

Examples doing change of variable in differential equations

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1 Change of variable in the independent variable

1.1 Example 1. Euler ODE

One way to solve Euler ODE

$$x^2 y''(x) + xy'(x) + y(x) = 0 \quad (1)$$

is to do a change of variable in the independent variable. Let $x = e^t$. Then

$$\frac{dx}{dt} = e^t \quad (2)$$

Also we have that $\ln x = t$, which means

$$\frac{dt}{dx} = \frac{1}{x} \quad (3)$$

To do this change of variable and obtain a new ode where now $y(x)$ becomes $y(t)$, we start by changing $y'(x)$ to $y'(t)$ and changing $y''(x)$ to $y''(t)$. Given that

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} \quad (4)$$

Substituting (3) into (4) gives

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{1}{x}$$

But $\frac{1}{x} = e^{-t}$. The above becomes

$$\frac{dy}{dx} = e^{-t} \frac{dy}{dt} \quad (5)$$

This was not too bad. Now we need to work on $y''(x)$.

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

Substituting (5) into the above gives

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(e^{-t} \frac{dy}{dt} \right)$$

Now here comes the important trick. We want $\frac{d}{dx}$ to be $\frac{d}{dt}$ in order to apply it on its argument. This is done by dividing the numerator and denominator of $\frac{d}{dx}$ by dt which becomes

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{\frac{d}{dt}}{\frac{dx}{dt}} \left(e^{-t} \frac{dy}{dt} \right) \\ &= \frac{dt}{dx} \frac{d}{dt} \left(e^{-t} \frac{dy}{dt} \right) \end{aligned}$$

But from (3) $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$. Hence the above becomes

$$\frac{d^2 y}{dx^2} = e^{-t} \frac{d}{dt} \left(e^{-t} \frac{dy}{dt} \right)$$

Now we can apply the product rule to finish the above

$$\begin{aligned} \frac{d^2 y}{dx^2} &= e^{-t} \left(-e^{-t} \frac{dy}{dt} + e^{-t} \frac{d^2 y}{dt^2} \right) \\ &= e^{-2t} \left(-\frac{dy}{dt} + \frac{d^2 y}{dt^2} \right) \\ &= e^{-2t} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) \quad (6) \end{aligned}$$

We have now converted both $y'(x)$ and $y''(x)$. what is left is to plug these back into the original ODE (1). Which becomes

$$\begin{aligned} x^2 y''(x) + xy'(x) + y(x) &= 0 \\ x^2 e^{-2t} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + x e^{-t} \frac{dy}{dt} + y(t) &= 0 \end{aligned}$$

But $x = e^t$ and $x^2 = e^{2t}$. The above becomes

$$\begin{aligned} \frac{d^2 y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + y(t) &= 0 \\ \frac{d^2 y}{dt^2} + y(t) &= 0 \end{aligned}$$

This is now constant coefficient ODE. It is standard one which has the solution

$$y(t) = A \cos(t) + B \sin(t)$$

Now we switch back to x . Since $\ln x = t$, then the above becomes

$$y(x) = A \cos(\ln x) + B \sin(\ln x)$$

This completes the solution.

2 Change of variable in the dependent variable

2.1 Example 1. Bessel ODE

Given the ode

$$y''(x) + \left(1 - \frac{3}{4x^2}\right)y(x) = 0 \quad (1)$$

We will use the change of variable in the dependent variable $y = ux^{\frac{1}{2}}$ to transform the above ODE to Bessel ODE

$$x^2u'' + xu' + (x^2 - 1)u = 0$$

Since $y = ux^{\frac{1}{2}}$ then

$$\frac{dy}{dx} = \frac{du}{dx}x^{\frac{1}{2}} + u\frac{x^{-\frac{1}{2}}}{2} \quad (2)$$

And

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{du}{dx}x^{\frac{1}{2}} + u\frac{x^{-\frac{1}{2}}}{2} \right) \\ &= \frac{d}{dx} \left(\frac{du}{dx}x^{\frac{1}{2}} \right) + \frac{d}{dx} \left(u\frac{x^{-\frac{1}{2}}}{2} \right) \\ &= \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{1}{2}\frac{du}{dx}x^{-\frac{1}{2}} + \frac{1}{2}\frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} \\ &= \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} \end{aligned} \quad (3)$$

Substituting (2,3) into (1) gives

$$\begin{aligned} \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} + \left(1 - \frac{3}{4x^2}\right)ux^{\frac{1}{2}} &= 0 \\ \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} + ux^{\frac{1}{2}} - \frac{3}{4}ux^{-\frac{3}{2}} &= 0 \\ \frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - ux^{-\frac{3}{2}} + ux^{\frac{1}{2}} &= 0 \end{aligned}$$

Multiplying both side by $x^{\frac{3}{2}}$ gives

$$\begin{aligned} x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - u + ux^2 &= 0 \\ x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - (1 - x^2)u &= 0 \\ x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - 1)u &= 0 \end{aligned}$$

Which is Bessel ode where $v = 1$.