

Note on finding the Laplacian in Polar, Cylindrical and Spherical coordinates using Tensor calculus

Nasser M. Abbasi

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1 Introduction

I wrote this note to help me learn tensors. The goal is to derive the Laplacian ∇^2 using tensor calculus for 2D Polar, 3D Cylindrical and in 3D Spherical coordinates.

The Laplacian in Cartesian coordinates is given by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The following diagram shows $\nabla^2 u$ in Polar, Cylindrical and Spherical coordinates

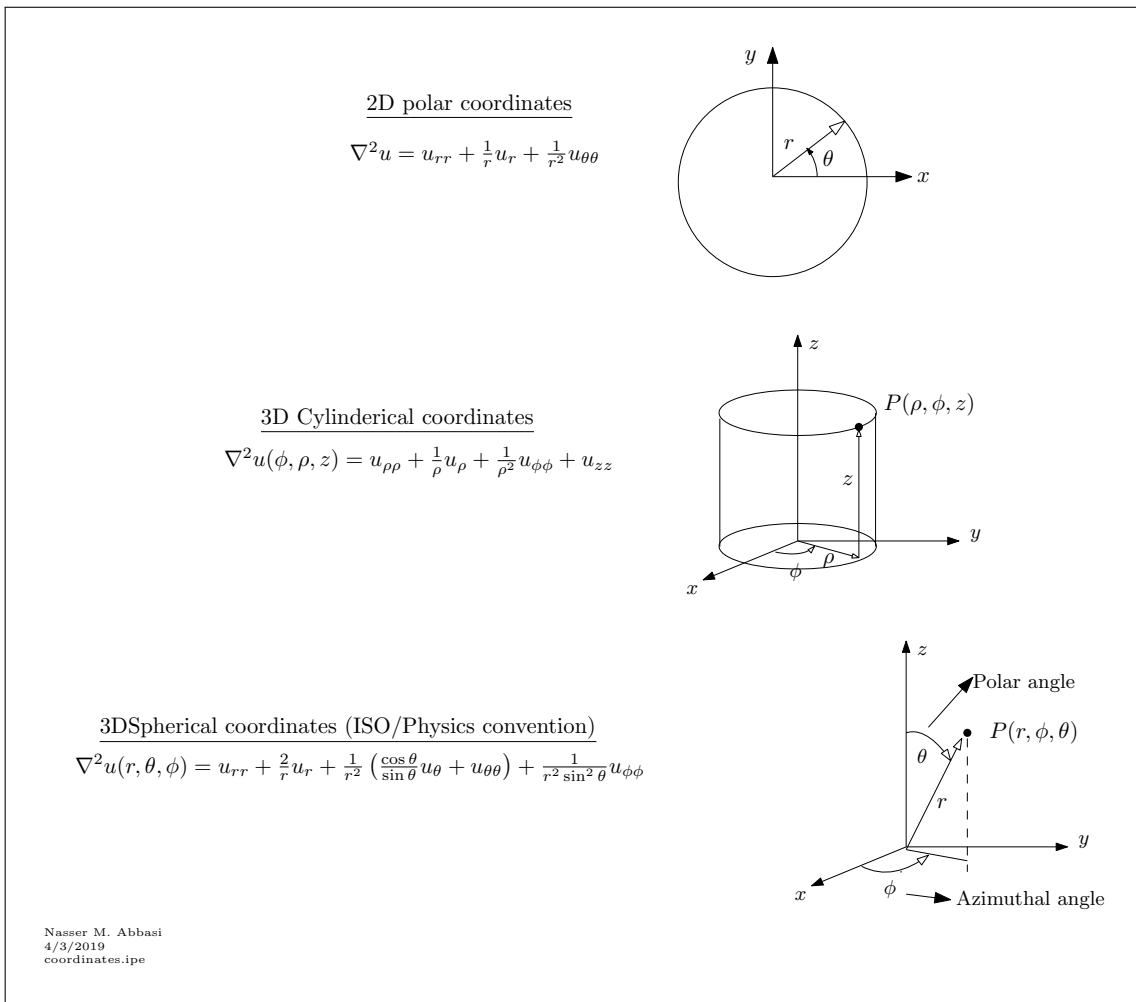


Figure 1: Laplacian in different coordinate systems

The Laplacian operator in any orthogonal coordinate system is given by

$$\nabla^2 = \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right) \quad (1)$$

