Analysis of the eigenvalues and eigenfunctions for $y''(x) + \lambda y(x) = 0$ for all possible homogeneous boundary conditions

Nasser M. Abbasi

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Contents

The eigenvalues and eigenfunctions for $y'' + \lambda y = 0$ over $0 < x < L$ for all possible combinations of homogeneous boundary conditions are derived analytically. For each boundary condition case, a plot of the first few normalized eigenfunctions are given as well as the numerical values of the first few eigenvalues for the special case when $L = \pi$.

1 Summary of result

This section is a summary of the results. It shows for each boundary conditions the eigenvalues found and the corresponding eigenfunctions, and the full solution. A partial list of the numerical values of the eigenvalues for $L = \pi$ is given and a plot of the first few normalized eigenfunctions.

1.1 case 1: boundary conditions $y(0) = 0, y(L) = 0$

eigenvalues			eigenfunctions
$\lambda < 0$	None		None
$\lambda = 0$	None		None
$\lambda > 0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$	$n = 1, 2, 3, \cdots$	$\Phi_n(x) = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$

Normalized eigenfunctions: For $L = 1$,

$$\Phi_n(x) = \sqrt{2} \sin\left(\sqrt{\lambda_n}\, x\right)$$

For $L = \pi$,

$$\Phi_n(x) = \sqrt{\frac{2}{\pi}} \sin\left(\sqrt{\lambda_n}\, x\right)$$

List of eigenvalues

$$\left\{\frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \frac{16\pi^2}{L^2}, \cdots\right\}$$

List of numerical eigenvalues when $L = \pi$

$$\{1, 4, 8, 16, 25, \cdots\}$$

This is a plot showing how the eigenvalues change in value

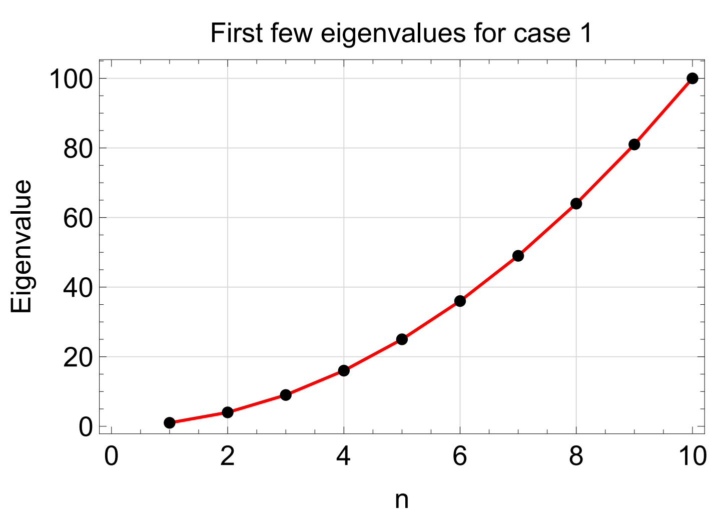

Figure 1: plot of eigenvalues change in value

This is a plot showing the corresponding normalized eigenfunctions for the first 4 eigenvalues. We see that the number of zeros for $\Phi_n(x)$ is $n-1$ inside the interval $0 < x < \pi$. (not counting the end points). Hence $\Phi_1(x)$ which correspond to $\lambda_1 = 1$ in this case, will have no zeros inside the interval. While $\Phi_2(x)$ which correspond to $\lambda_2 = 4$ in this case, will have one zero and so on.

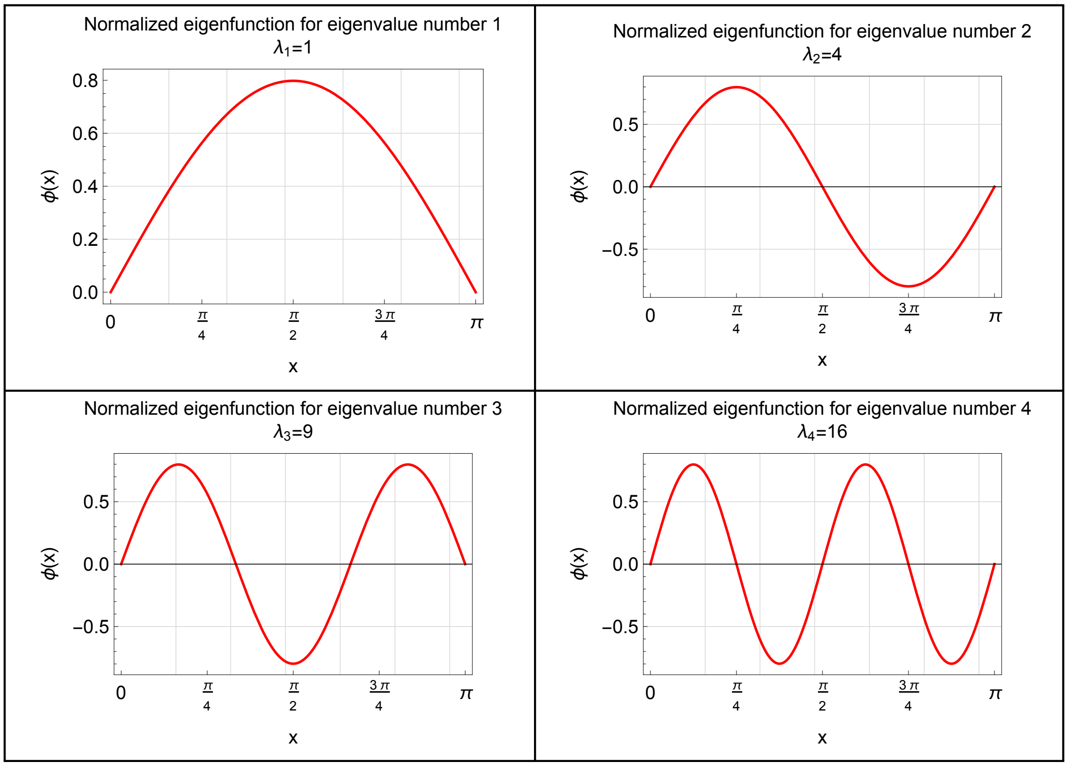

Figure 2: plot showing the corresponding normalized eigenfunction

1.2 case 2: boundary conditions $y(0) = 0, y'(L) = 0$

eigenvalues		eigenfunctions
$\lambda < 0$	None	None
$\lambda = 0$	None	None
$\lambda > 0$	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2$ $\quad n = 1, 3, 5, \cdots$	$\Phi_n(x) = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$

Normalized eigenfunctions: For $L = 1$,

$$\Phi_n(x) = \sqrt{2} \sin\left(\sqrt{\lambda_n}\, x\right)$$

For $L = \pi$,

$$\Phi_n(x) = \sqrt{\frac{2}{\pi}} \sin\left(\sqrt{\lambda_n}\, x\right)$$

List of eigenvalues

$$\left\{ \frac{\pi^2}{4L^2}, \frac{9\pi^2}{4L^2}, \frac{25\pi^2}{4L^2}, \frac{49\pi^2}{4L^2}, \cdots \right\}$$

List of numerical eigenvalues when $L = \pi$

$$\{0.25, 2.25, 6.25, 12.25, 20.25, \cdots\}$$

This is a plot showing how the eigenvalues change in value

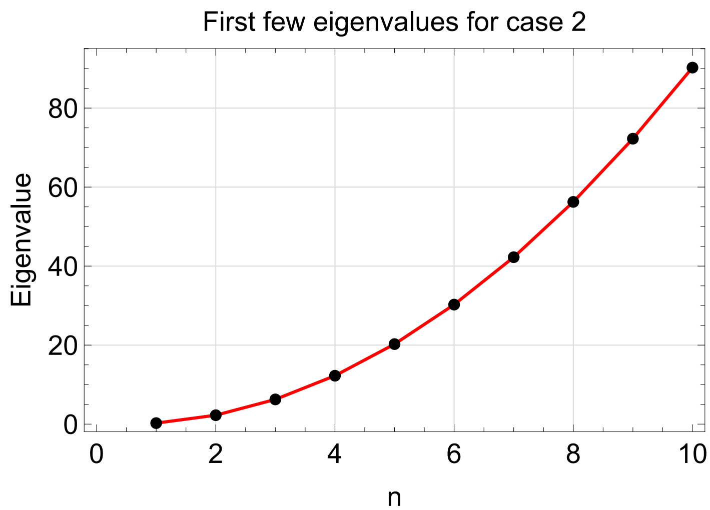

Figure 3: plot showing how the eigenvalues change in value

This is a plot showing the corresponding normalized eigenfunctions for the first 4 eigenvalues.

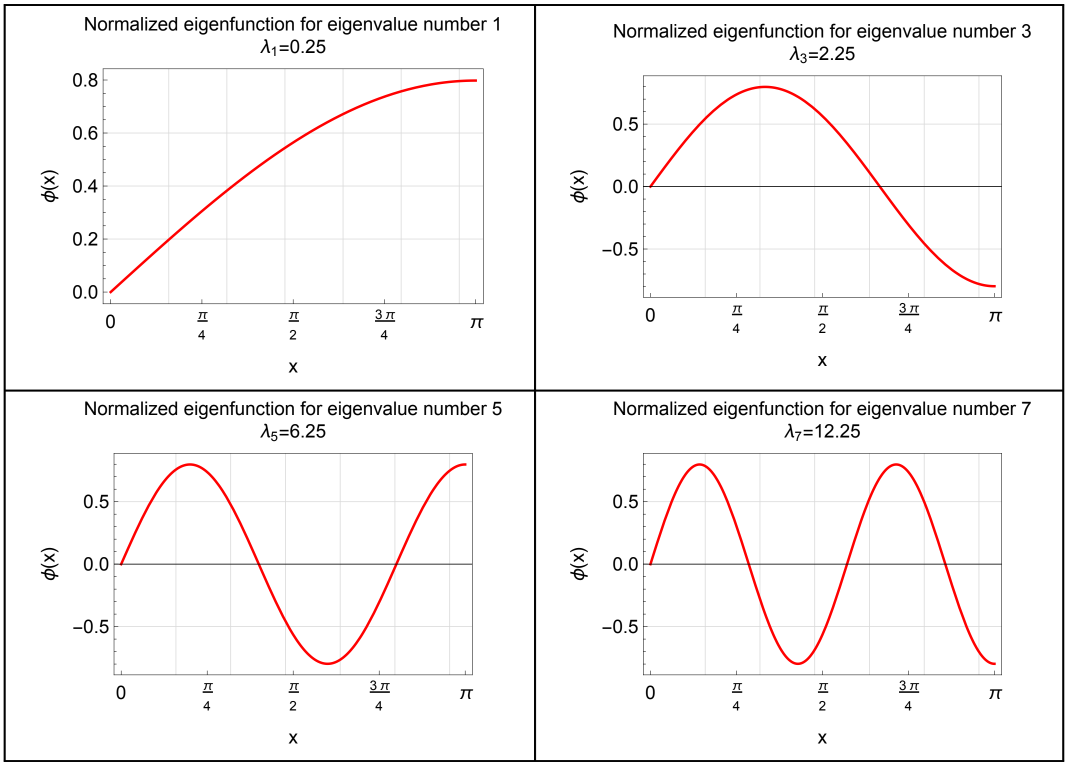

Figure 4: plot showing the corresponding normalized eigenfunctions

1.3 case 3: boundary conditions $y(0) = 0, y(L) + y'(L) = 0$

eigenvalues		eigenfunctions
$\lambda < 0$	None	None
$\lambda = 0$	None	None
$\lambda > 0$	roots of $\tan\left(\sqrt{\lambda}L\right) + \sqrt{\lambda} = 0$	$\Phi_n(x) = c_n \sin\left(\sqrt{\lambda_n}\, x\right)$

