

transfer from the LEO orbit to the GEO orbit

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The problem: We are given 2 non co-planer circular orbits of different radius. For example, a LEO orbit that is inclined at an angle β to the plane of another, larger, circular orbit (for example, a GEO orbit).

We wish to transfer from the LEO orbit to the GEO orbit.

We must correct the plane inclination to be able to transfer to the desired GEO orbit.

There are 3 possible ways to achieve this¹:

1. Correct all of the plane inclination before performing a Hohmann transfer from LEO to GEO. In other words, all of the correction is made at the perigee of the Hohmann ellipse. This is the most fuel costly maneuver.
2. Perform a Hohmann transfer to move from the LEO orbit to a GEO orbit. Then apply all of the plane inclination correction at the apogee of the Hohmann elliptical orbit. This is less costly than above procedure, and is a common maneuver.
3. Apply a small and partial plane inclination correction (say angle α) at the perigee of the Hohmann orbit, then apply the remaining inclination correction (angle $\beta - \alpha$) at the apogee. This is the case we will solve for here to find the optimal amount of angle α . For $\alpha = 0$, this case becomes the same as case (2) above.

See the following diagram to illustrate the geometry of the 3rd case above.

¹We assume that the ratio of the radius of LEO orbit to radius of the GEO orbit is < 11.9 .

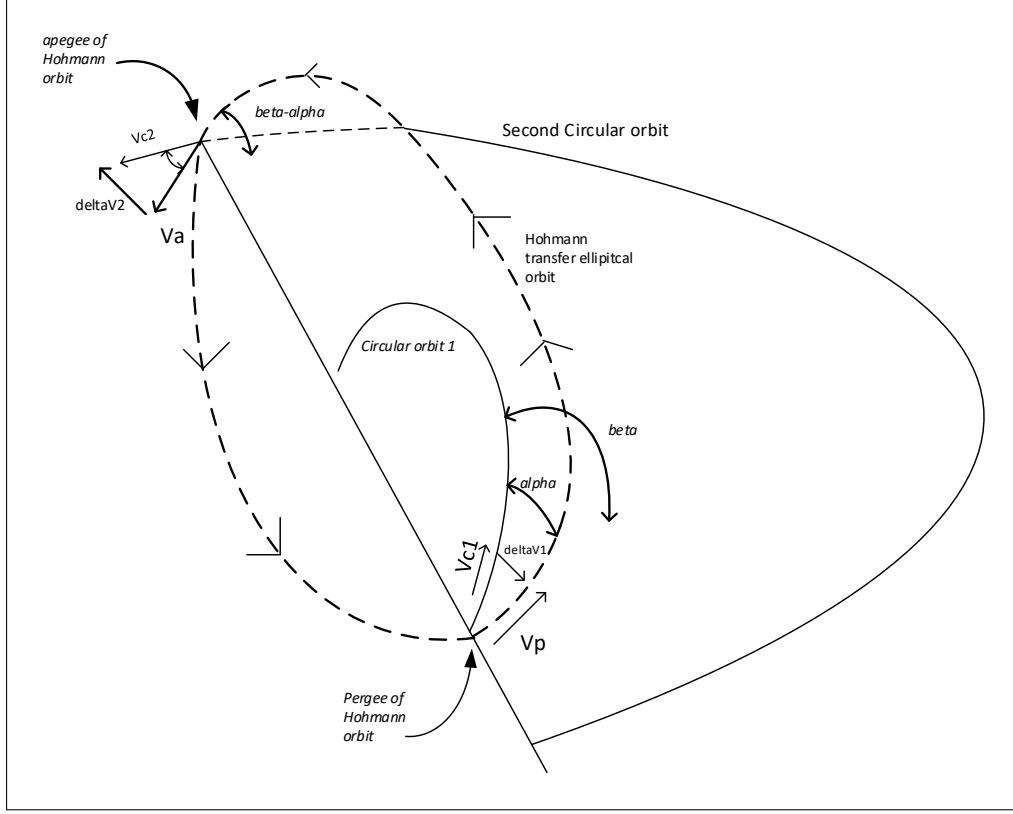


Figure 1: geometry of the 3rd case above

At the perigee of the Hohmann transfer, apply the law of the cosines to obtain

$$\Delta V_1 = \sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)}$$

$$\Delta V_2 = \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)}$$

$$\Delta V_{total} = \Delta V_1 + \Delta V_2$$

To find the minimum ΔV_{total} for a given α , take α as the independent variable, and minimize ΔV_{total} as a function of α .

Hence solve $\frac{\partial \Delta V_{total}}{\partial \alpha} = 0$

$$\text{Let } f(\alpha) = \Delta V_{total} = \sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)} + \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)}$$

$$\frac{\partial f}{\partial \alpha} = \frac{2V_{c1}V_p \sin(\alpha)}{\sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)}} - \frac{2V_{c2}V_a \sin(\beta - \alpha)}{\sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)}} = 0 \text{ for minimum}$$

so

$$\frac{2V_{c1}V_p \sin(\alpha) \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)} - 2V_{c2}V_a \sin(\beta - \alpha) \sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)}}{\sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)} \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)}} = 0$$

$$V_{c1}V_p \sin(\alpha) \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)} - V_{c2}V_a \sin(\beta - \alpha) \sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)} = 0 \quad (1)$$

let

$$F(\alpha) = V_{c1} V_p \sin(\alpha) \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)} - V_{c2}V_a \sin(\beta - \alpha) \sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)}$$

This is a non-linear equation in α . Solve for α using Newton root finding method.

$$F'(\alpha) = V_{c1} V_p \cos(\alpha) \sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)} + V_{c1} V_p \sin(\alpha) \left(\frac{-V_{c2}V_a \sin(\beta - \alpha)}{\sqrt{V_{c2}^2 + V_a^2 - 2V_{c2}V_a \cos(\beta - \alpha)}} \right) - \left[V_{c2}V_a \cos(\beta - \alpha) \sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)} + V_{c2}V_a \sin(\beta - \alpha) \frac{V_{c1}V_p \sin(\alpha)}{\sqrt{V_{c1}^2 + V_p^2 - 2V_{c1}V_p \cos(\alpha)}} \right]$$

To solve for α , I wrote a small MATLAB function that uses Newton root finding method to find the root of $F(\alpha)$ for a given r_1, r_2, β . where r_1 and r_2 are the radius of the smaller and the larger circular orbits respectively, and β is the angle (in degrees) of the inclination of the first circular orbit relative to the second circular orbit, measured positive clockwise. The function returns back the angle α for which ΔV_{total} is minimum. For an initial guess for α , and since α is expected to be small compared to β , I selected $\alpha_0 = 0.1 \beta$.

Apply Newton iterative root finding:

$$\alpha_{i+1} = \alpha_i - \frac{F(\alpha_i)}{F'(\alpha_i)}$$

The tricky part in this problem was finding a good initial guess for the root. When I selected α_0 to be 50% of β , Newton method did not converge. Selecting α_0 to be close to where one expects it to be (which is very small compared to β) worked and a root was found.

For the design project, we are given $\beta = 15^\circ$. Using this matlab function I found that $\alpha = 1.28891^\circ$ is the solution, for the given geometry and parameters. Hence, this is the angle I will use.