Dynamics cheat sheet

Nasser M. Abbasi

July 8, 2025 Compiled on July 8, 2025 at 5:44pm

Contents

Chapter 1

Vibration

1.1 Modal analysis for two degrees of freedom system

Detailed steps to perform modal analysis are given below for a standard undamped two degrees of freedom system. The main advantage of solving a multidegree system using modal analysis is that it decouples the equations of motion (assuming they are coupled) making solving them much simpler.

In addition it shows the fundamental shapes that the system can vibrate in, which gives more insight into the system. Starting with standard 2 degrees of freedom system

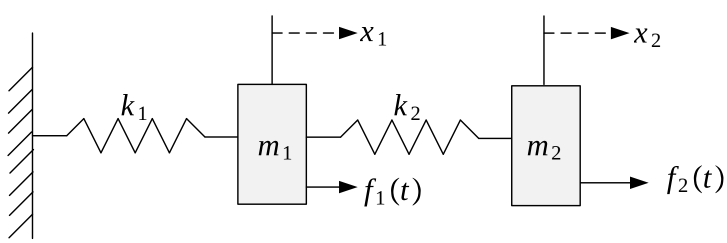
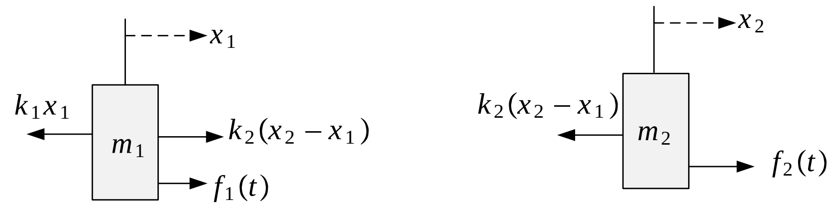


Figure 1.1: 2 degrees of freedom system

In the above the generalized coordinates are x_1 and x_2. Hence the system requires two equations of motion (EOM's).

1.1.1 Step one. Finding the equations of motion in normal coordinates space

The two EOM's are found using any method such as Newton's method or Lagrangian method. Using Newton's method, free body diagram is made of each mass and then $F = ma$ is written for each mass resulting in the equations of motion. In the following it is assumed that both masses are moving in the positive direction and that x_2 is larger than x_1 when these equations of equilibrium are written



$$\sum F = m_1 x_1'' \qquad \sum F = m_2 x_2''$$

$$-k_1 x_1 + k_2(x_2 - x_1) + f_1(t) = m_1 x_1'' \qquad -k_2(x_2 - x_1) + f_2(t) = m_2 x_2''$$

Figure 1.2: general 2 degrees of freedom system

Hence, from the above the equations of motion are

$$\begin{aligned} m_1 x_1'' + k_1 x_1 - k_2(x_2 - x_1) &= f_1(t) \\ m_2 x_2'' + k_2(x_2 - x_1) &= f_2(t) \end{aligned}$$

or

$$\begin{aligned} m_1 x_1'' + x_1(k_1 + k_2) - k_2 x_2 &= f_1(t) \\ m_2 x_2'' + k_2 x_2 - k_2 x_1 &= f_2(t) \end{aligned}$$

In Matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

The above two EOM are coupled in stiffness, but not mass coupled. Using short notations, the above is written as

$$[M]\{x''\} + [K]\{x\} = \{f\}$$

Modal analysis now starts with the goal to decouple the EOM and obtain the fundamental shape functions that the system can vibrate in. To make these derivations more general, the mass matrix and the stiffness matrix are written in general notations as follows

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

The mass matrix $[M]$ and the stiffness matrix $[K]$ must always come out to be symmetric. If they are not symmetric, then a mistake was made in obtaining them. As a general rule, the mass matrix $[M]$ is PSD (positive definite matrix) and the $[K]$ matrix is positive semi-definite matrix. The reason the $[M]$ is PSD is that $x^T[M]\{x\}$ represents the kinetic energy of the system, which is typically positive and not zero. But reading some other references [1] it is possible that $[M]$ can be positive semi-definite. It depends on the application being modeled.

1.1.2 Step 2. Solving the eigenvalue problem, finding the natural frequencies

The first step in modal analysis is to solve the eigenvalue problem $\det\left([K] - \omega^2[M]\right) = 0$ in order to determine the natural frequencies of the system. This equations leads to a polynomial in ω and the roots of this polynomial are the natural frequencies of the system. Since there are two degrees of freedom, there will be two natural frequencies ω_1, ω_2 for the system.

$$\begin{aligned} \det\left([K] - \omega^2[M]\right) &= 0 \\ \det\left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\right) &= 0 \\ \det\begin{bmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} \end{bmatrix} &= 0 \\ \left(k_{11} - \omega^2 m_{11}\right)\left(k_{22} - \omega^2 m_{22}\right) - \left(k_{12} - \omega^2 m_{12}\right)\left(k_{21} - \omega^2 m_{21}\right) &= 0 \\ \omega^4(m_{11}m_{22} - m_{12}m_{21}) + \omega^2(-k_{11}m_{22} + k_{12}m_{21} + k_{21}m_{12} - k_{22}m_{11}) + k_{11}k_{22} - k_{12}k_{21} &= 0 \end{aligned}$$

The above is a polynomial in ω^4. Let $\omega^2 = \lambda$ it becomes

$$\lambda^2(m_{11}m_{22} - m_{12}m_{21}) + \lambda(-k_{11}m_{22} + k_{12}m_{21} + k_{21}m_{12} - k_{22}m_{11}) + k_{11}k_{22} - k_{12}k_{21} = 0$$

This quadratic polynomial in λ which is now solved using the quadratic formula. Then the positive square root of each λ root to obtain ω_1 and ω_2 which are the roots of the original eigenvalue problem. Assuming from now that these roots are ω_1 and ω_2 the next step is to obtain the non-normalized shape vectors $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2$ also called the eigenvectors associated with ω_1 and ω_2

[1] `http://en.wikipedia.org/wiki/Fundamental_equation_of_constrained_motion`

1.1.3 Step 3. Finding the non-normalized eigenvectors

For each natural frequency ω_1 and ω_2 the corresponding shape function is found by solving the following two sets of equations for the vectors φ_1, φ_2

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_1^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_2^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For ω_1, let $\varphi_{11} = 1$ and solve for

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_1^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_{11} - \omega_1^2 m_{11} & k_{12} - \omega_1^2 m_{12} \\ k_{21} - \omega_1^2 m_{21} & k_{22} - \omega_1^2 m_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Which gives one equation now to solve for φ_{21} (the first row equation is only used)

$$\left(k_{11} - \omega_1^2 m_{11}\right) + \varphi_{21}\left(k_{12} - \omega_1^2 m_{12}\right) = 0$$

Hence

$$\varphi_{21} = \frac{-(k_{11} - \omega_1^2 m_{11})}{(k_{12} - \omega_1^2 m_{12})}$$

Therefore the first shape vector is

$$\boldsymbol{\varphi}_1 = \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \frac{-(k_{11} - \omega_1^2 m_{11})}{(k_{12} - \omega_1^2 m_{12})} \end{Bmatrix}$$

Similarly the second shape function is obtained. For ω_2, let $\varphi_{12} = 1$ and solve for

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_2^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_{11} - \omega_2^2 m_{11} & k_{12} - \omega_2^2 m_{12} \\ k_{21} - \omega_2^2 m_{21} & k_{22} - \omega_2^2 m_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Which gives one equation now to solve for φ_{22} (the first row equation is only used)

$$\left(k_{11} - \omega_2^2 m_{11}\right) + \varphi_{22}\left(k_{12} - \omega_2^2 m_{12}\right) = 0$$

Hence

$$\varphi_{22} = \frac{-(k_{11} - \omega_2^2 m_{11})}{(k_{12} - \omega_2^2 m_{12})}$$

Therefore the second shape vector is

$$\boldsymbol{\varphi}_2 = \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \frac{-(k_{11}-\omega_2^2 m_{11})}{(k_{12}-\omega_2^2 m_{12})} \end{Bmatrix}$$

Now that the two non-normalized shape vectors are found, the next step is to perform mass normalization

1.1.4 Step 4. Mass normalization of the shape vectors (or the eigenvectors)

Let

$$\mu_1 = \boldsymbol{\varphi}_1^T [M] \boldsymbol{\varphi}_1$$

This results in a scalar value μ_1, which is later used to normalize $\boldsymbol{\varphi}_1$. Similarly

$$\mu_2 = \boldsymbol{\varphi}_2^T [M] \boldsymbol{\varphi}_2$$

For example, to find μ_1

$$\begin{aligned}
\mu_1 &= \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix}^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} \\
&= \begin{Bmatrix} \varphi_{11} & \varphi_{21} \end{Bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} \\
&= \begin{Bmatrix} \varphi_{11} m_{11} + \varphi_{21} m_{21} & \varphi_{11} m_{12} + \varphi_{21} m_{22} \end{Bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} \\
&= \varphi_{11}(\varphi_{11} m_{11} + \varphi_{21} m_{21}) + \varphi_{21}(\varphi_{11} m_{12} + \varphi_{21} m_{22})
\end{aligned}$$

Similarly, μ_2 is found

$$\begin{aligned}
\mu_2 &= \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix}^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} \\
&= \begin{Bmatrix} \varphi_{12} & \varphi_{22} \end{Bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} \\
&= \begin{Bmatrix} \varphi_{12} m_{11} + \varphi_{22} m_{21} & \varphi_{12} m_{12} + \varphi_{22} m_{22} \end{Bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} \\
&= \varphi_{12}(\varphi_{12} m_{11} + \varphi_{22} m_{21}) + \varphi_{22}(\varphi_{12} m_{12} + \varphi_{22} m_{22})
\end{aligned}$$

Now that μ_1, μ_2 are obtained, the mass normalized shape vectors are found. They are called $\mathbf{\Phi}_1, \mathbf{\Phi}_2$

$$\mathbf{\Phi}_1 = \frac{\boldsymbol{\varphi}_1}{\sqrt{\mu_1}} = \frac{\begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix}}{\sqrt{\mu_1}} = \begin{Bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} \end{Bmatrix}$$

Similarly

$$\mathbf{\Phi}_2 = \frac{\boldsymbol{\varphi}_2}{\sqrt{\mu_2}} = \frac{\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix}}{\sqrt{\mu_2}} = \begin{Bmatrix} \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{Bmatrix}$$

1.1.5 Step 5, obtain the modal transformation matrix Φ

The modal transformation matrix is the 2×2 matrix made of of $\mathbf{\Phi}_1, \mathbf{\Phi}_2$ in each of its columns

$$\begin{aligned} [\Phi] &= [\mathbf{\Phi}_1 \mathbf{\Phi}_2] \\ &= \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix} \end{aligned}$$

Now the $[\Phi]$ is found, the transformation from the normal coordinates $\{x\}$ to modal coordinates, which is called $\{\eta\}$ is found

$$\begin{aligned} \{x\} &= [\Phi] \{\eta\} \\ \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \end{aligned}$$

The transformation from modal coordinates back to normal coordinates is

$$\begin{aligned} \{\eta\} &= [\Phi]^{-1} \{x\} \\ \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} &= \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix}^{-1} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \end{aligned}$$

However, $[\Phi]^{-1} = [\Phi]^T [M]$ therefore

$$\begin{aligned} \{\eta\} &= [\Phi]^T [M] \{x\} \\ \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} &= \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix}^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \end{aligned}$$

The next step is to apply this transformation to the original equations of motion in order to decouple them

1.1.6 Step 6. Applying modal transformation to decouple the original equations of motion

The EOM in normal coordinates is

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

Applying the above modal transformation $\{x\} = [\Phi]\{\eta\}$ on the above results in

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} [\Phi] \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} [\Phi] \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

pre-multiplying by $[\Phi]^T$ results in

$$[\Phi]^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} [\Phi] \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + [\Phi]^T \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} [\Phi] \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

The result of $[\Phi]^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} [\Phi]$ will always be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This is because mass normalized shape vectors are used. If the shape functions were not mass normalized, then the diagonal values will not be 1 as shown.

The result of $[\Phi]^T \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} [\Phi]$ will be $\begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$.

Let the result of $[\Phi]^T \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$ be $\begin{Bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \end{Bmatrix}$,Therefore, in modal coordinates the original EOM becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \end{Bmatrix}$$

The EOM are now decouples and each can be solved as follows

$$\eta_1''(t) + \omega_1^2 \eta_1(t) = \tilde{f}_1(t)$$
$$\eta_2''(t) + \omega_2^2 \eta_2(t) = \tilde{f}_2(t)$$

To solve these EOM's, the initial conditions in normal coordinates must be transformed to modal coordinates using the above transformation rules

$$\{\eta(0)\} = [\Phi]^T [M] \{x(0)\}$$
$$\{\eta'(0)\} = [\Phi]^T [M] \{x'(0)\}$$

Or in full form

$$\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix}^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$

and

$$\begin{Bmatrix} \eta_1'(0) \\ \eta_2'(0) \end{Bmatrix} = \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix}^T \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} x_1'(0) \\ x_2'(0) \end{Bmatrix}$$

Each of these EOM are solved using any of the standard methods. This will result is solutions $\eta_1(t)$ and $\eta_2(t)$

1.1.7 Step 7. Converting modal solution to normal coordinates solution

The solutions found above are in modal coordinates $\eta_1(t), \eta_2(t)$. The solution needed is $x_1(t), x_2(t)$. Therefore, the transformation $\{x\} = [\Phi]\{\eta\}$ is now applied to convert the solution to normal coordinates

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \begin{bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}} & \frac{\varphi_{12}}{\sqrt{\mu_2}} \\ \frac{\varphi_{21}}{\sqrt{\mu_1}} & \frac{\varphi_{22}}{\sqrt{\mu_2}} \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{\varphi_{11}}{\sqrt{\mu_1}}\eta_1(t) + \frac{\varphi_{12}}{\sqrt{\mu_2}}\eta_2(t) \\ \frac{\varphi_{21}}{\sqrt{\mu_1}}\eta_1(t) + \frac{\varphi_{22}}{\sqrt{\mu_2}}\eta_2(t) \end{Bmatrix} \end{aligned}$$

Hence

$$x_1(t) = \frac{\varphi_{11}}{\sqrt{\mu_1}}\eta_1(t) + \frac{\varphi_{12}}{\sqrt{\mu_2}}\eta_2(t)$$

and

$$x_2(t) = \frac{\varphi_{21}}{\sqrt{\mu_1}}\eta_1(t) + \frac{\varphi_{22}}{\sqrt{\mu_2}}\eta_2(t)$$

Notice that the solution in normal coordinates is a linear combination of the modal solutions. The terms $\frac{\varphi_{ij}}{\sqrt{\mu}}$ are just scaling factors that represent the contribution of each modal solution to the final solution. This completes modal analysis

1.1.8 Numerical solution using modal analysis

This is a numerical example that implements the above steps using a numerical values for $[K]$ and $[M]$. Let $k_1 = 1, k_2 = 2, m_1 = 1, m_2 = 3$ and let $f_1(t) = 0$ and $f_2(t) = \sin(5t)$. Let initial conditions be $x_1(0) = 0, x_1'(0) = 1, x_2(0) = 1.5, x_2'(0) = 3$, hence

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

and

$$\begin{Bmatrix} x_1'(0) \\ x_2'(0) \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ 3 \end{Bmatrix}$$

In normal coordinates, the EOM are

$$\begin{aligned} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} &= \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \end{Bmatrix} + \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} &= \begin{Bmatrix} 0 \\ \sin(5t) \end{Bmatrix} \end{aligned}$$

In this example $m_{11} = 1, m_{12} = 0, m_{21} = 0, m_{22} = 3$ and $k_{11} = 3, k_{12} = -2, k_{21} = -2, k_{22} = 2$ and $f_1(t) = 0$ and $f_2(t) = \sin(5t)$

step 2 is now applied which solves the eigenvalue problem in order to find the two natural frequencies

$$\begin{aligned} \det\left([K] - \omega^2[M]\right) &= 0 \\ \det\left(\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\right) &= 0 \\ \det\begin{bmatrix} 3-\omega^2 & -2 \\ -2 & 2-3\omega^2 \end{bmatrix} &= 0 \\ \left(3-\omega^2\right)\left(2-3\omega^2\right) - (-2)(-2) &= 0 \\ 3\omega^4 - 11\omega^2 + 2 &= 0 \end{aligned}$$

Let $\omega^2 = \lambda$ hence

$$3\lambda^2 - 11\lambda + 2 = 0$$

The solution is $\lambda_1 = 3.475$ and $\lambda_2 = 0.192$, therefore

$$\omega_1 = \sqrt{3.475} = 1.864$$

And

$$\omega_2 = \sqrt{0.192} = 0.438$$

step 3 is now applied which finds the non-normalized eigenvectors. For each natural frequency ω_1 and ω_2 the corresponding shape function is found by solving the following two sets of equations for the eigen vectors φ_1, φ_2

$$\left(\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} - \omega_1^2 \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $\omega_1 = 1.864$

$$\left(\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} - 1.864^2 \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -0.475 & -2 \\ -2 & -8.424 \end{bmatrix} \begin{Bmatrix} 1 \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This gives one equation to solve for φ_{21} (the first row equation is only used)

$$-0.475 - 2\varphi_{21} = 0$$

Hence

$$\varphi_{21} = \frac{0.475}{-2} = -0.237$$

The first eigen vector is

$$\boldsymbol{\varphi}_1 = \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.237 \end{Bmatrix}$$

Similarly for $\omega_2 = 0.438$

$$\left(\begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} - 0.438^2 \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 2.808 & -2 \\ -2 & 1.425 \end{bmatrix} \begin{Bmatrix} 1 \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This gives one equation to solve for φ_{22} (the first row equation is only used)

$$2.808 - 2\varphi_{22} = 0$$

Hence

$$\varphi_{22} = \frac{-2.808}{-2} = 1.404$$

The second eigen vector is

$$\boldsymbol{\varphi}_2 = \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.404 \end{Bmatrix}$$

Now step 4 is applied, which is mass normalization of the shape vectors (or the eigenvectors)

$$\mu_1 = \boldsymbol{\varphi}_1^T [M] \, \boldsymbol{\varphi}_1$$

Hence

$$\begin{aligned} \mu_1 &= \begin{Bmatrix} 1 \\ -0.237 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.237 \end{Bmatrix} \\ &= 1.169 \end{aligned}$$

Similarly, μ_2 is found

$$\mu_2 = \boldsymbol{\varphi}_2^T [M] \, \boldsymbol{\varphi}_2$$

Hence

$$\begin{aligned} \mu_2 &= \begin{Bmatrix} 1 \\ 1.404 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.404 \end{Bmatrix} \\ &= 6.914 \end{aligned}$$

Now that μ_1, μ_2 are found, the mass normalized eigen vectors are found. They are called $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2$

$$\begin{aligned} \boldsymbol{\Phi}_1 &= \frac{\boldsymbol{\varphi}_1}{\sqrt{\mu_1}} = \frac{\begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix}}{\sqrt{\mu_1}} = \frac{\begin{Bmatrix} 1 \\ -0.237 \end{Bmatrix}}{\sqrt{1.169}} \\ &= \begin{Bmatrix} 0.925 \\ -0.219 \end{Bmatrix} \end{aligned}$$

Similarly

$$\begin{aligned} \boldsymbol{\Phi}_2 &= \frac{\boldsymbol{\varphi}_2}{\sqrt{\mu_2}} = \frac{\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix}}{\sqrt{\mu_2}} = \frac{\begin{Bmatrix} 1 \\ 1.404 \end{Bmatrix}}{\sqrt{6.914}} \\ &= \begin{Bmatrix} 0.380 \\ 0.534 \end{Bmatrix} \end{aligned}$$

Therefore, the modal transformation matrix is

$$\begin{aligned} [\Phi] &= [\boldsymbol{\Phi}_1 \boldsymbol{\Phi}_2] \\ &= \begin{bmatrix} 0.925 & 0.380 \\ -0.219 & 0.534 \end{bmatrix} \end{aligned}$$

This result can be verified using Matlab's eig function as follows

```
K=[3 -2;-2 2]; M=[1 0;0 3];
[phi,lam]=eig(K,M)
phi =
-0.3803   -0.9249
-0.5340    0.2196
diag(sqrt(lam))
0.4380
1.8641
```

Matlab result agrees with the result obtained above. The sign difference is not important.

Now step 5 is applied. Matlab generates mass normalized eigenvectors by default.

Now that $[\Phi]$ is found, the transformation from the normal coordinates $\{x\}$ to modal coordinates, called $\{\eta\}$, is obtained

$$\{x\} = [\Phi]\{\eta\}$$
$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} 0.925 & 0.380 \\ -0.219 & 0.534 \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$$

The transformation from modal coordinates back to normal coordinates is

$$\{\eta\} = [\Phi]^{-1}\{x\}$$
$$\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \begin{bmatrix} 0.925 & 0.380 \\ -0.219 & 0.534 \end{bmatrix}^{-1} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

However, $[\Phi]^{-1} = [\Phi]^T [M]$ therefore

$$\{\eta\} = [\Phi]^T [M]\{x\}$$
$$\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \begin{bmatrix} 0.925 & 0.380 \\ -0.219 & 0.534 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$
$$= \begin{bmatrix} 0.925 & -0.657 \\ 0.38 & 1.6 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

The next step is to apply this transformation to the original equations of motion in order to decouple them.

Applying step 6 results in

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} &= [\Phi]^T \begin{Bmatrix} 0 \\ \sin(5t) \end{Bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 1.864^2 & 0 \\ 0 & 0.438^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} &= \begin{bmatrix} 0.925 & 0.380 \\ -0.219 & 0.534 \end{bmatrix}^T \begin{Bmatrix} 0 \\ \sin(5t) \end{Bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 3.47 & 0 \\ 0 & 0.192 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} &= \begin{Bmatrix} -0.219 \sin(5t) \\ 0.534 \sin(5t) \end{Bmatrix}
\end{aligned}
$$

The EOM are now decoupled and each EOM can be solved easily as follows

$$
\begin{aligned}
\eta_1''(t) + 3.47\eta_1(t) &= -0.219 \sin(5t) \\
\eta_2''(t) + 0.192\eta_2(t) &= 0.534 \sin(5t)
\end{aligned}
$$

To solve these EOM's, the initial conditions in normal coordinates must be transformed to modal coordinates using the above transformation rules

$$
\begin{aligned}
\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} &= \begin{bmatrix} 0.925 & -0.657 \\ 0.38 & 1.6 \end{bmatrix} \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} \\
\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} &= \begin{bmatrix} 0.925 & -0.657 \\ 0.38 & 1.6 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\
&= \begin{Bmatrix} -0.657 \\ 1.6 \end{Bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{Bmatrix} \eta_1'(0) \\ \eta_2'(0) \end{Bmatrix} &= \begin{bmatrix} 0.925 & -0.657 \\ 0.38 & 1.6 \end{bmatrix} \begin{Bmatrix} x_1'(0) \\ x_2'(0) \end{Bmatrix} \\
&= \begin{bmatrix} 0.925 & -0.657 \\ 0.38 & 1.6 \end{bmatrix} \begin{Bmatrix} 1.5 \\ 3 \end{Bmatrix} \\
&= \begin{Bmatrix} -0.584 \\ 5.37 \end{Bmatrix}
\end{aligned}
$$

Each of these EOM are solved using any of the standard methods. This results in solutions $\eta_1(t)$ and $\eta_2(t)$. Hence the following EOM's are solved

$$
\begin{aligned}
\eta_1''(t) + 3.47\eta_1(t) &= -0.219 \sin(5t) \\
\eta_1(0) &= -0.657 \\
\eta_1'(0) &= -0.584
\end{aligned}
$$

and also

$$
\begin{aligned}
\eta_2''(t) + 0.192\eta_2(t) &= 0.534\sin(5t) \\
\eta_2(0) &= 1.6 \\
\eta_2'(0) &= 5.37
\end{aligned}
$$

The solutions $\eta_1(t)$, $\eta_2(t)$ are found using basic methods shown in other parts of these notes. The last step is to transform back to normal coordinates by applying step 7

$$
\begin{aligned}
\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \begin{bmatrix} 0.925 & 0.380 \\ -0.219 & 0.534 \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \\
&= \begin{Bmatrix} 0.925\,\eta_1 + 0.38\eta_2 \\ 0.534\,\eta_2 - 0.219\,\eta_1 \end{Bmatrix}
\end{aligned}
$$

Hence

$$
x_1(t) = 0.925\eta_1(t) + 0.38\eta_2(t)
$$

and

$$
x_2(t) = 0.534\eta_1(t) - 0.219\eta_2(t)
$$

The above shows that the solution $x_1(t)$ and $x_2(t)$ has contributions from both nodal solutions.

1.2 Fourier series representation of a periodic function

Given a periodic function $f(t)$ with period T then its Fourier series approximation $\tilde{f}(t)$ using N terms is

$$
\begin{aligned}
\tilde{f}(t) &= \frac{1}{2}F_0 + \operatorname{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}t}\right) \\
&= \frac{1}{2}F_0 + \frac{1}{2}\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}t} + F_n^* e^{-in\frac{2\pi}{T}t} \\
&= \frac{1}{2}\sum_{n=-N}^{N} F_n e^{in\frac{2\pi}{T}t}
\end{aligned}
$$

Where

$$
\begin{aligned}
F_n &= \frac{2}{T}\int_0^T f(t)\, e^{-in\frac{2\pi}{T}t} dt \\
F_0 &= \frac{2}{T}\int_0^T f(t)\, dt
\end{aligned}
$$

Another way to write the above is to use the classical representation using cos and sin. The same coefficients (i.e. the same series) will result.

$$
\begin{aligned}
\tilde{f}(t) &= a_0 + \sum_{n=1}^{N} a_n \cos n\frac{2\pi}{T}t + \sum_{n=1}^{N} b_n \sin n\frac{2\pi}{T}t \\
a_0 &= \frac{1}{T}\int_0^T f(t)\,dt \\
a_n &= \frac{1}{T/2}\int_0^T f(t)\cos\left(n\frac{2\pi}{T}t\right)dt \\
b_n &= \frac{1}{T/2}\int_0^T f(t)\sin\left(n\frac{2\pi}{T}t\right)dt
\end{aligned}
$$

Just watch out in the above, that we divide by the full period when finding a_0 and divide by half the period for all the other coefficients. In the end, when we find $\tilde{f}(t)$ we can convert that to complex form. The complex form seems easier to use.

1.3 Generating Transfer functions for different vibration systems

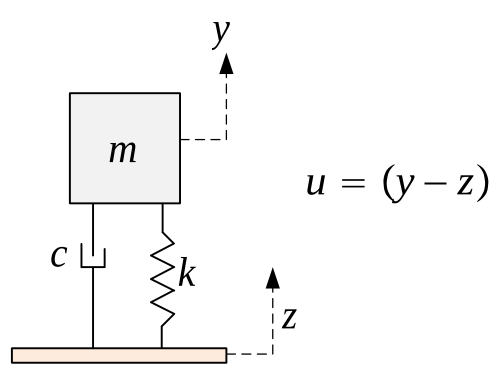


Figure 1.3: vibration model

1.3.1 Force transmissibility

Let steady state

$$
x_{ss} = \operatorname{Re}\left\{\frac{\hat{F}}{k}D(r,\zeta)\,e^{i\varpi t}\right\}
$$

Then

$$\begin{aligned}
f_{tr}(t) &= f_{spring} + f_{damper}\\
&= kx + c\dot{x}\\
&= \operatorname{Re}\left\{k\frac{\hat{F}}{k}D(r,\zeta)\,e^{i\varpi t}\right\} + \operatorname{Re}\left\{ci\varpi\frac{\hat{F}}{k}D(r,\zeta)\,e^{i\varpi t}\right\}\\
&= \operatorname{Re}\left\{\left(\hat{F} + ci\varpi\frac{\hat{F}}{k}\right)D(r,\zeta)\,e^{i\varpi t}\right\}
\end{aligned}$$

Hence

$$|f_{tr}(t)|_{\max} = \left|\hat{F}\right| |D| \sqrt{1 + c^2\frac{\varpi^2}{k^2}} = \left|\hat{F}\right| |D| \sqrt{1 + (2\zeta r)^2}$$

So TR or force transmissibility is

$$TR = \frac{|f_{tr}(t)|_{\max}}{\left|\hat{F}\right|} = |D| \sqrt{1 + (2\zeta r)^2}$$

If $r > \sqrt{2}$ then we want small ζ to reduce force transmitted to base. For $r < \sqrt{2}$, it is the other way round.

1.3.2 vibration isolation

We need transfer function between y and z. Equation of motion

$$\begin{aligned}
my'' &= -c(y' - z') - k(y - z)\\
my'' + cy' + ky &= cz' + kz
\end{aligned}$$

Let $z = \operatorname{Re}\{Ze^{i\omega t}\}, z' = \operatorname{Re}\{i\omega Ze^{i\omega t}\}$ and let $y = \operatorname{Re}\{Ye^{i\omega t}\}, y' = \operatorname{Re}\{i\omega Ye^{i\omega t}\}, y'' = \operatorname{Re}\{-\omega^2 Ye^{i\omega t}\}$, hence the above becomes

$$\begin{aligned}
m\operatorname{Re}\left\{-\omega^2 Ye^{i\omega t}\right\} + c\operatorname{Re}\left\{i\omega Ye^{i\omega t}\right\} + k\operatorname{Re}\left\{Ye^{i\omega t}\right\} &= c\operatorname{Re}\left\{i\omega Ze^{i\omega t}\right\} + k\operatorname{Re}\left\{Ze^{i\omega t}\right\}\\
Y &= \frac{ci\omega + k}{-\omega^2 m + ci\omega + k}Z\\
&= \frac{i2\zeta\omega_n m\omega + k}{-\omega^2 m + i2\zeta\omega_n m\omega + k}Z\\
&= \frac{i2\zeta\omega_n\omega + \omega_n^2}{-\omega^2 + i2\zeta\omega_n\omega + \omega_n^2}Z\\
&= \frac{i2\zeta r + 1}{(1 - r^2) + i2\zeta r}Z
\end{aligned}$$

Hence $|D(r,\zeta)| = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ and $\arg(D) = \tan^{-1}(2\zeta r) - \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ where $r = \frac{\omega}{\omega_n}$.

Hence for good vibration isolation we need $\frac{|Y|_{\max}}{|Z|}$ to be small. i.e. $|D|\sqrt{1+(2\zeta r)^2}$ to be small. **This is the same TR as for force isolation** above.

For small $|D|$, we need small ζ and small k (the small k is to make $r > \sqrt{2}$) see plot

```
In[1]:= parms = {z → 0.01};
        tf = Sqrt[1 + (2 z r)^2] / Sqrt[(1 - r^2)^2 + (2 z r)^2];
        Plot[tf /. parms, {r, 0.01, 3}, GridLines → Automatic]
```

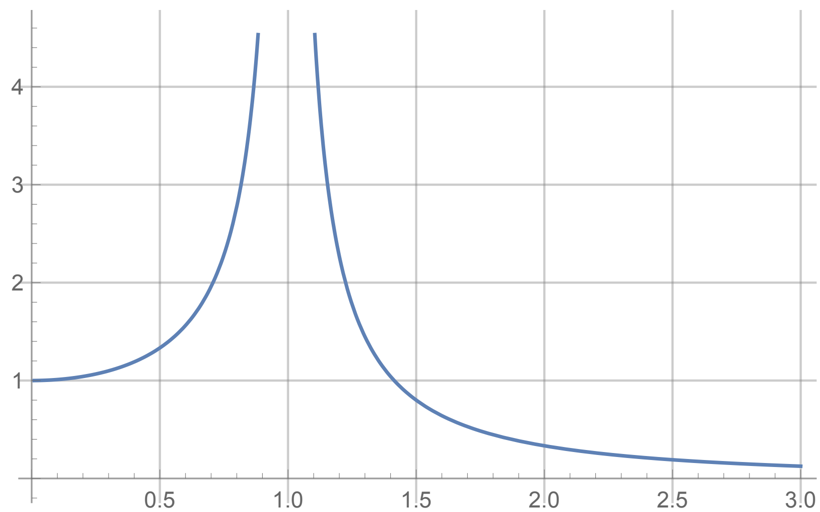


Figure 1.4: force transibility

In Matlab, the above can be plotted using

```
close all;
zeta = linspace(0.1, 0.7, 10);
r    = linspace(0, 3, 10);
D0   = @(r,z) (sqrt(1+(2*z*r).^2)./sqrt((1-r.^2).^2+(2*z*r).^2));
figure;
hold on;
for i = 1:length(zeta)
plot(r,D0(r,zeta(i)));
end
grid on;
```

1.3.3 Accelerometer

We need transfer function between u and z_a where now z_a is the amplitude of the ground acceleration. This device is used to measure base acceleration by relating it linearly to relative displacement of m to base.

Equation of motion. We use relative distance now.

$$\begin{aligned} m(u'' + z'') + cu' + ku &= 0 \\ mu'' + cu' + ku &= -mz'' \end{aligned}$$

Let $z'' = \operatorname{Re}\{Z_a e^{i\omega t}\}$. Notice we here jumped right away to the z'' itself and wrote it as $\operatorname{Re}\{Z_a e^{i\omega t}\}$ and we did not go through the steps as above starting from base motion. This is because we want the transfer function between relative motion u and acceleration of base.

Now, $u = \operatorname{Re}\{Ue^{i\omega t}\}, u' = \operatorname{Re}\{i\omega Ue^{i\omega t}\}, u'' = \operatorname{Re}\{-\omega^2 Ue^{i\omega t}\}$, hence the above becomes

$$\begin{aligned} m\operatorname{Re}\left\{-\omega^2 Ue^{i\omega t}\right\} + c\operatorname{Re}\left\{i\omega Ue^{i\omega t}\right\} + k\operatorname{Re}\left\{Ue^{i\omega t}\right\} &= -m\operatorname{Re}\left\{Z_a e^{i\omega t}\right\} \\ U &= \frac{-m}{-\omega^2 m + i\omega c + k} Z_a \\ &= \frac{-1}{-\omega^2 + i\omega 2\zeta\omega_n + \omega_n^2} Z_a \\ &= \frac{-1}{(\omega_n^2 - \omega^2) + i\omega 2\zeta\omega_n} Z_a \end{aligned}$$

Hence $|D(r,\zeta)| = \frac{-1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\omega\zeta\omega_n)^2}}$ and $\arg(D) = -180^0 - \tan^{-1}\left(\frac{2\omega\zeta\omega_n}{\omega_n^2-\omega^2}\right)$

When system is very stiff, which means ω_n very large compared to ω , then $D(r,\zeta) \approx \frac{-1}{\omega_n^2} Z_a$, hence by measuring u we estimate Z_a the amplitude of the ground acceleration since ω_n^2 is known. For accuracy, need $\omega_n > 5\omega$ at least.

1.3.4 Seismometer

Now we need to measure the base motion (not base acceleration like above). But we still use the relative displacement. Now the transfer function is between u and z where now z is the base motion amplitude.

Equation of motion. We use relative distance now.

$$\begin{aligned} m(u'' + z'') + cu' + ku &= 0 \\ mu'' + cu' + ku &= -mz'' \end{aligned}$$

Let $z = \operatorname{Re}\{Ze^{i\omega t}\}, z' = \operatorname{Re}\{i\omega Ze^{i\omega t}\}$,$z'' = \operatorname{Re}\{-\omega^2 Ze^{i\omega t}\}$,and let $u = \operatorname{Re}\{Ue^{i\omega t}\}, u' = \operatorname{Re}\{i\omega Ue^{i\omega t}\}, u'' = \operatorname{Re}\{-\omega^2 Ue^{i\omega t}\}$, hence the above becomes

Now, $u = \operatorname{Re}\{Ue^{i\omega t}\}, u' = \operatorname{Re}\{i\omega Ue^{i\omega t}\}, u'' = \operatorname{Re}\{-\omega^2 Ue^{i\omega t}\}$, hence the above becomes

$$\begin{aligned} m\operatorname{Re}\left\{-\omega^2 Ue^{i\omega t}\right\} + c\operatorname{Re}\left\{i\omega Ue^{i\omega t}\right\} + k\operatorname{Re}\left\{Ue^{i\omega t}\right\} &= -m\operatorname{Re}\left\{-\omega^2 Ze^{i\omega t}\right\} \\ U &= \frac{m\omega^2}{-\omega^2 m + i\omega c + k} Z \\ &= \frac{\omega^2}{-\omega^2 + i\omega 2\zeta\omega_n + \omega_n^2} Z \\ &= \frac{r^2}{(1-r^2) + i2\zeta r} Z \end{aligned}$$

Hence $|D(r,\zeta)| = \frac{r^2}{\sqrt{(1-r^2)+i2\zeta r}}$ and $\arg(D) = -\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$

Now if r is very large, which happens when $\omega_n \ll \omega$, then $\frac{1}{(1-r^2)+i2\zeta r} \Rightarrow \frac{1}{-r^2}$ since r^2 is the dominant factor. Therefore $U = \frac{r^2}{(1-r^2)+i2\zeta r} Z_a$ now becomes $U \simeq -Z_a$ therefore measuring the relative displacement U gives linear estimate of the ground motion. However, this device requires that ω_n be much smaller than ω, which means that m has to be massive. So this device is heavy compared to accelerometer.

1.3.5 Summary of vibration transfer functions

For good isolation of mass from ground motion, rule of thumb: Make damping low, and stiffness low (soft spring).

Isolate base from force. transmitted by machine

Equation used $f_{\text{tr}}(t) = f_{\text{spring}} + f_{\text{damper}}$

Transfer function $\frac{|f_{\text{tr}}(t)|_{\max}}{|\hat{F}|} = |D| \sqrt{1 + (2\zeta r)^2}$

Isolate machine from motion of base

Equation used. Use absolute mass position

$$my'' + cy' + ky = cz' + kz$$

Transfer function

$$\frac{|Y|_{\max}}{|Z|} = |D| \sqrt{1 + (2\zeta r)^2}$$

Accelerometer: Measure base acc. using relative displacement

Equation used. Use relative mass position

$$mu'' + cu' + ku = -mz''$$

Transfer function

$$\begin{aligned} U &= \frac{-1}{(\omega_n^2 - \omega^2) + i\omega 2\zeta\omega_n} Z_a \Rightarrow |D(r,\zeta)| \\ &= \frac{-1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\omega\zeta\omega_n)^2}} \end{aligned}$$

Seismometer: Measure base motion using relative displacement

Equation used. Use relative mass position

$$mu'' + cu' + ku = -mz''$$

Transfer function

$$\begin{aligned} U &= \frac{r^2}{(1 - r^2) + i2\zeta r} Z \to |D(r,\zeta)| \\ &= \frac{r^2}{\sqrt{(1 - r^2) + i2\zeta r}} \end{aligned}$$

1.4 Solution of Vibration equation of motion for different loading

1.4.1 common definitions

These definitions are used throughout the derivations below.

$$
\begin{aligned}
\xi &= \frac{c}{c_r} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m} \\
u_{st} &= \frac{F}{k} \text{ static deflection} \\
\omega_n &= \sqrt{\frac{k}{m}} \\
\omega_D &= \omega_n\sqrt{1-\xi^2}\text{note: not defined for } \xi > 1 \text{ since becomes complex} \\
r &= \frac{\varpi}{\omega_n} \\
T_d &= \frac{2\pi}{\omega_d} \text{ damped period of oscillation} \\
\tau &= \left\{\frac{-1}{\lambda_1}, \frac{-1}{\lambda 2}\right\} \text{ time constants where } \lambda_i \text{ are roots of characteristic equation} \\
\beta &= \frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2r\xi\right)^2}} \text{ magnification factor} \\
\beta_{\max} \text{ when } r &= \sqrt{1-2\xi^2} \\
\beta_{\max} &= \frac{1}{2\xi\sqrt{1-\xi^2}} \\
\frac{y_n}{y_{n+1}} &= e^{\frac{\zeta\omega_n 2\pi}{\omega_D}} \overset{\text{small damping}}{\Rightarrow} e^{\frac{\zeta 2\pi}{\sqrt{1-\zeta^2}}} \Rightarrow e^{\zeta 2\pi} \\
\ln\left(\frac{y_n}{y_{n+1}}\right) &= \zeta 2\pi \\
\frac{1}{M}\ln\left(\frac{y_n}{y_{n+M}}\right) &= \zeta 2\pi
\end{aligned}
$$

1.4.2 Harmonic loading $mu'' + cu' + ku = F \sin \varpi t$

1.4.2.1 Undamped Harmonic loading

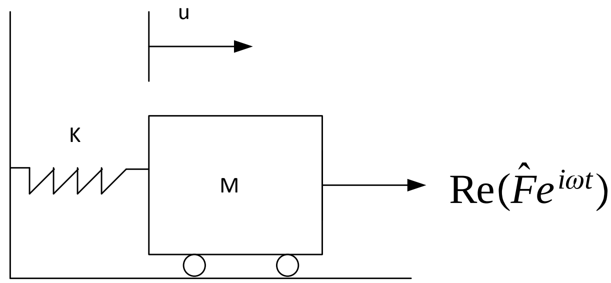


Figure 1.5: single degree no damping forced

$$mu'' + ku = F \sin \varpi t$$

Since there is no damping in the system, then there is no steady state solution. In other words, the particular solution is not the same as the steady state solution in this case. We need to find the particular solution using method on undetermined coefficients.