Normalized eigenfunctions: For $L = \pi$,

$$\Phi_1 = (0.729448) \sin\left(\sqrt{0.620}x\right)$$
$$\Phi_2 = (0.766385) \sin\left(\sqrt{2.794}x\right)$$
$$\vdots$$

The normalization constant in this case depends on the eigenvalue.

List of numerical eigenvalues when $L = \pi$ (since there is no analytical solution)

$$\{0.620, 2.794, 6.845, 12.865, 20.879, \cdots\}$$

This is a plot showing how the eigenvalues change in value

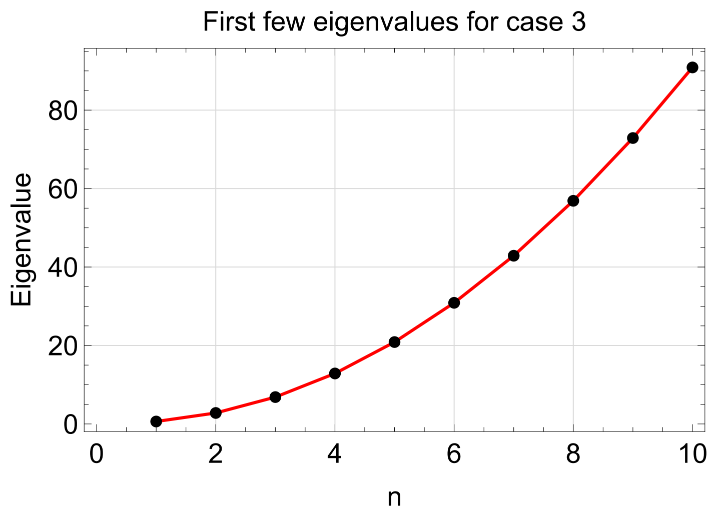

Figure 5: plot showing how the eigenvalues change in value

This is a plot showing the corresponding normalized eigenfunctions for the first 4 eigenvalues.

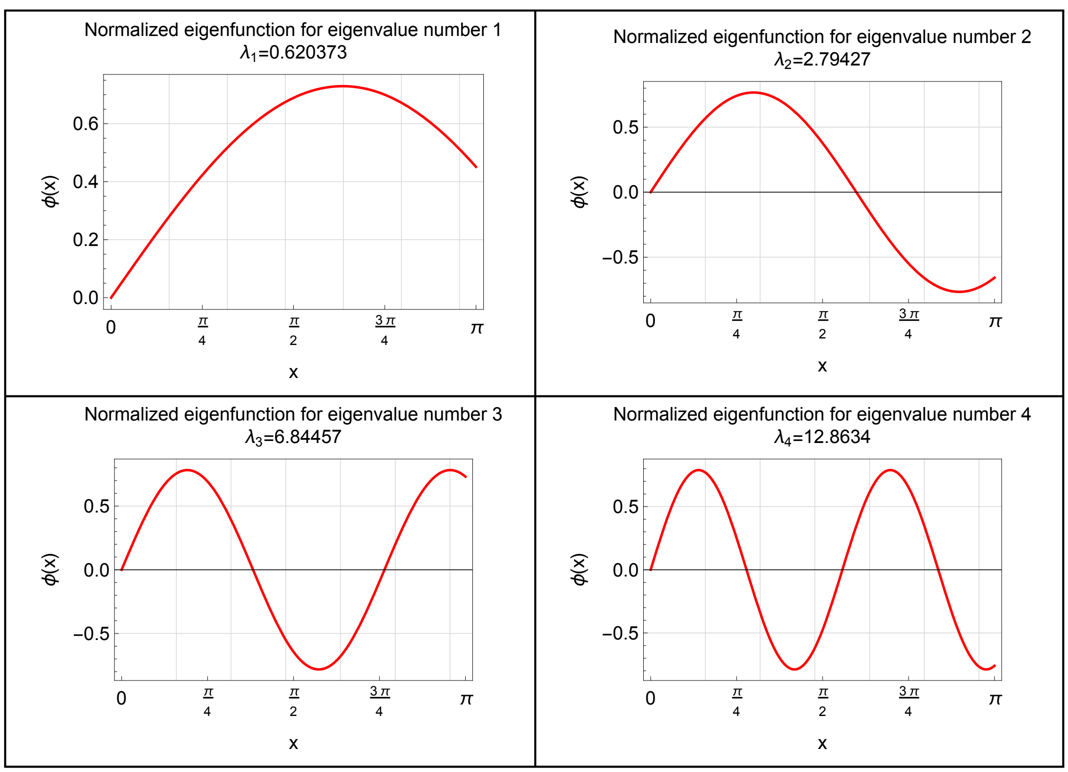

Figure 6: plot showing the corresponding normalized eigenfunctions

1.4 case 4: boundary conditions $y'(0) = 0, y(L) = 0$

eigenvalues			eigenfunctions
$\lambda < 0$	None		None
$\lambda = 0$	None		None
$\lambda > 0$	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2$	$n = 1, 3, 5, \cdots$	$\Phi_n(x) = c_n \cos\left(\sqrt{\lambda_n}\, x\right)$

Normalized eigenfunctions for $L = 1$

$$\tilde{\Phi}_n = \sqrt{2} \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

When $L = \pi$

$$\tilde{\Phi}_n = \sqrt{\frac{2}{\pi}} \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

List of eigenvalues

$$\left\{\frac{\pi^2}{4L^2}, \frac{9\pi^2}{4L^2}, \frac{25\pi^2}{4L^2}, \frac{49\pi^2}{4L^2}, \cdots\right\}$$

List of numerical eigenvalues when $L = \pi$

$$\{0.25, 2.25, 6.25, 12.25, 20.25, \cdots\}$$

This is a plot showing how the eigenvalues change in value

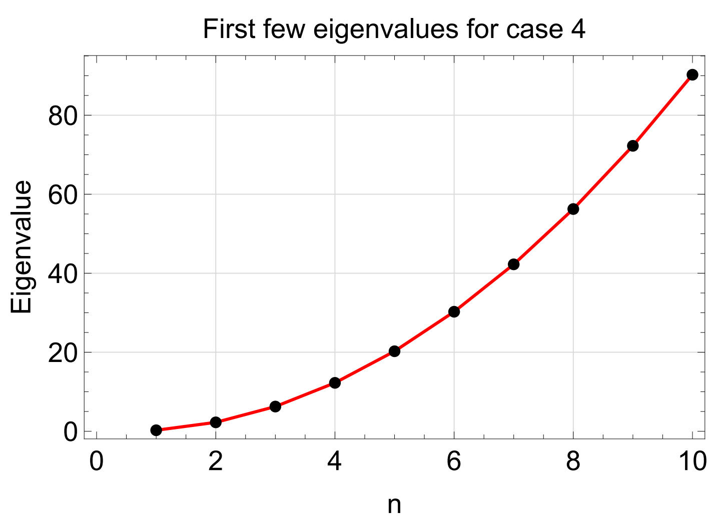

Figure 7: plot showing how the eigenvalues change in value

This is a plot showing the corresponding normalized eigenfunctions for the first 4 eigenvalues.

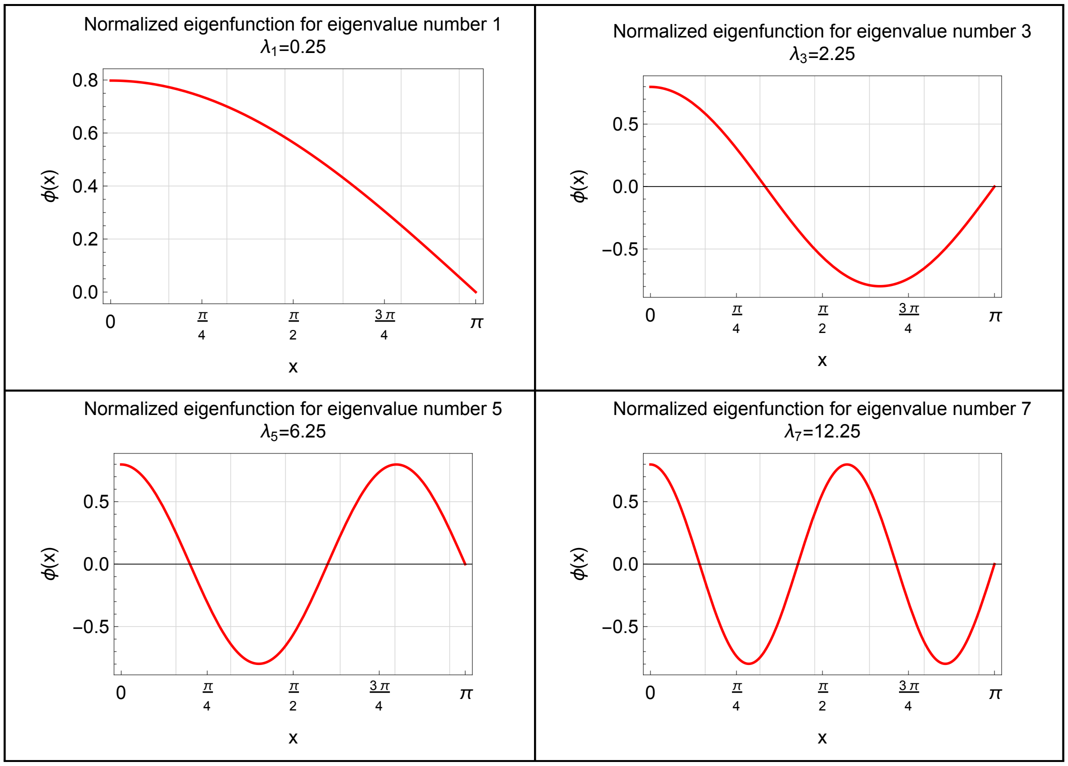

Figure 8: plot showing the corresponding normalized eigenfunctions

1.5 case 5: boundary conditions $y'(0) = 0, y'(L) = 0$

eigenvalues			eigenfunctions
$\lambda < 0$	None		None
$\lambda = 0$	Yes		constant say 1
$\lambda > 0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2$	$n = 1, 2, 3, \cdots$	$\Phi_n(x) = c_n \cos\left(\sqrt{\lambda_n}\, x\right)$

