Analytical solution to diffusion-convection PDE in 1D

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This is a diffusion-convection PDE.

$$\frac{\partial F(t,z)}{\partial t} = k\frac{\partial^2 F(t,z)}{\partial z^2} + v\frac{\partial F(t,z)}{\partial z} \tag{1}$$
$$t > 0$$
$$0 < z < L$$

Where k is the diffusion constant and v is the convection speed. Boundary conditions are

$$F(t,0) = 0$$
$$F(t,L) = 1$$

Initial conditions are

$$F(0,z) = \begin{cases} 0 & 0 \leq z < L \\ 1 & z = L \end{cases}$$

The first step is to convert the PDE to pure diffusion PDE using the transformation

$$F(t,z) = A(t,z)\,u(t,z)$$

Substituting this back in (1) gives

$$A_t u + A u_t = k(A_{zz}u + 2A_z u_z + A u_{zz}) + v(A_z u + A u_z)$$

Dividing by A and simplifying

$$\frac{A_t u}{A} + u_t = k\left(\frac{A_{zz}u + 2A_z u_z}{A} + u_{zz}\right) + v\left(\frac{A_z u}{A} + u_z\right)$$
$$u_t = k u_{zz} + k\frac{A_{zz} - \frac{A_t}{k} + vA_z}{A}u + \left(\frac{2kA_z + vA}{A}\right)u_z \tag{2}$$

To make (2) pure diffusion PDE, we want

$$k\frac{A_{zz}-\frac{A_t}{k}+vA_z}{A}u=0 \tag{3}$$

$$\left(\frac{2kA_z+vA}{A}\right)u_z=0 \tag{4}$$

From (4) $(2kA_z+vA)\,u_z=0$ or $2kA_z+vA=0$ or $\frac{\partial A}{\partial z}+\frac{v}{2k}A=0$ which has the solution

$$A(t,z)=C(t)\,e^{-\frac{v}{2k}z} \tag{5}$$

From (3) we want $k\left(A_{zz}-\frac{A_t}{k}+vA_z\right)=0$. Substituting the result just obtained for $A(t,z)$ in (3) gives

$$\begin{aligned}\frac{v^2}{4k^2}C(t)\,e^{-\frac{v}{2k}z}-\frac{dC(t)}{dt}\frac{1}{k}e^{-\frac{v}{2k}z}-\frac{v^2}{2k}C(t)\,e^{-\frac{v}{2k}z}&=0\\ \frac{v^2}{4k^2}C(t)-\frac{1}{k}C'(t)-\frac{v^2}{2k}C(t)&=0\\ C'(t)+\frac{v^2}{4k}C(t)&=0\end{aligned}$$

Hence

$$C(t)=C_1e^{\frac{-v^2}{4k}t}$$

For some constant C_1. The constant C_1 ends up canceling out at the very end. Hence we set it to 1 now instead of carrying along in all the derivation in order to simplify notations. Therefore $C(t)=e^{\frac{-v^2}{4k}t}$. Substituting this into (5) gives the transformation function

$$A(t,z)=e^{-\left(\frac{v^2t}{4k}+\frac{vz}{2k}\right)}$$

Using this in (2) gives the pure diffusion PDE to solve

$$u_t=ku_{zz} \tag{6}$$

Converting the original boundary conditions from F to u gives

$$\begin{aligned}F(t,0)&=0\\ A(t,0)\,u(t,0)&=0\\ e^{-\frac{v^2t}{4k}}u(t,0)&=0\\ u(t,0)&=0\end{aligned}$$

And

$$\begin{aligned}
F(t,L) &= 1 \\
A(t,L)\,u(t,L) &= 1 \\
e^{-\left(\frac{v^2 t}{4k}+\frac{vL}{2k}\right)} u(t,L) &= 1 \\
u(t,L) &= e^{\left(\frac{v^2 t}{4k}+\frac{vL}{2k}\right)}
\end{aligned}$$