Let $u = u_h + u_p$. By guessing that $u_p = c_1 \sin \varpi t$ then we find the solution to be

$$u = A \cos \omega_n t + B \sin \omega_n t + \frac{F}{k} \frac{1}{1 - r^2} \sin \varpi t$$

Applying initial conditions is always done on the full solution. Applying initial conditions gives

$$u(0) = A$$

$$\begin{aligned} u'(t) &= -A\omega \sin \omega_n t + B\omega \cos \omega_n t + \varpi \frac{F}{k} \frac{1}{1 - r^2} \cos \varpi t \\ u'(0) &= B\omega_n + \varpi \frac{F}{k} \frac{1}{1 - r^2} \\ B &= \frac{u'(0)}{\omega_n} - \frac{F}{k} \frac{r}{1 - r^2} \end{aligned}$$

Where $r = \frac{\varpi}{\omega_n}$

The complete solution is

$$u(t) = u(0) \cos \omega_n t + \left(\frac{u'(0)}{\omega_n} - \frac{F}{k} \frac{r}{1 - r^2} \right) \sin \omega_n t + \frac{F}{k} \frac{1}{1 - r^2} \sin \varpi t \tag{1}$$

Example: Given force $f(t) = 3\sin(5t)$ then $\varpi = 5$ rad/sec, and $\hat{F} = 3$. Let $m = 1, k = 1$, then $\omega_n = 1$ rad/sec. Hence $r = 5$, Let initial conditions be zero, then

$$\begin{aligned} u &= \left(-3\frac{5}{1-5^2}\right)\sin t + 3\frac{1}{1-5^2}\sin 5t \\ &= 0.625\,\sin t - 0.125\,\sin 5.0t \end{aligned}$$

1.4.2.1.1 Resonance forced vibration

When $\varpi \approx \omega$ we obtain resonance since $r \to 1$ in the solution given in Eq (1) above and as written the solution can not be used for analysis. To obtain a solution for resonance some calculus is needed. Eq (1) is written as

$$u(t) = u(0)\cos\omega t + \left(\frac{u'(0)}{\omega} - \frac{F}{k}\frac{\omega\varpi}{\omega^2 - \varpi^2}\right)\sin\omega t + \frac{F}{k}\frac{\omega^2}{\omega^2 - \varpi^2}\sin\varpi t \tag{1A}$$

When $\varpi \approx \omega$ but less than ω, letting

$$\omega - \varpi = 2\Delta \tag{2}$$

where Δ is very small positive quantity. And since $\varpi \approx \omega$ let

$$\omega + \varpi \approx 2\varpi \tag{3}$$

Multiplying Eq (2) and (3) gives

$$\omega^2 - \varpi^2 = 4\Delta\varpi \tag{4}$$

Eq (1A) can now be written in terms of Eqs (2,3) as

$$\begin{aligned} u(t) &= u(0)\cos\omega t + \left(\frac{u'(0)}{\omega} - \frac{F}{k}\frac{\omega\varpi}{4\Delta\varpi}\right)\sin\omega t + \frac{F}{k}\frac{\omega^2}{4\Delta\varpi}\sin\varpi t \\ &= u(0)\cos\omega t + \left(\frac{v_0}{\omega} - \frac{F}{k}\frac{\omega}{4\Delta}\right)\sin\omega t + \frac{F}{k}\frac{\omega^2}{4\Delta\varpi}\sin\varpi t \end{aligned}$$

Since $\varpi \approx \omega$ the above becomes

$$\begin{aligned} u(t) &= u(0)\cos\omega t + \left(\frac{u'(0)}{\omega} - \frac{F}{k}\frac{\omega}{4\Delta}\right)\sin\omega t + \frac{F}{k}\frac{\omega}{4\Delta}\sin\varpi t \\ &= u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t + \frac{F}{k}\frac{\omega}{4\Delta}(\sin\varpi t - \sin\omega t) \end{aligned}$$

Using $\sin\varpi t - \sin\omega t = 2\sin\left(\frac{\varpi-\omega}{2}t\right)\cos\left(\frac{\varpi+\omega}{2}t\right)$ the above becomes

$$u(t) = u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t + \frac{F}{k}\frac{\omega}{2\Delta}\left(\sin\left(\frac{\varpi-\omega}{2}t\right)\cos\left(\frac{\varpi+\omega}{2}t\right)\right)$$

From Eqs (2,3) the above can be written as

$$u(t) = u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t + \frac{F}{k}\frac{\omega}{2\Delta}(\sin(-\Delta t)\cos(\varpi t))$$

Since $\lim_{\Delta\to 0}\frac{\sin(\Delta t)}{\Delta} = t$ the above becomes

$$u(t) = u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t - \frac{F}{k}\frac{\omega t}{2}\cos(\omega t)$$

This is the solution to use for resonance.

1.4.2.2 Underdamped harmonic loading $c < c_r, \xi < 1$

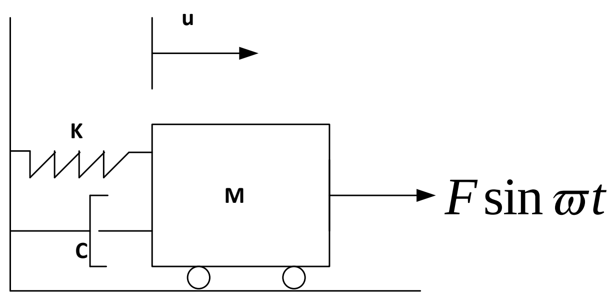


Figure 1.6: single degree damping forced

$$mu'' + cu' + ku = F\sin\varpi t$$
$$u'' + 2\xi\omega u' + \omega^2 u = \frac{F}{m}\sin\varpi t$$

The solution is

$$u(t) = u_h + u_p$$

where

$$u_h(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t)$$

and

$$u_p(t) = \frac{F}{\sqrt{(k - m\varpi)^2 + (c\varpi)^2}}\sin(\varpi t - \theta)$$

where

$$\tan\theta = \frac{c\varpi}{k - m\varpi^2} = \frac{2\xi r}{1 - r^2}$$

Very important note here in the calculations of $\tan\theta$ above, one should be careful on the sign of the denominator. When the forcing frequency $\varpi > \omega$ the denominator will become negative (the case of $\varpi = \omega$ is resonance and is handled separately). Therefore, one should use arctan that takes care of which quadrant the angle is. For example, in Mathematica use

```
ArcTan[1 - r^2, 2 Zeta r]]
```

and in Matlab use

```
atan2(2 Zeta r,1 - r^2)
```

Otherwise, wrong solution will result when $\varpi > \omega$ The full solution is

$$u(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}\sin\left(\varpi t - \theta\right) \tag{1}$$

Applying initial conditions gives

$$\begin{aligned}
A &= u(0) + \frac{F}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}\sin\theta \\
B &= \frac{u'(0)}{\omega_d} + \frac{u(0)\,\xi\omega}{\omega_d} + \frac{F}{k}\frac{1}{\omega_d\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}(\xi\omega\sin\theta - \varpi\cos\theta)
\end{aligned}$$

Another form of these equations is given as follows

$$u_p = \frac{p_0}{k}\frac{1}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}\left(\left(1-r^2\right)\sin\varpi t - 2\zeta r\cos\varpi t\right)$$

Hence the full solution is

$$u(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{k}\frac{1}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}\left(\left(1-r^2\right)\sin\varpi t - 2\zeta r\cos\varpi t\right) \tag{1.1}$$

Applying initial conditions now gives

$$\begin{aligned}
A &= u(0) + \frac{2Fr\xi}{k}\frac{1}{\left(1-r^2\right)^2 + \left(2\xi r\right)^2} \\
B &= \frac{u'(0)}{\omega_d} + \frac{u(0)\,\xi\omega_n}{\omega_d} - \frac{F(1-r^2)}{k\omega_d}\frac{\varpi}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2} + \frac{2Fr\zeta}{k\omega_d}\frac{\omega_n}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}
\end{aligned}$$

The above 2 sets of equations are equivalent. One uses the phase angle explicitly and the second ones do not. Also, the above assume the force is $F\sin\varpi t$ and not $F\cos\varpi t$. If the force is $F\cos\varpi t$ then in Eq 1.1 above, the term reverse places as in

$$u(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{k}\frac{1}{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}\left(\left(1-r^2\right)\cos\varpi t - 2\zeta r\sin\varpi t\right)$$

Applying initial conditions now gives

$$A = u(0) + \frac{F}{k}\frac{(1-r^2)}{(1-r^2)^2 + (2\xi r)^2}$$
$$B = \frac{u'(0)}{\omega_d} + \frac{u(0)\,\xi\omega_n}{\omega_d} + \frac{2Fr\zeta}{k\omega_d}\frac{\varpi}{(1-r^2)^2 + (2\zeta r)^2} - \frac{F(1-r^2)}{k\omega_d}\frac{\omega_n}{(1-r^2)^2 + (2\zeta r)^2}$$

When a system is damped, the problem with the divide by zero when $r = 1$ does not occur here as was the case with undamped system, since when when $\varpi \approx \omega$ or $r = 1$, the solution in Eq (1) becomes

$$u(t) = e^{-\xi\omega t}\left(\left(u(0) + \frac{F}{k}\frac{1}{2\xi}\sin\theta\right)\cos\omega_d t + \left(\frac{u'(0)}{\omega_d} + \frac{u(0)\,\xi\omega}{\omega_d} + \frac{F}{k}\frac{1}{2\omega_d\xi}(\xi\omega\sin\theta - \varpi\cos\theta)\right)\sin\omega_d t\right) + \frac{F}{k}\frac{1}{2}\sin(\varpi t - \theta)$$

and the problem with the denominator going to zero does not show up here. The amplitude when steady state response is maximum can be found as follows. The amplitude of steady state motion is $\frac{F}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$. This is maximum when the magnification factor $\beta = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$ is maximum or when $\sqrt{(1-r^2)^2 + (2\xi r)^2}$ or $\sqrt{\left(1-\left(\frac{\varpi}{\omega}\right)^2\right)^2 + \left(2\xi\frac{\varpi}{\omega}\right)^2}$ is minimum. Taking derivative w.r.t. ϖ and equating the result to zero and solving for ϖ gives

$$\varpi = \omega\sqrt{1-2\xi^2}$$

We are looking for positive ϖ, hence when $\varpi = \omega\sqrt{1-2\xi^2}$ the under-damped response is maximum.

1.4.2.3 critically damping harmonic loading $\xi = \frac{c}{c_r} = 1$

The solution is

$$u(t) = u_h + u_p$$

Where $u_h = (A + Bt)\,e^{-\omega t}$ and $u_p = \frac{F}{k}\frac{1}{\sqrt{(1-r^2)^2+(2r)^2}}\sin(\varpi t - \theta)$ where $\tan\theta = \frac{2r}{1-r^2}$(making sure to use correct arctan definition). Hence

$$u(t) = (A + Bt)\,e^{-\omega t} + \frac{F}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2r)^2}}\sin(\varpi t - \theta)$$

where A, B are found from initial conditions

$$A = u(0) + \frac{F}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2r)^2}}\sin\theta$$
$$B = u'(0) + u(0)\,\omega + \frac{F}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2r)^2}}(\omega\sin\theta - \varpi\cos\theta)$$

1.4.2.4 overdamped harmonic loading $\xi = \frac{c}{c_r} > 1$

The solution is

$$u(t) = u_h + u_p$$

where

$$u_h(t) = Ae^{p_1 t} + Be^{p_2 t}$$

and

$$u_p(t) = \frac{F}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}\sin\left(\varpi t - \theta\right)$$

hence

$$u = Ae^{p_1 t} + Be^{p_2 t} + \frac{F}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}}\sin\left(\varpi t - \theta\right)$$

where $\tan\theta = \frac{2\xi r}{1-r^2}$ and

$$\begin{aligned}
p_1 &= -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega\xi + \omega_n\sqrt{\xi^2-1}\\
p_2 &= -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega\xi - \omega_n\sqrt{\xi^2-1}
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
u(t) &= Ae^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t} + Be^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t} + \frac{F}{k}\beta\sin\left(\varpi t-\theta\right)\\
A &= \frac{u'(0) + u(0)\,\omega\xi + u(0)\,\omega\sqrt{\xi^2-1} + \frac{F}{k}\beta\left(\left(\xi+\sqrt{\xi^2-1}\right)\omega\sin\theta - \varpi\cos\theta\right)}{2\omega\sqrt{\xi^2-1}}\\
B &= -\frac{u'(0) + u(0)\,\omega\xi - u(0)\,\omega\sqrt{\xi^2-1} + \frac{F}{k}\beta\left(\left(\xi-\sqrt{\xi^2-1}\right)\omega\sin\theta - \varpi\cos\theta\right)}{2\omega\sqrt{\xi^2-1}}
\end{aligned}$$

1.4.2.5 Solution using frequency approach to harmonic loading

$$\begin{aligned}
my'' + cy' + ky &= \operatorname{Re}\left(\hat{F}e^{i\varpi t}\right)\\
x &= \operatorname{Re}\left\{\hat{X}e^{i\varpi t}\right\}\\
\hat{X} &= \frac{\hat{F}}{k}D(r,\zeta)\\
D(r,\zeta) &= \frac{1}{\left(1-r^2\right) + 2i\zeta r}\\
x &= \operatorname{Re}\left\{\frac{\hat{F}}{k}\left|D(r,\zeta)\right| e^{i(\varpi t-\theta)}\right\}\\
\theta &= \tan^{-1}\frac{2\zeta r}{1-r^2}
\end{aligned}$$

Let load be harmonic and represented in general as $\operatorname{Re}\left(\hat{F}e^{i\varpi t}\right)$ where $\hat{F}$ is the complex amplitude of the force.

Hence system is represented by

$$\begin{aligned} my'' + cy' + ky &= \operatorname{Re}\left(\hat{F}e^{i\varpi t}\right) \\ y'' + 2\zeta\omega_n y' + \omega_n^2 y &= \operatorname{Re}\left(\frac{\hat{F}}{m}e^{i\varpi t}\right) \end{aligned}$$

Let $y = \operatorname{Re}\left(\hat{Y}e^{i\varpi t}\right)$ Hence $y' = \operatorname{Re}\left(i\varpi\hat{Y}e^{i\varpi t}\right), y'' = \operatorname{Re}\left(-\varpi^2\hat{Y}e^{i\varpi t}\right)$, therefore the differential equation becomes

$$\begin{aligned} \operatorname{Re}\left(-\varpi^2\hat{Y}e^{i\varpi t}\right) + 2\zeta\omega_n \operatorname{Re}\left(i\varpi\hat{Y}e^{i\varpi t}\right) + \omega_n^2 \operatorname{Re}\left(\hat{Y}e^{i\varpi t}\right) &= \operatorname{Re}\left(\frac{\hat{F}}{m}e^{i\varpi t}\right) \\ \left(-\varpi^2 + 2\zeta\omega_n i\varpi + \omega_n^2\right)\hat{Y} &= \frac{\hat{F}}{m} \\ \hat{Y} &= \frac{\frac{\hat{F}}{m}}{\left(-\varpi^2 + 2\zeta\omega_n i\varpi + \omega_n^2\right)} \end{aligned}$$

Dividing numerator and denominator ω_n^2 gives

$$\hat{Y} = \frac{\hat{F}}{k}\frac{1}{(1-r^2) + i2\zeta r}$$

Where $r = \frac{\varpi}{\omega_n}$, hence the response is

$$y = \operatorname{Re}\left(\frac{\hat{F}}{k}\frac{1}{(1-r^2) + i2\zeta r}e^{i\varpi t}\right)$$

Therefore, the phase of the response is

$$\arg(y) = \arg\left(\hat{F}\right) - \tan^{-1}\left(\frac{2\zeta r}{(1-r^2)}\right) + \varpi t$$

Hence at $t = 0$ the phase of the response will be

$$\arg(y) = \arg\left(\hat{F}\right) - \tan^{-1}\left(\frac{2\zeta r}{(1-r^2)}\right)$$

So when $\hat{F}$ is real, the phase of the response is simply $-\tan^{-1}\left(\frac{2\zeta r}{(1-r^2)}\right)$

Undamped case

When $\zeta = 0$ the above becomes

$$
\begin{aligned}
y &= \operatorname{Re}\left(\frac{\hat{F}}{k}\frac{1}{\left(1-r^2\right)}e^{i\varpi t}\right) \\
&= \frac{\left|\hat{F}\right|}{k}\frac{1}{\left(1-r^2\right)}\cos\left(\varpi t + \arg\left(\hat{F}\right)\right)
\end{aligned}
$$

For real force this becomes

$$
y = \frac{F}{k}\frac{1}{\left(1-r^2\right)}\cos\left(\varpi t\right)
$$

The magnitude $\left|\hat{Y}\right| = \frac{F}{k}\frac{1}{(1-r^2)}$ and phase zero.

damped cases

$\zeta > 0$

$$
y = \operatorname{Re}\left(\frac{\hat{F}}{k}\frac{1}{\left(1-r^2\right)+i2\zeta r}e^{i\varpi t}\right)
$$

$$
\begin{aligned}
\left|\hat{Y}\right| &= \frac{\left|\hat{F}\right|}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}} \\
\arg\left(\hat{Y}\right) &= \phi = \arg\left(\hat{F}\right) - \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) + \varpi t
\end{aligned}
$$

Hence for real force and at $t = 0$ the phase of displacement is

$$
-\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)
$$

lag behind the load.

When $r < 1$ then ϕ goes from 0 to -90^0 Therefore phase of displacement is 0 to -90^0 behind force. The minus sign at the front was added since the complex number is in the denominator. Hence the response will always be lagging in phase relative for load.

For $r > 1$

Now $1 - r^2$ is negative, hence the phase will be from $-90°$ to $-180°$

When $r = 1$

$$
y = \operatorname{Re}\left(\frac{\hat{F}}{k}\frac{1}{i2\zeta}e^{i\varpi t}\right)
$$

$$\left|\hat{Y}\right| = \frac{\left|\hat{F}\right|}{k}\frac{1}{2\zeta}$$
$$\arg\left(\hat{Y}\right) = -90^{\circ}$$

Now phase is -90°

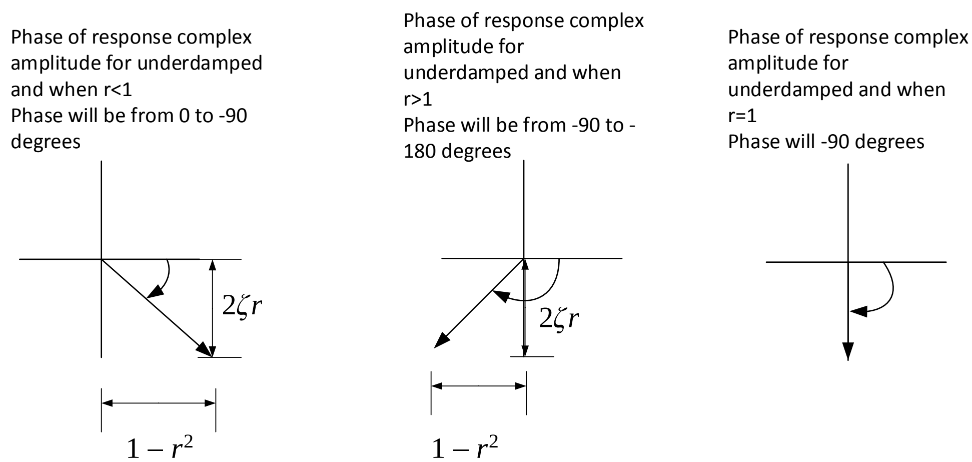


Figure 1.7: steady state

Examples. System has $\zeta = 0.1$ and $m = 1, k = 1$ subjected for force $3\cos(0.5t)$ find the steady state solution.

Answer $y(t) = \operatorname{Re}\left(\hat{Y}e^{i\varpi t}\right)$, $\omega_n = \sqrt{\frac{k}{m}} = 1$ rad/sec, hence $r = 0.5$ under the response is

$$\begin{aligned}
y(t) &= \operatorname{Re}\left(\left|\hat{Y}\right| e^{i\varpi t}\right)\\
&= \left|\hat{Y}\right| \cos(\varpi t)\\
&= \frac{F}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}\cos\left(.5t-\tan^{-1}\left(\frac{2(0.1)\,0.5}{1-0.5^2}\right)\right)\\
&= 3\frac{1}{\sqrt{\left(1-0.5^2\right)^2+\left(2\,(0.1)\,0.5\right)^2}}\cos\left(.5t-7.59^{\circ}\right)\\
&= 3.9649\cos\left(.5t-7.59^{\circ}\right)
\end{aligned}$$

```
In[4]:= Plot[{3 Cos[0.5 t], 3.9649 Cos[.5 t - 7.59 Degree]}, {t, 0, 20}, PlotStyle → {Dashed, Red},
          PlotLegends → {"load", "response"}]
```

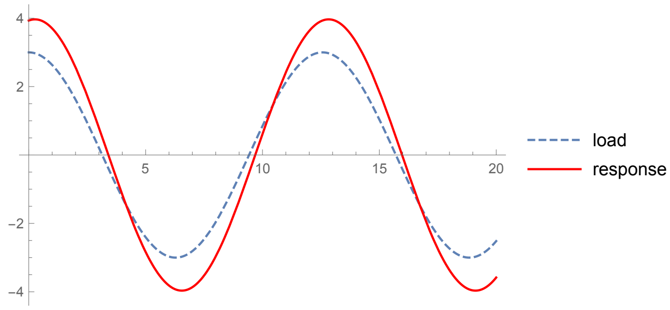


Figure 1.8: steady state plot

The equation of motion can also be written as $u'' + 2\zeta\omega u' + \omega^2 u = \frac{F}{m}\sin\varpi t$.

The following table gives the solutions for initial conditions are $u(0)$ and $u'(0)$ under all damping conditions. The roots shown are the roots of the quadratic characteristic equation $\lambda^2 + 2\zeta\omega\lambda + \omega^2\lambda = 0$. Special handling is needed to obtain the solution of the differential equation for the case of $\zeta = 0$ and $\varpi = \omega$ as described in the detailed section below.

1.4.2.6 Summary table

$\zeta=0$	roots $\begin{cases}-i\omega\\+i\omega\end{cases}$ $u(t)\begin{cases}\varpi=\omega\rightarrow u(0)\cos\varpi t+\frac{u'(0)}{\varpi}\sin\varpi t-\frac{F}{K}\frac{\varpi t}{2}\cos\left(\varpi t\right)\\ \varpi\neq\omega\rightarrow u(0)\cos\omega t+\left(\frac{u'(0)}{\omega}-\frac{F}{K}\frac{r}{1-r^2}\right)\sin\omega t+\frac{F}{K}\frac{1}{1-r^2}\sin\varpi t\end{cases}$
$\zeta<1$	roots $\begin{cases}-\xi\omega+i\omega_n\sqrt{1-\xi^2}\\-\xi\omega-i\omega_n\sqrt{1-\xi^2}\end{cases}$ $u(t)=e^{-\xi\omega t}(A\cos\omega_d t+B\sin\omega_d t)+\frac{F}{K}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin\left(\varpi t-\theta\right)$ $A=u_0+\frac{F}{K}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin\theta$ $B=\frac{v_0}{\omega_d}+\frac{u_0\xi\omega}{\omega_d}+\frac{F}{K}\frac{1}{\omega_d\sqrt{(1-r^2)^2+(2\xi r)^2}}(\xi\omega\sin\theta-\varpi\cos\theta)$
$\zeta=1$	roots $\begin{cases}-\omega\\-\omega\end{cases}$ $u(t)=(A+Bt)\,e^{-\omega t}+\frac{F}{K}\frac{1}{\sqrt{(1-r^2)^2+(2r)^2}}\sin\left(\varpi t-\theta\right)$ $A=u_0+\frac{F}{K}\frac{1}{\sqrt{(1-r^2)^2+(2r)^2}}\sin\theta$ $B=v_0+u_0\omega+\frac{F/k}{\sqrt{(1-r^2)^2+(2r)^2}}(\omega\sin\theta-\varpi\cos\theta)$
$\zeta>1$	roots $\begin{cases}-\omega_n\xi+\omega_n\sqrt{\xi^2-1}\\-\omega_n\xi-\omega_n\sqrt{\xi^2-1}\end{cases}$ $u(t)=Ae^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t}+Be^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}+\frac{F}{K}\beta\sin\left(\varpi t-\theta\right)$ $A=\frac{v_0+u_0\omega\xi+u_0\omega\sqrt{\xi^2-1}+\frac{F}{K}\beta\left(\left(\xi+\sqrt{\xi^2-1}\right)\omega\sin\theta-\varpi\cos\theta\right)}{2\omega\sqrt{\xi^2-1}}$ $B=-\frac{v_0+u_0\omega\xi-u_0\omega\sqrt{\xi^2-1}+\frac{F}{K}\beta\left(\left(\xi-\sqrt{\xi^2-1}\right)\omega\sin\theta-\varpi\cos\theta\right)}{2\omega\sqrt{\xi^2-1}}$

1.4.3 constant loading $mu''+cu'+ku=F$

1.4.3.1 Undamped Constant loading case $\zeta=0$

$$mu''+ku=F$$
$$u''+\omega^2u=F$$
$$u(t)=u_h+u_p$$

Where $u_h=A\cos\omega t+B\sin\omega t$ and $u_p=\frac{F}{k}$, the solution is

$$u(t)=A\cos\omega t+B\sin\omega t+\frac{F}{k}$$

Applying initial conditions gives

$$\begin{aligned}A &= u(0) - \frac{F}{k}\\ B &= \frac{u'(0)}{\omega}\end{aligned}$$

And complete solution is

$$u(t) = \frac{F}{k} + \left(u(0) - \frac{F}{k}\right)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t$$

1.4.3.2 underdamped constant loading $\zeta < 1$

The general solution is

$$u(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{k}$$

From initial conditions

$$\begin{aligned}A &= u(0) - \frac{F}{k}\\ B &= \frac{u'(0) + u(0)\,\xi\omega - \frac{F}{k}\xi\omega}{\omega_d}\end{aligned}$$

Hence the solution is

$$u(t) = e^{-\xi\omega t}\left(\left(u(0) - \frac{F}{k}\right)\cos\omega_d t + \left(\frac{u'(0) + u(0)\,\xi\omega - \frac{F}{k}\xi\omega}{\omega_d}\right)\sin\omega_d t\right) + \frac{F}{k}$$

1.4.3.3 Critical damping constant loading $\zeta = 1$

The general solution is

$$u(t) = (A + Bt)e^{-\omega t} + \frac{F}{k}$$

Where from initial conditions

$$\begin{aligned}A &= u(0) - \frac{F}{k}\\ B &= u'(0) + u(0)\omega - \frac{F}{k}\omega\end{aligned}$$

1.4.3.4 Over-damped constant loading $\zeta > 0$

The solution is

$$u(t) = Ae^{p_1 t} + Be^{p_2 t} + \frac{F}{k}$$

Where now

$$B = \frac{\frac{F}{k}p_1 - u_0 p_1 + u'(0)}{(p_2 - p_1)}$$
$$A = u(0) - \frac{F}{k} - B$$

Hence the solution is

$$u(t) = Ae^{p_1 t} + Be^{p_2 t} + \frac{F}{k}$$

Where

$$p_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega\xi + \omega_n\sqrt{\xi^2 - 1}$$
$$p_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega\xi - \omega_n\sqrt{\xi^2 - 1}$$

1.4.3.5 Summary table for constant loading solutions

$\zeta = 0$	roots$\begin{cases} -i\omega \\ +i\omega \end{cases}$ $u(t) = \left(u_0 - \frac{F}{k}\right)\cos\omega t + \frac{v_0}{\omega}\sin\omega t + \frac{F}{k}$
$\zeta < 1$	roots$\begin{cases} -\xi\omega + i\omega_n\sqrt{1-\xi^2} \\ -\xi\omega - i\omega_n\sqrt{1-\xi^2} \end{cases}$ $u(t) = e^{-\xi\omega t}\left(\left(u_0 - \frac{F}{k}\right)\cos\omega_d t + \left(\frac{v_0 + u_0\xi\omega - \frac{F}{k}\xi\omega}{\omega_d}\right)\sin\omega_d t\right) + \frac{F}{k}$
$\zeta = 1$	roots $\begin{cases} -\omega \\ -\omega \end{cases}$ $u(t) = \left(\left(u_0 - \frac{F}{k}\right) + \left(v_0 + u_0\omega - \frac{F}{k}\omega\right)t\right)e^{-\omega t} + \frac{F}{k}$
$\zeta > 1$	roots $\begin{cases} -\omega_n\xi + \omega_n\sqrt{\xi^2-1} \\ -\omega_n\xi - \omega_n\sqrt{\xi^2-1} \end{cases}$ $B = \frac{\frac{F}{k}p_1 - u_0 p_1 + v_0}{(p_2 - p_1)}$ $A = u_0 - \frac{F}{k} - B$ $p_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega_n\xi + \omega_n\sqrt{\xi^2-1}$ $p_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega_n\xi - \omega_n\sqrt{\xi^2-1}$ $u(t) = Ae^{p_1 t} + Be^{p_2 t} + \frac{F}{k}$

1.4.4 No loading (free vibration) $mu'' + cu' + ku = 0$

1.4.4.1 Undamped free vibration

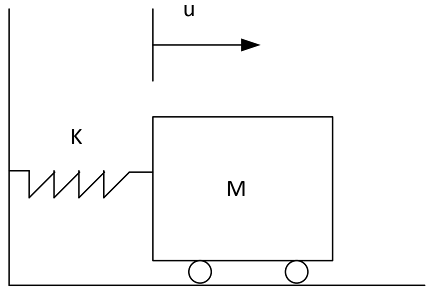


Figure 1.9: single degree mass no damping

$$mu'' + ku = 0$$

$$u'' + \omega^2 u = 0$$

The solution is

$$u(t) = u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t$$

1.4.4.2 under-damped free vibration $c < c_r, \xi < 1$

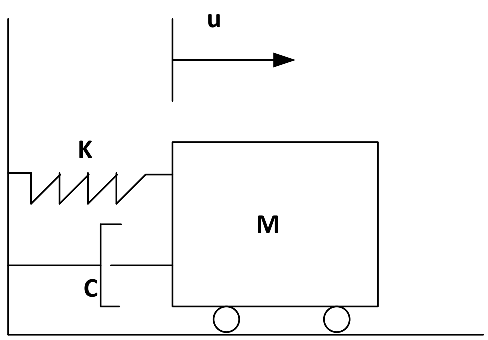


Figure 1.10: single degree mass damping

$$mu'' + cu' + ku = 0$$

$$u'' + 2\xi\omega u' + \omega^2 u = 0$$

The solution is

$$u = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t)$$

Applying initial conditions gives $A = u(0)$ and $B = \frac{u'(0)+u(0)\xi\omega}{\omega_d}$. Therefore the solution becomes

$$u(t) = e^{-\xi\omega t}\left(u(0)\cos\omega_d t + \frac{u'(0)+u(0)\,\xi\omega}{\omega_d}\sin\omega_d t\right)$$

1.4.4.3 critically damped free vibration $\xi = \frac{c}{c_r} = 1$

The solution is

$$\begin{aligned} u(t) &= (A+Bt)\,e^{-\left(\frac{c_r}{2m}\right)t} \\ &= (A+Bt)\,e^{-\omega t} \end{aligned}$$

where A, B are found from initial conditions $A = u(0)$,$B = u'(0) + u(0)\,\omega$, hence

$$u(t) = (u(0) + (u'(0) + u(0)\,\omega)\,t)\,e^{-\omega t}$$

1.4.4.4 over-damped free vibration $\xi = \frac{c}{c_r} > 1$

The solution is

$$u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

where A, B are found from initial conditions.

$$\begin{aligned} A &= \frac{u'(0) - u(0)\,\lambda_2}{2\omega\sqrt{\xi^2-1}} \\ B &= \frac{-u'(0) + u(0)\,\lambda_1}{2\omega\sqrt{\xi^2-1}} \end{aligned}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\begin{aligned} \lambda_1 &= -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\xi\omega + \omega\sqrt{\xi^2-1} \\ \lambda_2 &= -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\xi\omega - \omega\sqrt{\xi^2-1} \end{aligned}$$

1.4.4.5 Summary table for free vibration solutions

$\zeta = 0$ $u'' + \omega^2 u = 0$	$\text{roots}\begin{cases} -i\omega \\ +i\omega \end{cases}$ $u(t) = u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t$ $\begin{cases} u(t) = A\cos(\omega t - \phi) \\ A = \sqrt{u^2(0) + \left(\frac{u'(0)}{\omega}\right)^2} \\ \phi = \tan^{-1}\left(\frac{u'(0)/\omega}{u(0)}\right) \end{cases}$
$\zeta < 1$	$\text{roots}\begin{cases} -\xi\omega + i\omega\sqrt{1-\xi^2} \\ -\xi\omega - i\omega\sqrt{1-\xi^2} \end{cases}$ $u(t) = e^{-\xi\omega t}\left(u(0)\cos\omega_d t + \frac{u'(0)+u(0)\xi\omega}{\omega_d}\sin\omega_d t\right)$
$\zeta = 1$	$\text{roots}\begin{cases} -\omega \\ -\omega \end{cases}$ $u(t) = (u(0)(1+\omega t) + u'(0)t)\, e^{-\omega t}$
$\zeta > 1$	$\text{roots}\begin{cases} \lambda_1 = -\omega\xi + \omega\sqrt{\xi^2-1} \\ \lambda_2 = -\omega\xi - \omega\sqrt{\xi^2-1} \end{cases}$ $\begin{cases} u(t) = Ae^{\lambda_1\omega t} + Be^{\lambda_2\omega t} \\ A = \frac{u'(0)-u(0)\lambda_2}{2\omega\sqrt{\xi^2-1}} \\ B = \frac{-u'(0)+u(0)\lambda_1}{2\omega\sqrt{\xi^2-1}} \end{cases}$

1.4.4.6 Roots of characteristic equation

The roots of the characteristic equation for $u'' + 2\xi\omega u' + \omega^2 u = 0$ are given in this table

	roots	time constant τ
$\xi < 1$	$\{-\xi\omega + j\omega_n\sqrt{1-\xi^2}, -\xi\omega - i\omega_n\sqrt{1-\xi^2}\}$	$\frac{1}{\xi\omega}$
$\xi = 1$	$\{-\omega, -\omega\}$	$\frac{1}{\omega}$
$\xi > 1$	$\{-\omega_n\xi + \omega_n\sqrt{\xi^2-1}, -\omega_n\xi - \omega_n\sqrt{\xi^2-1}\}$	$\frac{1}{\omega_n\xi - \omega_n\sqrt{\xi^2-1}}, \frac{1}{\omega_n\xi + \omega_n\sqrt{\xi^2-1}}$(which to use? the bigger?)

1.4.5 impulse $F_0\delta(t)$ loading

1.4.5.1 impulse input

1.4.5.1.1 Undamped system with impulse

$$m\ddot{u} + ku = F_0\delta(t)$$

with initial conditions $u(0) = 0$ and $u'(0) = 0$.Assuming the impulse acts for a very short time period from 0 to t_1 seconds, where t_1 is small amount. Integrating the above differential equation gives

$$\int_0^{t_1} m\ddot{u}dt + \int_0^{t_1} kudt = \int_0^{t_1} F_0\delta(t)$$

Since t_1 is very small, it can be assumed that u changes is negligible, hence the above reduces to

$$\begin{aligned}
\int_0^{t_1} m\ddot{u}dt &= \int_0^{t_1} F_0\delta(t) \\
\int_0^{t_1} m\left(\frac{d\dot{u}}{dt}\right) dt &= \int_0^{t_1} F_0\delta(t) \\
\int_{\dot{u}(0)}^{\dot{u}(t_1)} d\dot{u} &= \frac{F_0}{m}\int_0^{t_1} \delta(t) \\
\dot{u}(t_1) - \dot{u}(0) &= \frac{F_0}{m}\int_0^{t_1} \delta(t) \\
\dot{u}(t_1) &= \frac{F_0}{m}\int_0^{t_1} \delta(t)
\end{aligned}$$

since we assumed $u'(0) = 0$ and since $\int_0^{t_1} \delta(t) = 1$ then the above reduces to

$$\dot{u}(t_1) = \frac{F_0}{m}$$

Therefore, the effect of the impulse is the same as if the system was a free system but with initial velocity given by $\frac{F_0}{m}$ and zero initial position. Hence the system is now solved as follows

$$m\ddot{u} + ku = 0$$

With $u(0) = 0$ and $u'(0) = \frac{F_0}{m}$. The solution is

$$u_{impulse}(t) = \frac{F_0}{m\omega}\sin\omega t$$

If the initial conditions were not zero, then the solution for these are added to the above. From earlier, it was found that the solution is $u(t) = u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t$, therefore, the full solution is

$$u(t) = \overbrace{u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t}^{\text{due to IC only}} + \overbrace{\frac{F_0}{m\omega}\sin\omega t}^{\text{due to impulse}}$$

1.4.5.1.2 under-damped with impulse $c < c_r, \xi < 1$

$$\begin{aligned}
m\ddot{u} + c\dot{u} + ku &= \delta(t) \\
\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u &= \delta(t)
\end{aligned}$$

with initial conditions $u(0) = 0$ and $u'(0) = 0$.Integrating gives

$$\int_0^{t_1} m\ddot{u}dt + \int_0^{t_1} c\dot{u}dt + \int_0^{t_1} kudt = \int_0^{t_1} F_0\delta(t)$$

Since t_1 is very small, it can be assumed that u changes is negligible as well as the change in velocity, hence the above reduces to the same result as in the case of undamped. Therefore, the system is solved as free system, but with initial velocity $u'(0) = F_0/m$ and zero initial position.