Normalized eigenfunction when $L = 1$

$$\tilde{\Phi}_n = \sqrt{2} \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots$$

When $L = \pi$

$$\tilde{\Phi}_n = \sqrt{\frac{2}{\pi}} \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots$$

For $\tilde{\Phi}_0$, When $L = 1$

$$\tilde{\Phi}_0 = 1$$

When $L = \pi$

$$\tilde{\Phi}_0 = \sqrt{\frac{1}{\pi}}$$

List of eigenvalues

$$\left\{0, \frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \frac{9\pi^2}{L^2}, \frac{16\pi^2}{L^2}, \cdots\right\}$$

List of numerical eigenvalues when $L = \pi$

$$\{0, 1, 4, 9, 16, \cdots\}$$

This is a plot showing how the eigenvalues change in value

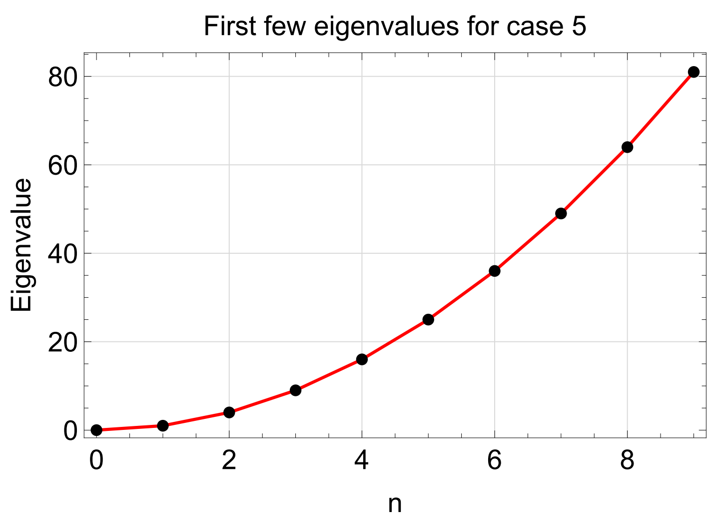

Figure 9: plot showing how the eigenvalues change in value

This is a plot showing the corresponding normalized eigenfunctions for the first 4 eigenvalues.

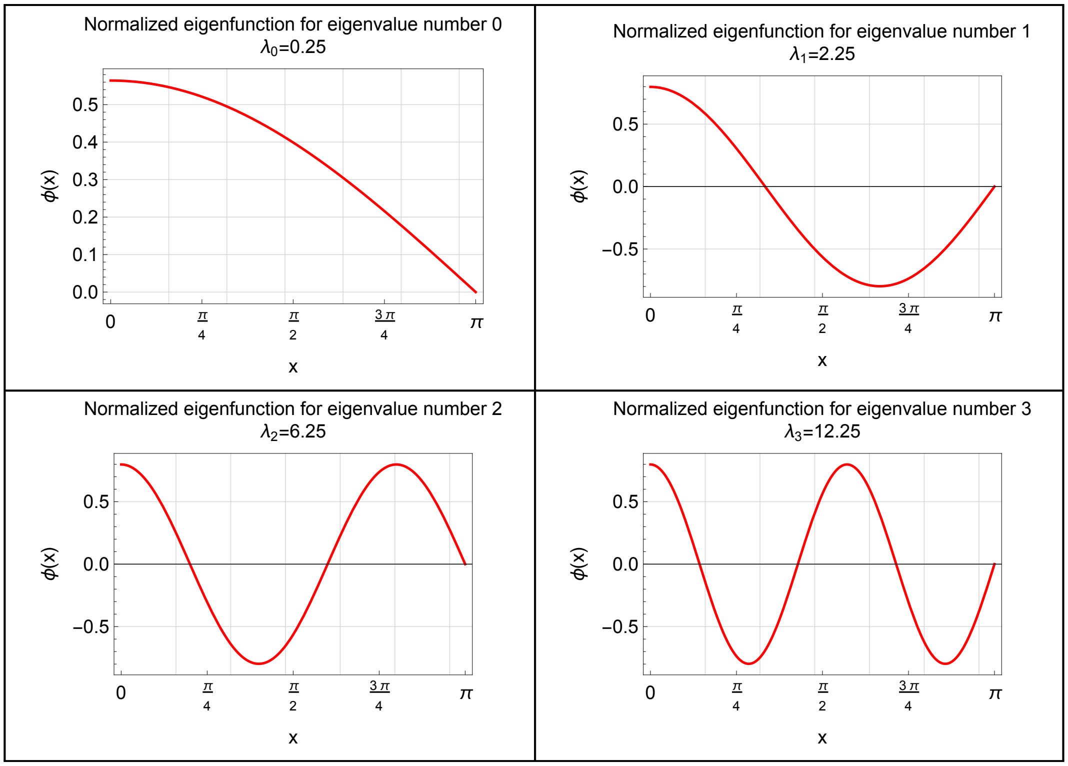

Figure 10: plot showing the corresponding normalized eigenfunctions

1.6 case 6: boundary conditions $y'(0) = 0, y(L) + y'(L) = 0$

eigenvalues		eigenfunctions
$\lambda < 0$	None	None
$\lambda = 0$	None	None
$\lambda > 0$	Roots of $\sqrt{\lambda}\tan\left(\sqrt{\lambda}L\right) = 1$	$\Phi_n(x) = c_n \cos\left(\sqrt{\lambda_n}\, x\right)$

Normalized eigenfunctions for $L = \pi$ are

$$\Phi_1 = (0.705925)\cos\left(\sqrt{0.147033x}\right)$$
$$\Phi_2 = (0.751226)\cos\left(\sqrt{1.48528x}\right)$$
$$\vdots$$

List of numerical eigenvalues when $L = \pi$ (There is no analytical solution for the roots)

$$\{0.147033, 1.48528, 4.576, 9.606, 16.622, \cdots\}$$

This is a plot showing how the eigenvalues change in value

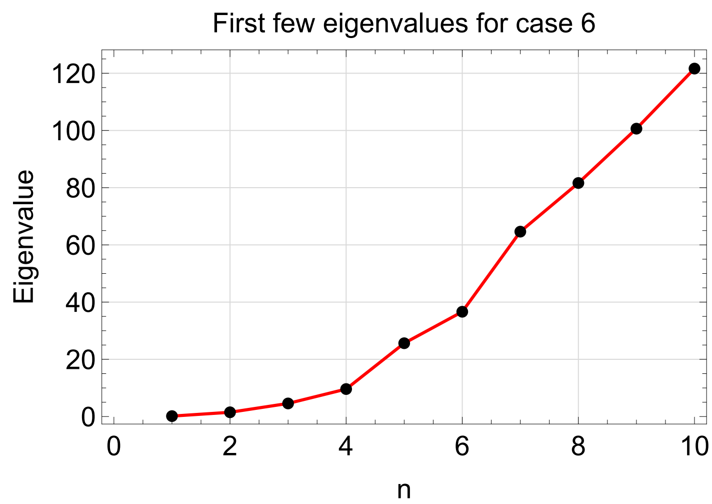

Figure 11: plot showing how the eigenvalues change in value

This is a plot showing the corresponding normalized eigenfunctions for the first 4 eigenvalues.

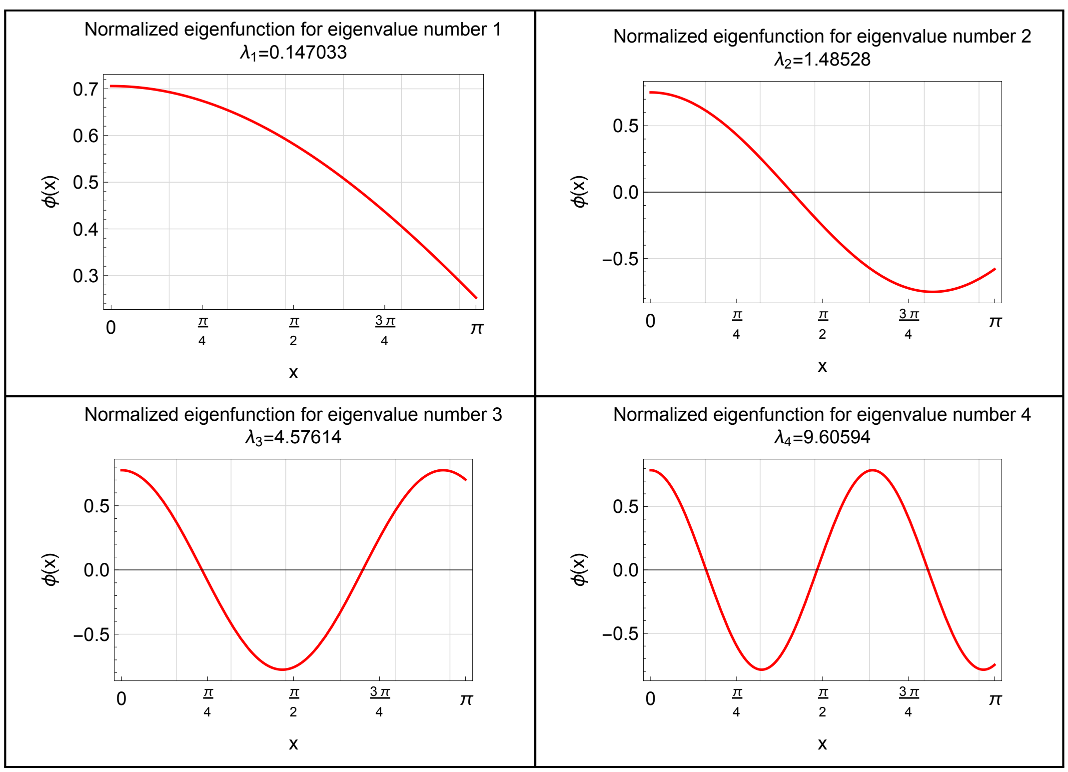

Figure 12: plot showing the corresponding normalized eigenfunctions

1.7 case 7: boundary conditions $y(0)+y'(0)=0, y(L)=0$

eigenvalues		eigenfunctions
$\lambda<0$	Root of $\tanh\left(\sqrt{-\lambda}L\right)=\sqrt{-\lambda}$(one root)	$\Phi(x)=\sinh\left(\sqrt{-\lambda}x\right)-\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$
$\lambda=0$	None	None
$\lambda>0$	Roots of $\tan\left(\sqrt{\lambda}L\right)=\sqrt{\lambda}$	$\Phi_n(x)=\sin\left(\sqrt{\lambda}x\right)-\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$