Where g is the metric tensor and $|g|$ is the determinant of g . The derivation of the above is not shown here. References contains the derivation of the above. Only the use of (1) is shown here.

2 2D Polar

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y$ and the coordinates in the other system (Polar) are $x^1 = r, x^2 = \theta$. The relation between these must be known and invertible also, meaning $\zeta \equiv \zeta(x)$ and $x \equiv x(\zeta)$. This relation can be found from geometry as

$$\begin{aligned}\zeta^1 &= r \cos \theta \\ \zeta^2 &= r \sin \theta\end{aligned}$$

The first step is to determine the metric tensor g for the Polar coordinates. This is given by

$$g_{kl} = \delta_{ij} \frac{\partial \zeta^i}{\partial x^k} \frac{\partial \zeta^j}{\partial x^l}$$

The above using Einstein summation notation. Since the coordinate system is orthogonal, g_{kl} will be diagonal, hence only g_{11}, g_{22} are non zero. This is not the case for all coordinates systems. For general curvilinear coordinates system, g can contain all components. But for the coordinates systems used here, g will always be diagonal.

$$\begin{aligned}g_{11} &= \frac{\partial \zeta^1}{\partial x^1} \frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1} \frac{\partial \zeta^2}{\partial x^1} \\ &= \frac{\partial \zeta^1}{\partial r} \frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r} \frac{\partial \zeta^2}{\partial r} \\ &= \left(\frac{\partial \zeta^1}{\partial r} \right)^2 + \left(\frac{\partial \zeta^2}{\partial r} \right)^2 \\ &= \cos^2 \theta + \sin^2 \theta \\ &= 1\end{aligned}$$

And

$$\begin{aligned}g_{22} &= \frac{\partial \zeta^1}{\partial x^2} \frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2} \frac{\partial \zeta^2}{\partial x^2} \\ &= \frac{\partial \zeta^1}{\partial \theta} \frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta} \frac{\partial \zeta^2}{\partial \theta} \\ &= \left(\frac{\partial \zeta^1}{\partial \theta} \right)^2 + \left(\frac{\partial \zeta^2}{\partial \theta} \right)^2 \\ &= (-r \sin \theta)^2 + (r \cos \theta)^2 \\ &= r^2\end{aligned}$$

Hence ds^2 in polar coordinates is

$$\begin{aligned}ds^2 &= g_{kl} dx^k dx^l \\ &= g_{11} (dx^1)^2 + g_{22} (dx^2)^2 \\ &= g_{11} (dr)^2 + g_{22} (d\theta)^2 \\ &= (dr)^2 + r^2 (d\theta)^2\end{aligned}$$

From the above we see that

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Hence $\det(g) = r^2$. We are now ready to apply (1)

$$\begin{aligned} \nabla^2 &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right) \\ &= \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_1} \left(\frac{\sqrt{r^2}}{g_{11}} \frac{\partial}{\partial x^1} \right) + \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_2} \left(\frac{\sqrt{r^2}}{g_{22}} \frac{\partial}{\partial x^2} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{r}{r^2} \frac{\partial}{\partial \theta} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \\ &= \frac{1}{r} \left(\frac{\partial}{\partial r} + r \frac{\partial^2}{\partial r^2} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \end{aligned}$$

Therefore

$$\begin{aligned} \nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\ &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \end{aligned}$$

