^2}\right) \\
&= \left(\frac{(4r^6+18r^5+20r^4-15r^3-18r^2+93r+114)}{6(r^3+6r^2+11r+6)^3}\right)_{r=0} \\
&= \frac{114}{6(6)^3} \\
&= \frac{19}{216}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_1 &= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r_2} \\
&= y_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^n \\
&= y_1 \ln(x) + \left(2x - \frac{19}{24}x^2 + \frac{19}{216}x^3 + \cdots\right)
\end{aligned}
$$

Therefore the complete solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{144}x^4 + \cdots\right) \\
&+ c_2\left(\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{144}x^4 + \cdots\right)\ln(x) + \left(2x - \frac{19}{24}x^2 + \frac{19}{216}x^3 + \cdots\right)\right)
\end{aligned}
$$

2.3 Frobenius series. Roots differ by integer. Good case (no log needed)

Local contents

2.3.1 Algorithm

In this case the solution is

$$
y = c_1 y_1 + c_2 y_2
$$

There are two sub cases that show up when roots differ by integer. First sub case is when the second solution y_2 is obtained similar to how y_1 is obtained. i.e. using standard Frobenius series but with the second root. The second sub case is the harder one, this is when y_2 fails to be obtained using the standard method due to b_N being undefined where N is the difference between the roots. In this sub case we need to use a modified Frobenius series method where, which is explained more using examples below. Therefore for sub case one (called the good case) we have

$$
\begin{aligned}
y_1 &= x^{r_1}\sum_{n=0}^{\infty} a_n x^n \\
y_2 &= C y_1 \ln(x) + x^{r_2}\sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

Where C above come out to be zero (good case). Hence the above simplifies to

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2}
\end{aligned}
$$

For the second subcase (called the bad case), C above is not zero. To determine if $C=0$ or not, we first find all the a_n, including a_N where N is difference between the roots. Then evaluate

$$\lim_{r\to r_2} a_N(r)$$

And if this exists, then $C=0$. In the above we have to keep $a_n(r)$ as function of r in symbolic form to do this.

2.3.2 Examples

Local contents

2.3.2.1 Example 1 $(x-x^2)\,y''+3y'+2y=3x^2$

$$(x-x^2)\,y''+3y'+2y=3x^2$$

Comparing the above to $y''+p(x)\,y'+q(x)\,y=0$ shows that $p(x)=\frac{3}{x(1-x)}, q(x)=\frac{2}{x(x-1)}$. Hence there are two singular points, one at $x=0$ and one at $x=1$. Let the expansion be around $x=0$. This means the solution will define up to $x=1$, which is the next nearest singular point.

$$p_0=\lim_{x\to 0} xp(x)=\lim_{x\to 0} x\frac{3}{x(1-x)}=3$$

And

$$q_0=\lim_{x\to 0} x^2\frac{2}{x(1-x)}=0$$

Hence $x_0=0$ is a regular singular point. The indicial equation is

$$\begin{aligned}
r(r-1)+p_0r+q_0&=0\\
r(r-1)+3r&=0\\
r^2-r+3r&=0\\
r^2+2r&=0\\
r(r+2)&=0
\end{aligned}$$

Therefore $r=0, r=-2$. They differ by an integer $N=2$. Therefore two linearly independent solutions can be constructed using

$$\begin{aligned}
y_1&=\sum_{n=0}^{\infty} a_n x^{n+r_1}\\
y_2&=Cy_1\ln(x)+\sum_{n=0}^{\infty} b_n x^{n+r_2}
\end{aligned}$$

Where C above can be zero depending on a condition given below. Now we will work out the solution for a general r. Let

$$\begin{aligned}
y&=\sum_{n=0}^{\infty} a_n x^{n+r}\\
y'&=\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\\
y''&=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}$$

The homogeneous ode becomes

$$(x-x^2)\,y''+3y'+2y=0$$

$$(x-x^2)\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+3\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+2\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-1}-\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)\,a_n x^{n+r-1}+\sum_{n=0}^{\infty}2a_n x^{n+r}=0$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-1}-\sum_{n=1}^{\infty}(n+r-1)\,(n+r-2)\,a_{n-1}x^{n+r-1}+\sum_{n=0}^{\infty}3(n+r)\,a_n x^{n+r-1}+\sum_{n=1}^{\infty}2a_{n-1}x^{n+r-1}=0 \tag{1A}$$

For $n = 0$

$$\begin{aligned}(n+r)(n+r-1)\, a_n x^{n+r-1} + 3(n+r)\, a_n x^{n+r-1} &= 0\\ (r(r-1)+3r)\, a_0 x^{r-1} &=\\ (r^2+2r)\, a_0 x^{r-1} &= 0\end{aligned} \tag{1B}$$

Since $a_0 \neq 0$, then $r = 0, r = -2$ as was found above. Hence $N = 2$ which is the difference between the two roots. The homogenous ode therefore satisfies

$$(x - x^2)\, y'' + 3y' + 2y = 0$$

The recurrence relation is when $n \geq 1$ from (1A) and is given by

$$(n+r)(n+r-1)\, a_n - (n+r-1)(n+r-2)\, a_{n-1} + 3(n+r)\, a_n + 2a_{n-1} = 0$$

Keeping larger a_n on the left and all lower a_n on the right gives

$$\begin{aligned}a_n &= \frac{-2 + (n+r-1)(n+r-2)}{(n+r)(n+r-1) + 3(n+r)} a_{n-1}\\ a_n &= \frac{n+r-3}{n+r+2} a_{n-1}\end{aligned} \tag{1}$$

Now we find $y_h = c_1 y_1 + c_2 y_2$. For $n = 1$ and letting $a_0 = 1$ then (1) gives

$$\begin{aligned}a_1 &= \frac{-2+r}{3+r} a_0\\ &= \frac{-2+r}{3+r}\\ &= \frac{-2}{3}\end{aligned}$$

For $n = 2$ Eq. (1) gives

$$\begin{aligned}a_2 &= \frac{-1+r}{4+r} a_1\\ &= \left(\frac{-1+r}{4+r}\right)\left(\frac{-2+r}{3+r}\right)\\ &= \left(\frac{-1}{4}\right)\left(\frac{-2}{3}\right) = \frac{1}{6}\end{aligned}$$

For $n = 3$ Eq. (1) gives

$$a_3 = \frac{3+r-3}{3+r+2} a_{n-1} = 0$$

And all other higher $a_n = 0$. Hence

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_0 + a_1 x + a_2 x^2\\ &= 1 - \frac{2}{3}x + \frac{1}{6}x^2\end{aligned}$$

Now we need to find y_2. We first check if y_2 can be found using standard method as was done above for y_1. We need to evaluate

$$\begin{aligned}\lim_{r \to r_2} a_N(r) &= \lim_{r \to -2} a_2(r)\\ &= \lim_{r \to -2} \left(\frac{-1+r}{4+r}\right)\left(\frac{-2+r}{3+r}\right)\\ &= \lim_{r \to -2} \left(\frac{-1-2}{4-2}\right)\left(\frac{-2-2}{3-2}\right)\\ &= 6\end{aligned}$$

Since limit exist then $C = 0$ and we do not need the log term.

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

From (1) and using b instead of a and using $r = r_2 = -2$ gives

$$\begin{aligned}b_n &= \frac{n+r-3}{n+r+2} b_{n-1}\\ &= \frac{n-2-3}{n-2+2} b_{n-1}\\ &= \frac{n-5}{n} b_{n-1}\end{aligned}$$

Hence for $n=1$ and using $b_0=1$ as we did for a_0 gives

$$b_1 = -4b_0 = -4$$

For $n=N=2$ which is the special case,

$$b_n = \frac{-3}{2} b_1 = 6$$

Since b_N is defined, we can continue and y_2 is found using same recurrence relation. Hence this is subcase one. For $n=3$

$$b_3 = \frac{-2}{3} b_2 = -4$$

For $n=4$

$$b_4 = \frac{-1}{4} b_3 = 1$$

And so on. Hence

$$\begin{aligned}
y_2 &= \frac{1}{x^2} \sum_{n=0}^{\infty} b_n x^n \\
&= \frac{1}{x^2} \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 \right) \\
&= \frac{1}{x^2} \left(1 - 4x + 6x^2 - 4x^3 + x^4 \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(1 - \frac{2}{3} x + \frac{1}{6} x^2 \right) + c_2 \left(\frac{1}{x^2} \left(1 - 4x + 6x^2 - 4x^3 + x^4 \right) \right)
\end{aligned}$$

Now we find y_p. From earlier we found in (1B) the balance equation which gives

$$\left(x - x^2 \right) y'' + 3y' + 2y = 3x^2$$

The balance relation is $\left(r^2 + 2r \right) c_0 x^{r-1} = 3x^2$. This implies $r - 1 = 2$ or $r = 3$. Therefore $\left(r^2 + 2r \right) c_0 = 3$ or $(9 + 6) c_0 = 3$ which gives $c_0 = \frac{3}{15} = \frac{1}{5}$. Hence

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}$$

To find c_n, the same recurrence relation given in (1) is used again but a replaced by c. This gives the recurrence relation to find coefficients of the particular solution as

$$c_n = \frac{n + r - 3}{n + r + 2} c_{n-1}$$

For $r = 3$ the above becomes

$$\begin{aligned}
c_n &= \frac{n + 3 - 3}{n + 3 + 2} c_{n-1} \\
&= \frac{n}{n + 5} c_{n-1}
\end{aligned}$$

For $n = 1$

$$c_1 = \frac{1}{6} c_0 = \frac{1}{6} \left(\frac{1}{5} \right) = \frac{1}{30}$$

For $n = 2$

$$c_2 = \frac{2}{2 + 5} c_1 = \frac{2}{7} \left(\frac{1}{30} \right) = \frac{1}{105}$$

And so on. Hence

$$\begin{aligned}
y_p &= x^3 \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \left(c_0 + c_1 x + c_2 x^2 + \cdots \right) \\
&= x^3 \left(\frac{1}{5} + \frac{1}{30} x + \frac{1}{105} x^2 + \cdots \right)
\end{aligned}$$

Hence the final solution

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1 \left(1 - \frac{2}{3} x + \frac{1}{6} x^2 \right) + c_2 \left(\frac{1}{x^2} \left(1 - 4x + 6x^2 - 4x^3 + x^4 \right) \right) + \left(\frac{1}{5} x^3 + \frac{1}{30} x^4 + \frac{1}{105} x^5 + \cdots \right)
\end{aligned}$$

If we try to find y_p by assuming $y_p = \sum_{n=0}^{\infty} c_n x^n$ and substituting into the ode and try to match coefficients, we can not always be successful. The above method using the balance equation always works and that is what I am using in my solver.

2.3.2.2 Example 2 $4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$

$$4x^2y'' + 4xy' + (4x^2 - 1)\, y = 0$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{4x^2-1}{4x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} \frac{4x^2-1}{4} = -\frac{1}{4}$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r - \frac{1}{4} &= 0 \\ r^2 - \frac{1}{4} &= 0 \\ r &= -\frac{1}{2}, \frac{1}{2} \end{aligned}$$

Therefore $r_1 = \frac{1}{2}, r_2 = -\frac{1}{2}$. They differ by an integer $N = 1$. Therefore two linearly independent solutions can be constructed using

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\ y_2 &= C y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{aligned}$$

We start by expanding

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

The ode becomes

$$\begin{aligned} 4x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + 4x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + 4x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 4(n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 4(n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} - \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (4(n+r)\,(n+r-1) + 4(n+r) - 1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} &= 0 \\ \sum_{n=0}^{\infty} (4n^2 + 8nr + 4r^2 - 1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} &= 0 \\ \sum_{n=0}^{\infty} \left(4(n+r)^2 - 1\right) a_n x^{n+r} + \sum_{n=0}^{\infty} 4a_n x^{n+r+2} &= 0 \end{aligned} \tag{1}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} \left(4(n+r)^2 - 1\right) a_n x^{n+r} + \sum_{n=2}^{\infty} 4a_{n-2} x^{n+r} = 0 \tag{2}$$

$n = 0$ is not used since was used to find roots. We let $a_0 = 1$. For $n = 1$ EQ. (2) gives

$$\left(4(1+r)^2 - 1\right) a_1 = 0 \tag{3}$$

$$\begin{aligned} \left(4\left(1 + \left(\frac{1}{2}\right)\right)^2 - 1\right) a_1 &= 0 \\ 8a_1 &= 0 \\ a_1 &= 0 \end{aligned}$$

The recurrence relation is when $n \geq 2$ from (2) is given by

$$\left(4(n+r)^2 - 1\right) a_n + 4a_{n-2} = 0$$

$$a_n = \frac{-4}{4\,(n+r)^2 - 1} a_{n-2} \tag{4}$$

For $n=2$ and from (4)

$$\begin{aligned}
a_2 &= \frac{-4}{4(2+r)^2-1}a_0\\
&= \frac{-4}{4(2+r)^2-1}\\
&= \frac{-4}{4\left(2+\frac{1}{2}\right)^2-1}\\
&= -\frac{1}{6}
\end{aligned}$$

For $n=3$ Eq (5) gives (and since $a_1=0$)

$$\begin{aligned}
a_3 &= \frac{-4}{4(3+r)^2-1}a_1\\
&= 0
\end{aligned}$$

And so on. Hence

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\\
&= x^{\frac{1}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)\\
&= \sqrt{x}\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots\right)
\end{aligned}$$

Now we need to find y_2. We first check if y_2 can be found using standard method as was done above for y_1. To find if $C=0$ or not, We need to evaluate

$$\begin{aligned}
\lim_{r\rightarrow r_2} a_N(r) &= \lim_{r\rightarrow -\frac{1}{2}} a_1(r)\\
&= \lim_{r\rightarrow -\frac{1}{2}} 0\\
&= 0
\end{aligned}$$

Since limit exist then $C=0$ and we do not need the log term.

$$y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Using same recursive relation for a_n above, but using b_n instead and use $r=\frac{-1}{2}$ instead of $r=\frac{1}{2}$ we now can find all b_n.

The recursive relation becomes

$$\begin{aligned}
b_n &= \frac{-4}{4\left(n-\frac{1}{2}\right)^2-1}b_{n-2}\\
&= -\frac{1}{n(n-1)}b_{n-2} \qquad (6)
\end{aligned}$$

For $n=2$ Eq (6) gives (and using $b_0=1$)

$$\begin{aligned}
b_2 &= -\frac{1}{2(2-1)}b_0\\
&= -\frac{1}{2}
\end{aligned}$$

For $n=3$ Eq (6) gives

$$\begin{aligned}
b_3 &= -\frac{1}{3(3-1)}b_1\\
&= 0
\end{aligned}$$

For $n=4$ Eq (6) gives

$$\begin{aligned}
b_4 &= -\frac{1}{4(4-1)}b_2\\
&= -\frac{1}{12}\left(-\frac{1}{2}\right)\\
&= \frac{1}{24}
\end{aligned}$$

And so on. Hence

$$
\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \\
&= \frac{1}{\sqrt{x}} \left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\
&= \frac{1}{\sqrt{x}} \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots\right)
\end{aligned}
$$

Therefore the final solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \sqrt{x} \left(1 - \frac{1}{6} x^2 + \frac{1}{120} x^4 + \cdots\right) + c_2 \frac{1}{\sqrt{x}} \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots\right)
\end{aligned}
$$

2.3.2.3 Example 3 $y'' + y' + y = \sqrt{x}$

$$
y'' + y' + y = \sqrt{x}
$$

Let the solution be $y = y_h + y_p$. We start by finding y_h which is solution to $y'' + y' + y = 0$. Comparing this ode to

$$
y'' + p(x)\, y' + q(x)\, y = 0
$$

Hence $p(x) = 1, q(x) = 1$. Therefore $p_0 = \lim_{x \to 0} x p(x) = \lim_{x \to 0} x = 0$ and $q_0 = \lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 = 0$. Hence the indicial equation is

$$
\begin{aligned}
r(r-1) &= 0 \\
r &= 0, 1
\end{aligned}
$$

Therefore $r_1 = 1, r_2 = 0$. Expansion around $x = 0$. Let

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

The ode becomes

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{1}
$$

Reindexing to lowest powers on x gives

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1} x^{n+r-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-2} = 0 \tag{2}
$$

$n = 0$ gives

$$
r(r-1)\, a_0 x^{r-2} = 0
$$

Since $a_0 \neq 0$, then $r_1 = 1, r_2 = 0$ as was found above. Since roots differ by integer $N = r_1 - r_2 = 1$, then the two linearly independent solutions can be constructed using

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\
y_2 &= C y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}
\end{aligned}
$$

Where C above can be zero depending on a condition given below. We start by finding y_1. When $n = 1$ then (2) gives

$$
\begin{aligned}
(1+r)(r)\, a_1 + r a_0 &= 0 \\
a_1 &= \frac{-a_0}{1+r}
\end{aligned} \tag{3}
$$

The recurrence relation is when $n \geq 2$ from (2) is given by

$$
\begin{aligned}
(n+r)(n+r-1)\, a_n + (n+r-1)\, a_{n-1} + a_{n-2} &= 0 \\
a_n &= \frac{-(n+r-1)\, a_{n-1} - a_{n-2}}{(n+r)(n+r-1)}
\end{aligned} \tag{4}
$$

Now, using $r=1$. For $n=1$ and from (3) and using $a_0=1$ gives

$$\begin{aligned}a_1 &= \frac{-a_0}{2}\\ a_1 &= \frac{-1}{2}\end{aligned}$$

From $n=2$ from (4) and using $r=1$ it becomes

$$a_2 = \frac{-2a_1 - a_0}{(2+1)(2)} = \frac{-2a_1 - a_0}{6} = \frac{-2\left(\frac{-1}{2}\right) - 1}{6} = 0$$

For $n=3$ then (5) gives

$$a_3 = \frac{-(3)\,a_2 - a_1}{(3+1)(3)} = \frac{-a_1}{12} = \frac{-\left(\frac{-1}{2}\right)}{12} = \frac{1}{24}$$

And so on. Hence

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1}\\ &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)\\ &= x\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{120}x^3 + \cdots\right)\end{aligned}$$

Now we need to find y_2. We first check if y_2 can be found using standard method as was done above for y_1. For this we look at

$$\begin{aligned}\lim_{r\to r_2} a_N(r) &= \lim_{r\to 0} a_1(r)\\ &= \lim_{r\to 0} \frac{-a_0}{1+r}\\ &= -a_0\end{aligned}$$

And see this is defined. Hence $C=0$ and we can find y_2 using same series expansion and using $b_0=1$ (we do not need the log term)

$$\begin{aligned}b_1 &= \frac{-b_0}{1+r}\\ &= \frac{-1}{1}\\ &= -1\end{aligned}$$

For $n \geq 2$ we have

$$b_n = \frac{-(n+r-1)\,b_{n-1} - b_{n-2}}{(n+r)(n+r-1)}$$

Which for $r=0$ becomes

$$b_n = \frac{-(n-1)\,b_{n-1} - b_{n-2}}{n\,(n-1)} \tag{5}$$

For $n=2$

$$\begin{aligned}b_2 &= \frac{-(2-1)\,b_1 - b_0}{2}\\ &= \frac{-(2-1)(-1) - 1}{2}\\ &= 0\end{aligned}$$

For $n=3$

$$\begin{aligned}b_3 &= \frac{-(3-1)\,b_2 - b_1}{3\,(3-1)}\\ &= \frac{1}{6}\end{aligned}$$

For $n=4$

$$\begin{aligned}b_4 &= \frac{-(3)\,b_3 - b_2}{4\,(3)}\\ &= \frac{-(3)\left(\frac{1}{6}\right)}{4\,(3)}\\ &= -\frac{1}{24}\end{aligned}$$

And so on. Hence

$$\begin{aligned}y_2 &= \sum_{n=0}^{\infty} b_n x^{n+0}\\ &= \left(b_0 + b_1 x + b_2 x^2 + \cdots\right)\\ &= 1 - x + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots\end{aligned}$$

Therefore y_h

$$\begin{aligned}y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{120}x^3 + \cdots\right) + c_2\left(1 - x + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots\right)\end{aligned}$$

Now we find y_p. Let

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting into the original ode gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,c_n x^{n+r-2} + \sum_{n=1}^{\infty}(n+r-1)\,c_{n-1}x^{n+r-2} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r-2} = \sqrt{x} \tag{6}$$

For $n = 0$

$$r(r-1)\,c_0 x^{r-2} = \sqrt{x}$$

For balance we need $r - 2 = \frac{1}{2}$ or $r = \frac{5}{2}$. Hence the coefficient is $r(r-1)\,c_0 = 1$ or $\frac{5}{2}\left(\frac{5}{2}-1\right)c_0 = 1$. Solving for c_0 gives

$$c_0 = \frac{4}{15}$$

Therefore

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+\frac{5}{2}}$$

For $n = 1$ EQ. (6) gives (where now we put zero on the right side, since there can be no more balance terms, as it was used for $n = 0$)

$$\begin{aligned}(1+r)(r)\,c_1 + r c_0 &= 0 \\ c_1 &= \frac{-c_0}{1+r} \\ &= \frac{-c_0}{1+\frac{5}{2}} \\ &= \frac{-\frac{4}{15}}{1+\frac{5}{2}} \\ &= -\frac{8}{105}\end{aligned}$$

Recursive relation is for $n \geq 2$ from EQ.(6) and rememebring to always have zero on right side from now on. This gives

$$\begin{aligned}(n+r)(n+r-1)\,c_n + (n+r-1)\,c_{n-1} + c_{n-2} &= 0 \\ c_n &= \frac{-c_{n-2} - (n+r-1)\,c_{n-1}}{(n+r)(n+r-1)} \\ &= \frac{-c_{n-2} - \left(n+\frac{5}{2}-1\right)c_{n-1}}{\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}-1\right)} \\ &= \frac{-c_{n-2} - \left(n+\frac{3}{2}\right)c_{n-1}}{\left(n+\frac{5}{2}\right)\left(n+\frac{3}{2}\right)}\end{aligned}$$

For $n = 2$

$$\begin{aligned}c_2 &= \frac{-c_{n-2} - \left(2+\frac{3}{2}\right)c_1}{\left(2+\frac{5}{2}\right)\left(2+\frac{3}{2}\right)} \\ &= \frac{-\frac{4}{15} - \left(2+\frac{3}{2}\right)\left(-\frac{8}{105}\right)}{\left(2+\frac{5}{2}\right)\left(2+\frac{3}{2}\right)} \\ &= 0\end{aligned}$$

And so on. Hence

$$\begin{aligned}y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^{n+\frac{5}{2}} \\ &= x^{\frac{5}{2}} \sum_{n=0}^{\infty} c_n x^n \\ &= x^{\frac{5}{2}}\left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\ &= x^{\frac{5}{2}}\left(\frac{4}{15} - \frac{8}{105}x + \frac{32}{10\,395}x^3 + \cdots\right)\end{aligned}$$

Hence the final solution

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 x\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{120}x^3+\cdots\right)+c_2\left(1-x+\frac{1}{6}x^3-\frac{1}{24}x^4+\cdots\right)+x^{\frac{5}{2}}\left(\frac{4}{15}-\frac{8}{105}x+\frac{32}{10\,395}x^3+\cdots\right)
\end{aligned}
$$

2.3.2.4 Example 4. $x^2y''+3xy'+4x^4y=0$

Given

$$
x^2y''+3xy'+4x^4y=0 \tag{1}
$$

Expanding around $x=0$. Writing the ode as

$$
y''+\frac{3}{x}y'+4x^2y=0
$$

Shows that $x=0$ is a singular point. But $\lim_{x\to 0} x\frac{3}{x}=3$. Hence the singularity is removable. This means $x=0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$
y=\sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
\begin{aligned}
x^2\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}+3x\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}+4x^4\sum_{n=0}^{\infty}a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)\,a_n x^{n+r}+\sum_{n=0}^{\infty}4a_n x^{n+r+4} &= 0 \qquad \text{(1A)}
\end{aligned}
$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$
\sum_{n=0}^{\infty}4a_n x^{n+r+4}=\sum_{n=4}^{\infty}4a_{n-4}x^{n+r}
$$

And now Eq (1A) becomes

$$
\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r}+\sum_{n=0}^{\infty}3(n+r)\,a_n x^{n+r}+\sum_{n=4}^{\infty}4a_{n-4}x^{n+r}=0 \tag{1B}
$$

$n=0$ gives the indicial equation

$$
\begin{aligned}
(n+r)\,(n+r-1)\,a_n+3(n+r)\,a_n &= 0 \\
(r)\,(r-1)\,a_0+3ra_0 &= 0 \\
((r)\,(r-1)+3r)\,a_0 &= 0
\end{aligned}
$$

Since $a_0\neq 0$ then the above becomes

$$
(r)\,(r-1)+3r=0
$$

Hence the roots of the indicial equation are $r_1=0, r_2=-2$. Or $r_1=r_2+N$ where $N=2$. We always take r_1 to be the larger of the roots.