List of numerical eigenvalues when $L=\pi$ (There is no analytical solution for the roots)

$$\{-0.992, 1.664, 5.631, 11.623, \cdots\}$$

This is a plot showing how the eigenvalues change in value

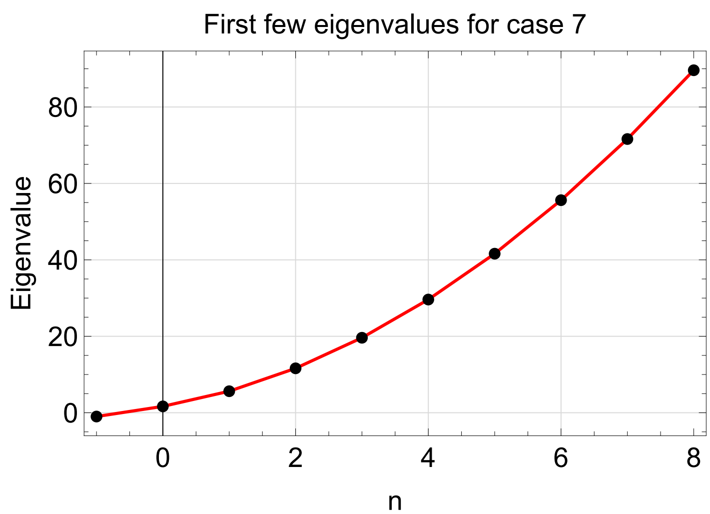

Figure 13: plot showing how the eigenvalues change in value

This is a plot showing the corresponding eigenfunctions for the first 4 eigenvalues.

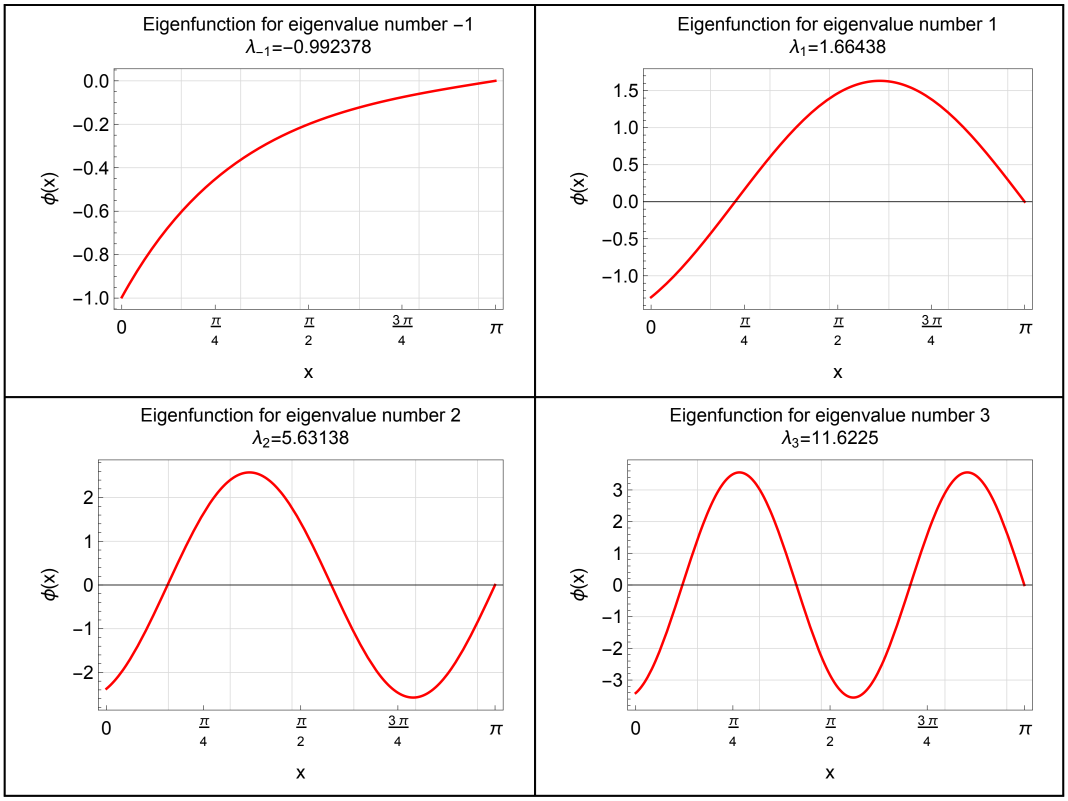

Figure 14: plot showing the corresponding eigenfunctions

1.8 case 8: boundary conditions $y(0)+y'(0)=0, y'(L)=0$

eigenvalues		eigenfunctions
$\lambda<0$	Root of $\tanh\left(\sqrt{-\lambda}L\right)=\frac{1}{\sqrt{-\lambda}}$(one root)	$\Phi_{-1}(x)=\sinh\left(\sqrt{-\lambda}x\right)-\sqrt{-\lambda}\cosh\left(\sqrt{-\lambda}x\right)$
$\lambda=0$	None	None
$\lambda>0$	Roots of $\tan\left(\sqrt{\lambda}L\right)=\frac{-1}{\sqrt{\lambda}}$	$\Phi_n(x)=\sin\left(\sqrt{\lambda}x\right)-\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$

List of numerical eigenvalues when $L=\pi$ (There is no analytical solution for the roots)

$$\{-1.007, 0.480, 3.392, 8.376, 24, 368, \cdots\}$$

This is a plot showing how the eigenvalues change in value

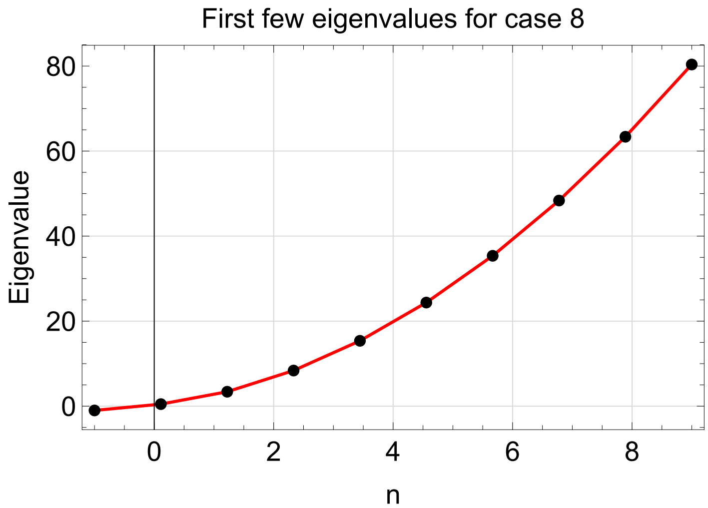

Figure 15: plot showing how the eigenvalues change in value

This is a plot showing the corresponding eigenfunctions for the first 4 eigenvalues.

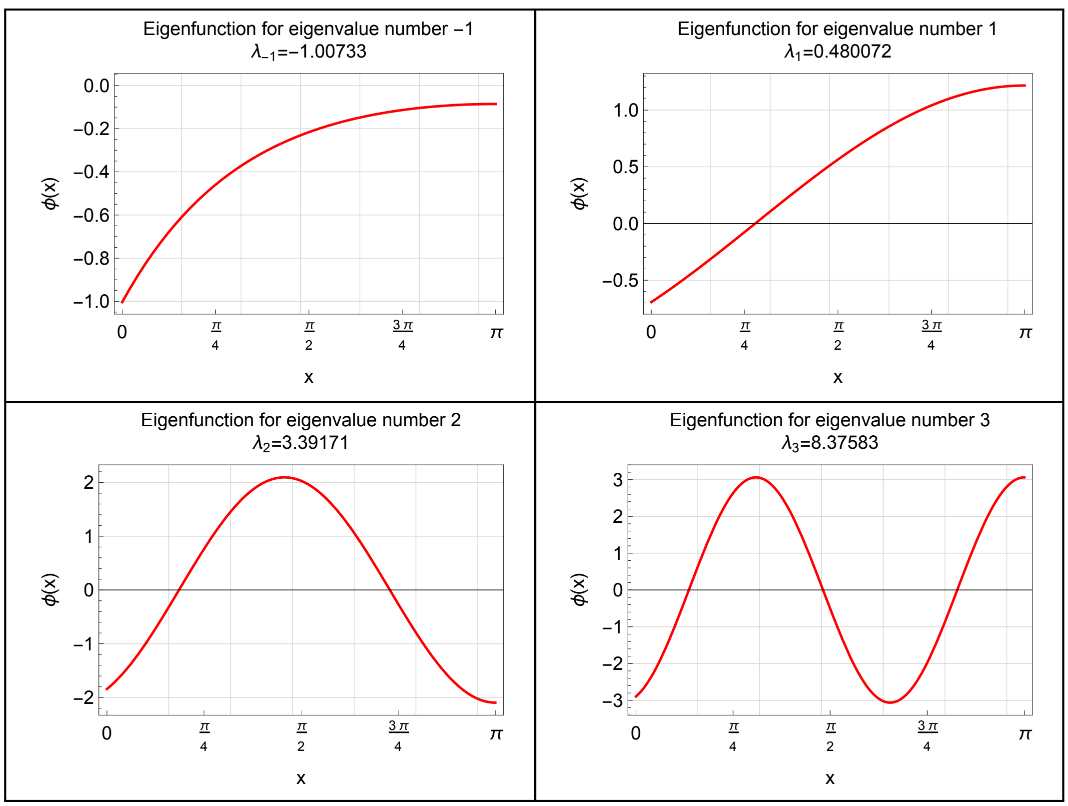

Figure 16: plot showing the corresponding eigenfunctions

1.9 case 9: boundary conditions $y(0)+y'(0)=0, y(L)+y'(L)=0$

eigenvalues			eigenfunctions
$\lambda<0$	-1		$\Phi_{-1}(x)=\sinh(x)-\cosh(x)$
$\lambda=0$	None		None
$\lambda>0$	$\lambda_n=\left(\frac{n\pi}{L}\right)^2$	$n=1,2,3,\cdots$	$\Phi_n(x)=\sin\left(\sqrt{\lambda_n}x\right)-\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right)$

