my Partial differential equations cheat sheet

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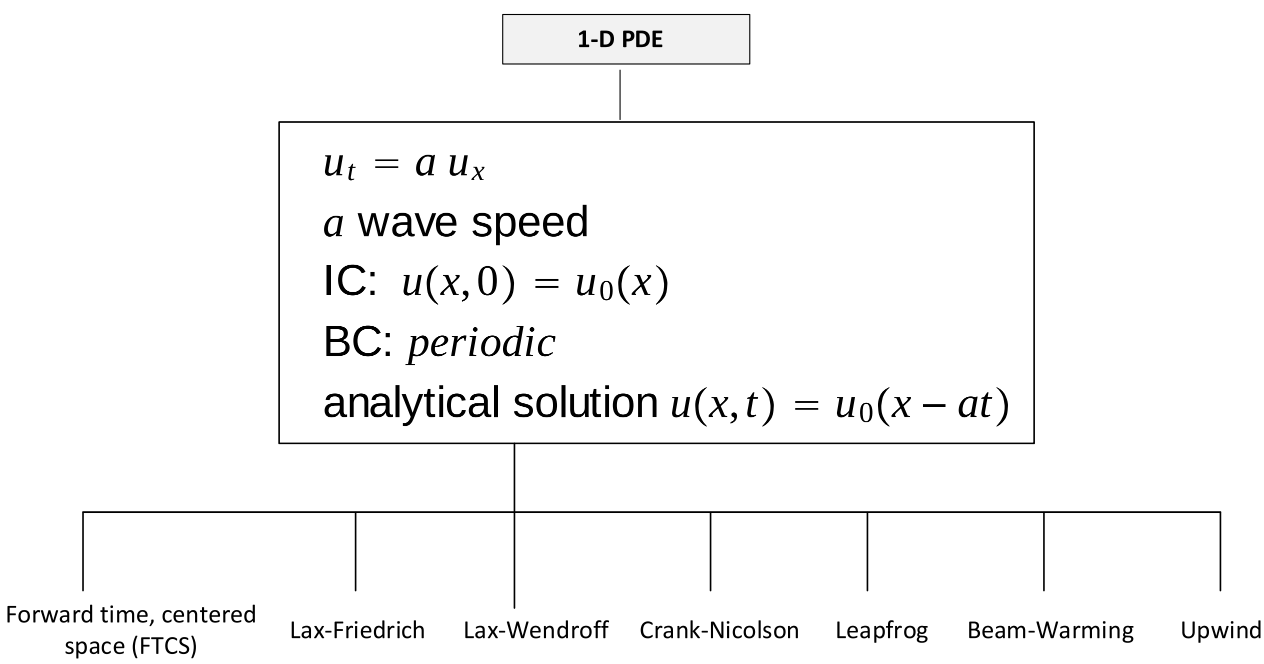
1 introduction

These are part of my study notes on PDE's.

Trying to classify PDE's, here is current diagram. It is very large, but it is meant to include a summary of many methods in one place. Easier to view in a browser than in the pdf.

Some diagrams I made



1-D PDE

1-D Parabolic (heat equation or diffusion)

Stiff, Implicit, small time step

$u_t = Du_{xx}$

D diffusion constant > 0

IC: $u(x,0) = u_0(x)$

BC: $u(0,t) = g(t), u(1,t) = f(t)$

FTCS (Forward-time, centered space)
(second order in space, first order in time)

$$\frac{u_i^{n+1}-u_i^n}{k} = D\frac{(u_{i-1}^n-2u_i^n+u_{i+1}^n)}{h^2}$$

$$u_i^{n+1} = u_i^n + \frac{Dk}{h^2}(u_{i-1}^n - 2u_i^n + u_{i+1}^n)$$

$$u_i^{n+1} = \frac{Dk}{h^2}u_{i-1}^n + \left(1-2\frac{Dk}{h^2}\right)u_i^n + \frac{Dk}{h^2}u_{i+1}^n$$

Let $r = \frac{Dk}{h^2}$

$$u_i^{n+1} = ru_{i-1}^n + (1-2r)u_i^n + ru_{i+1}^n$$

Crank Nicholson
(second order in space, second order in time)

$$\frac{u_i^{n+1}-u_i^n}{k} = \frac{D}{2}\left(\frac{(u_{i-1}^n-2u_i^n+u_{i+1}^n)}{h^2} + \frac{\left(u_{i-1}^{n+1}-2u_i^{n+1}+u_{i+1}^{n+1}\right)}{h^2}\right)$$

$$u_i^{n+1} - \frac{Dk}{2h^2}(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}) = u_i^n + \frac{Dk}{2h^2}(u_{i-1}^n - 2u_i^n + u_{i+1}^n)$$

$$-\frac{Dk}{2h^2}u_{i-1}^{n+1} + \left(1-\frac{Dk}{h^2}\right)u_i^{n+1} - \frac{Dk}{2h^2}u_{i+1}^{n+1} = \frac{Dk}{2h^2}u_{i-1}^n + \left(1-\frac{Dk}{h^2}\right)u_i^n + \frac{Dk}{2h^2}u_{i+1}^n$$