When this happens, the solution is given by

$$
y=c_1y_1+c_2y_2
$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$
y_1=\sum_{n=0}^{\infty}a_n x^{n+r} \tag{2}
$$

Where we take $a_0=1$ as it is arbitrary and where $r=r_1=0$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r=r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$
y_2=Cy_1\ln(x)+\sum_{n=0}^{\infty}b_n x^{n+r} \tag{3}
$$

We will show below how to find C and b_n. First, let us find $y_1(x)$. From Eq(2)

$$
\begin{aligned}
y_1' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y_1'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r = r_1 = 0$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r = r_1 = 0$. Substituting the above in (1) gives Eq (1B) above (We are following pretty much the same process we did to find the indicial equation here)

$$
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r} + \sum_{n=4}^{\infty} 4a_{n-4} x^{n+r} = 0 \tag{1B}
$$

Now we are ready to find a_n. Now we skip $n = 0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose.

For $n = 1$, Eq (1B) gives

$$
\begin{aligned}
(1+r)\,(1+r-1)\, a_1 + 3(1+r)\, a_1 &= 0 \\
((1+r)\,(1+r-1) + 3(1+r))\, a_1 &= 0 \\
\left(r^2 + 4r + 3\right) a_1 &= 0
\end{aligned}
$$

But $r = r_1 = 0$. The above becomes

$$
3a_1 = 0
$$

Hence $a_1 = 0$.

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	0	0

For $n = 2$, Eq (1B) gives

$$
\begin{aligned}
(2+r)\,(2+r-1)\, a_2 + 3(2+r)\, a_2 &= 0 \\
((2+r)\,(2+r-1) + 3(2+r))\, a_2 &= 0
\end{aligned}
$$

But $r = r_1 = 0$. The above becomes

$$
\begin{aligned}
((2)\,(1) + 3(2))\, a_2 &= 0 \\
8a_2 &= 0
\end{aligned}
$$

Hence $a_2 = 0$. The table becomes

n	$a_n(r)$	$a_n(r = 0)$
0	1	1
1	0	0
2	0	0

For $n = 3$, Eq (1B) gives

$$
(3+r)\,(3+r-1)\, a_3 + 3(3+r)\, a_3 = 0
$$

But $r = 0$. The above becomes

$$
\begin{aligned}
(3)\,(2)\, a_3 + 3(3)\, a_3 &= 0 \\
15a_3 &= 0
\end{aligned}
$$

Hence $a_3 = 0$ and the table becomes

n	$a_n(r)$	$a_n(r = 0)$
0	1	1
1	0	0
2	0	0
3	0	0

For $n \geq 4$ we obtain the recursion equation

$$(n+r)(n+r-1)a_n + 3(n+r)a_n + 4a_{n-4} = 0$$
$$((n+r)(n+r-1) + 3(n+r))a_n + 4a_{n-4} = 0$$
$$a_n(r) = -\frac{4a_{n-4}(r)}{(n+r)(n+r-1) + 3(n+r)} \tag{4}$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (4) for $n = 4$

$$a_4(r) = -\frac{4a_0(r)}{(4+r)(4+r-1) + 3(4+r)}$$

and $r = r_1 = 0$ and $a_0 = 1$, then the above becomes

$$a_4 = -\frac{4}{(4)(3) + 3(4)} = -\frac{1}{6}$$

The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{6}$

And for $n = 5$ from Eq(4)

$$a_5(r) = -\frac{4a_1(r)}{(n+r)(n+r-1) + 3(n+r)}$$
$$= 0$$

Since $a_1 = 0$. Similarly $a_6 = 0, a_7 = 0$. For $n = 8$

$$a_8(r) = -\frac{4a_4(r)}{(8+r)(8+r-1) + 3(8+r)}$$

But $a_4(r) = -\frac{4}{(4+r)(4+r-1)+3(4+r)}$. The above becomes

$$a_8(r) = \frac{4\frac{4}{(4+r)(4+r-1)+3(4+r)}}{(8+r)(8+r-1) + 3(8+r)} = \frac{16}{r^4 + 28r^3 + 284r^2 + 1232r + 1920}$$

When $r = r_1 = 0$ the above becomes

$$a_8(r) = \frac{1}{120}$$

And so on. The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{6}$
5	0	0
6	0	0
7	0	0
8	$\frac{16}{r^4+28r^3+284r^2+1232r+1920}$	$\frac{1}{120}$

Hence $y_1(x)$ is

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$y_1 = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$
$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots$$

Using values found for a_n in the above table, then (5) becomes

$$y_1 = 1 + a_4 x^4 + a_8 x^8 + \cdots$$
$$= 1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 + O(x^9)$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2 = Cy_1 \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} \tag{3}$$

The first thing to do is to determine if C is zero or not. This is done by finding

$$\lim_{r \to r_2} a_N(r)$$

If this limit exist, then $C = 0$, else we need to keep the log term. From the above above we see that $a_N(r) = a_2(r) = 0$. Recall that $N = 2$ since this was the difference between the two roots and $r_2 = -2$ (the smaller root). Therefore

$$\lim_{r \to r_2} 0 = \lim_{r \to 0} 0 = 0$$

Hence the limit exist. Therefore we do not need the log term. This means we can let $C = 0$. This is the easy case. Hence (3) becomes

$$\begin{aligned} y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r} \qquad \text{(3A)} \\ &= x^{-2} \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Since $r = r_2 = -2$. Let $b_0 = 1$. We have to remember now that $b_N = b_2 = 0$. This is the same we did when the log term was needed in the above example, since b_N is arbitrary, and used to generate $y_1(x)$. Common practice is to use $b_N = 0$. The rest of the b_n are found in similar way, from recursive relation as was done above. Substituting (3A) into $x^2y'' + 3xy' + 4x^4y = 0$ gives Eq. (1B) again, but with a_n replaced by b_n

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) b_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) b_n x^{n+r} + \sum_{n=4}^{\infty} 4b_{n-4} x^{n+r} = 0 \tag{1B}$$

For $n = 0$, we skip and let $b_0 = 1$. For $n = 1$ the above gives $b_1 = 0$. And $b_2 = 0$ since it is the special term b_N. And for $n = 3$, we get $b_3 = 0$. The table for b_n is now

n	$b_n(r)$	$b_n(r=-2)$
0	1	1
1	0	0
2	0	0
3	0	0

For $n \geq 4$, the recursion relation is

$$(n+r)(n+r-1) b_n + 3(n+r) b_n + 4b_{n-4} = 0$$

$$b_n(r) = -\frac{4b_{n-4}(r)}{(n+r)(n+r-1) + 3(n+r)}$$

For $n = 4$

$$\begin{aligned} b_4(r) &= -\frac{4b_0(r)}{(4+r)(4+r-1) + 3(4+r)} \\ &= -\frac{4}{(4+r)(4+r-1) + 3(4+r)} \qquad b_0 = 1 \end{aligned}$$

but $r = -2$. The above becomes

$$b_4 = -\frac{4}{(4-2)(4-2-1) + 3(4-2)} = -\frac{1}{2}$$

The table becomes

n	$b_n(r)$	$b_n(r=-2)$
0	1	1
1	0	0
2*	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{2}$

We will find that $b_5 = b_6 = b_7 = 0$. And for $n = 8$

$$b_8(r) = -\frac{4b_4(r)}{(8+r)(7+r) + 3(8+r)}$$

But $b_4(r) = -\frac{4}{(4+r)(4+r-1)+3(4+r)}$. Hence

$$b_8(r) = \frac{4\frac{4}{(4+r)(4+r-1)+3(4+r)}}{(8+r)(7+r) + 3(8+r)} = \frac{16}{r^4 + 28r^3 + 284r^2 + 1232r + 1920}$$

But $r=-2$.

$$b_8(r)=\frac{16}{(-2)^4+28(-2)^3+284(-2)^2+1232(-2)+1920}=\frac{1}{24}$$

The table becomes

n	$b_n(r)$	$b_n(r=-2)$
0	1	1
1	0	0
2*	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{2}$
5	0	0
6	0	0
7	0	0
8	$\frac{16}{r^4+28r^3+284r^2+1232r+1920}$	$\frac{1}{24}$

And so on. Hence the second solution is

$$\begin{aligned}
y_2(x)&=\sum_{n=0}^{\infty}b_nx^{n+r}\\
&=\sum_{n=0}^{\infty}b_nx^{n-2}\\
&=x^{-2}\sum_{n=0}^{\infty}b_nx^n\\
&=x^{-2}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6+b_7x^7+b_8x^8+\cdots\right)\\
&=x^{-2}\left(1-\frac{1}{2}x^4+\frac{1}{24}b_8x^8+O(x^9)\right)
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y&=c_1y_1+c_2y_2\\
&=c_1\left(1-\frac{1}{6}x^4+\frac{1}{120}x^8+O(x^9)\right)+c_2\left(x^{-2}\left(1-\frac{1}{2}x^4+\frac{1}{24}b_8x^8+O(x^9)\right)\right)
\end{aligned}$$

The following are important items to remember. Always let $b_N=0$ where N is the difference between the roots. When the log term is not needed (as in this problem), y_2 is found in very similar way to y_1 where $b_0=1$ and the recursion formula is used to find all b_n. But when the log term is needed (as in the above problem), it is a little more complicated and need to find C and b_1 values by comparing coefficients as was done).

This completes the solution.

2.3.2.5 Example 5. $xy''+2y'+xy=0, y(0)=1, y'(0)=0$

Solve

$$\begin{aligned}
xy''+2y'+xy&=0 &(1)\\
y(0)&=1\\
y'(0)&=0
\end{aligned}$$

Using power series method by expanding around $x=0$. We see that $x=0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y(x)=\sum_{n=0}^{\infty}a_nx^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned}
y'&=\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}\\
y''&=\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_nx^{n+r-2}
\end{aligned}$$

Substituting the above in (1) gives

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_nx^{n+r-2}+2\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}+x\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_nx^{n+r-1}+\sum_{n=0}^{\infty}2(n+r)\,a_nx^{n+r-1}+\sum_{n=0}^{\infty}a_nx^{n+r+1}=0 \tag{1A}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}2(n+r)a_n x^{n+r-1}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r-1}=0 \tag{2}$$

Indicial equation is found from $n=0$ which gives

$$r(r-1)a_0+2ra_0=0$$
$$(r^2+r)a_0=0$$

Since $a_0 \neq 0$ then $r=0$ or $r=-1$. Roots differ by integer. Let $r_1=0, r_2=-1$, hence the solutions are

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty}a_n x^{n+r_1}\\ &= \sum_{n=0}^{\infty}a_n x^n\\ y_2 &= Cy_1\ln(x)+\sum_{n=0}^{\infty}b_n x^{n+r}\\ &= Cy_1\ln(x)+\sum_{n=0}^{\infty}b_n x^{n-1}\end{aligned}$$

We do not know yet if C will be zero or not (it will be). We have to wait after finding y_1 and finding all a_n to find this. $N=1$ in this case, (which is difference between roots). So it depends if $\lim_{r\to r_2}a_1$ if this is defined or not. Now we find y_1. Using (2), for $n=1$

$$(1+r)(r)a_1+2(1+r)a_1=0$$

But $r_1=0$, hence $2a_1=0$ or $a_1=0$. We setup the table now to help find C

The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0

For $n=2$ EQ. (2) gives

$$\begin{aligned}(2+r)(1+r)a_2+2(2+r)a_2+a_0&=0\\ a_2&=\frac{-a_0}{(2+r)(1+r)+2(2+r)}\\ &=\frac{-1}{r^2+5r+6}\end{aligned}$$

But $r=0$ then

$$a_2=\frac{-1}{6}$$

The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	$\frac{-1}{r^2+5r+6}$	$\frac{-1}{6}$

If we continue we will find that $a_3=0, a_4=\frac{1}{120}, a_5=0,\cdots$. Hence

$$\begin{aligned}y_1&=\sum_{n=0}^{\infty}a_n x^n\\ &=a_0+a_1x+a_2x^2+a_3x^3+\cdots\\ &=1-\frac{1}{6}x^2+\frac{x^4}{120}+\cdots\end{aligned}$$

Now that we found y_1 we need to decide if $C=0$ or not. Since $N=1$ then we need

$$\lim_{r\to r_2}a_1(r)$$

But $a_1(r)=0$. Does not depend on r as we see from the above table. Hence $\lim_{r\to -1}a_1(r)=0$. Since limit exist then $C=0$. Hence

$$\begin{aligned}y_2&=Cy_1\ln(x)+\sum_{n=0}^{\infty}b_n x^{n+r}\\ &=\sum_{n=0}^{\infty}b_n x^{n-1}\end{aligned}$$

We skip $n=0$ since that was used to find roots and let $b_0=1$. To find all b_n we can use (2) but replace a_n by b_n and replace r by -1 which gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n-2}+\sum_{n=0}^{\infty}2(n+r)\,b_n x^{n-2}+\sum_{n=2}^{\infty}b_{n-2}x^{n-2}=0$$
$$\sum_{n=0}^{\infty}(n-1)(n-2)\,b_n x^{n-2}+\sum_{n=0}^{\infty}2(n-1)\,b_n x^{n-2}+\sum_{n=2}^{\infty}b_{n-2}x^{n-2}=0$$

For $n=1$

$$0b_1=0$$

Hence b_1 can be any value. Let $b_1=0$. Recursive relation for $n\geq 2$ which becomes

$$(n-1)(n-2)\,b_n+2(n-1)\,b_n+b_{n-2}=0$$
$$b_n=\frac{-b_{n-2}}{(n-1)(n-2)+2(n-1)}$$

For $n=2$

$$\begin{aligned}b_2&=\frac{-b_0}{2}\\&=-\frac{1}{2}\end{aligned}$$

For $n=3$

$$\begin{aligned}b_3&=\frac{-b_1}{(3-1)(3-2)+2(3-1)}\\&=0\end{aligned}$$

Since $b_1=0$. For $n=4$

$$\begin{aligned}b_4&=\frac{-b_2}{(4-1)(4-2)+2(4-1)}\\&=\frac{-\left(-\frac{1}{2}\right)}{(4-1)(4-2)+2(4-1)}\\&=\frac{1}{24}\end{aligned}$$

And so on. We find that

$$\begin{aligned}y_2&=\sum_{n=0}^{\infty}b_n x^{n-1}\\&=\frac{1}{x}\sum_{n=0}^{\infty}b_n x^n\\&=\frac{1}{x}\left(b_0+b_1x+b_2x^2+b_3x^3+\cdots\right)\\&=\frac{1}{x}\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots\right)\end{aligned}$$

Hence the solution is

$$\begin{aligned}y&=c_1y_1+c_2y_2\\&=c_1\left(1-\frac{1}{6}x^2+\frac{x^4}{120}+\cdots\right)+c_2\left(\frac{1}{x}\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots\right)\right)\\&=c_1\left(1-\frac{1}{6}x^2+\frac{x^4}{120}+\cdots\right)+c_2\left(\frac{1}{x}-\frac{1}{2}x+\frac{1}{24}x^3+\cdots\right)\end{aligned}\tag{3}$$

We now need to determine the c_1,c_2 from initial conditions $y(0)=1,y'(0)=0$. At $x=0$ we have

$$\begin{aligned}1&=c_1\lim_{x\to 0}\left(1-\frac{1}{6}x^2+\frac{x^4}{120}+\cdots\right)+\lim_{x\to 0}c_2\left(\frac{1}{x}-\frac{1}{2}x+\frac{1}{24}x^3+\cdots\right)\\&=c_1+c_2\lim_{x\to 0}\frac{1}{x}\end{aligned}$$

So we need to have $c_2=0$ since $\lim_{x\to 0}\frac{1}{x}$ is undefined. Hence $c_1=1$ and the solution now becomes

$$y=1-\frac{1}{6}x^2+\frac{x^4}{120}+\cdots$$

Now we need to verify this solution satisfies the second IC $y'(0)=0$. Taking derivative gives

$$y'=-\frac{1}{3}x+\frac{x^3}{30}+\cdots$$

At $x=0$, the above gives $y'=0$. Satisfied. Hence the solution is

$$y = 1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \cdots$$

Lets see what happens if the IC was just $y'(0)=0$. Taking derivative of (3) gives

$$y' = c_1\left(-\frac{1}{3}x + \frac{x^3}{30} + \cdots\right) + c_2\left(-\frac{1}{x^2} - \frac{1}{2} + \frac{3}{24}x^2 + \cdots\right)$$

At $x=0$ the above gives

$$0 = c_1(0) + c_2 \lim_{x\to 0}\left(-\frac{1}{x^2} - \frac{1}{2}\right)$$

Therefore we need to have $c_2=0$ since no limit exist. We see that c_1 can be any value. Hence the solution (3) becomes

$$y = c_1\left(1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \cdots\right)$$

Lets see what happens if the IC was just $y'(0)=A$ for $A\neq 0$. Then taking derivative as above gives

$$A = c_1(0) + c_2 \lim_{x\to 0}\left(-\frac{1}{x^2} - \frac{1}{2}\right)$$

Setting $c_2=0$ gives

$$A = 0$$

Which is contradiction. Therefore for the IC $y'(0)=A$ there is no solution. Finally, lets see what happens when the IC is $y(0)=A$ for $A\neq 0$. From (3)

$$A = c_1(1) + c_2 \lim_{x\to 0}\left(\frac{1}{x}\right)$$

Since limit is not defined, we make $c_2=0$ which means $c_1=A$. And the solution (3) becomes

$$y = A\left(1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \cdots\right)$$

Let verify this satisfies the IC $y(0)=A$. We see it does. When $x=0$ the above gives $A=A$.

2.4 Frobenius series. Roots differ by integer. Bad case (log is needed)

Local contents

2.4.1 Examples

Local contents

2.4.1.1 Example 1 $x^2y'' + xy' + (x^2-4)\,y = 0$

$$x^2y'' + xy' + (x^2-4)\,y = 0$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{x^2-4}{x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2 - 4 = -4$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r - 4 &= 0 \\ r^2 - 4 &= 0 \\ r &= 2, -2 \end{aligned}$$

Therefore $r_1 = 2, r_2 = -2$. Expansion around $x = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + (x^2-4) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} - 4 \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} 4 a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} ((n+r)\,(n+r-1) + (n+r) - 4)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0
\end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} ((n+r)\,(n+r-1) + (n+r) - 4)\, a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{2}$$

$n = 0$ gives

$$\begin{aligned}
(r(r-1) + r - 4)\, a_0 x^r &= 0 \\
(r^2 - 4)\, a_0 x^r &= 0
\end{aligned}$$

Since $a_0 \neq 0$, then $r^2 = 4$ or $r_1 = 2, r_2 = -2$ as was found above. Since roots differ by an integer $N = 4$ then the two linearly independent solutions can be constructed using

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\
y_2 &= C y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}
\end{aligned}$$

Where C can be zero depending on if $\lim_{r\to r_2} a_N(r)$ exist or not. We start by finding y_1. Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the ode gives (2) as was done above but with $r = r_1$. We start by $n = 1$ since $n = 0$ was used to find the roots.

For $n = 1$, EQ. (2) gives

$$((1+r)\,(r) + (1+r) - 4)\, a_1 = 0$$

Since $r = 2$ now, then

$$\begin{aligned}
((1+2)\,(2) + (1+2) - 4)\, a_1 &= 0 \\
5 a_1 &= 0 \\
a_1 &= 0
\end{aligned}$$

The recurrence relation is when $n \geq 2$ from (2) is given by

$$\begin{aligned}
((n+r)\,(n+r-1) + (n+r) - 4)\, a_n + a_{n-2} &= 0 \\
a_n &= \frac{-a_{n-2}}{((n+r)\,(n+r-1) + (n+r) - 4)}
\end{aligned} \tag{4}$$

Where in the above $r = r_1 = 2$. For $n = 2$ the above gives

$$\begin{aligned}
a_2 &= \frac{-a_0}{((2+r)\,(1+r) + (2+r) - 4)} \\
&= -\frac{1}{r}\frac{a_0}{r+4} \\
&= -\frac{1}{2}\frac{a_0}{2+4} \\
&= -\frac{1}{12} a_0
\end{aligned}$$

For $n = 3$

$$a_3 = \frac{-a_1}{((n+r)(n+r-1)+(n+r)-4)} = 0$$

Since $a_1 = 0$. For $n = 4$

$$\begin{aligned}
a_4 &= \frac{-a_2}{((4+r)(3+r)+(4+r)-4)} \\
&= -\frac{a_2}{(r+6)(r+2)} \\
&= -\frac{-\frac{1}{r}\frac{a_0}{r+4}}{(r+6)(r+2)} \\
&= \frac{1}{r}\frac{a_0}{(r+4)(r+6)(r+2)} \\
&= \frac{1}{2}\frac{a_0}{(2+4)(2+6)(2+2)} \\
&= \frac{1}{384}a_0
\end{aligned}$$

And so on. Hence

$$\begin{aligned}
y_1 &= x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right) \\
&= x^2\left(a_0 - \frac{1}{12}a_0 x^2 + \frac{1}{384}a_0 x^4 + \cdots\right) \\
&= a_0 x^2\left(1 - \frac{1}{12}x^2 + \frac{1}{384}x^4 + \cdots\right)
\end{aligned}$$

Or for $a_0 = 1$

$$y_1 = x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots$$

Now we find y_2. First we check if $C = 0$ or not. Since $N = 4$ then from the above we see

$$\begin{aligned}
a_N(r) &= a_4(r) \\
&= \frac{1}{r}\frac{a_0}{(r+4)(r+6)(r+2)}
\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{r\to r_2} a_4 &= \lim_{r\to -2} a_4 \\
&= \frac{1}{-2}\frac{a_0}{(-2+4)(-2+6)(-2+2)}
\end{aligned}$$

Which is not defined. Therefore C is not zero and we need the log term. The value of C is found when evaluating b_N below. Therefore we have

$$\begin{aligned}
y_2 &= C y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2} \\
&= C y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}$$

Hence

$$\begin{aligned}
y_2' &= C\left(y_1' \ln(x) + y_1 \frac{1}{x}\right) + \sum_{n=0}^{\infty} (n-2) b_n x^{n-3} \\
y_2'' &= C\left(y_1'' \ln(x) + y_1' \frac{1}{x} + y_1' \frac{1}{x} - y_1 \frac{1}{x^2}\right) + \sum_{n=0}^{\infty} (n-2)(n-3) b_n x^{n-4} \\
&= C\left(y_1'' \ln(x) + 2y_1' \frac{1}{x} - y_1 \frac{1}{x^2}\right) + \sum_{n=0}^{\infty} (n-2)(n-3) b_n x^{n-4}
\end{aligned}$$

Substituting the above in $x^2 y'' + x y' + (x^2 - 4) y = 0$ gives

$$x^2\left(C\left(y_1''\ln(x) + 2y_1'\frac{1}{x} - y_1\frac{1}{x^2}\right) + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-4}\right) + x\left(C\left(y_1'\ln(x) + y_1\frac{1}{x}\right) + \sum_{n=0}^{\infty}(n-2)b_n x^{n-3}\right) + (x^2$$

$$x^2 C\left(y_1''\ln(x) + 2y_1'\frac{1}{x} - y_1\frac{1}{x^2}\right) + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + xC\left(y_1'\ln(x) + y_1\frac{1}{x}\right) + \sum_{n=0}^{\infty}(n-2)b_n x^{n-2} + (x^2-4)$$

$$C(x^2 y_1''\ln(x) + 2xy_1' - y_1) + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + C(xy_1'\ln(x) + y_1) + \sum_{n=0}^{\infty}(n-2)b_n x^{n-2} + x^2 C y_1 \ln x - 4Cy_1\ln(x)$$

$$Cx^2 y_1''\ln(x) + 2Cxy_1' - Cy_1 + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + xCy_1'\ln(x) + Cy_1 + \sum_{n=0}^{\infty}(n-2)b_n x^{n-2} + x^2 C y_1\ln x - 4Cy_1$$

$$C\ln(x)\left[x^2 y_1'' + xy_1' + (x^2-4)y_1\right] + 2Cxy_1' + \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + \sum_{n=0}^{\infty}(n-2)b$$

