Examples solving second order linear ode using power series method for the case of a regular singular point

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1.1 Example 1. homogeneous ode example

Solve

$$x^2y''(x)+xy'(x)+x^2y(x)=0 \tag{1}$$

Using power series method by expanding around $x=0$. Writing the ode as

$$y''(x)+\frac{1}{x}y'(x)+y(x)=0$$

Shows that $x=0$ is a singular point. But $\lim_{x\to 0} x\frac{1}{x}=1$. Hence the singularity is removable. This means $x=0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y(x)=\sum_{n=0}^{\infty}a_nx^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned}
y'(x)&=\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}\\
y''(x)&=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}
\end{aligned}$$

Substituting the above in (1) gives

$$\begin{aligned}
x^2\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r-1}+x^2\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\
\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}&=0 \quad\text{(1A)}
\end{aligned}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty}a_nx^{n+r+2}=\sum_{n=2}^{\infty}a_{n-2}x^{n+r}$$

And now Eq (1A) becomes

$$\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_nx^{n+r}+\sum_{n=0}^{\infty}(n+r)\,a_nx^{n+r}+\sum_{n=2}^{\infty}a_{n-2}x^{n+r}=0$$

$n=0$ gives the indicial equation

$$\begin{aligned}
(n+r)\,(n+r-1)\,a_0x^r+(n+r)\,a_0x^r&=0\\
(r)\,(r-1)\,a_0x^r+(r)a_0x^r&=0\\
((r)\,(r-1)\,a_0+ra_0)\,x^r&=0\\
((r)\,(r-1)+r)\,a_0&=0
\end{aligned}$$

Since $a_0\neq 0$ then

$$\begin{aligned}
(r)\,(r-1)+r&=0\\
r^2-r+r&=0\\
r^2&=0
\end{aligned}$$

Hence the roots of the indicial equation are $r=0$ which is a double root. Hence $r_1=r_2=0$. When this happens, the solution is given by

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+r} \tag{2}$$

Where we take $a_0=1$ as it is arbitrary and where $r=r_1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r=r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x)=y_1(x)\ln(x)+\sum_{n=1}^{\infty}b_nx^{n+r} \tag{3}$$

Something important to notice. In the sum above, it starts from 1 and not from 0. The main issue is how to find b_n. Since that is the only thing we need to be able to complete the solution as $y_1(x)$ is easily found. It turns out that there is a relation between the b_n and the a_n. The b_n can be found by taking just derivative of a_n as function of r for each n and then evaluate the result at $r = r_1$. How this is done will be shown below. First we need to find $y_1(x)$. From (2)

$$y_1'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y_1''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r = r_1 = 0$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r = r_1 = 0$. Substituting the above in (1) gives (We are following pretty much the same process we did to find the indicial equation here)

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{4}$$
$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$
$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{5}$$

Now we are ready to find a_n. Now we skip $n = 0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose. We start from $n = 1$. Eq (5) gives

$$\begin{aligned}
(n+r)(n+r-1)\, a_1 + (n+r)\, a_1 &= 0 \\
(1+r)(1+r-1)\, a_1 + (1+r)\, a_1 &= 0 \\
((1+r)(1+r-1) + (1+r))\, a_1 &= 0 \\
(r+1)^2\, a_1 &= 0
\end{aligned}$$

But $r = r_1 = 0$. The above becomes $\underline{a_1 = 0}$. It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	0	0

For $n \geq 2$ we obtain the recursion equation

$$\begin{aligned}
(n+r)(n+r-1)\, a_n + (n+r)\, a_n + a_{n-2} &= 0 \\
a_n &= -\frac{a_{n-2}}{(n+r)(n+r-1) + (n+r)}
\end{aligned}$$

To more clearly indicate that a_n is function of r, we write the above as

$$a_n(r) = -\frac{a_{n-2}(r)}{(n+r)(n+r-1) + (n+r)} \tag{6}$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (6) for $n = 2$

$$a_2(r) = -\frac{a_0(r)}{(2+r)(2+r-1) + (2+r)}$$

But $a_0(r) = 1$. The above becomes

$$a_2(r) = -\frac{1}{(2+r)(2+r-1) + (2+r)} = -\frac{1}{(r+2)^2}$$

and $r = r_1 = 0$ then the above becomes

$$a_2 = -\frac{1}{(2)^2} = -\frac{1}{4}$$

The table now becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	0	0
2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$

And for $n=3$

$$a_3(r) = -\frac{a_1(r)}{(3+r)(3+r-1)+(3+r)}$$

But $a_1(r)=0$. Then $a_3(r)=0$. The table becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	0	0
2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
3	0	0

For $n=4$ Eq (6) gives

$$a_4(r) = -\frac{a_2(r)}{(4+r)(4+r-1)+(4+r)}$$

But $a_2(r)$ from the table is $-\frac{1}{(2+r)(2+r-1)+(2+r)}$. Hence

$$a_4(r) = -\frac{-\frac{1}{(2+r)(2+r-1)+(2+r)}}{(4+r)(4+r-1)+(4+r)} = \frac{1}{(r^2+6r+8)^2}$$

The above becomes at $r=r_1=0$

$$a_4 = \frac{1}{(8)^2} = \frac{1}{64}$$

The Table now becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	0	0
2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
3	0	0
4	$\frac{1}{(r^2+6r+8)^2}$	$\frac{1}{64}$

For $n=5$ Eq (6) gives

$$a_5(r) = -\frac{a_3(r)}{(n+r)(n+r-1)+(n+r)}$$

But $a_3(r)=0$, hence $a_5(r)=0$. The table becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	0	0
2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
3	0	0
4	$\frac{1}{(r^2+6r+8)^2}$	$\frac{1}{64}$
5	0	0

For $n=6$ Eq (6) gives

$$a_6(r) = -\frac{a_4(r)}{(6+r)(6+r-1)+(6+r)}$$

But from the table $a_4(r) = \frac{1}{(r^2+6r+8)^2}$, so the above becomes

$$a_6(r) = -\frac{\frac{1}{(r^2+6r+8)^2}}{(6+r)(6+r-1)+(6+r)} = -\frac{1}{(r+6)^2(r^2+6r+8)^2}$$