List of eigenvalues

$$\left\{-1,\frac{\pi^2}{L^2},\frac{4\pi^2}{L^2},\frac{9\pi^2}{L^2},\frac{16\pi^2}{L^2},\ldots\right\}$$

List of numerical eigenvalues when $L=\pi$

$$\{-1,1,4,9,16,\cdots\}$$

This is a plot showing how the eigenvalues change in value

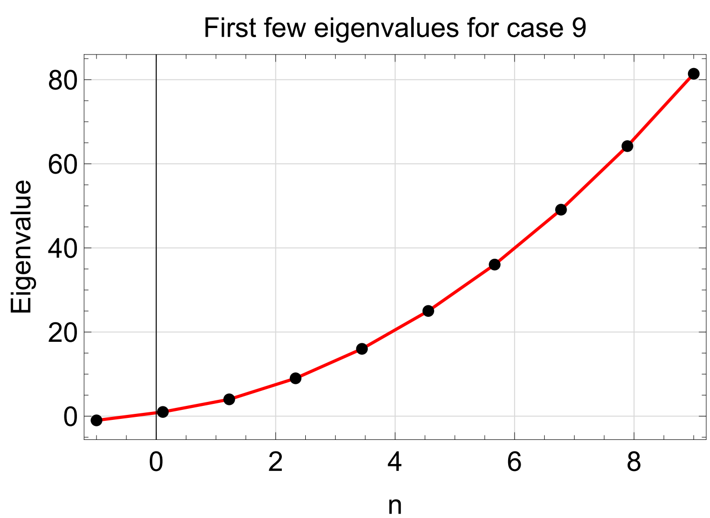

Figure 17: plot showing how the eigenvalues change in value

This is a plot showing the corresponding eigenfunctions for the first 4 eigenvalues.

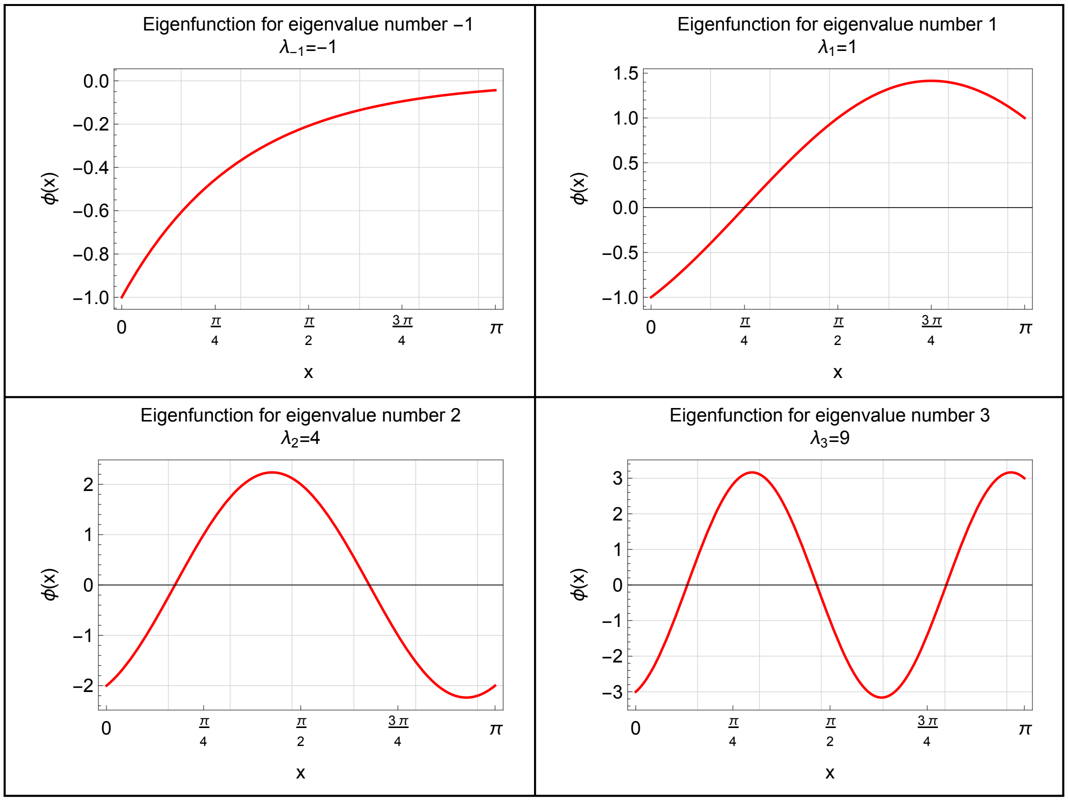

Figure 18: plot showing the corresponding eigenfunctions

2 Derivations

2.1 case 1: boundary conditions $y(0) = 0, y(L) = 0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$r^2 + \lambda = 0$$
$$r = \pm\sqrt{-\lambda}$$

Let $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

First B.C. $y(0) = 0$ gives

$$0 = c_1$$

The solution becomes

$$y(x) = c_2 \sinh(\mu x)$$

The second B.C. $y(L) = 0$ results in

$$0 = c_2 \sinh(\mu L)$$

But $\sinh(\mu L) \neq 0$ since $\mu L \neq 0$, hence $c_2 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y(0) = 0$ gives

$$0 = c_1$$

The solution becomes

$$y(x) = c_2 x$$

Applying the second B.C. $y(L) = 0$ gives

$$0 = c_2 L$$

Therefore $c_2 = 0$, leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Let $\lambda > 0$, The solution is

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

First B.C. $y(0) = 0$ gives

$$0 = c_1$$

The solution becomes

$$y(x) = c_2 \sin\left(\sqrt{\lambda} x\right)$$

Second B.C. $y(L) = 0$ gives

$$0 = c_2 \sin\left(\sqrt{\lambda} L\right)$$

Non-trivial solution implies $\sin\left(\sqrt{\lambda} L\right) = 0$ or $\sqrt{\lambda} L = n\pi$ for $n = 1, 2, 3, \cdots$. Therefore

$$\begin{aligned} \sqrt{\lambda_n} &= \frac{n\pi}{L} \qquad n = 1, 2, 3, \cdots \\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots \end{aligned}$$

The corresponding eigenfunctions are

$$\Phi_n = c_n \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots$$

The normalized $\tilde{\Phi}_n$ eigenfunctions are now found. In this problem the weight function is $r(x) = 1$, therefore solving for c_n from

$$
\begin{aligned}
\int_0^L r(x)\,\Phi_n^2 dx &= 1\\
\int_0^L c_n^2 \sin^2\left(\sqrt{\lambda_n} x\right) dx &= 1\\
c_n^2 \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\left(2\sqrt{\lambda_n} x\right)\right) dx &= 1\\
\int_0^L \frac{1}{2} dx - \int_0^L \frac{1}{2}\cos\left(2\sqrt{\lambda_n} x\right) dx &= \frac{1}{c_n^2}\\
\frac{1}{2}L - \frac{1}{2}\left(\frac{\sin\left(2\sqrt{\lambda_n} x\right)}{2\sqrt{\lambda_n}}\right)_0^L &= \frac{1}{c_n^2}\\
\frac{1}{2}L - \frac{1}{4\sqrt{\lambda_n}}\sin\left(2\sqrt{\lambda_n} L\right) &= \frac{1}{c_n^2}\\
2\sqrt{\lambda_n} L - \sin\left(2\sqrt{\lambda_n} L\right) &= \frac{4\sqrt{\lambda_n}}{c_n^2}
\end{aligned}
$$

Hence

$$
c_n = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} L - \sin\left(2\sqrt{\lambda_n} L\right)}}
$$

For example, when $L = 1$ the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{L} = n\pi$)

$$
\begin{aligned}
c_n &= \sqrt{\frac{4n\pi}{2n\pi - \sin(2n\pi)}}\\
&= \sqrt{\frac{4n\pi}{2n\pi}}\\
c_n &= \sqrt{2}
\end{aligned}
$$

For $L = \pi$, the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{\pi} = n$)

$$
\begin{aligned}
c_n &= \sqrt{\frac{4n}{2n\pi - \sin(2n\pi)}}\\
&= \sqrt{\frac{4n}{2n\pi}}\\
c_n &= \sqrt{\frac{2}{\pi}}
\end{aligned}
$$

The normalization c_n value depends on the length. When $L = 1$

$$\tilde{\Phi}_n = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$

When $L = \pi$

$$\tilde{\Phi}_n = \sqrt{\frac{2}{\pi}}\sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$

2.2 case 2: boundary conditions $y(0) = 0, y'(L) = 0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$r^2 + \lambda = 0$$
$$r = \pm\sqrt{-\lambda}$$

Let $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

First B.C. gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sinh(\mu x)$$

Second B.C. gives

$$y'(x) = \mu c_2 \cosh(\mu x)$$
$$0 = \mu c_2 \cosh(\mu L)$$

But $\cosh(\mu L)$ can not be zero, hence only other choice is $c_2 = 0$, leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 x$$

Second B.C. gives

$$y'(x) = c_2$$
$$0 = c_2$$

Leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Let $\lambda > 0$, the solution is

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

First B.C. gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sin\left(\sqrt{\lambda} x\right)$$

Second B.C. gives

$$\begin{aligned} y'(x) &= \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} x\right) \\ 0 &= \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} L\right) \end{aligned}$$

Non-trivial solution implies $\cos\left(\sqrt{\lambda} L\right) = 0$ or $\sqrt{\lambda} L = \frac{n\pi}{2}$ for $n = 1, 3, 5, \cdots$. Therefore

$$\begin{aligned} \sqrt{\lambda_n} L &= \frac{n\pi}{2} \\ \sqrt{\lambda_n} &= \frac{n\pi}{2L} \qquad n = 1, 3, 5, \cdots \end{aligned}$$

The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 3, 5, \cdots$$

The corresponding eigenfunctions are

$$\Phi_n = c_n \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

The normalized $\tilde{\Phi}_n$ eigenfunctions are now found. Since the weight function is $r(x) = 1$, therefore solving for c_n from

$$\begin{aligned} \int_0^L r(x)\, \Phi_n^2 dx &= 1 \\ \int_0^L c_n^2 \sin^2\left(\sqrt{\lambda_n} x\right) dx &= 1 \end{aligned}$$