But $x^2y_1'' + xy_1' + (x^2-4)\,y_1 = 0$ since y_1 is solution to the ode. The above simplifies to

$$2Cxy_1' + \sum_{n=0}^{\infty}(n-2)(n-3)\,b_n x^{n-2} + \sum_{n=0}^{\infty}(n-2)\,b_n x^{n-2} + \sum_{n=0}^{\infty} b_n x^n - \sum_{n=0}^{\infty} 4b_n x^{n-2} = 0 \tag{5}$$

The above is what we will use to determine C and all the b_n. But $y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$ then

$$y_1' = \sum_{n=0}^{\infty}(n+2)\,a_n x^{n+1}$$

Eq (5) now becomes

$$\begin{aligned}
2Cx\sum_{n=0}^{\infty}(n+2)\,a_n x^{n+1} + \sum_{n=0}^{\infty}(n-2)(n-3)\,b_n x^{n-2} + \sum_{n=0}^{\infty}(n-2)\,b_n x^{n-2} + \sum_{n=0}^{\infty} b_n x^n - \sum_{n=0}^{\infty} 4b_n x^{n-2} &= 0\\
C\sum_{n=0}^{\infty}2(n+2)\,a_n x^{n+2} + \sum_{n=0}^{\infty}(n-2)(n-3)\,b_n x^{n-2} + \sum_{n=0}^{\infty}(n-2)\,b_n x^{n-2} + \sum_{n=0}^{\infty} b_n x^n - \sum_{n=0}^{\infty} 4b_n x^{n-2} &= 0
\end{aligned}$$

Adjusting all the powers of x to the lowest one gives

$$C\sum_{n=4}^{\infty}2(n-2)\,a_{n-4}x^{n-2} + \sum_{n=0}^{\infty}(n-2)(n-3)\,b_n x^{n-2} + \sum_{n=0}^{\infty}(n-2)\,b_n x^{n-2} + \sum_{n=2}^{\infty} b_{n-2} x^{n-2} - \sum_{n=0}^{\infty} 4b_n x^{n-2} = 0 \tag{6}$$

For $n=0$

$$\begin{aligned}
(n-2)(n-3)\,b_0 + (n-2)\,b_0 - 4b_0 &= 0\\
(-2)(-3)\,b_0 + (-2)\,b_0 - 4b_0 &= 0\\
0b_0 &= 0
\end{aligned}$$

Hence b_0 is arbitrary. We always take $b_0 = 1$. For $n=1$

$$\begin{aligned}
(-1)(-2)\,b_1 + (-1)\,b_1 - 4b_1 &= 0\\
-3b_1 &= 0\\
b_1 &= 0
\end{aligned}$$

For $n=2$, EQ. (6) gives

$$\begin{aligned}
b_0 - 4b_2 &= 0\\
b_2 &= \frac{b_0}{4}\\
&= \frac{1}{4}
\end{aligned}$$

For $n=3$

$$\begin{aligned}
b_3 + b_1 - 4b_3 &= 0\\
-3b_3 &= 0\\
b_3 &= 0
\end{aligned}$$

Now we get to $n=4$ which is the special case since $N=4$. This will generate the value of C. From (6) and for $n=4$

$$\begin{aligned}
4Ca_0 + 2b_4 + 2b_4 + b_2 - 4b_4 &= 0\\
4Ca_0 &= -b_2\\
C &= -\frac{b_2}{4a_0}\\
&= -\frac{\frac{1}{4}}{4}\\
&= -\frac{1}{16}
\end{aligned}$$

We also notice that b_4 is not used as it cancels. We always set $b_N = b_4 = 0$ in this case. Now that we found C we can use recursive relation to all higher values of b_n. Form (6) and for all $n>4$ we have

$$\begin{aligned}
2C(n-2)\,a_{n-4} + (n-2)(n-3)\,b_n + (n-2)\,b_n + b_{n-2} - 4b_n &= 0\\
b_n &= \frac{-2C(n-2)\,a_{n-4} - b_{n-2}}{(n-2)(n-3) + (n-2) - 4}
\end{aligned} \tag{7}$$

In the above, we already know all the a_n values since we solved for y_1 before. We also know C. For $n=5$ EQ. (7) gives

$$\begin{aligned}
b_5 &= \frac{-2C(3)\,a_1 - b_3}{(3)(2) + (3) - 4}\\
&= \frac{-6Ca_1 - b_3}{5}
\end{aligned}$$

But $b_3 = 0$ and $a_1 = 0$ hence $b_5 = 0$. For $n = 6$ EQ. (7) gives

$$\begin{aligned} b_6 &= \frac{-2C(4)\,a_2 - b_4}{(4)\,(3) + (4) - 4} \\ &= \frac{-8Ca_2 - b_4}{12} \end{aligned}$$

but $b_4 = 0, a_2 = -\frac{1}{12}a_0 = -\frac{1}{12}$ since $a_0 = 1$ and $C = \frac{1}{16}$. The above gives

$$\begin{aligned} b_6 &= \frac{-8\left(-\frac{1}{16}\right)\left(-\frac{1}{12}\right)}{12} \\ &= -\frac{1}{288} \end{aligned}$$

And so on. Hence

$$\begin{aligned} y_2 &= Cy_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n-2} \\ &= -\frac{1}{12}y_1 \ln(x) + x^{-2}\left(b_0 + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + \cdots\right) \\ &= -\frac{1}{12}y_1 \ln(x) + x^{-2}\left(1 + \frac{1}{4}x^2 - \frac{1}{288}x^6 + \cdots\right) \\ &= -\frac{1}{12}y_1 \ln(x) + \left(\frac{1}{x^2} + \frac{1}{4} - \frac{1}{288}x^4 + \cdots\right) \end{aligned}$$

Therefore the final solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + x^{-2}\left(1 + \frac{1}{4}x^2 - \frac{1}{288}x^6 + \cdots\right)\right) \end{aligned} \tag{8}$$

Let us now try different IC on the above.

<u>IC $y(0) = 0$</u> Then (8) becomes

$$\begin{aligned} 0 &= c_1 \lim_{x\to 0}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + \left(\frac{1}{x^2} + \frac{1}{4} - \frac{1}{288}x^4 + \cdots\right)\right) \\ &= c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + \left(\frac{1}{x^2} + \frac{1}{4} - \frac{1}{288}x^4 + \cdots\right)\right) \end{aligned}$$

We see the second solution is not defined at $x = 0$. Hence we let $c_2 = 0$ and c_1 remains arbitrary. So the solution (8) becomes

$$y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)$$

<u>IC $y(0) = 1$</u> Then (8) becomes

$$\begin{aligned} 1 &= c_1 \lim_{x\to 0}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + \left(\frac{1}{x^2} + \frac{1}{4} - \frac{1}{288}x^4 + \cdots\right)\right) \\ 1 &= c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + \left(\frac{1}{x^2} + \frac{1}{4} - \frac{1}{288}x^4 + \cdots\right)\right) \end{aligned}$$

We see the second solution is not defined at $x = 0$. Hence we let $c_2 = 0$ and the above gives

$$1 = 0$$

Hence <u>no solution</u> exist with this initial conditions.

<u>IC $y'(0) = 0$</u> Taking derivative of (8) gives

$$y' = c_1\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right) + c_2\left(-\frac{1}{12}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)\ln(x) + \frac{1}{x}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{-1}{x^3} - \right.\right.$$

Then at $x = 0$ the above gives

$$\begin{aligned} 0 &= c_1 \lim_{x\to 0}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right) + c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)\ln(x) + \frac{1}{x}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) \right. \\ &= c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)\ln(x) + \frac{1}{x}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{-1}{x^3} - \frac{4}{288}x^3 + \cdots\right)\right) \end{aligned}$$

Since limit is not defined, we set $c_2 = 0$. Therefore the solution (8) becomes

$$y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)$$

IC $y'(0)=1$ Taking derivative of (8) gives

$$y' = c_1\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right) + c_2\left(-\frac{1}{12}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)\ln(x) + \frac{1}{x}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{-1}{x^3} - \right.\right.$$

Then at $x=0$ the above gives

$$1 = c_1 \lim_{x\to 0}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right) + c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)\ln(x) + \frac{1}{x}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) \right.$$

$$1 = c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)\ln(x) + \frac{1}{x}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{-1}{x^3} - \frac{4}{288}x^3 + \cdots\right)\right)$$

Since limit is not defined, we set $c_2=0$ and the above becomes

$$1 = 0$$

Hence no solution exist with this initial conditions.

IC $y(0)=0, y'(0)=0$

Using $y(0)=0$ the solution (8) becomes

$$0 = c_2 \lim_{x\to 0}\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + x^{-2}\left(1 + \frac{1}{4}x^2 - \frac{1}{288}x^6 + \cdots\right)\right)$$

Since limit is not defined, we set $c_2=0$ and the solution now becomes

$$y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)$$

Taking derivatives gives

$$y' = c_1\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)$$

At $x=0$, using the second IC gives

$$0 = c_1 \lim_{x\to 0}\left(2x - \frac{4}{12}x^3 + \frac{6}{384}x^5 + \cdots\right)$$
$$0 = c_1 0$$

Hence c_1 is arbitrary. It can be any value. Therefore the solution is

$$y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)$$

Notice that even though we had two initial conditions, the final solution still has one arbitrary constant in it.

2.4.1.2 Example 2 $x^2y'' + xy' + (x^2-4)\,y = 1$

$$x^2y'' + xy' + (x^2-4)\,y = 1$$

This is same as last example but with non zero on right side. We can find y_h as

$$y_h = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + x^{-2}\left(1 + \frac{1}{4}x^2 - \frac{1}{288}x^6 + \cdots\right)\right)$$

Let solution be $y = y_h + y_p$. We just need to find y_p. Assuming

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting this into the ode gives EQ. (2) in the above example but with c_n replacing a_n and with 1 now on the right side

$$\sum_{n=0}^{\infty}\left((n+r)(n+r-1) + (n+r) - 4\right)c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = 1$$

For $n=0$

$$\left(r(r-1) + r - 4\right)c_0 x^r = 1$$

For balance we need $r=0$. Hence the coefficient is

$$\left(r(r-1) + r - 4\right)c_0 = 1$$
$$-4c_0 = 1$$
$$c_0 = -\frac{1}{4}$$

Hence

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^n \\
y_p' &= \sum_{n=0}^{\infty} n c_n x^{n-1} \\
&= \sum_{n=1}^{\infty} n c_n x^{n-1} \\
y_p'' &= \sum_{n=1}^{\infty} n(n-1)\, c_n x^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2}
\end{aligned}
$$

Substituting this into the ode gives

$$
\begin{aligned}
x^2 \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} + (x^2-4) \sum_{n=0}^{\infty} c_n x^n &= 1 \\
\sum_{n=2}^{\infty} n(n-1)\, c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} - 4 \sum_{n=0}^{\infty} c_n x^n &= 1
\end{aligned}
$$

Adjust all powers to lowest gives

$$
\sum_{n=2}^{\infty} n(n-1)\, c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n - \sum_{n=0}^{\infty} 4 c_n x^n = 1
$$

For $n > 0$ the right side is zero since no balance (no x^n terms with $n > 0$). The above can then be written as

$$
\sum_{n=2}^{\infty} n(n-1)\, c_n x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n - \sum_{n=0}^{\infty} 4 c_n x^n = 0
$$

For $n = 1$

$$
\begin{aligned}
c_1 - 4c_1 &= 0 \\
c_1 &= 0
\end{aligned}
$$

The recursive relation is for $n \geq 2$ gives

$$
\begin{aligned}
n(n-1)\, c_n + n c_n + c_{n-2} - 4 c_n &= 0 \\
c_n &= \frac{-c_{n-2}}{n(n-1) + n - 4} \qquad (1)
\end{aligned}
$$

For $n = 2$

$$
\begin{aligned}
c_2 &= \frac{-c_0}{n(n-1) + n - 4} \\
&= \frac{\frac{1}{4}}{0} \\
&= \infty
\end{aligned}
$$

The series does not converge. Hence <u>no solution in series</u> exist.

2.4.1.3 Example 3 $x^2 y'' + x y' + (x^2 - 4)\, y = \frac{1}{x}$

$$
x^2 y'' + x y' + (x^2 - 4)\, y = \frac{1}{x}
$$

This is same example as above but with $\frac{1}{x}$ instead of 1 in the RHS. This is same as last example but with non zero on right side. We can find y_h as

$$
y_h = c_1 \left(x^2 - \frac{1}{12} x^4 + \frac{1}{384} x^6 + \cdots \right) + c_2 \left(-\frac{1}{12} \left(x^2 - \frac{1}{12} x^4 + \frac{1}{384} x^6 + \cdots \right) \ln(x) + x^{-2} \left(1 + \frac{1}{4} x^2 - \frac{1}{288} x^6 + \cdots \right) \right)
$$

Let solution be $y = y_h + y_p$. We just need to find y_p. Assuming

$$
y_p = \sum_{n=0}^{\infty} c_n x^{n+r}
$$

Substituting this into the ode gives EQ. (2) in the above example but with c_n replacing a_n and with 1 now on the right side

$$
\sum_{n=0}^{\infty} \left((n+r)(n+r-1) + (n+r) - 4 \right) c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = x^{-1}
$$

For $n = 0$

$$\left(r(r-1)+r-4\right) c_0 x^r = x^{-1}$$

For balance we need $r = -1$. Hence the coefficient is

$$\begin{aligned}
\left(r(r-1)+r-4\right) c_0 &= 1\\
\left(-1(-2)-1-4\right) c_0 &= 1\\
-3c_0 &= 1\\
c_0 &= -\frac{1}{3}
\end{aligned}$$

Hence

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n-1}\\
y_p' &= \sum_{n=0}^{\infty} (n-1)\, c_n x^{n-2}\\
y_p'' &= \sum_{n=1}^{\infty} (n-1)\,(n-2)\, c_n x^{n-3}
\end{aligned}$$

Substituting this into the ode gives

$$\begin{aligned}
x^2 \sum_{n=1}^{\infty} (n-1)\,(n-2)\, c_n x^{n-3} + x\sum_{n=0}^{\infty} (n-1)\, c_n x^{n-2} + \left(x^2-4\right) \sum_{n=0}^{\infty} c_n x^{n-1} &= \frac{1}{x}\\
\sum_{n=1}^{\infty} (n-1)\,(n-2)\, c_n x^{n-1} + \sum_{n=0}^{\infty} (n-1)\, c_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^{n-1} - \sum_{n=0}^{\infty} 4 c_n x^{n-1} &= \frac{1}{x}\\
\sum_{n=1}^{\infty} (n-1)\,(n-2)\, c_n x^{n-1} + \sum_{n=0}^{\infty} (n-1)\, c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} 4 c_n x^{n-1} &= \frac{1}{x}
\end{aligned}$$

Adjust all powers to lowest gives

$$\sum_{n=1}^{\infty} (n-1)\,(n-2)\, c_n x^{n-1} + \sum_{n=0}^{\infty} (n-1)\, c_n x^{n-1} + \sum_{n=2}^{\infty} c_{n-2} x^{n-1} - \sum_{n=0}^{\infty} 4 c_n x^{n-1} = \frac{1}{x}$$

For $n > 0$ the right side is zero since no balance (no x^n terms with $n > 0$). The above can then be written as

$$\sum_{n=1}^{\infty} (n-1)\,(n-2)\, c_n x^{n-1} + \sum_{n=0}^{\infty} (n-1)\, c_n x^{n-1} + \sum_{n=2}^{\infty} c_{n-2} x^{n-1} - \sum_{n=0}^{\infty} 4 c_n x^{n-1} = 0$$

For $n = 1$

$$\begin{aligned}
-4c_1 &= 0\\
c_1 &= 0
\end{aligned}$$

The recursive relation is for $n \geq 2$ gives

$$\begin{aligned}
(n-1)\,(n-2)\, c_n + (n-1)\, c_n + c_{n-2} - 4c_n &= 0\\
c_n &= \frac{-c_{n-2}}{(n-1)\,(n-2)+(n-1)-4} \qquad (1)
\end{aligned}$$

For $n = 2$

$$\begin{aligned}
c_2 &= \frac{-c_0}{1-4}\\
&= \frac{\frac{1}{3}}{-3}\\
&= -\frac{1}{9}
\end{aligned}$$

For $n = 3$

$$\begin{aligned}
c_n &= \frac{-c_1}{(n-1)\,(n-2)+(n-1)-4}\\
&= 0
\end{aligned}$$

Since $c_1 = 0$. For $n = 4$

$$\begin{aligned}
c_4 &= \frac{-c_1}{(3)\,(2)+(4-1)-4}\\
&= \frac{\frac{1}{9}}{(3)\,(2)+(4-1)-4}\\
&= \frac{1}{45}
\end{aligned}$$

And so on. Hence

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= x^{-1} \sum_{n=0}^{\infty} c_n x^n \\
&= x^{-1}\left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\
&= \frac{1}{x}\left(-\frac{1}{3} - \frac{1}{9}x^2 + \frac{1}{45}x^4 - \cdots\right) \\
&= \left(-\frac{1}{3x} - \frac{1}{9}x + \frac{1}{45}x^3 - \cdots\right)
\end{aligned}
$$

Therefore the complete solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2\left(-\frac{1}{12}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right)\ln(x) + x^{-2}\left(1 + \frac{1}{4}x^2 - \frac{1}{288}x^6 + \cdots\right)\right) + \left(-\frac{1}{3x}
\end{aligned}
$$

2.4.1.4 Example 4. $x^2y'' - xy = 0$

Solve

$$
x^2 y'' - xy = 0 \tag{1}
$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$
y'' - \frac{1}{x}y = 0
$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x^2 \frac{1}{x} = 0$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} - x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0 \qquad (1\text{A})
\end{aligned}
$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$
\sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
$$

And now Eq (1A) becomes

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \tag{1B}
$$

$n = 0$ gives the indicial equation

$$
\begin{aligned}
(n+r)(n+r-1)\, a_n x^r &= 0 \\
(r)(r-1)\, a_0 x^r &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the above becomes

$$
(r)(r-1)\, x^r = 0
$$

Since this is true for all x, then

$$
(r)(r-1) = 0
$$

Hence the roots of the indicial equation are $r_1 = 1, r_2 = 0$. Or $r_1 = r_2 + N$ where $N = 1$. We always take r_1 to be the larger of the roots.

When this happens, the solution is given by

$$y = c_1 y_1(x) + c_2 y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{2}$$

Where we take $a_0 = 1$ as it is arbitrary and where $r = r_1 = 1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2 = C y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3}$$

We will show below how to find C and b_n. First, let us find $y_1(x)$. From Eq(2)

$$\begin{aligned} y_1' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y_1'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r = r_1 = 1$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r = r_1 = 1$. Substituting the above in (1) gives Eq (1B) above (We are following pretty much the same process we did to find the indicial equation here)

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \tag{1B}$$

Now we are ready to find a_n. Now we skip $n = 0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose. We start from $n = 1$. For $n \geq 1$ we obtain the recursion equation

$$\begin{aligned} (n+r)(n+r-1) a_n - a_{n-1} &= 0 \\ a_n &= \frac{a_{n-1}}{(n+r)(n+r-1)} \end{aligned}$$

To more clearly indicate that a_n is function of r, we write the above as

$$a_n(r) = \frac{a_{n-1}(r)}{(n+r)(n+r-1)} \tag{4}$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (4) for $n = 1$

$$a_1(r) = \frac{a_0(r)}{(1+r)(r)} = \frac{1}{(1+r)(r)} \qquad a_0 = 1$$

and $r = r_1 = 1$ then the above becomes

$$a_1 = \frac{1}{2}$$

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$

And for $n = 2$ from Eq(4)

$$a_2(r) = \frac{a_1(r)}{(2+r)(1+r)}$$

But $a_1(r) = \frac{1}{(1+r)(r)}$. Then

$$a_2(r) = \frac{\frac{1}{(1+r)(r)}}{(2+r)(1+r)} = \frac{1}{r(r+1)^2(r+2)}$$

When $r = r_1 = 1$ the above becomes

$$a_2 = \frac{1}{(2)^2(3)} = \frac{1}{12}$$

The table becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$
2	$\frac{1}{r(r+1)^2(r+2)}$	$\frac{1}{12}$

For $n=3$ Eq (4) gives

$$a_3(r)=\frac{a_2(r)}{(3+r)(2+r)}$$

Using the value of $a_2(r)$ from the the above becomes

$$a_3(r)=\frac{\frac{1}{r(r+1)^2(r+2)}}{(3+r)(2+r)}=\frac{1}{r(r+1)^2(r+2)^2(r+3)}$$

When $r=r_1=1$ the above becomes

$$a_3=\frac{1}{(2)^2(3)^2(4)}=\frac{1}{144}$$

The Table now becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$
2	$\frac{1}{r(r+1)^2(r+2)}$	$\frac{1}{12}$
3	$\frac{1}{r(r+1)^2(r+2)^2(r+3)}$	$\frac{1}{144}$

And so on. Hence $y_1(x)$ is

$$y_1=\sum_{n=0}^{\infty}a_n x^{n+r}$$

But $r=r_1=1$. Therefore

$$\begin{aligned}y_1&=\sum_{n=0}^{\infty}a_n x^{n+1}\qquad(5)\\&=x\sum_{n=0}^{\infty}a_n x^n\\&=x\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots\right)\\&=x\left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+\cdots\right)\end{aligned}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2=Cy_1(x)\ln(x)+\sum_{n=0}^{\infty}b_n x^{n+r}\qquad(3)$$

The first thing to do is to determine if C is zero or not. This is done by finding $\lim_{r\to r_2}a_N(r)$. If this limit exist, then $C=0$, else we need to keep the log term. From the above above we see that $a_N=a_1=\frac{1}{(1+r)(r)}$. Recall that $N=1$ since this was the difference between the two roots and $r_2=0$ (the smaller root). Therefore

$$\lim_{r\to r_2}a_1(r)=\lim_{r\to 0}\frac{1}{(1+r)(r)}$$

Which does not exist. Therefore we need to keep the log term. In this case, we replace Eq. (3) back in the original ODE.

$$\begin{aligned}y_2'&=Cy_1'\ln(x)+Cy_1\frac{1}{x}+\sum_{n=0}^{\infty}(n+r)b_nx^{n+r-1}\\y_2''&=Cy_1''\ln(x)+Cy_1'\frac{1}{x}+Cy_1'\frac{1}{x}-Cy_1\frac{1}{x^2}+\sum_{n=0}^{\infty}(n+r)(n+r-1)b_nx^{n+r-2}\\&=Cy_1''\ln(x)+2Cy_1'\frac{1}{x}-Cy_1\frac{1}{x^2}+\sum_{n=0}^{\infty}(n+r)(n+r-1)b_nx^{n+r-2}\end{aligned}$$

Substituting the above in $x^2y''-xy=0$ gives

$$\begin{gathered}x^2\left(Cy_1''\ln(x)+2Cy_1'\frac{1}{x}-Cy_1\frac{1}{x^2}+\sum_{n=0}^{\infty}(n+r)(n+r-1)b_nx^{n+r-2}\right)-x\left(Cy_1\ln(x)+\sum_{n=0}^{\infty}b_nx^{n+r}\right)=0\\Cx^2y_1''\ln(x)+2x^2Cy_1'\frac{1}{x}-Cy_1+x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)b_nx^{n+r-2}-Cxy_1\ln(x)-x\sum_{n=0}^{\infty}b_nx^{n+r}=0\\Cx^2y_1''\ln(x)+2xCy_1'-Cy_1+\sum_{n=0}^{\infty}(n+r)(n+r-1)b_nx^{n+r}-Cxy_1\ln(x)-\sum_{n=0}^{\infty}b_nx^{n+r+1}=0\\C\ln(x)\left(x^2y_1''-xy_1\right)+2xCy_1'-Cy_1+\sum_{n=0}^{\infty}(n+r)(n+r-1)b_nx^{n+r}-\sum_{n=0}^{\infty}b_nx^{n+r+1}=0\end{gathered}$$