Let $r = \frac{Dk}{2h^2}$

$$-ru_{i-1}^{n+1} + (1-2r)u_i^{n+1} - ru_{i+1}^{n+1} = ru_{i-1}^n + (1-2r)u_i^n + ru_{i+1}^n$$

$$A\mathbf{u}^{n+1} = B\mathbf{u}^n$$

1D PDE, parabolic, heat/ diffusion, C-N

$$\begin{bmatrix} (1-2r) & -r & & \\ -r & (1-2r) & -r & \\ & -r & \ddots & -r \\ & & -r & (1-2r) \end{bmatrix} \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} (1-2r) & r & & \\ r & (1-2r) & r & \\ & r & \ddots & r \\ & & r & (1-2r) \end{bmatrix} \begin{bmatrix} u_2^n \\ u_3^n \\ \vdots \\ u_{N-1}^n \end{bmatrix}$$

second order linear partial differential equation

elliptic
No characteristic curves. diffusion process reached equilibrium, steady state temperature distribution. Numerically solved by relaxation methods.
$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$
$B^2 - 4AC < 0$
$A = 1, C = 1, B = 0$

parabolic

hyperbolic

$F = 0$
a function that satisfies Laplace is called harmonic

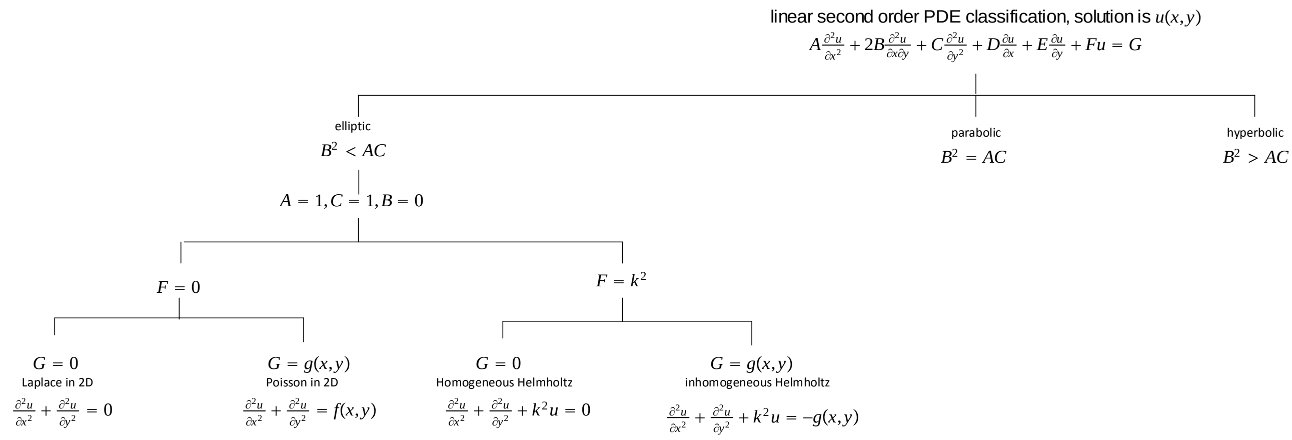
$F = k^2$

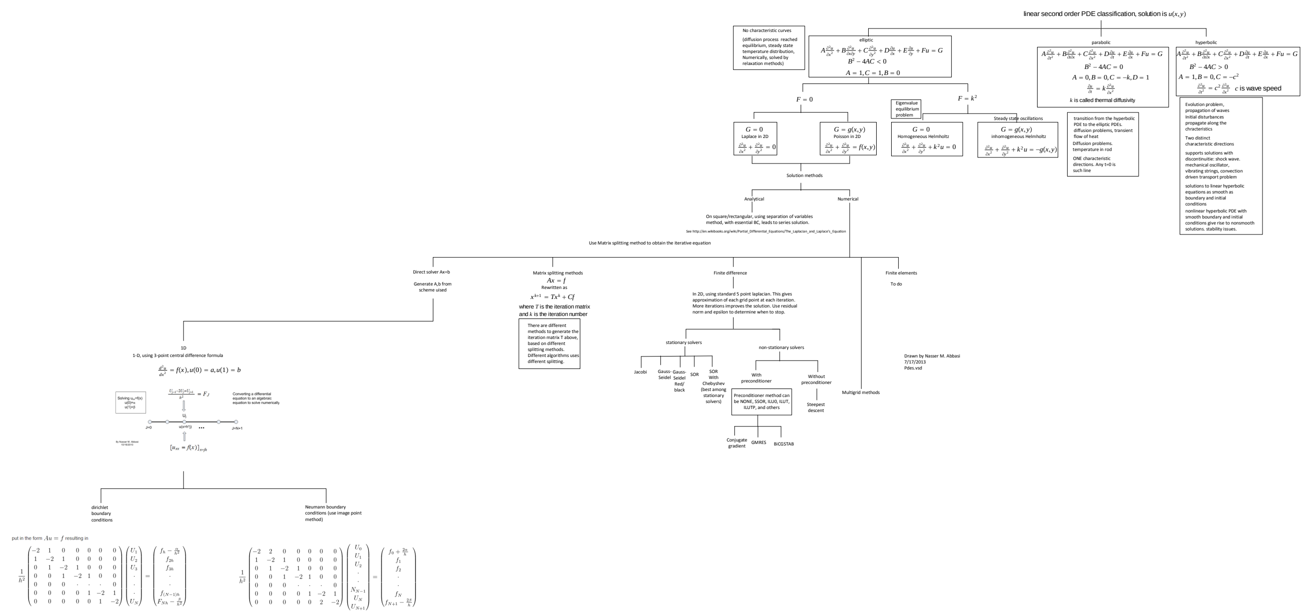
$G = 0$
Laplace in 2D
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$G = g(x,y)$
Poisson in 2D
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y)$