Initial conditions are $u(0) = 0$ and $u'(0) = 0$ then the solution is

$$u_{impulse} = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t)$$

applying initial conditions gives $A = 0$ and $B = \frac{\left(\frac{F_0}{m}\right)}{\omega_d}$, hence

$$u_{impulse}(t) = e^{-\xi\omega t}\left(\frac{F_0}{m\omega_d}\sin\omega_d t\right)$$

If the initial conditions were not zero, then the solution for these are added to the above. From earlier, it was found that the solution is $u(t) = e^{-\xi\omega t}\left(u(0)\cos\omega_d t + \frac{u'(0)+u(0)\xi\omega}{\omega_d}\sin\omega_d t\right)$, therefore, the full solution is

$$u(t) = \overbrace{e^{-\xi\omega t}\left(u(0)\cos\omega_d t + \frac{u'(0)+u(0)\xi\omega}{\omega_d}\sin\omega_d t\right)}^{\text{due to IC only}} + \overbrace{e^{-\xi\omega t}\left(\frac{F_0}{m\omega_d}\sin\omega_d t\right)}^{\text{due to impulse}}$$

1.4.5.1.3 critically damped with impulse input $\xi = \frac{c}{c_r} = 1$

with initial conditions $u(0) = 0$ and $u'(0) = 0$ then the solution is

$$\begin{aligned} u(t) &= (A + Bt)\, e^{-\left(\frac{c_r}{2m}\right)t} \\ &= (A + Bt)\, e^{-\omega t} \end{aligned}$$

where A, B are found from initial conditions $A = u(0) = 0$ and $B = u'(0) + u(0)\,\omega = \frac{F_0}{m}$, hence the solution is

$$u_{impulse}(t) = \frac{F_0 t}{m} e^{-\omega t}$$

If the initial conditions were not zero, then the solution for these are added to the above. From earlier, it was found that the solution is $u(t) = (u_0(1+\omega t) + u'(0)\,t)\,e^{-\omega t}$, therefore, the full solution is

$$u(t) = \overbrace{(u(0)\,(1+\omega t) + u'(0)\,t)\,e^{-\omega t}}^{\text{due to IC only}} + \overbrace{\frac{F_0 t}{m}e^{-\omega t}}^{\text{due to impulse}}$$

1.4.5.1.4 over-damped with impulse input $\xi = \frac{c}{c_r} > 1$

With initial conditions are $u(0) = 0$ and $u'(0) = 0$ the solution is

$$u_{impulse}(t) = Ae^{\lambda_1 \omega t} + Be^{\lambda_2 \omega t}$$

where A, B are found from initial conditions and

$$\begin{aligned}
\lambda_1 &= -\omega\xi + \omega\sqrt{\xi^2 - 1} \\
\lambda_2 &= -\omega\xi - \omega\sqrt{\xi^2 - 1} \\
A &= \frac{u'(0) - u(0)\,\lambda_2}{2\omega\sqrt{\xi^2 - 1}} \\
B &= \frac{-u'(0) + u(0)\,\lambda_1}{2\omega\sqrt{\xi^2 - 1}}
\end{aligned}$$

Hence the solution is

$$u_{impulse}(t) = Ae^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t} + Be^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t}$$

where

$$\begin{aligned}
A &= \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2 - 1}} \\
B &= \frac{-\frac{F_0}{m}}{2\omega\sqrt{\xi^2 - 1}}
\end{aligned}$$

Hence

$$u_{impulse}(t) = \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2 - 1}} e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t} - \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2 - 1}} e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t}$$

If the initial conditions were not zero, then the solution for these are added to the above. From earlier, it was found that the solution is $u(t) = Ae^{p_1 t} + Be^{p_2 t}$, therefore, the full solution is

$$u(t) = Ae^{\lambda_1 \omega t} + Be^{\lambda_2 \omega t} + \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2 - 1}} e^{\lambda_1 \omega t} - \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2 - 1}} e^{\lambda_2 \omega t}$$

$$\begin{aligned}
A &= \frac{u'(0) - u(0)\,\lambda_2}{2\omega\sqrt{\xi^2 - 1}} \\
B &= \frac{-u'(0) + u(0)\,\lambda_1}{2\omega\sqrt{\xi^2 - 1}}
\end{aligned}$$

1.4.5.1.5 Summary table

$\zeta = 0$ $u'' + \omega^2 u = 0$	$\text{roots}\begin{cases} -i\omega \\ +i\omega \end{cases}$ $u(t) = \overbrace{u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t}^{\text{transient}} + \overbrace{\frac{F_0}{m\omega}\sin\omega t}^{\text{steady state}}$
$\zeta < 1$	$\text{roots}\begin{cases} -\xi\omega + i\omega\sqrt{1-\xi^2} \\ -\xi\omega - i\omega\sqrt{1-\xi^2} \end{cases}$ $u(t) = \overbrace{e^{-\xi\omega t}\left(u(0)\cos\omega_d t + \frac{u'(0)+u(0)\xi\omega}{\omega_d}\sin\omega_d t\right)}^{\text{transient}} + \overbrace{e^{-\xi\omega t}\left(\frac{F_0}{m\omega_d}\sin\omega_d t\right)}^{\text{steady state}}$
$\zeta = 1$	$\text{roots}\begin{cases} -\omega \\ -\omega \end{cases}$ $u(t) = (u(0)\,(1+\omega t) + u'(0)\,t)\,e^{-\omega t} + \frac{F_0 t}{m} e^{-\omega t}$
$\zeta > 1$	$\text{roots}\begin{cases} \lambda_1 = -\omega\xi + \omega\sqrt{\xi^2-1} \\ \lambda_2 = -\omega\xi - \omega\sqrt{\xi^2-1} \end{cases}$ $\begin{cases} u(t) = Ae^{\lambda_1\omega t} + Be^{\lambda_2\omega t} + \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2-1}}e^{\lambda_1\omega t} - \frac{\frac{F_0}{m}}{2\omega\sqrt{\xi^2-1}}e^{\lambda_2\omega t} \\ A = \frac{u'(0)-u(0)\lambda_2}{2\omega\sqrt{\xi^2-1}} \\ B = \frac{-u'(0)+u(0)\lambda_1}{2\omega\sqrt{\xi^2-1}} \end{cases}$

The impulse response can be implemented in Mathematica as

```
parms = {m -> 10, c -> 1.2, k -> 4.3, a -> 1};
tf = TransferFunctionModel[a/(m s^2 + c s + k) /. parms, s]
sol = OutputResponse[tf, DiracDelta[t], t];
Plot[sol, {t, 0, 60}, PlotRange -> All, Frame -> True,
FrameLabel -> {{z[t], None}, {Row[{t, " (sec)"}], eq}},
GridLines -> Automatic]
```

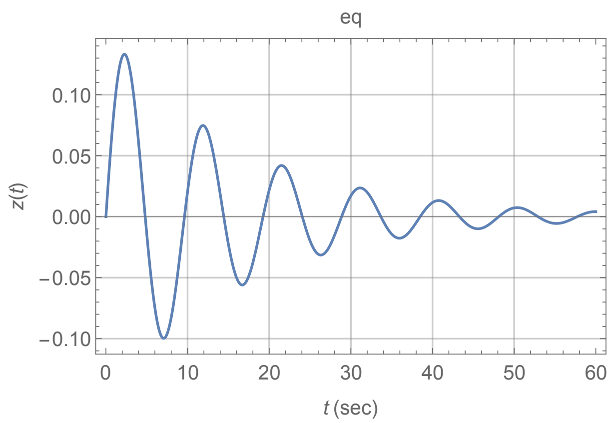


Figure 1.11: impulse Response Diagram

1.4.5.2 Impulse sin function

Now assume the input is as follows

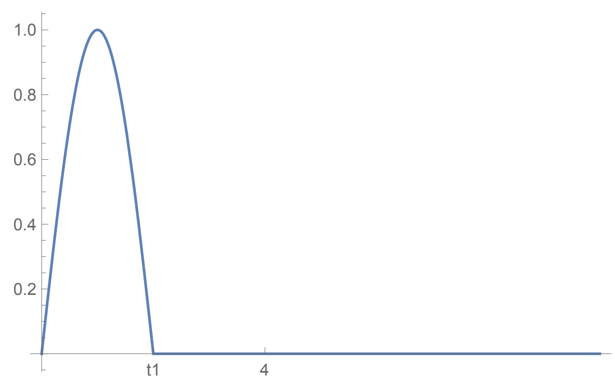


Figure 1.12: input function

given by $F(t) = F_0 \sin(\varpi t)$ where $\varpi = \frac{2\pi}{2t_1} = \frac{\pi}{t_1}$

1.4.5.2.1 undamped system with sin impulse

$$m\ddot{u} + ku = \begin{cases} F_0 \sin(\varpi t) & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$

with $u(0) = u_0$ and $\dot{u}(0) = v_0$. For $0 \leq t \leq t_1$ the solution is

$$u(t) = u_0 \cos \omega t + \left(\frac{v_0}{\omega} - u_{st} \frac{r}{1 - r^2} \right) \sin \omega t + u_{st} \frac{1}{1 - r^2} \sin\left(\frac{\pi}{t_1} t \right)$$

where $r = \frac{\varpi}{\omega} = \frac{\pi/t_1}{\omega} = \frac{T}{2t_1}$ where T is the natural period of the system. $u_{st} = \frac{F_0}{k}$, hence the above becomes

$$u(t) = u_0 \cos\omega t + \left(\frac{v_0}{\omega} - \frac{F_0}{k} \frac{\left(\frac{\pi/t_1}{\omega}\right)}{1 - \left(\frac{\pi/t_1}{\omega}\right)^2} \right) \sin\omega t + \frac{F_0}{k} \frac{1}{1 - \left(\frac{\pi/t_1}{\omega}\right)^2} \sin\left(\pi \frac{t}{t_1}\right) \tag{1}$$

When $u_0 = 0$ and $v_0 = 0$ then

$$u(t) = -\frac{F_0}{k} \frac{\left(\frac{\pi/t_1}{\omega}\right)}{1 - \left(\frac{\pi/t_1}{\omega}\right)^2} \sin\omega t + \frac{F_0}{k} \frac{1}{1 - \left(\frac{\pi/t_1}{\omega}\right)^2} \sin\left(\pi \frac{t}{t_1}\right)$$

$$u(t) = \frac{F_0}{k} \frac{1}{1 - \left(\frac{\pi/t_1}{\omega}\right)^2} \left(\sin\left(\pi \frac{t}{t_1}\right) - \frac{\pi/t_1}{\omega} \sin\omega t \right)$$

The above Eq (1) gives solution during the time $0 \leq t \leq t_1$

Now after $t = t_1$ the force will disappear, the differential equation becomes

$$m\ddot{u} + ku = 0 \qquad t > t_1$$

but with the initial conditions evaluate at $t = t_1$. From (1)

$$\begin{aligned} u(t_1) &= u_0 \cos\omega t_1 + \left(\frac{v_0}{\omega} - u_{st} \frac{r}{1 - r^2} \right) \sin\omega t_1 + u_{st} \frac{1}{1 - r^2} \sin\varpi t_1 \\ &= u_0 \cos\omega t_1 + \left(\frac{v_0}{\omega} - u_{st} \frac{r}{1 - r^2} \right) u_{st} \frac{r}{1 - r^2} \sin\omega t_1 \end{aligned} \tag{2}$$

since $\sin\varpi t_1 = 0$. taking derivative of Eq (1)

$$\dot{u}(t) = -\omega u_0 \sin\omega t + \omega \left(\frac{v_0}{\omega} - u_{st} \frac{r}{1 - r^2} \right) \cos\omega t + \varpi \frac{1}{1 - r^2} \cos\varpi t$$

and at $t = t_1$ the above becomes

$$\begin{aligned} \dot{u}(t_1) &= -\omega u_0 \sin\omega t_1 + \omega \left(\frac{v_0}{\omega} - u_{st} \frac{r}{1 - r^2} \right) \cos\omega t_1 + \varpi \frac{1}{1 - r^2} \cos\varpi t_1 \\ &= -\omega u_0 \sin\omega t_1 + \omega \left(\frac{v_0}{\omega} - u_{st} \frac{r}{1 - r^2} \right) \cos\omega t_1 - \varpi \frac{1}{1 - r^2} \end{aligned} \tag{3}$$

since $\cos\varpi t_1 = -1$. Now (2) and (3) are used as initial conditions to solve $m\ddot{u} + ku = 0$. The solution for $t > t_1$ is

$$u(t) = u(t_1) \cos\omega t + \frac{\dot{u}(t_1)}{\omega} \sin\omega t$$

Resonance with undamped sin impulse When $\varpi \approx \omega$ and $t \leq t_1$ we obtain resonance since $r \to 1$ in the solution shown up and as written the solution can't be used for analysis in this case. To obtain a solution for resonance some calculus is needed. Eq (1) is written as

$$
\begin{aligned}
u(t) &= u_0 \cos\omega t + \left(\frac{v_0}{\omega} - u_{st} \frac{\frac{\varpi}{\omega}}{1 - \left(\frac{\varpi}{\omega}\right)^2} \right) \sin\omega t + u_{st} \frac{1}{1 - \left(\frac{\varpi}{\omega}\right)^2} \sin\varpi t \\
&= u_0 \cos\omega t + \left(\frac{v_0}{\omega} - u_{st} \frac{\omega\varpi}{\omega^2 - \varpi^2} \right) \sin\omega t + u_{st} \frac{\omega^2}{\omega^2 - \varpi^2} \sin\varpi t \qquad (1A)
\end{aligned}
$$

Now looking at case when $\varpi \approx \omega$ but less than ω, hence let

$$\omega - \varpi = 2\Delta \tag{2}$$

where Δ is very small positive quantity. and we also have

$$\omega + \varpi \approx 2\varpi \tag{3}$$

Multiplying Eq (2) and (3) with each others gives

$$\omega^2 - \varpi^2 = 4\Delta\varpi \tag{4}$$

Going back to Eq (1A) and rewriting it as

$$
\begin{aligned}
u(t) &= u_0 \cos\omega t + \left(\frac{v_0}{\omega} - u_{st} \frac{\omega\varpi}{4\Delta\varpi} \right) \sin\omega t + u_{st} \frac{\omega^2}{4\Delta\varpi} \sin\varpi t \\
&= u_0 \cos\omega t + \left(\frac{v_0}{\omega} - u_{st} \frac{\omega}{4\Delta} \right) \sin\omega t + u_{st} \frac{\omega^2}{4\Delta\varpi} \sin\varpi t
\end{aligned}
$$

Since $\varpi \approx \omega$ the above becomes

$$
\begin{aligned}
u(t) &= u_0 \cos\omega t + \left(\frac{v_0}{\omega} - u_{st} \frac{\omega}{4\Delta} \right) \sin\omega t + u_{st} \frac{\omega}{4\Delta} \sin\varpi t \\
&= u_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t + u_{st} \frac{\omega}{4\Delta} (\sin\varpi t - \sin\omega t)
\end{aligned}
$$

now using $\sin\varpi t - \sin\omega t = 2\sin\left(\frac{\varpi-\omega}{2} t\right)\cos\left(\frac{\varpi+\omega}{2} t\right)$ the above becomes

$$u(t) = u_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t + u_{st} \frac{\omega}{2\Delta} \left(\sin\left(\frac{\varpi - \omega}{2} t\right) \cos\left(\frac{\varpi + \omega}{2} t\right) \right)$$

From Eq(2) $\varpi - \omega = -2\Delta$ and $\omega + \varpi \approx 2\varpi$ hence the above becomes

$$u(t) = u_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t + u_{st} \frac{\omega}{2\Delta} (\sin(-\Delta t) \cos(\varpi t))$$

or since $\varpi \approx \omega$

$$u(t) = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t - u_{st} \frac{\omega}{2\Delta} (\sin(\Delta t) \cos(\omega t))$$

Now $\lim_{\Delta \to 0} \frac{\sin(\Delta t)}{\Delta} = t$ hence the above becomes

$$u(t) = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t - u_{st} \frac{\omega t}{2} \cos(\omega t)$$

This can also be written as

$$\begin{aligned} u(t) &= u_0 \cos \varpi t + \frac{v_0}{\varpi} \sin \varpi t - u_{st} \frac{\varpi t}{2} \cos(\varpi t) \\ &= u_0 \cos\left(\frac{\pi}{t_1} t\right) + \frac{v_0}{\varpi} \sin\left(\frac{\pi}{t_1} t\right) - u_{st} \left(\frac{\pi}{2t_1} t\right) \cos\left(\frac{\pi}{t_1} t\right) \end{aligned} \tag{1}$$

since $\varpi \approx \omega$ in this case. This is the solution to use for resonance and for $t \leq t_1$

Hence for $t > t_1$, the above equations is used to determine initial conditions at $t = t_1$

$$u(t_1) = u_0 \cos \varpi t_1 + \frac{v_0}{\varpi} \sin \varpi t_1 - u_{st} \frac{\varpi t_1}{2} \cos(\varpi t_1)$$

but $\cos \varpi t_1 = \cos \frac{\pi}{t_1} t_1 = -1$ and $\sin \varpi t_1 = 0$ and $\frac{\varpi t_1}{2} = \frac{\pi}{2}$, hence the above becomes

$$u(t_1) = -u_0 + u_{st} \frac{\pi}{2}$$

Taking derivative of Eq (1) gives

$$\dot{u}(t) = -\varpi u_0 \sin \varpi t + v_0 \cos \varpi t + u_{st} \frac{\varpi^2 t}{2} \sin(\varpi t) - u_{st} \frac{\varpi}{2} \cos(\varpi t)$$

and at $t = t_1$

$$\begin{aligned} \dot{u}(t_1) &= -\varpi u_0 \sin \varpi t_1 + v_0 \cos \varpi t_1 + u_{st} \frac{\varpi^2 t_1}{2} \sin(\varpi t_1) - u_{st} \frac{\varpi}{2} \cos(\varpi t_1) \\ &= -v_0 + u_{st} \frac{\varpi}{2} \end{aligned}$$

Now the solution for $t > t_1$ is

$$\begin{aligned} u(t) &= u(t_1) \cos \omega t + \frac{\dot{u}(t_1)}{\omega} \sin \omega t \\ &= \left(-u(0) + u_{st} \frac{\pi}{2}\right) \cos \omega t + \frac{-u'(0) + u_{st} \frac{\pi}{2t_1}}{\omega} \sin \omega t \end{aligned}$$

1.4.5.2.2 under-damped with sin impulse $c < c_r, \xi < 1$

$$m\ddot{u} + c\dot{u} + ku = \begin{cases} F_0 \sin(\varpi) & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$

or

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \begin{cases} F_0 \sin(\varpi) & 0 \leq t \leq t_1 \\ 0 & t > t_1 \end{cases}$$

$$m\ddot{u} + c\dot{u} + ku = F \sin \varpi t$$

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \frac{F}{m} \sin \varpi t$$

For $t \leq t_1$Initial conditions are $u(0) = u_0$ and $\dot{u}(0) = v_0$ and $u_{st} = \frac{F}{k}$ then the solution from above is

$$u(t) = e^{-\xi\omega t}(A \cos \omega_d t + B \sin \omega_d t) + \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\varpi t - \theta) \tag{1}$$

Applying initial conditions gives

$$\begin{aligned} A &= u_0 + \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin \theta \\ B &= \frac{v_0}{\omega_d} + \frac{u_0 \xi \omega}{\omega_d} + \frac{u_{st}}{\omega_d \sqrt{(1-r^2)^2 + (2\xi r)^2}} (\xi\omega \sin \theta - \varpi \cos \theta) \end{aligned}$$

For $t > t_1$. From (1)

$$u(t_1) = e^{-\xi\omega t_1}(A \cos \omega_d t_1 + B \sin \omega_d t_1) + \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\varpi t_1 - \theta) \tag{2}$$

Taking derivative of (1) gives

$$\begin{aligned} \dot{u}(t) = -\xi\omega e^{-\xi\omega t}(A \cos \omega_d t + B \sin \omega_d t) + e^{-\xi\omega t}(-A\omega_d \sin \omega_d t + \omega_d B \cos \omega_d t) \\ + \varpi \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \cos(\varpi t - \theta) \end{aligned}$$

at $t = t_1$

$$\begin{aligned} \dot{u}(t_1) = -\xi\omega e^{-\xi\omega t_1}(A \cos \omega_d t_1 + B \sin \omega_d t_1) + e^{-\xi\omega t_1}(-A\omega_d \sin \omega_d t_1 + \omega_d B \cos \omega_d t_1) \\ + \varpi \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \cos(\varpi t_1 - \theta) \end{aligned} \tag{3}$$

Now for $t > t_1$ the equation becomes

$$m\ddot{u} + c\dot{u} + ku = 0$$

which has the solution

$$u = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t)$$

where $A = u(t_1)$ and $B = \frac{\dot{u}(t_1)+u(t_1)\xi\omega}{\omega_d}$

1.4.5.2.3 critically damped with sin impulse $\xi = \frac{c}{c_r} = 1$

For $t \leq t_1$Initial conditions are $u(0) = u_0$ and $\dot{u}(0) = v_0$ then the solution is from above

$$u(t) = (A + Bt)\,e^{-\omega t} + \frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2r\right)^2}}\sin\left(\varpi t - \theta\right) \tag{1}$$

Where $\tan\theta = \frac{c\varpi}{k-m\varpi^2} = \frac{2\xi r}{1-r^2}$. A, B are found from initial conditions

$$\begin{aligned} A &= u_0 + \frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2r\right)^2}}\sin\theta \\ B &= v_0 + u_0\omega + \frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2r\right)^2}}(\omega\sin\theta - \varpi\cos\theta) \end{aligned}$$

For $t > t_1$ the solution is

$$u(t) = \left(u(t_1) + \left(\dot{u}(t_1) + u(t_1)\,\omega\right)t\right)e^{-\omega t} \tag{2}$$

To find $u(t_1)$, from Eq(1)

$$u(t_1) = (A + Bt)\,e^{-\omega t_1} + \frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2r\right)^2}}\sin\left(\varpi t_1 - \theta\right)$$

taking derivative of (1) gives

$$\dot{u}(t) = -\omega(A + Bt)\,e^{-\omega t} + Be^{-\omega t} + \varpi\frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2r\right)^2}}\sin\left(\varpi t - \theta\right) \tag{3}$$

at $t = t_1$

$$\dot{u}(t_1) = -\omega(A + Bt_1)\,e^{-\omega t_1} + Be^{-\omega t_1} + \varpi\frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2r\right)^2}}\sin\left(\varpi t_1 - \theta\right) \tag{4}$$

Hence Eq (2) can now be evaluated using Eq(3,4)

1.4.5.2.4 over-damped with sin impulse $\xi = \frac{c}{c_r} > 1$

For $t \leq t_1$Initial conditions are $u(0) = u_0$ and $\dot{u}(0) = v_0$ then the solution is

$$u = Ae^{p_1 t} + Be^{p_2 t} + \frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}} \sin\left(\varpi t - \theta\right)$$

where $\tan\theta = \frac{2\xi r}{1-r^2}$(make sure you use correct quadrant, see not above on arctan) and

$$\begin{aligned} p_1 &= -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ &= -\omega\xi + \omega\sqrt{\xi^2 - 1} \end{aligned}$$

and

$$\begin{aligned} p_2 &= -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ &= -\omega\xi - \omega\sqrt{\xi^2 - 1} \end{aligned}$$

leading to the solution where $\tan\theta = \frac{2\xi r}{1-r^2}$ and

$$\begin{aligned} p_1 &= -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega\xi + \omega_n\sqrt{\xi^2 - 1} \\ p_2 &= -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\omega\xi - \omega_n\sqrt{\xi^2 - 1} \end{aligned}$$

is

$$\begin{aligned} u(t) &= Ae^{\left(-\xi + \sqrt{\xi^2-1}\right)\omega t} + Be^{\left(-\xi - \sqrt{\xi^2-1}\right)\omega t} + \frac{F}{k}\beta \sin\left(\varpi t - \theta\right) \\ A &= \frac{u'(0) + u(0)\,\omega\xi + u(0)\,\omega\sqrt{\xi^2-1} + \frac{F}{k}\beta\left(\left(\xi + \sqrt{\xi^2-1}\right)\omega\sin\theta - \varpi\cos\theta\right)}{2\omega\sqrt{\xi^2-1}} \\ B &= -\frac{u'(0) + u(0)\,\omega\xi - u(0)\,\omega\sqrt{\xi^2-1} + \frac{F}{k}\beta\left(\left(\xi - \sqrt{\xi^2-1}\right)\omega\sin\theta - \varpi\cos\theta\right)}{2\omega\sqrt{\xi^2-1}} \\ \beta &= \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\xi r\right)^2}} \end{aligned}$$

For $t > t_1$. From Eq(1) and at $t = t_1$

$$u(t_1) = Ae^{\left(-\xi + \sqrt{\xi^2-1}\right)\omega t_1} + Be^{\left(-\xi - \sqrt{\xi^2-1}\right)\omega t_1} + D\sin\left(\varpi t_1 - \theta\right) \tag{2}$$

Taking derivative of Eq (1)

$$\dot{u}(t) = \omega Ae^{\left(-\xi + \sqrt{\xi^2-1}\right)\omega t} + \omega Be^{\left(-\xi - \sqrt{\xi^2-1}\right)\omega t} + \varpi D\cos\left(\varpi t - \theta\right)$$

At $t = t_1$

$$\dot{u}(t_1) = \omega A e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t_1} + \omega B e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t_1} + \varpi D \cos\left(\varpi t_1 - \theta\right) \tag{3}$$

Equation of motion now is

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = 0$$

which has solution for over-damped given by

$$u(t) = A e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + B e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}$$

where

$$A = -\frac{\dot{u}(t_1) + u(t_1)\,\omega_n\left(\xi - \sqrt{\xi^2-1}\right)}{2\omega_n\sqrt{\xi^2-1}}$$
$$B = \frac{\dot{u}(t_1) + u(t_1)\,\xi\omega_n\left(\xi + \sqrt{\xi^2-1}\right)}{2\omega_n\sqrt{\xi^2-1}}$$

Input is given by $F(t) = F_0 \sin\left(\varpi t\right)$ where $\varpi = \frac{2\pi}{2t_1} = \frac{\pi}{t_1}$

```
t1 = 2;
Plot[(UnitStep[t] - UnitStep[t - 2]) Sin[Pi/t1 t], {t, 0, 10},
PlotRange -> All, Ticks -> {{0, {2, "t1"}, 4}, Automatic}]
```

1.4.5.2.5 Summary table

$\zeta = 0$	$\text{roots}\begin{cases} -i\omega \\ +i\omega \end{cases}$ $u(t)\begin{cases} \varpi = \omega \to \begin{cases} u(0)\cos\omega t + \frac{u'(0)}{\omega}\sin\omega t - \frac{F_0}{k}\frac{\omega t}{2}\cos\left(\omega t\right) & 0 \leq t \leq t_1 \\ \left(-u(0) + \frac{F_0}{k}\frac{\pi}{2}\right)\cos\omega t + \frac{-u'(0)+\frac{F_0}{k}\frac{\pi}{2t_1}}{\omega}\sin\omega t & t > t_1 \end{cases} \\ \varpi \neq \omega \to \begin{cases} u(0)\cos\omega t + \left(\frac{u'(0)}{\varpi} - \frac{(F_0/k)\left(\frac{\pi/t_1}{\omega}\right)}{1-\left(\frac{\pi/t_1}{\omega}\right)^2}\right)\sin\omega t + \frac{F_0/k}{1-\left(\frac{\pi/t_1}{\omega}\right)^2}\sin\left(\frac{\pi t}{t_1}\right) & 0 \leq t \leq t_1 \\ u(t_1)\cos\omega t + \frac{u'(t_1)}{\omega}\sin\omega t & t > t_1 \end{cases} \end{cases}$

$\zeta < 1$	$\text{roots}\begin{cases}-\xi\omega + i\omega_n\sqrt{1-\xi^2}\\ -\xi\omega - i\omega_n\sqrt{1-\xi^2}\end{cases}$ time constant $\tau = \frac{1}{\zeta\omega_n}$ $u(t) = \begin{cases} e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin\left(\pi\frac{t}{t_1}-\theta\right) & 0\leq t\leq t_1\\ e^{-\xi\omega t}\left(u(t_1)\cos\omega_d t + \frac{u'(t_1)+u(t_1)\xi\omega}{\omega_d}\sin\omega_d t\right) & t > t_1\end{cases}$ $A = u(0) + \frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin\theta$ $B = \frac{u'(0)}{\omega_d} + \frac{u(0)\xi\omega}{\omega_d} + \frac{F_0}{k}\frac{1}{\omega_d\sqrt{(1-r^2)^2+(2\xi r)^2}}(\xi\omega\sin\theta - \varpi\cos\theta)$
$\zeta = 1$	$\text{roots}\begin{cases}-\omega\\ -\omega\end{cases}$ $u(t) = \begin{cases}(A+Bt)\,e^{-\omega t} + \frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2r)^2}}\sin\left(\pi\frac{t}{t_1}-\theta\right) & 0\leq t\leq t_1\\ u(t) = (u(t_1) + (u'(t_1)+u(t_1)\,\omega)\,t)\,e^{-\omega t} & t > t_1\end{cases}$ $A = u(0) + \frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2r)^2}}\sin\theta$ $B = u'(0) + u(0)\,\omega + \frac{F_0}{k}\frac{1}{\sqrt{(1-r^2)^2+(2r)^2}}\left(\omega\sin\theta - \frac{\pi}{t_1}\cos\theta\right)$
$\zeta > 1$	$\text{roots}\begin{cases}-\omega_n\xi + \omega_n\sqrt{\xi^2-1}\\ -\omega_n\xi - \omega_n\sqrt{\xi^2-1}\end{cases}$ $u(t) = \begin{cases}Ae^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega t} + Be^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega t} + \frac{F}{k}\beta\sin\left(\pi\frac{t}{t_1}-\theta\right) & 0\leq t\leq t_1\\ u(t) = A_1 e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + B_1 e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t} & t > t_1\end{cases}$ $A = \frac{u'(0)+u(0)\omega\xi+u(0)\omega\sqrt{\xi^2-1}+\frac{F}{k}\beta\left(\left(\xi+\sqrt{\xi^2-1}\right)\omega\sin\theta-\frac{\pi}{t_1}\cos\theta\right)}{2\omega\sqrt{\xi^2-1}}$ $B = -\frac{u'(0)+u(0)\omega\xi-u(0)\omega\sqrt{\xi^2-1}+\frac{F}{k}\beta\left(\left(\xi-\sqrt{\xi^2-1}\right)\omega\sin\theta-\frac{\pi}{t_1}\cos\theta\right)}{2\omega\sqrt{\xi^2-1}}$ $\beta = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$ $A_1 = -\frac{\dot{u}(t_1)+u(t_1)\omega_n\left(\xi-\sqrt{\xi^2-1}\right)}{2\omega_n\sqrt{\xi^2-1}}$ $B_1 = \frac{\dot{u}(t_1)+u(t_1)\xi\omega_n\left(\xi+\sqrt{\xi^2-1}\right)}{2\omega_n\sqrt{\xi^2-1}}$

1.4.6 Tree view look at the different cases

This tree illustrates the different cases that needs to be considered for the solution of single degree of freedom system with harmonic loading.

There are 12 cases to consider. Resonance needs to be handled as special case when damping is absent due to the singularity in the standard solution when the forcing frequency is the same as the natural frequency. When damping is present, there is no resonance, however, there is what is called practical response which occur when the forcing frequency is almost the same as the natural frequency.

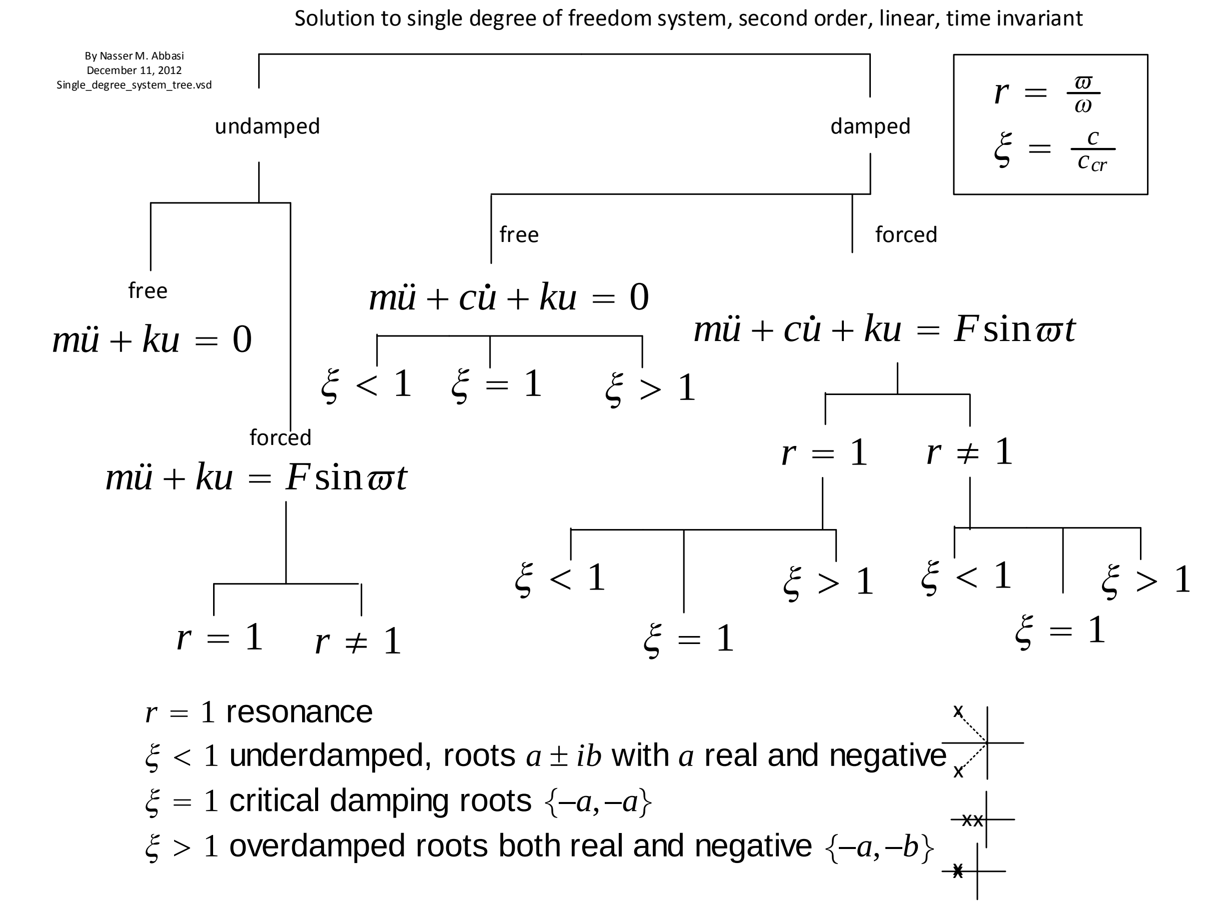


Figure 1.13: single degree system tree

The following is another diagram made sometime ago which contains more useful information and is kept here for reference.

One degree freedom linear system unforced response

$$y''(t) + \frac{c}{m}y'(t) + \frac{k}{m}y(t) = 0$$

$$y''(t) + 2\zeta\omega_n y'(t) + \omega_n^2 y(t) = 0$$

Solution roots

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Some defintions

$$\omega_n = \sqrt{\frac{k}{m}} \text{ (rad/sec)}$$

ζ damping ratio

$$\zeta = \frac{c}{c_r} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

Defined only for under damped case(else zero)

3 cases

$\zeta < 1$

Underdamped 2 roots, complex conjugate

$$y(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

$$A = y(0),\ B = \frac{y'(0)+A\zeta\omega_n}{\omega_d}$$

$$T_d = \frac{2\pi}{\omega_d}$$ Damped period of oscillation

$$\tau = \frac{1}{\xi\omega_n}$$ Time constant

$\zeta = 1$

Critical damped (one real root, double multiplicity)

$$y(t) = (A+Bt)e^{-\xi\omega_n t}$$

$$A = y(0),\ B = y'(0) + A\zeta\omega_n$$

$\zeta > 1$

Over damped, 2 real roots, distinct

$$y(t) = Ae^{\left(-\zeta-\sqrt{\xi^2-1}\right)\omega_n t} + Be^{\left(-\zeta+\sqrt{\xi^2-1}\right)\omega_n t}$$

$$A = -\frac{y'(0)+y(0)\omega_n\left(\zeta-\sqrt{\xi^2-1}\right)}{2\omega_n\sqrt{\xi^2-1}}$$

$$B = \frac{y'(0)+y(0)\omega_n\left(\zeta+\sqrt{\xi^2-1}\right)}{2\omega_n\sqrt{\xi^2-1}}$$

Nasser M. Abbasi
May 25, 2011
One_DOF_system.vsd

Figure 1.14: one DOF system

1.4.7 Cycles for the peak to decay by half its original value

This table shows many cycles it takes for the peak to decay by half its original value as a function of the damping ζ. For example, we see that when $\zeta = 2.7\%$ then it takes 4 cycles for the peak (i.e. displacement) to reduce to half its value.

```
data = Table[{i, (1/i Log[2]/(2*Pi)*100)}, {i, 1, 20}];
TableForm[N@data,
TableHeadings -> {None, {Column[{"number of cycles",
"needed for peak", "to decay by half"}], "\[Zeta] (%)"}}]
```

number of cycles needed for peak to decay by half	ξ (%)
1.	11.0318
2.	5.51589
3.	3.67726
4.	2.75795
5.	2.20636
6.	1.83863
7.	1.57597
8.	1.37897
9.	1.22575
10.	1.10318
11.	1.00289
12.	0.919315
13.	0.848598
14.	0.787984
15.	0.735452
16.	0.689486
17.	0.648928
18.	0.612877
19.	0.58062
20.	0.551589

Figure 1.15: peak table

1.4.8 references

1. Vibration analysis by Robert K. Vierck
2. Structural dynamics theory and computation, 5th edition by Mario Paz, William Leigh
3. Dynamic of structures, Ray W. Clough and Joseph Penzien
4. Theory of vibration,volume 1, by A.A.Shabana
5. Notes on Diffy Qs, Differential equations for engineers, by Jiri Lebl, online PDF book, chapter 2.6, oct 1,2012 http://www.jirka.org/diffyqs/

Chapter 2

Dynamics equations, kinematics, velocity and acceleration diagrams

2.1 Derivation of rotation formula

This formula is very important. Will show its derivation now in details. It is how to express vectors in rotating frames.

Consider this diagram

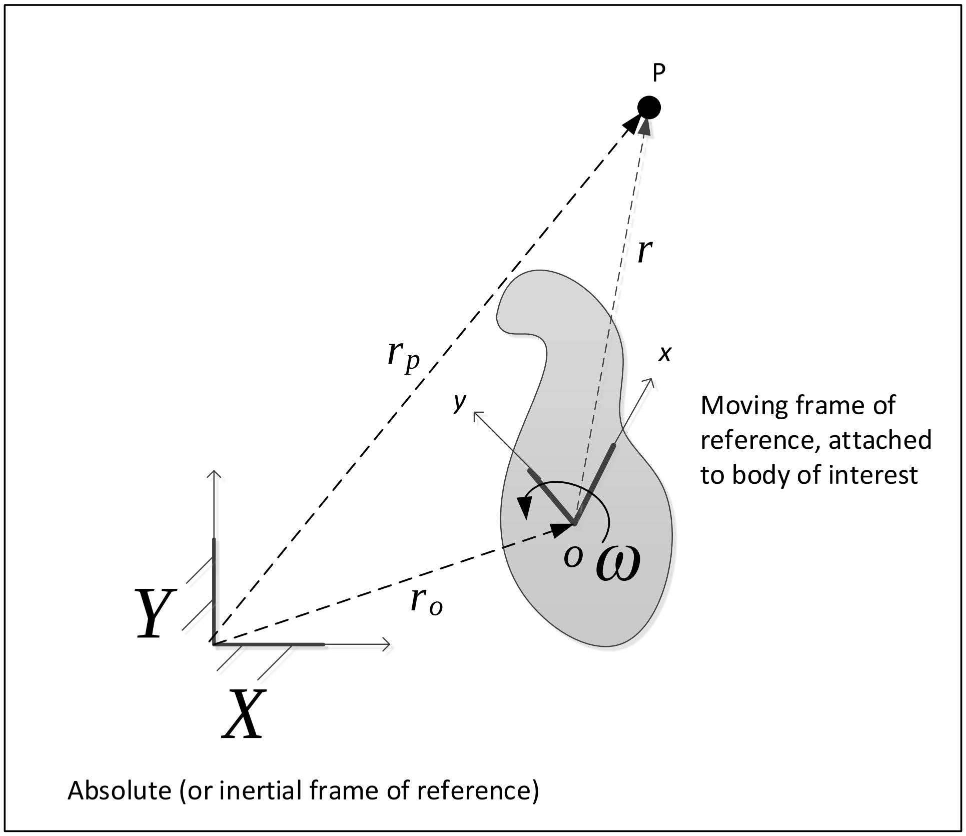


Figure 2.1: rotating frames

In the above, the small axis x, y is a frame attached to some body which rotate around this axis with angular velocity ω (measured by the inertial frame of course). All laws derived below are based on the following one rule

$$\left.\frac{d}{dt}r\right|_{absolute} = \left.\frac{d}{dt}r\right|_{relative} + \omega \times r \tag{1}$$

Lets us see how to apply this rule. Let us express the position vector of the particle r_p. We can see by normal vector additions that the position vector of particle is

$$r_p = r_o + r \tag{2}$$

Notice that nothing special is needed here, since we have not yet looked at rate of change with time. The complexity (i.e. using rule (1)) appears only when we want to look at velocities and accelerations. This is when we need to use the above rule (1). Let us now find the velocity of the particle. From above

$$\dot{r}_p = \dot{r}_o + \dot{r}$$

Every time we take derivatives, we stop and look. For any vector that originates from the moving frame, we must apply rule (1) to it. That is all. In the above, only r needs rule (1) applied to it, since that is the only vector measure from the moving frame. Replacing $\dot{r}_p$ by V_p and $\dot{r}_o$ by V_o, meaning the velocity of P and o, Hence the above becomes

$$V_p = V_o + \dot{r}$$

and now we apply rule (1) to expand $\dot{r}$

$$V_p = V_o + (V_{rel} + \omega \times r) \tag{3}$$

where V_{rel} is just $\frac{d}{dt}r\Big|_{relative}$

The above is the final expression for the velocity of the particle V_p using its velocity as measured by the moving frame in order to complete the expression.

So the above says that the absolute velocity of the particle is equal to the absolute velocity of the base of the moving frame + something else and this something else was $(V_{rel} + \omega \times r)$

Now we will find the absolute acceleration of P. Taking time derivatives of (3) gives

$$\dot{V}_p = \dot{V}_o + \left(\dot{V}_{rel} + \dot{\omega} \times r + \omega \times \dot{r}\right) \tag{4}$$

As we said above, each time we take time derivatives, we stop and look for vectors which are based on the moving frame, and apply rule (1) to them. In the above, $\dot{V}_{rel}$ and $\dot{r}$ qualify. Apply rule (1) to $\dot{V}_{rel}$ gives

$$\dot{V}_{rel} = a_{rel} + \omega \times V_{rel} \tag{5}$$

where a_{rel} just means the acceleration relative to moving frame. And applying rule (1) to $\dot{r}$ gives

$$\dot{r} = V_{rel} + \omega \times r \tag{6}$$

Replacing (5) and (6) into (4) gives

$$\begin{aligned} a_p &= a_o + (a_{rel} + \omega \times V_{rel} + \dot{\omega} \times r + \omega \times (V_{rel} + \omega \times r)) \\ &= a_o + a_{rel} + (\omega \times V_{rel}) + (\dot{\omega} \times r) + (\omega \times V_{rel}) + (\omega \times (\omega \times r)) \\ &= a_o + a_{rel} + 2(\omega \times V_{rel}) + (\dot{\omega} \times r) + (\omega \times (\omega \times r)) \end{aligned} \tag{7}$$

Eq (7) says that the absolute acceleration a_p of P is the sum of the acceleration of the base a_o of the moving frame plus the relative acceleration a_{rel} of the particle to the moving frame plus $2(\omega \times V_{rel}) + (\dot{\omega} \times r) + (\omega \times (\omega \times r))$

Hence, using Eq(3) and Eq(7) gives us the expressions we wanted for velocity and acceleration.

2.2 Miscellaneous hints

1. When finding the generalized force for the user with the Lagrangian method (the hardest step), using the virtual work method, if the force (or virtual work by the force) ADDS energy to the system, then make the sign of the force positive otherwise the sign is negative.
2. For damping force, the sign is always negative.
3. External forces such as linear forces applied, torque applied, in general, are positive.
4. Friction force is negative (in general) as friction takes energy from the system like damping.