But $x^2y_1'' - xy_1 = 0$ since y_1 is solution to the ode. The above simplifies to

$$C(2xy_1' - y_1) + \sum_{n=0}^{\infty} (n+r)(n+r-1) b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0 \tag{6}$$

The above is what we will use to determine C and all the b_n. Remembering that $r = r_2 = 0$ in the above, since this is for the second solution associated with the second root which we found above to be zero. But we found $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$ then

$$y_1' = \sum_{n=0}^{\infty} (n+1) a_n x^n$$

Eq (6) now becomes

$$C\left(2x \sum_{n=0}^{\infty} (n+1) a_n x^n\right) - C\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) + \sum_{n=0}^{\infty} (n+r)(n+r-1) b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0$$

$$2C \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} - C \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} (n+r)(n+r-1) b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0$$

But $r = r_2 = 0$. The above becomes

$$2C \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} - C \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} n(n-1) b_n x^n - \sum_{n=0}^{\infty} b_n x^{n+1} = 0$$

Adjusting the index of terms above, so so all x powers are the same gives

$$2C \sum_{n=1}^{\infty} n a_{n-1} x^n - C \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} n(n-1) b_n x^n - \sum_{n=1}^{\infty} b_{n-1} x^n = 0 \tag{7}$$

$n = 0$ is skipped, since b_0 is arbitrary and can be taken as say

$$b_0 = 1$$

At $n = 1$, Eq(7) gives

$$2Ca_0 - Ca_0 - b_0 = 0$$

But $a_0 = 1, b_0 = 1$ hence the above becomes

$$C = 1$$

For $b_N = b_1$ we are free to select any value since it is arbitrary. The standard way is to choose

$$b_1 = 0$$

Now we find the rest of the b_n terms. From Eq(7), for $n = 2$, it gives

$$2C(2a_1) - Ca_1 + 2b_2 - b_1 = 0$$

But $C = 1, b_1 = 0$ and $a_1 = \frac{1}{2}$ from table. Hence the above becomes

$$\begin{aligned} 2\left(2\frac{1}{2}\right) - \frac{1}{2} + 2b_2 &= 0 \\ 2 - \frac{1}{2} + 2b_2 &= 0 \\ b_2 &= -\frac{3}{4} \end{aligned}$$

And for $n = 3$ from Eq. (7) it gives

$$2C(3a_2) - Ca_2 + (3)(2) b_3 - b_2 = 0$$

But $C = 1, b_2 = -\frac{3}{4}, a_2 = \frac{1}{12}$. The above becomes

$$\begin{aligned} 2\left(3\left(\frac{1}{12}\right)\right) - \frac{1}{12} + (3)(2) b_3 + \frac{3}{4} &= 0 \\ b_3 &= -\frac{7}{36} \end{aligned}$$

And so on. Hence the second solution is, for $r = 0, C = 1$

$$\begin{aligned} y_2(x) &= Cy_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^n \\ &= y_1(x) \ln(x) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) \\ &= y_1(x) \ln(x) + \left(1 + (0)x - \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right) \\ &= \left(x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \cdots\right) \ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right) \\ &= x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + O(x^4)\right) \ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + O(x^4)\right) \end{aligned}$$

Some observations: b_N is always taken as zero. Where N is the difference between the roots. In this case it is $b_1 = 0$. Now that we found y_1, y_2 then the general solution is

$$
\begin{aligned}
y &= C_1 y_1 + C_2 y_2 \\
&= C_1 x\left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+O(x^4)\right)+C_2\left(x\left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+O(x^4)\right)\ln x+\left(1+\frac{3}{4}x^2-\frac{7}{36}x^3+O(x^4)\right)\right)
\end{aligned}
$$

This completes the solution.

2.4.1.5 Example 5. $x^{\frac{3}{2}}y''+y=0$

Solve

$$x^{\frac{3}{2}}y''+y=0$$

Since $x=0$ is regular singular point, then Frobenius power series must be used. Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^{\frac{n}{2}+r}$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)a_n x^{\frac{n}{2}+r-1}\\
y'' &= \sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)\left(\frac{n}{2}+r-1\right)a_n x^{\frac{n}{2}+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)\left(\frac{n}{2}+r-1\right)a_n x^{\frac{n}{2}+r-2}\right)+\left(\sum_{n=0}^{\infty}a_n x^{\frac{n}{2}+r}\right) &= 0\\
\sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)\left(\frac{n}{2}+r-1\right)a_n x^{\frac{n}{2}+r-\frac{1}{2}}+\sum_{n=0}^{\infty}a_n x^{\frac{n}{2}+r} &= 0 \qquad \text{(1A)}
\end{aligned}
$$

$n=0$ gives the indicial equation

$$
\begin{aligned}
(r)(r-1)a_0x^{r-\frac{1}{2}}+a_0x^r &= 0\\
\left((r)(r-1)x^{r-\frac{1}{2}}+x^r\right)a_0 &= 0\\
(r)(r-1)x^{r-\frac{1}{2}}+x^r &= 0\\
\left((r)(r-1)x^{-\frac{1}{2}}+1\right)x^r &= 0\\
(r)(r-1)\frac{1}{\sqrt{x}}+1 &= 0
\end{aligned}
$$

Not possible to obtain indicial equation in r only. How to handle this? Maple can't solve this using series solution either.

2.4.1.6 Example 6 $xy''+y=x$

$$xy''+y=x \tag{1}$$

Let solution be $y=y_h+y_p$. We always start by finding y_h then find y_p using balance method. $x=0$ is regular singular point. Hence Frobenius series gives

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\\
y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
\begin{aligned}
x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}a_n x^{n+r} &= x\\
\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r} &= x \qquad \text{(1A)}
\end{aligned}
$$

Adjusting indices to all powers of x are the same gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=x \tag{1B}$$

The indicial equation is found from only the terms with the expansion of the dependent variable y. This means by making the LHS of (3) vanish. We only consider

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0 \tag{1C}$$

For $n=0$

$$(r)(r-1)a_n x^{r-1}=0$$

EQ (1D) is used to find r. Since $a_0\neq 0$ then (1D) gives

$$(r)(r-1)=0$$

Hence roots are $r_1=1, r_2=0$. Hence the two basis solution for y_h are

$$\begin{aligned}
y_1 &= x^{r_1}\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}a_n x^{n+1}\\
y_2 &= C_1 y_1\ln x+x^{r_2}\sum_{n=0}^{\infty}b_n x^n\\
&= C_1 y_1\ln x+\sum_{n=0}^{\infty}b_n x^n
\end{aligned}$$

To find y_1, we can find the recursive equation to be for $n>0$

$$a_n=-\frac{a_{n-1}}{n(n+1)}$$

Which results in

$$y_1=x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\frac{x^5}{2880}+\cdots$$

Finding y_2 is a little more involved because we need to determine C. This can be found to be $C=-1$. Using this we can find

$$y_2=\left(-x+\frac{x^2}{2}-\frac{x^3}{12}+\frac{x^4}{144}-\frac{x^5}{2880}+\cdots\right)\ln x+\left(1-\frac{3}{4}x^2+\frac{7x^3}{36}-\cdots\right)$$

Hence

$$\begin{aligned}
y_h &= c_1y_1+c_2y_2\\
&= c_1\left(x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\frac{x^5}{2880}+\cdots\right)+c_2\left(\left(-x+\frac{x^2}{2}-\frac{x^3}{12}+\frac{x^4}{144}-\frac{x^5}{2880}+\cdots\right)\ln x+\left(1-\frac{3}{4}x^2+\frac{7x^3}{36}-\cdots\right)\right)
\end{aligned} \tag{2}$$

What is left is to find y_p. Let

$$y_p=\sum_{n=0}^{\infty}c_n x^{n+r}$$

Substituting this into the ode $xy''+y=x$ and simplifying as we did above results in

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r-1}=x \tag{3A}$$

For $n=0$

$$(r)(r-1)c_0x^{r-1}=x$$

Hence for balance we need $r-1=1$ or $r=2$. Therefore $(r)(r-1)c_0=1$ and solving for c_0 gives $c_0=\frac{1}{2}$. Therefore (3A) becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_n x^{n+1}+\sum_{n=1}^{\infty}c_{n-1}x^{n+1}=x \tag{3B}$$

Now that we used $n=0$ to find r by matching against the RHS which is x, from now on, for all $n>0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x on the right. For $n=1$ then (3B) gives

$$6c_1x^2+c_0x^2=0$$

Hence $6c_1+c_0=0$ or $c_1=-\frac{c_0}{6}=\frac{-1}{12}$. For $n=2$, EQ(3B) gives

$$12c_2x^3+c_1x^3=0$$

Hence $12c_2 + c_1 = 0$ or $c_2 = -\frac{c_1}{12} = -\frac{\frac{-1}{12}}{12} = \frac{1}{144}$. For $n = 3$, EQ(3B) gives

$$20c_3x^4 + c_2x^4 = 0$$

Hence $20c_3 + c_2 = 0$ or $c_3 = -\frac{c_2}{20} = -\frac{\frac{1}{144}}{20} = -\frac{1}{2880}$ and so one. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= c_0x^2 + c_1x^3 + c_2x^4 + c_3x^5 + \cdots \\ &= \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + \cdots \end{aligned} \tag{4}$$

Hence the final solution is

$$y = y_h + y_p$$

Where y_h is given by (2) and y_p is given by (4).

2.4.1.7 Example 7. $xy'' - 2y' + y = \cos x$

$$xy'' - 2y' + y = \cos x \tag{1}$$

Let solution be $y = y_h + y_p$. We always start by finding y_h then find y_p using balance method. $x = 0$ is regular singular point. Hence Frobenius series gives

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

Substituting the above in (1) gives

$$x\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} - 2\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = \cos x$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2(n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = \cos x \tag{1A}$$

Adjusting indices to all powers of x are the same gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2(n+r)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = \cos x \tag{1B}$$

The indicial equation is found from only the terms with the expansion of the dependent variable y. This means by making the LHS of (3) vanish. We only consider

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2(n+r)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0 \tag{1C}$$

For $n = 0$

$$\begin{aligned} (r)(r-1)\, a_0 x^{r-1} - 2ra_0x^{r-1} &= 0 \\ (r(r-1) - 2r)\, a_0 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the above gives

$$\begin{aligned} r(r-1) - 2r &= 0 \\ r^2 - 3r &= 0 \\ r(r-3) &= 0 \end{aligned}$$

Hence roots are $r_1 = 3, r_2 = 0$. Since the difference is integer, the two basis solution for y_h are

$$\begin{aligned} y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_n x^{n+3} \\ y_2 &= C_1 y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \\ &= C_1 y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

To find y_1, we can find the recursive equation to be for $n>0$. From (1C) and for $n>0$

$$
\begin{aligned}
(n+r)(n+r-1)a_n x^{n+r-1} - 2(n+r)a_n x^{n+r-1} + a_{n-1}x^{n+r-1} &= 0\\
(n+r)(n+r-1)a_n - 2(n+r)a_n + a_{n-1} &= 0
\end{aligned}
$$

But for $r=3$ the above becomes

$$
\begin{aligned}
(n+3)(n+2)a_n - 2(n+3)a_n + a_{n-1} &= 0\\
a_n &= -\frac{a_{n-1}}{(n+3)(n+2)-2(n+3)}\\
&= \frac{a_{n-1}}{n(n+3)}
\end{aligned}
$$

Iterating over few values of n gives this table (where we always use $a_0=1$ by default)

n	$a_n(r)$	$a_n(r_1=3)$
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$\frac{-1}{720}$
a_4	$-\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$

Hence

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1}\\
&= \sum_{n=0}^{\infty} a_n x^{n+3}\\
&= x^3 \sum_{n=0}^{\infty} a_n x^n\\
&= x^3\left(a_0 - a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)\\
&= x^3\left(1 - \frac{x}{4} + \frac{1}{40}x^2 - \frac{1}{720}x^3 + \frac{1}{20160}x^4 + \cdots\right)
\end{aligned}
$$

Finding $y_2 = Cy_1\ln x + \sum_{n=0}^{\infty} b_n x^n$ is a little more involved because we need to determine C first. This decided if ln term is needed or not. Let $r_1 - r_2 = N$ where we always use r_1 as the larger root. Hence $N = 3-0 = 3$. To find if C is zero or not, we take $\lim_{r\to r_2} a_3$. If this limit exists then $C=0$ else we need to keep ln and $C\neq 0$. From the table above, we see $a_3 = -\frac{1}{(r^4-5r^2+4)r(r+3)}$. Since $r_2=0$ (the smaller root), then $\lim_{r\to 0} a_3$ is not defined. Hence we need to keep the ln term. This means

$$
\begin{aligned}
y_2 &= Cy_1\ln x + \sum_{n=0}^{\infty} b_n x^n\\
y_2' &= Cy_1'\ln x + \frac{Cy_1}{x} + \sum_{n=0}^{\infty} n b_n x^{n-1}\\
y_2'' &= Cy_1''\ln x + \frac{Cy_1'}{x} + \frac{Cy_1'}{x} - \frac{Cy_1}{x^2} + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-2}\\
&= Cy_1''\ln x + \frac{2Cy_1'}{x} - \frac{Cy_1}{x^2} + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-2}
\end{aligned}
$$

Substituting the above into the given ode (with zero on RHS) $xy'' - 2y' + y = 0$ gives

$$
\begin{aligned}
x\left(Cy_1''\ln x + \frac{2Cy_1'}{x} - \frac{Cy_1}{x^2} + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-2}\right) - 2\left(Cy_1'\ln x + \frac{Cy_1}{x} + \sum_{n=0}^{\infty} n b_n x^{n-1}\right) + Cy_1\ln x + \sum_{n=0}^{\infty} b_n x^n &= 0\\
xCy_1''\ln x + 2Cy_1' - \frac{Cy_1}{x} + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - 2Cy_1'\ln x - 2\frac{Cy_1}{x} + \sum_{n=0}^{\infty} -2nb_n x^{n-1} + Cy_1\ln x + \sum_{n=0}^{\infty} b_n x^n &= 0\\
C\ln x(xy_1'' - 2y_1' + y_1) + 2Cy_1' - \frac{Cy_1}{x} + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - 2\frac{Cy_1}{x} + \sum_{n=0}^{\infty} -2nb_n x^{n-1} + \sum_{n=0}^{\infty} b_n x^n &= 0
\end{aligned}
$$

But $xy_1'' - 2y_1' + y_1 = 0$ since y_1 is solution. The above simplifies to

$$
2Cy_1' - \frac{Cy_1}{x} + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - 2\frac{Cy_1}{x} + \sum_{n=0}^{\infty} -2nb_n x^{n-1} + \sum_{n=0}^{\infty} b_n x^n = 0
$$

But $y_1=\sum_{n=0}^{\infty} a_n x^{n+3}$. Hence $y_1'=\sum_{n=0}^{\infty}(n+3)\,a_n x^{n+2}$ and the above now becomes

$$2C\sum_{n=0}^{\infty}(n+3)\,a_n x^{n+2}-\frac{C}{x}\sum_{n=0}^{\infty} a_n x^{n+3}+\sum_{n=0}^{\infty} n(n-1)\,b_n x^{n-1}-2\frac{C}{x}\sum_{n=0}^{\infty} a_n x^{n+3}+\sum_{n=0}^{\infty}-2nb_n x^{n-1}+\sum_{n=0}^{\infty} b_n x^n=0$$

$$2C\sum_{n=0}^{\infty}(n+3)\,a_n x^{n+2}-C\sum_{n=0}^{\infty} a_n x^{n+2}+\sum_{n=0}^{\infty} n(n-1)\,b_n x^{n-1}-2C\sum_{n=0}^{\infty} a_n x^{n+2}+\sum_{n=0}^{\infty}-2nb_n x^{n-1}+\sum_{n=0}^{\infty} b_n x^n=0$$

$$\sum_{n=0}^{\infty}2C(n+3)\,a_n x^{n+2}+\sum_{n=0}^{\infty}-3Ca_n x^{n+2}+\sum_{n=0}^{\infty} n(n-1)\,b_n x^{n-1}+\sum_{n=0}^{\infty}-2nb_n x^{n-1}+\sum_{n=0}^{\infty} b_n x^n=0$$

Making all sums the same with powers on x being $n-1$ gives

$$\sum_{n=3}^{\infty}2Cna_{n-3}x^{n-1}+\sum_{n=3}^{\infty}-3Ca_{n-3}x^{n-1}+\sum_{n=0}^{\infty} n(n-1)\,b_n x^{n-1}+\sum_{n=0}^{\infty}-2nb_n x^{n-1}+\sum_{n=1}^{\infty} b_{n-1}x^{n-1}=0 \quad (2)$$

For $n=0$ we choose $b_0=1$ as we did for a_0. For $n=1$ the above gives

$$\begin{aligned}-2b_1+b_0&=0\\-2b_1+b_0&=0\\b_1&=\frac{b_0}{2}\\&=\frac{1}{2}\end{aligned}$$

For $n=2$, EQ. (2) gives

$$\begin{aligned}2(1)\,b_2x-4b_2x+b_1x&=0\\-2b_2+b_1&=0\\b_2&=\frac{b_1}{2}\\&=\frac{1}{4}\end{aligned}$$

For $n=3$ and since this is where $N=3$, then this is special case. It becomes

$$\begin{aligned}2Cna_{n-3}-3Ca_{n-3}+n(n-1)\,b_n-2nb_n+b_{n-1}&=0\\6Ca_0-3Ca_0+6b_3-6b_n+b_2&=0\\3Ca_0+b_2&=0\\C&=\frac{-b_2}{3a_0}\\&=-\frac{\frac{1}{4}}{3}\\&=-\frac{1}{12}\end{aligned}$$

Where we used $a_0=1$ in the above. Notice that for $n=3$, b_3 is not used since it does not show up. We are free to choose $b_N=0$ or $b_3=0$. Now that we found C then we update EQ. (2) and it becomes

$$\sum_{n=3}^{\infty}-\frac{1}{6}na_{n-3}x^{n-1}+\sum_{n=3}^{\infty}\frac{1}{4}a_{n-3}x^{n-1}+\sum_{n=0}^{\infty} n(n-1)\,b_n x^{n-1}+\sum_{n=0}^{\infty}-2nb_n x^{n-1}+\sum_{n=1}^{\infty} b_{n-1}x^{n-1}=0 \quad (3)$$

The recursive relation now for $n>3$ is

$$\begin{aligned}-\frac{1}{6}na_{n-3}+\frac{1}{4}a_{n-3}+n(n-1)\,b_n-2nb_n+b_{n-1}&=0\\b_n&=\frac{\frac{1}{6}na_{n-3}-\frac{1}{4}a_{n-3}-b_{n-1}}{n\,(n-1)-2n}\end{aligned}$$

For b_4 the above gives

$$b_4=\frac{\frac{4}{6}a_1-\frac{1}{4}a_1-b_3}{4\,(3)-8}$$

But a_1 from the earlier table is $-\frac{1}{4}$ and $b_3=0$. Hence

$$\begin{aligned}b_4&=\frac{\frac{4}{6}\left(-\frac{1}{4}\right)-\frac{1}{4}\left(-\frac{1}{4}\right)}{4\,(3)-8}\\&=-\frac{5}{192}\end{aligned}$$

And for $n=5$ the recursion relation gives

$$\begin{aligned}b_5&=\frac{\frac{1}{6}na_{n-3}-\frac{1}{4}a_{n-3}-b_{n-1}}{n\,(n-1)-2n}\\&=\frac{\frac{5}{6}a_2-\frac{1}{4}a_2-b_4}{5\,(4)-10}\end{aligned}$$

But $a_2 = \frac{1}{40}, b_4 = -\frac{5}{192}$ and the above becomes

$$
\begin{aligned}
b_5 &= \frac{\frac{5}{6}\left(\frac{1}{40}\right) - \frac{1}{4}\left(\frac{1}{40}\right) + \frac{5}{192}}{10} \\
&= \frac{13}{3200}
\end{aligned}
$$

And so on. Now that we found all b_n and found C then we know y_2

$$
\begin{aligned}
y_2 &= Cy_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \\
&= -\frac{1}{12} \ln(x)\, y_1 + \left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\
&= -\frac{1}{12} \ln(x) \left(x^3\left(1 - \frac{x}{4} + \frac{1}{40}x^2 - \frac{1}{720}x^3 + \frac{1}{20160}x^4 + \cdots\right)\right) + \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 + 0x^3 - \frac{5}{192}x^4 + \frac{13}{3200}x^5 + \cdots\right) \\
&= -\frac{1}{12} \ln(x) \left(x^3 - \frac{x^4}{4} + \frac{1}{40}x^5 - \frac{1}{720}x^6 + \frac{1}{20160}x^6 + \cdots\right) + \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{5}{192}x^4 + \frac{13}{3200}x^5 + \cdots\right)
\end{aligned}
$$

Hence

$$
y_h = c_1 y_1 + c_2 y_2
$$

Or

$$
\begin{aligned}
y_h &= c_1\left(x^3 - \frac{x^4}{4} + \frac{1}{40}x^5 - \frac{1}{720}x^6 + \frac{1}{20160}x^6 + \cdots\right) \\
&+ c_2\left(-\frac{1}{12} \ln(x) \left(x^3 - \frac{x^4}{4} + \frac{1}{40}x^5 - \frac{1}{720}x^6 + \frac{1}{20160}x^6 + \cdots\right) + \left(1 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{5}{192}x^4 + \frac{13}{3200}x^5 + \cdots\right)\right)
\end{aligned}
$$

The above was the easy part. Now we need to find the particular solution. Since right side is not a single term of a polynomial, we expand $\cos x$ and add all the particular solutions from each term in the series expansion of $\cos x$. Now we have

$$
xy'' - 2y' + y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots
$$

Starting with the first term 1. We have

$$
xy'' - 2y' + y = 1 \tag{4}
$$

Let the particular solution associated with 1 on the right side be

$$
y_p = \sum_{n=0}^{\infty} c_n x^{n+r}
$$

Just be careful here, the r used above has nothing to do with the r that was used in finding y_h. Substituting this into the ode (4) gives (1C) but with a_n now being c_n

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-1} + \sum_{n=0}^{\infty} -2(n+r)\, c_n x^{n+r-1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} = 1 \tag{5}
$$

For $n = 0$

$$
\begin{aligned}
(r)(r-1)\, c_0 x^{r-1} - 2(r)\, c_0 x^{r-1} &= 1 \\
(r(r-1) - 2r)\, c_0 x^{r-1} &= 1
\end{aligned}
$$

For balance we must have $r - 1 = 0$ since there is no x on the right side. This means $r = 1$. Which implies that

$$
\begin{aligned}
(r(r-1) - 2r)\, c_0 &= 1 \\
-2c_0 &= 1 \\
c_0 &= -\frac{1}{2}
\end{aligned}
$$

Now that we found r, c_0 then y_p becomes

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} c_n x^{n+1}
\end{aligned}
$$

And (5) becomes

$$
\sum_{n=0}^{\infty} (n+1)(n)\, c_n x^n + \sum_{n=0}^{\infty} -2(n+1)\, c_n x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 1 \tag{6}
$$

Recursion relation for $n > 0$ is the following. Important note here, when finding Recursion we set right side back to zero since we can no longer have a match for any x. So the recursive relation is from (6) but with zero on right side.

$$
\begin{aligned}
(n+1)(n)\,c_n - 2(n+1)\,c_n + c_{n-1} &= 0 \\
c_n &= \frac{-c_{n-1}}{(n+1)(n) - 2(n+1)} \qquad (7)
\end{aligned}
$$

For $n = 1$ the above gives

$$
\begin{aligned}
c_1 &= \frac{-c_0}{2-4} \\
&= \frac{-\left(-\frac{1}{2}\right)}{-2} \\
&= -\frac{1}{4}
\end{aligned}
$$

For $n = 2$ then (7) gives

$$
\begin{aligned}
c_2 &= \frac{-c_1}{(2+1)(2) - 2(2+1)} \\
&= \frac{-\frac{1}{4}}{6-6} \\
&= \infty
\end{aligned}
$$

This indicates the series solution for this particular solution does not converge. When this happens we throw the white towel and give up and say that no particular solution exist. No need to try other terms in expansion of $\cos x$. If one particular solution to any term does this, then this means the problem can not be solved using series method. Maple also fail to solve this using series method. Mathematica solves it, but using asymptotic expansion.