As was done earlier, the above results in

$$c_n = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} L - \sin\left(2\sqrt{\lambda_n} L\right)}} \qquad n = 1, 3, 5, \cdots$$

For $L = 1$ the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{2L} = \frac{n\pi}{2}$)

$$\begin{aligned} c_n &= \sqrt{\frac{4\frac{n\pi}{2}}{2\frac{n\pi}{2} - \sin\left(2\frac{n\pi}{2}\right)}} \\ &= \sqrt{\frac{2n\pi}{n\pi}} \\ c_n &= \sqrt{2} \end{aligned}$$

For $L = \pi$, the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{2\pi} = \frac{n}{2}$)

$$\begin{aligned} c_n &= \sqrt{\frac{4\frac{n}{2}}{2\frac{n}{2}\pi - \sin\left(2\frac{n}{2}\pi\right)}} \\ &= \sqrt{\frac{2n}{n\pi}} \\ c_n &= \sqrt{\frac{2}{\pi}} \end{aligned}$$

Therefore, for $L = 1$

$$\tilde{\Phi}_n = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 3, 5, \cdots$$

For $L = \pi$

$$\tilde{\Phi}_n = \sqrt{\frac{2}{\pi}}\sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 3, 5, \cdots$$

2.3 case 3: boundary conditions $y(0) = 0, y(L) + y'(L) = 0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Let $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

First B.C. $y(0) = 0$ gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sinh(\mu x)$$

Second B.C. $y(L) + y'(L) = 0$ gives

$$0 = c_2(\sinh(\mu L) + \mu \cosh(\mu x))$$

But $\sinh(\mu L) \neq 0$ since $\mu L \neq 0$ and $\cosh(\mu x)$ can not be zero, hence $c_2 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y(0) = 0$ gives

$$0 = c_1$$

The solution becomes

$$y(x) = c_2 x$$

Second B.C. $y(L) + y'(L) = 0$ gives

$$\begin{aligned} 0 &= c_2 L + c_2 \\ &= c_2(1 + L) \end{aligned}$$

Therefore $c_2 = 0$, leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Let $\lambda > 0$, The solution is

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

First B.C. $y(0) = 0$ gives

$$0 = c_1$$

The solution becomes

$$y(x) = c_2 \sin\left(\sqrt{\lambda} x\right)$$

Second B.C. $y(L) + y'(L) = 0$ gives

$$0 = c_2\left(\sin\left(\sqrt{\lambda} L\right) + \sqrt{\lambda} \cos\left(\sqrt{\lambda} L\right)\right)$$

For non-trivial solution, we want $\sin\left(\sqrt{\lambda} L\right) + \sqrt{\lambda} \cos\left(\sqrt{\lambda} L\right) = 0$ or $\tan\left(\sqrt{\lambda} L\right) + \sqrt{\lambda} = 0$ Therefore the eigenvalues are given by the solution to

$$\tan\left(\sqrt{\lambda} L\right) + \sqrt{\lambda} = 0$$

And the corresponding eigenfunction is

$$\Phi_n = c_n \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 2, 3, \cdots$$

The normalized $\tilde{\Phi}_n$ eigenfunctions are now found. Since the weight function is $r(x) = 1$, therefore solving for c_n from

$$\int_0^L r(x)\, \Phi_n^2 dx = 1$$
$$\int_0^L c_n^2 \sin^2\left(\sqrt{\lambda_n} x\right) dx = 1$$

As was done earlier, the above results in

$$c_n = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} L - \sin\left(2\sqrt{\lambda_n} L\right)}} \qquad n = 1, 2, 3, \cdots$$

Since there is no closed form solution to λ_n as it is a root of nonlinear equation $\tan\left(\sqrt{\lambda} L\right) + \sqrt{\lambda} = 0$, the normalized constant is found numerically. For $L = \pi$, the first few roots are

$$\lambda_n = \{0.620, 2.794, 6.845, 12.865, 20.879, \cdots\}$$

In this case, the normalization constants depends on n and are not the same as in earlier cases. The following small program was written to find the first 10 normalization constants and to verify that each will make $\int_0^L c_n^2 \sin^2\left(\sqrt{\lambda_n} x\right) dx = 1$

The normalized constants are found to be (for $L = \pi$)

$$c_n = \{0.729448, 0.766385, 0.782173, 0.788879, 0.792141, 0.79393, 0.795006, 0.7957, 0.796171, 0.796506\}$$

```
In[137]:= L = Pi;
          eig = lam /. NSolve[Tan[Sqrt[lam] L] + Sqrt[lam] == 0 && 0 < lam < 110, lam];
          c[lam_] := Sqrt[(4 Sqrt[lam]) / (2 Sqrt[lam] Pi - Sin[2 Sqrt[lam] Pi])];
          normalizedC = c[#] & /@ eig

Out[140]= {0.729448, 0.766385, 0.782173, 0.788879, 0.792141, 0.79393, 0.795006, 0.7957, 0.796171, 0.796506}

In[141]:= MapThread[Integrate[#1^2 * Sin[Sqrt[#2] x]^2, {x, 0, Pi}] &, {normalizedC, eig}]

Out[141]= {1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}
```

Figure 19: normalized constants

The above implies that the first normalized eigenfunction is

$$\Phi_1 = (0.729448)\sin\left(\sqrt{0.620}x\right)$$

And the second one is

$$\Phi_2 = (0.766385)\sin\left(\sqrt{2.794}x\right)$$

And so on.

2.4 case 4: boundary conditions $y'(0) = 0, y(L) = 0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Let $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$\begin{aligned} y &= c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \\ y' &= c_1\mu \sinh(\mu x) + c_2\mu \cosh(\mu x) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2\mu \\ c_2 &= 0 \end{aligned}$$

Hence solution becomes

$$y(x) = c_1 \cosh(\mu x)$$

Second B.C. $y(L) = 0$ gives

$$0 = c_1 \cosh(\mu L)$$

But $\cosh(\mu L)$ can not be zero, hence $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y'(0) = 0$ gives

$$0 = c_2$$

The solution becomes

$$y(x) = c_1$$

Second B.C. $y(L)=0$ gives

$$0=c_1$$

Therefore $c_1=0$, leading to trivial solution. Therefore $\lambda=0$ is not eigenvalue.

Let $\lambda>0$, The solution is

$$y(x)=c_1\cos\left(\sqrt{\lambda}x\right)+c_2\sin\left(\sqrt{\lambda}x\right)$$
$$y'(x)=-c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)+c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$$

First B.C. $y'(0)=0$ gives

$$0=c_2\sqrt{\lambda}$$
$$c_2=0$$

The solution becomes

$$y(x)=c_1\cos\left(\sqrt{\lambda}x\right)$$

Second B.C. $y(L)=0$ gives

$$0=c_1\cos\left(\sqrt{\lambda}L\right)$$

For non-trivial solution, we want $\cos\left(\sqrt{\lambda}L\right)=0$ or $\sqrt{\lambda}L=\frac{n\pi}{2}$ for odd $n=1,3,5,\cdots$ Therefore

$$\lambda_n=\left(\frac{n\pi}{2L}\right)^2 \qquad n=1,3,5,\cdots$$

The corresponding eigenfunctions are

$$\Phi_n=c_n\cos\left(\sqrt{\lambda_n}x\right) \qquad n=1,3,5,\cdots$$

The normalized $\tilde{\Phi}_n$ eigenfunctions are now found. In this problem the weight function is $r(x)=1$, therefore solving for c_n from

$$
\begin{aligned}
\int_0^L r(x)\,\Phi_n^2 dx &= 1 \\
\int_0^L c_n^2 \cos^2\left(\sqrt{\lambda_n} x\right) dx &= 1 \\
c_n^2 \int_0^L \left(\frac{1}{2} + \frac{1}{2}\cos\left(2\sqrt{\lambda_n} x\right)\right) dx &= 1 \\
\int_0^L \frac{1}{2} dx + \int_0^L \frac{1}{2}\cos\left(2\sqrt{\lambda_n} x\right) dx &= \frac{1}{c_n^2} \\
\frac{1}{2}L + \frac{1}{2}\left(\frac{\sin\left(2\sqrt{\lambda_n} x\right)}{2\sqrt{\lambda_n}}\right)_0^L &= \frac{1}{c_n^2} \\
\frac{1}{2}L + \frac{1}{4\sqrt{\lambda_n}}\sin\left(2\sqrt{\lambda_n} L\right) &= \frac{1}{c_n^2} \\
2\sqrt{\lambda_n} L + \sin\left(2\sqrt{\lambda_n} L\right) &= \frac{4\sqrt{\lambda_n}}{c_n^2}
\end{aligned}
$$

Hence

$$
c_n = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} L + \sin\left(2\sqrt{\lambda_n} L\right)}}
$$

For example, when $L = 1$ the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{2L} = \frac{n\pi}{2}$)

$$
\begin{aligned}
c_n &= \sqrt{\frac{4\frac{n\pi}{2}}{2\frac{n\pi}{2} + \sin\left(2\frac{n\pi}{2}\right)}} \\
&= \sqrt{\frac{2n\pi}{n\pi}} \\
c_n &= \sqrt{2}
\end{aligned}
$$

Which is the same when the eigenfunction was $\sin\left(\frac{n\pi}{2L}x\right)$. For $L = \pi$, the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{2L} = \frac{n}{2}$)

$$
\begin{aligned}
c_n &= \sqrt{\frac{4\frac{n}{2}}{2\frac{n}{2}\pi + \sin\left(2\frac{n}{2}\pi\right)}} \\
&= \sqrt{\frac{2n}{2n\pi}} \\
c_n &= \sqrt{\frac{2}{\pi}}
\end{aligned}
$$

The normalization c_n value depends on the length. When $L = 1$

$$\tilde{\Phi}_n = \sqrt{2} \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

When $L = \pi$

$$\tilde{\Phi}_n = \sqrt{\frac{2}{\pi}} \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

2.5 case 5: boundary conditions $y'(0) = 0, y'(L) = 0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Let $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$\begin{aligned} y &= c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \\ y' &= c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2 \mu \\ c_2 &= 0 \end{aligned}$$

Hence solution becomes

$$y(x) = c_1 \cosh(\mu x)$$

Second B.C. $y'(L) = 0$ gives

$$0 = c_1 \mu \sinh(\mu L)$$

But $\sinh(\mu L)$ can not be zero since $\mu L \neq 0$, hence $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y'(0) = 0$ gives

$$0 = c_2$$

The solution becomes

$$y(x) = c_1$$

Second B.C. $y'(L)=0$ gives

$$0=0$$

Therefore c_1 can be any value. Therefore $\lambda=0$ is an eigenvalue and the corresponding eigenfunction is any constant, say 1.