At $r=r_1=0$ the above becomes

$$a_6(r) = -\frac{1}{(6)^2(8)^2} = -\frac{1}{2304}$$

The table becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	0	0
2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
3	0	0
4	$\frac{1}{(r^2+6r+8)^2}$	$\frac{1}{64}$
5	0	0
6	$-\frac{1}{(r+6)^2(r^2+6r+8)^2}$	$-\frac{1}{2304}$

And so on. Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \cdots \end{aligned} \tag{6A}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

To find b_n, we will use the following

$$b_n(r) = \frac{d}{dr}\left(a_n(r)\right) \tag{7}$$

Notice that n starts from 1. Hence

$$b_1(r) = \frac{d}{dr}\left(a_1(r)\right)\bigg|_{r=r_1}$$

What the above says, is that we first take derivative of $a_n(r)$ w.r.t. r and evaluate the result at the root of the indicial equation. Using the table above, we obtain (recalling that $r_1 = 0$ in this example)

n	$a_n(r)$	$a_n(r = r_1)$	$b_n(r) = \frac{d}{dr}(a_n(r))$	$b_n(r = r_1)$
0	1	1	N/A since b starts from $n = 1$	N/A
1	0	0	0	0
2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{d}{dr}\left(-\frac{1}{(r+2)^2}\right) = \frac{2}{(r+2)^3}$	$\frac{2}{(2)^3} = \frac{1}{4}$
3	0	0	0	0
4	$\frac{1}{(r^2+6r+8)^2}$	$\frac{1}{64}$	$\frac{d}{dr}\left(\frac{1}{(r^2+6r+8)^2}\right) = -2\frac{2r+6}{(r^2+6r+8)^3}$	$-2\frac{6}{(8)^3} = -\frac{3}{128}$
5	0	0	0	0
6	$-\frac{1}{(r+6)^2(r^2+6r+8)^2}$	$-\frac{1}{2304}$	$\frac{d}{dr}\left(-\frac{1}{(r+6)^2(r^2+6r+8)^2}\right) = 2\frac{3r^2+24r+44}{(r^3+12r^2+44r+48)^3}$	$2\frac{44}{(48)^3} = \frac{11}{13\,824}$

We have found all b_n terms. Hence

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

And since $r = r_1 = 0$ then

$$y_2(x) = y_1(x)\ln(x) + \left(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots\right)$$

But from the above table, we see that $b_1 = 0, b_2 = \frac{1}{4}, b_3 = 0, b_4 = -\frac{3}{128}, b_5 = 0, b_6 = \frac{11}{13\,824}$. The above becomes

$$y_2(x) = y_1(x)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13\,824}x^6 + O\left(x^8\right)\right)$$

And we know what $y_1(x)$ is from Eq (6A). Hence

$$y_2(x) = \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O\left(x^8\right)\right)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13\,824}x^6 + O\left(x^8\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O\left(x^8\right)\right) \\ &+ c_2\left(\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O\left(x^8\right)\right)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13\,824}x^6 + O\left(x^8\right)\right)\right) \end{aligned}$$

This completes the solution. This method is easier that reduction of order, which would involve having to compute integrals (I should make another example showing that method also). The only difficulty in this method, is to make sure when finding the b_n is to have access to the a_n with r being unevaluated form in order to take derivatives correctly. This was done above by keeping a table of these quantities updated.

1.2 Example 2. inhomogeneous ode example

Solve

$$x^2y''(x) + xy'(x) + x^2y(x) = \sin(x) \tag{1}$$

Let the solution be $y = y_h + y_p$ where y_h is the solution to the homogeneous ode which we found above. And y_p is a particular solution. Let

$$y_p = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots$$

Then

$$\begin{aligned} y_p' &= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots \\ y_p'' &= 2c_2 + 6c_3x + 12c_4x^2 + \cdots \end{aligned}$$

Substituting the above in (1) gives

$$x^2\left(2c_2 + 6c_3x + 12c_4x^2 + \cdots\right) + x\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots\right) + x^2\left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\right) = \sin(x)$$

Now we replace $\sin(x)$ by its Taylor series expansion around $x = 0$, The above becomes

$$x^2\left(2c_2 + 6c_3x + 12c_4x^2 + \cdots\right) + x\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots\right) + x^2\left(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots$$

Expanding terms gives

$$\left(2c_2x^2 + 6c_3x^3 + 12c_4x^4 + \cdots\right) + \left(c_1x + 2c_2x^2 + 3c_3x^3 + 4c_4x^4 + \cdots\right) + \left(c_0x^2 + c_1x^3 + c_2x^4 + c_3x^5 + c_4x^6 + \cdots\right) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \cdots$$

Or

$$x^0\left(c_0\right) + x\left(c_1\right) + x^2\left(4c_2 + c_0\right) + x^3\left(9c_3 + c_1\right) + x^4\left(16c_4 + c_2\right) + x^5\left(25c_5 + c_3\right)\cdots = (0)x^0 + x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$$

Where c_0 was added explicitly since the RHS starts at x^1. Now comparing terms gives

$$\begin{aligned} c_0 &= 0 \\ c_1 &= 1 \\ 4c_2 + c_0 &= 0 \\ 9c_3 + c_1 &= -\frac{1}{6} \\ 16c_4 + c_2 &= 0 \\ 25c_5 + c_3 &= \frac{1}{120} \end{aligned}$$

The solution is $\left[c_0 = 0, c_1 = 1, c_2 = 0, c_3 = -\frac{7}{54}, c_4 = 0, c_5 = \frac{149}{27\,000}\right]$. Substituting the solution in y_p gives

$$\begin{aligned} y_p &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots \\ &= 0 + x + 0 - \frac{7}{54}x^3 + 0 + \frac{149}{27000}x^5 + \cdots \\ &= x - \frac{7}{54}x^3 + \frac{149}{27000}x^5 + \cdots \\ &= x\left(1 - \frac{7}{54}x^2 + \frac{149}{27000}x^4 + \cdots\right) \end{aligned}$$

If we want more terms in y_p, we just increase the terms in the polynomial trial y_p we started with. The final solution is

$$y = y_h + y_p$$

Using y_h from the above section gives

$$\begin{aligned} y = {} & c_1\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O\left(x^8\right)\right) \\ & + c_2\left(\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O\left(x^8\right)\right)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O\left(x^8\right)\right)\right) \\ & + x\left(1 - \frac{7}{54}x^2 + \frac{149}{27000}x^4 + \cdots + O\left(x^8\right)\right) \end{aligned}$$