Hence No solution exist.

2.4.1.8 Example 8 $xy'' + xy' + y = 1, y(0) = 1$

$$
\begin{aligned}
xy'' + xy' + y &= 1 \\
y(0) &= 1
\end{aligned}
$$

Let solution be $y = y_h + y_p$. We always start by finding y_h then find y_p using balance method. $x = 0$ is regular singular point. Hence Frobenius series gives

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in homogeneous ode gives

$$
\begin{aligned}
x\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \qquad (1A)
\end{aligned}
$$

Adjusting indices to all powers of x to lowest power

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-1)\,a_{n-1} x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0 \qquad (1B)
$$

For $n = 0$

$$
(r)(r-1)\,a_0 = 0
$$

Since $a_0 \neq 0$ then the above gives roots are $r_1 = 1, r_2 = 0$. Since the difference is integer, the two basis solution for y_h are

$$
\begin{aligned}
y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2 &= C y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n \\
&= C y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

Substituting y_1 into the ode gives (1B) but with $r = 1$ now. We will leave r as symbol to generate the table. This results in (as was done in the other examples)

	$a_n(r)$	$a_n(r=0)$
a_0	1	1
a_1	$-\frac{1}{r}$	-1
a_2	$\frac{1}{r(r+1)}$	$\frac{1}{2}$
a_3	$\frac{-1}{r(r+1)(r+2)}$	$\frac{-1}{6}$
$\vdots$	$\vdots$	$\vdots$

Hence y_1 is

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= x \sum_{n=0}^{\infty} a_n x^n \\
&= x\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \\
&= x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)
\end{aligned}
$$

To find y_2 we first have to find if the log term is needed or not.

$$
\begin{aligned}
\lim_{r \to r_2} a_N(r) &= \lim_{r \to 0} a_1(r) \\
&= \lim_{r \to 0} -\frac{1}{r}
\end{aligned}
$$

Since limit does not exist, then we have to use the log term. This means

$$
\begin{aligned}
y_2 &= C y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \\
y_2' &= C\left(y_1' \ln x + y_1 \frac{1}{x}\right) + \sum_{n=0}^{\infty} n b_n x^{n-1} \\
y_2'' &= C\left(y_1'' \ln x + y_1' \frac{1}{x} + y_1' \frac{1}{x} - y_1 \frac{1}{x^2}\right) + \sum_{n=0}^{\infty} n(n-1) b_n x^{n-2}
\end{aligned}
$$

Substituting this in the ode and simplifying and using fact that $x y_1'' + x y_1' + y_1 = 0$ gives

$$
C \sum_{n=1}^{\infty} 2 a_{n-1} n x^{n-1} + C \sum_{n=2}^{\infty} a_{n-2} n x^{n-1} - C \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=0}^{\infty} n(n-1) x^{n-1} b_n + \sum_{n=1}^{\infty} (n-1) b_{n-1} x^{n-1} + \sum_{n=1}^{\infty} b_{n-1} x^{n-1} = 0
$$

For $n = 0$ we choose $b_0 = 1$ always. For $n = 1$ (which is the special case also, since $N = 1$) the above gives

$$
\begin{aligned}
2 C a_0 - C a_0 + b_0 &= 0 \\
C &= -\frac{b_0}{a_0} \\
&= -1
\end{aligned}
$$

And we set $b_1 = 0$ (always for the special case $n = N$). Now using the recursive relative for $n > 1$ gives

$$
\begin{aligned}
2 C n a_{n-1} + C n a_{n-2} - C a_{n-1} + n(n-1) b_n + (n-1) b_{n-1} + b_{n-1} &= 0 \\
-2 n a_{n-1} - n a_{n-2} + a_{n-1} + n(n-1) b_n + (n-1) b_{n-1} + b_{n-1} &= 0 \\
b_n &= \frac{-(n-1) b_{n-1} - b_{n-1} + 2 n a_{n-1} + n a_{n-2} - a_{n-1}}{n(n-1)} \\
&= \frac{-n b_{n-1} + (2n-1) a_{n-1} + n a_{n-2}}{n(n-1)}
\end{aligned}
$$

For $n=2$

$$
\begin{aligned}
b_2 &= \frac{-2b_1+3a_1+a_0}{2}\\
&= \frac{-3+1}{2}\\
&= -1
\end{aligned}
$$

And do on. we find

$$
\begin{aligned}
y_2 &= Cy_1 \ln x + \sum_{n=0}^{\infty} b_n x^n\\
&= -y_1 \ln x + \left(b_0 + b_1 x + b_2 x^2 + \cdots\right)\\
&= -y_1 \ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 + \cdots\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \qquad (2)\\
&= c_1\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + c_2\left(-\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right)\ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 + \cdots\right)\right)\\
&= c_1\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + c_2\left(\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + \cdots\right)\ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 + \cdots\right)\right)
\end{aligned}
$$

Now we find y_p. Let

$$
y_p = \sum_{n=0}^{\infty} c_n x^{n+r}
$$

Substituting this into the original ode gives (1B) but with c_n in place of a_n

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=1}^{\infty} (n+r-1) c_{n-1} x^{n+r-1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} = 1 \qquad (1C)
$$

For $n=0$

$$
(r)(r-1) c_0 x^{r-1} = x^0
$$

For balance we need $r-1=0$ or $r=1$. Hence coefficient is

$$
0c_0 = 1
$$

Not possible to solve for c_0. No particular solution exist. No solution exist.

But sometimes this workaround ends up giving a solution. When the ode has form $Ay'' + By' + y = a$ where a is constant, as in this case, we can do change of variable $y = u + a$ first. The ode generated is then $Au'' + Bu' + u + a = a$ or $Au'' + Bu' + u = 0$. Now we solve this and obtain solution given above in (2) but now with u instead of y. In this example $a = 1$.

$$
u = c_1\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + c_2\left(\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + \cdots\right)\ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 + \cdots\right)\right)
$$

Replacing $u = y - 1$ the above becomes

$$
y - 1 = c_1\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + c_2\left(\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + \cdots\right)\ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 + \cdots\right)\right) \qquad (3)
$$

Now we apply IC. At $x=0$, $y(0)=1$ and the above gives

$$
\begin{aligned}
1 - 1 &= c_1 \lim_{x\to 0}\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + c_2 \lim_{x\to 0}\left(\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + \cdots\right)\ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 +
\right.\right.\\
0 &= 0 + c_2
\end{aligned}
$$

Hence $c_2 = 0$. The solution (3) becomes

$$
\begin{aligned}
y - 1 &= c_1\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right)\\
y &= c_1\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + 1 \qquad (4)
\end{aligned}
$$

This trick worked only because a happened to be same as what initial conditions $y(0) = a$. If there were different, then no solution exist. Lets see. Let IC be $y(0) = 5$. Then let $u = y - 1$ and the equation below (3) above now becomes

$$
\begin{aligned}
5 - 1 &= c_1 \lim_{x\to 0}\left(x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} + \cdots\right) + c_2 \lim_{x\to 0}\left(\left(-x + x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + \cdots\right)\ln x + \left(1 - x^2 + \frac{3}{4}x^3 - \frac{11}{36}x^4 +
\right.\right.\\
4 &= 0 + c_2\\
c_2 &= 4
\end{aligned}
$$

And the solution (3) now becomes

$$y-1=c_1\left(x-x^2+\frac{x^3}{2}-\frac{x^4}{6}+\frac{x^5}{24}+\cdots\right)+4\left(\left(-x+x^2-\frac{x^3}{2}+\frac{x^4}{6}-\frac{x^5}{24}+\cdots\right)\ln x+\left(1-x^2+\frac{3}{4}x^3-\frac{11}{36}x^4+\cdots\right)\right)$$
$$y=c_1\left(x-x^2+\frac{x^3}{2}-\frac{x^4}{6}+\frac{x^5}{24}+\cdots\right)+4\left(\left(-x+x^2-\frac{x^3}{2}+\frac{x^4}{6}-\frac{x^5}{24}+\cdots\right)\ln x+\left(1-x^2+\frac{3}{4}x^3-\frac{11}{36}x^4+\cdots\right)\right)+$$

Which does not verify the ode. So solution for this ode is given by (4).

2.5 Frobenius series. Indicial equation with root that differ by non integer

Local contents

2.5.1 Algorithm

If one of the roots is an integer, and the ode is inhomogeneous. ode, then we do not need to split the solution into y_h, y_p and can use the integer root to find y_p directly. If both roots are non-integer, we have to split the problem into y_h, y_p. This is because it will not be possible to match powers on x from the left side to the right side. Because the RHS will be polynomial in x, but the LHS will not be polynomial in x because of the non integer powers on x.In this case the solution is

$$y=c_1y_1+c_2y_2$$

Where

$$y_1=\sum_{n=0}^{\infty}a_nx^{n+r_1}$$
$$y_2=\sum_{n=0}^{\infty}b_nx^{n+r_2}$$

And r_1, r_2 are roots of the indicial equation. a_0, b_0 are set to 1 as arbitrary.

2.5.2 Examples

Local contents

2.5.2.1 Example 1 $2x^2y''+3xy'-xy=x^2+2x$

$$2x^2y''+3xy'-xy=x^2+2x$$

Comparing the ode to

$$y''+p(x)\,y'+q(x)\,y=0$$

Hence $p(x)=\frac{3}{2x}, q(x)=\frac{-1}{2x}$. There is one singular point at $x=0$. Therefore $p_0=\lim_{x\to 0}xp(x)=\lim_{x\to 0}\frac{3}{2}=\frac{3}{2}$ and $q_0=\lim_{x\to 0}x^2q(x)=\lim_{x\to 0}-\frac{x}{2}=0$. Hence the indicial equation is

$$\begin{aligned} r(r-1)+p_0r+q_0&=0\\ r(r-1)+\frac{3}{2}r+0&=0\\ r(2r+1)&=0\\ r&=0,-\frac{1}{2}\end{aligned}$$

Therefore $r_1=0, r_2=-\frac{1}{2}$.

Expansion around $x=x_0=0$. This is regular singular point. Hence Frobenius is needed. First we find y_h.

Let $y=\sum_{n=0}^{\infty} a_n x^{n+r}$, hence

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

The ode becomes

$$
2x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - x \sum_{n=0}^{\infty} a_n x^{n+r} = 0
$$

$$
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0
$$

Re indexing to lowest powers on x gives

$$
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)\, a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0
$$

When $n=0$

$$
\begin{aligned}
2(r)\,(r-1)\, a_0 x^r + 3(r)\, a_0 x^r &= 0 \\
(r(2r+1))\, a_0 x^r &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then $r(2r+1)=0$ and $r=0, r=-\frac{1}{2}$ as was found above. Therefore the homogenous ode satisfies

$$
2x^2 y'' + 3xy' - xy = (r(2r+1))\, a_0 x^r
$$

Where the RHS will be zero when $r=0$ or $r=-\frac{1}{2}$. For $n \geq 1$ the recurrence relation is

$$
\begin{aligned}
2(n+r)\,(n+r-1)\, a_n + 3(n+r)\, a_n - a_{n-1} &= 0 \\
a_n &= \frac{a_{n-1}}{2\,(n+r)\,(n+r-1) + 3\,(n+r)} \\
&= \frac{a_{n-1}}{2n^2 + 4nr + n + 2r^2 + r} \qquad (1)
\end{aligned}
$$

For $r=0$ the above becomes

$$
a_n = \frac{a_{n-1}}{2n^2 + n}
$$

For $n=1$ and letting $a_0 = 1$

$$
a_1 = \frac{1}{3}
$$

For $n=2$

$$
a_2 = \frac{a_1}{8+2} = \frac{a_1}{10} = \frac{1}{30}
$$

For $n=3$

$$
a_3 = \frac{a_2}{18+3} = \frac{a_2}{21} = \frac{1}{21\,(30)} = \frac{1}{630}
$$

And so on. Hence

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + \cdots \\
&= 1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \cdots
\end{aligned}
$$

And for $r=-\frac{1}{2}$ the recurrence relation (2) becomes, and using b instead of a

$$
b_n = \frac{b_{n-1}}{2n^2 + 4n\left(-\frac{1}{2}\right) + n + \frac{1}{2} - \frac{1}{2}} = -\frac{b_{n-1}}{n - 2n^2}
$$

For $n=1$ and using $b_0 = 1$

$$
b_1 = -\frac{b_0}{1-2} = 1
$$

For $n=2$

$$
b_2 = -\frac{b_1}{2-8} = -\frac{1}{2-8} = \frac{1}{6}
$$

For $n=3$

$$
b_3 = -\frac{b_2}{3-18} = -\frac{\frac{1}{6}}{3-18} = \frac{1}{90}
$$

And so on. Hence

$$
\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2} \\
&= \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} b_n x^n \\
&= \frac{1}{\sqrt{x}} \left(b_0 + b_1 x + b_2 x^2 + \cdots\right) \\
&= \frac{1}{\sqrt{x}} \left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \cdots\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \cdots\right) + c_2 \frac{1}{\sqrt{x}}\left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \cdots\right)
\end{aligned}
$$

Now we find y_p. Since ode satisfies

$$
2x^2 y'' + 3xy' - xy = (r(2r+1))\, a_0 x^r
$$

To find y_p, and relabeling r as m and a as c so not to confuse terms used for y_h. Then the above becomes

$$
2x^2 y'' + 3xy' - xy = (m(2m+1))\, c_0 x^m
$$

The RHS is $x^2 + 2x$. We balance each term at a time, this finds a particular solution for each term on the RHS, then these particular solutions are added at the end. For the input $2x$ the balance equation is

$$
(m(2m+1))\, c_0 x^m = 2x
$$

This implies that

$$
m = 1
$$

Therefore $(m(2m+1))\, c_0 = 2$, or $c_0(1(2+1)) = 2$ or $3c_0 = 2$ or

$$
c_0 = \frac{2}{3}
$$

The recurrence relation now becomes (using m for r and c_0 for a_0)

$$
c_n = \frac{c_{n-1}}{2n^2 + 4nm + n + 2m^2 + m}
$$

For $m = 1$ the above becomes

$$
c_n = \frac{c_{n-1}}{2n^2 + 5n + 3}
$$

For $n = 1$ and using $c_0 = \frac{2}{3}$

$$
c_1 = \frac{\frac{2}{3}}{2 + 5 + 3} = \frac{1}{15}
$$

For $n = 2$

$$
c_2 = \frac{c_1}{8 + 10 + 3} = \frac{\frac{1}{15}}{8 + 10 + 3} = \frac{1}{315}
$$

For $n = 3$

$$
c_3 = \frac{c_2}{18 + 15 + 3} = \frac{\frac{1}{315}}{18 + 15 + 3} = \frac{1}{11\,340}
$$

And so on. Hence

$$
\begin{aligned}
y_{p_1} &= \sum_{n=0}^{\infty} c_n x^{n+m} = x \sum_{n=0}^{\infty} c_n x^n \\
&= x\left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\
&= x\left(\frac{2}{3} + \frac{1}{15}x + \frac{1}{315}x^2 + \frac{1}{11\,340}x^3 + \cdots\right) \\
&= \left(\frac{2}{3}x + \frac{1}{15}x^2 + \frac{1}{315}x^3 + \frac{1}{11\,340}x^4 + \cdots\right)
\end{aligned}
$$

The second term x^2 is now balanced x^2. The balance equation is

$$
(m(2m+1))\, c_0 x^m = x^2
$$

Therefore $m = 2$ and $(m(2m+1))\, c_0 = 1$. Hence

$$
\begin{aligned}
(2(4+1))\, c_0 &= 1 \\
c_0 &= \frac{1}{10}
\end{aligned}
$$

The recurrence relation becomes for $m = 2$

$$c_n = \frac{c_{n-1}}{2n^2 + 4nm + n + 2m^2 + m}$$

For $m = 2$ the above becomes

$$c_n = \frac{c_{n-1}}{2n^2 + 9n + 10}$$

For $n = 1$ and using $c_0 = \frac{1}{10}$

$$c_1 = \frac{\frac{1}{10}}{2 + 9 + 10} = \frac{1}{210}$$

For $n = 2$

$$c_2 = \frac{c_1}{8 + 18 + 10} = \frac{\frac{1}{210}}{8 + 18 + 10} = \frac{1}{7560}$$

For $n = 3$

$$c_3 = \frac{c_2}{18 + 27 + 10} = \frac{\frac{1}{7560}}{18 + 27 + 10} = \frac{1}{415\,800}$$

And so on. Hence

$$\begin{aligned}
y_{p_2} &= \sum_{n=0}^{\infty} c_n x^{n+m} = x^2 \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \left(c_0 + c_1 x + c_2 x^2 + \cdots\right) \\
&= x^2 \left(\frac{1}{10} + \frac{1}{210} x + \frac{1}{7560} x^2 + \frac{1}{415\,800} x^3 + \cdots\right) \\
&= \left(\frac{1}{10} x^2 + \frac{1}{210} x^3 + \frac{1}{7560} x^4 + \frac{1}{415\,800} x^5 + \cdots\right)
\end{aligned}$$

The particular solution is the sum of all the particular solutions found above, which is

$$\begin{aligned}
y_p &= y_{p_1} + y_{p_2} \\
&= \left(\frac{2}{3} x + \frac{1}{15} x^2 + \frac{1}{315} x^3 + \frac{1}{11\,340} x^4 + \cdots\right) + \left(\frac{1}{10} x^2 + \frac{1}{210} x^3 + \frac{1}{7560} x^4 + \frac{1}{415\,800} x^5 + \cdots\right) \\
&= \frac{2}{3} x + \left(\frac{1}{15} + \frac{1}{10}\right) x^2 + \left(\frac{1}{315} + \frac{1}{210}\right) x^3 + \left(\frac{1}{11\,340} + \frac{1}{7560}\right) x^4 + \cdots \\
&= \frac{2}{3} x + \frac{1}{6} x^2 + \frac{1}{126} x^3 + \frac{1}{4536} x^4 + \cdots
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1 \left(1 + \frac{1}{3} x + \frac{1}{30} x^2 + \frac{1}{630} x^3 + \cdots\right) + c_2 \frac{1}{\sqrt{x}} \left(1 + x + \frac{1}{6} x^2 + \frac{1}{90} x^3 + \cdots\right) + \frac{2}{3} x + \frac{1}{6} x^2 + \frac{1}{126} x^3 + \frac{1}{4536} x^4 + \cdots
\end{aligned}$$

2.5.2.2 Example 2 $2xy'' + (x + 1) y' + 3y = 5$

$$2xy'' + (x + 1) y' + 3y = 5$$

Comparing the ode to

$$y'' + p(x) y' + q(x) y = 0$$

Hence $p(x) = \frac{(x+1)}{2x}, q(x) = \frac{3}{2x}$. There is one singular point at $x = 0$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{(x+1)}{2} = \frac{1}{2}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{3x}{2} = 0$. Hence the indicial equation is

$$\begin{aligned}
r(r - 1) + p_0 r + q_0 &= 0 \\
r(r - 1) + \frac{1}{2} r + 0 &= 0 \\
r(2r - 1) &= 0 \\
r &= 0, \frac{1}{2}
\end{aligned}$$

Therefore $r_1 = 0, r_2 = \frac{1}{2}$.

Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. Let

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}
\end{aligned}$$

The homogenous ode becomes

$$2x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+(x+1)\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+3\sum_{n=0}^{\infty}a_nx^{n+r}=0$$
$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=0}^{\infty}3a_nx^{n+r}=0$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty}2(n+r)(n+r-1)a_nx^{n+r-1}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+\sum_{n=1}^{\infty}3a_{n-1}x^{n+r-1}=0$$

For $n=0$

$$(2(r)(r-1)a_0+ra_0)x^{r-1}=0$$
$$(2r(r-1)+r)a_0=0$$

Since $a_0\neq 0$ then the first term above will vanish only when $2r(r-1)+r=0$ or $r(2r-1)=0$. Hence $r=0, r=\frac{1}{2}$ as was found above. For $n\geq 1$

$$2(n+r)(n+r-1)a_n+(n+r-1)a_{n-1}+(n+r)a_n+3a_{n-1}=0$$
$$a_n=-\frac{n+r+2}{(n+r)(2r+2n-1)}a_{n-1} \tag{1}$$

Therefore the differential equation satisfies

$$2xy''+(x+1)y'+3y=r(2r-1)a_0x^{r-1} \tag{2}$$

The RHS above will be zero when $r=0$ or $r=\frac{1}{2}$. When $r=0$ the recurrence relation (1) becomes

$$a_n=-\frac{n+2}{(n)(2n-1)}a_{n-1}$$

Which gives (for $a_0=1$) (working out few terms using the above)

$$y_1=1-3x+2x^2-\frac{2}{3}x^3+\cdots$$

And when $r=\frac{1}{2}$ the recurrence relation is (using b in place of a and letting $b_0=1$ also)

$$b_n=-\frac{n+\frac{5}{2}}{\left(n+\frac{1}{2}\right)(1+2n-1)}b_{n-1}$$

Which gives (working out few terms)

$$y_2=\sqrt{x}\left(1-\frac{7x}{6}+21\frac{x^2}{40}+\cdots\right)$$

Hence the solution is

$$\begin{aligned}y_h&=c_1y_1+c_2y_2\\&=c_1\left(1-3x+2x^2-\frac{2}{3}x^3+\cdots\right)+c_2\left(\sqrt{x}\left(1-\frac{7x}{6}+21\frac{x^2}{40}+\cdots\right)\right)\end{aligned}$$

Now we find y_p. From (2), and relabeling r as m and a as c so not to confuse terms used

$$2xy''+(x+1)y'+3y=m(2m-1)c_0x^{m-1}$$

Therefore we need to balance $m(2m-1)c_0x^{m-1}=5$ since the RHS is 5. This implies $m-1=0$ or $m=1$. Therefore $m(2m-1)c_0=5$ or $(2-1)c_0=5$ which gives $c_0=5$. Hence

$$\begin{aligned}y_p&=\sum_{n=0}^{\infty}c_nx^{n+m}\\&=x\sum_{n=0}^{\infty}c_nx^n\end{aligned}$$

To find c_n, the same recurrence relation (1) is used by with r replaced by m and a replaced by c. This gives

$$c_n=-\frac{n+m+2}{(n+m)(2m+2n-1)}c_{n-1}$$

For $m=1$ the above becomes

$$c_n=-\frac{n+3}{(n+1)(1+2n)}c_{n-1}$$

For $n=1$

$$c_1 = -\frac{1+3}{(1+1)(1+2)}c_0 = -\frac{2}{3}c_0 = -\frac{2}{3}(5) = -\frac{10}{3}$$

For $n=2$

$$c_2 = -\frac{2+3}{(2+1)(1+4)}c_1 = -\frac{1}{3}c_1 = -\frac{1}{3}\left(-\frac{10}{3}\right) = \frac{10}{9}$$

For $n=3$

$$c_1 = -\frac{3+3}{(3+1)(1+6)}c_2 = -\frac{3}{14}\left(\frac{10}{9}\right) = -\frac{2}{3}(5) = -\frac{5}{21}$$

And so on. Hence

$$\begin{aligned}
y_p &= x\sum_{n=0}^{\infty} c_n x^n \\
&= x\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) \\
&= x\left(5 - \frac{10}{3}x + \frac{10}{9}x^2 - \frac{5}{21}x^3 + \cdots\right) \\
&= \left(5x - \frac{10}{3}x^2 + \frac{10}{9}x^3 - \frac{5}{21}x^4 + \cdots\right)
\end{aligned}$$