3 3D Spherical

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$. And the coordinates in the Spherical system are $x^1 = \phi, x^2 = r, x^3 = \theta$. The relation between these is known as (Note that the following depends on convention used for which is θ and which is ϕ . Physics convention as shown in the diagram above is used here).

$$\begin{aligned} \zeta^1 &= r \sin \theta \cos \phi \\ \zeta^2 &= r \sin \theta \sin \phi \\ \zeta^3 &= r \cos \theta \end{aligned}$$

The first step is to determine the metric tensor g for the Spherical coordinates. This is given by

$$g_{kl} = \delta_{ij} \frac{\partial \zeta^i}{\partial x^k} \frac{\partial \zeta^j}{\partial x^l}$$

Since the coordinate system are orthogonal, g_{kl} will be diagonal. Hence only g_{11}, g_{22}, g_{33} are non zero.

$$\begin{aligned}
g_{11} &= \frac{\partial \zeta^1}{\partial x^1} \frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1} \frac{\partial \zeta^2}{\partial x^1} + \frac{\partial \zeta^3}{\partial x^1} \frac{\partial \zeta^3}{\partial x^1} \\
&= \frac{\partial \zeta^1}{\partial \phi} \frac{\partial \zeta^1}{\partial \phi} + \frac{\partial \zeta^2}{\partial \phi} \frac{\partial \zeta^2}{\partial \phi} + \frac{\partial \zeta^3}{\partial \phi} \frac{\partial \zeta^3}{\partial \phi} \\
&= \left(\frac{\partial \zeta^1}{\partial \phi} \right)^2 + \left(\frac{\partial \zeta^2}{\partial \phi} \right)^2 + \left(\frac{\partial \zeta^3}{\partial \phi} \right)^2 \\
&= (-r \sin \theta \sin \phi)^2 + (r \sin \theta \cos \phi)^2 + (0)^2 \\
&= r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi \\
&= r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi) \\
&= r^2 \sin^2 \theta
\end{aligned}$$

And

$$\begin{aligned}
g_{22} &= \frac{\partial \zeta^1}{\partial x^2} \frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2} \frac{\partial \zeta^2}{\partial x^2} + \frac{\partial \zeta^3}{\partial x^2} \frac{\partial \zeta^3}{\partial x^2} \\
&= \frac{\partial \zeta^1}{\partial r} \frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r} \frac{\partial \zeta^2}{\partial r} + \frac{\partial \zeta^3}{\partial r} \frac{\partial \zeta^3}{\partial r} \\
&= \left(\frac{\partial \zeta^1}{\partial r} \right)^2 + \left(\frac{\partial \zeta^2}{\partial r} \right)^2 + \left(\frac{\partial \zeta^3}{\partial r} \right)^2 \\
&= (\sin \theta \cos \phi)^2 + (\sin \theta \sin \phi)^2 + (\cos \theta)^2 \\
&= \sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta \\
&= \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta \\
&= \sin^2 \theta + \cos^2 \theta \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
g_{33} &= \frac{\partial \zeta^1}{\partial x^3} \frac{\partial \zeta^1}{\partial x^3} + \frac{\partial \zeta^2}{\partial x^3} \frac{\partial \zeta^2}{\partial x^3} + \frac{\partial \zeta^3}{\partial x^3} \frac{\partial \zeta^3}{\partial x^3} \\
&= \frac{\partial \zeta^1}{\partial \theta} \frac{\partial \zeta^1}{\partial \theta} + \frac{\partial \zeta^2}{\partial \theta} \frac{\partial \zeta^2}{\partial \theta} + \frac{\partial \zeta^3}{\partial \theta} \frac{\partial \zeta^3}{\partial \theta} \\
&= \left(\frac{\partial \zeta^1}{\partial \theta} \right)^2 + \left(\frac{\partial \zeta^2}{\partial \theta} \right)^2 + \left(\frac{\partial \zeta^3}{\partial \theta} \right)^2 \\
&= (r \cos \theta \cos \phi)^2 + (r \cos \theta \sin \phi)^2 + (-r \sin \theta)^2 \\
&= r^2 \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^2 \theta \\
&= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\
&= r^2
\end{aligned}$$