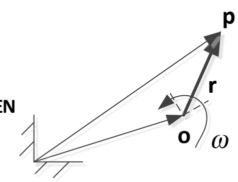
2.3 Formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

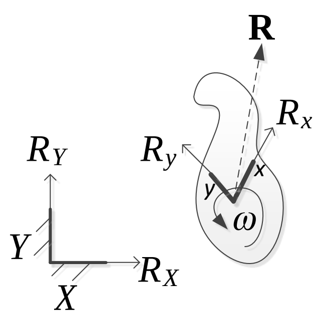
$$V_p = V_o + \omega \times r + V_{rel}$$

Velocity of p relative to o

$$a_p = a_0 + \dot{\omega} \times r + \omega \times (\omega \times r) + 2\omega \times V_{rel} + a_{rel}$$

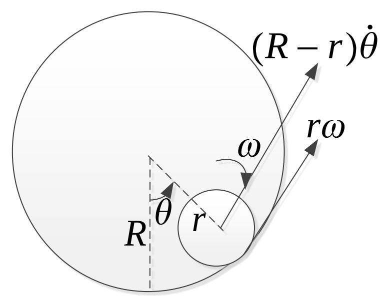


IF rigid body is rotating AND translation at the same time **THEN**
Take moments around its cg only
ELSE
can also take moments about any other points on it (but change I)
END IF



BODY FIXED COORDINATES
This gives the relation between the rate of change with respect to time of a vector expressed in a frame of reference which is body fixed (y,x) here, to the rate of change with respect to time of the same vector expressed using an inertial frame coordinates XY.
The omega here is the angular velocity of the rigid body rotation, or the body fixed coordinates, with respect to the inertial frame.

$$\frac{d}{dt}\mathbf{R}_{X,Y} = \frac{d}{dt}\mathbf{R}_{x,y} + \omega \times \mathbf{R}_{X,Y}$$



The small disk rotates at angular speed of ω
The condition of no slip can be seen to be
$(R-r)\dot{\theta} = r\omega$

For the sign of generalized force:
If work done by force takes away energy from system, then the sign is negative, else positive. So Friction will always have negative sign, so will damping force.

$$\frac{d}{dt}\cos(x(t)) = -\sin(x(t))x'(t)$$
$$\frac{d}{dt}\left(\frac{df}{dx(t)}\right) = \left(\frac{d}{dx(t)}\left(\frac{df}{dx(t)}\right)\right)\frac{d}{dt}x(t)$$
$$\frac{d}{dt}\cos(x(t)) = \left(\frac{d}{dx(t)}(\cos x(t))\right)\frac{d}{dt}x(t)$$

Figure 2.2: Formulas 1

$F = ma$	$\tau = I\ddot{\theta}$
linear momentum $p = mv$	angular momentum $H = I_{cg}\dot{\theta}$
$F = \frac{d}{dt}p$	$\tau = \frac{d}{dt}(H)$
particle kinetic energy $T = \frac{1}{2}mv^2$	rigid body $T = \frac{1}{2}Mv_{cg}^2 + \frac{1}{2}I_{cg}\dot{\theta}^2$

$$my'' + 2\zeta\omega_n y' + \omega_n^2 y = f(y,t)$$
$$y'' + cy' + \frac{k}{m}y = f(y,t)$$
$$\omega_n = \sqrt{\frac{k}{m}}, c = \frac{2\zeta\omega_n}{m}$$

For small epsilon

$$\frac{1}{1+\varepsilon} = 1 - \varepsilon$$
$$\frac{1}{1-\varepsilon} = 1 + \varepsilon$$

conservation of angular momentum $\frac{d}{dt}(H)$ =constant

$x'' + \omega_n^2 x = 0$

If $\omega_n > 0$ then sinusodial solution, ok

if $\omega_n < 0$ then solution blows up, exponential

Figure 2.3: Formulas 2

2.4 Velocity and acceleration diagrams

2.4.1 Spring pendulum

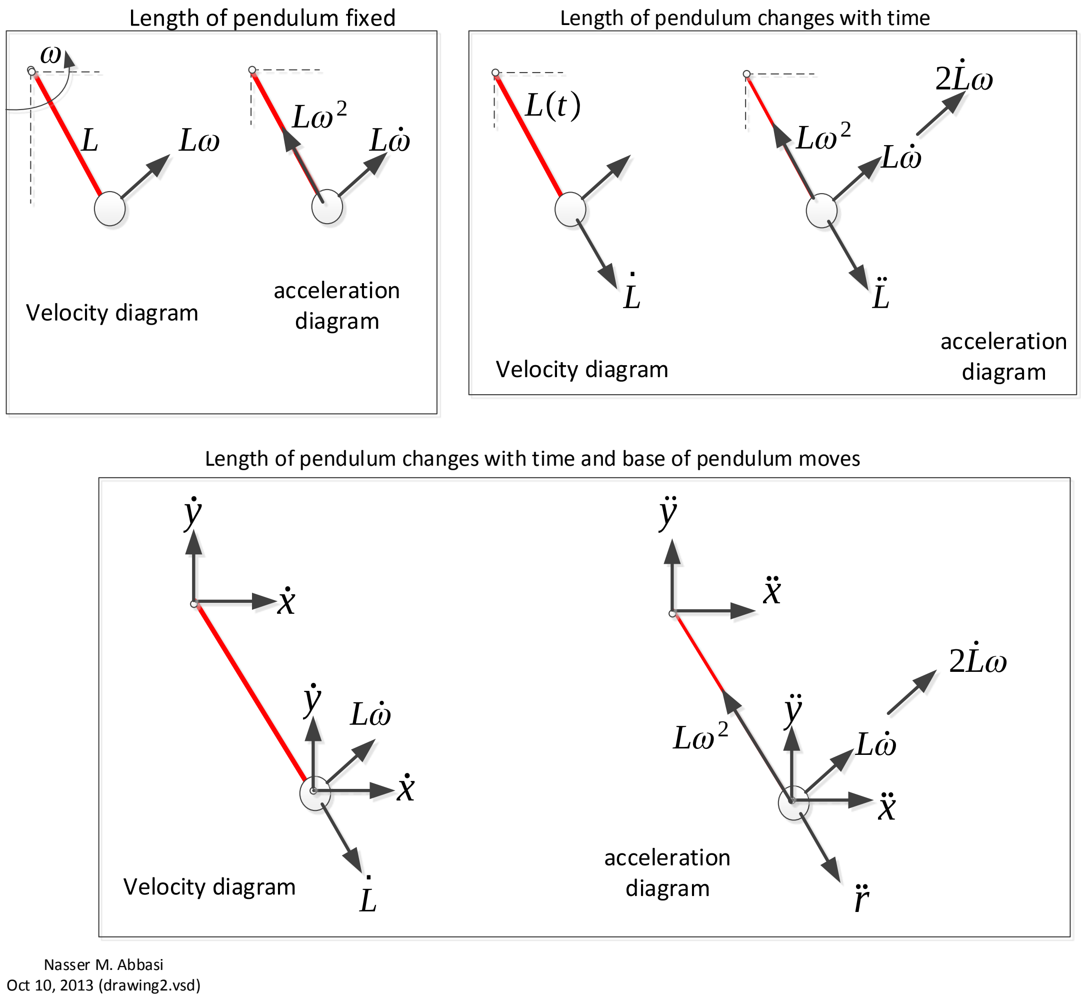


Figure 2.4: Spring pendulum

2.4.2 pendulum with blob moving in slot

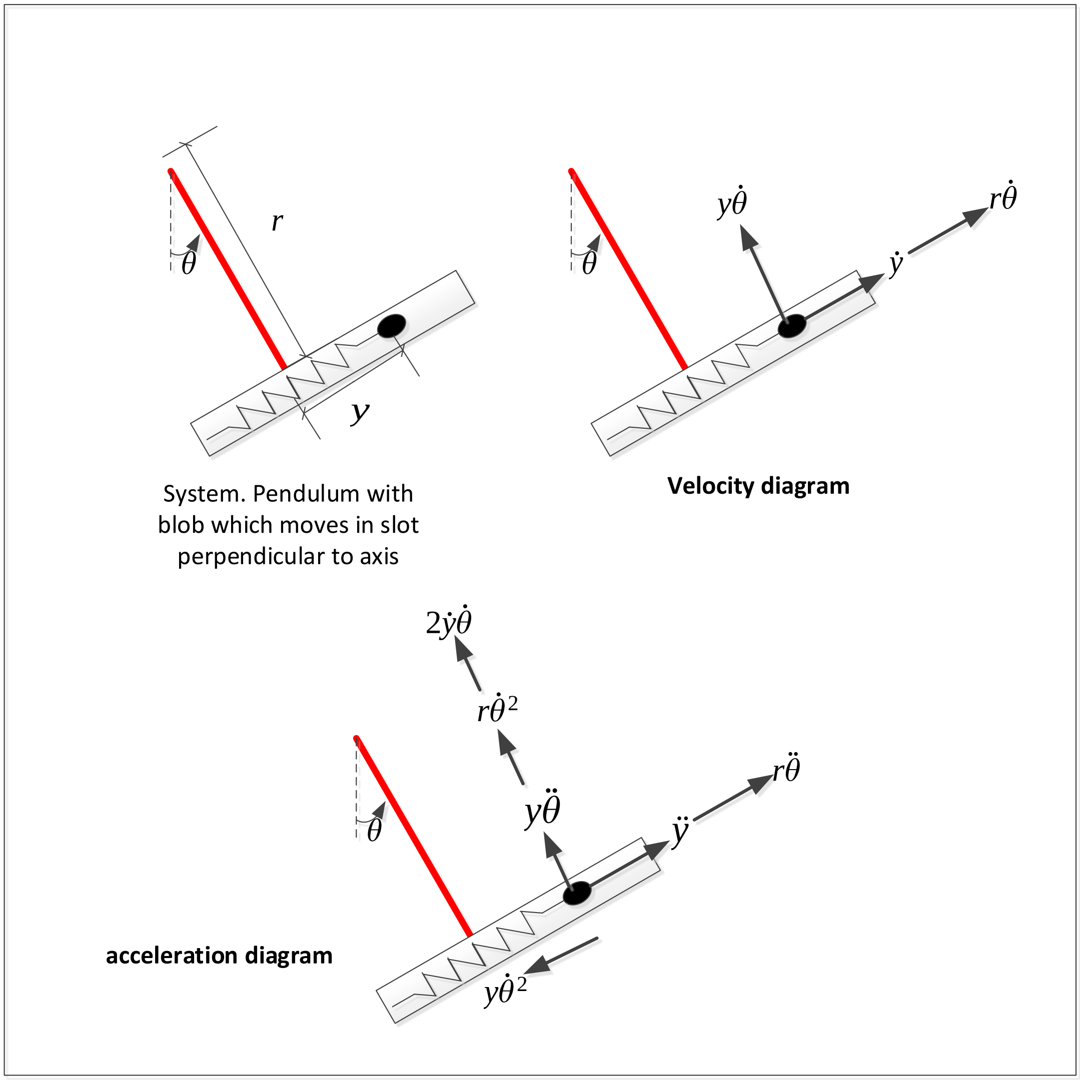


Figure 2.5: pendulum with blob

2.4.3 spring pendulum with block moving in slot

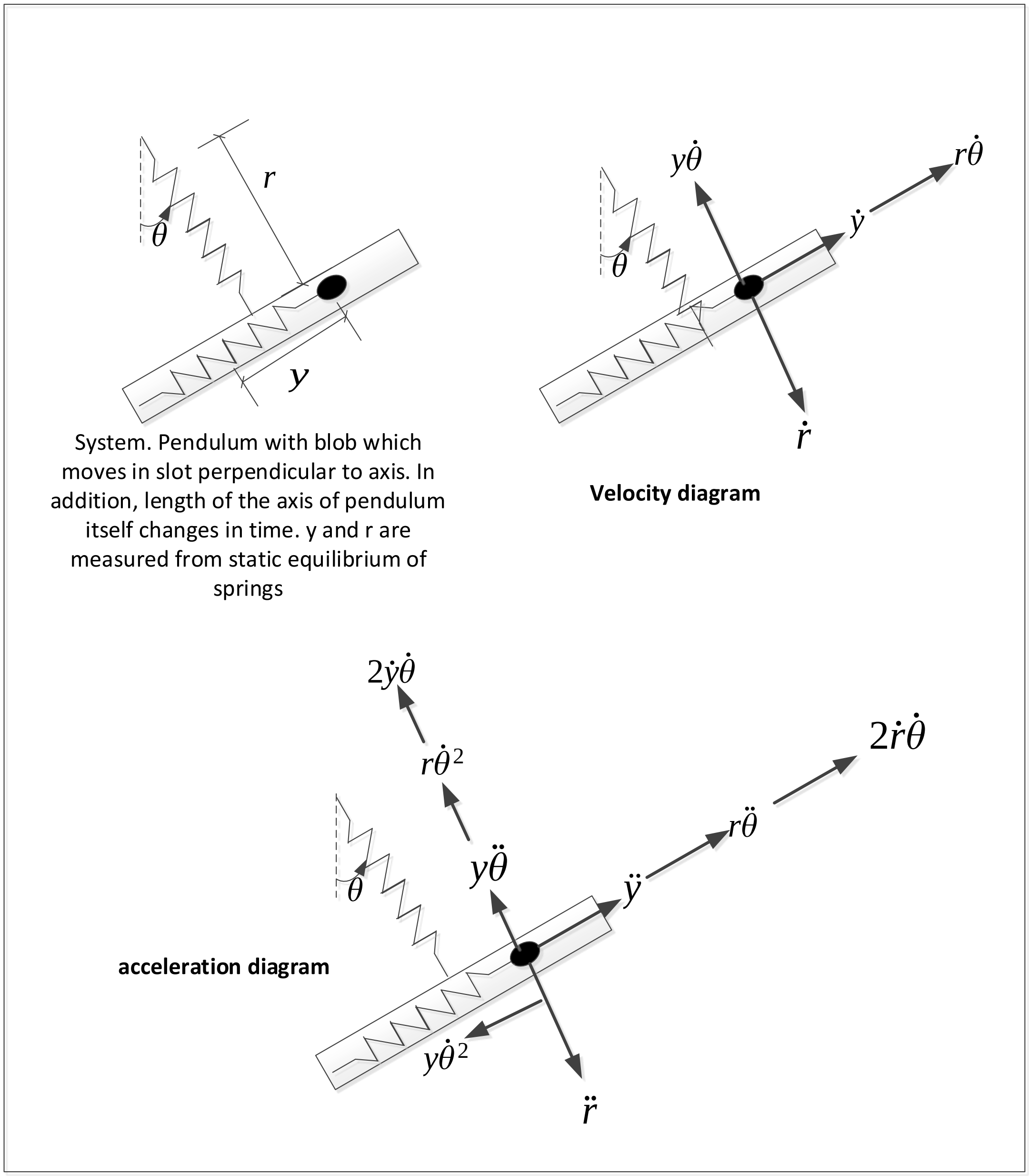


Figure 2.6: spring pendulum with block moving in slot

2.4.4 double pendulum

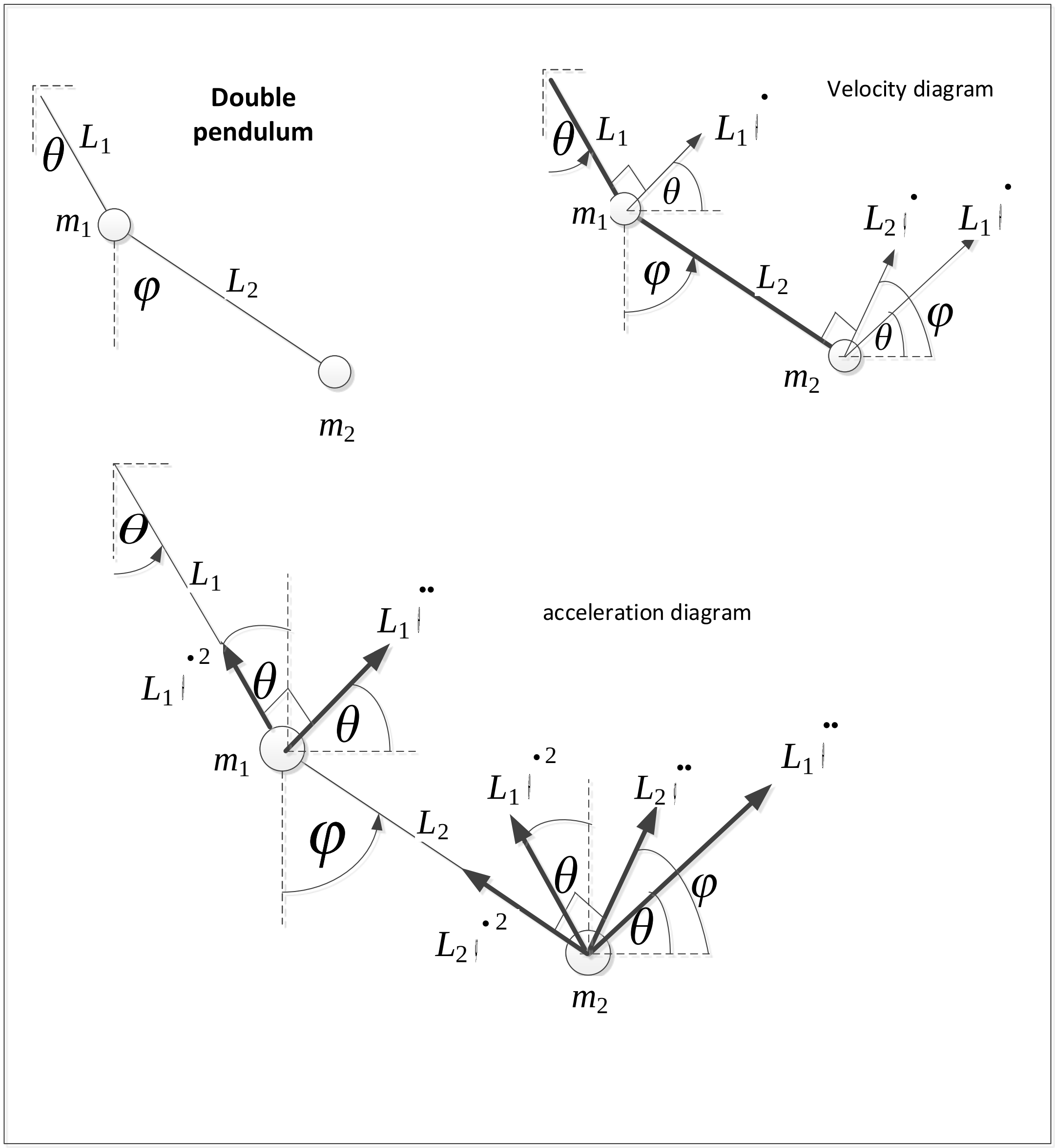


Figure 2.7: double pendulum

2.5 Velocity and acceleration of rigid body 2D

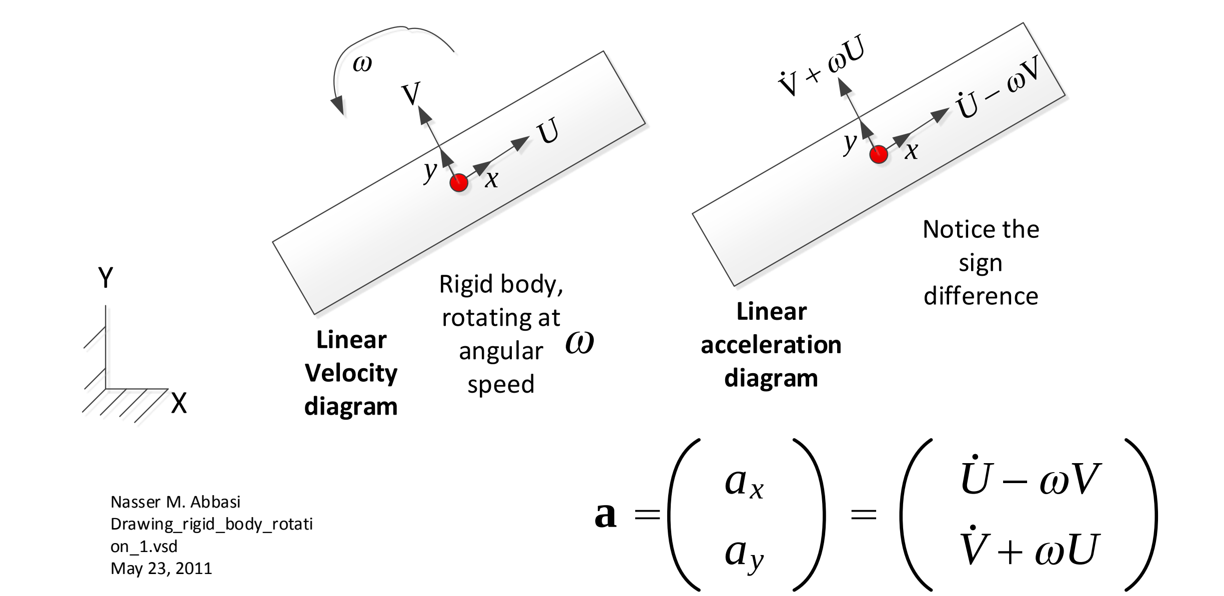


Figure 2.8: Velocity and acceleration of rigid body 2D

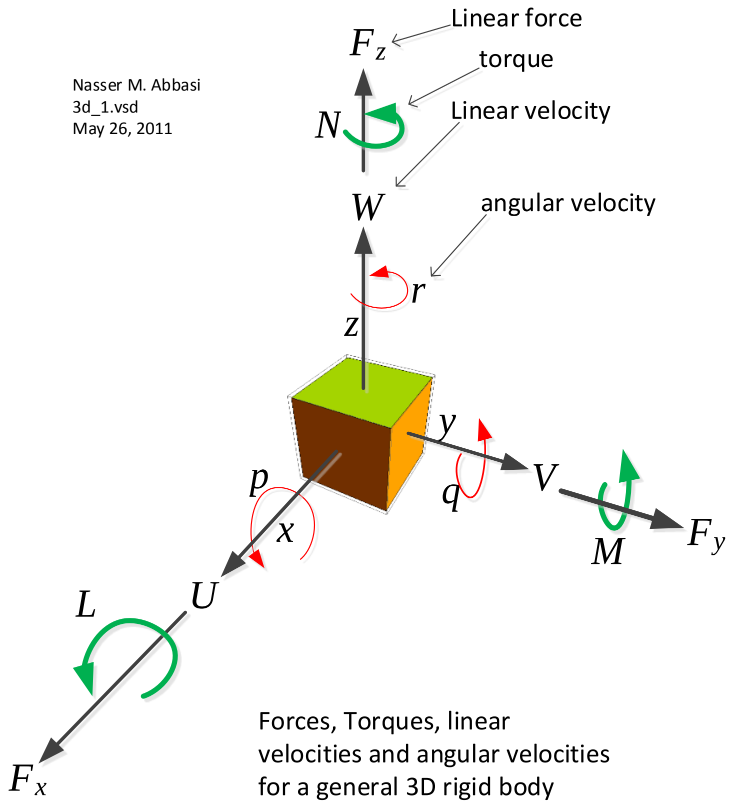
Finding linear acceleration of center of mass of a rigid body under pure rotation using fixed body coordinates.

In the above U is the speed of the center of mass in the direction of the x axis, where this axis is fixed on the body itself. Similarly, V is the speed of the center of mass in the direction of the y axis, where the y axis is attached to the body itself.

Just remember that all these speeds (i.e. U,V) and accelerations (a_x, a_y) are still being measured by an observer in the inertial frame. It is only that the directions of the velocity components of the center of mass is along an axis fixed on the body. Only the direction. But actual speed measurements are still done by a stationary observer. Since clearly if the observer was sitting on the body itself, then they will measure the speeds to be zero in that case.

2.6 Velocity and acceleration of rigid body 3D

2.6.1 Using Vehicle dynamics notations



$$\mathbf{v} = \begin{pmatrix} U \\ V \\ W \end{pmatrix} \qquad \boldsymbol{\omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \qquad \boldsymbol{\tau} = \begin{pmatrix} L \\ M \\ N \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}$$

$$\mathbf{F} = \underset{\text{linear momentum}}{\frac{d}{dt}\mathbf{p}} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d}{dt}\mathbf{v} = m\overbrace{\begin{pmatrix} \dot{U} + qW - rV \\ \dot{V} + rU - pW \\ \dot{W} + pV - qU \end{pmatrix}}^{\mathbf{a}}$$

$$\boldsymbol{\tau} = \underset{\text{Angular momentum}}{\frac{d}{dt}\mathbf{H}} = \frac{d}{dt}(\mathbf{I}\boldsymbol{\omega})$$

Derivation of this is much more complicated than with the case of linear motion (F=ma), since m is scalar there, but for rotation, I is matrix. See next page for the derivation

Figure 2.9: Vehicle dynamics notations

2.6.2 3D Not Using vehicle dynamics notations

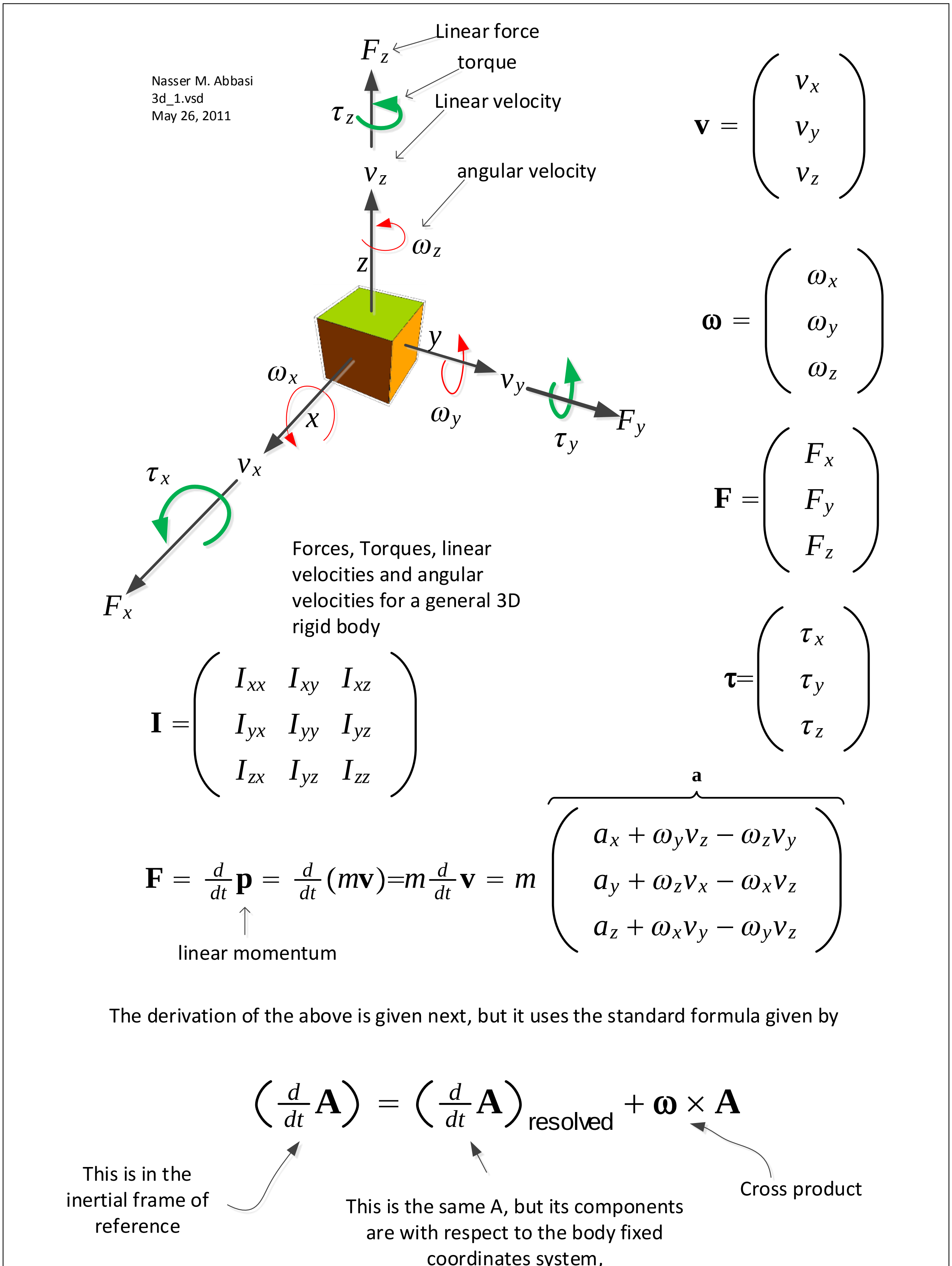


Figure 2.10: 3D Not Using vehicle dynamics notations

2.6.2.1 Derivation for $F = ma$ in 3D

$$
\begin{aligned}
\boldsymbol{F} &= \frac{d}{dt}\boldsymbol{p} \\
&= \frac{d}{dt}(m\boldsymbol{v}) \\
&= m\frac{d}{dt}\boldsymbol{v} \\
&= m\left[\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \otimes \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}\right] \\
&= m\left[\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + \det\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ v_x & v_y & v_z \end{vmatrix}\right] \\
&= m\left[\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + \begin{pmatrix} \omega_y v_z - \omega_z v_y \\ -(\omega_x v_z - \omega_z v_x) \\ \omega_x v_y - \omega_y v_x \end{pmatrix}\right] \\
&= m\begin{pmatrix} a_x + \omega_y v_z - \omega_z v_y \\ a_y - \omega_x v_z + \omega_z v_x \\ a_z + \omega_x v_y - \omega_y v_x \end{pmatrix}
\end{aligned}
$$

2.6.2.2 Derivation for $\tau = I\omega$ in 3D

Let $A = I\omega$ then using the rule

$$
\begin{aligned}
\boldsymbol{\tau} &= \left(\frac{d}{dt}\boldsymbol{A}\right) \\
&= \left(\frac{d}{dt}\boldsymbol{A}\right)_{\text{resolved}} + \boldsymbol{\omega} \times \boldsymbol{A}
\end{aligned}
$$

Then $\tau = I\omega$ can be found for the general case

$$
\begin{aligned}
\boldsymbol{\tau} &= \frac{d}{dt}\left[\overbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}^{\boldsymbol{A}}\right] + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \overbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}^{\boldsymbol{A}} \\
&= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}\begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z \end{pmatrix} \\
&= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}\begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} + \det\left|\begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) & (I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) & (I_{zx}\omega_x + I_{yz}\omega_y + I_{zz} \end{matrix}\right. \\
&= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}\begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} + \begin{pmatrix} \omega_y(I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z) - \omega_z(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) \\ \omega_x(I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) \\ \omega_x(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) \end{pmatrix}
\end{aligned}
$$

2.6.2.3 Derivation for $\tau = I\omega$ in 3D using principle axes

The above derivation simplifies now since we will be using principle axes. In this case, all cross products of moments of inertia vanish.

$$
I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}
$$

Hence

$$
\begin{aligned}
\boldsymbol{\tau} &= \frac{d}{dt}\left[\overbrace{\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}^{\boldsymbol{A}}\right] + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \overbrace{\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}^{\boldsymbol{A}} \\
&= \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}\begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{pmatrix} \\
&= \begin{pmatrix} I_{xx}\alpha_x \\ I_{yy}\alpha_y \\ I_{zz}\alpha_z \end{pmatrix} + \det\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x & I_{yy}\omega_y & I_{zz}\omega_z \end{vmatrix} \\
&= \begin{pmatrix} I_{xx}\alpha_x \\ I_{yy}\alpha_y \\ I_{zz}\alpha_z \end{pmatrix} + \begin{pmatrix} \omega_y(I_{zz}\omega_z) - \omega_z(I_{yy}\omega_y) \\ -\omega_x(I_{zz}\omega_z) + \omega_z(I_{xx}\omega_x) \\ \omega_x(I_{yy}\omega_y) - \omega_y(I_{xx}\omega_x) \end{pmatrix} \\
&= \begin{pmatrix} I_{xx}\alpha_x \\ I_{yy}\alpha_y \\ I_{zz}\alpha_z \end{pmatrix} + \begin{pmatrix} \omega_y\omega_z(I_{zz} - I_{yy}) \\ \omega_x\omega_z(I_{xx} - I_{zz}) \\ \omega_x\omega_y(I_{yy} - I_{xx}) \end{pmatrix}
\end{aligned}
$$

So, we can see how much simpler it became when using principle axes. Compare the above to

$$
\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{pmatrix}\begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} + \begin{pmatrix} \omega_y(I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z) - \omega_z(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) \\ \omega_x(I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z) - \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) \\ \omega_x(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) \end{pmatrix}
$$

So, always use principle axes for the body fixed coordinates system!

2.6.3 Acceleration terms due to rotation and acceleration

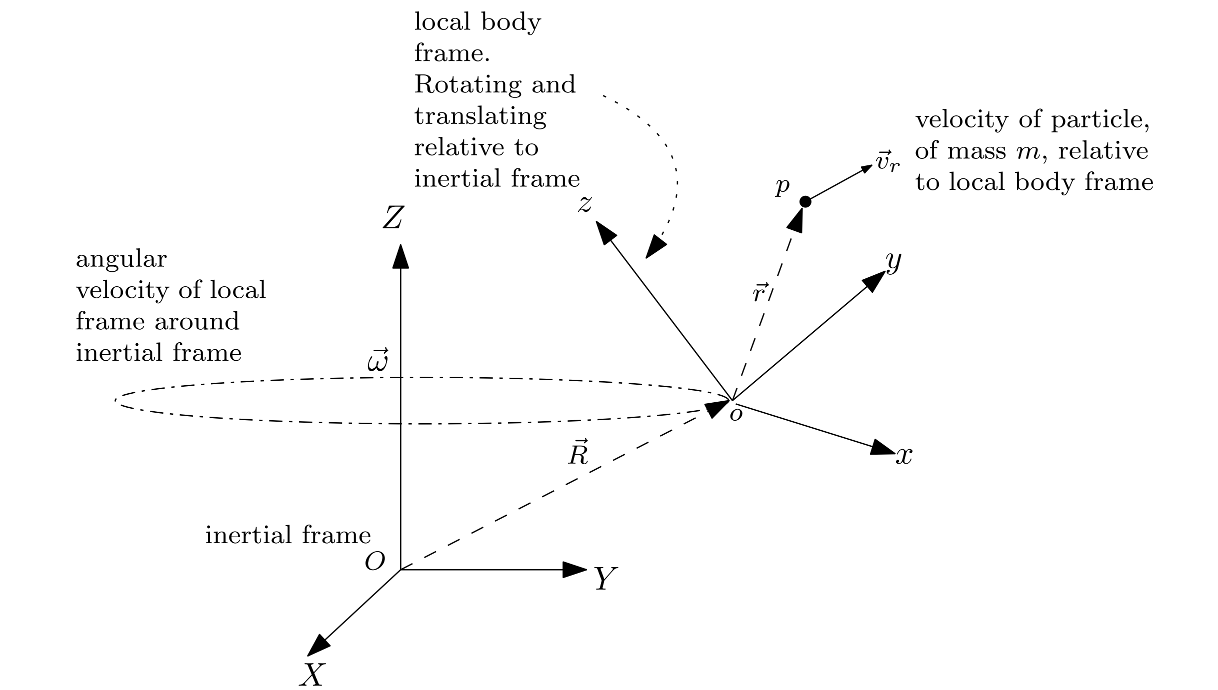


Figure 2.11: Acceleration terms due to rotation 1.

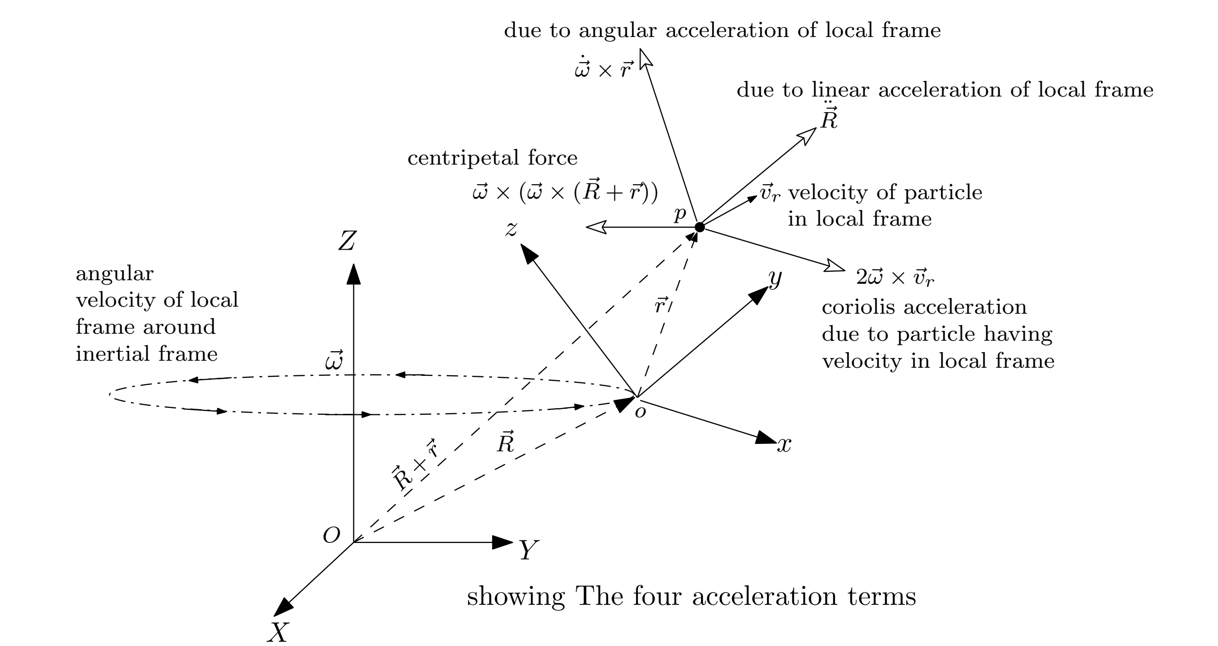


Figure 2.12: Acceleration terms due to rotation 2.

2.7 Wheel spinning precession

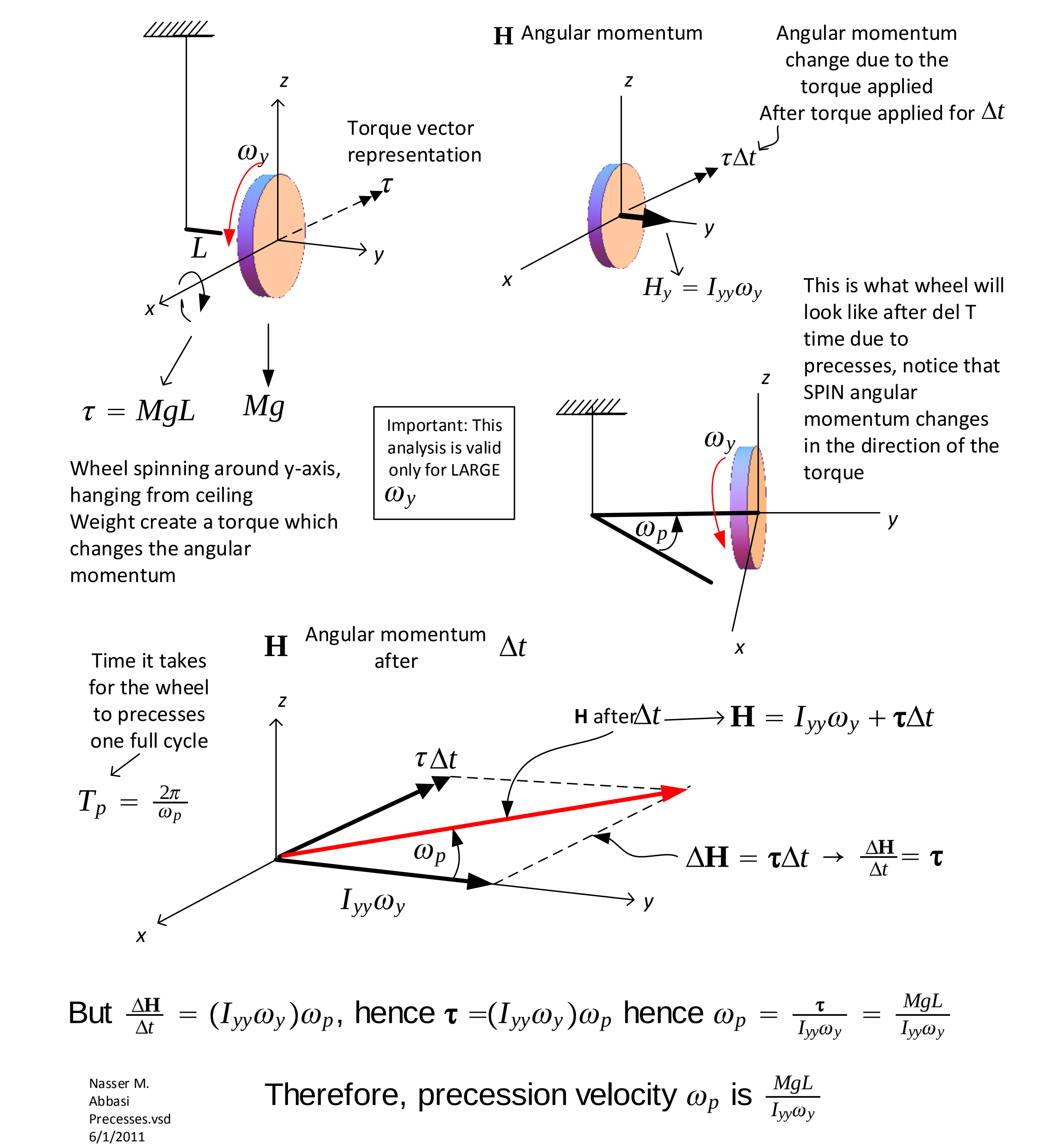


Figure 2.13: Wheel spinning precession

2.8 References

1. Structural Dynamics 5th edition. Mario Paz, William Leigh

2.9 Misc. items

The Jacobian matrix for a system of differential equations, such as

$$
\begin{aligned}
x'(t) &= f(x,y,z) \\
y'(t) &= g(x,y,z) \\
z'(t) &= h(x,y,z)
\end{aligned}
$$

is given by

$$
J = \begin{pmatrix} \frac{df}{dx} & \frac{df}{dy} & \frac{df}{dz} \\ \frac{dg}{dx} & \frac{dg}{dy} & \frac{dg}{dz} \\ \frac{dh}{dx} & \frac{dh}{dy} & \frac{dh}{dz} \end{pmatrix}
$$

For example, for the given the following 3 set of coupled differential equations in n^3

$$
\begin{aligned}
x'(t) &= -y(t) - z(t) \\
y'(t) &= x(t) + ay(t) \\
z'(t) &= b + z(t)\,(x(t) - c)
\end{aligned}
$$

then the Jacobian matrix is

$$
J = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z(t) & 0 & x(t) - c \end{pmatrix}
$$

Now to find stability of this system, we evaluate this matrix at $t = t_0$ where $x(t_0)\,, y(t_0)\,, z(t_0)$ is a point in this space (may be stable point or initial conditions, etc...) and then J become all numerical now. Then we can evaluate the eigenvalues of the resulting matrix and look to see if all eigenvalues are negative. If so, this tells us that the point is a stable point. I.e. the system is stable.