Hence the final solution

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1\left(1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots\right) + \sqrt{x}c_2\left(1 - \frac{7x}{6} + 21\frac{x^2}{40} + \cdots\right) + \left(5x - \frac{10}{3}x^2 + \frac{10}{9}x^3 - \frac{5}{21}x^4 + \cdots\right)
\end{aligned}$$

2.5.2.3 Example 3 $2xy'' + (x+1)y' + 3y = x$

$$2xy'' + (x+1)y' + 3y = x$$

This is the same problem as above but different RHS. As shown above, we obtained that the differential equation satisfies

$$2xy'' + (x+1)y' + 3y = r(2r-1)a_0 x^{r-1}$$

To find y_p, and using m in place of r and c in place of a so not to confuse terms with the y_h terms, then the above becomes

$$2xy'' + (x+1)y' + 3y = m(2m-1)c_0 x^{m-1}$$

The RHS above will be zero when $m=0$ or $m=\frac{1}{2}$. We now need to balance the RHS against given RHS which is x. Hence

$$m(2m-1)c_0 x^{m-1} = x$$

To balance this we need $m-1=1$ or $m=2$. Hence $2(4-1)c_0 = 1$ or $c_0 = \frac{1}{6}$. Using the recurrence relation we found above, which is for $n \geq 1$ (again, calling r as m so not to confuse y_h terms with y_p terms), we obtain

$$c_n = -\frac{n+m+2}{(n+r)(2m+2n-1)}c_{n-1}$$

But now using $m=2$

$$c_n = -\frac{n+4}{(n+2)(4+2n-1)}c_{n-1}$$

Hence for $n=1$

$$\begin{aligned}
c_1 &= -\frac{1+4}{(1+2)(4+2-1)}c_0 \\
&= -\frac{1}{3}c_0 \\
&= -\frac{1}{3}\left(\frac{1}{6}\right) = -\frac{1}{18}
\end{aligned}$$

for $n=2$

$$\begin{aligned}
c_2 &= -\frac{6}{(2+2)(4+4-1)}c_1 \\
&= -\frac{3}{14}c_1 = -\frac{3}{14}\left(-\frac{1}{18}\right) = \frac{1}{84}
\end{aligned}$$

For $n=3$

$$\begin{aligned}
c_3 &= -\frac{3+4}{(3+2)(4+6-1)}c_2 \\
&= -\frac{7}{45}c_2 = -\frac{7}{45}\left(\frac{1}{84}\right) = -\frac{1}{540}
\end{aligned}$$

For $n = 4$

$$\begin{aligned} c_4 &= -\frac{4+4}{(4+2)(4+8-1)} c_3 \\ &= -\frac{4}{33} c_3 = -\frac{4}{33}\left(-\frac{1}{540}\right) = \frac{1}{4455} \end{aligned}$$

And so on. Hence

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 (c_0 + c_1 x + c_2 x^2 + \cdots) \\ &= x^2 \left(\frac{1}{6} - \frac{1}{18}x + \frac{1}{84}x^2 - \frac{1}{540}x^3 + \frac{1}{4455}x^4 + \cdots\right) \end{aligned}$$

Hence the solution is (y_h was found in the earlier problem)

$$\begin{aligned} y &= y_h + y_p \\ &= c_1\left(1 - 3x + 2x^2 - \frac{2}{3}x^3 + \cdots\right) + c_2\left(\sqrt{x}\left(1 - \frac{7x}{6} + 21\frac{x^2}{40} + \cdots\right)\right) + x^2\left(\frac{1}{6} - \frac{1}{18}x + \frac{1}{84}x^2 - \frac{1}{540}x^3 + \frac{1}{4455}x^4 + \cdots\cdots\right) \end{aligned}$$

2.5.2.4 Example 4 $x^2y'' + (x+1)y' + y = 5$

$$x^2y'' + (x+1)y' + y = 5$$

Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. Comparing the ode to

$$y'' + p(x)y' + q(x)y = 0$$

Hence $p(x) = \frac{x+1}{x^2}, q(x) = \frac{1}{x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{x+1}{x}$ which is not defined. Hence not possible to solve this using series solution.

2.5.2.5 Example 5 $2x^2y'' - xy' + (1-x^2)y = x^2$

$$2x^2y'' - xy' + (1-x^2)y = x^2$$

Comparing the ode to

$$y'' + p(x)y' + q(x)y = 0$$

Hence $p(x) = \frac{-x}{2x^2} = -\frac{1}{2x}, q(x) = \frac{(1-x^2)}{2x^2}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{-1}{2} = \frac{-1}{2}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{(1-x^2)}{2} = \frac{1}{2}$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) - \frac{1}{2}r + \frac{1}{2} &= 0 \\ r^2 - \frac{3}{2}r + \frac{1}{2} &= 0 \\ r &= 1, \frac{1}{2} \end{aligned}$$

Therefore $r_1 = 0, r_2 = -\frac{1}{2}$. Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. First we find y_h. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, hence

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

The homogenous ode becomes

$$\begin{aligned} 2x^2y'' - xy' + (1-x^2)y &= 0 \\ 2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + (1-x^2) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0 \end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2}x^{n+r} = 0$$

When $n=0$

$$\begin{aligned}(2(n+r)(n+r-1)a_0 - (n+r)a_0 + a_0)x^r &= 0\\ (2r(r-1) - r + 1)a_0 x^r &= 0\\ (2r^2 - 3r + 1)a_0 x^r &= 0\end{aligned}$$

Since $a_0 \neq 0$ then $2r^2 - 3r + 1 = 0$, hence $r = 1, r = \frac{1}{2}$ as was found above. Therefore the homogenous ode satisfies

$$2x^2y'' - xy' + (1-x^2)y = (2r^2 - 3r + 1)a_0 x^r$$

Where the RHS will be zero when $r = 1, r = \frac{1}{2}$. When $n = 1$

$$\begin{aligned}2(1+r)(1+r-1)a_1 - (1+r)a_1 + a_1 &= 0\\ (2(1+r)(1+r-1) - (1+r) + 1)a_1 &= 0\\ r(2r+1)a_1 &= 0\end{aligned}$$

Hence $a_1 = 0$. For $n \geq 2$ the recurrence relation is

$$\begin{aligned}2(n+r)(n+r-1)a_n - (n+r)a_n + a_n - a_{n-2} &= 0\\ a_n &= \frac{a_{n-2}}{2(n+r)(n+r-1) - (n+r) + 1}\\ &= \frac{a_{n-2}}{2(n+r)(n+r-1) - (n+r) + 1} \qquad (1)\end{aligned}$$

For $r = 1$ the above becomes

$$a_n = \frac{a_{n-2}}{n(2n+1)}$$

For $n = 2$ and letting $a_0 = 1$

$$a_2 = \frac{a_0}{2(4+1)} = \frac{1}{10}$$

For $n = 3$

$$a_3 = \frac{a_1}{n(2n+1)} = 0$$

For $n = 4$

$$a_4 = \frac{a_2}{4(8+1)} = \frac{\frac{1}{10}}{4(8+1)} = \frac{1}{360}$$

And so on. Hence

$$\begin{aligned}y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r} = x\sum_{n=0}^{\infty} a_n x^n\\ &= x(a_0 + a_1 x + a_2 x^2 + \cdots)\\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + \cdots\right)\end{aligned}$$

And for $r = \frac{1}{2}$ the recurrence relation (1) becomes, and using b instead of a

$$\begin{aligned}b_n &== \frac{b_{n-2}}{2(n+r)(n+r-1) - (n+r) + 1}\\ &= \frac{b_{n-2}}{2\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}-1\right) - \left(n+\frac{1}{2}\right) + 1}\\ &= \frac{b_{n-2}}{n(2n-1)}\end{aligned}$$

Notice also that $b_1 = 0$ just like $a_1 = 0$ from above. Now, for $n = 2$ and using $b_0 = 1$

$$b_2 = \frac{b_0}{2(4-1)} = \frac{1}{6}$$

For $n = 3$

$$b_2 = -\frac{b_1}{2-8} = -\frac{1}{2-8} = \frac{1}{6}$$

For $n = 3$

$$b_3 = \frac{b_1}{n(2n-1)} = 0$$

For $n = 4$

$$b_n = \frac{b_2}{4(8-1)} = \frac{\frac{1}{6}}{4(8-1)} = \frac{1}{168}$$

And so on. Hence

$$
\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2} \\
&= \sqrt{x} \sum_{n=0}^{\infty} b_n x^n \\
&= \sqrt{x}(b_0 + b_1 x + b_2 x^2 + \cdots) \\
&= \sqrt{x}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + \cdots\right)\right) + c_2\sqrt{x}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right) \\
&= c_1\left(x + \frac{x^3}{10} + \frac{x^5}{360} + \cdots\right) + c_2\sqrt{x}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)
\end{aligned}
$$

Now we find y_p. Since ode satisfies

$$
2x^2 y'' - xy' + \left(1 - x^2\right) y = \left(2r^2 - 3r + 1\right) a_0 x^r
$$

To find y_p, and relabeling r as m and a as c so not to confuse terms used for y_h. Then the above becomes

$$
2x^2 y'' - xy' + \left(1 - x^2\right) y = \left(2m^2 - 3m + 1\right) c_0 x^m
$$

The RHS is x^2. Hence the balance equation is

$$
\left(2m^2 - 3m + 1\right) c_0 x^m = x^2
$$

This implies that

$$
m = 2
$$

Therefore $\left(2m^2 - 3m + 1\right) c_0 = 1$, or $(8 - 6 + 1)\, c_0 = 1$ or

$$
c_0 = \frac{1}{3}
$$

The recurrence relation (1) from above now becomes (using m for r and c_0 for a_0)

$$
c_n = \frac{c_{n-2}}{2\,(n+m)\,(n+m-1) - (n+m) + 1}
$$

For $m = 2$ the above becomes

$$
\begin{aligned}
c_n &= \frac{c_{n-2}}{2\,(n+2)\,(n+1) - (n+2) + 1} \\
&= \frac{c_{n-2}}{2n^2 + 5n + 3}
\end{aligned}
$$

For $n = 1$ we use $c_1 = 0$ the same as was found for a_1, b_1. For $n \geq 2$ the above is used. Hence for $n = 2$

$$
c_2 = \frac{c_0}{8 + 10 + 3} = \frac{\frac{1}{3}}{8 + 10 + 3} = \frac{1}{63}
$$

For $n = 3$

$$
c_3 = \frac{c_1}{18 + 15 + 3} = 0
$$

For $n = 4$

$$
c_4 = \frac{c_2}{32 + 20 + 3} = \frac{\frac{1}{63}}{32 + 20 + 3} = \frac{1}{3465}
$$

And so on. Hence

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} = x^2 \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 (c_0 + c_1 x + c_2 x^2 + \cdots) \\
&= x^2 \left(\frac{1}{3} + \frac{1}{63}x^2 + \frac{1}{3465}x^4 + \cdots\right) \\
&= \frac{1}{3}x^2 + \frac{1}{63}x^4 + \frac{1}{3465}x^6 + \cdots
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1\left(x + \frac{x^3}{10} + \frac{x^5}{360} + \cdots\right) + c_2\sqrt{x}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right) + \left(\frac{1}{3}x^2 + \frac{1}{63}x^4 + \frac{1}{3465}x^6 + \cdots\right)
\end{aligned}
$$

Alternative way to find y_p is the the following. Let $y_p = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ then $y_p' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$ and $y_p'' = 2c_2 + 6c_3 x + \cdots$. Hence the ode becomes

$$\begin{aligned}
2x^2(2c_2 + 6c_3 x + \cdots) - x(c_1 + 2c_2 x + 3c_3 x^2 + \cdots) + (1 - x^2)\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) &= x^2 \\
c_0 + x(-c_1 + c_1) + x^2(4c_2 - 2c_2 + c_2 - c_0) + x^3(\cdots) &= x^2
\end{aligned}$$

Hence $c_0 = 0, 4c_2 - 2c_2 + c_2 - c_0 = 1$ or $3c_2 - c_0 = 1$ or $c_2 = \frac{1}{3}$. We need to keep adding more equations and solving them simultaneously. This method is not as easy to use as the method used above, which uses the balance equation to find to y_p. Also this method could fail, since in practice we should not use undetermined coefficients method (which is what this does) on an ode with variable coefficients. So I will not use this any more.

2.5.2.6 Example 6 $2xy'' + y' + y = 0$

$$2xy'' + y' + y = 0$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{1}{2x}, q(x) = \frac{1}{2x}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{x}{2} = 0$. Hence the indicial equation is

$$\begin{aligned}
r(r-1) + p_0 r + q_0 &= 0 \\
r(r-1) + \frac{1}{2} r &= 0 \\
r(2r-1) &= 0 \\
r &= 0, \frac{1}{2}
\end{aligned}$$

Therefore $r_1 = 0, r_2 = \frac{1}{2}$. Expansion around $x = x_0 = 0$. This is regular singular point. Hence Frobenius is needed. First we find y_h. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, hence

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
xy'' + y' + y &= 0 \\
2x \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0
\end{aligned}$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

When $n = 0$

$$\begin{aligned}
2(r)\,(r-1)\, a_0 x^{r-1} + r a_0 x^{r-1} &= 0 \\
(2r(r-1) + r)\, a_0 x^{r-1} &= 0 \\
(r(2r-1))\, a_0 x^{r-1} &= 0
\end{aligned}$$

Since $a_0 \neq 0$ then $r(2r-1) = 0$, hence $r = 0, r = \frac{1}{2}$ as was found above. Therefore the homogenous ode satisfies

$$2xy'' + y' + y = (r(2r-1))\, a_0 x^{r-1}$$

Where the RHS will be zero when $r = 1, r = \frac{1}{2}$. For $n \geq 1$ the recurrence relation is

$$\begin{aligned}
2(n+r)\,(n+r-1)\, a_n + (n+r)\, a_n &= -a_{n-1} \\
a_n &= \frac{-a_{n-1}}{2\,(n+r)\,(n+r-1) + (n+r)} \\
&= \frac{-a_{n-1}}{2n^2 + 4nr - n + 2r^2 - r} \qquad (1)
\end{aligned}$$

For $r=0$ the above becomes

$$a_n = \frac{-a_{n-1}}{n(2n-1)}$$

For $n=1$ and using $a_0 = 1$

$$a_1 = \frac{-a_0}{n(2n-1)} = -1$$

For $n=2$

$$a_2 = \frac{-a_1}{2(3)} = \frac{1}{6}$$

For $n=3$

$$a_3 = \frac{-a_2}{3(5)} = \frac{-\frac{1}{6}}{15} = -\frac{1}{90}$$

And so on. Hence

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\ &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= 1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots \end{aligned}$$

To find y_2, using (1) but replacing a by b and using $r = \frac{1}{2}$ and letting $b_0 = 1$ and following the above process gives

$$b_n = \frac{-b_{n-1}}{2n^2 + 4n\left(\frac{1}{2}\right) - n + 2\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -\frac{b_{n-1}}{2n^2 + n}$$

For $n=1$

$$b_1 = -\frac{b_0}{3} = -\frac{1}{3}$$

For $n=2$

$$b_2 = -\frac{b_1}{8+2} = -\frac{b_1}{10} = -\frac{-\frac{1}{3}}{10} = \frac{1}{30}$$

And so on. Hence we obtain

$$\begin{aligned} y_2 &= \sqrt{x} \sum_{n=0}^{\infty} b_n x^n \\ &= \sqrt{x}(b_0 + b_1 x + b_2 x^2 + \cdots) \\ &= \sqrt{x}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 + \cdots\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_1 \\ &= c_1\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) + c_2\left(\sqrt{x}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 + \cdots\right)\right) \end{aligned}$$

2.5.2.7 Example 7 $4xy'' + 3y' + 3y = \sqrt{x}$

$$4xy'' + 3y' + 3y = \sqrt{x}$$

Comparing the ode to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = \frac{3}{4x}, q(x) = \frac{3}{4x}$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} \frac{3}{4} = \frac{3}{4}$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{3x}{4} = 0$. Hence $x=0$ is regular singular point. The indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + \frac{3}{4}r + 0 &= 0 \\ r(r-1) + \frac{3}{4}r &= 0 \\ r &= \frac{1}{4}, 0 \end{aligned}$$

Frobenius is now used. Roots differ by non integer. First we find y_h. Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2} \end{aligned}$$

The homogenous ode becomes

$$4x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + 3 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

Re indexing to lowest powers on x gives

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 3(n+r) a_n x^{n+r-1} + \sum_{n=1}^{\infty} 3a_{n-1} x^{n+r-1} = 0$$

When $n = 0$

$$\begin{aligned} 4(n+r)(n+r-1) a_n x^{n+r-1} + 3(n+r) a_n x^{n+r-1} &= 0 \\ 4r(r-1) a_0 + 3r a_0 &= 0 \\ (4r(r-1) + 3r) a_0 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then $4r(r-1) + 3r = 0$, hence $r = 0, r = \frac{1}{4}$ as was found above. Therefore the homogenous ode satisfies

$$4xy'' + 3y' + 3y = (4r(r-1) + 3r) a_0 x^{r-1}$$

Hence the balance equation is that we will use to find the particular solution is

$$(4m(m-1) + 3m) c_0 x^{m-1} = \sqrt{x}$$

We will get back to the above after finding y_h. Going over the same steps as before, we find the recurrence relation

$$a_n = -\frac{3a_{n-1}}{4n^2 + 8nr + 4r^2 - n - r}$$

For $r = \frac{1}{4}, n > 0$ and similarly

$$b_n = -\frac{3a_{n-1}}{4n^2 + 8nr + 4r^2 - n - r}$$

For $r = 0, n > 0$. Finding few terms using the above gives the solution as

$$\begin{aligned} y_h &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{4}} \left(1 - \frac{3}{5}x + \frac{1}{10}x^2 - \frac{1}{130}x^3 + \cdots\right) + c_2 \left(1 - x + \frac{3}{14}x^2 - \frac{3}{154}x^3 + \cdots\right) \end{aligned}$$

Now we need to find y_p. From the balance equation

$$(4m(m-1) + 3m) c_0 x^{m-1} = \sqrt{x}$$

Hence $m - 1 = \frac{1}{2}$ or $m = \frac{3}{2}$. And $(4m(m-1) + 3m) c_0 = 1$, hence $\left(4\left(\frac{3}{2}\right)\left(\frac{3}{2} - 1\right) + 3\left(\frac{3}{2}\right)\right) c_0 = 1$, which gives $c_0 = \frac{2}{15}$. Therefore

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^{\frac{3}{2}} (c_0 + c_1 x + c_2 x^2 + \cdots) \\ &= x^{\frac{3}{2}} \left(\frac{2}{15} + c_1 x + c_2 x^2 + \cdots\right) \end{aligned}$$

We now just need to determined c_n for $n > 0$. For this we use the same recurrence relation as found above. We can use a_n or b_n as they are the same, but change a_n to c_n and r to c (so not to confuse notations). This gives

$$c_n = -\frac{3c_{n-1}}{4n^2 + 8nm + 4m^2 - n - m}$$

For $n > 0$ and $m = \frac{3}{2}$. Hence for $n = 1$ the above gives

$$\begin{aligned} c_1 &= -\frac{3c_0}{4 + 8\left(\frac{3}{2}\right) + 4\left(\frac{3}{2}\right)^2 - 1 - \frac{3}{2}} \\ &= -\frac{3\left(\frac{2}{15}\right)}{4 + 8\left(\frac{3}{2}\right) + 4\left(\frac{3}{2}\right)^2 - 1 - \frac{3}{2}} \\ &= -\frac{4}{225} \end{aligned}$$

For $n = 2$

$$\begin{aligned} c_1 &= -\frac{3c_1}{4(2)^2 + 8(2)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - 2 - \left(\frac{3}{2}\right)} \\ &= -\frac{3\left(-\frac{4}{225}\right)}{4(2)^2 + 8(2)\left(\frac{3}{2}\right) + 4\left(\frac{3}{2}\right)^2 - 2 - \left(\frac{3}{2}\right)} \\ &= \frac{8}{6825} \end{aligned}$$

And so on. Hence

$$
\begin{aligned}
y_p &= x^{\frac{3}{2}}\left(\frac{2}{15} + c_1 x + c_2 x^2 + \cdots\right) \\
&= x^{\frac{3}{2}}\left(\frac{2}{15} - \frac{4}{225}x + \frac{8}{6825}x^2 - \frac{16}{348075}x^3 + \cdots\right)
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 x^{\frac{1}{4}}\left(1 - \frac{3}{5}x + \frac{1}{10}x^2 - \frac{1}{130}x^3 + \cdots\right) + c_2\left(1 - x + \frac{3}{14}x^2 - \frac{3}{154}x^3 + \cdots\right) + x^{\frac{3}{2}}\left(\frac{2}{15} - \frac{4}{225}x + \frac{8}{6825}x^2 - \frac{16}{348075}x^3 + \cdots\right)
\end{aligned}
$$

2.5.2.8 Example 8. $2x^2y''(x) - xy'(x) + (1 - x^2)\, y(x) = 0$

With expansion around $x = 0$. This is a regular singular ODE. Let

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above into the ode and simplifying gives

$$
\begin{aligned}
2x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + (1-x^2) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0
\end{aligned}
$$

The next step is to make all powers of x to be $n + r$. This results in

$$
\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{1}
$$

The indicial equation is obtained from $n = 0$

$$
\begin{aligned}
2(n+r)\,(n+r-1)\, a_n - (n+r)\, a_n + a_n &= 0 \\
a_n(2(n+r)\,(n+r-1) - (n+r) + 1) &= 0 \\
2(n+r)\,(n+r-1) - (n+r) + 1 &= 0 \\
2(0+r)\,(0+r-1) - (0+r) + 1 &= 0 \\
2r^2 - 3r + 1 &= 0
\end{aligned}
$$

The roots are

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots differ by non integer, then the solutions are given by

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} = \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2 &= \sum_{n=0}^{\infty} a_n x^{n+r_2} = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}
\end{aligned}
$$

We start by finding y_1. EQ. (1) gives for $r = 1$

$$
\sum_{n=0}^{\infty} 2(n+1)\,(n)\, a_n x^{n+1} - \sum_{n=0}^{\infty} (n+1)\, a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+1} = 0
$$

For $n=1$ (we skip $n=0$ as that was used to find r) and we always let $a_0=1$ as it is arbitrary. Hence

$$\begin{aligned}
2(n+1)(n)a_n x^{n+1} - (n+1)a_n x^{n+1} + a_n x^{n+1} &= 0 \\
2(n+1)(n)a_n - (n+1)a_n + a_n &= 0 \\
4a_1 - 2a_1 + a_1 &= 0 \\
3a_1 &= 0 \\
a_1 &= 0
\end{aligned}$$

For $n \geq 2$, we have recursion relation. From it we can find that $a_2 = \frac{1}{10}, a_3 = 0, a_4 = \frac{1}{360}$ and so on. Hence

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots \\
&= x + \frac{1}{10}x^3 + \frac{1}{360}x^5 + \cdots
\end{aligned}$$

Now we do the same for y_2. EQ. (1) now becomes for $r_2 = \frac{1}{2}$

$$\sum_{n=0}^{\infty} 2\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}-1\right)a_n x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} - \sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{1}{2}} = 0$$

For $n=1$ (we skip $n=0$ as that was used to find r) and we always let $a_0=1$ as it is arbitrary. Hence

$$\begin{aligned}
2\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}-1\right)a_1 - \left(1+\frac{1}{2}\right)a_1 + a_1 &= 0 \\
a_1 &= 0
\end{aligned}$$

For $n \geq 2$, we have recursion relation. From it we can find that $a_2 = \frac{1}{6}, a_3 = 0, a_4 = \frac{1}{168}$ and so on. Hence

$$\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
&= x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n \\
&= x^{\frac{1}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right) \\
&= x^{\frac{1}{2}}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x + \frac{1}{10}x^3 + \frac{1}{360}x^5 + \cdots\right) + c_2\left(x^{\frac{1}{2}}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)\right)
\end{aligned}$$

2.5.2.9 Example 9. $2x^2y''(x) - xy'(x) + (1-x^2)\,y(x) = 1 + x$

With expansion around $x=0$. This is a regular singular ODE. This is same ode solved in example 1, but now with non-zero on the right side. Hence solution is given by

$$y = y_h + y_p$$

Where we found y_h to be

$$y_h = c_1\left(x + \frac{1}{10}x^3 + \frac{1}{360}x^5 + \cdots\right) + c_2\left(x^{\frac{1}{2}}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)\right)$$

To find y_p, we see there are two terms on the right side. we find y_{p_1} that corresponds to 1 on the right side and then y_{p_2} that corresponds to x on the right side. We will find that y_{p_2} does not exist. Hence no solution exist using series. Let us now find y_{p_1}. The ode now is

$$2x^2y''(x) - xy'(x) + (1-x^2)\,y(x) = 1$$

Let

$$y_{p_1} = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Using same expansion as in example 1, but now using c_n instead of a_n gives EQ. (1) from example 1 as

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 1 \tag{1A}$$

For $n=0$

$$\begin{aligned}
2r(r-1)\,c_0x^r - rc_0x^r + c_0x^r &= x^0\\
(2r(r-1)\,c_0 - rc_0 + c_0)\,x^r &= x^0\\
(2r(r-1) - r + 1)\,c_0x^r &= x^0\\
\left(2r^2 - 3r + 1\right)c_0x^r &= x^0
\end{aligned}$$

We see that for balance, then $r=0$. Which implies $\left(2r^2-3r+1\right)c_0 = 1$ or $c_0 = 1$. Hence

$$y_{p_1} = \sum_{n=0}^{\infty} c_n x^n$$

Now that we used $n=0$ to find r by matching against the RHS which is 1, from now on, for all $n>0$ we will use (1A) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^n on the right.