Hence ds^2 in Spherical coordinates is

$$\begin{aligned}
ds^2 &= g_{kl} dx^k dx^l \\
&= g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 \\
&= g_{11}(d\phi)^2 + g_{22}(dr)^2 + g_{33}(d\theta)^2 \\
&= r^2 \sin^2 \theta (d\phi)^2 + (dr)^2 + r^2 (d\theta)^2
\end{aligned}$$

From the above we see that

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} r^2 \sin^2 \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

Hence $\det(g) = r^4 \sin^2 \theta$. We are now ready to apply (1)

$$\begin{aligned}
\nabla^2 &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right) \\
&= \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_1} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{g_{11}} \frac{\partial}{\partial x^1} \right) + \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_2} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{g_{22}} \frac{\partial}{\partial x^2} \right) + \frac{1}{\sqrt{r^4 \sin^2 \theta}} \frac{\partial}{\partial x_3} \left(\frac{\sqrt{r^4 \sin^2 \theta}}{g_{33}} \frac{\partial}{\partial x^3} \right) \\
&= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{r^2 \sin \theta}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(\frac{r^2 \sin \theta}{1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{r^2 \sin \theta}{r^2} \frac{\partial}{\partial \theta} \right) \\
&= \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \\
&= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2} \left(2r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right) + \frac{1}{r^2 \sin \theta} \left(\cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial^2}{\partial \theta^2} \right) \\
&= \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \left(\frac{\cos \theta}{\sin \theta} \frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \\
&= u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left(\frac{\cos \theta}{\sin \theta} u_\theta + u_{\theta\theta} \right) + \frac{1}{r^2 \sin^2 \theta} u_{\phi\phi}
\end{aligned}$$

4 3D Cylindrical

The coordinates in the Cartesian system are $\zeta^1 = x, \zeta^2 = y, \zeta^3 = z$. And the coordinates in the Cylindrical system are $x^1 = \phi, x^2 = r, x^3 = z$. The relation between these is known as

$$\begin{aligned}\zeta^1 &= r \cos \phi \\ \zeta^2 &= r \sin \phi \\ \zeta^3 &= z\end{aligned}$$

The first step is to determine the metric tensor g for the Spherical coordinates. This is given by

$$g_{kl} = \delta_{ij} \frac{\partial \zeta^i}{\partial x^k} \frac{\partial \zeta^j}{\partial x^l}$$

Since the coordinate system are orthogonal, g_{kl} will be diagonal. Hence only g_{11}, g_{22}, g_{33} are non zero.

$$\begin{aligned}g_{11} &= \frac{\partial \zeta^1}{\partial x^1} \frac{\partial \zeta^1}{\partial x^1} + \frac{\partial \zeta^2}{\partial x^1} \frac{\partial \zeta^2}{\partial x^1} + \frac{\partial \zeta^3}{\partial x^1} \frac{\partial \zeta^3}{\partial x^1} \\ &= \frac{\partial \zeta^1}{\partial \phi} \frac{\partial \zeta^1}{\partial \phi} + \frac{\partial \zeta^2}{\partial \phi} \frac{\partial \zeta^2}{\partial \phi} + \frac{\partial \zeta^3}{\partial \phi} \frac{\partial \zeta^3}{\partial \phi} \\ &= \left(\frac{\partial \zeta^1}{\partial \phi} \right)^2 + \left(\frac{\partial \zeta^2}{\partial \phi} \right)^2 + \left(\frac{\partial \zeta^3}{\partial \phi} \right)^2 \\ &= (-r \sin \phi)^2 + (r \cos \phi)^2 + (0)^2 \\ &= r^2\end{aligned}$$