If X is $N(0,1)$ distributed then $mu + sigma * X$ is $N(mu, sigma^2)$ distributed.

Chapter 3

Astrodynamics

3.1 Ellipse main parameters

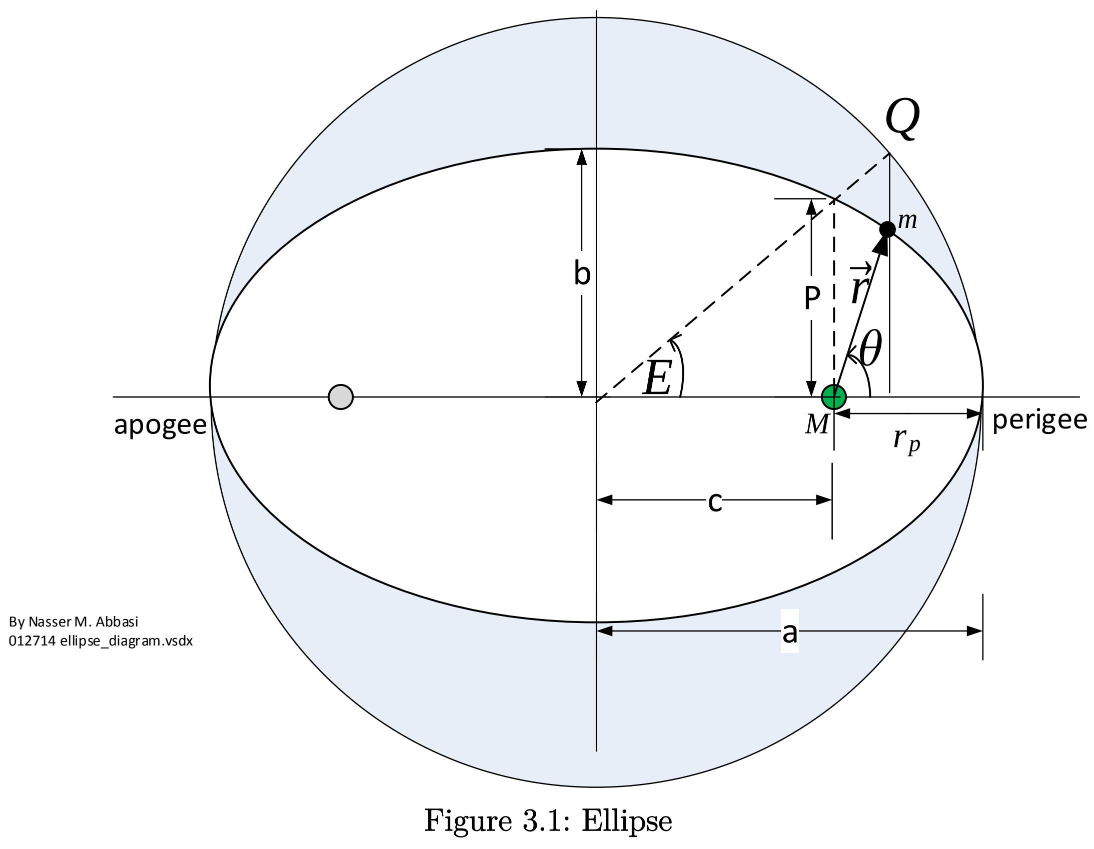


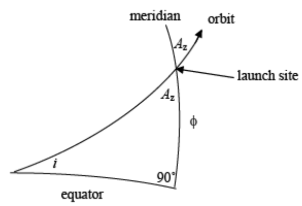
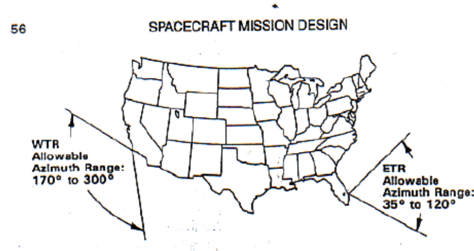
Figure 3.1: Ellipse

3.2 Table of common equations

The following table contains the common relations to use for elliptic motion. Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

term to find	relation
conversion between E and θ	$\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$ $\cos E=\frac{e+\cos\theta}{1+e\cos\theta}$ $\cos\theta=\frac{e-\cos E}{e\cos E-1}$
position of satellite at time t Solve for E, then find θ. τ here is time at *perigee* and n is mean satellite speed.	$E-e\sin E=n(t-\tau)$
eccentricity e	$e=\frac{c}{a}=\frac{r_a-r_p}{r_a+r_p}=\sqrt{1+\frac{2\mathcal{E}h^2}{\mu^2}}$
Major axes a	$a=\frac{r_p(1+e)}{1-e^2}$ $=\frac{r_a(1-e)}{1-e^2}$ $=-\frac{\mu}{2\mathcal{E}}$ $=\sqrt{b^2+c^2}$ $=\frac{p}{1-e^2}$
Minor axes b	$b=a\sqrt{1-e^2}$
r_p	$r_p=\frac{a(1-e^2)}{1+e}$ $=a(1-e)$
r_a	$r_a=\frac{a(1-e^2)}{1-e}$ $=a(1+e)\quad \frac{r_p}{r_a}=\frac{1-e}{1+e}$ $=\frac{h^2}{\mu}\frac{1}{1-e}$
p	$p=a(1-e^2)=\frac{h^2}{\mu}=r_p(1+e)=r_a(1-e)$
specific angular momentum h	$h=r_pv_p=r_av_a=\vec{r}\times\vec{v}=\sqrt{p\mu}$ $h=\sqrt{\mu r}\quad$ (circular orbit)
Total Energy $\mathcal{E}$	$\mathcal{E}=\frac{v^2}{2}-\frac{\mu}{r}=-\frac{\mu}{2a}$

velocity v	$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$ (vis-viva) $v_{escape} = \sqrt{\frac{2\mu}{r}}$ (escape velocity for parabola) $v_{radial} = \sqrt{\frac{\mu}{p}} e \sin\theta$ $v_{normal} = \sqrt{\frac{\mu}{p}}(1 + e\cos\theta)$
$v_{perigee}$ (closest)	$v_p = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)}$ $= \sqrt{\frac{\mu}{p}}(1+e)$ $= \sqrt{\mu\left(\frac{2}{r_p} - \frac{1}{a}\right)}$
v_{apogee} (furthest)	$v_a = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$ $= \sqrt{\frac{\mu}{p}}(1-e)$ $= \sqrt{\mu\left(\frac{2}{r_a} - \frac{1}{a}\right)}$
magnitude of $\vec{r}$	$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}$ $r\cos\theta = a(\cos E - e)$ $r = a(1 - e\cos E)$ (eq 4.2-14 Bate book)
period T	$T = \frac{2}{h}\pi ab = 2\pi\sqrt{\frac{a^3}{\mu}}$
mean satellite speed n	$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$
eccentric anomaly E	$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$
area sweep rate	$\frac{dA}{dt} = \frac{h}{2}$
equation of motion	$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$

spherical coordinates relation	$\cos(i) = \sin(A_z)\cos(\phi)$ where i is the inclination and A_z is the azimuth and ϕ is latitude [1]   Fig. 4.3 Acceptable launch azimuth range for the United States.

Notice in the above, that the period T of satellite depends only on a (for same μ)

In the above, $\mu = GM$ where M is the mass of the body at the focus of the ellipse and G is the gravitational constant. h is the specific mass angular momentum (moment of linear momentum) of the satellite. Hence the units of $\frac{h^2}{\mu}$ is length.

To draw the locus of the satellite (the small body moving around the ellipse, all what we need is the eccentricity e and a, the major axes length. Then by changing the angle θ the path of the satellite is drawn. I have a demo on this here

See `http://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html` for earth facts

This table below is from my class EMA 550 handouts (astrodynamics, spring 2014)

[1] Image is from my class notes, page 7-9, EMA 550, Univ. Of Wisconsin, Madison by professor S. Sandrik

Earth

Mass = 5.974×10^{24} kg

Equatorial radius = 6378 km

$\mu_{Earth} = Gm_{Earth} = 3.986 \times 10^5$ km^3/s^2

Mean distance from the Sun = 1 AU = 1.495978×10^8 km

Sun

Mass = 1.989×10^{30} kg

Mean radius = 695,990 km

$\mu_{Sun} = Gm_{Sun} = 1.327 \times 10^{11}$ km^3/s^2

	Mean distance from the Sun (AU)	Orbit eccentricity	Orbit inclination to the ecliptic plane (deg)	Mass (units of M_{Earth})	Equatorial radius (km)	Sphere of influence radius (km)
Mercury	0.3871	0.2056	7.005	0.0553	2440	1.13×10^5
Venus	0.7233	0.006777	3.395	0.8149	6052	6.17×10^5
Earth	1.000	0.01671	0.000	1.000	6378	9.24×10^5
Mars	1.524	0.09339	1.850	0.1074	3396	5.74×10^5
Jupiter	5.203	0.04839	1.304	317.9	71,492	4.83×10^7
Saturn	9.537	0.05386	2.486	95.18	60,268	3.47×10^7
Uranus	19.19	0.04726	0.7726	14.53	25,559	5.19×10^7
Neptune	30.07	0.008590	1.770	17.14	24,764	8.67×10^7
Pluto	39.48	0.2488	17.14	0.0022	1195	3.17×10^7

Moon

Mass = 7.3483×10^{22} kg

Mean planetary radius = 1738 km

$\mu_{Moon} = Gm_{Moon} = 4902.8$ km^3/s^2

Mean distance from the Earth = 384,400 km

Orbit eccentricity = 0.05490

Orbit inclination to ecliptic = 5.15°

Orbit inclination to the Earth's equatorial plane ranges from 18° to 29°

Sphere of influence radius: 6.61×10^4 km

Universal Constant of Gravitation

$G = 6.674 \times 10^{-11}$ m^3/(kg s^2)

Figure 3.2: Astrodynamics constants

3.3 Flight path angle for ellipse γ

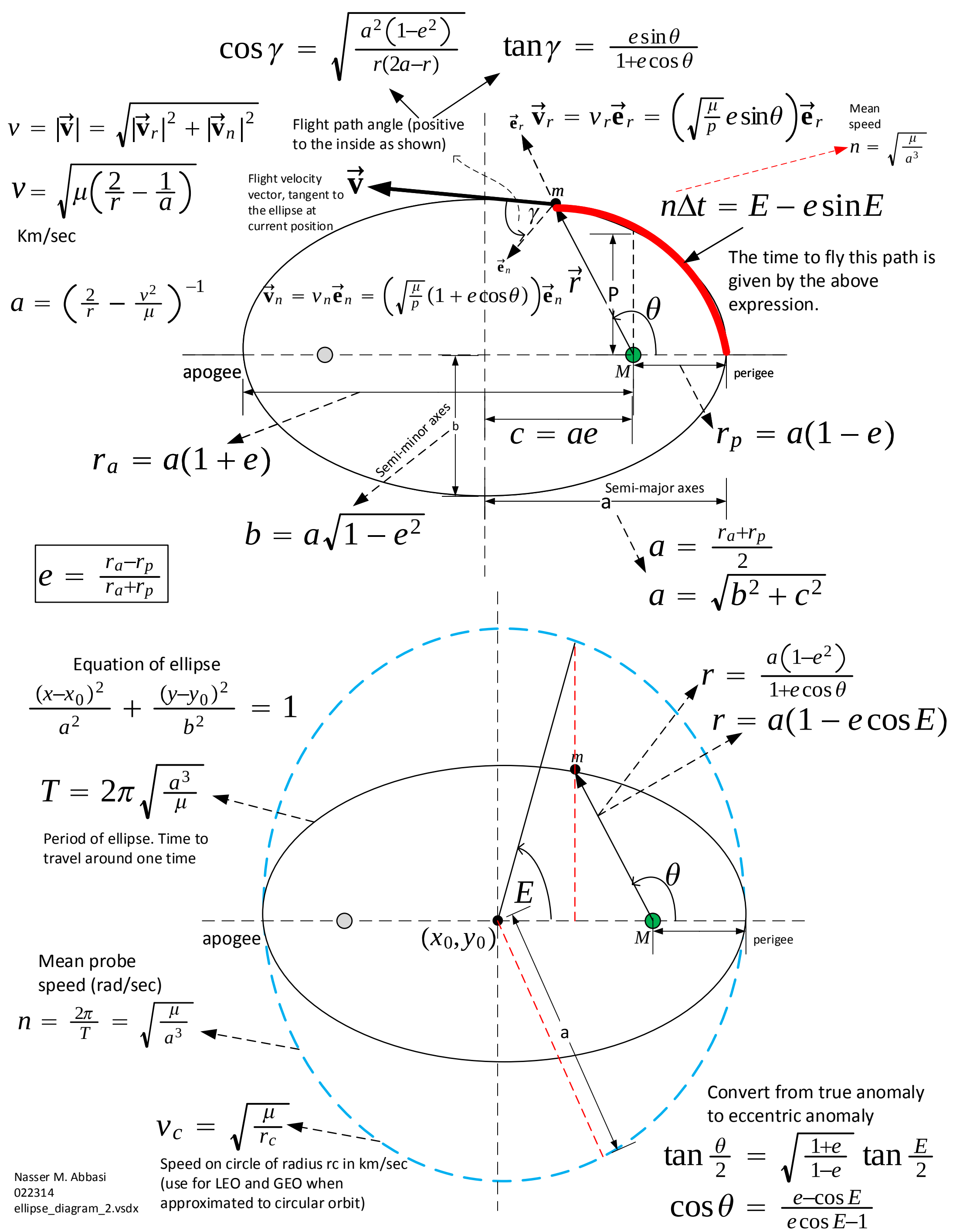


Figure 3.3: Flight path angle

$$
\begin{aligned}
v = |\vec{\boldsymbol{v}}| &= \sqrt{|\vec{\boldsymbol{v}}_r|^2 + |\vec{\boldsymbol{v}}_n|^2} \\
&= \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}
\end{aligned}
$$

To find γ, if r is given then use

$$
\cos\gamma = \sqrt{\frac{ap}{r\,(2a-r)}} = \sqrt{\frac{a^2\,(1-e^2)}{r\,(2a-r)}}
$$

If θ is given, then use

$$
\tan\gamma = \frac{e\sin\theta}{1+e\cos\theta}
$$

3.4 Parabolic trajectory

This diagram shows the parabolic trajectory

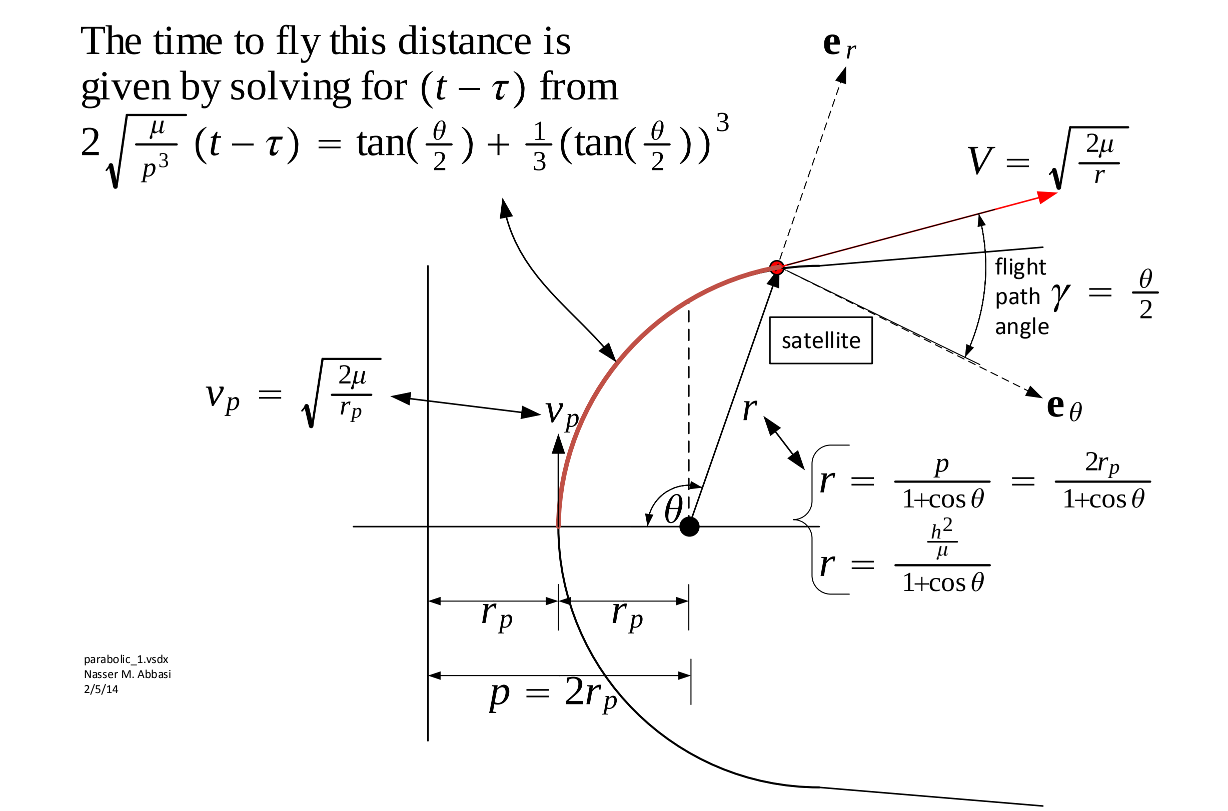


Figure 3.4: parabolic trajectory

3.5 Hyperbolic trajectory

This diagram shows the hyperbolic trajectory

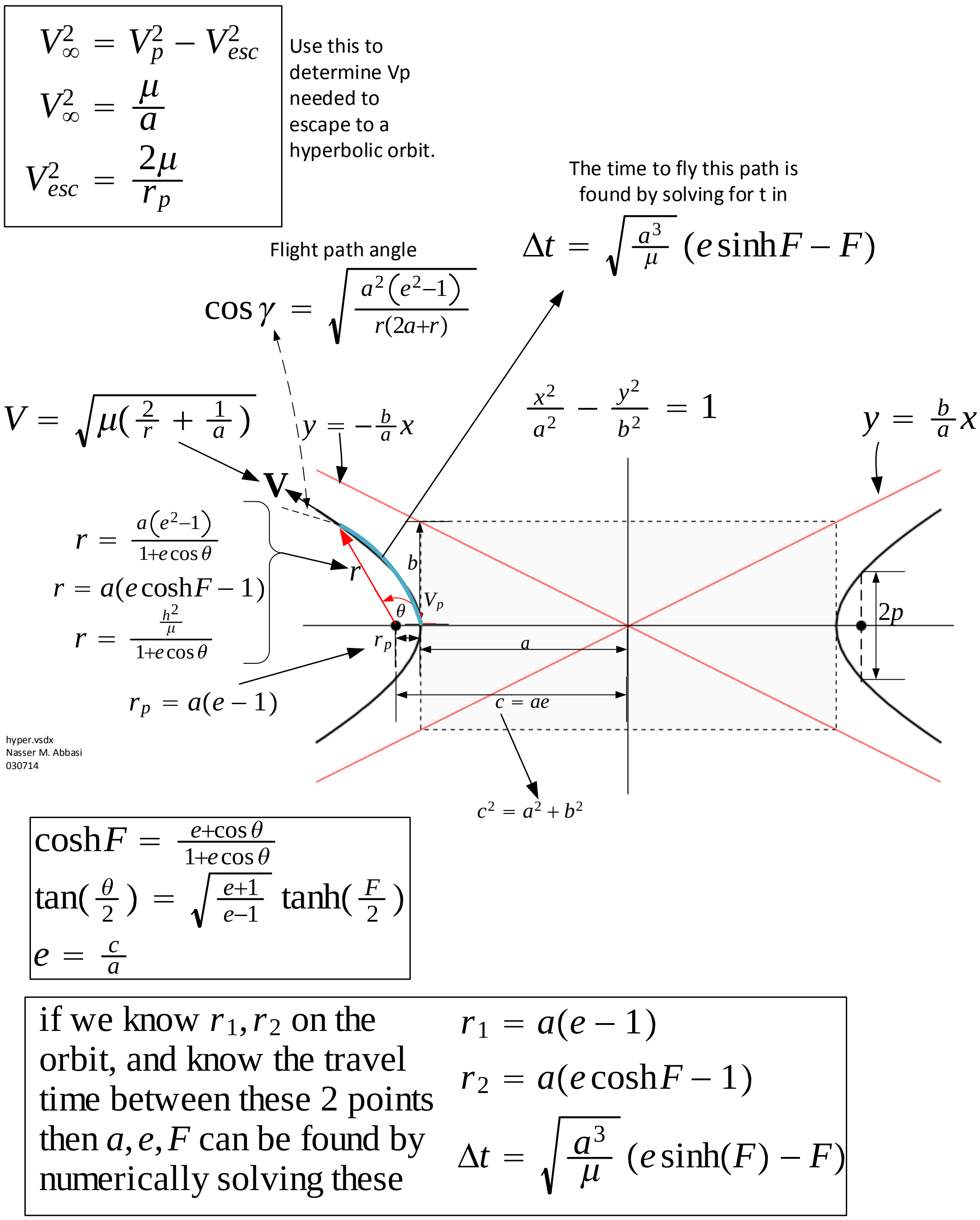


Figure 3.5: hyperbolic trajectory

This diagram below from *Orbital mechanics for Engineering students, second edition*, by Howard D. Curtis, page 109

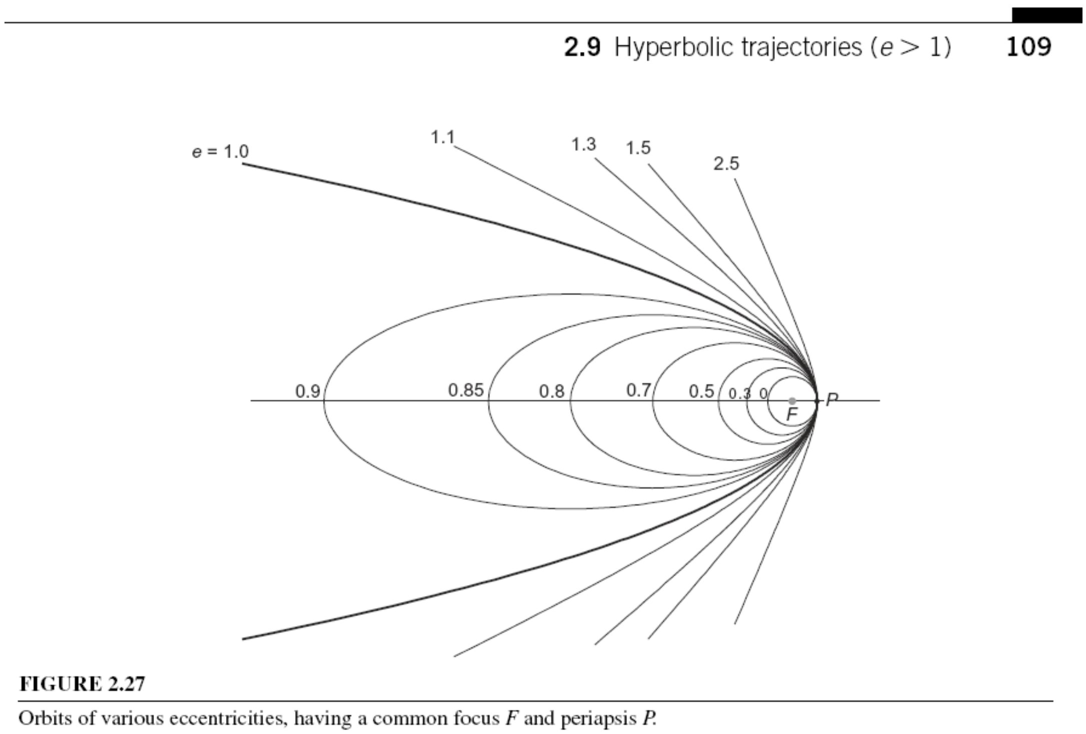


Figure 3.6: diagram below from Orbital mechanics textbook

3.6 showing that energy is constant

Showing that energy $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$ is constant.

Most of such relations starts from the same place. The equation of motion of satellite under the assumption that its mass is much smaller than the mass of the large body (say earth) it is rotating around. Hence we can use $\nu = GM$ and the equation of motion reduces to

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0$$

In the above equation, the vector $\vec{r}$ is the relative vector from the center of the earth to the center of the satellite. The reason the center of earth is used as the origin of the inertial frame of reference is due to the assumption that $M \gg m$ where M is the mass of earth (or the body at the focal of the ellipse) and m is the mass of the satellite. Hence the median center of mass between the earth and the satellite is taken to be the center of earth. This is an approximation, but a very good approximation.

The first step is to dot product the above equation with $\dot{\vec{r}}$ giving

$$\dot{\vec{r}} \cdot \ddot{\vec{r}} + \dot{\vec{r}} \cdot \frac{\mu}{r^3}\vec{r} = 0 \tag{1}$$

And there is the main trick. We look ahead and see that $\dot{\vec{r}} \cdot \ddot{\vec{r}} = \dot{r}\ddot{r}$ but $\dot{r}\ddot{r} = \frac{d}{dt}\left(\frac{\dot{r}^2}{2}\right)$ and we also see that $\dot{\vec{r}} \cdot \frac{\mu}{r^3}\vec{r} = \mu\frac{\dot{r}}{r^2}$ but $\mu\frac{\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{-\mu}{r}\right)$ Hence equation 1 above can be written as

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{r}{\mu}\right) = 0$$

Hence

$$\mathcal{E} = \frac{v^2}{2} - \frac{r}{\mu}$$

Where $\mathcal{E}$ is a constant, which is the total energy of the satellite.

3.7 Earth satellite Transfer orbits

3.7.1 Hohmann transfer

This diagram shows the Hohmann transfer

$$\begin{aligned} a &= \frac{r_1 + r_2}{2} \\ V_1 &= \sqrt{\frac{\mu}{r_1}} \\ V_2 &= \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)} \\ \Delta V_1 &= V_2 - V_1 \\ V_3 &= \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a}\right)} \\ V_4 &= \sqrt{\frac{\mu}{r_2}} \\ \Delta V_2 &= V_4 - V_3 \\ \Delta V &= |\Delta V_1| + |\Delta V_2| \end{aligned}$$

Total Velocity
change needed

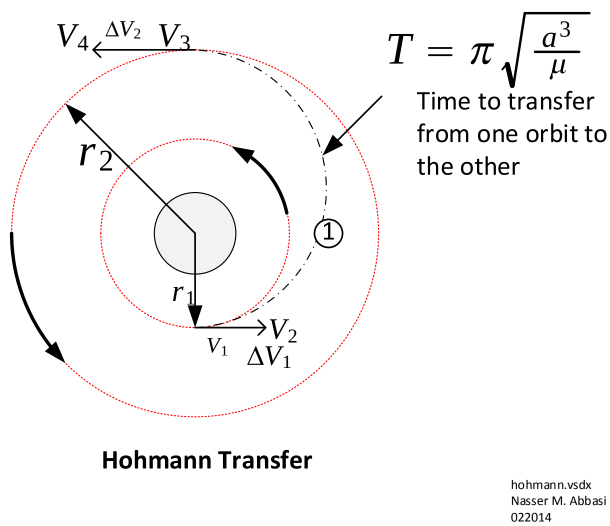
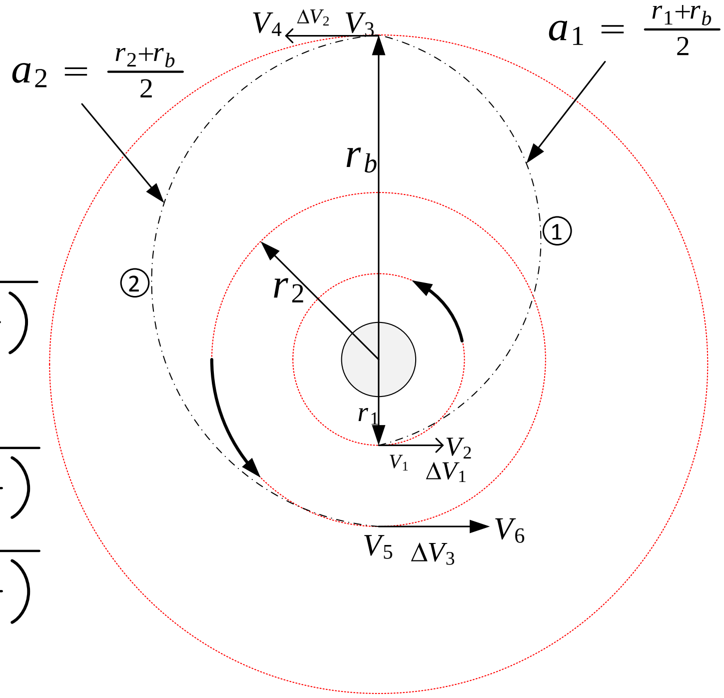


Figure 3.7: Hohmann transfer

3.7.2 Bi-Elliptic transfer orbit

$$
\begin{aligned}
a_1 &= \frac{r_1 + r_b}{2}\\
a_2 &= \frac{r_2 + r_b}{2}\\
V_1 &= \sqrt{\frac{\mu}{r_1}}\\
V_2 &= \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a_1}\right)}\\
\Delta V_1 &= V_2 - V_1\\
V_3 &= \sqrt{\mu\left(\frac{2}{r_b} - \frac{1}{a_1}\right)}\\
V_4 &= \sqrt{\mu\left(\frac{2}{r_b} - \frac{1}{a_2}\right)}\\
\Delta V_2 &= V_4 - V_3\\
V_5 &= \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a_2}\right)}\\
V_6 &= \sqrt{\frac{\mu}{r_2}}\\
\Delta V_3 &= V_6 - V_5\\
\Delta V &= |\Delta V_1| + |\Delta V_2| + |\Delta V_3|
\end{aligned}
$$

Total Velocity
change needed



Bi-Elliptic Transfer

$$T = \pi\sqrt{\frac{a_1^3}{\mu}} + \pi\sqrt{\frac{a_2^3}{\mu}}$$

Time to transfer
from one orbit to
the other

bi_ellptic.vsdx
Nasser M. Abbasi
022314

Figure 3.8: Bi-Elliptic transfer orbit

3.7.3 semi-tangential elliptical transfer

$$a = \frac{r_1 + r_b}{2}$$

$$V_1 = \sqrt{\frac{\mu}{r_1}}$$

$$V_2 = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{a}\right)}$$

$$\Delta V_1 = V_2 - V_1$$

$$V_3 = \sqrt{\mu\left(\frac{2}{r_2} - \frac{1}{a}\right)}$$

$$V_4 = \sqrt{\frac{\mu}{r_2}}$$

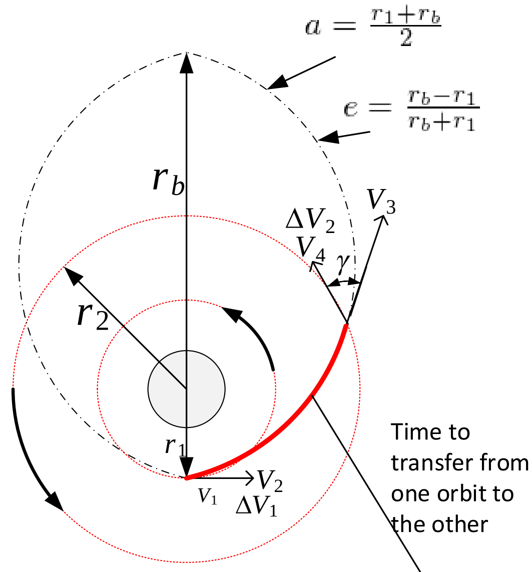
$$e = \frac{r_b - r_1}{r_b + r_1}$$

$$\cos\gamma = \sqrt{\frac{a^2(1-e^2)}{r_2(2a - r_2)}}$$

$$\Delta V_2 = \sqrt{V_4^2 + V_3^2 - 2V_4V_3\cos\gamma}$$

$$\Delta V = |\Delta V_1| + |\Delta V_2|$$

Total Velocity
change needed



Semi-tangential transfer

$$r_2 = a(1 - e\cos E)$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$\Delta t = \frac{1}{n}(E - e\sin E)$$

semi_tangential.vsdx
Nasser M. Abbasi
022314

Figure 3.9: semi-tangential elliptical transfer

3.8 Rocket engines, Hohmann transfer, plane change at equator

two cases: Hohmann transfer, 2 burns, or semi-tangential. All burns at equator.

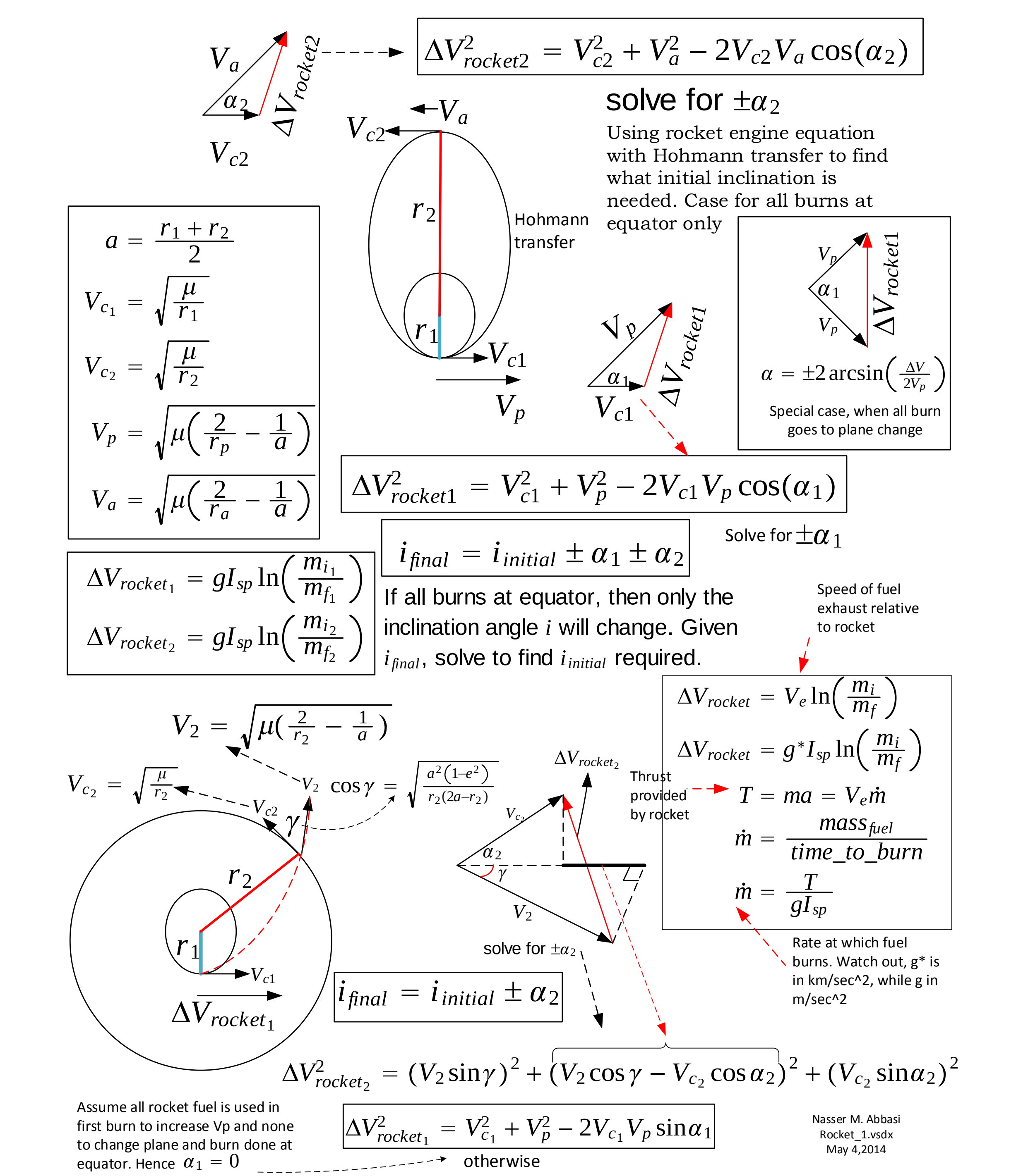


Figure 3.10: Hohmann transfer cases

3.8.0.1 Rocket equation with plane change not at equator

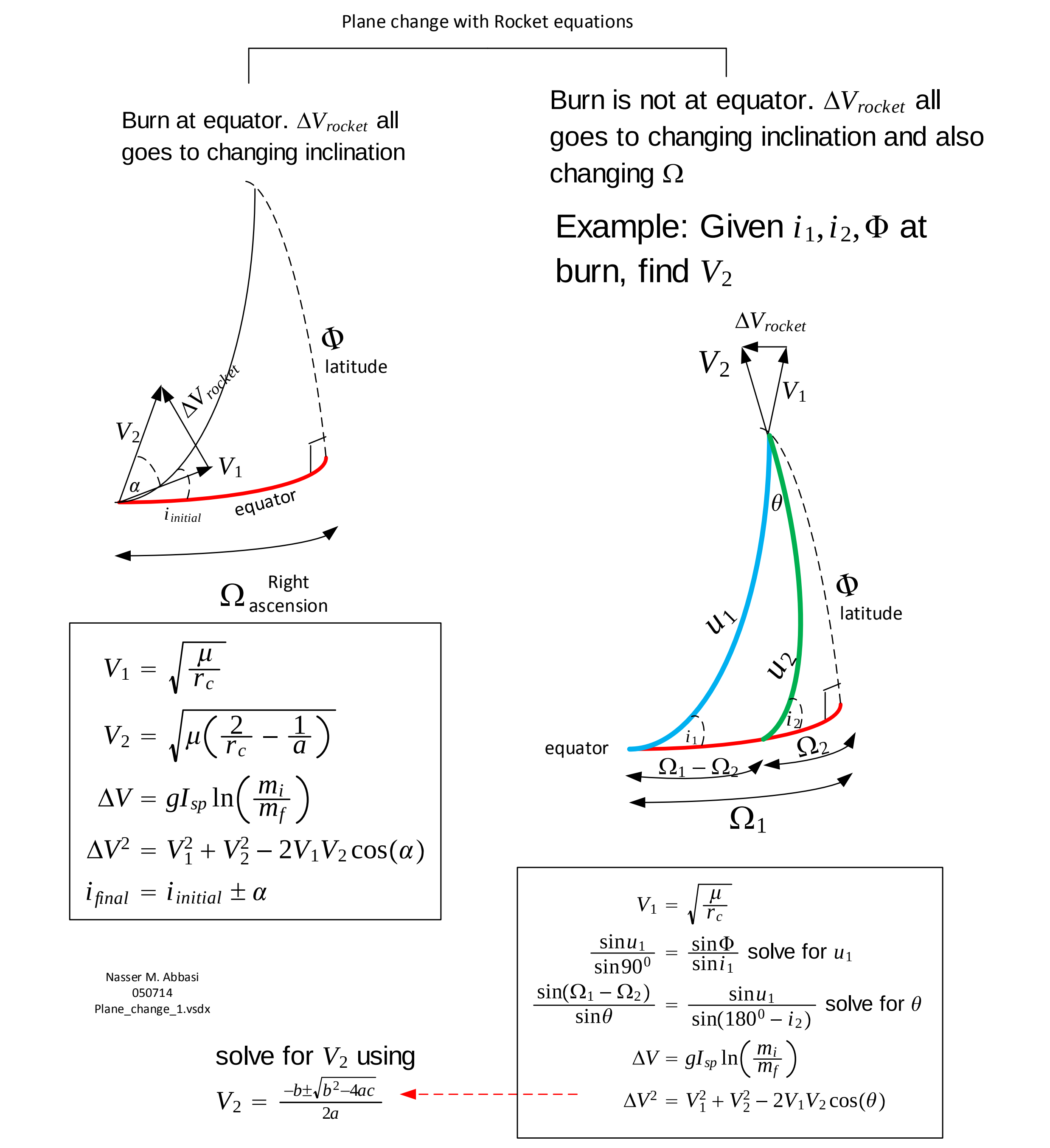


Figure 3.11: Rocket equation with plane change not at equator

3.9 Spherical coordinates

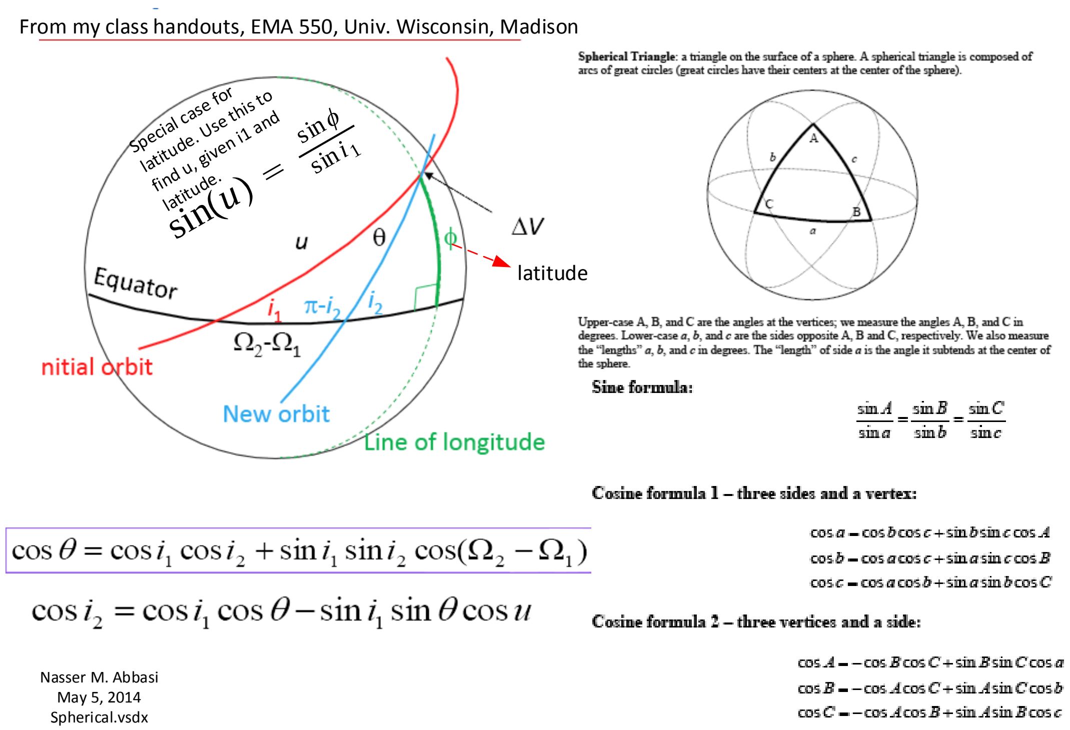


Figure 3.12: Spherical coordinates

3.10 interplanetary transfer orbits

3.10.0.1 interplanetary hohmann transfer orbit, case one

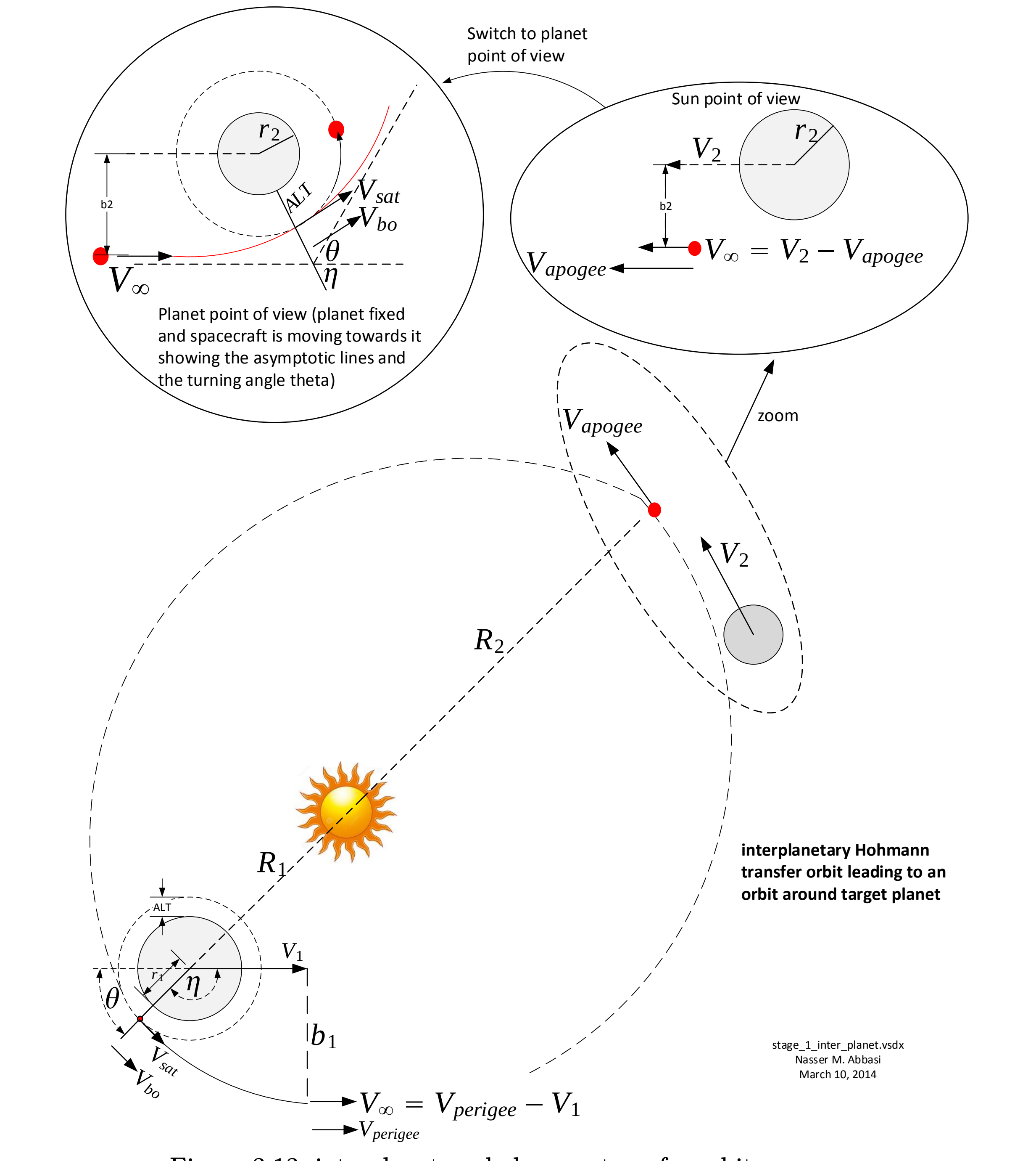


Figure 3.13: interplanetary hohmann transfer orbit, case one

The following are the steps to accomplish the above. The first stage is getting into the Hohmann orbit from planet 1, then reaching the sphere of influence of the second planet. Then we either do a fly-by or do a parking orbit around the second planet. These steps below show how to reach the second planet and do a parking orbit around it.