1.3 Example 3. homogeneous ode example

Solve

$$\left(e^{x}-1\right)y''(x)+e^{x}y'(x)+y(x)=0 \tag{1}$$

Using power series method by expanding around $x=0$. Writing the ode as

$$y''(x)+\frac{e^{x}}{(e^{x}-1)}y'(x)+\frac{1}{(e^{x}-1)}y(x)=0$$

Shows that $x=0$ is a singular point. But $\lim_{x\to 0}x\frac{e^{x}}{(e^{x}-1)}=1$ and $\lim_{x\to 0}x^{2}\frac{1}{(e^{x}-1)}=0$ Hence the singularity is removable. This means $x=0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y(x)=\sum_{n=0}^{\infty}a_{n}x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned}
y'(x)&=\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}\\
y''(x)&=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}
\end{aligned}$$

Substituting the above in (1) gives

$$\left(e^{x}-1\right)\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}+e^{x}\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\sum_{n=0}^{\infty}a_{n}x^{n+r}=0$$

Expanding e^{x} in Taylor series around x gives $e^{x}=1+x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\cdots$. The above becomes

$$\begin{aligned}
&\left(x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\cdots\right)\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}\\
&\quad+\left(1+x+\frac{1}{2}x^{2}+\frac{1}{6}x^{3}+\frac{1}{24}x^{4}+\cdots\right)\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\sum_{n=0}^{\infty}a_{n}x^{n+r}=0 \qquad (1)
\end{aligned}$$

Expanding gives (and keeping only terns up to x^{4} gives

$$\begin{aligned}
&x\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}+\frac{1}{2}x^{2}\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}+\frac{1}{6}x^{3}\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}\\
&+\frac{1}{24}x^{4}\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+x\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\frac{1}{2}x^{2}\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}\\
&\quad+\frac{1}{6}x^{3}\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\frac{1}{24}x^{4}\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\sum_{n=0}^{\infty}a_{n}x^{n+r}=0
\end{aligned}$$

Moving the x inside the sum, the above becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-1}+\sum_{n=0}^{\infty}\frac{1}{2}(n+r)\,(n+r-1)\,a_{n}x^{n+r}+\sum_{n=0}^{\infty}\frac{1}{6}(n+r)\,(n+r-1)\,a_{n}x^{n+r+1}\\
&+\sum_{n=0}^{\infty}\frac{1}{24}(n+r)\,(n+r-1)\,a_{n}x^{n+r+2}+\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r}+\sum_{n=0}^{\infty}\frac{1}{2}(n+r)\,a_{n}x^{n+r+1}\\
&\quad+\sum_{n=0}^{\infty}\frac{1}{6}(n+r)\,a_{n}x^{n+r+2}+\sum_{n=0}^{\infty}\frac{1}{24}(n+r)\,a_{n}x^{n+r+3}+\sum_{n=0}^{\infty}a_{n}x^{n+r}=0
\end{aligned}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}x^{n+r-1}+\sum_{n=1}^{\infty}\frac{1}{2}(n+r-1)\,(n+r-2)\,a_{n-1}x^{n+r-1}+\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)\,(n+r-3)\,a_{n-2}x^{n+r-1}\\
&+\sum_{n=3}^{\infty}\frac{1}{24}(n+r-3)\,(n+r-4)\,a_{n-3}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)\,a_{n}x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-1)\,a_{n-1}x^{n+r-1}+\sum_{n=2}^{\infty}\frac{1}{2}(n+r-2)\,a_{n-2}x^{n+r-1}\\
&\quad+\sum_{n=3}^{\infty}\frac{1}{6}(n+r-3)\,a_{n-3}x^{n+r-1}+\sum_{n=4}^{\infty}\frac{1}{24}(n+r-4)\,a_{n-4}x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0 \qquad (2)
\end{aligned}$$

The case $n = 0$ gives the indicial equation

$$\begin{aligned}(n+r)(n+r-1)+(n+r) &= 0\\ (r)(r-1)+(r) &= 0\\ r^2 &= 0\end{aligned}$$

Hence the roots of the indicial equation are $r = 0$ which is a double root. Hence $r_1 = r_2 = 0$. When this happens, the solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{3}$$

Where we take $a_0 = 1$ as it is arbitrary and where $r = r_1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} \tag{3}$$

Something important to notice. In the sum above, it starts from 1 and not from 0. The main issue is how to find b_n. Since that is the only thing we need to be able to complete the solution as $y_1(x)$ is easily found. It turns out that there is a relation between the b_n and the a_n. The b_n can be found by taking just derivative of a_n as function of r for each n and then evaluate the result at $r = r_1$. How this is done will be shown below. First we need to find $y_1(x)$. We take Eq(3) and substitute it in the original ODE. This will result in Eq (2) which we found above so no need to repeat that. We just need to remember that now we now what r is. It has a numerical value unlike the above phase where we still did not know its value.

Now we are ready to find a_n. We skip $n = 0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose. We start from $n = 1$.

For $n = 1$ only, using Eq (2) gives

$$\begin{aligned}(n+r)(n+r-1)a_1 + \frac{1}{2}(n+r-1)(n+r-2)a_0 + (n+r)a_1 + (n+r-1)a_0 + a_0 &= 0\\ (1+r)(1+r-1)a_1 + \frac{1}{2}(1+r-1)(1+r-2)a_0 + (1+r)a_1 + (1+r-1)a_0 + a_0 &= 0\\ ((1+r)(1+r-1)+(1+r))a_1 + \left(\frac{1}{2}(1+r-1)(1+r-2)+(1+r-1)+1\right)a_0 &= 0\end{aligned}$$

But $a_0 = 1$. The above becomes

$$\begin{aligned}((1+r)(1+r-1)+(1+r))a_1 &= -\left(\frac{1}{2}(1+r-1)(1+r-2)+(1+r-1)+1\right)\\ a_1 &= \frac{-\left(\frac{1}{2}(1+r-1)(1+r-2)+(1+r-1)+1\right)}{((1+r)(1+r-1)+(1+r))} = -\frac{(r^2+r+2)}{2r^2+4r+2}\end{aligned}$$