Then (1A) becomes (using $r=0$)

$$\sum_{n=0}^{\infty} 2n(n-1)\,c_nx^n - \sum_{n=0}^{\infty} nc_nx^n + \sum_{n=0}^{\infty} c_nx^n - \sum_{n=2}^{\infty} c_{n-2}x^n = 0 \tag{2}$$

For $n=1$ the above gives

$$-c_1x + c_1x = 0$$

c_1 is arbitrary Let $c_1 = 0$. For $n=2$ EQ. (2) gives

$$\begin{aligned}
4c_2x^2 - 2c_2x^2 + c_2x^2 - c_0x^2 &= 0\\
(4c_2 - 2c_2 + c_2 - c_0)\,x^2 &= 0\\
(3c_2 - 1)\,x^2 &= 0
\end{aligned}$$

Hence $3c_2 - 1 = 0$ since there is no x^2 term on the right side. Hence $c_2 = \frac{1}{3}$. For $n=3$ EQ. (2) gives

$$\begin{aligned}
(12c_3 - 3c_3 + c_3 - c_1)\,x^3 &= 0\\
(10c_3)\,x^3 &= 0
\end{aligned}$$

Hence $c_3 = 0$. For $n=4$ EQ. (2) gives

$$\begin{aligned}
24c_4x^4 - 4c_4x^4 + c_4x^4 - c_2x^4 &= 0\\
\left(21c_4 - \frac{1}{3}\right)x^4 &= 0
\end{aligned}$$

Hence $21c_4 - \frac{1}{3} = 0$ or $c_4 = \frac{1}{63}$ and so on. We find that

$$\begin{aligned}
y_{p_1} &= \sum_{n=0}^{\infty} c_nx^n\\
&= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4 + \cdots
\end{aligned}$$

Now we find y_{p_2}. EQ. (1A) now is

$$\sum_{n=0}^{\infty} 2(n+r)\,(n+r-1)\,c_nx^{n+r} - \sum_{n=0}^{\infty} (n+r)\,c_nx^{n+r} + \sum_{n=0}^{\infty} c_nx^{n+r} - \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = x$$

For $n=0$

$$\begin{aligned}
2(n+r)\,(n+r-1)\,c_nx^{n+r} - (n+r)\,c_nx^{n+r} + c_nx^{n+r} &= x\\
2r(r-1)\,c_0x^r - rc_0x^r + c_0x^r &= x\\
(2r(r-1) - r + 1)\,c_0x^r &= x\\
\left(2r^2 - 3r + 1\right)c_0x^r &= x
\end{aligned}$$

For balance we need $r=1$. This results in

$$\begin{aligned}
\left(2r^2 - 3r + 1\right)c_0 &= 1\\
(2 - 3 + 1)\,c_0 &= 1\\
0c_0 &= 1
\end{aligned}$$

Not possible. We see why there is no series solution. It is not possible to solve for y_{p2}.

2.6 Frobenius series where indicial equation roots are complex conjugate

Local contents

2.6.1 Algorithm

In this case the solution is

$$y = c_1 y_1 + c_2 y_2$$

Where

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} \\ y_2 &= \sum_{n=0}^{\infty} b_n x^{n+r_2} \end{aligned}$$

Where r_1, r_2 are roots of the indicial equation. a_0, b_0 are set to 1 as arbitrary.

2.6.2 Examples

Local contents

2.6.2.1 Example 1 $x^2y'' + xy' + y = 1$

$$x^2y'' + xy' + y = 1$$

Comparing the ode to

$$y'' + p(x)\, y' + q(x)\, y = 0$$

Hence $p(x) = \frac{1}{x}, q(x) = \frac{1}{x^2}$. There is one singular point at $x_0 = 0$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} 1 = 1$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) + r + 1 &= 0 \\ r^2 + 1 &= 0 \\ r &= \pm i \end{aligned}$$

Hence $r_1 = i, r_2 = -i$. Expansion around $x = 0$. This is regular singular point. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \end{aligned}$$

Solving first for the homogenous ode.

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \end{aligned}$$

For $n = 0$

$$(r(r-1) + r + 1)\, a_0 x^r = 0 \tag{1}$$

Since $a_0 \neq 0$, then $(r(r-1) + r + 1) = 0$ or $r^2 + 1 = 0$. Therefore $r = \pm i$ as was found above. The homogenous ode therefore satisfies

$$x^2y'' + xy' + y = \left(r^2 + 1\right) a_0 x^r$$

Since when $r = \pm i$, the RHS is zero. For $n \geq 1$ the recurrence relation is

$$\begin{aligned} (n+r)\,(n+r-1)\, a_n + (n+r)\, a_n + a_n &= 0 \\ ((n+r)\,(n+r-1) + (n+r) + 1)\, a_n &= 0 \\ \left(n^2 + 2nr + r^2 + 1\right) a_n &= 0 \end{aligned} \tag{2}$$

Let $a_0 = 1$. For $r = i$. For $n = 1$

$$(1 + 2i - 1 + 1)\, a_1 = 0$$

Hence $a_1 = 0$. Similarly all $a_n = 0$ for $n \geq 1$. Hence

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^{n+i} \\ &= x^i (a_0 + a_1 x + \cdots) \\ &= a_0 x^i \\ &= x^i \end{aligned}$$

For $r = -i$. For $n = 1$ and using b instead of a, we obtain (also using $b_0 = 1$)

$$(1 - 2i + 1 + 1)\, b_n = 0$$

Hence $b_1 = 0$. Similarly all $b_n = 0$ for $n \geq 1$. Hence

$$\begin{aligned} y_2 &= \sum_{n=0}^{\infty} b_n x^{n-i} \\ &= x^{-i} (b_0 + b_1 x + \cdots) \\ &= b_0 x^{-i} \\ &= x^{-i} \end{aligned}$$

Therefore

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^i + c_2 x^{-i} \end{aligned}$$

To find y_p since the ode satisfies

$$x^2 y'' + x y' + y = (r^2 + 1)\, a_0 x^r$$

Relabel $r = m, a_0 = c_0$ to avoid confusion with terms used above, then we balance RHS, hence

$$(m^2 + 1)\, c_0 x^m = 1$$

This implies $m = 0$ and $c_0 = 1$. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the recurrence relation (2) found above, but now using the values found $m = 0$ and $c_0 = 1$, then (2) becomes

$$\begin{aligned} (n^2 + 2nm + m^2 + 1)\, c_n &= 0 \\ (n^2 + 1)\, c_n &= 0 \end{aligned}$$

Hence all $c_n = 0$ except for c_0. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 \\ &= 1 \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 x^i + c_2 x^{-i} + 1 \end{aligned}$$

2.6.2.2 Example 2. $x^2y'' + x^2y' + y = 0$

$$x^2 y'' + x^2 y' + y = 0 \tag{1}$$

With expansion around $x = 0$. This is a regular singular ODE. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

Substituting the above in (1) gives

$$x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + x^2 \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{1A}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the middle term above as follows

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r+1} = \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1} x^{n+r}$$

And now Eq (1A) becomes

$$\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{3A}$$

$n = 0$ gives the indicial equation

$$\begin{aligned}
(n+r)\,(n+r-1)\, a_0 x^r + a_0 x^r &= 0 \\
(r)\,(r-1)\, a_0 x^r + (r)\, a_0 x^r &= 0 \\
(r(r-1)\, a_0 + a_0)\, x^r &= 0 \\
(r(r-1)+1)\, a_0 x^r &= 0
\end{aligned} \tag{3B}$$

EQ (3B) is used to solve for r. Since $a_0 \neq 0$ then (3B) gives

$$\begin{aligned}
(r)\,(r-1) + 1 &= 0 \\
r^2 - r + 1 &= 0
\end{aligned}$$

The roots are

$$\begin{aligned}
r_1 &= \frac{1}{2} + \frac{1}{2} i\sqrt{3} \\
r_2 &= \frac{1}{2} - \frac{1}{2} i\sqrt{3}
\end{aligned}$$

The roots will always be complex conjugate of each others (since second order ode) and the real part will always be equal. Let the roots be

$$r_{1,2} = \alpha \pm i\beta$$

When this happens, the solution is given similar to the case when the roots differ by non integer, except now the solution and the coefficients will be complex. Let the solution be

$$y = c_1 y_1 + c_2 y_2$$

Now $y_1(x)$ is solved for. The solution is

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

Starting with Eq (2A) which was derived above gives

$$\sum_{n=0}^{\infty} (n+r_1)\,(n+r_1-1)\, a_n x^{n+r_1} + \sum_{n=1}^{\infty} (n+r_1-1)\, a_{n-1} x^{n+r_1} + \sum_{n=0}^{\infty} a_n x^{n+r_1} = 0$$

The case of $n = 0$ is skipped since this was used to find the roots and $a_0 \neq 0$. $n \geq 1$ gives the recursion equation

$$\begin{aligned}
(n+r_1)\,(n+r_1-1)\, a_n + (n+r_1-1)\, a_{n-1} + a_n &= 0 \\
(n+r_1)\,(n+r_1-1)\, a_n + a_n &= -(n+r_1-1)\, a_{n-1} \\
a_n &= \frac{-(n+r_1-1)\, a_{n-1}}{(n+r_1)\,(n+r_1-1)+1}
\end{aligned} \tag{3}$$

For $n=1$ Eq. (3) gives

$$a_1 = \frac{-r_1 a_{n-1}}{(1+r_1)(r_1)+1}$$

But $r_1 = \alpha \pm i\beta = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$. and $a_0 = 1$. Hence the above becomes

$$a_1 = \frac{-\left(\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)}{\left(1+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)+1} = \frac{-1}{2}$$

For $n=2$ Eq. (3) gives

$$a_2 = \frac{-(1+r_1)\,a_1}{(2+r_1)(1+r_1)+1} = \frac{-(1+r_1)\left(\frac{-1}{2}\right)}{(2+r_1)(1+r_1)+1} = \frac{-\left(1+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(\frac{-1}{2}\right)}{\left(2+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(1+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)+1} = \frac{9}{56} - \frac{1}{56}i\sqrt{3}$$

For $n=3$ Eq. (3) gives

$$a_3 = \frac{-(3+r_1-1)\,a_2}{(3+r_1)(2+r_1)+1} = \frac{-(3+r_1-1)\left(\frac{9}{56}-\frac{1}{56}i\sqrt{3}\right)}{(3+r_1)(2+r_1)+1} = \frac{-\left(2+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(\frac{9}{56}-\frac{1}{56}i\sqrt{3}\right)}{\left(3+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(2+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)+1} = \frac{1}{112}i\sqrt{3} - \frac{13}{336}$$

And so on. Hence the first solution is

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n = x^{\frac{1}{2}+\frac{1}{2}i\sqrt{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \tag{4}$$

but

$$x^{\frac{1}{2}+\frac{1}{2}i\sqrt{3}} = x^{\frac{1}{2}} x^{\frac{1}{2}i\sqrt{3}} = x^{\frac{1}{2}} e^{\ln\left(x^{\frac{1}{2}i\sqrt{3}}\right)} = x^{\frac{1}{2}} e^{i\ln\left(x^{\frac{\sqrt{3}}{2}}\right)} = x^{\frac{1}{2}}\left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right) + i\sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)$$

Substituting the above in (4) and using values found for a_n gives

$$y_1(x) = x^{\frac{1}{2}}\left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right) + i\sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)\left(1 - \frac{1}{2}x + \left(\frac{9}{56} - \frac{1}{56}i\sqrt{3}\right)x^2 + \left(-\frac{13}{336} + \frac{1}{112}i\sqrt{3}\right)x^3 + \cdots\right) \tag{5}$$

Since the roots are complex conjugate of each others, then the second solution is

$$y_2(x) = x^{\frac{1}{2}}\left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right) - i\sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)\left(1 - \frac{1}{2}x + \left(\frac{9}{56} + \frac{1}{56}i\sqrt{3}\right)x^2 + \left(-\frac{13}{336} - \frac{1}{112}i\sqrt{3}\right)x^3 + \cdots\right) \tag{6}$$

The final solution is therefore

$$y = c_1 y_1 + c_2 y_2$$

2.7 Ordinary point for second order ode

Local contents

2.7.1 Algorithm

ode internal name "second_order_taylor_series_method_ordinary_point"

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$.

Expansion point is an ordinary point. Using standard power series. For an ordinary point, and for inhomogeneous. ode, always generate the full solution directly from the summation. Do not split the problem into y_h, y_p. To be able to do this, we have to express the RHS as Taylor series (expand it around the same expansion point). If the RHS is already a polynomial in x then there is nothing to do as it is already in Taylor series form. Examples below show how to do this. When the RHS is not zero, do not attempt to find recurrence relation as the RHS will get in the way, If the RHS is zero, then find recurrence relation.

$$y'' = f(x, y, y')$$

In this method, we let Let $y = \sum_{n=0}^{\infty} a_n x^n$ and replace this in the above ode and solve for a_n using recurrence relation. Examples below show how these methods work.

2.7.2 Examples

Local contents

2.7.2.1 Example 1 $y'' + xy' + y = 2x + x^2 + x^4$

Solved using Taylor series method.

$$
\begin{aligned}
y'' + xy' + y = 2x + x^2 + x^4 \\
y'' &= -xy' - y + 2x + x^2 + x^4 \\
y'' &= f(x, y, y')
\end{aligned}
$$

Hence

$$
y = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)!} F_n|_{x_0, y_0, y_0'}
$$

Where

$$
\begin{aligned}
F_0 &= f(x, y, y') \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0
\end{aligned}
$$

Hence

$$
\begin{aligned}
F_1 &= \frac{\partial(-xy' - y + 2x + x^2 + x^4)}{\partial x} + \frac{\partial(-xy' - y + 2x + x^2 + x^4)}{\partial y} y' + \frac{\partial(-xy' - y + 2x + x^2 + x^4)}{\partial y'} y'' \\
&= (4x^3 + 2x - y' + 2) - y' - xy'' \\
&= 2x - 2y' - xy'' + 4x^3 + 2
\end{aligned}
$$

But $y'' = f(x, y, y')$, the above becomes

$$
F_1 = 2x - 2y' + x^2y' + xy - 2x^2 + 3x^3 - x^5 + 2
$$

And

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x}(2x - 2y' + x^2y' + xy - 2x^2 + 3x^3 - x^5 + 2) + \\
&\quad + \left(\frac{\partial}{\partial y}(2x - 2y' + x^2y' + xy - 2x^2 + 3x^3 - x^5 + 2)\right) y' \\
&\quad + \left(\frac{\partial}{\partial y'}(2x - 2y' + x^2y' + xy - 2x^2 + 3x^3 - x^5 + 2)\right) y'' \\
&= (y - 4x + 2xy' + 9x^2 - 5x^4 + 2) + xy' + (-2 + x^2) y'' \\
&= y - 4x - 2y'' + 3xy' + x^2y'' + 9x^2 - 5x^4 + 2
\end{aligned}
$$

But $y'' = f(x, y, y')$, the above becomes

$$
\begin{aligned}
F_2 &= y - 4x - 2(-xy' - y + 2x + x^2 + x^4) + 3xy' + x^2(-xy' - y + 2x + x^2 + x^4) + 9x^2 - 5x^4 + 2 \\
&= 3y - 8x + 5xy' - x^2y - x^3y' + 7x^2 + 2x^3 - 6x^4 + x^6 + 2
\end{aligned}
$$

And

$$
\begin{aligned}
F_3 &= \frac{d}{dx}(F_2) \\
&= \frac{\partial}{\partial x} F_2 + \left(\frac{\partial F_2}{\partial y}\right) y' + \left(\frac{\partial F_2}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x}(3y - 8x + 5xy' - x^2y - x^3y' + 7x^2 + 2x^3 - 6x^4 + x^6 + 2) \\
&\quad + \left(\frac{\partial}{\partial y}(3y - 8x + 5xy' - x^2y - x^3y' + 7x^2 + 2x^3 - 6x^4 + x^6 + 2)\right) y' \\
&\quad + \left(\frac{\partial}{\partial y'}(3y - 8x + 5xy' - x^2y - x^3y' + 7x^2 + 2x^3 - 6x^4 + x^6 + 2)\right) y'' \\
&= 14x + 5y' - 3x^2y' - 2xy + 6x^2 - 24x^3 + 6x^5 - 8 + (3 - x^2) y' + (5x - x^3) y''
\end{aligned}
$$

But $y''=f(x,y,y')$, the above becomes

$$\begin{aligned}F_3 &= 14x+5y'-3x^2y'-2xy+6x^2-24x^3+6x^5-8+\left(3-x^2\right)y'+\left(5x-x^3\right)\left(-xy'-y+2x+x^2+x^4\right)\\&= 14x+8y'+x^3y-9x^2y'+x^4y'-7xy+16x^2-19x^3-2x^4+10x^5-x^7-8\end{aligned}$$

And so on. Evaluating each of the above at $x=0, y=y_0, y'=y'_0$ gives

$$\begin{aligned}F_0 &= \left(-xy'-y+2x+x^2+x^4\right)_{x=0,y_0,y'_0} = -y_0\\F_1 &= \left(2x-2y'+x^2y'+xy-2x^2+3x^3-x^5+2\right)_{x=0,y_0,y'_0} = \left(-2y'_0+2\right)\\F_2 &= 3y-8x+5xy'-x^2y-x^3y'+7x^2+2x^3-6x^4+x^6+2 = 3y_0+2\\F_3 &= 14x+8y'+x^3y-9x^2y'+x^4y'-7xy+16x^2-19x^3-2x^4+10x^5-x^7-8 = 8y'_0-8\end{aligned}$$

Hence

$$\begin{aligned}y(x) &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0}\\&= y_0+xy'_0+\frac{x^2}{2}F_0+\frac{x^3}{6}F_1+\frac{x^4}{24}F_2+\frac{x^5}{5!}F_3+\cdots\\&= y_0+xy'_0+\frac{x^2}{2}(-y_0)+\frac{x^3}{6}(-2y'_0+2)+\frac{x^4}{24}(3y_0+2)+\frac{x^5}{5!}(8y'_0-8)+\cdots\\&= y_0\left(1-\frac{x^2}{2}+\frac{1}{8}x^4+\cdots\right)+y'_0\left(x-\frac{x^3}{3}+\frac{1}{15}x^4\cdots\right)+\left(\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{1}{15}x^4\right)\\&= c_1\left(1-\frac{x^2}{2}+\frac{1}{8}x^4+\cdots\right)+c_2\left(x-\frac{x^3}{3}+\frac{1}{15}x^4\cdots\right)+\left(\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{1}{15}x^4\right)\end{aligned}$$

Solved using power series method.

$$y''+xy'+y=2x+x^2+x^4$$

Comparing the homogenous ode to $y''+p(x)\,y'+q(x)\,y=0$ shows that $p(x)=x, q(x)=1$. These are defined everywhere. Let the expansion point be $x_0=0$. This is ordinary point since $p(x), q(x)$ are defined at x_0. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Hence $y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum_{n=1}^{\infty}(n)(n-1)\,a_nx^{n-2}=\sum_{n=2}^{\infty}(n)(n-1)\,a_nx^{n-2}$. The homogenous ode becomes

$$\sum_{n=2}^{\infty}(n)(n-1)\,a_nx^{n-2}+x\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{n=0}^{\infty}a_nx^n=2x+x^2+x^4$$

$$\sum_{n=2}^{\infty}(n)(n-1)\,a_nx^{n-2}+\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=2x+x^2+x^4$$

Adjust all sums to lowest power on x gives

$$\sum_{n=2}^{\infty}(n)(n-1)\,a_nx^{n-2}+\sum_{n=3}^{\infty}(n-2)\,a_{n-2}x^{n-2}+\sum_{n=2}^{\infty}a_{n-2}x^{n-2}=2x+x^2+x^4$$

$n=2$ gives x^0 on the LHS with no match on the RHS. Hence

$$\begin{aligned}2a_2+a_0&=0\\a_2&=-\frac{1}{2}a_0\end{aligned}$$

$n=3$ gives x^1 on the LHS with one match on the RHS. Hence

$$\begin{aligned}6a_3+2a_1&=2\\a_3&=\frac{2-2a_1}{6}\\&=\frac{1}{3}-\frac{1}{3}a_1\end{aligned}$$

$n=4$ gives x^2 on the LHS with one match on the RHS. Hence

$$\begin{aligned}12a_4+3a_2&=1\\a_4&=\frac{1-3a_2}{12}\\&=\frac{1-3\left(-\frac{1}{2}a_0\right)}{12}\\&=\frac{1}{8}a_0+\frac{1}{12}\end{aligned}$$

$n=5$ gives x^3 on the LHS with no match on the RHS. Hence

$$\begin{aligned}20a_5+4a_3&=0\\a_5&=\frac{-4a_3}{20}\\&=\frac{-4\left(\frac{1}{3}-\frac{1}{3}a_1\right)}{20}\\&=\frac{1}{15}a_1-\frac{1}{15}\end{aligned}$$

$n=6$ gives x^4 on the LHS with one match on the RHS. Hence

$$\begin{aligned} 30a_6 + 5a_4 &= 1 \\ a_6 &= \frac{1-5a_4}{30} \\ &= \frac{1-5\left(\frac{1}{8}a_0+\frac{1}{12}\right)}{30} \\ &= \frac{7}{360} - \frac{1}{48}a_0 \end{aligned}$$

And for $n \geq 7$ we have recurrence relation

$$\begin{aligned} (n)(n-1)a_n + (n-2)a_{n-2} + a_{n-2} &= 0 \\ a_n &= -\frac{n-1}{n(n-1)}a_{n-2} \end{aligned}$$

Hence for $n=7$

$$\begin{aligned} a_7 &= -\frac{6}{42}a_5 \\ &= -\frac{6}{42}\left(\frac{1}{15}a_1 - \frac{1}{15}\right) \\ &= \frac{1}{105} - \frac{1}{105}a_1 \end{aligned}$$

For $n=8$

$$\begin{aligned} a_8 &= -\frac{7}{(8)(7)}a_6 \\ &= -\frac{7}{(8)(7)}\left(\frac{7}{360} - \frac{1}{48}a_0\right) \\ &= \frac{1}{384}a_0 - \frac{7}{2880} \end{aligned}$$

And so on. Hence

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + a_1 x - \frac{1}{2}a_0 x^2 + \left(\frac{1}{3} - \frac{1}{3}a_1\right)x^3 + \left(\frac{1}{8}a_0 + \frac{1}{12}\right)x^4 + \left(\frac{1}{15}a_1 - \frac{1}{15}\right)x^5 + \left(\frac{7}{360} - \frac{1}{48}a_0\right)x^6 + \left(\frac{1}{105} - \frac{1}{105}a_1\right)x^7 + \cdots \\ &= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots\right) + a_1\left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 - \cdots\right) + \left(\frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{1}{15}x^5 + \frac{7}{360}x^6 + \frac{1}{105}x^7 + \cdots \end{aligned}$$

Which is the same answer given using the Taylor series method. We see that the Taylor series method is much simpler, but requires using the computer to calculate the derivatives as they become very complicated as more terms are needed.