The input is the following.

1. μ_1 planet one standard gravitational parameter
2. μ_2 planet two standard gravitational parameter
3. μ_{sun} standard gravitational parameter for the sun 1.327×10^8 km
4. r_1 planet one radius
5. r_2 planet two radius
6. alt_1 original satellite altitude above planet one. For example, for LEO use 300 km
7. alt_2 satellite altitude above second planet. (since goal is to send satellite for circular orbit around second planet)
8. R_1 mean distance of center of first planet from the sun. For earth use $AU = 1.495978 \times 10^8$ km
9. R_2 mean distance of center of second planet from the sun. For Mars use 1.524 AU
10. SOI_1 sphere of influence for first planet. For earth use 9.24×10^8 km
11. SOI_2 sphere of influence for second planet.

Given the above input, there are the steps to achieve the above maneuver

1. Find the burn out distance of the satellite $r_{bo} = r_1 + alt_1$
2. Find satellite speed around planet earth (relative to planet) $V_{sat} = \sqrt{\frac{\mu_1}{r_{bo}}}$
3. Find Hohmann ellipse $a = \frac{R_1+R_2}{2}$
4. Find speed of satellite at perigee relative to sun $V_{perigee} = \sqrt{\mu_{sun}\left(\frac{2}{R_1} - \frac{1}{a}\right)}$
5. Find speed of earth (first planet) relative to sun $V_1 = \sqrt{\frac{\mu_{sun}}{R_1}}$
6. Find escape velocity from first planet $V_{\infty,out} = V_{perigee} - V_1$
7. Find burn out speed at first planet by solving the energy equation $\frac{V_{bo}^2}{2} - \frac{\mu_1}{r_{bo}} = \frac{V_{\infty,out}^2}{2} - \frac{\mu_1}{SOI_1}$ for V_{bo}
8. Find ΔV_1 needed at planet one $\Delta V_1 = V_{bo} - V_{sat}$
9. Find e the eccentricity of the escape hyperbola $e = \sqrt{1 + \frac{V_\infty^2 V_{bo}^2 r_{bo}^2}{\mu_1^2}}$

10. Find the angle with the path of planet one velocity vector $\eta = \arccos\left(-\frac{1}{e}\right)$
11. Find the dusk-line angle $\theta = 180^0 - \eta$

The above completes the first stage, now the satellite is in the Hohmann transfer orbit. Assuming it reached the orbit of the second planet ahead of it as shown in the diagram above. Now we start the second stage to land the satellite on a parking orbit around the second planet at altitude alt_2 above the surface of the second planet. These are the steps needed.

1. Find the apogee speed of the satellite $V_{apogree} = \sqrt{\mu_{sun}\left(\frac{2}{R_2} - \frac{1}{a}\right)}$
2. Find speed of second planet $V_2 = \sqrt{\frac{\mu_{sun}}{R_2}}$
3. Find V_∞ entering the second planet sphere of influence $V_{\infty,in} = V_2 - V_{apogree}$
4. Find burn in radius where the satellite will be closest to the second planet. $r_{bo} = r_1 + alt_2$
5. Find burn out speed at second planet by solving the energy equation $\frac{V_{bo}^2}{2} - \frac{\mu_2}{r_{bo}} = \frac{V_{\infty,in}^2}{2} - \frac{\mu_2}{SOI_2}$for V_{bo}
6. Find impact parameter b on entry to second planet SOI $b = \frac{r_{bo}V_{bo}}{V_{\infty,in}}$
7. Find the required satellite speed around the second planet $V_{sat} = \sqrt{\frac{\mu_2}{r_{bo}}}$
8. Find ΔV_2 needed at planet two $\Delta V_2 = V_{sat} - V_{bo}$
9. Find e the eccentricity of the approaching hyperbola on second planet $e = \sqrt{1 + \frac{V_\infty^2 V_{bo}^2 r_{bo}^2}{\mu_2^2}}$
10. Find the angle with the path of planet two velocity vector $\eta = \arccos\left(-\frac{1}{e}\right)$
11. Find the dusk-line angle for second planet $\theta = 2\eta - 180^0$

3.10.1 rendezvous orbits

3.10.1.1 Two satellite, walking rendezvous using Hohmann transfer

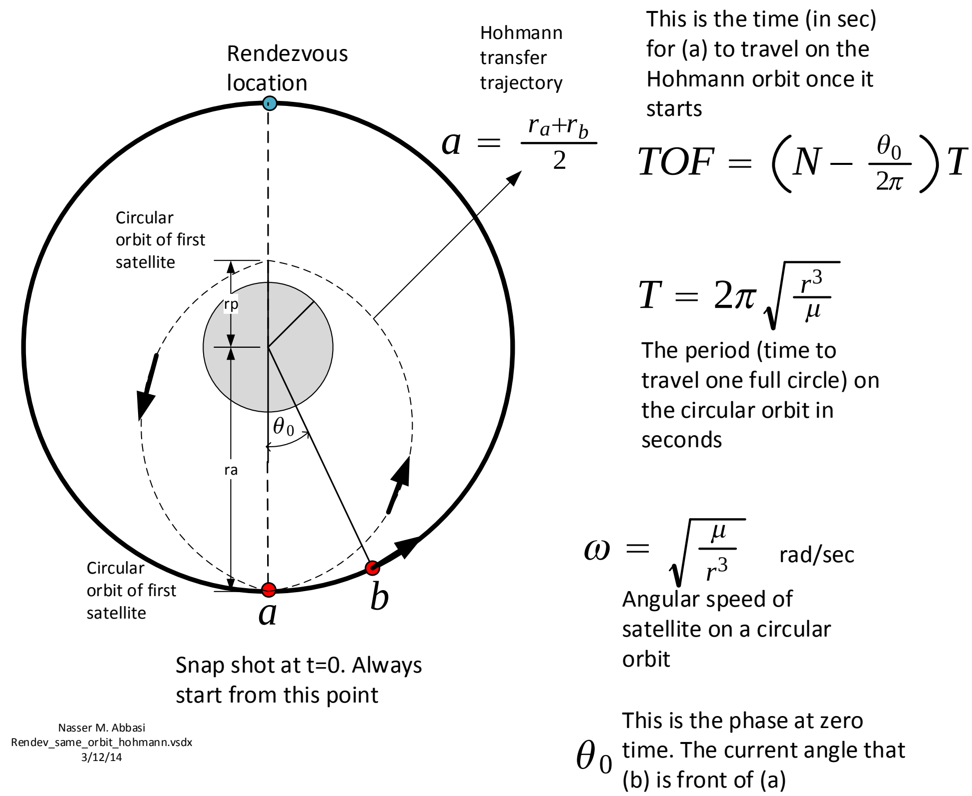


Figure 3.14: Two satellite, walking rendezvous using Hohmann transfer

Algorithm 1 Hohmann Walking Rendezvous Orbit, case 1

function HOHMANN_WALKING_RENDEZVOUS($\theta_0, r, altitude, \mu$)
  $\theta_0 := \theta_0 \frac{\pi}{180}$ ▷ convert from degrees to radian
  $r_a := r + \text{altitude}$
  $T := 2\pi\sqrt{\frac{r_a^3}{\mu}}$ ▷ period of circular orbit
  $N := 1$
  done:=**false**
  while not(done) **do**
    TOF $:= \left(N - \frac{\theta_0}{2\pi}\right) T$
    $a :=$ **solve** $\left(TOF = 2\pi\sqrt{\frac{a^3}{\mu}}\right)$ for a
    $r_p := 2a - r_a$
    if $rp < r$ **then**
      $N = N + 1$
    else
      done:=**true**
    end if
  end while
  $V_{befor} := \sqrt{\frac{\mu}{h}}$
  $V_{after} := \sqrt{\mu\left(\frac{2}{h} - \frac{1}{a}\right)}$
  $\Delta V := 2(V_{after} - V_{before})$
  return (TOF, ΔV)
end function

An example implementation is below

```
hohmannRendezvousSameOrbit[\[Theta]00_, r_, alt_, mu_] :=
Module[{\[Theta]0 = \[Theta]00*Pi/180, n = 1, delT, v1, v2, period, a,
rp, ra, done = False, vBefore, vAfter},
ra     = r + alt;
period = 2 Pi Sqrt[ra^3/mu];
While[Not[done],
delT = (n - \[Theta]0 /(2 Pi)) period ;
a = First@Select[a /. NSolve[delT == 2 Pi Sqrt[a^3/mu], a],
Element[#, Reals] &];
rp = 2 a - ra;
If[rp < r,(*we hit the earth, try again*)
n = n + 1,
done = True
]
];
vBefore = Sqrt[mu/h];
```

```
vAfter  = Sqrt[mu (2/h - 1/a)];
{delT, 2 (vAfter - vBefore)}  (*return value*)
]
```

And calling the above

```
mu = 324859;
alt = 1475.776;
r = 6052;
\[Theta]0 = 3.80562; (*degree*)
hohmannRendezvousSameOrbit[\[Theta]0, r, alt, mu]
```

gives

```
{7123.89, -0.0467913}
```

3.10.1.2 Two satellite, separate orbits, rendezvous using Hohmann transfer, coplaner

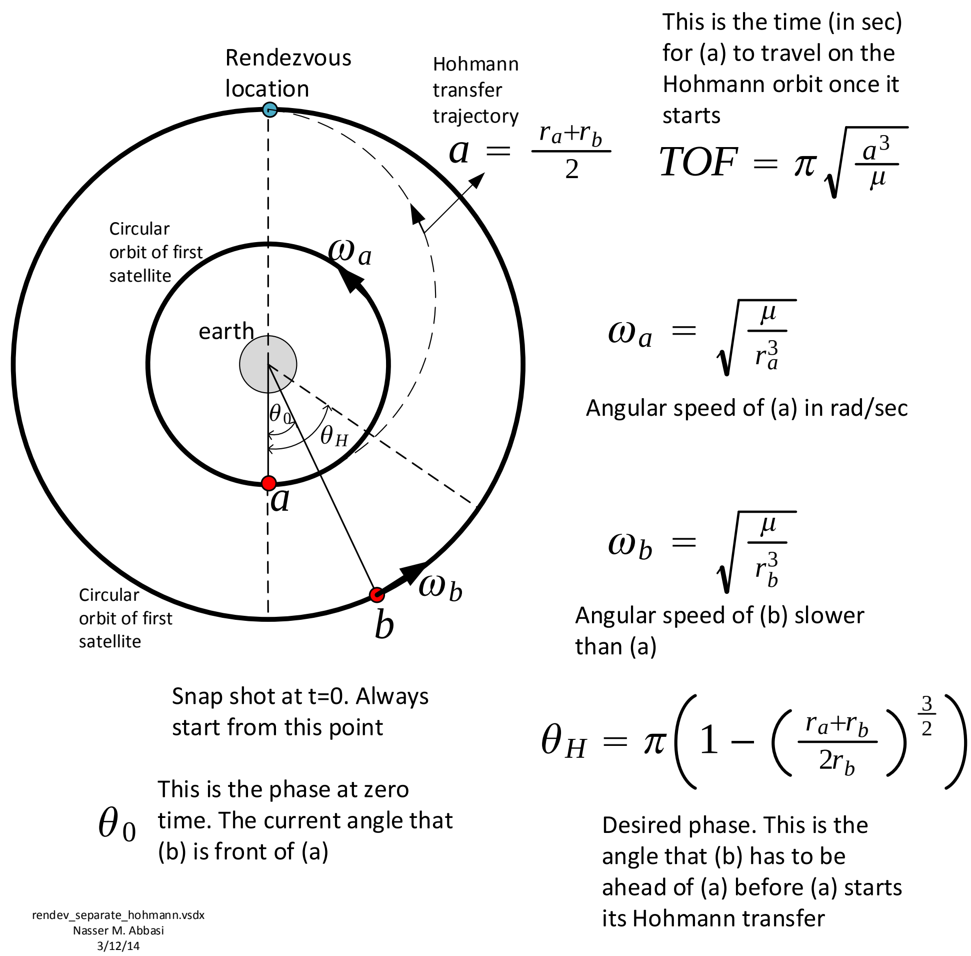


Figure 3.15: Two satellite, separate orbits, rendezvous using Hohmann transfer, coplaner

Algorithm 2 Hohmann rendezvous algorithm, case 1

function HOHMANN_RENDEZVOUS_1(θ_0, r_a, r_b, μ)
- $\theta_0 := \theta_0 \frac{\pi}{180}$ ▷ convert from degrees to radian
- $a := \frac{r_a + r_b}{2}$ ▷ Hohmann orbit semi-major axes
- $\text{TOF} := \pi \sqrt{\frac{a^3}{\mu}}$ ▷ time of flight on Hohmann orbit
- $\theta_H := \pi \left(1 - \left(\frac{r_a + r_b}{2 r_b}\right)^{3/2}\right)$ ▷ required phase angle before starting Hohmann transfer
- $\omega_a := \sqrt{\frac{\mu}{r_a^3}}$ ▷ angular speed of lower rad/sec
- $\omega_b := \sqrt{\frac{\mu}{r_b^3}}$ ▷ angular speed of higher satellite rad/sec
- **if** $\theta_0 \leq \theta_H$ **then** ▷ adjust initial angle if needed
  - $\theta_0 := \theta_0 + 2\pi$
- **end if**
- wait_time := $\frac{\theta_0 - \theta_H}{\omega_a - \omega_b}$ ▷ how long to wait before starting Hohmann transfer
- wait_time := wait_time + TOF ▷ now ready to go, add Hohmann transfer time
- **return** wait_time

end function

An example implementation is below (in Maple)

```
hohmann_rendezvous_1:= proc({
theta::numeric:=0,
r1::numeric:=0,
r2::numeric:=0,
mu::numeric:=3.986*10^5})
local theta0,thetaH,TOF,a,omega1,omega2,wait_time;
theta0  := evalf(theta*Pi/180);
a       := (r1+r2)/2;
TOF     := Pi*(sqrt(a^3/mu));
omega1  := sqrt(mu/r1^3);
omega2  := sqrt(mu/r2^3);
thetaH  := evalf(Pi*(1-((r1+r2)/(2*r2))^(3/2)));
if theta0 <= thetaH then
theta0 := theta0+2*Pi;
fi;
wait_time := TOF+(theta0-thetaH)/(omega1-omega2);
eval(wait_time);
end proc:
```

And calling the above for two different cases gives (times in hrs)

```
TOF:=hohmann_rendezvous_1(r1=6678,r2=6878,theta=0):
```

```
evalf(TOF/(60*60));
35.23480353
```

And

```
TOF:=hohmann_rendezvous_1(r1=6678,r2=6878,theta=280):
evalf(TOF/(60*60));
27.49212919
```

3.10.1.3 Two satellite, separate orbits, rendezvous using bi-elliptic transfer, coplaner

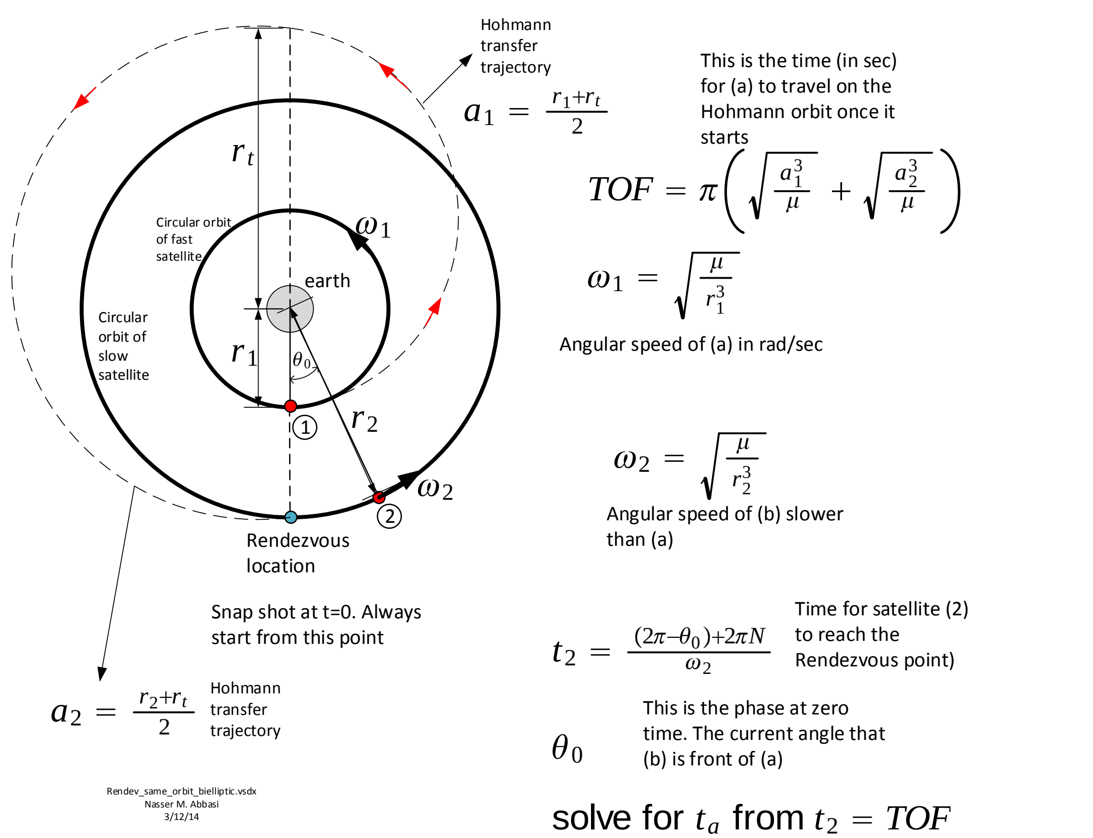


Figure 3.16: Two satellite, separate orbits, rendezvous using bi-elliptic transfer, coplaner

In this transfer, the lower (fast satellite) does not have to wait for phase lock as in the case with Hohmann transfer. The transfer can starts immediately. There is a free parameter N that one select depending on fuel cost requiments or any limitiation on the first transfer orbit semi-major axes distance required. One can start with $N = 0$ and adjust as needed.

Algorithm 3 Hohmann rendezvous algorithm, case 2

function HOHMANN__RENDEZVOUS__2$(\theta_0, r_1, r_2, N, \mu)$
    $\theta_0 := \theta_0 \frac{\pi}{180}$ ▷ convert from degrees to radian
    $\theta_H := \pi\left(1 - \left(\frac{r_1+r_2}{2r_2}\right)^{3/2}\right)$ ▷ Find Hohmann ideal phase angle before transfer
    if $\theta_0 = \theta_H$ **and** $N = 0$ **then** ▷ adjust for special case
        $a := \frac{r_2+r_1}{2}$
        $\text{TOF} := \pi\left(\sqrt{\frac{a^3}{\mu}}\right)$
    else
        $\omega_2 := \sqrt{\frac{\mu}{r_2^3}}$ ▷ angular speed of slower satellite in rad/sec
        $t_2 := \frac{(2\pi-\theta_0)+2\pi N}{\omega_2}$ ▷ find time of light of the slower satellite
        $a_1 := \frac{r_t+r_1}{2}$
        $a_2 := \frac{r_t+r_2}{2}$
        $\text{TOF} := \pi\left(\sqrt{\frac{a_1^3}{\mu}+\frac{a_2^3}{\mu}}\right)$ ▷ time of flight for the fast satellite
        $r_t :=$ **solve** $(t_2 = TOF)$ for r_t ▷ Solve numerically for r_t
    end if
    return TOF
end function

An example implementation is below in Maple

```
hohmann_rendezvous_2:= proc({
theta::numeric:=0,
r1::numeric:=0,
r2::numeric:=0,
N::nonnegint:=0,
mu::numeric:=3.986*10^5})
local theta0,thetaH,TOF;
theta0  := theta*Pi/180;
thetaH  := Pi*(1-((r1+r2)/(2*r2))^(3/2));
if theta0 = thetaH and N = 0 then
proc()
local a:=(r1+r2)/2;
TOF:= Pi*(sqrt(a^3/mu));
end proc()
else
proc()
local t2,a1,a2,rt,omega2;
omega2 := sqrt(mu/r2^3);
t2     := ((2*Pi-theta0)+2*Pi*N)/omega2;
a1     := (rt+r1)/2;
```

```
a2     := (rt+r2)/2;
TOF    := Pi*(sqrt(a1^3/mu)+sqrt(a2^3/mu));
rt     := op(select(is, [solve(t2=TOF,rt)], real));
end proc()
fi;
eval(TOF);
end proc:
```

And calling the above for two different cases gives

```
TOF:=hohmann_rendezvous_2(theta=0,r1=6678,r2=6878,N=0):
evalf(TOF/(60*60)); #in hrs
1.576892101
TOF:=hohmann_rendezvous_2(theta=160,r1=6678,r2=6878,N=1):
evalf(TOF/(60*60));  #in hrs
2.452943266
```

3.10.2 Semi-tangential transfers, elliptical, parabolic and hyperbolic

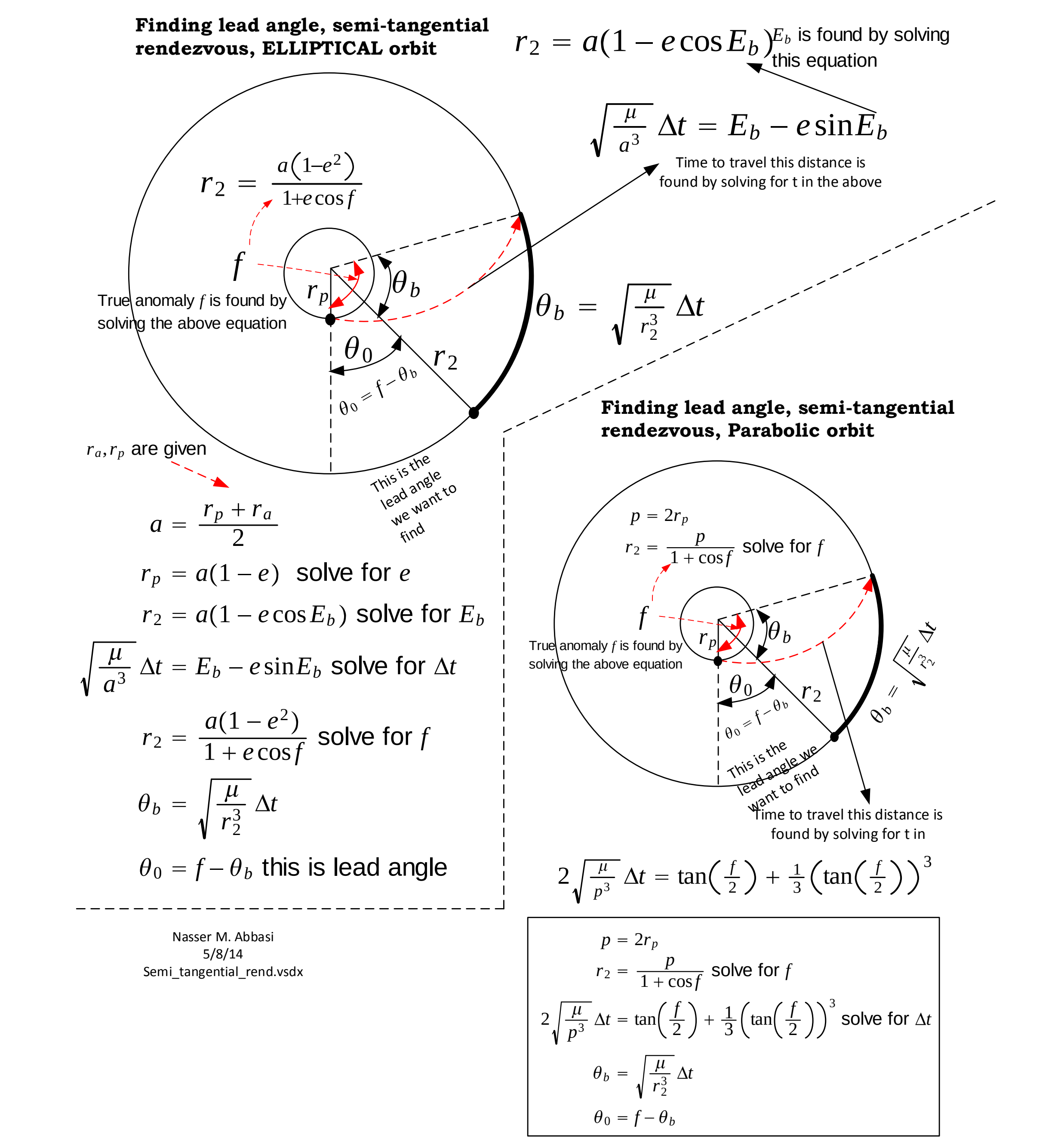


Figure 3.17: Semi-tangential transfers, elliptical, parabolic and hyperbolic

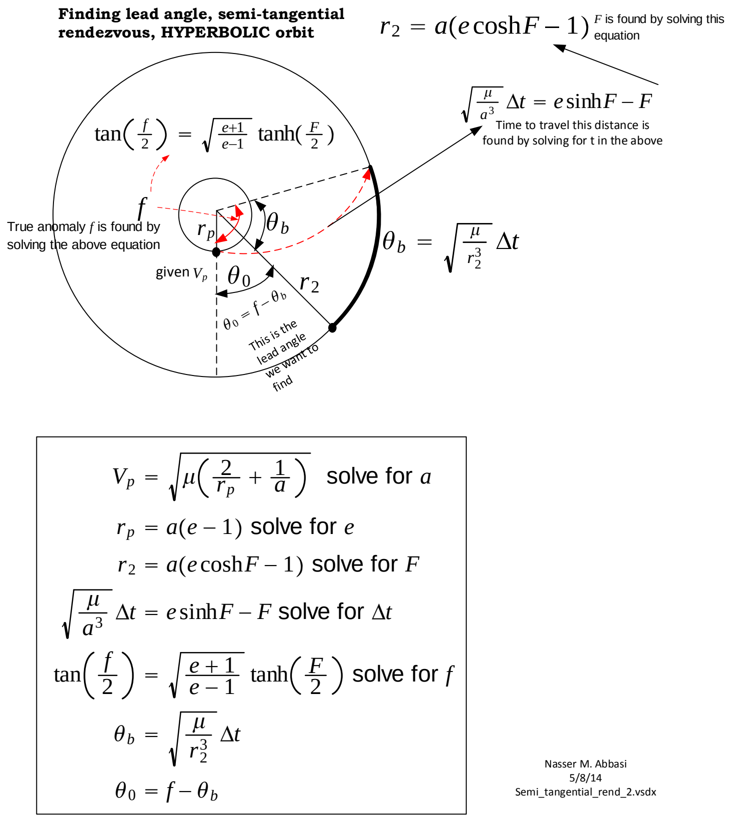


Figure 3.18: Semi-tangential transfers, elliptical, parabolic and hyperbolic (2)

3.10.3 Lagrange points

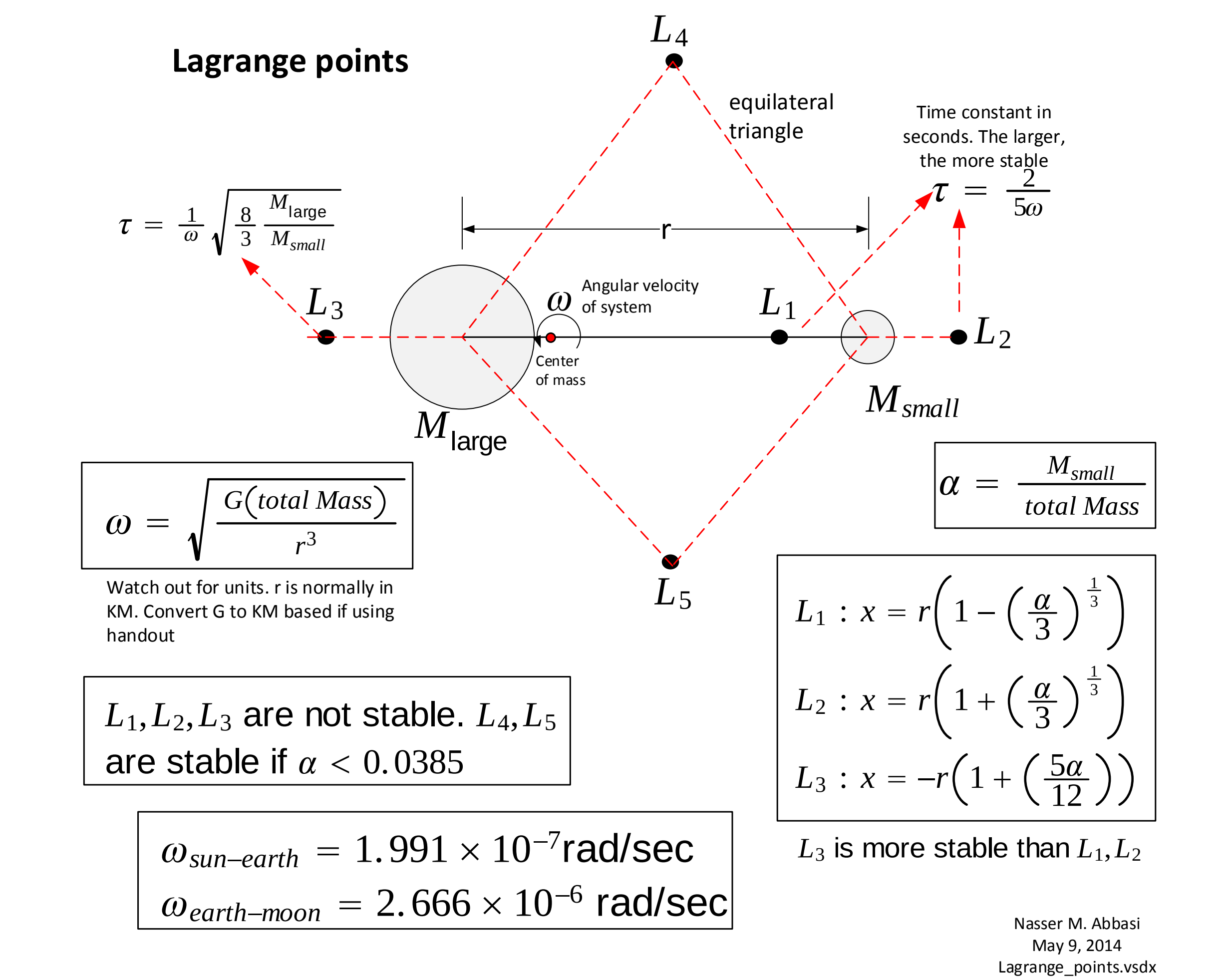


Figure 3.19: Lagrange points

3.10.4 Orbit changing by low contiuous thrust

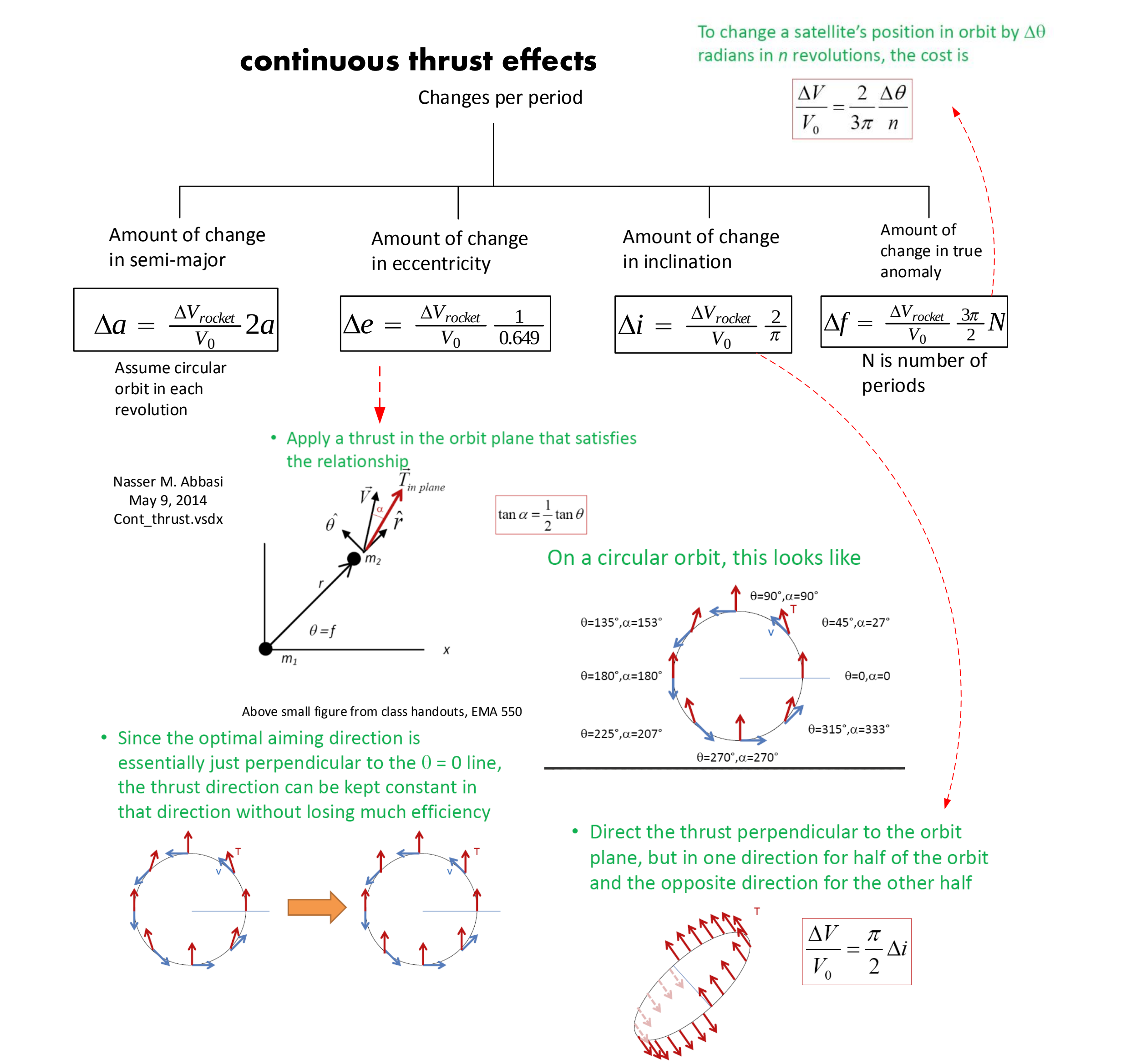


Figure 3.20: Orbit changing by low contiuous thrust

3.10.5 References

1. *Orbital mechanics for Engineering students, second edition*, by Howard D. Curtis
2. Orbital Mechanics, Vladimir Chobotov, second edition, AIAA
3. Fundamentals of Astrodynamics, Bates, Muller and White. Dover1971

Chapter 4

Dynamic of flights

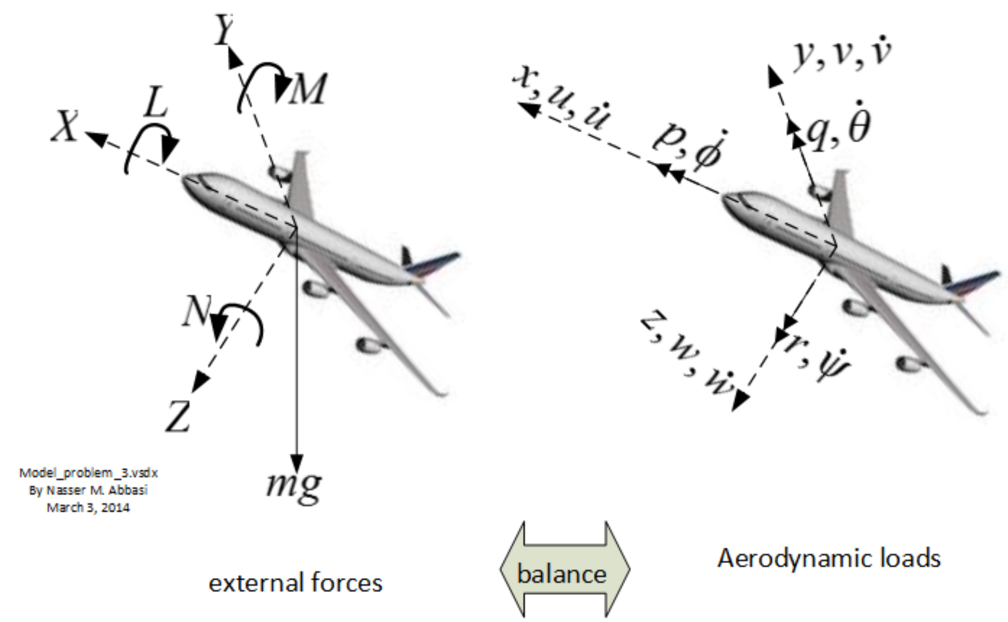


Figure 4.1: Dynamic of flights

4.1 Wing geometry

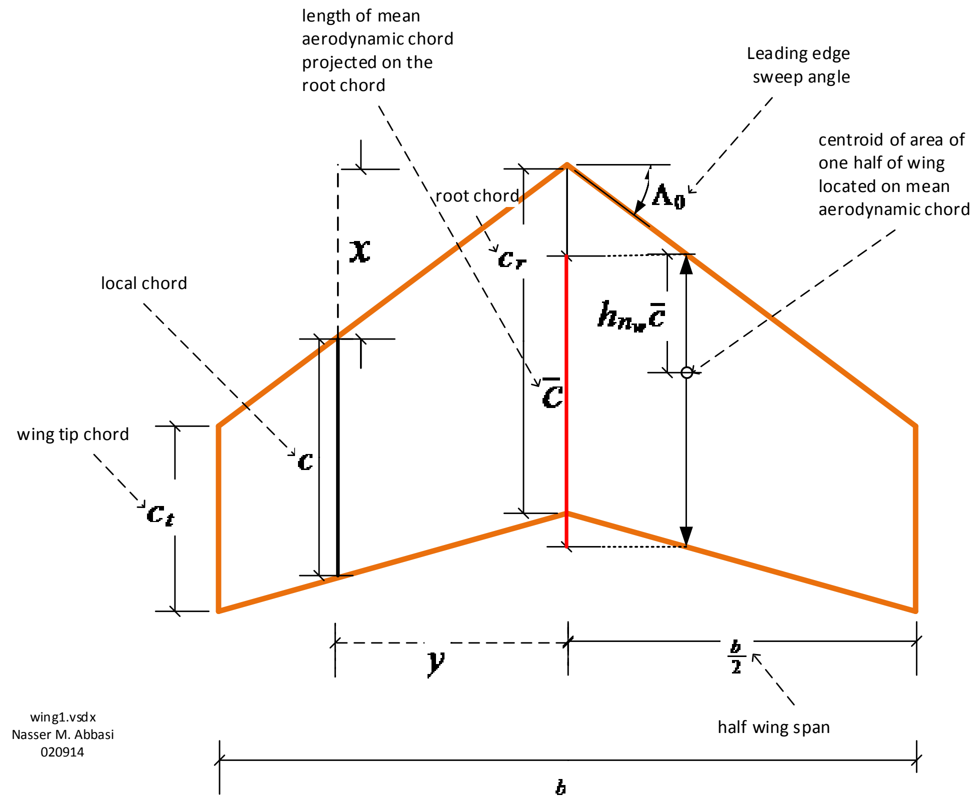


Figure 4.2: Wing geometry

C_r below is the core chord of the wing.