Which at $r = 0$ gives

$$a_1 = -1$$

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1

For $n = 2$ only, using Eq (2) gives

$$\begin{aligned}(n+r)(n+r-1)a_2 + \frac{1}{2}(n+r-1)(n+r-2)a_1 + \frac{1}{6}(n+r-2)(n+r-3)a_0 + (n+r)a_2 + (n+r-1)a_1 + \frac{1}{2}(n+r\\ (2+r)(2+r-1)a_2 + \frac{1}{2}(2+r-1)(2+r-2)a_1 + \frac{1}{6}(2+r-2)(2+r-3)a_0 + (2+r)a_2 + (2+r-1)a_1 + \frac{1}{2}(2+r\\ ((2+r)(2+r-1)+(2+r))a_2 + \left(\frac{1}{2}(2+r-1)(2+r-2)+(2+r-1)+1\right)a_1 + \left(\frac{1}{6}(2+r-2)(2+r-3)+\frac{1}{2}(2\\ (r+2)^2 a_2 + \left(\frac{1}{2}r^2+\frac{3}{2}r+2\right)a_1 +\end{aligned}$$

But $a_0 = 1$ and $a_1 = -\frac{(r^2+r+2)}{2r^2+4r+2}$. The above becomes

$$(r+2)^2 a_2 + \left(\frac{1}{2}r^2 + \frac{3}{2}r + 2\right)\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) + \frac{1}{6}r(r+2) = 0$$

$$(r+2)^2 a_2 = \frac{r^4+4r^3+17r^2+26r+24}{2(r+1)^2}$$

$$a_2 = \frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$$

At $r = 0$ the above becomes

$$a_2 = \frac{24}{12(2)^2} = \frac{1}{2}$$

The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$

For $n = 3$ only, using Eq (2) gives

$$(n+r)(n+r-1)a_3 + \frac{1}{2}(n+r-1)(n+r-2)a_2 + \frac{1}{6}(n+r-2)(n+r-3)a_1$$
$$+\frac{1}{24}(n+r-3)(n+r-4)a_0 + (n+r)a_3 + (n+r-1)a_2 + \frac{1}{2}(n+r-2)a_1 + \frac{1}{6}(n+r-3)a_0 + a_2 = 0$$

Or

$$(3+r)(2+r)a_3 + \frac{1}{2}(2+r)(1+r)a_2 + (3+r)a_3 + (2+r)a_2 + \frac{1}{2}(1+r)a_1 + a_2 = 0$$

Or

$$((3+r)(2+r) + (3+r))a_3 + \left(\frac{1}{2}(2+r)(1+r) + (2+r) + 1\right)a_2 + \frac{1}{2}(1+r)a_1 = 0$$

$$(r+3)^2 a_3 + \left(\frac{1}{2}r^2 + \frac{5}{2}r + 4\right)a_2 + \frac{1}{2}(1+r)a_1 = 0$$

But $a_1 = -\frac{(r^2+r+2)}{2r^2+4r+2}, a_2 = \frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$. The above becomes

$$\begin{aligned}(r+3)^2 a_3 &= -\left(\frac{1}{2}r^2 + \frac{5}{2}r + 4\right)a_2 - \frac{1}{2}(1+r)a_1 \\ &= -\left(\frac{1}{2}r^2 + \frac{5}{2}r + 4\right)\left(\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}\right) - \frac{1}{2}(1+r)\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) \\ &= -\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2} \\ a_3 &= -\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}\end{aligned}$$

For $r = 0$ the above reduces to

$$a_3 = -\frac{144}{24(2)^2(3)^2} = -\frac{1}{6}$$

The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$
3	$-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}$	$-\frac{1}{6}$

And so on. Recursion starts at $n \geq 5$ but we have enough terms, so we stop here. $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$\begin{aligned}y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= 1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + \cdots\end{aligned} \tag{6A}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

To find b_n, we will use the following

$$b_n(r) = \frac{d}{dr}\left(a_n(r)\right) \tag{7}$$

Notice that n starts from 1. Hence

$$b_1(r) = \frac{d}{dr}\left(a_1(r)\right)\bigg|_{r=r_1}$$

What the above says, is that we first take derivative of $a_n(r)$ w.r.t. r and evaluate the result at the root of the indicial equation. Using the table above, we obtain (recalling that $r_1 = 0$ in this example)

n	$a_n(r)$	$a_n(r = r_1)$	$b_n(r) = \frac{d}{dr}(a_n(r))$
0	1	1	N/A since b starts from $n = 1$
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1	$\frac{d}{dr}\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) = -\frac{(r-3)}{2(r+1)^3}$
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$	$\frac{d}{dr}\left(\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}\right) = -\frac{(-r^4+7r^3+27r^2+53r+46)}{6(r^2+3r+2)^3}$
3	$-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}$	$-\frac{1}{6}$	$\frac{d}{dr}\left(-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}\right) = \frac{(-9r^7-44r^6+24r^5+662r^4\cdots}{24(r^3+6\cdots}$

We have found all b_n terms. Hence

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

And since $r = r_1 = 0$ then

$$y_2(x) = y_1(x)\ln(x) + \left(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots\right)$$

But from the above table, we see that $b_1 = \frac{3}{2}, b_2 = -\frac{23}{24}, b_3 = \frac{3}{8}$, The above becomes

$$y_2(x) = y_1(x)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O\left(x^4\right)\right)$$

And we know what $y_1(x)$ is from Eq (6A). Hence

$$y_2(x) = \left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O\left(x^4\right)\right)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O\left(x^4\right)\right)$$

Therefore the general solution is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O\left(x^4\right)\right) + c_2\left(\left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O\left(x^4\right)\right)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O\left(x^4\right)\right)\right) \end{aligned}$$

This completes the solution.

2 Indicial equation roots differ by an integer

2.1 Example 1. homogeneous ode example where log term is needed

Solve

$$x^2 y''(x) - x y(x) = 0 \tag{1}$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$y''(x) - \frac{1}{x}y(x) = 0$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x^2 \frac{1}{x} = 0$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$y'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above in (1) gives

$$\begin{aligned}x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}-x\sum_{n=0}^{\infty}a_nx^{n+r}&=0\\ \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r+1}&=0\end{aligned}\tag{1A}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty}a_nx^{n+r+1}=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}$$

And now Eq (1A) becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}=0\tag{1B}$$

$n=0$ gives the indicial equation

$$\begin{aligned}(n+r)(n+r-1)a_nx^r&=0\\ (r)(r-1)a_0x^r&=0\end{aligned}$$

Since $a_0\neq 0$ then the above becomes

$$(r)(r-1)x^r=0$$

Since this is true for all x, then

$$(r)(r-1)=0$$

Hence the roots of the indicial equation are $r_1=1,r_2=0$. Or $r_1=r_2+N$ where $N=1$. We always take r_1 to be the larger of the roots.