And for the initial conditions

$$\begin{aligned}
F(0,z) &= \begin{cases} 0 & 0 \leq z < L \\ 1 & z = L \end{cases} \\
A(0,z)\,u(0,z) &= \begin{cases} 0 & 0 \leq z < L \\ 1 & z = L \end{cases} \\
e^{-\frac{vz}{2k}} u(0,z) &= \begin{cases} 0 & 0 \leq z < L \\ 1 & z = L \end{cases} \\
u(0,z) &= \begin{cases} 0 & 0 \leq z < L \\ e^{\frac{vL}{2k}} & z = L \end{cases}
\end{aligned}$$

Therefore the new PDE to solve is

$$u_t = k u_{zz}$$

With time varying boundary conditions

$$\begin{aligned}
u(t,0) &= 0 \\
u(t,L) &= e^{\left(\frac{v^2 t}{4k}+\frac{vL}{2k}\right)}
\end{aligned}$$

And initial conditions

$$u(0,z) = \begin{cases} 0 & 0 \leq z < L \\ e^{\frac{vL}{2k}} & z = L \end{cases}$$

To solve this using separation of variables, the boundary conditions has to be homogenous. Therefore we use standard method to handle this as follows. Let

$$u(t,z) = \phi(t,z) + u_E(t,z) \tag{7}$$

Where $u_E(t,z)$ is the steady state solution which needs to only satisfy the boundary conditions and $\phi(t,z)$ satisfies the PDE but with homogenous boundary conditions. Therefore

$$\begin{aligned} u_E(t,z) &= u(t,0) + z\left(\frac{u(t,L)-u(t,0)}{L}\right) \\ u_E(t,z) &= \frac{z}{L}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)} \end{aligned}$$

And (7) becomes

$$u(t,z) = \phi(t,z) + \frac{z}{L}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}$$

Substituting the above in (6) gives

$$\begin{aligned} \frac{\partial\phi}{\partial t} + \frac{z}{L}\frac{v^2}{4k}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)} &= k\frac{\partial^2\phi}{\partial z^2} \\ \frac{\partial\phi}{\partial t} &= k\frac{\partial^2\phi}{\partial z^2} - \frac{z}{L}\frac{v^2}{4k}e^{\left(\frac{v^2t}{4}+\frac{vL}{2}\right)} \end{aligned}$$

Or

$$\frac{\partial\phi}{\partial t} = k\frac{\partial^2\phi}{\partial z^2} + Q(t,z) \tag{8}$$

This is diffusion PDE with homogenous B.C. with source term

$$Q(t,z) = -\frac{d}{dt}u_E(t,z)$$

Now we find $\phi(t,z)$. Since this solution needs to satisfy homogenous boundary conditions, we know the solution to pure diffusion on bounded domain with source present is by given by the following eigenfunction expansion

$$\phi(t,z) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\sqrt{\lambda_n}z\right) \tag{8A}$$

Where eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \cdots$ and $\sin\left(\sqrt{\lambda_n}z\right)$ are the eigenfunction. Substituting the above in (8) in order to obtain an ODE to solve for $b_n(t)$ gives

$$\sum_{n=1}^{\infty} b_n'(t)\sin\left(\sqrt{\lambda_n}z\right) = k\sum_{n=1}^{\infty} -b_n(t)\,\lambda_n\sin\left(\sqrt{\lambda_n}z\right) + Q(t,z) \tag{9}$$

Expanding $Q(t,z)$ in terms of eigenfunctions

$$Q(t,z)=\sum_{n=1}^{\infty} q_n(t)\sin\left(\sqrt{\lambda_n}z\right)$$

Applying orthogonality

$$\int_0^L Q(t,z)\sin\left(\sqrt{\lambda_n}z\right)dz=q_n(t)\,\frac{L}{2} \tag{9A}$$

But

$$\begin{aligned}\int_0^a Q(t,z)\sin\left(\sqrt{\lambda_n}z\right)dz &= \int_0^L -\frac{z}{L}\frac{v^2}{4k}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}\sin\left(\sqrt{\lambda_n}z\right)dz\\ &= -\frac{v^2}{4kL}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}\int_0^L z\sin\left(\sqrt{\lambda_n}z\right)dz\\ &= -\frac{v^2}{4kL}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}\frac{(-1)^{n+1}L^2}{n\pi}\\ &= \frac{(-1)^n Lv^2e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}}{4kn\pi}\\ &= \frac{(-1)^n v^2e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}}{4k\sqrt{\lambda_n}}\end{aligned}$$

