

# Analytical solution to specific Stokes first problem PDE

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Solve

$$\begin{aligned}\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} \\ 0 < x < L \\ t > 0\end{aligned}\tag{1}$$

Initial conditions

$$u(0, x) = 0$$

Boundary conditions

$$\begin{aligned}u(0, t) &= \sin(t) \\ u(t, L) &= 0\end{aligned}$$

Let

$$u = v + u_E\tag{2}$$

where  $u_E(x, t)$  is steady state solution that only needs to satisfy boundary conditions and  $v(x, t)$  satisfies the PDE itself but with homogenous B.C. At steady state, the PDE becomes

$$\begin{aligned}0 &= k \frac{d^2 u_E}{dx^2} \\ u_E(0) &= \sin(t) \\ u_E(L) &= 0\end{aligned}$$

The solution is  $u_E(x) = \left(\frac{L-x}{L}\right) \sin(t)$ . Hence (2) becomes

$$u(x, t) = v(x, t) + \left(\frac{L-x}{L}\right) \sin(t)$$

Substituting the above in (1) gives

$$\begin{aligned}\frac{\partial v}{\partial t} + \left(\frac{L-x}{L}\right) \cos(t) &= k \frac{\partial^2 v}{\partial x^2} \\ \frac{\partial v}{\partial t} &= k \frac{\partial^2 v}{\partial x^2} + \left(\frac{x-L}{L}\right) \cos(t) \\ \frac{\partial v}{\partial t} &= k \frac{\partial^2 v}{\partial x^2} + Q(x, t)\end{aligned}\tag{3}$$

With boundary conditions  $u_0(0, t) = 0, u(L, t) = 0$ . This is now in standard form and separation of variables can be used to solve it.

$$Q(x, t) = \left( \frac{x-L}{L} \right) \cos(t)$$

Now acts as a source term. The eigenfunctions are known to be  $\Phi_n(x) = \sin(\sqrt{\lambda_n}x)$  where  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ . Hence by eigenfunction expansion, the solution to (3) is

$$v(x, t) = \sum_{n=1}^{\infty} B_n(t) \Phi_n(x) \quad (3A)$$

Substituting this into (3) gives

$$\sum_{n=1}^{\infty} \frac{dB_n(t)}{dt} \Phi_n(x) = k \sum_{n=1}^{\infty} B_n(t) \Phi_n''(x) + Q(x, t) \quad (4)$$

Expanding  $Q(x, t)$  using same basis (eigenfunctions) gives

$$Q(x, t) = \sum_{n=1}^{\infty} q_n(t) \Phi_n(x)$$

Applying orthogonality

$$\begin{aligned} \int_0^L Q(x, t) \Phi_m(x) dx &= \int_0^L \sum_{n=1}^{\infty} q_n(t) \Phi_n(x) \Phi_m(x) dx \\ &= \sum_{n=1}^{\infty} q_n(t) \int_0^L \Phi_n(x) \Phi_m(x) dx \end{aligned}$$

But  $\sum_{n=1}^{\infty} \int_0^L \Phi_n(x) \Phi_m(x) dx = \int_0^L \Phi_m^2(x) dx = \frac{L}{2}$  since  $\Phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$  and the above simplifies to

$$\begin{aligned} \int_0^L Q(x, t) \Phi_n(x) dx &= \frac{L}{2} q_n(t) \\ q_n(t) &= \frac{2}{L} \int_0^L Q(x, t) \sin\left(\frac{n\pi}{L}x\right) dx \end{aligned}$$

But  $Q(x, t) = \left(\frac{x-L}{L}\right) \cos(t)$ , hence

$$\begin{aligned} q_n(t) &= \frac{2}{L} \int_0^L \left(\frac{x-L}{L}\right) \cos(t) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{-2}{n\pi} \cos(t) \end{aligned}$$

Therefore  $Q(x, t) = \sum_{n=1}^{\infty} q_n(t) \Phi_n(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi} \cos(t) \sin\left(\frac{n\pi}{L}x\right)$  and (4) becomes