When this happens, the solution is given by

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+r}\tag{2}$$

Where we take $a_0=1$ as it is arbitrary and where $r=r_1=1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r=r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x)=Cy_1(x)\ln(x)+\sum_{n=0}^{\infty}b_nx^{n+r}\tag{3}$$

We will show below how to find C and b_n. First, let us find $y_1(x)$. From Eq(2)

$$\begin{aligned}y_1'(x)&=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\\ y_1''(x)&=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}\end{aligned}$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r=r_1=1$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r=r_1=1$. Substituting the above in (1) gives Eq (1B) above (We are following pretty much the same process we did to find the indicial equation here)

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}-\sum_{n=1}^{\infty}a_{n-1}x^{n+r}=0\tag{1B}$$

Now we are ready to find a_n. Now we skip $n=0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0=1}$ is an arbitrary value to choose. We start from $n=1$. For $n\geq 1$ we obtain the recursion equation

$$\begin{aligned}(n+r)(n+r-1)a_n-a_{n-1}&=0\\ a_n&=\frac{a_{n-1}}{(n+r)(n+r-1)}\end{aligned}$$

To more clearly indicate that a_n is function of r, we write the above as

$$a_n(r) = \frac{a_{n-1}(r)}{(n+r)(n+r-1)} \tag{4}$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (4) for $n = 1$

$$a_1(r) = \frac{a_0(r)}{(1+r)(r)} = \frac{1}{(1+r)(r)} \qquad a_0 = 1$$

and $r = r_1 = 1$ then the above becomes

$$a_1 = \frac{1}{2}$$

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$

And for $n = 2$ from Eq(4)

$$a_2(r) = \frac{a_1(r)}{(2+r)(1+r)}$$

But $a_1(r) = \frac{1}{(1+r)(r)}$. Then

$$a_2(r) = \frac{\frac{1}{(1+r)(r)}}{(2+r)(1+r)} = \frac{1}{r(r+1)^2(r+2)}$$

When $r = r_1 = 1$ the above becomes

$$a_2 = \frac{1}{(2)^2(3)} = \frac{1}{12}$$

The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$
2	$\frac{1}{r(r+1)^2(r+2)}$	$\frac{1}{12}$

For $n = 3$ Eq (4) gives

$$a_3(r) = \frac{a_2(r)}{(3+r)(2+r)}$$

Using the value of $a_2(r)$ from the the above becomes

$$a_3(r) = \frac{\frac{1}{r(r+1)^2(r+2)}}{(3+r)(2+r)} = \frac{1}{r(r+1)^2(r+2)^2(r+3)}$$

When $r = r_1 = 1$ the above becomes

$$a_3 = \frac{1}{(2)^2(3)^2(4)} = \frac{1}{144}$$

The Table now becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$
2	$\frac{1}{r(r+1)^2(r+2)}$	$\frac{1}{12}$
3	$\frac{1}{r(r+1)^2(r+2)^2(r+3)}$	$\frac{1}{144}$

And so on. Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 1$. Therefore

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \qquad (5) \\ &= x \sum_{n=0}^{\infty} a_n x^n \\ &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \\ &= x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + \cdots\right) \end{aligned}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3}$$

The first thing to do is to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_N(r)$. If this limit exist, then $C = 0$, else we need to keep the log term. From the above above we see that $a_N(r) = a_1(r) = \frac{1}{(1+r)(r)}$. Recall that $N = 1$ since this was the difference between the two roots and $r_2 = 0$ (the smaller root). Therefore

$$\lim_{r\to r_2} \frac{1}{(1+r)(r)} = \lim_{r\to 0} \frac{1}{(1+r)(r)}$$

Which does not exist. Therefore we need to keep the log term. In this case, we replace Eq. (3) back in the original ODE.

$$\begin{aligned} y_2'(x) &= Cy_1'\ln(x) + Cy_1\frac{1}{x} + \sum_{n=0}^{\infty}(n+r)\,b_n x^{n+r-1} \\ y_2''(x) &= Cy_1''\ln(x) + Cy_1'\frac{1}{x} + Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r-2} \\ &= Cy_1''\ln(x) + 2Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r-2} \end{aligned}$$

Substituting the above in $x^2y''(x) - xy(x) = 0$ gives

$$\begin{aligned} x^2\left(Cy_1''\ln(x) + 2Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r-2}\right) - x\left(Cy_1\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r}\right) &= 0 \\ Cx^2y_1''\ln(x) + 2x^2Cy_1'\frac{1}{x} - Cy_1 + x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r-2} - Cxy_1\ln(x) - x\sum_{n=0}^{\infty} b_n x^{n+r} &= 0 \\ Cx^2y_1''\ln(x) + 2xCy_1' - Cy_1 + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r} - Cxy_1\ln(x) - \sum_{n=0}^{\infty} b_n x^{n+r+1} &= 0 \\ C\ln(x)\left(x^2y_1'' - xy_1\right) + 2xCy_1' - Cy_1 + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} &= 0 \end{aligned}$$

But $x^2y_1'' - xy_1 = 0$ since y_1 is solution to the ode. The above simplifies to

$$C\left(2xy_1' - y_1\right) + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0 \tag{6}$$

The above is what we will use to determine C and all the b_n. Remembering that $r = r_2 = 0$ in the above, since this is for the second solution associated with the second root which we found above to be zero. But we found $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$ then

$$y_1' = \sum_{n=0}^{\infty}(n+1)\,a_n x^n$$

Eq (6) now becomes

$$\begin{aligned} C\left(2x\sum_{n=0}^{\infty}(n+1)\,a_n x^n\right) - C\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} &= 0 \\ 2C\sum_{n=0}^{\infty}(n+1)\,a_n x^{n+1} - C\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty}(n+r)(n+r-1)\,b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} &= 0 \end{aligned}$$