Hence from (9A) we find

$$q_n(t)=\frac{(-1)^n v^2e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}}{2Lk\sqrt{\lambda_n}}$$

Using the above in (9) gives

$$\begin{aligned}\sum_{n=1}^{\infty} b_n'(t)\sin\left(\sqrt{\lambda_n}z\right) &= k\sum_{n=1}^{\infty}-b_n(t)\,\lambda_n\sin\left(\sqrt{\lambda_n}z\right)+\sum_{n=1}^{\infty}q_n(t)\sin\left(\sqrt{\lambda_n}z\right)\\ b_n'(t)+k\lambda_n b_n(t) &= q_n(t)\end{aligned}$$

To solve the above ODE, the integrating factor is $\mu=e^{k\lambda_n t}$, therefore

$$\begin{aligned}b_n(t)\,e^{k\lambda_n t} &= \int_0^t q_n(\tau)\,e^{k\lambda_n\tau}d\tau+C_n\\ b_n(t) &= e^{-k\lambda_n t}\int_0^t q_n(\tau)\,e^{k\lambda_n\tau}d\tau+C_ne^{-k\lambda_n t}\\ &= \int_0^t q_n(\tau)\,e^{k\lambda_n(\tau-t)}d\tau+C_ne^{-k\lambda_n t}\end{aligned}$$

Using the above in (7) gives

$$u(t,z)=\frac{z}{L}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}+\sum_{n=1}^{\infty}\left(\int_0^t q_n(\tau)\,e^{k\lambda_n(\tau-t)}d\tau+C_ne^{-k\lambda_nt}\right)\sin\left(\sqrt{\lambda_n}z\right) \tag{10}$$

C_n is now found from initial conditions. At $t=0$ the above becomes

$$u(0,z)-\frac{z}{L}e^{\frac{vL}{2k}}=\sum_{n=1}^{\infty}C_n\sin\left(\sqrt{\lambda_n}z\right)$$

Applying orthogonality

$$\int_0^L\left(u(0,z)-\frac{z}{L}e^{\frac{vL}{2k}}\right)\sin\left(\sqrt{\lambda_m}z\right)dz=\int_0^L\sum_{n=1}^{\infty}C_n\sin\left(\sqrt{\lambda_n}z\right)\sin\left(\sqrt{\lambda_m}z\right)dz$$

$$\int_0^L u(0,z)\sin\left(\sqrt{\lambda_n}z\right)dz-\int_0^L\frac{z}{L}e^{\frac{vL}{2k}}\sin\left(\sqrt{\lambda_n}z\right)dz=C_n\frac{L}{2}$$

But $\int_0^L u(0,z)\sin\left(\sqrt{\lambda_n}z\right)dz=0$ since $u(0,z)$ is zero everywhere except at the end point. And

$$-\int_0^L\frac{z}{L}e^{\frac{vL}{2k}}\sin\left(\sqrt{\lambda_n}z\right)dz=\frac{(-1)^n\,e^{\frac{vL}{2k}}}{\sqrt{\lambda_n}}$$

Therefore

$$\frac{(-1)^n\,e^{\frac{vL}{2k}}}{\sqrt{\lambda_n}}=C_n\frac{L}{2}$$

$$C_n=\frac{2}{L}\frac{(-1)^n\,e^{\frac{vL}{2k}}}{\sqrt{\lambda_n}} \tag{10A}$$

And the solution (10) becomes

$$u(t,z)=\frac{z}{L}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}+\sum_{n=1}^{\infty}\left(\int_0^t q_n(\tau)\,e^{k\lambda_n(\tau-t)}d\tau+C_ne^{-k\lambda_nt}\right)\sin\left(\sqrt{\lambda_n}z\right) \tag{11}$$

But

$$\begin{aligned}\int_0^t q_n(\tau)\,e^{k\lambda_n(\tau-t)}d\tau&=\int_0^t\frac{(-1)^n\,v^2e^{\left(\frac{v^2\tau}{4k}+\frac{vL}{2k}\right)}}{2Lk\sqrt{\lambda_n}}e^{k\lambda_n(\tau-t)}d\tau\\&=\frac{2(-1)^n\,v^2e^{-k\lambda_nt+\frac{vL}{2k}}\left(e^{k\lambda_nt+\frac{tv^2}{4k}}-1\right)}{n\pi\left(4k^2\lambda_n+v^2\right)}\end{aligned}$$