Let $\lambda>0$, The solution is

$$\begin{aligned}y(x)&=c_1\cos\left(\sqrt{\lambda}x\right)+c_2\sin\left(\sqrt{\lambda}x\right)\\ y'(x)&=-c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right)+c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)\end{aligned}$$

First B.C. $y'(0)=0$ gives

$$\begin{aligned}0&=c_2\sqrt{\lambda}\\ c_2&=0\end{aligned}$$

The solution becomes

$$y(x)=c_1\cos\left(\sqrt{\lambda}x\right)$$

Second B.C. $y'(L)=0$ gives

$$0=-c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)$$

For non-trivial solution, we want $\sin\left(\sqrt{\lambda}L\right)=0$ or $\sqrt{\lambda}L=n\pi$ for $n=1,2,3,\cdots$ Therefore

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2\qquad n=1,2,3,\cdots$$

And the corresponding eigenfunctions are

$$\Phi_n(x)=c_n\cos\left(\sqrt{\lambda}x\right)\qquad n=1,2,3,\cdots$$

The normalized $\tilde{\Phi}_n$ eigenfunctions are now found. In this problem the weight function is $r(x)=1$, therefore solving for c_n from

$$\begin{aligned}\int_0^L r(x)\,\Phi_n^2dx&=1\\ \int_0^L c_n^2\cos^2\left(\sqrt{\lambda_n}x\right)dx&=1\end{aligned}$$

As before, the above simplifies to

$$c_n=\sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}L+\sin\left(2\sqrt{\lambda_n}L\right)}}$$

For example, when $L = 1$ the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{L} = n\pi$)

$$c_n = \sqrt{\frac{4n\pi}{2n\pi + \sin(2n\pi)}}$$
$$c_n = \sqrt{2}$$

For $L = \pi$, the normalization constant becomes (since now $\sqrt{\lambda_n} = \frac{n\pi}{L} = n$)

$$c_n = \sqrt{\frac{4n}{2n\pi + \sin(2n\pi)}}$$
$$c_n = \sqrt{\frac{2}{\pi}}$$

The normalization c_n value depends on the length. When $L = 1$

$$\tilde{\Phi}_n = \sqrt{2}\cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

When $L = \pi$

$$\tilde{\Phi}_n = \sqrt{\frac{2}{\pi}}\cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

For $n = 0$, corresponding to the λ_0 eigenvalue, since the eigenfunction is taken as the constant 1, then

$$\int_0^L c_0^2 dx = 1$$
$$c_0 = \sqrt{\frac{1}{L}}$$

Therefore, When $L = 1$

$$\tilde{\Phi}_0 = 1$$

When $L = \pi$

$$\tilde{\Phi}_0 = \sqrt{\frac{1}{\pi}}$$

2.6 case 6: boundary conditions $y'(0)=0, y(L)+y'(L)=0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$
\begin{aligned}
r^2 + \lambda &= 0 \\
r &= \pm\sqrt{-\lambda}
\end{aligned}
$$

Let $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$
\begin{aligned}
y &= c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \\
y' &= c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x)
\end{aligned}
$$

First B.C. $y'(0) = 0$ gives

$$
\begin{aligned}
0 &= c_2 \mu \\
c_2 &= 0
\end{aligned}
$$

Hence solution becomes

$$
y(x) = c_1 \cosh(\mu x)
$$

Second B.C. $y(L) + y'(L) = 0$ gives

$$
0 = c_1(\cosh(\mu L) + \mu \sinh(\mu L))
$$

But $\sinh(\mu L)$ can not be negative since its argument is positive here. And $\cosh \mu L$ is always positive. In addition $\cosh(\mu L) + \mu \sinh(\mu L)$ can not be zero since $\sinh(\mu L)$ can not be zero as $\mu L \neq 0$ and $\cosh(\mu L)$ is not zero. Therefore $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$
y(x) = c_1 + c_2 x
$$

First B.C. $y'(0) = 0$ gives

$$
0 = c_2
$$

The solution becomes

$$
y(x) = c_1
$$

Second B.C. $y(L) + y'(L) = 0$ gives

$$
0 = c_1
$$

This gives trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Let $\lambda > 0$, The solution is

$$
\begin{aligned}
y(x) &= c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \\
y'(x) &= -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right)
\end{aligned}
$$

First B.C. $y'(0) = 0$ gives

$$
\begin{aligned}
0 &= c_2\sqrt{\lambda} \\
c_2 &= 0
\end{aligned}
$$

The solution becomes

$$y(x) = c_1 \cos\left(\sqrt{\lambda}x\right)$$

Second B.C. $y(L) + y'(L) = 0$ gives

$$
\begin{aligned}
0 &= c_1 \cos\left(\sqrt{\lambda}L\right) - c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right) \\
&= c_1\left(\cos\left(\sqrt{\lambda}L\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)\right)
\end{aligned}
$$

For non-trivial solution, we want $\cos\left(\sqrt{\lambda}L\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right) = 0$ or $\sqrt{\lambda}\tan\left(\sqrt{\lambda}L\right) = 1$ Therefore the eigenvalues are the solution to

$$\sqrt{\lambda}\tan\left(\sqrt{\lambda}L\right) = 1$$

And the corresponding eigenfunctions are

$$\Phi_n = \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Where λ_n are the roots of $\sqrt{\lambda}\tan\left(\sqrt{\lambda}L\right) = 1$.

The normalized $\tilde{\Phi}_n$ eigenfunctions are now found. Since the weight function is $r(x) = 1$, therefore solving for c_n from

$$
\begin{aligned}
\int_0^L r(x)\,\Phi_n^2 dx &= 1 \\
\int_0^L c_n^2 \cos^2\left(\sqrt{\lambda_n}x\right) dx &= 1
\end{aligned}
$$

As was done earlier, the above results in

$$c_n = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}L + \sin\left(2\sqrt{\lambda_n}L\right)}} \qquad n = 1, 2, 3, \cdots$$

Since there is no closed form solution to λ_n as it is a root of nonlinear equation $\sqrt{\lambda}\tan\left(\sqrt{\lambda}L\right) = 1$, the normalized constant is found numerically. For $L = \pi$, the first few roots are

$$\lambda_n = \{0.147033, 1.48528, 4.57614, 9.60594, 25.6247, 36.6282, 64.6318, 81.6328, 100.634, 121.634, \cdots\}$$

In this case, the normalization constants depends on n and are not the same as in earlier cases. The following small program was written to find the first 10 normalization constants and to verify that each will make $\int_0^L c_n^2 \cos^2\left(\sqrt{\lambda_n} x\right) dx = 1$

The normalized constants are found to be (for $L = \pi$)

$$c_n = \{0.705925, 0.751226, 0.776042, 0.786174, 0.790773, 0.793157, 0.794531, \cdots\}$$

```
In[247]:= ClearAll[lam, c, eig];
          L = Pi;
          eig = lam /. NSolve[Sqrt[lam] Tan[Sqrt[lam] L] == 1 && 0 < lam < 110, lam];
          c[lam_] := Sqrt[4 Sqrt[lam] / (2 Sqrt[lam] Pi + Sin[2 Sqrt[lam] Pi])];
          normalizedC = c[#] & /@ eig
Out[251]= {0.705925, 0.751226, 0.776042, 0.786174, 0.790773, 0.793157, 0.794531, 0.795388, 0.795957, 0.796352, 0.796638}

In[252]:= MapThread[Integrate[#1^2 * Cos[Sqrt[#2] x]^2, {x, 0, Pi}] &, {normalizedC, eig}]
Out[252]= {1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}
```

Figure 20: normalized constants

The above implies that the first normalized eigenfunction is

$$\Phi_1 = (0.705925)\cos\left(\sqrt{0.147033}x\right)$$

And the second one is

$$\Phi_2 = (0.751226)\cos\left(\sqrt{1.48528}x\right)$$

And so on.

2.7 case 7: boundary conditions $y(0) + y'(0) = 0, y(L) = 0$

Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$r^2 + \lambda = 0$$
$$r = \pm\sqrt{-\lambda}$$

<u>Let $\lambda < 0$</u>

In this case $-\lambda$ is positive and $\sqrt{-\lambda}$ is positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$
$$y' = c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x)$$

First B.C. $y(0) + y'(0) = 0$ gives

$$0 = c_1 + c_2\mu \tag{1}$$

Second B.C. $y(L) = 0$ gives

$$0 = c_1 \cosh(\mu L) + c_2 \sinh(\mu L)$$

From (1) $c_1 = -c_2\mu$ and the above now becomes

$$\begin{aligned} 0 &= -c_2\mu \cosh(\mu L) + c_2 \sinh(\mu L) \\ &= c_2(\sinh(\mu L) - \mu \cosh(\mu L)) \end{aligned}$$

For non-trivial solution, we want $\sinh(\mu L) - \mu \cosh(\mu L) = 0$. This means $\tanh(\mu L) = \mu$. Therefore $\lambda < 0$ is an eigenvalue and these are given by $\lambda_n = -\mu_n^2$, where μ_n is the solution to

$$\tanh(\mu L) = \mu$$

Or equivalently, the roots of

$$\tanh\left(\sqrt{-\lambda}L\right) = \sqrt{-\lambda}$$

There is only one negative root when solving the above numerically, which is $\lambda_{-1} = 0.992$.The corresponding eigenfunction is

$$\Phi_{-1} = c_{-1}\left(\sinh\left(\sqrt{-\lambda_{-1}}x\right) - \sqrt{-\lambda_{-1}}\cosh\left(\sqrt{-\lambda_{-1}}x\right)\right)$$

<u>Let $\lambda = 0$</u>, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y(0) + y'(0) = 0$ gives

$$0 = c_1 + c_2$$

The solution becomes

$$y(x) = c_1(1 - x)$$

Second B.C. $y(L)$ gives

$$0 = c_1(1 - L)$$

This gives trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

<u>Let $\lambda > 0$</u>, The solution is

$$\begin{aligned} y(x) &= c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right) \\ y'(x) &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) \end{aligned}$$

First B.C. $y(0) + y'(0) = 0$ gives

$$0 = c_1 + c_2\sqrt{\lambda}$$

The solution now becomes

$$\begin{aligned}y(x) &= -c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)+c_2\sin\left(\sqrt{\lambda}x\right)\\ &= c_2\left(\sin\left(\sqrt{\lambda}x\right)-\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)\right)\end{aligned}$$

Second B.C. $y(L)=0$ the above becomes

$$0 = c_2\left(\sin\left(\sqrt{\lambda}L\right)-\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)\right)$$

For non-trivial solution, we want $\sin\left(\sqrt{\lambda}L\right)-\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)=0$ or $\tan\left(\sqrt{\lambda}L\right)-\sqrt{\lambda}=0$ or

$$\sqrt{\lambda}=\tan\left(\sqrt{\lambda}L\right)$$

Therefore the eigenvalues are the solution to the above (must be done numerically) And the corresponding eigenfunctions are

$$\Phi_n(x) = c_n\left(\sin\left(\sqrt{\lambda_n}x\right)-\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right)\right)$$

for each root λ_n.

