

Deriving trig identities

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To derive trig identities (something useful in the exam), we will use Euler relation as starting point, which is $e^{ix} = \cos x + i \sin x$.

1 $\cos(A + B)$ and $\sin(A + B)$

$$e^{i(A+B)} = \cos(A + B) + i \sin(A + B) \quad (1)$$

But $e^{i(A+B)} = e^{iA}e^{iB}$ therefore

$$\begin{aligned} e^{iA}e^{iB} &= (\cos A + i \sin A)(\cos B + i \sin B) \\ &= \cos A \cos B + i \cos A \sin B + i \sin A \cos B - \sin A \sin B \\ &= (\cos A \cos B - \sin A \sin B) + i(\cos A \sin B + \sin A \cos B) \end{aligned} \quad (2)$$

Now (1) is the same as (2). Hence the real part and the imaginary parts must be the same. Therefore

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (3)$$

$$\sin(A + B) = \cos A \sin B + \sin A \cos B \quad (4)$$

2 $\cos(A - B)$ and $\sin(A - B)$

This can be derived in similar way to the above using $e^{i(A-B)} = \cos(A - B) + i \sin(A - B)$ and so on. But more easily, it can be derived from (3,4) directly by just changing replacing B by $-B$ everywhere and then changing $\sin(-B)$ to $-\sin B$ and leaving $\cos B$ the same since $\cos(-B) = \cos B$. This is because $\cos$ is even and $\sin$ is odd, then (3) becomes

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (3A)$$

$$\sin(A - B) = -\cos A \sin B + \sin A \cos B \quad (4A)$$

So we really just need to find (3) to find the 4 formulas for addition and subtractions of angles.

3 $\cos(2A)$ and $\sin(2A)$

These also can be found from (3,4). By replacing B with A resulting in

$$\cos(A + A) = \cos A \cos A - \sin A \sin A$$

$$\sin(A + A) = \cos A \sin A + \sin A \cos A$$

Therefore

$$\cos(2A) = \cos^2 A - \sin^2 A \quad (3C)$$

$$\sin(2A) = 2 \cos A \sin A \quad (4C)$$

Or we could use Euler formula, but the above is simpler. To use Euler formula, we write

$$e^{i(2A)} = \cos(2A) + i \sin(2A) \quad (5)$$

But $e^{i(2A)} = e^{iA}e^{iA}$ therefore

$$\begin{aligned} e^{iA}e^{iA} &= (\cos A + i \sin A)(\cos A + i \sin A) \\ &= \cos^2 A + 2i \cos A \sin A - \sin^2 A \\ &= (\cos^2 A - \sin^2 A) + i(2 \cos A \sin A) \end{aligned} \tag{6}$$

Comparing (5,6) shows that

$$\begin{aligned} \cos(2A) &= \cos^2 A - \sin^2 A \\ \sin(2A) &= 2 \cos A \sin A \end{aligned}$$

Which is the same as (3C,4C) above.

4 $\cos\left(\frac{x}{2}\right)$

From the double angle formula (3C)

$$\cos(2A) = \cos^2 A - \sin^2 A$$

But $\cos^2 A + \sin^2 A = 1$ then $\sin^2 A = 1 - \cos^2 A$ and the above becomes

$$\begin{aligned} \cos(2A) &= \cos^2 A - (1 - \cos^2 A) \\ &= 2 \cos^2 A - 1 \end{aligned}$$

Hence

$$\cos^2 A = \frac{\cos(2A) + 1}{2}$$

Let $A = \frac{x}{2}$ then the above becomes

$$\begin{aligned} \cos^2\left(\frac{x}{2}\right) &= \frac{\cos(x) + 1}{2} \\ \cos\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{\cos(x) + 1}{2}} \end{aligned}$$

The sign depends on the quadrant of $\frac{x}{2}$.