Hence (11) becomes

$$u(t,z)=\frac{z}{L}e^{\left(\frac{v^2t}{4k}+\frac{vL}{2k}\right)}+\sum_{n=1}^{\infty}\left(\frac{2(-1)^n v^2 e^{-k\lambda_n t+\frac{vL}{2k}}\left(e^{k\lambda_n t+\frac{tv^2}{4k}}-1\right)}{n\pi\left(4k^2\lambda_n+v^2\right)}+\frac{2}{L}\frac{(-1)^n e^{\frac{vL}{2k}}}{\sqrt{\lambda_n}}e^{-k\lambda_n t}\right)\sin\left(\sqrt{\lambda_n}z\right) \tag{12}$$

We now convert back to $F(t,z)$. Since $F(t,z)=A(t,z)\,u(t,z)$ and $A(t,z)=e^{-\left(\frac{v^2t}{4}+\frac{vz}{2}\right)}$ then the final solution is

$$F(t,z)=e^{-\left(\frac{v^2t}{4k}+\frac{vz}{2k}\right)}u(t,z)$$

The following is animation of the solution for 30 seconds, side-by-side with numerical solution.

The following is the code used

```
ClearAll[t, z, L0, k, lam, n, v, f]
L0 = 10;
v = 1/2;
k = 1/2;
ode = D[f[t, z], t] == k*D[f[t, z], {z, 2}] + v*D[f[t, z], z];
ic = f[0, z] == Piecewise[{{1, z == L0}, {0, True}}];
bc = {f[t, 0] == 0, f[t, L0] == 1};
sol = NDSolve[{ode, bc, ic}, f, {t, 0, 100}, {z, 0, 10}];
(*analytical*)
lam[n_] := (n^2*Pi^2)/L0^2;
max = 100;
u[t_, z_] := Module[{},
   Sum[((2*(-1)^n*v^2*Exp[(L0*v)/(2*k) - k*lam[n]*t]*
      (Exp[(t*v^2)/(4*k) + k*lam[n]*t] - 1))/(n*
      Pi*(4*k^2*lam[n] + v^2)) +
      (2/L0)*(((-1)^n*Exp[(L0*v)/(2*k)])/Sqrt[lam[n]])*
      Exp[(-lam[n])*k*t])*
      Sin[Sqrt[lam[n]]*z], {n, 1, max}] + (z/L0)*
      Exp[(v^2*t)/(4*k) + (v*L0)/(2*k)]];

fAnalytical[t_, z_] := Module[{}, Exp[-((v^2*t)/(4*k) + (v*z)/(2*k))]*u[t, z]];

Manipulate[Grid[{{Row[{"t=", t}], SpanFromLeft},
    {Plot[Evaluate[f[t, z] /. sol], {z, 0, 10},
     PlotRange -> {{0, 10}, {-0.2, 1}},
     ImageSize -> 300, PlotLabel -> "NDSolve solution"],
     Plot[Evaluate[fAnalytical[t, z]], {z, 0, 10},
     PlotRange -> {{0, 10}, {-0.2, 1}},
```

```
    ImageSize -> 300, PlotLabel -> "Analytical solution",
    PlotStyle -> Red]}}],

{{t, 0, "t"}, 0, 30, 0.01},
TrackedSymbols :> {t}]
```

References

- Paper "Analytical Solution to the One-Dimensional Advection-Diffusion Equation with Temporally Dependent Coefficients". Dilip Kumar Jaiswal, Atul Kumar, Raja Ram Yadav. Journal of Water Resource and Protection, 2011, 3, 76-84
- Paper "An analytical solution of the diffusion convection equation over a finite domain". Mohammad Farrukh N. Mohsen and Mohammed H. Baluch, Appl. Math. Modelling, 1983, Vol. 7, August 285.
- Lecture 20: Heat conduction with time dependent boundary conditions using Eigenfunction Expansions. Introductory lecture notes on Partial Differential Equations By Anthony Peirce.