Examples doing change of variable in differential equations

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1 Change of variable in the independent variable

1.1 Example 1. Euler ODE

One way to solve Euler ODE

$$x^2y''(x) + xy'(x) + y(x) = 0 \tag{1}$$

is to do a change of variable in the independent variable. Let $x = e^t$. Then

$$\frac{dx}{dt} = e^t \tag{2}$$

Also we have that $\ln x = t$, which means

$$\frac{dt}{dx} = \frac{1}{x} \tag{3}$$

To do this change of variable and obtain a new ode where now $y(x)$ becomes $y(t)$, we start by changing $y'(x)$ to $y'(t)$ and changing $y''(x)$ to $y''(t)$. Given that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \tag{4}$$

Substituting (3) into (4) gives

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{1}{x}$$

But $\frac{1}{x} = e^{-t}$. The above becomes

$$\frac{dy}{dx} = e^{-t}\frac{dy}{dt} \tag{5}$$

This was not too bad. Now we need to work on $y''(x)$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Substituting (5) into the above gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right)$$

Now here comes the important trick. We want $\frac{d}{dx}$ to be $\frac{d}{dt}$ in order to apply it on its argument. This is done by dividing the numerator and denominator of $\frac{d}{dx}$ by dt which becomes

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(e^{-t}\frac{dy}{dt}\right)\\ &= \frac{dt}{dx}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\end{aligned}$$

But from (3) $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$. Hence the above becomes

$$\frac{d^2y}{dx^2} = e^{-t}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)$$

Now we can apply the product rule to finish the above

$$\begin{aligned}\frac{d^2y}{dx^2} &= e^{-t}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)\\ &= e^{-2t}\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right)\\ &= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \qquad (6)\end{aligned}$$

We have now converted bot $y'(x)$ and $y''(x)$. what is left is to plug these back into the original ODE (1). Which becomes

$$
\begin{aligned}
x^2 y''(x) + x y'(x) + y(x) &= 0 \\
x^2 e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + x e^{-t}\frac{dy}{dt} + y(t) &= 0
\end{aligned}
$$

But $x = e^t$ and $x^2 = e^{2t}$. The above becomes

$$
\begin{aligned}
\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + y(t) &= 0 \\
\frac{d^2y}{dt^2} + y(t) &= 0
\end{aligned}
$$

This is now constant coefficient ODE. It is standard one which has the solution

$$
y(t) = A\cos(t) + B\sin(t)
$$

Now we switch back to x. Since $\ln x = t$, then the above becomes

$$
y(x) = A\cos(\ln x) + B\sin(\ln x)
$$

This completes the solution.

2 Change of variable in the dependent variable

2.1 Example 1. Bessel ODE

Given the ode

$$
y''(x) + \left(1 - \frac{3}{4x^2}\right) y(x) = 0 \tag{1}
$$

We will use the change of variable in the dependent variable $y = u x^{\frac{1}{2}}$ to transform the above ODE to Bessel ODE

$$
x^2 u'' + x u' + \left(x^2 - 1\right) u = 0
$$

Since $y = u x^{\frac{1}{2}}$ then

$$
\frac{dy}{dx} = \frac{du}{dx} x^{\frac{1}{2}} + u \frac{x^{\frac{-1}{2}}}{2} \tag{2}
$$

And

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx} x^{\frac{1}{2}} + u \frac{x^{\frac{-1}{2}}}{2}\right) \\
&= \frac{d}{dx}\left(\frac{du}{dx} x^{\frac{1}{2}}\right) + \frac{d}{dx}\left(u \frac{x^{\frac{-1}{2}}}{2}\right) \\
&= \frac{d^2u}{dx^2} x^{\frac{1}{2}} + \frac{1}{2}\frac{du}{dx} x^{-\frac{1}{2}} + \frac{1}{2}\frac{du}{dx} x^{\frac{-1}{2}} - \frac{1}{4} u x^{-\frac{3}{2}} \\
&= \frac{d^2u}{dx^2} x^{\frac{1}{2}} + \frac{du}{dx} x^{-\frac{1}{2}} - \frac{1}{4} u x^{-\frac{3}{2}} \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives

$$
\begin{aligned}
\frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} + \left(1 - \frac{3}{4x^2}\right)ux^{\frac{1}{2}} &= 0 \\
\frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - \frac{1}{4}ux^{-\frac{3}{2}} + ux^{\frac{1}{2}} - \frac{3}{4}ux^{-\frac{3}{2}} &= 0 \\
\frac{d^2u}{dx^2}x^{\frac{1}{2}} + \frac{du}{dx}x^{-\frac{1}{2}} - ux^{-\frac{3}{2}} + ux^{\frac{1}{2}} &= 0
\end{aligned}
$$

Multiplying both side by $x^{\frac{3}{2}}$ gives

$$
\begin{aligned}
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - u + ux^2 &= 0 \\
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - \left(1 - x^2\right)u &= 0 \\
x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(x^2 - 1\right)u &= 0
\end{aligned}
$$

Which is Bessel ode where $v = 1$.