2.8 case 8: boundary conditions $y(0)+y'(0)=0, y'(L)=0$

Let the solution be $y=Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned}r^2+\lambda &= 0\\ r &= \pm\sqrt{-\lambda}\end{aligned}$$

<u>Let $\lambda<0$</u>

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda}=\mu$ where $\mu>0$. Hence the roots are $\pm\mu$. This gives the solution

$$\begin{aligned}y &= c_1\cosh(\mu x)+c_2\sinh(\mu x)\\ y' &= c_1\mu\sinh(\mu x)+c_2\mu\cosh(\mu x)\end{aligned}$$

First B.C. $y(0)+y'(0)=0$ gives

$$0 = c_1+c_2\mu \tag{1}$$

Second B.C. $y'(L)=0$ gives

$$0 = c_1\mu\sinh(\mu L)+c_2\mu\cosh(\mu L)$$

From (1) $c_1=-c_2\mu$ and the above becomes

$$\begin{aligned}0 &= -c_2\mu^2\sinh(\mu L)+c_2\mu\cosh(\mu L)\\ &= c_2\mu(-\mu\sinh(\mu L)+\cosh(\mu L))\end{aligned}$$

For non-trivial solution, we want $-\mu \sinh(\mu L) + \cosh(\mu L) = 0$. This means $-\mu \tanh(\mu L) + 1 = 0$. Or $\tanh(\mu L) = \frac{1}{\mu}$, therefore $\lambda < 0$ is eigenvalues and these are given by $\lambda_n = -\mu_n^2$, where μ_n is the solution to

$$\begin{aligned} \tanh(\mu L) &= \frac{1}{\mu} \\ \tanh\left(\sqrt{-\lambda} L\right) &= \frac{1}{\sqrt{-\lambda}} \end{aligned}$$

This has one root, found numerically which is $\lambda_{-1} = -1$. Hence $\sqrt{-\lambda} = 1$. The corresponding eigenfunction is

$$\begin{aligned} \Phi_{-1}(x) &= c_{-1}(-\mu \cosh(\mu x) + \sinh(\mu x)) \\ &= c_{-1}(-\cosh(x) + \sinh(x)) \end{aligned}$$

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y(0) + y'(0) = 0$ gives

$$0 = c_1 + c_2$$

The solution becomes

$$\begin{aligned} y(x) &= c_1(1 - x) \\ y' &= -c_1 \end{aligned}$$

Second B.C. $y'(L)$ gives

$$0 = -c_1$$

This gives trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Let $\lambda > 0$, The solution is

$$\begin{aligned} y(x) &= c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \\ y'(x) &= -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right) \end{aligned}$$

First B.C. $y(0) + y'(0) = 0$ gives

$$0 = c_1 + c_2 \sqrt{\lambda}$$

The solution becomes

$$\begin{aligned} y(x) &= -c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \\ &= c_2\left(\sin\left(\sqrt{\lambda} x\right) - \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right)\right) \end{aligned}$$

Second B.C. $y'(L) = 0$ gives

$$0 = c_2\left(\sqrt{\lambda} \cos\left(\sqrt{\lambda} L\right) + \lambda \sin\left(\sqrt{\lambda} L\right)\right)$$

For non-trivial solution, we want $\lambda \sin\left(\sqrt{\lambda}L\right)+\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)=0$ or $\lambda \tan\left(\sqrt{\lambda}L\right)=-\sqrt{\lambda}$
Therefore the eigenvalues are the solution to

$$\tan\left(\sqrt{\lambda}L\right)=\frac{-\sqrt{\lambda}}{\lambda}=\frac{-1}{\sqrt{\lambda}}$$

And the corresponding eigenfunction is

$$\Phi_n(x)=c_n\left(\sin\left(\sqrt{\lambda}x\right)-\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)\right)$$

2.9 case 9: boundary conditions $y(0)+y'(0)=0, y(L)+y'(L)=0$

Let the solution be $y=Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2+\lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Let $\lambda<0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda}=\mu$ where $\mu>0$. Hence the roots are $\pm\mu$. This gives the solution

$$y=c_1\cosh(\mu x)+c_2\sinh(\mu x)$$

Hence

$$y'=\mu c_1\sinh(\mu x)+\mu c_2\cosh(\mu x)$$

Left B.C. gives

$$0=c_1+\mu c_2 \tag{1}$$

Right B.C. gives

$$\begin{aligned} 0 &= c_1\cosh(\mu L)+c_2\sinh(\mu L)+\mu c_1\sinh(\mu L)+\mu c_2\cosh(\mu L) \\ &= \cosh(\mu L)(c_1+\mu c_2)+\sinh(\mu L)(c_2+\mu c_1) \end{aligned}$$

Using (1) in the above, it simplifies to

$$0=\sinh(\mu L)(c_2+\mu c_1)$$

But from (1) again, we see that $c_1=-\mu c_2$ and the above becomes

$$\begin{aligned} 0 &= \sinh(\mu L)(c_2-\mu(\mu c_2)) \\ &= \sinh(\mu L)\left(c_2-\mu^2 c_2\right) \\ &= c_2\sinh(\mu L)\left(1-\mu^2\right) \end{aligned}$$

But $\sinh(\mu^2 L) \neq 0$ since $\mu^2 L \neq 0$ and so either $c_2 = 0$ or $(1-\mu^2) = 0$. $c_2 = 0$ results in trivial solution, therefore $(1-\mu^2) = 0$ or $\mu^2 = 1$ but $\mu^2 = -\lambda$, hence $\lambda = -1$ is the eigenvalue. Corresponding eigenfunction is

$$y = c_1 \cosh(x) + c_2 \sinh(x)$$

Using (1) the above simplifies to

$$\begin{aligned} y &= -\mu c_2 \cosh(x) + c_2 \sinh(x) \\ &= c_2(-\mu \cosh(x) + \sinh(x)) \end{aligned}$$

But $\mu = \sqrt{-\lambda} = 1$, hence the eigenfunction is

$$\boxed{y(x) = c_2(-\cosh(x) + \sinh(x))}$$

Let $\lambda = 0$ Solution now is

$$y = c_1 x + c_2$$

Therefore

$$y' = c_1$$

Left B.C. $0 = y(0) + y'(0)$ gives

$$0 = c_2 + c_1 \tag{2}$$

Right B.C. $0 = y(L) + y'(L)$ gives

$$\begin{aligned} 0 &= (c_1 L + c_2) + c_1 \\ 0 &= c_1(1 + L) + c_2 \end{aligned}$$

But from (2) $c_1 = -c_2$ and the above becomes

$$\begin{aligned} 0 &= -c_2(1 + L) + c_2 \\ 0 &= -c_2 L \end{aligned}$$

Which means $c_2 = 0$ and therefore the trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Assuming $\lambda > 0$ Solution is

$$y = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \tag{A}$$

Hence

$$y' = -\sqrt{\lambda} c_1 \sin\left(\sqrt{\lambda} x\right) + \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} x\right)$$

Left B.C. gives

$$0 = c_1 + \sqrt{\lambda} c_2 \tag{3}$$

Right B.C. gives

$$
\begin{aligned}
0 &= c_1\cos\left(\sqrt{\lambda}L\right) + c_2\sin\left(\sqrt{\lambda}L\right) - \sqrt{\lambda}c_1\sin\left(\sqrt{\lambda}L\right) + \sqrt{\lambda}c_2\cos\left(\sqrt{\lambda}L\right) \\
&= \cos\left(\sqrt{\lambda}L\right)\left(c_1 + \sqrt{\lambda}c_2\right) + \sin\left(\sqrt{\lambda}L\right)\left(c_2 - \sqrt{\lambda}c_1\right)
\end{aligned}
$$

Using (3) in the above, it simplifies to

$$
0 = \sin\left(\sqrt{\lambda}L\right)\left(c_2 - \sqrt{\lambda}c_1\right)
$$

But from (3), we see that $c_1 = -\sqrt{\lambda}c_2$. Therefore the above becomes

$$
\begin{aligned}
0 &= \sin\left(\sqrt{\lambda}L\right)\left(c_2 - \sqrt{\lambda}\left(-\sqrt{\lambda}c_2\right)\right) \\
&= \sin\left(\sqrt{\lambda}L\right)\left(c_2 + \lambda c_2\right) \\
&= c_2\sin\left(\sqrt{\lambda}L\right)(1+\lambda)
\end{aligned}
$$

Only choice for non trivial solution is either $(1+\lambda) = 0$ or $\sin\left(\sqrt{\lambda}L\right) = 0$. But $(1+\lambda) = 0$ implies $\lambda = -1$ but we said that $\lambda > 0$. Hence other choice is

$$
\begin{aligned}
\sin\left(\sqrt{\lambda}L\right) &= 0 \\
\sqrt{\lambda}L &= n\pi \qquad n = 1,2,3,\cdots \\
\lambda_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots
\end{aligned}
$$

The above are the eigenvalues. The corresponding eigenfunction is from (A)

$$
\Phi_n(x) = c_{1_n}\cos\left(\sqrt{\lambda_n}x\right) + c_{2_n}\sin\left(\sqrt{\lambda_n}x\right)
$$

But $c_{1_n} = -\sqrt{\lambda_n}c_{2_n}$ and the above becomes

$$
\begin{aligned}
\Phi_n(x) &= -\sqrt{\lambda_n}c_{2_n}\cos\left(\sqrt{\lambda_n}x\right) + c_2\sin\left(\sqrt{\lambda_n}x\right) \\
&= C_n\left(-\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right) + \sin\left(\sqrt{\lambda_n}x\right)\right)
\end{aligned}
$$