But $r = r_2 = 0$. The above becomes

$$2C\sum_{n=0}^{\infty}(n+1)\,a_n x^{n+1} - C\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} n(n-1)\,b_n x^n - \sum_{n=0}^{\infty} b_n x^{n+1} = 0$$

Adjusting the index of terms above, so so all x powers are the same gives

$$2C\sum_{n=1}^{\infty} na_{n-1}x^n - C\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=0}^{\infty} n(n-1)\,b_n x^n - \sum_{n=1}^{\infty} b_{n-1}x^n = 0 \tag{7}$$

$n = 0$ is skipped, since b_0 is arbitrary and can be taken as say

$$b_0 = 1$$

At $n=1$, Eq(7) gives

$$2Ca_0 - Ca_0 - b_0 = 0$$

But $a_0 = 1, b_0 = 1$ hence the above becomes

$$C = 1$$

For $b_N = b_1$ we are free to select any value since it is arbitrary. The standard way is to choose

$$b_1 = 0$$

Now we find the rest of the b_n terms. From Eq(7), for $n=2$, it gives

$$2C\left(2a_1\right) - Ca_1 + 2b_2 - b_1 = 0$$

But $C = 1, b_1 = 0$ and $a_1 = \frac{1}{2}$ from table. Hence the above becomes

$$\begin{aligned} 2\left(2\frac{1}{2}\right) - \frac{1}{2} + 2b_2 &= 0 \\ 2 - \frac{1}{2} + 2b_2 &= 0 \\ b_2 &= -\frac{3}{4} \end{aligned}$$

,And for $n=3$ from Eq. (7), it gives

$$2C\left(3a_2\right) - Ca_2 + (3)(2)b_3 - b_2 = 0$$

But $C = 1, b_2 = -\frac{3}{4}, a_2 = \frac{1}{12}$. The above becomes

$$\begin{aligned} 2\left(3\left(\frac{1}{12}\right)\right) - \frac{1}{12} + (3)\,(2)b_3 + \frac{3}{4} &= 0 \\ b_3 &= -\frac{7}{36} \end{aligned}$$

And so on. Hence the second solution is, for $r = 0, C = 1$

$$\begin{aligned} y_2(x) &= Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= y_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^n \\ &= y_1(x)\ln(x) + \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right) \\ &= y_1(x)\ln(x) + \left(1 + (0)x - \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right) \\ &= \left(x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \cdots\right)\ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right) \\ &= x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + O\left(x^4\right)\right)\ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + O\left(x^4\right)\right) \end{aligned}$$

Some observations: b_N is always taken as zero. Where N is the difference between the roots. In this case it is $b_1 = 0$. Now that we found y_1, y_2 then the general solution is

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 \\ &= C_1 x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + O\left(x^4\right)\right) + C_2\left(x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + O\left(x^4\right)\right)\ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + O\left(x^4\right)\right)\right) \end{aligned}$$

This completes the solution.

2.2 Example 2. homogeneous ode example where log term is not needed

Solve

$$x^2 y'' + 3xy' + 4x^4 y = 0 \tag{1}$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$y''(x) + \frac{3}{x}y' + 4x^2 y = 0$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x\frac{3}{x} = 3$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
x^2 \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + 4x^4 \sum_{n=0}^{\infty} a_n x^{n+r} = 0
$$

$$
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3\,(n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 4 a_n x^{n+r+4} = 0 \tag{1A}
$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$
\sum_{n=0}^{\infty} 4 a_n x^{n+r+4} = \sum_{n=4}^{\infty} 4 a_{n-4} x^{n+r}
$$

And now Eq (1A) becomes

$$
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3\,(n+r)\, a_n x^{n+r} + \sum_{n=4}^{\infty} 4 a_{n-4} x^{n+r} = 0 \tag{1B}
$$

$n = 0$ gives the indicial equation

$$
\begin{aligned}
(n+r)\,(n+r-1)\, a_n + 3\,(n+r)\, a_n &= 0 \\
(r)\,(r-1)\, a_0 + 3 r a_0 &= 0 \\
((r)\,(r-1) + 3r)\, a_0 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the above becomes

$$
(r)\,(r-1) + 3r = 0
$$

Hence the roots of the indicial equation are $r_1 = 0, r_2 = -2$. Or $r_1 = r_2 + N$ where $N = 2$. We always take r_1 to be the larger of the roots.

When this happens, the solution is given by

$$
y(x) = c_1 y_1(x) + c_2 y_2(x)
$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$
y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{2}
$$

Where we take $a_0 = 1$ as it is arbitrary and where $r = r_1 = 0$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$
y_2(x) = C y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3}
$$

We will show below how to find C and b_n. First, let us find $y_1(x)$. From Eq(2)

$$
\begin{aligned}
y_1'(x) &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y_1''(x) &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r = r_1 = 0$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r = r_1 = 0$. Substituting the above in (1) gives Eq (1B) above (We are following pretty much the same process we did to find the indicial equation here)

$$
\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} 3\,(n+r)\, a_n x^{n+r} + \sum_{n=4}^{\infty} 4 a_{n-4} x^{n+r} = 0 \tag{1B}
$$

Now we are ready to find a_n. Now we skip $n = 0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose.

For $n = 1$, Eq (1B) gives

$$\begin{aligned}(1+r)(1+r-1)a_1 + 3(1+r)a_1 &= 0\\ ((1+r)(1+r-1) + 3(1+r))a_1 &= 0\\ \left(r^2 + 4r + 3\right)a_1 &= 0\end{aligned}$$

But $r = r_1 = 0$. The above becomes

$$3a_1 = 0$$

Hence $a_1 = 0$.