Even though the expansion point is ordinary, we can also solve this using Frobenius series as follows. Comparing the ode $y'' + xy' + y = 0$ to

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Hence $p(x) = x, q(x) = 1$. Therefore $p_0 = \lim_{x\to 0} xp(x) = \lim_{x\to 0} x^2 = 0$ and $q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 = 0$. Hence the indicial equation is

$$\begin{aligned} r(r-1) + p_0 r + q_0 &= 0 \\ r(r-1) &= 0 \\ r &= 1, 0 \end{aligned}$$

Hence $r_1 = 1, r_2 = 0$. All ordinary points will have the same roots. Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2} + x\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \end{aligned}$$

Reindex to lowest powers gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=2}^{\infty}(n+r-2)a_{n-2}x^{n+r-2}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r-2}=0 \tag{1}$$

For $n=0$

$$r(r-1)a_0x^{r-2}=0$$

The homogenous ode therefore satisfies

$$y''+xy'+y=r(r-1)a_0x^r \tag{2}$$

For $n=1$, Eq (1) gives

$$(1+r)(r)a_1=0$$

For $r=1$ we see that $a_1=0$. But for $r=0$ then the above gives $0b_1=0$. This means b_1 can be any value and we choose $b_1=0$ in this case.

For $n\geq 2$ we obtain the recurrence relation

$$(n+r)(n+r-1)a_n+(n+r-2)a_{n-2}+a_{n-2}=0$$

$$a_n=\frac{-(n+r-2)a_{n-2}-a_{n-2}}{(n+r)(n+r-1)}=\frac{-(n+r-1)a_{n-2}}{(n+r)(n+r-1)} \tag{3}$$

Now we find y_1 which is associated with $r=1$. From (3) and for $r=1$ it becomes

$$a_n=-\frac{n}{(n+1)n}a_{n-2}=-\frac{1}{n+1}a_{n-2} \tag{4}$$

For $n=2$ and using $a_0=1$

$$a_2=-\frac{1}{3}a_0=-\frac{1}{3}$$

For $n=3$

$$a_3=-\frac{1}{4}a_1=0$$

All odd a_n will be zero. For $n=4$

$$a_4=-\frac{1}{5}a_2=-\frac{1}{5}\left(-\frac{1}{3}\right)=\frac{1}{15}$$

And so on. Hence

$$\begin{aligned}
y_1&=\sum a_n x^{n+r_1}\\
&=x\sum a_n x^n\\
&=x\left(a_0+a_1x+a_2x^2+\cdots\right)\\
&=x\left(1-\frac{1}{2}x^2+\frac{1}{10}x^4-\cdots\right)\\
&=x-\frac{1}{3}x^3+\frac{1}{15}x^5-\cdots
\end{aligned}$$

Now we find y_2 associated with $r=0$. From (3) this becomes (using b instead of a) and $r=0$

$$\begin{aligned}
b_n&=\frac{-(n+r-1)b_{n-2}}{(n+r)(n+r-1)}\\
&=\frac{-(n-1)b_{n-2}}{(n)(n-1)}\\
&=-\frac{b_{n-2}}{n}
\end{aligned} \tag{5}$$

From above, we found that $b_1=0$. Now we use (5) to find all b_n for $n\geq 2$. For $n=2$

$$b_2=-\frac{b_0}{2}=-\frac{1}{2}$$

For $n=3$

$$b_3=-\frac{b_1}{3}=0$$

For $n=4$

$$b_4=-\frac{b_2}{4}=\frac{1}{8}$$

And so on. Hence

$$\begin{aligned}
y_2&=\sum b_n x^{n+r_2}\\
&=\sum b_n x^n\\
&=\left(b_0+b_1x+b_2x^2+\cdots\right)\\
&=\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\cdots\right)
\end{aligned}$$

Hence the solution y_h is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \cdots\right) + c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right)
\end{aligned}
$$

We see this is the same y_h obtained using standard power series. This shows that we can also use Frobenius series to solve for ordinary point. The roots will always be $r_1 = 1, r_2 = 0$. But this requires more work than using standard power series. The main advantage of using Frobenius series for ordinary point comes in when the RHS has no series expansion at $x = 0$. For example, if the RHS in this ode was say $\sqrt{x}$ then we must use Frobenius to be able to solve it as standard power series will fail, since $\sqrt{x}$ has no series representation at $x = 0$. Examples below shows how to do this.

2.7.2.2 Example 2 $\frac{1}{x^5}y'' + y' + y = 0$

$$
\frac{1}{x^5}y'' + y' + y = 0
$$

Solved using Taylor series method.

$$
\begin{aligned}
y'' &= -x^5(y' + y) \\
&= -x^5 y - x^5 y' \\
y'' &= f(x, y, y')
\end{aligned}
$$

Hence

$$
y = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)!} F_n\big|_{x_0, y_0, y_0'}
$$

Where

$$
\begin{aligned}
F_0 &= f(x, y, y') \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0
\end{aligned}
$$

Hence

$$
\begin{aligned}
F_1 &= \frac{\partial(-x^5 y - x^5 y')}{\partial x} + \frac{\partial(-x^5 y - x^5 y')}{\partial y} y' + \frac{\partial(-x^5 y - x^5 y')}{\partial y'} y'' \\
&= (-5x^4 y - 5x^4 y') - x^5 y' - x^5 y''
\end{aligned}
$$

But $y'' = f(x, y, y')$, the above becomes

$$
\begin{aligned}
F_1 &= (-5x^4 y - 5x^4 y') - x^5 y' - x^5(-x^5 y - x^5 y') \\
&= x^{10} y - 5x^4 y - 5x^4 y' - x^5 y' + x^{10} y'
\end{aligned}
$$

And

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x}(x^{10} y - 5x^4 y - 5x^4 y' - x^5 y' + x^{10} y') + \\
&+ \left(\frac{\partial}{\partial y}(x^{10} y - 5x^4 y - 5x^4 y' - x^5 y' + x^{10} y')\right) y' \\
&+ \left(\frac{\partial}{\partial y'}(x^{10} y - 5x^4 y - 5x^4 y' - x^5 y' + x^{10} y')\right) y'' \\
&= (10x^9 y - 20x^3 y - 20x^3 y' - 5x^4 y' + 10x^9 y') + x^4(x^6 - 5) y' + (-5x^4 - x^5 + x^{10}) y''
\end{aligned}
$$

But $y'' = f(x, y, y')$, the above becomes

$$
\begin{aligned}
F_2 &= (10x^9 y - 20x^3 y - 20x^3 y' - 5x^4 y' + 10x^9 y') + x^4(x^6 - 5) y' + (-5x^4 - x^5 + x^{10})(-x^5(y' + y)) \\
&= -x^3(20y + 20y' + 10xy' - 15x^6 y - x^7 y + x^{12} y - 15x^6 y' - 2x^7 y' + x^{12} y')
\end{aligned}
$$

And

$$
\begin{aligned}
F_3 &= \frac{d}{dx}(F_2) \\
&= \frac{\partial}{\partial x}F_2 + \left(\frac{\partial F_2}{\partial y}\right)y' + \left(\frac{\partial F_2}{\partial y'}\right)y'' \\
&= \frac{\partial}{\partial x}\left(-x^3\left(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y'\right)\right) \\
&+ \left(\frac{\partial}{\partial y}\left(-x^3\left(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y'\right)\right)\right)y' \\
&+ \left(\frac{\partial}{\partial y'}\left(-x^3\left(20y+20y'+10xy'-15x^6y-x^7y+x^{12}y-15x^6y'-2x^7y'+x^{12}y'\right)\right)\right)y'' \\
&= -5x^2\left(12y+12y'+8xy'-27x^6y-2x^7y+3x^{12}y-27x^6y'-4x^7y'+3x^{12}y'\right) + x^3\left(-x^{12}+x^7+15x^6-20\right)y' + \left(-20x^3
\end{aligned}
$$

But $y'' = f(x,y,y')$, the above becomes

$$
\begin{aligned}
F_3 &= -5x^2\left(12y+12y'+8xy'-27x^6y-2x^7y+3x^{12}y-27x^6y'-4x^7y'+3x^{12}y'\right) + x^3\left(-x^{12}+x^7+15x^6-20\right)y' + \left(-20x^3 \right. \\
&= -x^2\left(60y+60y'+60xy'-155x^6y-20x^7y+30x^{12}y+2x^{13}y-x^{18}y-155x^6y'-45x^7y'-x^8y'+30x^{12}y'+3x^{13}y'-x^{1}\right.
\end{aligned}
$$

And so on. Since the derivatives become very complicated, the result was done on the computer which results in (Evaluating each of the above at $x=0, y=y_0, y'=y_0'$)

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0 \\
F_5 &= -120y_0' - 120y_0 \\
F_6 &= -720y_0' \\
F_7 &= 0 \\
F_8 &= 0 \\
F_9 &= 0 \\
F_{10} &= 0 \\
F_{11} &= 6652800y_0' + 6652800y_0 \\
F_{12} &= 79833600y_0' + 11404800y_0 \\
F_{13} &= 111196800y_0' \\
F_{14} &= 0 \\
&\vdots
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y(x) &= y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \\
&= y_0 + xy_0' + \frac{x^7}{7!}(-120y_0'-120y_0) - \frac{x^8}{8!}(720y_0') + \frac{x^{13}}{13!}(6652800y_0'+6652800y_0) \\
&+ \frac{x^{14}}{14!}(79833600y_0'+11404800y_0) + \frac{x^{15}}{15!}(111196800y_0') + \cdots \\
&= y_0\left(1-\frac{120}{7!}x^7+\frac{6652800}{13!}x^{13}+\frac{11404800}{14!}x^{14}-\cdots\right) + y_0'\left(x-\frac{120}{7!}x^7-\frac{720}{8!}x^8+\frac{6652800}{13!}x^{13}+\frac{79833600}{14!}x^{14}+\frac{11119}{1}\right. \\
&= y_0\left(1-\frac{1}{42}x^7+\frac{1}{936}x^{13}+\frac{1}{7644}x^{14}+\cdots\right) + y_0'\left(x-\frac{1}{42}x^7-\frac{1}{56}x^8+\frac{1}{936}x^{13}+\frac{1}{1092}x^{14}+\frac{1}{11760}x^{15}+\cdots\right)
\end{aligned}
$$

Solved using power series method

Expansion around $x=0$. This is ordinary point. Since RHS is zero, we will find recurrence relation.

Let $y=\sum_{n=0}^{\infty}a_nx^n$. Hence $y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum_{n=1}^{\infty}(n)(n-1)a_nx^{n-2}=\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}$. The ode becomes

$$
x^{-5}y'' + y' + y = 0
$$

Hence

$$
\begin{aligned}
\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty}na_nx^{n-1} + \sum_{n=0}^{\infty}a_nx^n &= 0 \\
\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-7} + \sum_{n=1}^{\infty}na_nx^{n-1} + \sum_{n=0}^{\infty}a_nx^n &= 0
\end{aligned}
$$

Reindex so all powers start at lowest powers $n-7$

$$\sum_{n=2}^{\infty}(n)(n-1)a_n x^{n-7}+\sum_{n=7}^{\infty}(n-6)a_{n-6}x^{n-7}+\sum_{n=7}^{\infty}a_{n-7}x^{n-7}=0 \tag{1}$$

For $n=2,3,4,5,6$ it generates $a_2=0,a_3=0,a_4=0,a_5=0,a_6=0$ since there is only one term in each one of these and the RHS is zero.

For $n\geq 7$ we have the recurrence relation

$$(n)(n-1)a_n+(n-6)a_{n-6}+a_{n-7}=0 \tag{2}$$

$$a_n=-\frac{(n-6)a_{n-6}+a_{n-7}}{(n+2)(n+1)}$$

Hence for $n=7$

$$a_7=-\frac{a_1+a_0}{42}$$

For $n=8$

$$a_8=-\frac{2a_2+a_1}{(6+2)(6+1)}=\frac{-a_1}{56}$$

For $n=9$

$$a_9=-\frac{(7-4)a_3+a_2}{(7+2)(7+1)}=0$$

For $n=10$

$$a_{10}=-\frac{(8-4)a_4+a_3}{(8+2)(8+1)}=0$$

For $n=11$

$$a_{11}=-\frac{(9-4)a_5+a_4}{(9+2)(9+1)}=0$$

For $n=12$

$$a_{12}=-\frac{(n-4)a_6+a_5}{(n+2)(n+1)}=0$$

For $n=13$

$$a_{13}=-\frac{(11-4)a_7+a_6}{(11+2)(11+1)}=-\frac{(11-4)a_7}{(11+2)(11+1)}=-\frac{7}{156}a_7=-\frac{7}{156}\left(-\frac{a_1+a_0}{42}\right)=\frac{1}{936}a_0+\frac{1}{936}a_1$$

And so on. Hence

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_n x^n\\&=a_0+a_1x+a_7x^7+a_{13}x^{13}+\cdots\end{aligned}$$

Notice that all terms $a_n=0$ for $n=2\cdots 6$. The above becomes

$$\begin{aligned}y&=a_0+a_1x+\left(-\frac{1}{42}a_0-\frac{1}{42}a_1\right)x^7+\left(\frac{1}{936}a_0+\frac{1}{936}a_1\right)x^{13}+\cdots\\&=a_0\left(1-\frac{1}{42}x^7+\frac{1}{936}x^{13}+\cdots\right)+a_1\left(x-\frac{1}{42}x^7+\frac{1}{936}x^{13}+\cdots\right)\end{aligned}$$

2.7.2.3 Example 3 $\frac{1}{x^2}y''+y'+y=\sin x$

$$\frac{1}{x^2}y''+y'+y=\sin x$$

Expansion around $x=0$. This is ordinary point. Since RHS is not zero, do not find recurrence relation. Let $y=\sum_{n=0}^{\infty}a_nx^n$. Hence $y'=\sum_{n=0}^{\infty}na_nx^{n-1}=\sum_{n=1}^{\infty}na_nx^{n-1}$ and $y''=\sum_{n=1}^{\infty}(n)(n-1)a_nx^{n-2}=\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}$. The ode becomes

$$y''+x^2y'+x^2y=x^2\sin x$$

Hence

$$\begin{aligned}\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}+x^2\sum_{n=1}^{\infty}na_nx^{n-1}+x^2\sum_{n=0}^{\infty}a_nx^n&=x^2\sin x\\\sum_{n=2}^{\infty}(n)(n-1)a_nx^{n-2}+\sum_{n=1}^{\infty}na_nx^{n+1}+\sum_{n=0}^{\infty}a_nx^{n+2}&=x^2\sin x\end{aligned}$$

Reindex so all powers to start from n. This results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=x^2\sin x$$

To be able to continue, we have to expand $\sin x$ as Taylor series around x. The above becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=x^2\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\cdots\right)$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=x^3-\frac{1}{6}x^5+\frac{1}{120}x^7-\frac{1}{5040}x^9+\cdots$$

For $n=0$

$$\begin{aligned}2a_2&=0\\a_2&=0\end{aligned}$$

For $n=1$

$$\begin{aligned}(3)(2)a_3&=0\\a_3&=0\end{aligned}$$

For $n=2$

$$\begin{aligned}(2+2)(2+1)a_4+(2-1)a_1+a_0&=0\\12a_4+a_1+a_0&=0\\a_4&=\frac{-a_1-a_0}{12}\end{aligned}$$

For $n=3$ (now we pick one term from the RHS which match on x^3)

$$\begin{aligned}20a_5+2a_2+a_1&=1\\a_5&=\frac{1-a_1}{20}\end{aligned}$$

For $n=4$

$$\begin{aligned}30a_6+3a_3+a_2&=0\\a_6&=0\end{aligned}$$

For $n=5$

$$\begin{aligned}42a_7+4a_4+a_3&=-\frac{1}{6}\\a_7&=\frac{-\frac{1}{6}-4a_4}{42}=\frac{-\frac{1}{6}-4\left(\frac{-a_1-a_0}{12}\right)}{42}=\frac{1}{126}a_0+\frac{1}{126}a_1-\frac{1}{252}\end{aligned}$$

And so on. Hence

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_nx^n\\&=a_0+a_1x+a_2x^2+a_3x^3+\cdots\\&=a_0+a_1x+\left(\frac{-a_1-a_0}{12}\right)x^4+\left(\frac{1-a_1}{20}\right)x^5+\left(\frac{1}{126}a_0+\frac{1}{126}a_1-\frac{1}{252}\right)x^7+\cdots\\&=a_0\left(1-\frac{1}{12}x^4+\frac{1}{126}x^7+\cdots\right)+a_1\left(x-\frac{1}{12}x^4-\frac{1}{20}x^5+\frac{1}{126}x^7+\cdots\right)+\left(\frac{1}{20}x^5-\frac{1}{252}x^7+\cdots\right)\end{aligned}$$

2.7.2.4 Example 4. $y''=\frac{1}{x}$

$$y''=\frac{1}{x}\tag{1}$$

With expansion around $x=0$. This is an ordinary point for the ode itself. Let

$$y(x)=\sum_{n=0}^{\infty}a_nx^n$$

Therefore

$$\begin{aligned}y'(x)&=\sum_{n=0}^{\infty}na_nx^{n-1}\\&=\sum_{n=1}^{\infty}na_nx^{n-1}\\y''(x)&=\sum_{n=1}^{\infty}n(n-1)a_nx^{n-2}\\&=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}\end{aligned}$$

Hence the ode becomes

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{1}{x} \tag{2}$$

The solution is given by $y = y_h + y_p$, where y_h is solution to

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 0$$

Recursive equation is

$$n(n-1)\, a_n = 0 \qquad n \geq 2$$

Hence all $a_n = 0$ for $n \geq 2$, therefore

$$\begin{aligned} y_h &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x \end{aligned}$$

Now we need to find y_p. From (2), and now we replace a_n by c_n

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} = \frac{1}{x}$$

$n = 2$ gives

$$2c_2 x^0 = x^{-1}$$

Hence for balance $0 = -1$ which is not possible. Hence no y_p exist using series method. Solution exist by direct integration.

3 Handling initial conditions for series solution

In series solutions, initial conditions must be specified at same location as the expansion point. For ordinary point, this is the easy case, as solution can always be written as

$$\begin{aligned} y(x) &= c_1(\cdots) \\ y(x) &= y(x_0)\,(\cdots) \end{aligned}$$

Where $\cdots$ is the series found using power series or taylor expansion. For second order it becomes

$$\begin{aligned} y(x) &= c_1(\cdots) + c_2(\cdots) \\ &= y(x_0)\,(\cdots) + y'(x_0)\,(\cdots) \end{aligned}$$

Where x_0 above is the expansion point, typically zero. So if we are given IC $y(0) = y_0, y'(0) = y_0$ we just plugin in these values in the above. But for regular point, this is not the case. If the solution found was

$$y(x) = c_1(\cdots) + c_2(\cdots)$$

And we are given IC $y(0) = y_0, y'(0) = y_0$, we can not replace c_1 by $y(0)$ and replace c_2 by $y'(0)$ as with ordinary point.

We have to set up two equations and solve for c_1, c_2 as we do with normal non series solutions. Only issue is that if we find one part of the above solution not defined at x_0 then this part is removed. For an example, lets say we obtained this solution for regular singular point

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \cdots\right) + c_2\left(\frac{1}{x} - \frac{1}{2}x + \frac{1}{24}x^3 + \cdots\right) \end{aligned}$$

And the initial conditions were $y(0) = 1, y'(0) = 0$. The at $x = 0$ we have

$$\begin{aligned} 1 &= c_1 \lim_{x\to 0}\left(1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \cdots\right) + \lim_{x\to 0} c_2\left(\frac{1}{x} - \frac{1}{2}x + \frac{1}{24}x^3 + \cdots\right) \\ &= c_1 + c_2 \lim_{x\to 0}\frac{1}{x} \end{aligned}$$

Since the second basis solution diverges, then we need to have $c_2 = 0$ since $\lim_{x\to 0}\frac{1}{x}$ is undefined. This gives $c_1 = 1$ and the solution now becomes

$$y = 1 - \frac{1}{6}x^2 + \frac{x^4}{120} + \cdots$$

This needs to be verified that it satisfies both initial conditions, which it does. If both basis solution diverges at the IC given, then no solution exist.

The following is another example. The ode $x^2y'' = xy' + (x^2 - 4)\,y = x^3$ has the series solution

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 + y_p \\
&= c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2\left(\ln(x)\left(9x^2 - \frac{3}{4}x^4 + \cdots\right) - \left(\frac{144}{x^2} - 36 + \frac{1}{2}x^4 + \cdots\right)\right) + \left(\frac{1}{5}x^3 - \frac{1}{105}x^2 + \cdots\right)
\end{aligned}
$$

Given IC $y(0) = 0, y'(0) = 1$, then using $y(0) = 0$ gives

$$
\begin{aligned}
0 &= c_1 \lim_{x\to 0}\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + c_2 \lim_{x\to 0}\left(\ln(x)\left(9x^2 - \frac{3}{4}x^4 + \cdots\right) - \left(\frac{144}{x^2} - 36 + \frac{1}{2}x^4 + \cdots\right)\right) + \lim_{x\to 0}\left(\frac{1}{5}x^3 - \frac{1}{105}\right. \\
&= c_2 \lim_{x\to 0}\left(\ln(x)\left(9x^2 - \frac{3}{4}x^4 + \cdots\right) - \left(\frac{144}{x^2} - 36 + \frac{1}{2}x^4 + \cdots\right)\right)
\end{aligned}
$$

Therefore we set $c_2 = 0$ since the limit is not defined. This makes the solution now as

$$
y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{1}{5}x^3 - \frac{1}{105}x^5 + \cdots\right)
$$

Taking derivative gives

$$
y' = c_1\left(2x - \frac{4}{12}x^4 + \frac{6}{384}x^5 + \cdots\right) + \left(\frac{3}{5}x^2 - \frac{5}{105}x^4 + \cdots\right)
$$

Using $y'(0) = 1$ gives

$$
\begin{aligned}
1 &= c_1 \lim_{x\to 0}\left(2x - \frac{4}{12}x^4 + \frac{6}{384}x^5 + \cdots\right) + \lim_{x\to 0}\left(\frac{3}{5}x^2 - \frac{5}{105}x^4 + \cdots\right) \\
&= 0
\end{aligned}
$$

Which gives $1 = 0$. Therefore there is <u>no solution</u>.

To handle IC for series solution, we start with general solution found $y = c_1y_1 + c_2y_2 + y_p$ and apply the above step by step in order to determine what happens. Let us change the IC for the above example to $y(0) = 0, y'(0) = 0$ and see what happens. We found that applying $y(0) = 0$ gives

$$
y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{1}{5}x^3 - \frac{1}{105}x^5 + \cdots\right)
$$

Taking derivative gives

$$
y' = c_1\left(2x - \frac{4}{12}x^4 + \frac{6}{384}x^5 + \cdots\right) + \left(\frac{3}{5}x^2 - \frac{5}{105}x^4 + \cdots\right)
$$

Using $y'(0) = 0$ gives

$$
\begin{aligned}
0 &= c_1 \lim_{x\to 0}\left(2x - \frac{4}{12}x^4 + \frac{6}{384}x^5 + \cdots\right) + \lim_{x\to 0}\left(\frac{3}{5}x^2 - \frac{5}{105}x^4 + \cdots\right) \\
0 &= 0c_1
\end{aligned}
$$

Which means c_1 is arbitrary. This means the final solution remains

$$
y = c_1\left(x^2 - \frac{1}{12}x^4 + \frac{1}{384}x^6 + \cdots\right) + \left(\frac{1}{5}x^3 - \frac{1}{105}x^5 + \cdots\right)
$$

Even though we had two initial conditions, the final solution still has one arbitrary constant. This happens quite often when the the expansion point is singular as in this example.