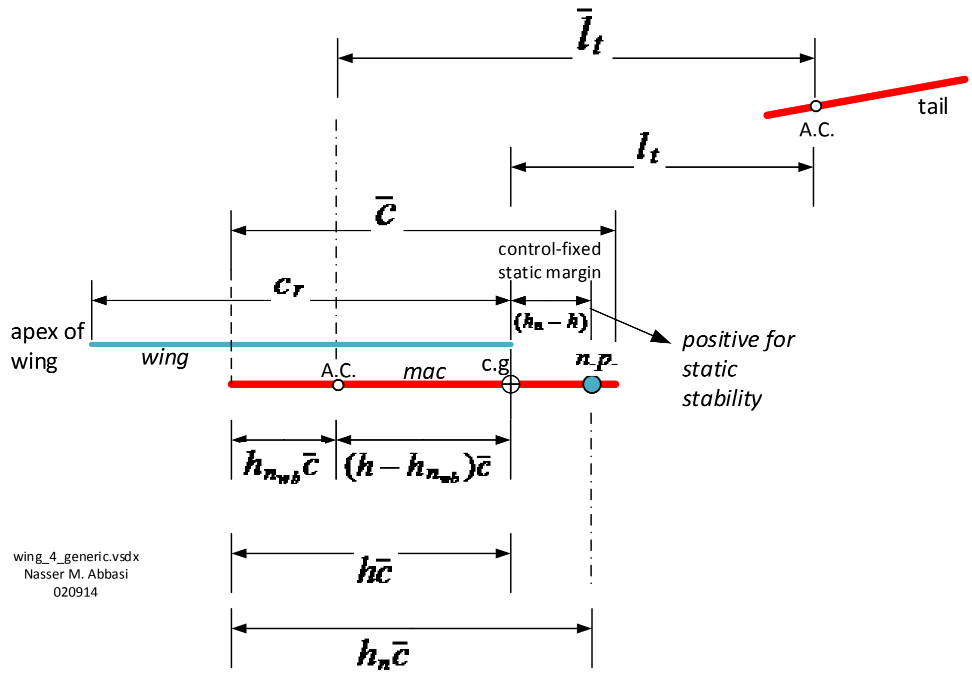


Figure 4.3: core chord of the wing

This is a diagram to use to generate equations of longitudinal equilibrium.

This distance is called the stick-fixed static margin $k_m = (h_n - h)\,\bar{c}$ Must be positive for static stability

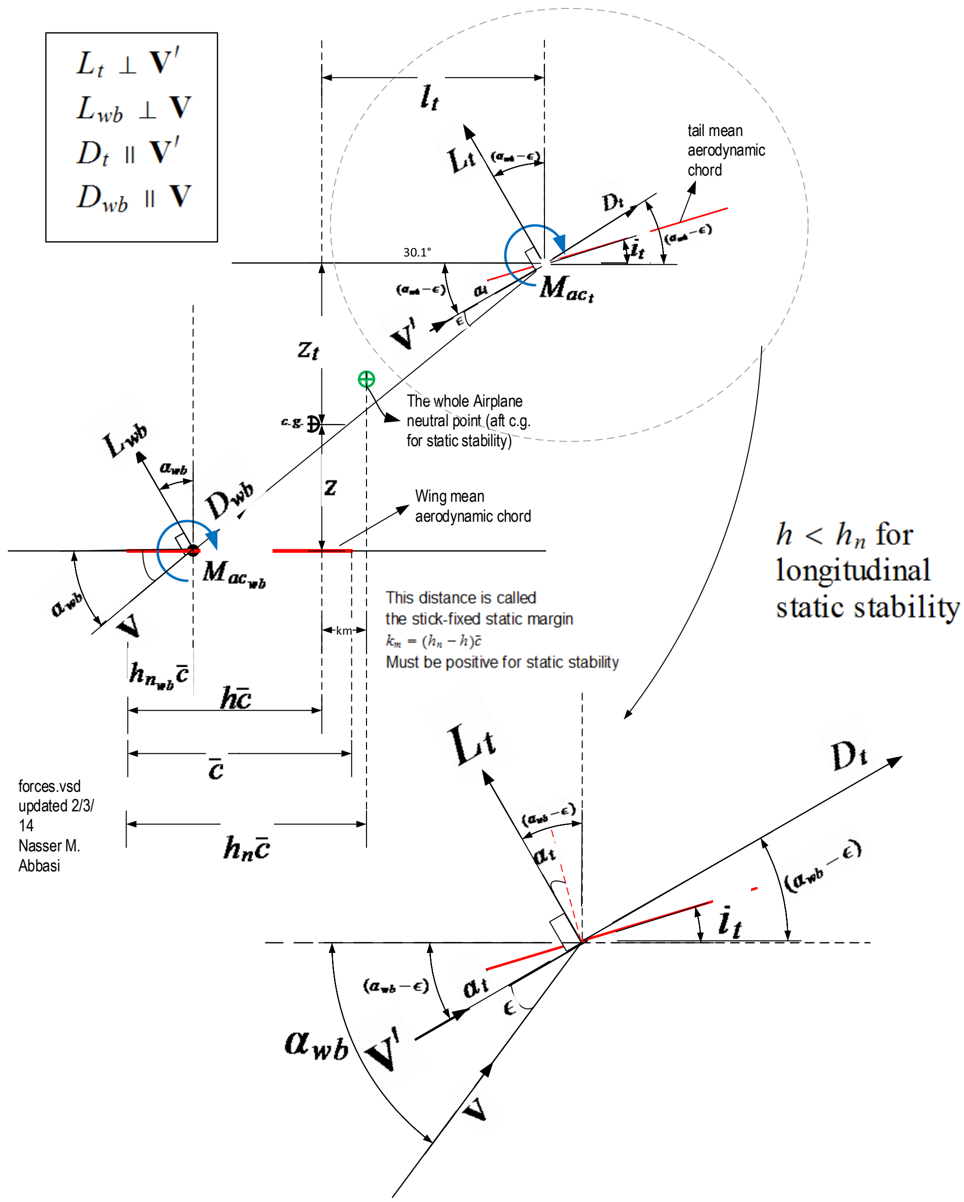


Figure 4.4: equations of longitudinal equilibrium

4.2 Summary of main equations

This table contain some definitions and equations that can be useful.

#	equation	meaning/use
1	$$\begin{aligned} C_L &= \frac{\partial C_L}{\partial \alpha}\alpha \\ &= C_{L_\alpha}\alpha \\ &= a\alpha \end{aligned}$$	C_L is lift coefficient. α is angle of attack. a is slope $\frac{\partial C_L}{\partial \alpha}$ which is the same as C_{L_α}

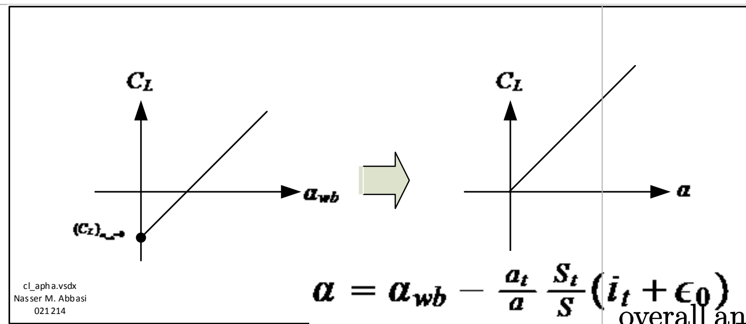
2	$C_{L_w} = C_{L_{w_\alpha}} \alpha$	wing lift coefficient
3	$C_D = C_{D_{\min}} + kC_L^2$	drag coefficient
4	$C_{m_w} = C_{m_{ac_w}} + (C_{L_w} + C_{D_{\min}} \alpha_w)(h - h_{n_w}) + (C_L \alpha_w - C_{D_w}) \frac{z}{\bar{c}}$	pitching moment coefficient due to wing only about the C.G. of the airplane assuming small α_w. This is simplified more by assuming $C_{D_w} \alpha_w \ll C_{L_w}$ and $(C_L \alpha_w - C_{D_w}) \ll 1$
5	$C_{m_w} = C_{m_{ac_w}} + C_{L_w}(h - h_{n_w})$	simplified wing Pitching moment
6	$\begin{aligned} C_{m_{wb}} &= C_{m_{ac_{wb}}} + C_{L_{wb}}(h - h_{n_w}) \\ &= C_{m_{ac_{wb}}} + \frac{\partial C_{L_{wb}}}{\partial \alpha_{wb}} \alpha_{wb}(h - h_{n_w}) \\ &= C_{m_{ac_{wb}}} + a_{wb} \alpha_{wb}(h - h_{n_w}) \end{aligned}$	simplified pitching moment coefficient due to wing and body about the C.G. of the airplane. α_{wb} is the angle of attack
7	$C_{L_t} = \frac{L_t}{\frac{1}{2}\rho V^2 S_t}$	C_{L_t} is the lift coefficient generated by tail. S_t is the tail area. V is airplane air speed
8	$L = L_{wb} + L_t$	total lift of airplane. L_{wb} is lift due to body and wing and L_t is lift due to tail
9	$C_L = C_{L_{wb}} + \frac{S_t}{S} C_{L_t}$	coefficient of total lift of airplane. $C_{L_{wb}}$ is coefficient of lift due to wing and body. C_{L_t} is lift coefficient due to tail. S is the total wing area. S_t is tail area
10	$M_t = -l_t L_t = -l_t C_{L_t} \frac{1}{2} \rho V^2 S_t$	pitching moment due to tail about C.G. of airplane
11	$C_{m_t} = \frac{M_t}{\frac{1}{2}\rho V^2 S_t \bar{c}} = -\frac{l_t}{\bar{c}} \frac{S_t}{S} C_{L_t} = -V_H C_{L_t}$	pitching moment coefficient due to tail. $V_H = \frac{l_t}{\bar{c}} \frac{S_t}{S}$ is called *tail volume*

12	$\begin{aligned} V_H &= \frac{l_t}{\bar{c}}\frac{S_t}{S} \\ \bar{V}_H &= \frac{\bar{l}_t}{\bar{c}}\frac{S_t}{S} \end{aligned}$	introducing $\bar{V}_H$ *bar tail volume* which is V_H but uses $\bar{l}_t$ instead of l_t. Important note. V_H depends on location of C.G., but $\bar{V}_H$ does not. $\bar{l}_t = l_t + (h - h_{n_{wb}})\,\bar{c}$
13	$C_{m_t} = -\bar{V}_H C_{L_t} + C_{L_t}\frac{S_t}{S}(h - h_{n_{wb}})$	pitching moment coefficient due to tail expressed using $\bar{V}_H$. This is the one to use.
14	C_{m_p}	pitching moment coefficient due to propulsion about airplane C.G.
15	$C_m = C_{m_{wb}} + C_{m_t} + C_{m_p}$	total airplane pitching moment coefficient about airplane C.G.
16	$\begin{aligned} C_m &= C_{m_{wb}} + C_{m_t} + C_{m_p} \\ &= \left[C_{m_{ac_{wb}}} + C_{L_{wb}}(h - h_{n_w})\right] + \left[-\bar{V}_H C_{L_t} + C_{L_t}\frac{S_t}{S}(h - h_{n_{wb}})\right] + C_{m_p} \\ &= C_{m_{ac_{wb}}} + \overbrace{\left(C_{L_{wb}} + C_{L_t}\frac{S_t}{S}\right)}^{C_L}(h - h_{n_w}) - \bar{V}_H C_{L_t} + C_{m_p} \\ &= C_{m_{ac_{wb}}} + C_L(h - h_{n_w}) - \bar{V}_H C_{L_t} + C_{m_p} \end{aligned}$	simplified total Pitching moment coefficient about airplane C.G.
17	$\begin{aligned} \frac{\partial C_m}{\partial \alpha} &= \frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha} + \frac{\partial C_L}{\partial \alpha}(h - h_{n_w}) - \bar{V}_H\frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha} \\ C_{m_\alpha} &= \frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha} + C_{L_\alpha}(h - h_{n_w}) - \bar{V}_H\frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha} \end{aligned}$	derivative of total pitching moment coefficient C_m w.r.t airplane angle of attack α
18	$h_n = h_{n_{wb}} - \frac{1}{\frac{\partial C_L}{\partial \alpha}}\left(\frac{\partial C_{m_{ac_{wb}}}}{\partial \alpha} - \bar{V}_H\frac{\partial C_{L_t}}{\partial \alpha} + \frac{\partial C_{m_p}}{\partial \alpha}\right)$	location of airplane neutral point of airplane found by setting $C_{m_\alpha} = 0$ in the above equation
19	$\begin{aligned} \frac{\partial C_m}{\partial \alpha} &= \frac{\partial C_L}{\partial \alpha}(h - h_n) \\ C_{m_\alpha} &= C_{L_\alpha}(h - h_n) \end{aligned}$	rewrite of C_{m_α} in terms of h_n. Derived using the above two equations.
20	$k_n = h_n - h$	static margin. Must be Positive for static stability

4.2.0.1 Writing the equations in linear form

The following equations are derived from the above set of equation using what is called the linear form. The main point is to bring into the equations the expression for C_{L_t} written in term of α_{wb}. This is done by expressing the tail angle of attack α_t in terms of α_{wb} via the downwash angle and the i_t angle. $\frac{\partial C_{L_{wb}}}{\partial \alpha_{wb}}$ in the above equations are replaced by a_{wb} and $\frac{\partial C_{L_t}}{\partial \alpha_t}$ is replaced by a_t. This replacement says that it is a linear relation between C_L and the corresponding angle of attack. The main of this rewrite is to obtain an expression for C_m in terms of α_{wb} where α_t is expressed in terms of α_{wb}, hence α_t do not show explicitly. The linear form of the equations is what from now on.

#	equation	meaning/use
1	$C_{L_{wb}} = \frac{\partial C_{L_{wb}}}{\partial \alpha_{wb}} \alpha_{wb} = a_{wb} \alpha_{wb}$ $C_{L_t} = a_t \alpha_t$ $C_{mp} = C_{m_{0p}} + \frac{\partial C_{mp}}{\partial \alpha} \alpha$	a_{wb} is constant, represents $\frac{\partial C_{L_{wb}}}{\partial \alpha_{wb}}$ and $C_{m_{0p}}$ is propulsion pitching moment coeff. at zero angle of attack α
2	$\alpha_t = \alpha_{wb} - i_t - \epsilon$ $\epsilon = \epsilon_0 + \frac{\partial \epsilon}{\partial \alpha} \alpha_{wb}$	main relation that associates α_{wb} with α_t. α_{wb} is the wing-body angle of attack, ϵ is downwash angle at tail, and i_t is tail angle with horizontal reference (see diagram)
3	$C_{L_t} = a_t \alpha_t = a_t \left[\alpha_{wb} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) - i_t - \epsilon_0 \right]$	Lift due to tail expressed using α_{wb} and ϵ (notice that α_t do not show explicitly)
4	$a = a_{wb} \left[1 + \frac{a_t}{a_{wb}} \frac{S_t}{S} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) \right]$	a defined for use with overall lift coefficient
5	$C_L = \overbrace{C_{L_{wb}}}^{a_{wb}\alpha_{wb}} + \frac{S_t}{S} C_{L_t} = a_{wb}\alpha_{wb} + \frac{S_t}{S} a_t \left[\alpha_{wb} \left(1 - \frac{\partial \epsilon}{\partial \alpha} \right) - i_t - \epsilon_0 \right] = a\alpha = (C_L)_{\alpha_{wb}=0} + a\alpha_{wb}$	overall airplane lift using linear relations

6	$\alpha = \alpha_{wb} - \frac{a_t}{a}\frac{S_t}{S}(i_t + \epsilon_0)$ 	overall angle of attack α as function of the wing and body angle of attack α_{wb} and tail angles
7	$C_m = C_{m0} + \frac{\partial C_m}{\partial \alpha}\alpha = C_{m0} + C_{m_\alpha}\alpha$ $C_m = \bar{C}_{m0} + \frac{\partial C_m}{\partial \alpha}\alpha_{wb} = \bar{C}_{m0} + C_{m_\alpha}\alpha_{wb}$	overall airplane pitch moment. Two versions one uses α_{wb} and one uses α
8	$C_{m_\alpha} = a(h - h_{n_{wb}}) - a_t\bar{V}_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) + \frac{\partial C_{mp}}{\partial \alpha}$ $C_{m_\alpha} = a_{wb}(h - h_{n_{wb}}) - a_t V_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) + \frac{\partial C_{mp}}{\partial \alpha}$	Two versions of $\frac{\partial C_m}{\partial \alpha}$ one for α_{wb} and one one uses α
9	$C_{m_0} = C_{m_{ac_{wb}}} + C_{m_{0_p}} + a_t\bar{V}_H(\epsilon_0 + i_t)\left[1 - \frac{a_t}{a}\frac{S_t}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]$ $\bar{C}_{m_0} = C_{m_{ac_{wb}}} + \bar{C}_{m_{0_p}} + a_t V_H(\epsilon_0 + i_t)$	C_{m_0} is total pitching moment coef. at zero lift (does not depend on C.G. location) but $\bar{C}_{m_0}$ is total pitching moment coef. at $\alpha_{wb} = 0$ (not at zero lift). This depends on location of C.G.
10	$\bar{C}_{m_{0_p}} = C_{m_{0_p}} + (\alpha - \alpha_{wb})\frac{\partial C_{mp}}{\partial \alpha}$	
11	$h_n = h_{n_{wb}} + \frac{a_t}{a}\bar{V}_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) - \frac{1}{a}\frac{\partial C_{mp}}{\partial \alpha}$ $= h_{n_{wb}} + \frac{a_t}{a_{wb}\left[1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]}\bar{V}_H\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right) - \frac{1}{a_{wb}\left[1 + \frac{a_t}{a_{wb}}\frac{S_t}{S}\left(1 - \frac{\partial \epsilon}{\partial \alpha}\right)\right]}\frac{\partial C_{mp}}{\partial \alpha}$	Used to determine h_n

4.2.1 definitions

1. Remember that for symmetric airfoil, when the chord is parallel to velocity vector, then the angle of attack is zero, and also the left coefficient is zero. But this is only for symmetric airfoil. For the common campbell airfoil shape, when the chord is parallel to the velocity vector, which means the angle of attack is zero, there will still be lift (small lift, but it is there). What this means, is that the chord line has to tilt down more to get zero lift. This extra tilting down makes the angle of attack negative. If we now draw a line from the right edge of the airfoil parallel to the velocity vector, this line is called the zero lift line (ZLL) see diagram below.
   Just remember, that angle of attack (which is always the angle between the chord and the velocity vector, the book below calls it the geometrical angle of attack) is negative for zero lift. This is when the airfoil is not symmetric. For symmetric airfoil, ZLL and the chord line are the same. This angle is small, -3^0 or so. Depending on shape. See *Foundations of Aerodynamics*, 5th ed, by Chow and Kuethe, here is the diagram.

5.7 **Properties of the Cambered Airfoil** **147**

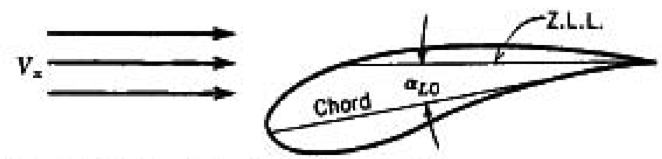


Fig. 5.11. Orientation of airfoil at zero lift.

camber, $c_{m_{c/4}}$ = constant and is negative, as is predicted by Eq. (5.31). (5) The angle of zero lift α_{L0} is zero for the symmetrical section and negative for positive camber, as is indicated by Eq. (5.34).

In Fig. 5.11, an airfoil is shown set at a geometric angle of attack equal to the angle of zero lift. A line on the airfoil parallel to the flight path V_∞ and passing through the trailing edge when the airfoil is set at the orientation of zero lift is called the *zero-lift line* (Z.L.L.) of the airfoil. For symmetrical airfoils, the zero-lift line coincides with the chord line.

The *absolute angle of attack* is defined as the angle included between the flight path and the zero-lift line and is given the symbol α_a. From Fig. 5.12,

$$\alpha_a = \alpha - \alpha_{L0} \tag{5.35}$$

Figure 4.5: diagram from 5th ed, by Chow and Kuethe

2. stall from `http://en.wikipedia.org/wiki/Stall_(flight)`

```
In fluid dynamics, a stall is a reduction in the lift coefficient
generated by a foil as angle of attack increases.[1] This occurs when
the critical angle of attack of the foil is exceeded. The critical
angle of attack is typically about 15 degrees, but it
may vary significantly depending on the fluid, foil, and Reynolds number.
```

3. Aerodynamics in road vehicle wiki page
4. some demos relating to airplane control `http://demonstrations.wolfram.com/ControllingAirplaneFlight/`

   `http://demonstrations.wolfram.com/ThePhysicsOfFlight/`
5. `http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm`
6. Lectures Helicopter Aerodynamics and Dynamics by Prof. C. Venkatesan, Department of Aerospace Engineering, IIT Kanpur `http://www.youtube.com/watch?v=DKWj2WzYXtQ&list=PLAE677E56C97A7C7D`
7. `http://avstop.com/ac/apgeneral/terminology.html` has easy to understand definitions airplane geometry. "The MAC is the mean average chord of the wing"
8. `http://www.tdmsoftware.com/afd/afd.html` airfoil design software

4.3 images and plots collected

These are diagrams and images collected from different places. References is given next to each image.

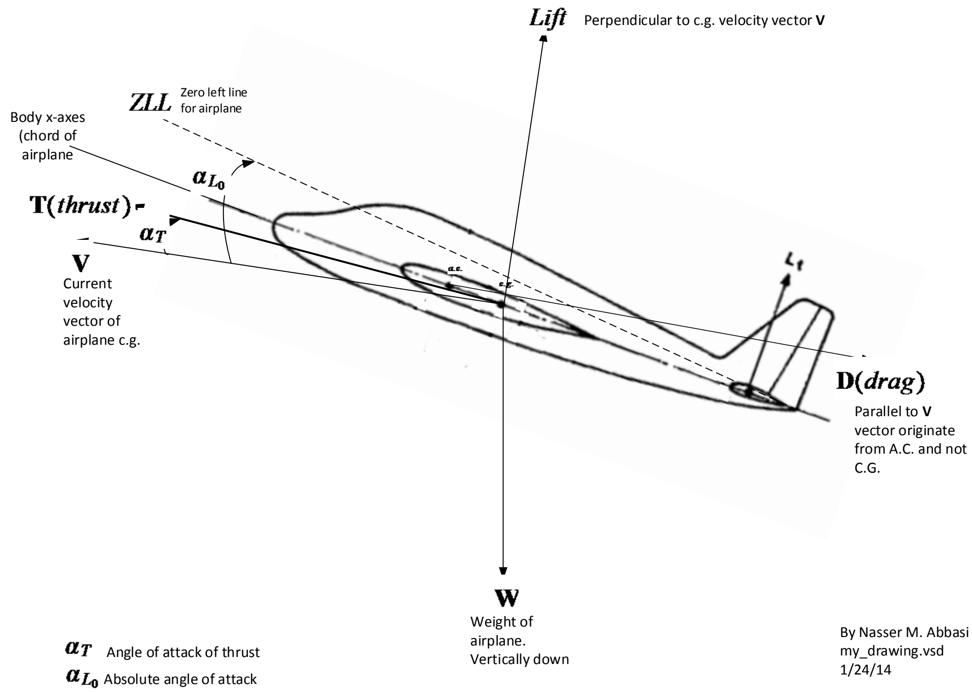


Figure 4.6: Main forces on airplane

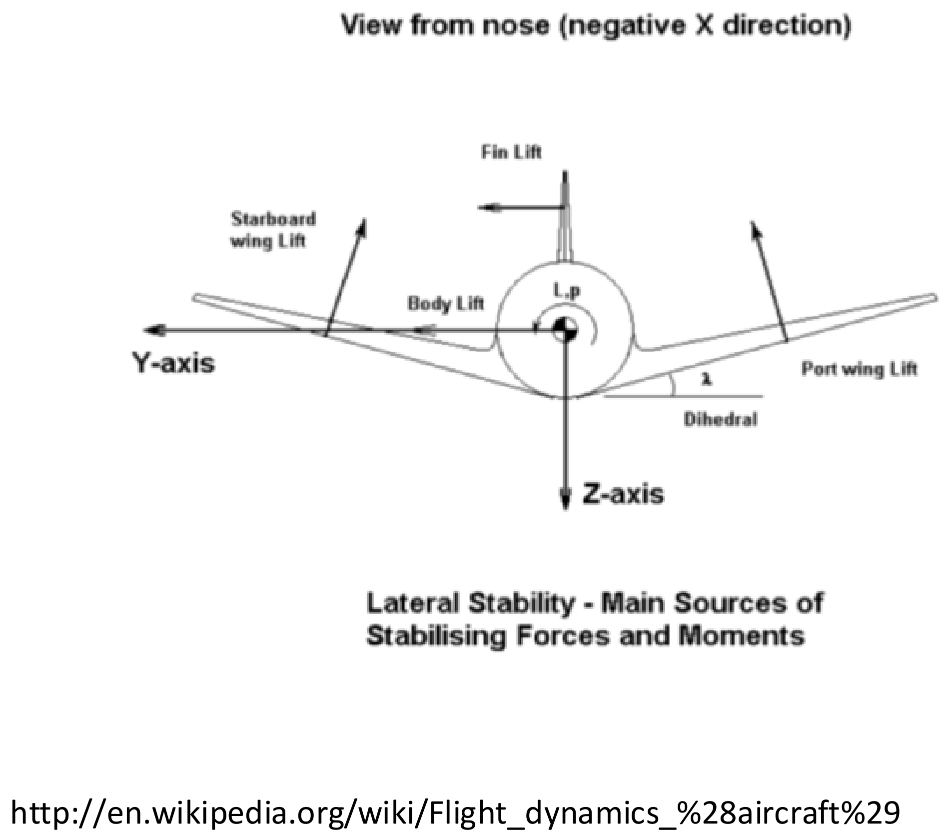


Figure 4.7: Local stability

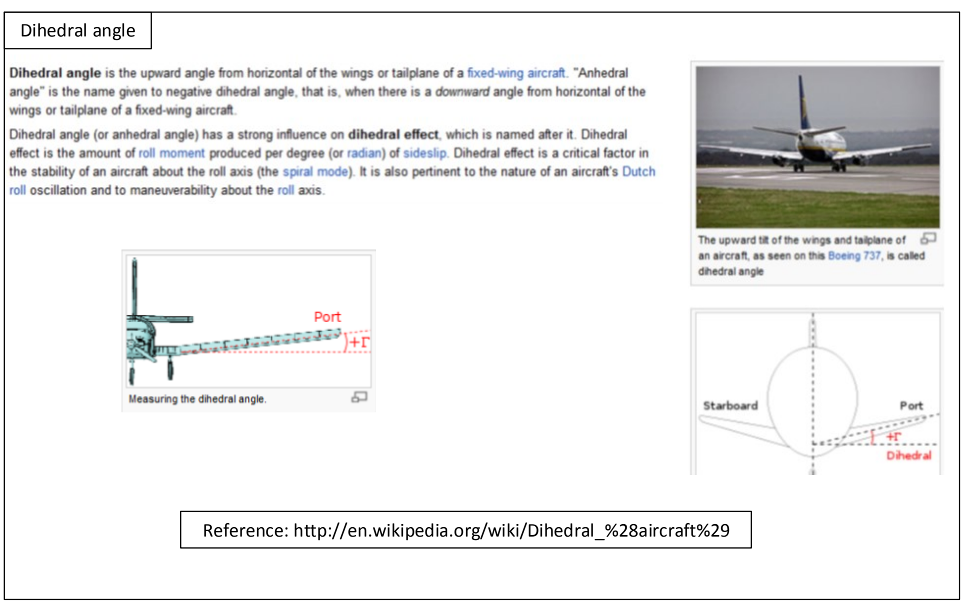


Figure 4.8: diherdal angle

definitions

From Performance, Stability, Dynamics, and Control of Airplanes
By Baudu N. Pamadi

The Reynolds number is the ratio of inertial forces to viscous forces.

$$Re = \frac{\text{Inertia force}}{\text{Viscous force}} \tag{1.1}$$

$$= \frac{\text{Change in momentum/time}}{\text{Shear stress} \times \text{Area}} \tag{1.2}$$

$$= \frac{\frac{d}{dt}(m V_\infty)}{\mu \frac{\partial V}{\partial y} A} \tag{1.3}$$

The Mach number is defined as the ratio of velocity of the body V_∞ to the speed of sound a.

$$M = \frac{V_\infty}{a} \tag{1.9}$$

In fluid dynamics, the **drag coefficient** (commonly denoted as: c_d, c_x or c_w) is a dimensionless quantity that is used to quantify the drag or resistance of an object in a fluid environment such as air or water. It is used in the drag equation, where a lower drag coefficient indicates the object will have less aerodynamic or hydrodynamic drag. The drag coefficient is always associated with a particular surface area.[1]

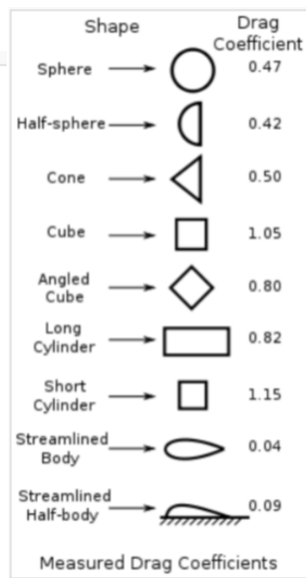
The drag coefficient C_d is defined as:

$$c_\mathrm{d} = \frac{2F_\mathrm{d}}{\rho v^2 A}$$

I do not see viscosity here?

where:

F_d is the drag force, which is by definition the force component in the direction of the flow velocity,[6]
ρ is the mass density of the fluid,[7]
v is the speed of the object relative to the fluid and
A is the reference area.



The drag equation:

$$F_d = \tfrac{1}{2}\rho v^2 c_d A$$

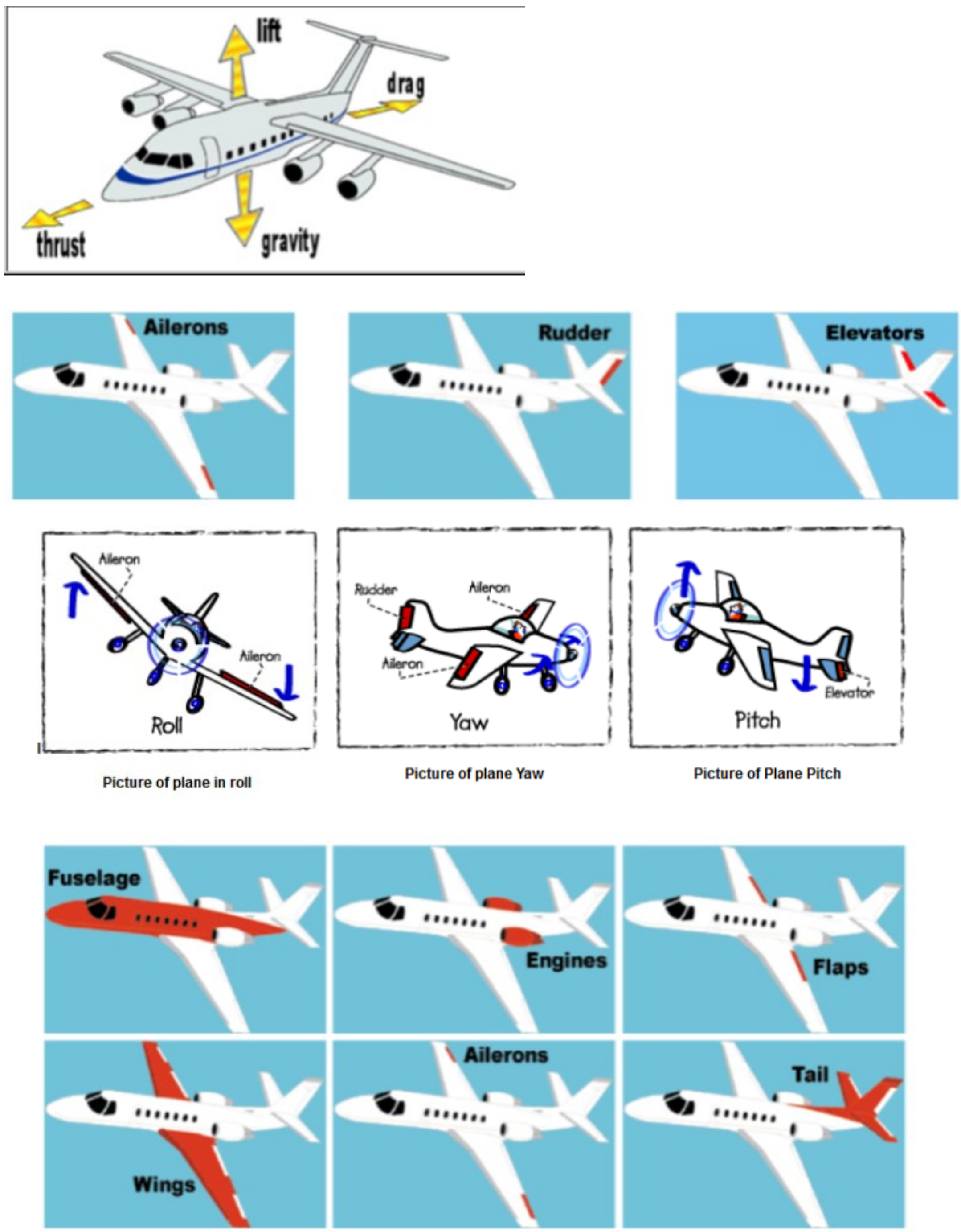
is essentially a statement that the drag force on any object is proportional to the density of the fluid and proportional to the square of the relative speed between the object and the fluid.