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	0	0

For $n = 2$, Eq (1B) gives

$$\begin{aligned}(2+r)(2+r-1)a_2 + 3(2+r)a_2 &= 0\\ ((2+r)(2+r-1) + 3(2+r))a_2 &= 0\end{aligned}$$

But $r = r_1 = 0$. The above becomes

$$\begin{aligned}((2)(1) + 3(2))a_2 &= 0\\ 8a_2 &= 0\end{aligned}$$

Hence $a_2 = 0$. The table becomes

n	$a_n(r)$	$a_n(r = 0)$
0	1	1
1	0	0
2	0	0

For $n = 3$, Eq (1B) gives

$$(3+r)(3+r-1)a_3 + 3(3+r)a_3 = 0$$

But $r = 0$. The above becomes

$$\begin{aligned}(3)(2)a_3 + 3(3)a_3 &= 0\\ 15a_3 &= 0\end{aligned}$$

Hence $a_3 = 0$ and the table becomes

n	$a_n(r)$	$a_n(r = 0)$
0	1	1
1	0	0
2	0	0
3	0	0

For $n \geq 4$ we obtain the recursion equation

$$\begin{aligned}(n+r)(n+r-1)a_n + 3(n+r)a_n + 4a_{n-4} &= 0\\ ((n+r)(n+r-1) + 3(n+r))a_n + 4a_{n-4} &= 0\\ a_n(r) &= -\frac{4a_{n-4}(r)}{(n+r)(n+r-1) + 3(n+r)} \qquad (4)\end{aligned}$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (4) for $n = 4$

$$a_4(r) = -\frac{4a_0(r)}{(4+r)(4+r-1) + 3(4+r)}$$

and $r = r_1 = 0$ and $a_0 = 1$, then the above becomes

$$a_4 = -\frac{4}{(4)(3) + 3(4)} = -\frac{1}{6}$$

The table becomes

n	$a_n(r)$	$a_n(r = 0)$
0	1	1
1	0	0
2	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{6}$

And for $n=5$ from Eq(4)

$$\begin{aligned} a_5(r) &= -\frac{4a_1(r)}{(n+r)(n+r-1)+3(n+r)} \\ &= 0 \end{aligned}$$

Since $a_1=0$. Similarly $a_6=0, a_7=0$. For $n=8$

$$a_8(r) = -\frac{4a_4(r)}{(8+r)(8+r-1)+3(8+r)}$$

But $a_4(r) = -\frac{4}{(4+r)(4+r-1)+3(4+r)}$. The above becomes

$$a_8(r) = \frac{4\frac{4}{(4+r)(4+r-1)+3(4+r)}}{(8+r)(8+r-1)+3(8+r)} = \frac{16}{r^4+28r^3+284r^2+1232r+1920}$$

When $r=r_1=0$ the above becomes

$$a_8(r) = \frac{1}{120}$$

And so on. The table becomes

n	$a_n(r)$	$a_n\,(r=0)$
0	1	1
1	0	0
2	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{6}$
5	0	0
6	0	0
7	0	0
8	$\frac{16}{r^4+28r^3+284r^2+1232r+1920}$	$\frac{1}{120}$

Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r=r_1=0$. Therefore

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \qquad (5) \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + \cdots \end{aligned}$$

Using values found for a_n in the above table, then (5) becomes

$$\begin{aligned} y_1(x) &= 1 + a_4x^4 + a_8x^8 + \cdots \\ &= 1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 + O\left(x^9\right) \end{aligned}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \qquad (3)$$

The first thing to do is to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_N(r)$. If this limit exist, then $C=0$, else we need to keep the log term. From the above above we see that $a_N(r) = a_2(r) = 0$. Recall that $N=2$ since this was the difference between the two roots and $r_2=-2$ (the smaller root). Therefore

$$\lim_{r\to r_2} 0 = \lim_{r\to 0} 0 = 0$$

Hence the limit exist. Therefore we do not need the log term. This means we can let $C=0$. This is the easy case. Hence (3) becomes

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \qquad (3A) \\ &= x^{-2}\sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Since $r=r_2=-2$. Let $b_0=1$. We have to remember now that $b_N=b_2=0$. This is the same we did when the log term was needed in the above example, since b_N is arbitrary, and used to generate $y_1(x)$. Common practice is to use $b_N=0$. The rest of the b_n are found in similar way, from recursive relation as was done above. Substituting (3A) into $x^2y'' + 3xy' + 4x^4y = 0$ gives Eq. (1B) again, but with a_n replaced by b_n

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) b_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r) b_n x^{n+r} + \sum_{n=4}^{\infty} 4b_{n-4} x^{n+r} = 0 \tag{1B}$$

For $n = 0$, we skip and let $b_0 = 1$. For $n = 1$ the above gives $b_1 = 0$. And $b_2 = 0$ since it is the special term b_N. And for $n = 3$, we get $b_3 = 0$. The table for b_n is now

n	$b_n(r)$	$b_n\ (r=-2)$
0	1	1
1	0	0
2	0	0
3	0	0

For $n \geq 4$, the recursion relation is

$$(n+r)(n+r-1) b_n + 3(n+r) b_n + 4b_{n-4} = 0$$
$$b_n(r) = -\frac{4b_{n-4}(r)}{(n+r)(n+r-1) + 3(n+r)}$$

For $n = 4$

$$\begin{aligned} b_4(r) &= -\frac{4b_0(r)}{(4+r)(4+r-1) + 3(4+r)} \\ &= -\frac{4}{(4+r)(4+r-1) + 3(4+r)} \qquad b_0 = 1 \end{aligned}$$

but $r = -2$. The above becomes

$$b_4 = -\frac{4}{(4-2)(4-2-1) + 3(4-2)} = -\frac{1}{2}$$

The table becomes

n	$b_n(r)$	$b_n\ (r=-2)$
0	1	1
1	0	0
2*	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{2}$

We will find that $b_5 = b_6 = b_7 = 0$. And for $n = 8$

$$b_8(r) = -\frac{4b_4(r)}{(8+r)(7+r) + 3(8+r)}$$

But $b_4(r) = -\frac{4}{(4+r)(4+r-1)+3(4+r)}$. Hence

$$b_8(r) = \frac{4\frac{4}{(4+r)(4+r-1)+3(4+r)}}{(8+r)(7+r) + 3(8+r)} = \frac{16}{r^4 + 28r^3 + 284r^2 + 1232r + 1920}$$

But $r = -2$.

$$b_8(r) = \frac{16}{(-2)^4 + 28(-2)^3 + 284(-2)^2 + 1232(-2) + 1920} = \frac{1}{24}$$

The table becomes

n	$b_n(r)$	$b_n\ (r=-2)$
0	1	1
1	0	0
2*	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{2}$
5	0	0
6	0	0
7	0	0
8	$\frac{16}{r^4+28r^3+284r^2+1232r+1920}$	$\frac{1}{24}$

And so on. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2} \\
&= x^{-2} \sum_{n=0}^{\infty} b_n x^n \\
&= x^{-2} \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + \cdots\right) \\
&= x^{-2} \left(1 - \frac{1}{2} x^4 + \frac{1}{24} b_8 x^8 + O\left(x^9\right)\right)
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= C_1 y_1 + C_2 y_2 \\
&= C_1 \left(1 - \frac{1}{6} x^4 + \frac{1}{120} x^8 + O\left(x^9\right)\right) + C_2 \left(x^{-2} \left(1 - \frac{1}{2} x^4 + \frac{1}{24} b_8 x^8 + O\left(x^9\right)\right)\right)
\end{aligned}
$$

The following are important items to remember. Always let $b_N = 0$ where N is the difference between the roots. When the log term is not needed (as in this problem), y_2 is found in very similar way to y_1 where $b_0 = 1$ and the recursion formula is used to find all b_n. But when the log term is needed (as in the above problem), it is a little more complicated and need to find C and b_1 values by comparing coefficients as was done).