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{dB_n(t)}{dt} \Phi_n(x) &= k \sum_{n=1}^{\infty} B_n(t) \Phi_n''(x) - \sum_{n=1}^{\infty} \frac{2}{n\pi} \cos(t) \sin\left(\frac{n\pi}{L}x\right) \\ \frac{dB_n(t)}{dt} \sin\left(\frac{n\pi}{L}x\right) &= kB_n(t) \left(-\frac{n^2\pi^2}{L^2} \sin\left(\frac{n\pi}{L}x\right)\right) - \frac{2}{n\pi} \cos(t) \sin\left(\frac{n\pi}{L}x\right) \\ \frac{dB_n(t)}{dt} + B_n(t) k \frac{n^2\pi^2}{L^2} &= -\frac{2}{n\pi} \cos(t) \end{aligned}$$

This is an ODE in  $B_n(t)$  whose solution is

$$B_n(t) = C_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} - \frac{2L^2 (kn^2\pi^2 \cos t + L^2 \sin t)}{n\pi (L^4 + k^2n^4\pi^4)}$$

From (3A)  $v(x, t)$  now becomes

$$v(x, t) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} \sin\left(\frac{n\pi}{L}x\right) - \frac{2L^2 (kn^2\pi^2 \cos t + L^2 \sin t)}{n\pi (L^4 + k^2n^4\pi^4)} \sin\left(\frac{n\pi}{L}x\right) \quad (5)$$

To find  $C_n$ , from initial conditions, at  $t = 0$  the above becomes

$$0 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) - \frac{2L^2 (kn^2\pi^2)}{n\pi (L^4 + k^2n^4\pi^4)} \sin\left(\frac{n\pi}{L}x\right)$$

Hence

$$C_n = \frac{2L^2 (kn^2\pi^2)}{n\pi (L^4 + k^2n^4\pi^4)}$$

Therefore (5) becomes

$$v(x, t) = \sum_{n=1}^{\infty} \left( \frac{2L^2 (kn^2\pi^2)}{n\pi (L^4 + k^2n^4\pi^4)} e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} - \frac{2L^2 (kn^2\pi^2 \cos t + L^2 \sin t)}{n\pi (L^4 + k^2n^4\pi^4)} \right) \sin\left(\frac{n\pi}{L}x\right)$$

And since  $u = v + u_E$  then the solution is

$$u(x, t) = \left( \sum_{n=1}^{\infty} \left( \frac{2L^2 (kn^2\pi^2)}{n\pi (L^4 + k^2n^4\pi^4)} e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} - \frac{2L^2 (kn^2\pi^2 \cos t + L^2 \sin t)}{n\pi (L^4 + k^2n^4\pi^4)} \right) \sin\left(\frac{n\pi}{L}x\right) \right) + \left( \frac{L-x}{L} \right) \sin(t)$$

To simulate

```
ClearAll[t, x, n]
k = 1; L0 = 5; max = 400;
u[x_, t_] =
Sum[(((2*L0^2*(k*n^2*Pi^2))/(n*Pi*(L0^4 + k^2*n^4*Pi^4)))*
Exp[(-k)*((n^2*Pi^2)/L0^2)*t] -
(2*L0^2*(k*n^2*Pi^2*Cos[t] + L0^2*Sin[t]))/(n*
Pi*(L0^4 + k^2*Pi^4*n^4))*Sin[((n*Pi)/L0)*x],
{n, 1, max}] + ((L0 - x)/L0)*Sin[t];
Manipulate[Grid[{{"Analytical solution"},
{Plot[Evaluate[u[x,t]], {x,0,5}, PlotRange->{{0,5},{-1.1,1.1}},
ImageSize->400]}]},
{{t,0,"t"},0,100,.01}
]
```

Here is the animation from the above

Here is the numerical solution to compare with

```

ClearAll["Global`*"];
pdeset = {Derivative[1, 0][U][t, x] == Derivative[0, 2][U][t, x]}
ics = {U[0, x] == 0};
bcs = {U[t, 0] == Sin[t], U[t, 5] == 0};
ibcAll = {ics, bcs};
numericalSol = NDSolve[{pdeset, ibcAll}, U, {t, 0, 100}, {x, 0, 5}];
Manipulate[Grid[{"Numerical solution"},
{Plot[Evaluate[U[t, x] /. numericalSol], {x, 0, 5},
PlotRange -> {{0, 5}, {-1, 1}}, ImageSize -> 400]}],
{{t, 0, "t"}, 0, 100, .01}
]

```

Here is the animation from the above

Reference: stokes second problem question and answer