And

$$\begin{aligned}g_{22} &= \frac{\partial \zeta^1}{\partial x^2} \frac{\partial \zeta^1}{\partial x^2} + \frac{\partial \zeta^2}{\partial x^2} \frac{\partial \zeta^2}{\partial x^2} + \frac{\partial \zeta^3}{\partial x^2} \frac{\partial \zeta^3}{\partial x^2} \\ &= \frac{\partial \zeta^1}{\partial r} \frac{\partial \zeta^1}{\partial r} + \frac{\partial \zeta^2}{\partial r} \frac{\partial \zeta^2}{\partial r} + \frac{\partial \zeta^3}{\partial r} \frac{\partial \zeta^3}{\partial r} \\ &= \left(\frac{\partial \zeta^1}{\partial r} \right)^2 + \left(\frac{\partial \zeta^2}{\partial r} \right)^2 + \left(\frac{\partial \zeta^3}{\partial r} \right)^2 \\ &= (\cos \phi)^2 + (\sin \phi)^2 + (1)^2 \\ &= 1\end{aligned}$$

And

$$\begin{aligned}g_{33} &= \frac{\partial \zeta^1}{\partial x^3} \frac{\partial \zeta^1}{\partial x^3} + \frac{\partial \zeta^2}{\partial x^3} \frac{\partial \zeta^2}{\partial x^3} + \frac{\partial \zeta^3}{\partial x^3} \frac{\partial \zeta^3}{\partial x^3} \\ &= \frac{\partial \zeta^1}{\partial z} \frac{\partial \zeta^1}{\partial z} + \frac{\partial \zeta^2}{\partial z} \frac{\partial \zeta^2}{\partial z} + \frac{\partial \zeta^3}{\partial z} \frac{\partial \zeta^3}{\partial z} \\ &= \left(\frac{\partial \zeta^1}{\partial z} \right)^2 + \left(\frac{\partial \zeta^2}{\partial z} \right)^2 + \left(\frac{\partial \zeta^3}{\partial z} \right)^2 \\ &= (0)^2 + (0)^2 + (1)^2 \\ &= 1\end{aligned}$$

Hence ds^2 in Cylindrical coordinates is

$$\begin{aligned}
 ds^2 &= g_{kl} dx^k dx^l \\
 &= g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + g_{33} (dx^3)^2 \\
 &= g_{11} (d\phi)^2 + g_{22} (dr)^2 + g_{33} (dz)^2 \\
 &= r^2 (d\phi)^2 + (dr)^2 + (dz)^2
 \end{aligned}$$

From the above we see that

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $\det(g) = r^2$. We are now ready to apply (1)

$$\begin{aligned}
 \nabla^2 &= \frac{1}{\sqrt{\det(g)}} \frac{\partial}{\partial x_i} \left(\frac{\sqrt{\det(g)}}{g_{ii}} \frac{\partial}{\partial x^i} \right) \\
 &= \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_1} \left(\frac{\sqrt{r^2}}{g_{11}} \frac{\partial}{\partial x^1} \right) + \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_2} \left(\frac{\sqrt{r^2}}{g_{22}} \frac{\partial}{\partial x^2} \right) + \frac{1}{\sqrt{r^2}} \frac{\partial}{\partial x_3} \left(\frac{\sqrt{r^2}}{g_{33}} \frac{\partial}{\partial x^3} \right) \\
 &= \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{1}{r} \frac{\partial}{\partial \phi} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial z} \left(r \frac{\partial}{\partial z} \right) \\
 &= \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \\
 &= \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \\
 &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\phi\phi} + u_{zz}
 \end{aligned}$$

4.1 References

1. Lecture notes, Physics 5041. UMN Spring 2019 by Professor Kapusta
2. Appendix A, Einstein's Theory, A rigorous introduction. By Gron and Naess. Springer publisher.