This completes the solution.

3 Indicial equation roots are complex conjugate

3.1 Example 1. homogeneous ode example

$$
x^2 y'' + x^2 y' + y = 0 \tag{1}
$$

With expansion around $x = 0$. This is a regular singular ODE. Let

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x^2 \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
$$

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{1A}
$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the middle term above as follows

$$
\sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1} = \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r}
$$

And now Eq (1A) becomes

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{2A}
$$

$n = 0$ gives the indicial equation

$$
\begin{aligned}
(n+r)(n+r-1) a_0 x^r + a_0 x^r &= 0 \\
(r)(r-1) a_0 x^r + (r) a_0 x^r &= 0 \\
((r)(r-1) a_0 + a_0) x^r &= 0 \\
((r)(r-1) + 1) a_0 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then

$$\begin{aligned}(r)(r-1)+1&=0\\ r^2-r+1&=0\end{aligned}$$

The roots are

$$\begin{aligned}r_1&=\frac{1}{2}+\frac{1}{2}i\sqrt{3}\\ r_2&=\frac{1}{2}-\frac{1}{2}i\sqrt{3}\end{aligned}$$

The roots will always be complex conjugate of each others (since second order ode) and the real part will always be equal. Let the roots be

$$r_{1,2}=\alpha\pm i\beta$$

When this happens, the solution is given similar to the case when the roots differ by non integer, except now the solution and the coefficients will be complex. Let the solution be

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

Now $y_1(x)$ is solved for. The solution is

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n$$

Starting with Eq (2A) which was derived above gives

$$\sum_{n=0}^{\infty}(n+r_1)(n+r_1-1)a_nx^{n+r_1}+\sum_{n=1}^{\infty}(n+r_1-1)a_{n-1}x^{n+r_1}+\sum_{n=0}^{\infty}a_nx^{n+r_1}=0$$

The case of $n=0$ is skipped since this was used to find the roots and $a_0=1$ is assumed.

$n\geq 1$ gives the recursion equation

$$\begin{aligned}(n+r_1)(n+r_1-1)a_n+(n+r_1-1)a_{n-1}+a_n&=0\\ (n+r_1)(n+r_1-1)a_n+a_n&=-(n+r_1-1)a_{n-1}\\ a_n&=\frac{-(n+r_1-1)a_{n-1}}{(n+r_1)(n+r_1-1)+1}\end{aligned}\tag{3}$$

For $n=1$ Eq. (3) gives

$$a_1=\frac{-r_1a_{n-1}}{(1+r_1)(r_1)+1}$$

But $r_1=\alpha\pm i\beta=\frac{1}{2}+\frac{1}{2}i\sqrt{3}$. and $a_0=1$. Hence the above becomes

$$a_1=\frac{-\left(\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)}{\left(1+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)+1}=\frac{-1}{2}$$

For $n=2$ Eq. (3) gives

$$a_2=\frac{-(1+r_1)a_1}{(2+r_1)(1+r_1)+1}=\frac{-(1+r_1)\left(\frac{-1}{2}\right)}{(2+r_1)(1+r_1)+1}=\frac{-\left(1+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(\frac{-1}{2}\right)}{\left(2+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(1+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)+1}=\frac{9}{56}-\frac{1}{56}i\sqrt{3}$$

For $n=3$ Eq. (3) gives

$$a_3=\frac{-(3+r_1-1)a_2}{(3+r_1)(2+r_1)+1}=\frac{-(3+r_1-1)\left(\frac{9}{56}-\frac{1}{56}i\sqrt{3}\right)}{(3+r_1)(2+r_1)+1}=\frac{-\left(2+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(\frac{9}{56}-\frac{1}{56}i\sqrt{3}\right)}{\left(3+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)\left(2+\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)+1}=\frac{1}{112}i\sqrt{3}-\frac{13}{336}$$

And so on. Hence the first solution is

$$y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_nx^n=x^{\frac{1}{2}+\frac{1}{2}i\sqrt{3}}\left(a_0+a_1x+a_2x^2+a_3x^3+\cdots\right)\tag{4}$$

but

$$x^{\frac{1}{2}+\frac{1}{2}i\sqrt{3}}=x^{\frac{1}{2}}x^{\frac{1}{2}i\sqrt{3}}=x^{\frac{1}{2}}e^{\ln\left(x^{\frac{1}{2}i\sqrt{3}}\right)}=x^{\frac{1}{2}}e^{i\ln\left(x^{\frac{\sqrt{3}}{2}}\right)}=x^{\frac{1}{2}}\left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right)+i\sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)$$

Substituting the above in (4) and using values found for a_n gives

$$y_1(x)=x^{\frac{1}{2}}\left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right)+i\sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)\left(1-\frac{1}{2}x+\left(\frac{9}{56}-\frac{1}{56}i\sqrt{3}\right)x^2+\left(-\frac{13}{336}+\frac{1}{112}i\sqrt{3}\right)x^3+\cdots\right)\tag{5}$$

Since the roots are complex conjugate of each others, then the second solution is

$$y_2(x)=x^{\frac{1}{2}}\left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right)-i\sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)\left(1-\frac{1}{2}x+\left(\frac{9}{56}+\frac{1}{56}i\sqrt{3}\right)x^2+\left(-\frac{13}{336}-\frac{1}{112}i\sqrt{3}\right)x^3+\cdots\right)\tag{6}$$

The final solution is therefore

$$y(x)=c_1y_1(x)+c_2y_2(x)$$