$G = 0$
homogeneous Helmholtz
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$
eigenvalue equilibrium

$G = g(x,y)$
inhomogeneous Helmholtz
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = g(x,y)$
steady state oscillation





To derive the PDE, we start by setting up the state quantities and the flow quantities, and relate these to each others by the use of the constitutive law. Then substiting this into the local conservation law, lead to the PDE.

state quantity	constitutive law	flow quantity
density tempreture pressure specific internal energy entropy	$\Longleftrightarrow$	tension velocity momentum heat flux

2 linear PDE's

2.1 Elliptic

Some properties
1. Solution to the PDE represents steady state of u.
2. Only boundary conditions are used to solve. No initial conditions.
3. Relation to complex analytic functions: If $f(z)=\phi(x,y)+i\psi(x,y)$ is analytic, then $\phi(x,y)$ and $\psi(x,y)$ are solutions to Laplace pde's
4. Solutions to Laplace PDE are called harmonic functions.
5. constitutive law: Either consider them as stationary process, or take the time dependent pde, and set those terms in that which depend on time to zero.

Examples of elliptic PDE's
1. Laplace $u_{xx}=0$ or in general $\nabla^2 u=0$
2. Poisson $u_{xx}=-f(x)$
3. Helmholtz in 1D $u_{xx}+\lambda u(x)=-f(x)$
4. Helmholtz in 2D $u_{xx}+u_{yy}+\lambda u(x,y)=-f(x,y)$

2.2 Parabolic

Some properties
1. Diffusion. Material spread is one specific example of diffusion. Here the state variable is the concentration of the diffusing matrial. The flow quantity is its flux. The constitutive law is Fickś law.
2. Heat spread. Here the state variable is the temprature, and the flow quantity is the heat flux. The constitutive law is Fourierś law.
3. Stiff PDE, hence requires small time step, solved using implicit methods, not explicit for stability.

4. Numerically, use Crank-Nicleson, in 2D, can use ADI.
5. Requires initial and boundary conditions to solve.

Examples of parabolic PDE's

1. Diffusion. $u_t - Du_{xx} = 0$ where D is the diffusion constant, must be positive quantity. For heat PDE, D is the thermal diffusivity $D = \kappa/c_p\rho$ where κ is thermal conductivity, c_p is specific heat capacity, ρ is density of medium.
2. In higher spatial dimension $u_t - D\nabla^2 u = 0$
3. Foller-Plank, Black-Sholes PDEs
4. Diffusion-Reaction $u_t - Du_{xx} = F(u(x,t))$ where $F(u(x,t))$ is the reaction term, which can be stiff or not. Examples
   a) Fischer equation, nonlinear PDE for modeling population growth. $u_t - Du_{xx} = ru(x,t)(1 - \frac{u(x,t)}{K})$ where K is carrying capacity, and r is growth rate.

2.3 Hyperbolic

Some properties

1. Advection PDE (or Transport or convection?). $u_t + au_x = 0$, Transport or drift of conserved substance (pollutant) in Fluid or Gas where a is speed of fluid. Analytic solution is $u(x,t) = f(x - at)$ where $f(x) = u(x,0)$ is the initial conditions.
2. The state variable is the concentration u of the contaminant, and the flow quantity is its flux ϕ. The constitutive law is $\phi = cu$.
3. Wave equation $u_{tt} = c^2 u_{xx}$. Analytic solution is $u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\mathrm{d}y$ where $f(x) = u(x,0)$ and $g(x) = u_t(x,0)$.

Examples

1. Advection, Wave (See above)
2. non-homogenouse advection and wave: $u_t + au_x = f(x,t)$ and $u_{tt} = c^2 u_{xx} + f(x,t)$.
3. Klein-Gordon $u_{tt} = c^2 u_{xx} - bu$
4. Telegraphy $u_{tt} + ku_t = c^2 u_{xx} + bu$

3 hints

reference

> Characteristics are curves in the space of the independent variables along which the governing PDE has only total differentials

4 references

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8. `http://gwu.geverstine.com/pde.pdf` table on classification, diagram for discriminant sign