C_d is not a constant but varies as a function of speed, flow direction, object position, object size, fluid density and fluid viscosity. Speed, kinematic viscosity and a characteristic length scale of the object are incorporated into a dimensionless quantity called the Reynolds number or Re. C_d is thus a function of Re. In compressible flow, the speed of sound is relevant and C_d is also a function of Mach number Ma.

http://en.wikipedia.org/wiki/Drag_coefficient

Figure 4.9: Definitions

This below from `http://www.grc.nasa.gov/WWW/k-12/UEET/StudentSite/dynamicsofflight.html`



http://www.grc.nasa.gov/WWW/k-12/UEET/StudentSite/dynamicsofflight.html

Figure 4.10: Drag coefficient

`http://www.grc.nasa.gov/WWW/k-12/airplane/alr.html`

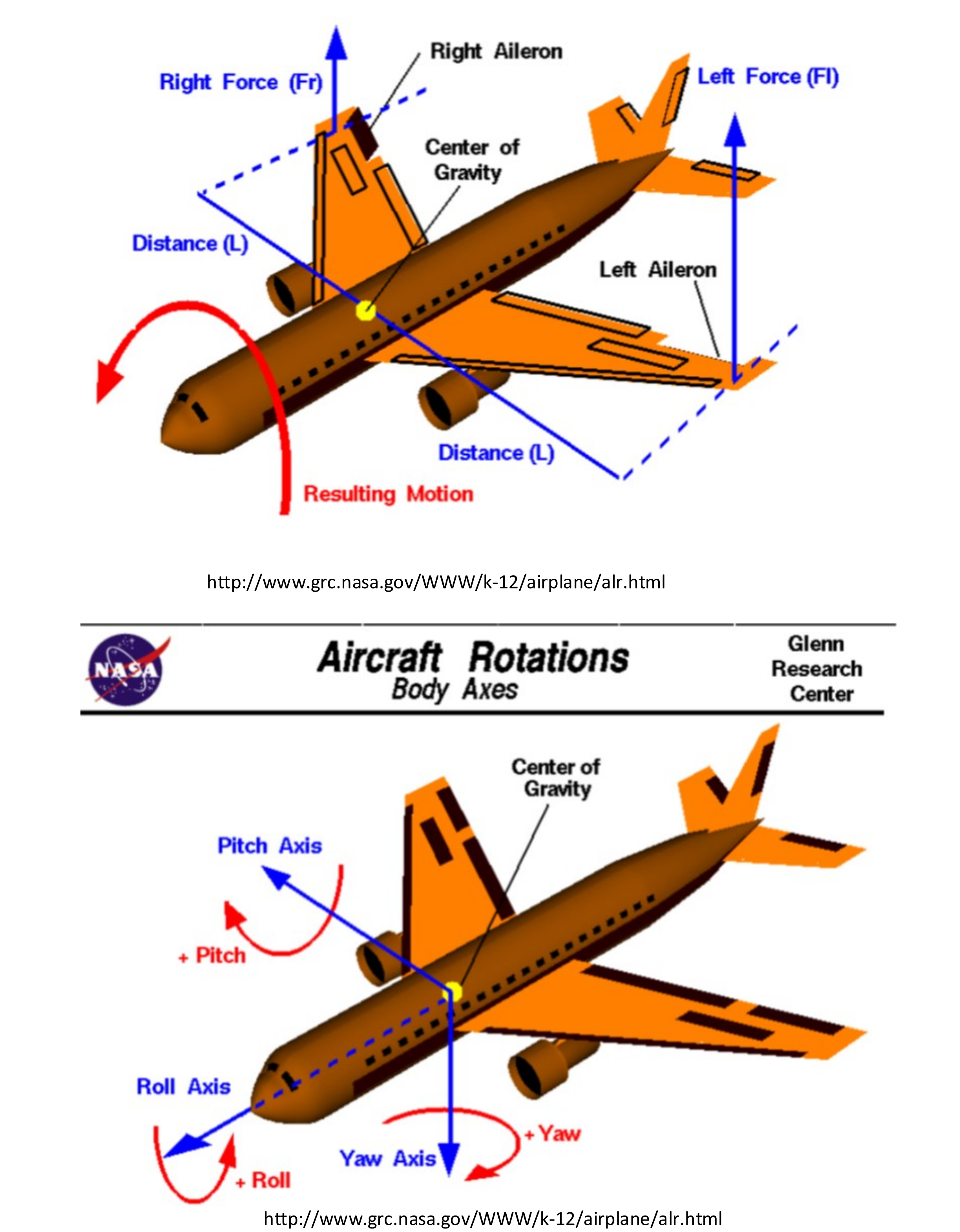


Figure 4.11: Roll, Yaw and Pitch

From `http://en.wikipedia.org/wiki/Lift_coefficient` and `http://en.wikipedia.o` rg/wiki/File:Aeroforces.svg

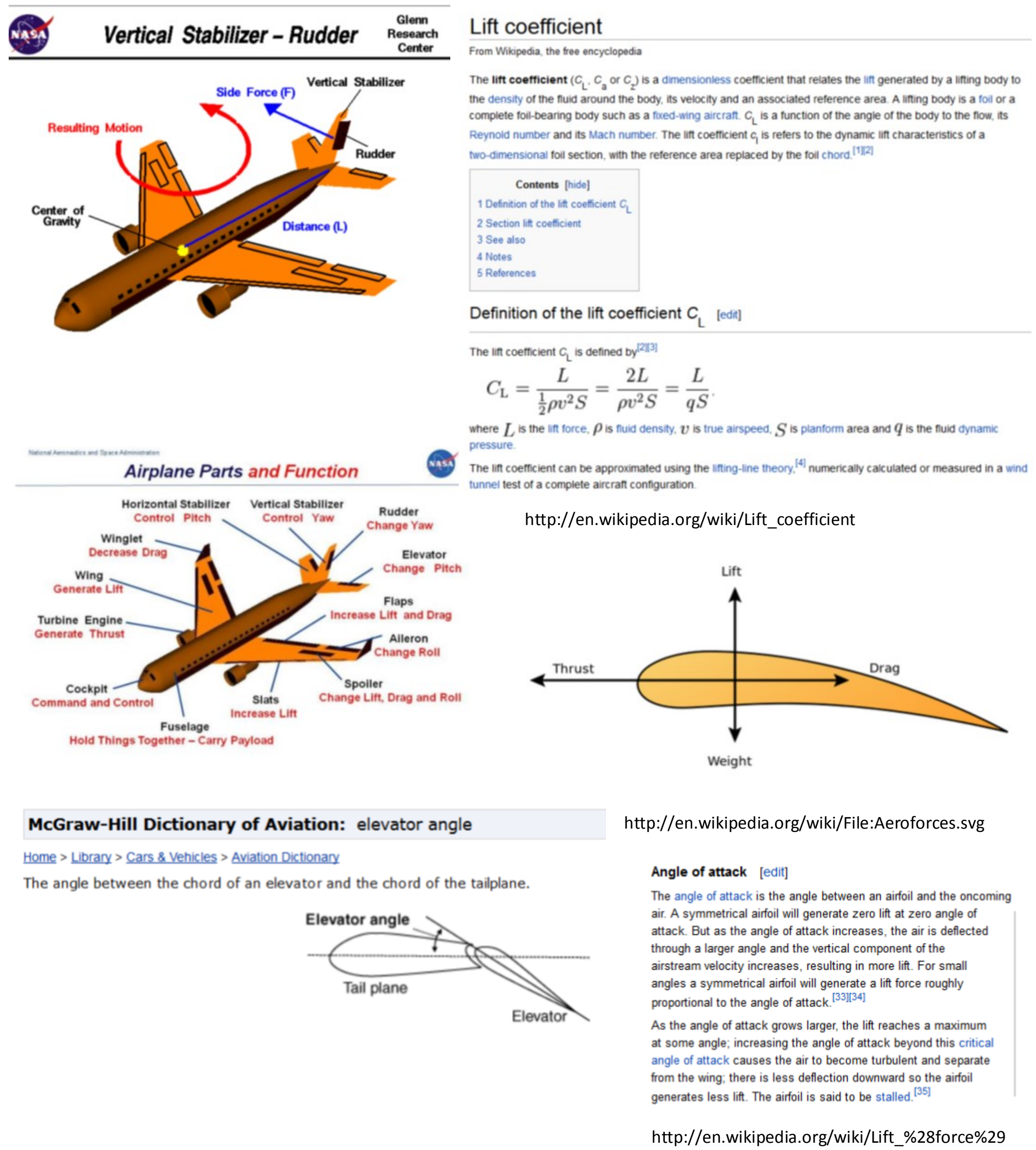
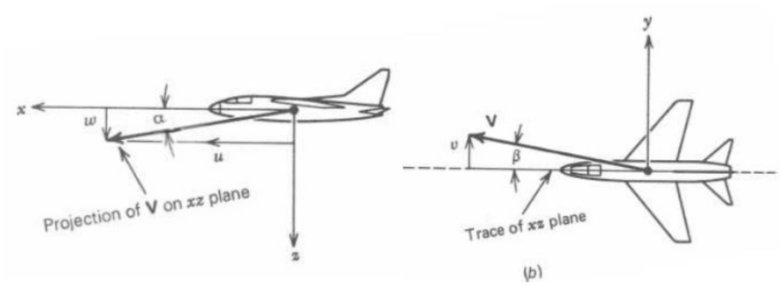


Figure 4.12: Forces diagram

from http://adg.stanford.edu/aa241/drag/sweepncdc.html



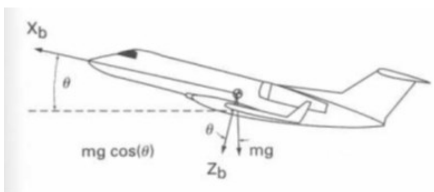


Figure 2: Definition of the pitch attitude angle, θ.

Sweptback Wings

Almost all high speed subsonic and supersonic aircraft have sweptback wings. The amount of sweep is measured by the angle between a lateral axis perpendicular to the airplane centerline and a constant percentage chord line along the semi-span of the wing. The latter is usually taken as the quarter chord line both because subsonic lift due to angle of attack acts at the quarter chord and because the crest is usually close to the quarter chord.

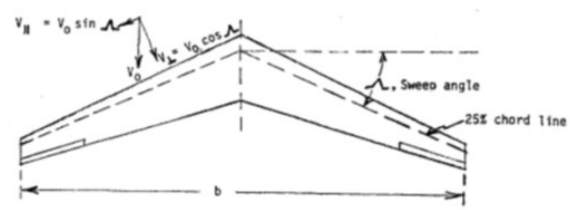


Figure 8. Velocity Components Affecting a Sweptback Wing

http://adg.stanford.edu/aa241/drag/sweepncdc.html

Angles Nomenclature

α = **angle of attack**, between the velocity projection in x, z-plane and the body-fixed x-axis

β = **sideslip angle**, between the velocity vector and the body-fixed x-axis

θ = **pitch attitude angle**, between plane-fixed x-axis and earth-fixed x_E-axis

$\gamma = \theta - \alpha$ = **climb (or path) angle**

ϕ = **roll angle**

ψ = **heading (or yaw) angle**

Figure 4.13: From sweepncdc website

Images from `http://adamone.rchomepage.com/cg_calc.htm` and Flight dynamics principles by Cook, 1997.

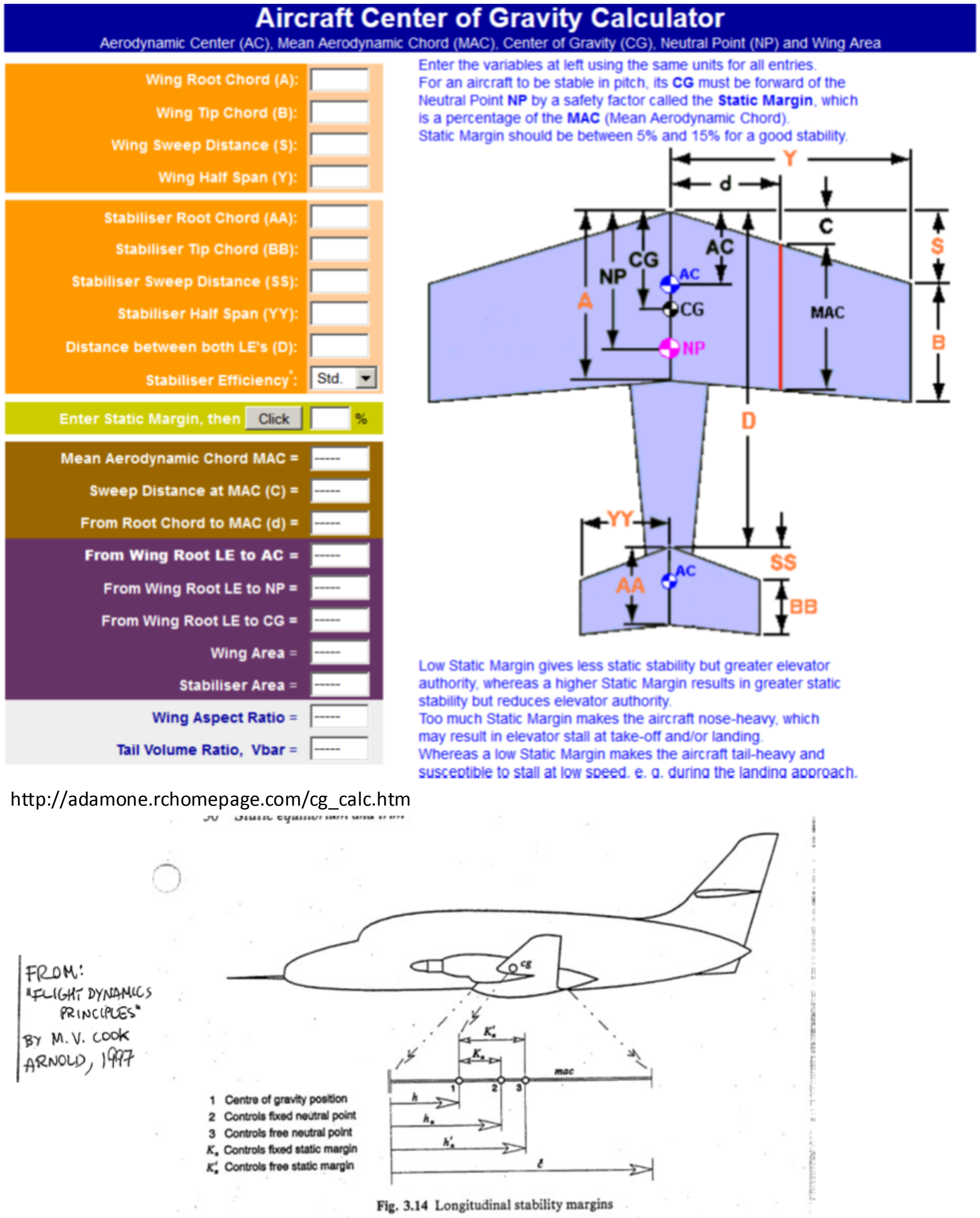


Figure 4.14: From Flight dynamics principles by Cook

From `http://chrusion.com/BJ7/SuperCalc7.html`

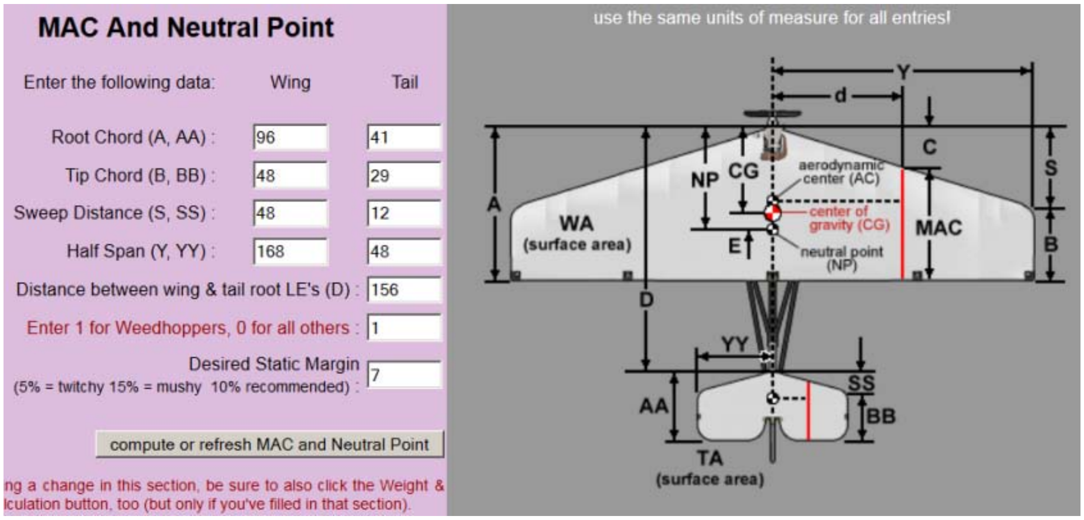


Figure 4.15: From chrusion site

From `http://www.willingtons.com/aircraft_center_of_gravity_calcu.html`

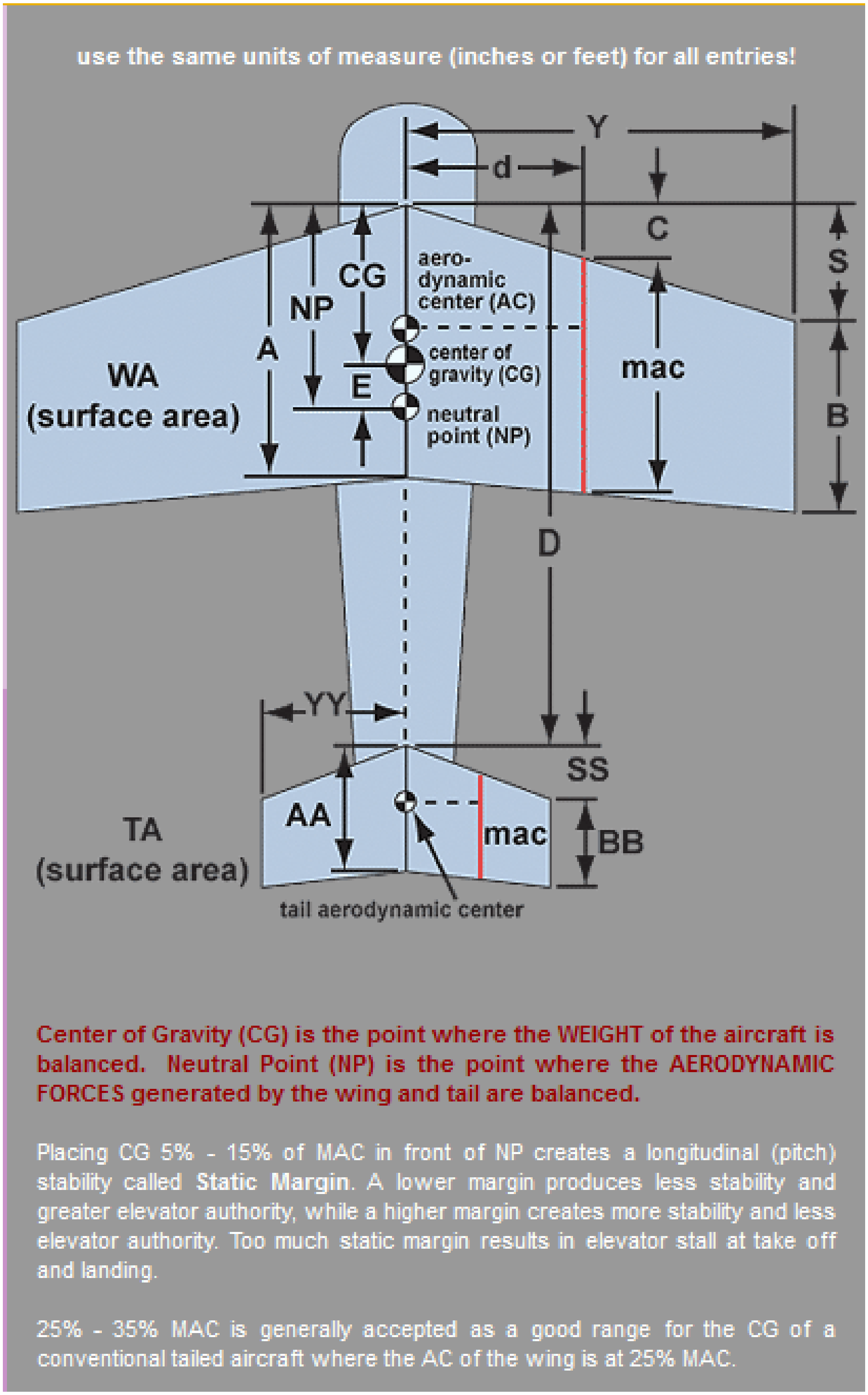


Figure 4.16: center of gravity

From `http://www.solar-city.net/2010/06/airplane-control-surfaces.html` nice diagram that shows clearly how the elevator causes the pitching motion (nose up/down). From same page, it says "*The purpose of the flaps is to generate more lift at slower airspeed, which enables the airplane to fly at a greatly reduced speed with a lower risk of stalling.*"

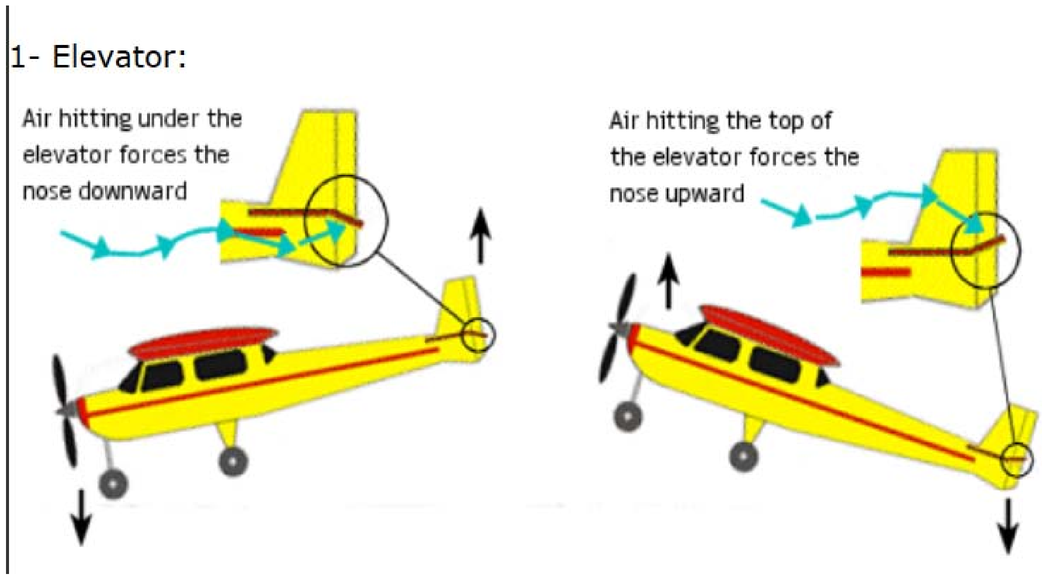


Figure 4.17: airplane control surfaces

Images from flight dynamics principles, by Cook, 1997.

154 *Lateral-directional dynamics*

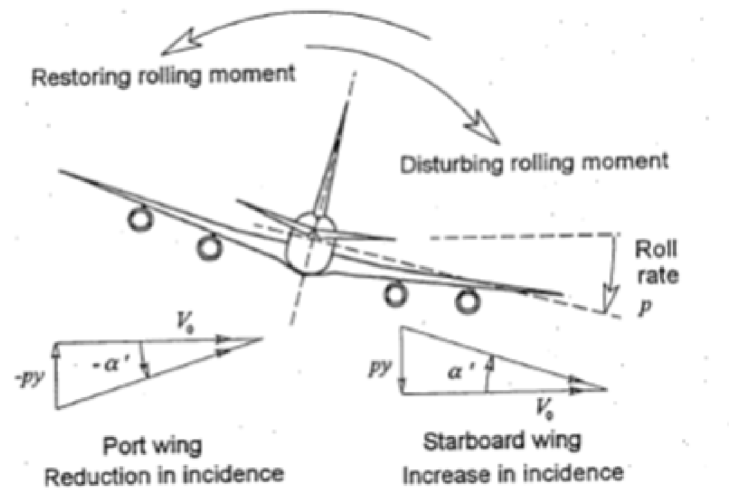


Fig. 7.3 The roll subsidence mode

Page 184, flight dynamics principles, by Cook

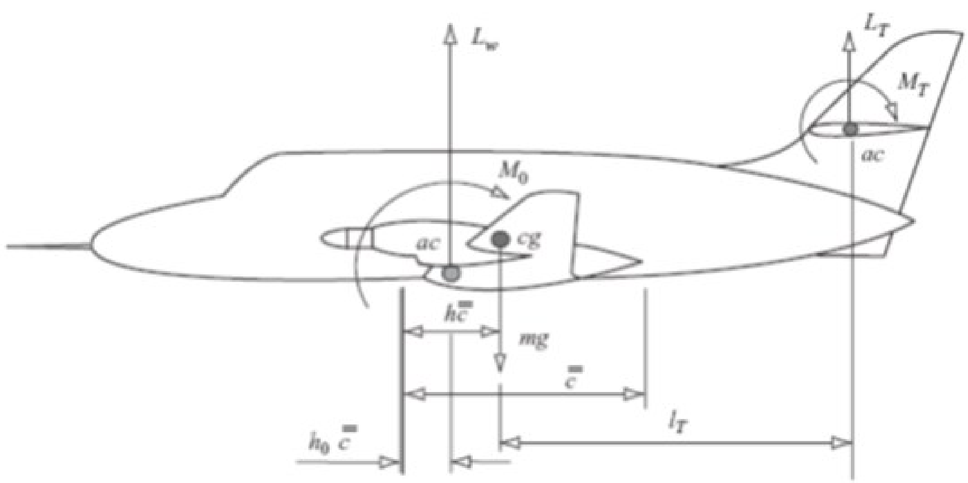


Figure 3.8 *Simple pitching moment model.*

Page 41, flight dynamics principles, by Cook

Figure 4.18: from flight dynamics principles, by Cook

Images from Performance, stability, dynamics and control of Airplanes. By Pamadi, AIAA press. Page 169. and `http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm`

STATIC STABILITY AND CONTROL 169

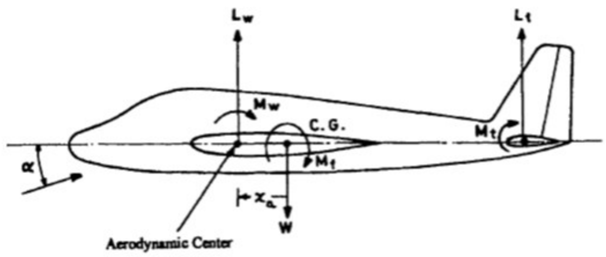


Fig. 3.4 Forces and moments acting on an airplane in level flight.

From: Performance, stability, dynamics and control of Airplanes. By Pamadi, AIAA press. Page 169

The following defines these forces in relation to straight-and-level, unaccelerated flight.

Thrust is the forward force produced by the power-plant/propeller. It opposes or overcomes the force of drag. As a general rule, it is said to act parallel to the longitudinal axis. However, this is not always the case as will be explained later.

Drag is a rearward, retarding force, and is caused by disruption of airflow by the wing, fuselage, and other protruding objects. Drag opposes thrust, and acts rear-ward parallel to the relative wind.

Weight is the combined load of the airplane itself, the crew, the fuel, and the cargo or baggage. Weight pulls the airplane downward because of the force of gravity. It opposes lift, and acts vertically downward through the airplane's center of gravity.

Lift opposes the downward force of weight, is pro-duced by the dynamic effect of the air acting on the wing, and acts perpendicular to the flightpath through the wing's center of lift.

In steady flight, the sum of these opposing forces is equal to zero. There can be no unbalanced forces in steady, straight flight (Newton's Third Law). This is true whether flying level or when climbing or descending. This is not the same thing as saying that the four forces are all equal. It simply means that the opposing forces are equal to, and thereby cancel the effects of, each other.

each other; not to lift/weight. To be correct about it, it must be said that in steady flight:

- **The sum of all upward forces (not just lift) equals the sum of all downward forces (not just weight).**
- **The sum of all forward forces (not just thrust) equals the sum of all backward forces (not just drag).**

http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm

Figure 4.19: from Performance, stability, dynamics and control

Image from `http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm`

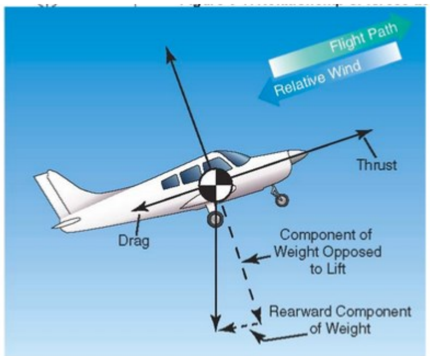


Figure 3-2. Force vectors during a stabilized climb.

In level flight the aerodynamic properties of the wing produce a required lift, but this can be obtained only at the expense of a certain penalty. The name given to this penalty is induced drag. Induced drag is inherent whenever a wing is producing lift and, in fact, this type of drag is inseparable from the production of lift. Consequently, it is always present if lift is produced.

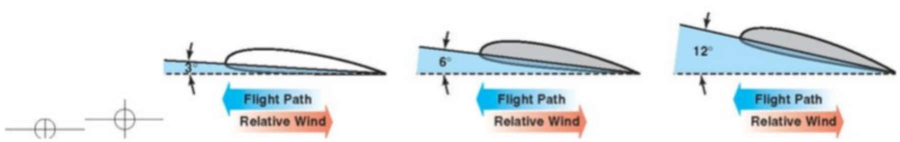
THRUST

Before the airplane begins to move, thrust must be exerted. It continues to move and gain speed until thrust and drag are equal. In order to maintain a con-stant airspeed, thrust and drag must remain equal, just as lift and weight must be equal to maintain a constant altitude. If in level flight, the engine power is reduced, the thrust is lessened, and the airplane slows down. As long as the thrust is less than the drag, the airplane continues to decelerate until its airspeed is insufficient to support it in the air.

Likewise, if the engine power is increased, thrust becomes greater than drag and the airspeed increases. As long as the thrust continues to be greater than the drag, the airplane continues to accel-erate. When drag equals thrust, the airplane flies at a constant airspeed.

Straight-and-level flight may be sustained at speeds from very slow to very fast. The pilot must coordi-nate angle of attack and thrust in all speed regimes if the airplane is to be held in level flight. Roughly, these regimes can be grouped in three categories: low-speed flight, cruising flight, and high-speed flight.

When the airspeed is low, the angle of attack must be relatively high to increase lift if the balance between lift and weight is to be maintained. [Figure 3-3] If thrust decreases and airspeed decreases, lift becomes



Level (High Speed) Level (Cruise Speed) Level (Low Speed)

Figure 3-3. Angle of attack at various speeds.

less than weight and the airplane will start to descend. To maintain level flight, the pilot can increase the angle of attack an amount which will generate a lift force again equal to the weight of the airplane and while the airplane will be flying more slowly, it will still maintain level flight if the pilot has properly coordinated thrust and angle of attack.

http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm

Good discussion on angle of attack

Figure 4.20: from pilots handbook

Image from `http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm`

The profile drag of a streamlined object held in a fixed position relative to the airflow increases approximately as the square of the velocity; thus, doubling the airspeed increases the drag four times, and tripling the airspeed increases the drag nine times.

The amount of induced drag varies inversely as the square of the airspeed.

From the foregoing discussion, it can be noted that parasite drag increases as the square of the airspeed, and induced drag varies inversely as the square of the airspeed.

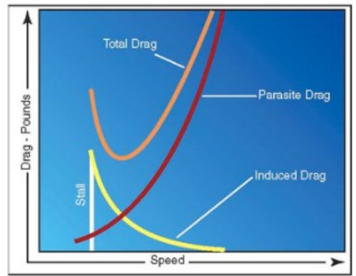


Figure 3-5. Drag versus speed.

The location of the center of gravity (CG) is determined by the general design of each particular airplane. The designers determine how far the center of pressure (CP) will travel. They then fix the center of gravity forward of the center of pressure for the corresponding flight speed in order to provide an adequate restoring moment to retain flight equilibrium.

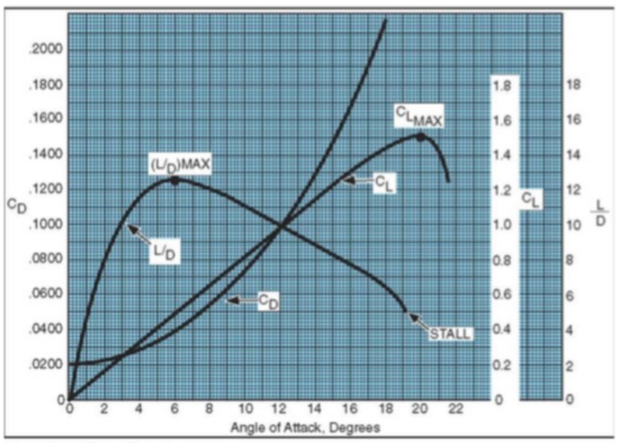


Figure 3-6. Lift coefficients at various angles of attack.

The pilot can control the lift. Any time the control wheel is more fore or aft, the angle of attack is changed. As angle of attack increases, lift increases (all other factors being equal). When the airplane reaches the maximum angle of attack, lift begins to diminish rapidly. This is the stalling angle of attack, or burble point.

Before proceeding further with lift and how it can be controlled, velocity must be interjected. The shape of the wing cannot be effective unless it continually keeps "attacking" new air. If an airplane is to keep flying, it must keep moving. Lift is proportional to the square of the airplane's velocity. For example, an airplane traveling at 200 knots has four times the lift as the same airplane traveling at 100 knots, if the angle of attack and other factors remain constant.

lift varies directly with the wing area

a wing with a planform area of 200 square feet lifts twice as much at the same angle of attack as a wing with an area of 100 square feet.

http://www.americanflyers.net/aviationlibrary/pilots_handbook/chapter_3.htm

Figure 4.21: from pilots handbook (2)

Image from FAA pilot handbook and `http://www.youtube.com/watch?v=8uT55aei1NI`

Empennage

The empennage includes the entire tail group and consists of fixed surfaces such as the vertical stabilizer and the horizontal stabilizer. The movable surfaces include the rudder, the elevator, and one or more trim tabs. *[Figure 2-10]*

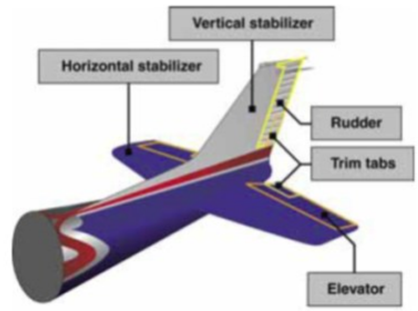


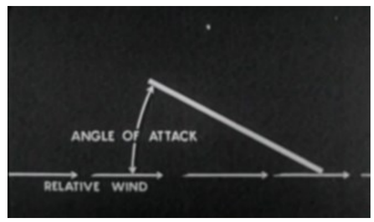
Figure 2-10. *Empennage components.*

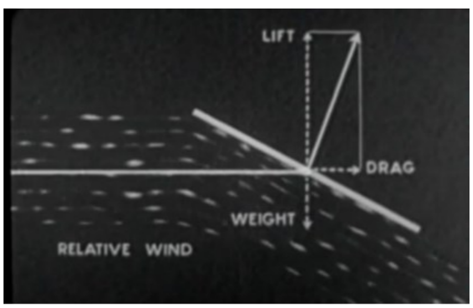
From FAA pilot's handbook

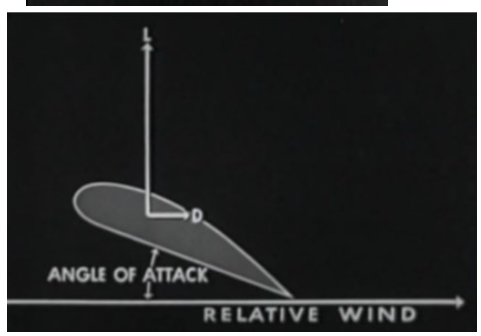
The downward backward flow from the top surface of an airfoil creates a downwash. This downwash meets the flow from the bottom of the airfoil at the trailing edge. Applying Newton's third law, the reaction of this downward backward flow results in an upward forward force on the airfoil.

From FAA pilot's handbook

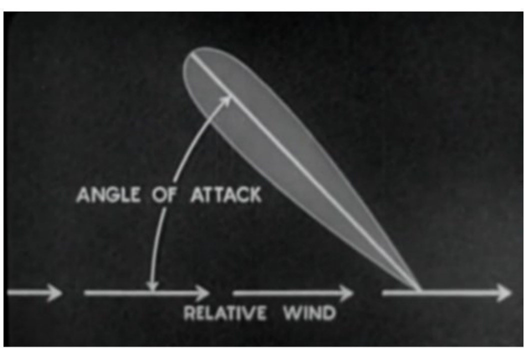
I do not understand the above







http://www.youtube.com/watch?v=8uT55aei1NI



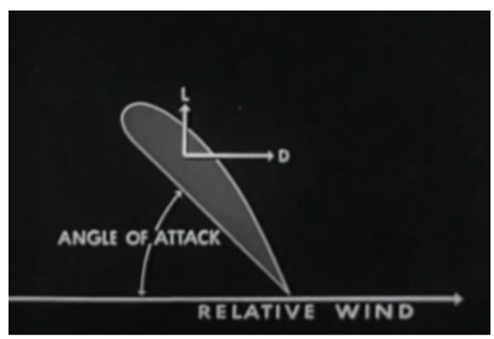
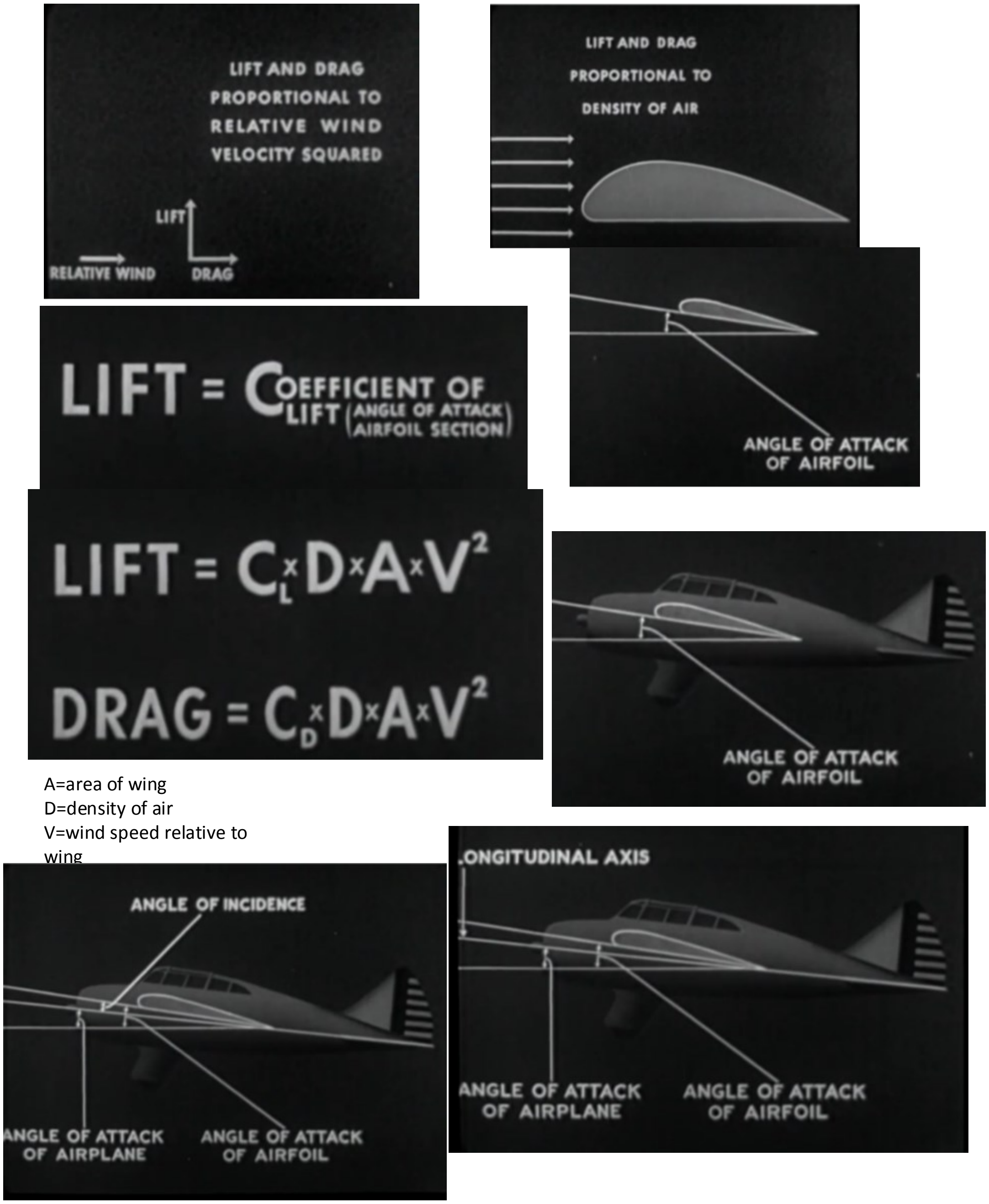


Figure 4.22: from FAA pilot handbook

Image http://www.youtube.com/watch?v=8uT55aei1NI and http://www.youtube.com/user/DAMSQAZ?feature=watch http://www.youtube.com/watch?v=8uT55aei1NI

http://www.youtube.com/user/DAMSQAZ?feature=watch



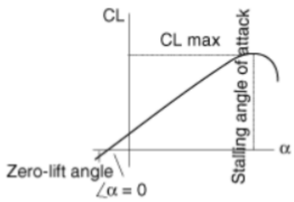
A=area of wing
D=density of air
V=wind speed relative to wing

Figure 4.23: from youtube

zero-lift angle

The angle of attack at which an airfoil does not produce any lift. Its value is generally less than zero unless the airfoil is symmetrical.

This from Foundations of aerodynamics, 5th ed. By Chow and Kuethe



In Fig. 5.11, an airfoil is shown set at a geometric angle of attack equal to the angle of zero lift. A line on the airfoil parallel to the flight path V_∞ and passing through the trailing edge when the airfoil is set at the orientation of zero lift is called the *zero-lift line* (Z.L.L.) of the airfoil. For symmetrical airfoils, the zero-lift line coincides with the chord line.

The *absolute angle of attack* is defined as the angle included between the flight path and the zero-lift line and is given the symbol α_a. From Fig. 5.12,

$$\alpha_a = \alpha - \alpha_{L0} \tag{5.35}$$

http://www.answers.com/topic/zero-lift-angle

These are from text :foundation of aerodynamics by Kuethe and Chow

1. the center of pressure is at ¼ chord for all values of lift coeff.
2. center of pressure (c.p.) of a force is defined as the point about which the moment vanishes.
3. The geometric angle of attack is defined as the angle between the flight path and the chord line of the airfoil. When the geometric angle of attack is zero, the lift coeff. Is zero.
4. The point about which the moment coeff. Is independent of the angle of attack is called the aerodynamic center of the section. (a.c.)
5. a.c. is at the ¼ chord line point.
6. The value of the angle of attack that makes the lift coeff. Zero is called the angle of zero lift (Z.L.L.)

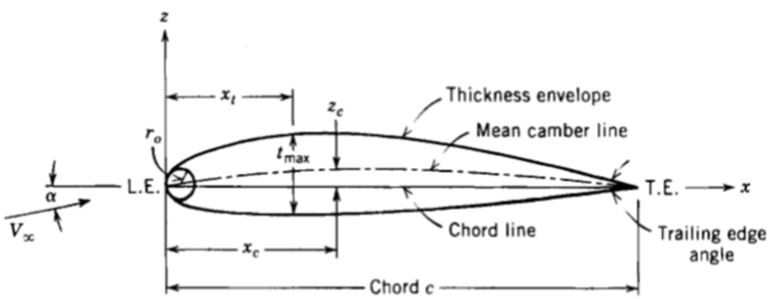


Fig. 5.1. Airfoil geometrical variables.

There are from Foundations of aerodynamics, 5th ed. By Chow and Kuethe

5.7 Properties of the Cambered Airfoil 147

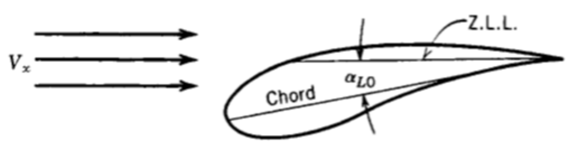


Fig. 5.11. Orientation of airfoil at zero lift.

Figure 4.24: from youtube (2)

4.4 Some strange shaped airplanes

Image http://edition.cnn.com/2014/01/16/travel/inside-airbus-beluga/index.html?hpt=ibu_c2

Figure 4.25: airbus beluga 1

Figure 4.26: Concorde

Image from http://edition.cnn.com/2014/01/16/travel/inside-airbus-beluga/index.html?hpt=ibu_c2

Figure 4.27: airbus beluga (2)

Image from http://www.nasa.gov/centers/dryden/Features/super_guppy.html

Figure 4.28: NASA SGT super guppy

Image from http://www.aerospaceweb.org/question/aerodynamics/q0130.shtml "Boeing Pelican ground effect vehicle"

Figure 4.29: pelican 01

4.5 links

1. `https://3dwarehouse.sketchup.com/search.html?redirect=1&tags=airplane`

4.6 references

1. Etkin and Reid, Dynamics of flight, 3rd edition.
2. Cook, Flight Dynamics principles, third edition.
3. Lecture notes, EMA 523 flight dynamics and control, University of Wisconsin, Madison by Professor Riccardo Bonazza
4. Kuethe and Chow, Foundations of Aerodynamics, 4th edition