5 $\sin\left(\frac{x}{2}\right)$

From the double angle formula (3C)

$$\cos(2A) = \cos^2 A - \sin^2 A$$

But $\cos^2 A + \sin^2 A = 1$ then $\cos^2 A = 1 - \sin^2 A$ and the above becomes

$$\begin{aligned} \cos(2A) &= 1 - \sin^2 A - \sin^2 A \\ &= 1 - 2 \sin^2 A \end{aligned}$$

Hence

$$\sin^2 A = \frac{1 - \cos(2A)}{2}$$

Let $A = \frac{x}{2}$ then the above becomes

$$\begin{aligned}\sin^2\left(\frac{x}{2}\right) &= \frac{1 - \cos(x)}{2} \\ \sin\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 - \cos(x)}{2}}\end{aligned}$$

The sign depends on the quadrant of $\frac{x}{2}$.

6 $\sin(\alpha) + \sin(\beta)$

This can be found by adding (4) and (4A). Let

$$\begin{aligned}A + B &= \alpha \\ A - B &= \beta\end{aligned}\tag{7}$$

Then (4)+(4A) now becomes

$$\begin{aligned}\sin(\alpha) + \sin(\beta) &= (\cos A \sin B + \sin A \cos B) - \cos A \sin B + \sin A \cos B \\ &= 2 \sin A \cos B\end{aligned}\tag{8}$$

Now we solve for A, B from (7). Which gives

$$\begin{aligned}A &= \frac{\alpha + \beta}{2} \\ B &= \frac{\alpha - \beta}{2}\end{aligned}$$

Substituting the above in (8) gives

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

7 $\cos(\alpha) + \cos(\beta)$

This can be found by adding (3) and (3A). Let

$$\begin{aligned}A + B &= \alpha \\ A - B &= \beta\end{aligned}\tag{7}$$

Then (3)+(3A) now becomes

$$\begin{aligned}\cos(\alpha) + \cos(\beta) &= (\cos A \cos B - \sin A \sin B) + (\cos A \cos B + \sin A \sin B) \\ &= 2 \cos A \cos B\end{aligned}\tag{9}$$

Now we solve for A, B from (7). Which gives

$$A = \frac{\alpha + \beta}{2}$$

$$B = \frac{\alpha - \beta}{2}$$

Substituting the above in (9) gives

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

8 $\sin(\alpha) - \sin(\beta)$

This can be found from (4)-(4A). Let

$$A + B = \alpha \tag{7}$$

$$A - B = \beta$$

Then (4)-(4A) now becomes

$$\begin{aligned} \sin(\alpha) - \sin(\beta) &= (\cos A \sin B + \sin A \cos B) + \cos A \sin B - \sin A \cos B \\ &= 2 \cos A \sin B \end{aligned} \tag{10}$$

Now we solve for A, B from (7). Which gives

$$A = \frac{\alpha + \beta}{2}$$

$$B = \frac{\alpha - \beta}{2}$$

Substituting the above in (10) gives

$$\sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

9 $\cos(\alpha) - \cos(\beta)$

This can be found from (3)-(3A). Let

$$A + B = \alpha \tag{7}$$

$$A - B = \beta$$

Then (3)-(3A) now becomes

$$\begin{aligned} \cos(\alpha) - \cos(\beta) &= (\cos A \cos B - \sin A \sin B) - (\cos A \cos B + \sin A \sin B) \\ &= -2 \sin A \sin B \end{aligned} \tag{11}$$

Now we solve for A, B from (7). Which gives

$$A = \frac{\alpha + \beta}{2}$$

$$B = \frac{\alpha - \beta}{2}$$

Substituting the above in (11) gives

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

10 $\cos(A) \cos(B)$

Adding (3)+(3A) gives

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \cos(A + B) + \cos(A - B) &= 2 \cos A \cos B\end{aligned}$$

Hence

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$$

11 $\sin(A) \cos(B)$

Adding (4)+(4A) gives

$$\begin{aligned}\sin(A + B) &= \cos A \sin B + \sin A \cos B \\ \sin(A - B) &= -\cos A \sin B + \sin A \cos B \\ \sin(A + B) + \sin(A - B) &= 2 \sin A \cos B\end{aligned}$$

Hence

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$$

12 $\sin(A) \sin(B)$

(3)-(3A) gives

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \cos(A + B) - \cos(A - B) &= -2 \sin A \sin B\end{aligned}$$

Hence

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$